Univerzita Karlova v Praze
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# DIPLOMOVÁ PRÁCE

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# Moduly s minimální množinou generátorů

## Modules with a minimal generating set

Katedra algebry

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Moduly s minimální množinou generátorů

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Abstrakt: Minimální množinou generátorů modulu máme na mysli podmnožinu, která je generující, ale žádná její vlastní podmnožina modul negeneruje. Pro moduly, které nejsou konečně generované, nemusí minimální množina generátorů existovat. Moduly mající minimální množinu generátorů nazýváme slabě obázované. V této práci poskytneme úplnou charakterizaci slabě obázovaných modulů nad Dedekindovými obory. Jako aplikaci tohoto výsledku dokážeme, že třída slabě obázovaných modulů není uzavřena na extenze, a že komplement této třídy není uzavřen na konečnou direktní sumu. Také ukážeme příklad abelovské grupy, která je slabě obázovaná, právě když platí CH. Dále se zabýváme okruhy, nad kterými jsou všechny moduly slabě obázované. Dokážeme, že Baerův regulární okruh má tuto vlastnost, jedině pokud je polojednoduchý, a také že $\aleph_0$-noetherovský komutativní regulární semiartinovský okruh tuto vlastnost má. Poslední část textu se věnuje problému Nashiera a Nicholse - obsahuje každá množina generátorů libovolného modulu nad perfektním okruhem minimální množinu generátorů?

Klíčová slova: modul, minimální množina generátorů, slabá baze

Title: Modules with a minimal generating set

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Abstract: By a minimal generating set of a module we mean a subset which generates the module but any of its proper subsets does not. If the module is not finitely generated, an existence of a minimal generating set is not guaranteed. We say that a module is weakly based if it has a minimal generating set. In the presented thesis, we provide a characterization of weakly based modules over Dedekind domains. As an application of this, we show that the class of weakly based modules is not closed under extensions and the complement of this class is not closed under finite direct sums. Also, we show an example of an abelian group which is weakly based if and only if CH holds. Then we treat rings such that all modules are weakly based. We prove that a Baer regular ring has this property if and only if it is semisimple, and we show that any $\aleph_0$-noetherian commutative semiartinian ring has this property. Final part of the text concerns the problem of Nashier and Nichols - does any generating set of any module over a perfect ring contain a minimal generating set?

Keywords: module, minimal generating set, weak basis

# Contents

# List of Symbols

| | |
|---|---|
| $\aleph_\alpha, \omega_\alpha$ | The $\alpha$-th infinite cardinal number |
| $\mathrm{Ann}_R(M)$ | The $R$-annihilator of module $M$ |
| $\mathrm{Ann}_R(x)$ | The $R$-annihilator of element $x$ |
| $\mathrm{card}(X)$ | The cardinality of set $X$ |
| $\mathrm{codim}_k(W)$ | The codimension of vector subspace over division ring $k$ |
| $\dim_k(V)$ | The dimension of vector space $V$ over division ring $k$ |
| $\mathrm{Ext}^1_{\mathrm{R}}(M, N)$ | See [Rot08, p. 186 and §7] |
| $\mathrm{gen}(A)$ | The minimum cardinality of a generating set of algebra $A$ |
| $\mathrm{Hom}_{\mathrm{R}}(M, N)$ | The Hom-bifunctor over $R$ applied to $R$-modules $M$ and $N$ |
| $\mathrm{Ker}(\varphi)$ | The kernel of module homomorphism $\varphi$ |
| $\mathrm{len}(S)$ | The length of semisimple module $S$ |
| $\mathbb{N}$ | The set $\{1, 2, 3, \ldots\}$ |
| $\mathbb{Q}$ | The abelian group of rational numbers |
| $\mathbb{Z}_n$ | The abelian group quotient $\mathbb{Z}/n\mathbb{Z}$ |
| $\mathbb{Z}_{(p)}$ | Localization of $\mathbb{Z}$ at $p\mathbb{Z}$ |
| $\mathbb{Z}_{p^\infty}$ | The Prüfer $p$-group |
| Mod-$R$ | The category of left modules over ring $R$ |
| m-Spec$(R)$ | The maximal spectrum of a commutative ring $R$ |
| $\omega$ | The smallest infinite ordinal number |
| $\oplus$ | Direct sum of modules |
| $\otimes_R$ | Tensor product of modules over ring $R$ |
| $\phi M$ | The torsion-free quotient $M/\tau M$ |
| $\mathrm{Soc}(R)$ | The left socle of ring $R$ |
| $\mathrm{Soc}_\alpha(R)$ | The $\alpha$-th left socle of ring $R$ |

| | |
|---|---|
| $\mathrm{Span}(X)$ | The smallest subalgebra containing set $X$ |
| $\mathrm{Spec}(M,<)$ | See Chapter 3 |
| $\mathrm{Spec}(M,=)$ | See Chapter 3 |
| $\mathrm{Spec}(M,x)$ | See Chapter 3 |
| $\mathrm{Spec}(R)$ | The (Zariski) spectrum of commutative ring $R$ |
| $\tau M$ | The maximal torsion submodule of $M$ |
| $f[A]$ | The $f$-image of set $A$ |
| $f^{-1}[B]$ | The $f$-preimage of set $B$ |
| $L(R)$ | The left Loewy length of semiartinian ring $R$ |
| $M^A$ | Direct product of $A$ copies of module $M$ |
| $M^{(A)}$ | Direct sum of $A$ copies of module $M$ |
| $M_P$ | Localization of module $M$ at prime ideal $P$ |

# Chapter 1

# Introduction

Given a universal algebra $A$, we can ask whether there is a set of generators $X$ of $A$ such that any proper subset of $X$ no longer generates $A$. Let us call such generating set a *minimal generating set* (the minimality is meant with respect to the set-theoretic inclusion). It is then a natural task to find out which algebras possess a minimal generating set. Of course, we can always find a generating set of *minimal cardinality*. This already shows that any algebra admitting a finite set of generators has a minimal generating set. On the other hand, the Prüfer $p$-group is an example of an abelian group which has no minimal generating sets (see Example 3.3), while any free basis of a free abelian group of any rank is a minimal generating set.

In this thesis we confine ourselves to the case of modules over rings. For these algebras, the notion of a minimal generating set is connected to a certain generalization of linear independence, which we call *weak independence*. This concept gives an alternative description of minimal generating sets of modules and it is treated, together with presenting our notation, terminology, and some basic observations, in Chapter 2. In the rest of the Introduction, we shall prefer to address minimal generating sets of modules as *weak bases* instead.

Minimal generating sets of groups, rings and fields were studied in [HR07]. More detailed study of abelian groups having a minimal generating set was done by the advisor in [Rů10]; this paper was the ground motivation for our research. In many cases, determining whether some module (abelian group in particular) has a weak basis is not trivial, at least until one develops some tools for this task. To give some motivational examples: the direct sum of $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p^n}$ and the Prüfer $p$-group; $\prod_{n\in\mathbb{N}} \mathbb{Z}_{p^n}$, $\mathbb{Z}^\omega$, and infinite products in general; torsion-free abelian groups of rank 1; and others. For this reason, we tried to interlace the text with a reasonable amount of examples. Another thing we found motivational was the study of the closure properties of the class of weakly based modules over some ring. Two questions arose - is any extension of weakly based modules also weakly based ? And can the direct sum of two non-weakly based module be weakly based ? The answers ("No, but yes if the extension is pure" and "Yes", respectively) turned out to be the converses of what we anticipated; we were not able to find counterexamples until we had (large parts of) the characterization of weakly based abelian groups.

In Chapter 3, we present some basic results concerning weak bases of modules, and mainly, we prove several sufficient (Lemma 3.11) and necessary (Lemma 3.17, Corollaries 3.18, 3.20, and 3.22) criteria for a module to possess a weak basis (we call such modules *weakly based*). Those will readily give a module-theoretic characterization of weakly based modules over a simple ring (Theorem 3.23).

Chapter 4 concerns weakly based modules over Dedekind domains. We present a complete and usable characterization of such modules (Theorem 4.44). We show some applications of this result, namely we (dis)prove several closure properties of the class of weakly based modules (Theorem 4.52), and we provide an example of an abelian group such that the existence of its weak basis is undecidable in ZFC (Example 4.46). Chapters 3 and 4 consist mainly of joint work from preprints [HRa], [HRb].

In the fifth Chapter we define a *strongly weakly based* module as a module such that any generating set contains a weak basis. We comment strongly weakly based modules over Dedekind domains and characterize them in the local case (Theorem 5.27). Also, we show that a free module of infinite rank over a non-perfect ring is not strongly weakly based (Proposition 5.22). The main topic of this Chapter are rings, such that all modules are weakly based or even strongly weakly based (we say that such ring has the left (strong) weak basis property). In treating the first task, we reprove the known fact that all (left) perfect rings have the (left) weak basis property, generalize the example of a non-perfect ring with this property from [NN91a], and show that a Baer regular ring has the weak basis property if and only if it is semisimple (Proposition 5.18). We also prove that any $\aleph_0$-noetherian commutative regular semiartinian ring has the weak basis property (Corollary 5.13), showing an example of such ring of an infinite Loewy length. Concerning the strong weak basis property, we present the question of Nashier and Nichols - do the rings with the strong weak basis property coincide with perfect rings ? We provide a positive answer only to a very special case.

The last Chapter is reserved for a short list of open problems.

# Chapter 2

# Preliminaries

## 2.1 Notation and terminology

Any ordinal number (including the natural numbers) will be viewed as a set of smaller ordinals. By $\aleph_\alpha$ or $\omega_\alpha$ we denote the $\alpha$-th infinite cardinal. Instead of $\omega_0$ we write just $\omega$. Each ordinal number $\alpha$ can be written in a unique way as $\alpha = \lambda + n$ where $\lambda$ is a limit ordinal and $n \in \omega$. We say that $\alpha$ is *even* if $n$ is even.

Let $X, Y$ be sets and $f : X \to Y$ a function. If $A$ is a subset of $X$, we write $f[A]$ for the $f$-image of $A$ in $Y$. Similarly, if $B$ is a subset of $Y$, we denote by $f^{-1}[B]$ the $f$-preimage of $B$ in $X$.

Let $A$ be a (universal) algebra and $X$ a non-empty subset of $A$. We denote by $\mathrm{Span}(X)$ the smallest subalgebra of $A$ containing set $X$, i.e. the intersection of all subalgebras $B$ of $A$ such that $X \subseteq B$. We denote by $\mathrm{gen}(A)$ the minimum of the set $\{\mathrm{card}(X) \mid X \subseteq A, \mathrm{Span}(X) = A\}$, i.e. the smallest cardinality among the generating sets of $A$. We say that $A$ is *$\varkappa$-generated* if $\mathrm{gen}(A) \leq \varkappa$ and that $A$ is *finitely generated* if $\mathrm{gen}(A) < \aleph_0$.

Any ring considered is always associative and unital (but not necessarily commutative). By $R$-module (or just module) we mean a left unital module over ring $R$. Given a set $A$ and an $R$-module $M$, we use the symbol $M^A$ for the product module $\prod_{a \in A} M$ and $M^{(A)}$ for the direct sum $\bigoplus_{a \in A} M$.

Let $M, N$ be $R$-modules and $\varphi : M \to N$ a homomorphism. We say that a subset $X$ of $M$ *lifts* a subset $Y$ of $N$ *via* $\varphi$ if $\varphi_{\restriction X}$ is a bijection of $X$ onto $Y$. If $K$ is a submodule of $M$ and $Z$ is a subset of $M/K$, we say that $X$ *lifts* $Z$ *over* $K$ if $X$ lifts $Z$ via the canonical projection of $M$ onto $M/K$.

Let $R$ be an integral domain and $M$ an $R$-module. We denote by $\tau M$ the maximal torsion submodule of $M$ and by $\phi M$ the torsion-free quotient $M/\tau M$. By $\mathbb{Z}_n$, for some $n \in \mathbb{N}$, we mean the abelian group quotient $\mathbb{Z}/n\mathbb{Z}$.

Let $R$ be a commutative ring and $I$ an ideal. Module $M$ is *$I$-divisible* provided that $M = IM$. If $R$ is an integral domain and $M = rM$ for any non-zero $r \in R$, then we say that $M$ is *divisible*.

Let $R$ be a commutative ring and $P$ a prime ideal. We denote by $R_P$ the localization of $R$ at $P$. To avoid confusion, we will denote the localization of $\mathbb{Z}$ at $p\mathbb{Z}$ by $\mathbb{Z}_{(p)}$. Furthermore, if $M$ is an $R$-module, we denote by $M_P$ the localization of $M$ at $P$, that is, $M_P = M \otimes_R R_P$.

## 2.2 Basic concept

**Definition.** Let $A$ be an algebra and $X$ a non-empty subset of $A$. We say that:

- $X$ is a *generating set* of $A$ ($X$ generates $A$) provided that $\mathrm{Span}(X) = A$,
- $X$ is $\mathcal{S}$*-independent* in $A$ provided that for each $x \in X$, $x \notin \mathrm{Span}(X \setminus \{x\})$ (following [Grä68, pages 26 and 46]),
- $X$ is a *minimal generating set* of $A$ provided that $X$ is $\mathcal{S}$-independent in $A$ and $X$ generates $A$.

It is easily seen that minimal generating sets of an algebra are exactly the generating sets minimal with respect to the ordering by set inclusion. In other words, a generating set is a minimal generating set if and only if any of its proper subset no longer generates the algebra.

The notions of a minimal generating set and of a generating set of minimal cardinality should not be confused. While any algebra has a generating set of minimal cardinality, existence of a minimal generating set is not guaranteed in general (which will be demonstrated throughout this thesis for modules over rings). Also, a minimal generating set of an algebra $A$ can be of cardinality greater than $\mathrm{gen}(A)$ (the set $\{2, 3\}$ is a minimal generating set of the abelian group $\mathbb{Z}$, but $\mathrm{gen}(\mathbb{Z}) = 1$).

For finitely generated algebras the question of existence of a minimal generating sets is trivial:

**Lemma 2.1.** *Any finitely generated algebra possesses a minimal generating set.*

*Proof.* Let $A$ be a finitely generated algebra and $X$ a generating subset of $A$ of cardinality $\mathrm{gen}(A)$. Since $\mathrm{gen}(A) < \aleph_0$, any proper subset of $X$ has cardinality strictly smaller than $\mathrm{gen}(A)$. Hence, any proper subset of $A$ is not a generating set of $A$, and thus $X$ is a minimal generating set of $A$. $\square$

From now on we confine ourselves to the case of modules over rings. We characterize the notion of $\mathcal{S}$-independent subset of an $R$-module as a weakened version of the standard $R$-linear independence.

**Definition.** Let $R$ be a ring, $M$ a left $R$-module and $X$ a subset of $M$. We say that:

- $X$ is *$R$-linearly independent* if for any $n \in \omega$, any choice of pairwise distinct elements $x_0, \dots, x_n \in X$, and any $r_0, \dots, r_n \in R$ the relation
$$r_0 x_0 + \cdots + r_n x_n = 0$$
implies that $r_i = 0$ for each $i = 0, \dots, n$.
- $X$ is a *(free) basis* of $M$ if $X$ is $R$-linearly independent and generates $M$.
- $X$ is *weakly independent* if for any $n \in \omega$, any choice of pairwise distinct elements $x_0, \dots, x_n \in X$, and any $r_0, \dots, r_n \in R$ the relation
$$r_0 x_0 + \cdots + r_n x_n = 0$$
implies that $r_i$ is a non-unit (i.e. not invertible in $R$) for each $i = 0, \dots, n$.

- $X$ is a *weak basis* of $M$ if $X$ is weakly independent and generates $M$. We say that an $R$-module $M$ is *weakly based* if $M$ possesses some weak basis.

Note that any $R$-linearly independent subset of a module is *a fortiori* weakly independent. If $R$ is a division ring, then the notions of weak independence and $R$-linear independence coincide because any non-zero element of $R$ is a unit.

In case of modules a span of an empty set makes good sense ($\operatorname{Span}(\emptyset) = 0$) and we can thus extend the definition of $\mathcal{S}$-independence and minimal generating set to all subsets of a module.

**Lemma 2.2.** *Let $R$ be a ring and $M$ an (left) $R$-module. A subset of $M$ is $\mathcal{S}$-independent if and only if it is weakly independent.*

*Proof.* ($\Rightarrow$) Let $X$ be an $\mathcal{S}$-independent subset of $M$. Suppose for a contradiction that there are $n \in \omega$, pairwise distinct $x_0, \dots, x_n \in X$, and $r_0, \dots, r_n$ such that

$$r_0 x_0 + \dots + r_n x_n = 0 \tag{2.1}$$

and $r_0$ is a unit. Let $s$ be an element of $R$ such that $sr_0 = 1$. Multiplying (2.1) by $s$ we get that $x_0 \in \operatorname{Span}(\{x_1, \cdots, x_n\})$. But this is a contradiction with $\mathcal{S}$-independence of $X$.

($\Leftarrow$) Let $X$ be a weakly independent subset of $M$. Towards a contradiction, suppose that there is $x \in X$ such that $x_0 \in \operatorname{Span}(X \setminus \{x\})$. It follows that there are $n \in \omega$, $x_1, \dots, x_n \in X$ pairwise distinct, and $r_1, \dots, r_n \in R$ such that $x_0 = r_0 x_0 + \dots + r_n x_n$. This is a contradiction with weak independence of $X$ because 1 is, of course, a unit of $R$. $\square$

In the rest of this thesis we will prefer the terminology of weak independence (weak bases) over $\mathcal{S}$-independence (minimal generating sets). The following very simple observations will be used throughout this thesis.

**Lemma 2.3.** *Let $M, N$ be $R$-modules and $\varphi : M \to N$ a homomorphism. Then any subset $X$ of $M$ which lifts a weakly independent subset of $N$ via $\varphi$ is itself weakly independent.*

*Proof.* Towards a contradiction, suppose that there is a vanishing linear combination $x_0 + r_1 x_1 + \dots + r_n x_n = 0$ of pairwise distinct elements $x_0, \dots, x_n$ of $X$ witnessing the weak dependency of $X$. Applying $\varphi$ on this equality, we get $\varphi(x_0) + r_1\varphi(x_1) + \dots + r_n\varphi(x_n) = 0$. Since $\varphi|_X$ is a bijection, we arrived to a contradiction with $\varphi[X]$ being a weakly independent set. $\square$

**Lemma 2.4.** *Let $R$ be a ring and $M$ an $R$-module. Suppose that we have a collection $\{X_i, i \in I\}$ of subsets of $M$ such that for each $i \in I$ the set $X_i$ lifts a weakly independent set over $\operatorname{Span}(\bigcup_{j \in I\setminus\{i\}} X_j)$. Then $X = \bigcup_{i \in I} X_i$ is a weakly independent subset of $M$.*

*Proof.* Towards a contradiction, suppose that there is $x \in X$ such that $x \in \operatorname{Span}(X \setminus \{x\})$. Let $i \in I$ be such that $x \in X_i$. Denote by $\bar{X}_i$ the image of $X_i$ in the canonical projection onto $M/\operatorname{Span}(\bigcup_{j \in I\setminus\{i\}} X_j)$. Denoting $\bar{x}$ the image of $x$ under the same projection, we infer that $\bar{x} \in \operatorname{Span}(\bar{X}_i \setminus \{\bar{x}\})$, a contradiction with $\bar{X}_i$ being weakly independent. $\square$

# Chapter 3

# General rings

In the first part of this Chapter, we gather several simple facts about weak bases of modules valid for general rings. Then we prove some general sufficient and necessary conditions for a module to be weakly based. Those will not provide a complete characterization, but will serve as a basis for investigation of weakly based modules over Dedekind domains. We also prove a full characterization of weakly based modules for a certain class of mostly non-commutative rings, including simple rings.

We start with a basic necessary condition which will provide us with many examples of modules that are not weakly based:

**Lemma 3.1.** *Let $R$ be a ring and $M$ a non-zero $R$-module. If $M$ is weakly based, then $M$ has a maximal submodule.*

*Proof.* Suppose that $X$ is a weak basis of $M$. Since $M$ is non-zero, we have that $X$ is non-empty. Pick an element $x$ of $X$ and put $C_x = M/\operatorname{Span}(X \setminus \{x\})$. It follows from the weak independence of $X$ that $C_x$ is a non-zero cyclic module. We have that $M$ has a projection onto a non-zero cyclic $R$-module. Standard evocation of Zorn's lemma shows that $C_x$ has a maximal submodule, and thus also $M$ has a maximal submodule. □

**Corollary 3.2.** *Let $R$ be a non-field integral domain and $M$ a non-zero divisible $R$-module. Then $M$ is not weakly based.*

*Proof.* By Lemma 3.1, it suffices to show that $M$ does not have a maximal submodule. Suppose that there is a maximal submodule of $M$. Then there is a projection of $M$ onto a simple module $V$. It is easily seen that a factor of a divisible module is also divisible, thus $V$ is divisible. On the other hand, $V$ is isomorphic to $R/P$ for some maximal ideal $P$ of $R$. Since $R$ is not a field, $P$ is non-zero. It follows that there is non-zero $p \in P$ such that $pV = 0$, a contradiction with $V$ being divisible and non-zero. □

**Example 3.3.** Corollary 3.2 shows that both the abelian group of rational numbers $\mathbb{Q}$ and the Prüfer group $\mathbb{Z}_{p^\infty}$ are examples of $\mathbb{Z}$-modules which are not weakly based.

Next we prove that the class of weakly based modules is always closed under arbitrary direct sums (many other closure properties of the class of weakly based modules do not hold in general, see Theorem 4.52).

**Lemma 3.4.** *Let $R$ be a ring, $I$ be a set and $M_\iota$ be a weakly based $R$-module for each $\iota \in I$. Then $\bigoplus_{\iota\in I} M_\iota$ is a weakly based $R$-module.*

*Proof.* Pick a weak basis $X_\iota$ of $M_\iota$ for each $\iota \in I$. Then $X = \bigcup_{\iota\in I} X_\iota$ is obviously a weak basis of $\bigoplus_{\iota\in I} M_\iota$. □

**Lemma 3.5.** (Nakayama) *Let $R$ be a ring and denote by $J$ the (Jacobson) radical of $R$. Let $M$ be a weakly based $R$-module. Then $M = JM$ implies that $M = 0$.*

*Proof.* Since $M$ is weakly based, there is a weak basis $X$ of $M$. Suppose that $M = JM$ and choose an element $x \in X$. We know that $x \in JM = J\operatorname{Span}(X)$, and thus there are $j_0, \dots, j_n \in J$ and $x_0, \dots, x_n \in X$ pairwise distinct such that $x = j_0x_0 + \cdots + j_nx_n$. If $x \notin \{x_0, \dots, x_n\}$, we have a contradiction with the weak independence of $X$, therefore $x = x_i$ for some $i = 0, \dots, n$. But then $(1 - j_i)x_i = j_0x_0 + \cdots + j_{i-1}x_{i-1} + j_{i+1}x_{i+1} + \cdots + j_nx_n$ where $x_0, \dots, x_n$ are pairwise distinct. Since $j_i$ lies in the radical, we have that $(1 - j_i)$ is a unit of $R$ and we again obtain a contradiction with weak independence of $X$. It follows that $X$ does not contain any non-zero element, and thus $M = 0$. □

**Lemma 3.6.** *Let $R$ be a ring and $M$ an $R$-module. If $X$ is weakly independent subset of $M$, then $X$ lifts a weakly independent set over $JM$ where $J$ is the (Jacobson) radical of $R$.*

*Proof.* Towards a contradiction, suppose that there is $x \in X$ such that $x \in \operatorname{Span}(X \setminus \{x\}) + JM$. Putting $C_x = M/\operatorname{Span}(X \setminus \{x\})$, we have that $C_x = JC_x$. The module $C_x$ is finitely generated, and thus weakly based. It follows from Lemma 3.5 that $C_x = 0$. But then $x \in \operatorname{Span}(X \setminus \{x\})$ which is a contradiction with $X$ being weakly independent. □

**Example 3.7.** There is a projective module which is not weakly based.

*Proof.* It is shown in [GS84] that there exists (necessarily non-commutative) ring $R$ such that there is an infinitely generated projective module $P$ with $P/JP$ being finitely generated (where $J$ is the radical of $R$). It follows from Lemma 3.6 that $P$ is not weakly based, since a finitely generated module does not admit an infinite weak basis (by similar argumentation as in the proof of Lemma 3.8). □

**Lemma 3.8.** *Let $R$ be a ring and $M$ an infinitely generated $R$-module. Then any weak basis of $M$ has cardinality* $\operatorname{gen}(M)$*.*

*Proof.* Let $X$ be a weak basis of $M$ and $Y$ some generating set of $M$ of cardinality $\operatorname{gen}(M)$. Since $\operatorname{Span}(X) = M$, there is for each $y \in Y$ a finite subset $F_y$ of $X$ such that $y \in \operatorname{Span}(F_y)$. Put $X' = \bigcup_{y\in Y} F_y$. We have $M = \operatorname{Span}(Y) \subset \operatorname{Span}(X')$, and thus $X = X'$. Indeed, otherwise, for any $x \in X \setminus X'$ we would have $x \in M = \operatorname{Span}(X')$, a contradiction with weak independence of $X$. Hence, $\operatorname{card}(X) = \operatorname{card}(\bigcup_{y\in Y} F_y) \le \aleph_0 \cdot \operatorname{card}(Y) = \operatorname{card}(Y) = \operatorname{gen}(M)$. □

Lemma 3.8 does not in general apply to finitely generated modules. For example, the abelian group $\mathbb{Z}$ possesses weak bases $\{1\}$ and $\{2, 3\}$[1]. However, if the ring is local, then the cardinality of minimal generating sets becomes invariant even for finitely generated modules:

[1]More generally, as shown in [HR07, p. 4], there is a weak basis of $\mathbb{Z}$ of cardinality $n$ for any $n \in \mathbb{N}$.

**Lemma 3.9.** *A ring $R$ is local if and only if for each $R$-module $M$ all weak bases of $M$ have cardinality* $\operatorname{gen}(M)$.

*Proof.* ($\Rightarrow$) Let $R$ be a local ring, $M$ an $R$-module and $X$ a weak basis of $M$. Denote by $J$ the radical of $R$. By [AF93, Proposition 15.15], $R/J$ is a division ring. It follows from Lemma 3.6 that $X$ lifts a weak basis over $JM$. Denote by $\bar{X}$ the image of $X$ in projection of $M$ onto $M/JM$. We have that $\operatorname{card}(X) = \operatorname{card}(\bar{X})$. Since $\bar{X}$ is a weak basis of the $R/J$-module $M/JM$ and $R/J$ is a division ring, we have that $\bar{X}$ is a basis of the vector space $M/JM$, and we are done by linear algebra (or alternatively, it is well known that all modules over a division ring are free and that division rings satisfy the invariant basis property).

($\Leftarrow$) Suppose that $R$ is not local and choose two distinct left maximal ideals $P, Q$ of $R$. Pick $p \in P$ and $q \in Q$ such that $p+q = 1$. We have that $\operatorname{Span}(\{p, q\}) = R$ and no proper subset of $\{p, q\}$ generates $R$, so $\{p, q\}$ is a weak basis of $R$. But $R$ is a cyclic module, thus $\operatorname{gen}(R) = 1$. □

**Lemma 3.10.** *Let $R$ be a local ring, $F$ a free $R$-module and $X$ a subset of $F$. Then $X$ is a weak basis if and only if $X$ is a free basis.*

*Proof.* ($\Rightarrow$) Denote by $J$ the maximal ideal of $R$. Let $G = R^{(X)}$ be a free $R$-module on set $X$ and let $\pi : G \to F$ be the projection extending identity on $X$. Since $F$ is projective, $\pi$ splits, and thus $G = F \oplus K$ where $K = \operatorname{Ker}(\pi)$. Let $k$ be an element of $K \subseteq G$. Because $G = \operatorname{Span}(X)$, there are (unique) elements $x_0, \ldots, x_n \in X$ pairwise distinct and $r_0, \ldots, r_n \in R$ such that $k = r_0x_0 + \cdots + r_nx_n$. Then we have $0 = \pi(k) = r_0x_0 + \cdots + r_nx_n$ in $F$, and therefore $r_i$ is a non-unit for all $i = 0, \ldots, n$ by the weak independence of $X$. It follows that $K \subseteq JG = JR^{(X)}$. Since $K$ is a direct summand of $G$, we have $K = JK$. On the other hand, $K$ is a direct summand of $G$, thus a projective module, and by [Pas04, Theorem 10.8], $K$ is a free module. Together we have that $K = 0$, $\pi$ is an isomorphism, and therefore $X$ is a free basis of $F$.

($\Leftarrow$) Obvious. □

Lemma 3.1 says that a non-zero weakly based module has a projection onto a simple module. We strengthen this necessary condition significantly by showing that any weakly based module has a projection onto a direct sum of $\operatorname{gen}(M)$ simple modules. Before that we introduce some convenient notation and terminology.

**Definition.** Let $R$ be a ring and $S$ an $R$-module. We say that $S$ is *semisimple* if $S$ is isomorphic to a direct sum of simple $R$-modules. We define the *length* of $S$, $\operatorname{len}(S)$, to be the count of simple modules in some decomposition of $S$ as a direct sum of simple modules (this is an invariant of $S$ by [Bou58, §3 Théorème 2]).

**Definition.** Let $R$ be a commutative ring.

- We denote by $\text{m-Spec}(R)$ the set of all maximal ideals of $R$.
- Suppose that $X$ is a weak basis of an $R$-module $M$ and let

$$C_x = M/\operatorname{Span}(X \setminus \{x\})$$

for each $x \in X$. We define

$$\operatorname{Spec}(M, x) = \{P \in \text{m-Spec}(R) \mid PC_x \neq C_x\}.$$

Note that since $C_x$ is a non-zero cyclic module, there is a projection of $C_x$ onto a simple module $V$. Because $R$ is commutative, $\operatorname{Ann}(V)$ is a maximal ideal, and we have that $\operatorname{Ann}(V) \in \operatorname{Spec}(M, x)$. In particular, $\operatorname{Spec}(M, x)$ is non-empty for any $x \in X$.

The following lemma provides quite a strong necessary condition for a module to be weakly based. The second part of the lemma will play an important role in characterizing weakly based modules over Dedekind domains.

**Lemma 3.11.** *Let $R$ be a ring and $M$ an $R$-module. Suppose that $M$ has a weak basis $X$. Then:*

1. *There is a projection of $M$ onto a semisimple $R$-module $S$ with $\operatorname{len}(S) = \operatorname{card}(X)$.*
2. *Suppose that $R$ is commutative and choose a maximal ideal $P_x \in \operatorname{Spec}(M, x)$ for each $x \in X$. Then there is a projection of $M$ onto $S = \bigoplus_{x \in X} R/P_x$.*

*Proof.* Again we set $C_x = M/\operatorname{Span}(X \setminus \{x\})$ for each $x \in X$. We denote the canonical projection of $M$ onto $C_x$ by $\varphi_x$. Let $\varphi : M \to \prod_{x \in X} C_x$ be a product of maps $\varphi_x$ over $X$. It follows directly from the definition of $\varphi_x$ and from weak independence of set $X$ that $\varphi_x(y) = 0$ for any distinct elements $x, y \in X$. Also, $\operatorname{Span}(\{\varphi(x)\}) = C_x$ for each $x \in X$. From this we infer that

$$\varphi[M] = \varphi[\operatorname{Span}(X)] = \operatorname{Span}(\varphi[X]) = \sum_{x \in X} \operatorname{Span}(\{\varphi(x)\}) = \bigoplus_{x \in X} C_x.$$

We found a projection of $M$ onto a direct sum of $\operatorname{card}(X)$ cyclic modules, and hence there is a projection of $M$ onto a direct sum of $\operatorname{card}(X)$ simple modules.

The second part follows from the fact that $C_x/P_xC_x \simeq R/P_x$ if $R$ is commutative. $\square$

**Remark 3.12.** Let $R$ be a commutative ring, $M$ an $R$-module and $P \in \text{m-Spec}(R)$. Then $M/PM$ is naturally an $R/P$-module, and thus a vector space over $R/P$. We denote the dimension of $M/PM$ over $R/P$ by $\dim_{R/P}(M/PM)$, or even $\dim(M/PM)$ if no confusion can arise.

Let $S$ be a semisimple $R$-module. Since $R$ is commutative, any simple $R$-module is of form $R/P$ for some $P \in \text{m-Spec}(R)$. In particular, any simple $R$-module $V$ is isomorphic to $R/\operatorname{Ann}(V)$. It follows that $S \simeq \bigoplus_{P \in \text{m-Spec}\, R} S/PS$. Notably, $\operatorname{len}(S) = \sum_{P \in \text{m-Spec}(R)} \dim(S/PS)$. Note that this does not hold for general rings. Indeed, if $P$ is a left maximal ideal which is not two-sided, then there is a projection $R \to R/P$, while $R/PR = 0$.

**Definition.** Let $R$ be a commutative ring and $M$ an $R$-module. In the spirit of Remark 3.12, we define:

$$\operatorname{Spec}(M, =) = \{P \in \text{m-Spec}(R) \mid \dim(M/PM) = \operatorname{gen}(M)\},$$

$$\operatorname{Spec}(M,<) = \{P \in \text{m-Spec}(R) \mid 0 < \dim(M/PM) < \operatorname{gen}(M)\}.$$

We infer that if a module over a commutative ring is weakly based, it is in some sense not “too divisible”.

**Corollary 3.13.** *Let $R$ be a commutative ring and $M$ a weakly based $R$-module. Then*

$$\sum_{P \in \text{m-Spec}(R)} \dim_{R/P}(M/PM) \geq \operatorname{gen}(M). \tag{3.1}$$

*Proof.* Since $M$ is weakly based, Lemma 3.11 provides us with a projection of $M$ onto a semisimple module $S$ with $\operatorname{len}(S) \geq \operatorname{gen}(M)$. For each $P \in \text{m-Spec}(R)$ the projection from $M$ onto $S/PS$ factors through $M/PM$. It follows that $S/PS$ is isomorphic to an $R/P$-subspace of $M/PM$, and thus $\dim(M/PM) \geq \dim(S/PS)$. Since any simple $R$-module is isomorphic to $R/P$ for some $P \in \text{m-Spec}(R)$, we infer that

$$\sum_{P \in \text{m-Spec}(R)} \dim(M/PM) \geq \sum_{P \in \text{m-Spec}\, R} \dim(S/PS) = \operatorname{len}(S) = \operatorname{gen}(M).$$

□

The following set-theoretic concept will be useful in our examples.

**Definition.** Let $X$ be an infinite set. We say that a system $\mathcal{A}$ of subsets of $X$ is *almost disjoint* if for each $Y \in \mathcal{A}$ we have that $\operatorname{card}(Y) = \operatorname{card}(X)$ and for any $Y' \in \mathcal{A}$ different than $Y$ the cardinality of the intersection of $Y$ with $Y'$ is strictly smaller than $\operatorname{card}(X)$.

**Lemma 3.14.** *Let $X$ be an infinite set of regular cardinality. Then there is an almost disjoint system $\mathcal{A}$ of subsets of $X$ such that* $\operatorname{card}(\mathcal{A}) > \operatorname{card}(X)$.

*Proof.* Let $\mathcal{P}$ be a partition of $X$ into $\operatorname{card}(X)$ disjoint sets of cardinality $\operatorname{card}(X)$. It is easy to see that the set of all almost disjoint systems on $X$ containing $\mathcal{P}$ is inductive, therefore we can call forth Zorn's Lemma in order to get a maximal almost disjoint system $\mathcal{A}$ on $X$ containing $\mathcal{P}$. Suppose that $\operatorname{card}(\mathcal{A}) = \operatorname{card}(X)$. Let $\mathcal{A} = \{Y_\alpha \mid \alpha < \operatorname{card}(X)\}$ be a well-ordering of $\mathcal{A}$. We can construct by induction a sequence of elements $Y = \{y_\alpha \mid \alpha < \operatorname{card}(X)\}$ such that $y_\alpha \in Y_\alpha \setminus \bigcup_{\beta<\alpha} Y_\beta$. Indeed, regularity of $\operatorname{card}(X)$ ensures that

$$\operatorname{card}\Big(\bigcup_{\beta<\alpha}(Y_\alpha \cap Y_\beta)\Big) < \operatorname{card}(X) = \operatorname{card}(Y_\alpha).$$

It follows that $Y$ is not found in $\mathcal{A}$, but $\mathcal{A} \cup \{Y\}$ is almost disjoint. This contradicts the maximality of $\mathcal{A}$. □

**Example 3.15.** Let $R$ be a commutative noetherian ring such that m-Spec$(R)$ is a of some infinite regular cardinality. Then the $R$-module $H = \prod_{P \in \text{m-Spec}(R)} R/P$ is not weakly based.

*Proof.* Let us identify elements of $H$ with sequences of form $\mathbf{x} = (x_P + P)_{P \in \text{m-Spec}(R)}$ where $x_P \in R$.

**Claim 3.16.** *Let $Q$ be a maximal ideal of $R$. We claim that* $\dim(H/QH) = 1$.

*Proof of Claim 3.16.* It is enough to show that

$$QH = \{\mathbf{x} = (x_P + P)_{P \in \text{m-Spec}(R)} \mid x_P \in Q\}.$$

The inclusion of $QH$ to the right hand side set is clear. Suppose that $\mathbf{x} = (x_P + P)_{P \in \text{m-Spec}(R)}$ is an element of $H$ such that $x_P \in Q$ for all $P \in \text{m-Spec}(R)$. We need to show that $\mathbf{x} \in QM$.

Assume that we are able to find an element $q \in Q$ such that $q \notin P$ for any $P \in \text{m-Spec}(R), P \neq Q$. Then for each $P \in \text{m-Spec}(R) \setminus \{Q\}$, there is $y_P \in R$ such that $qy_P + P = x_P$. Put $y_Q = 0$ and $\mathbf{y} = (y_P + P)_{P \in \text{m-Spec}(R)}$. It follows that $q\mathbf{y} = \mathbf{x}$, and thus $\mathbf{x} \in QH$.

It remains to show that $Q \setminus \bigcup\{\text{m-Spec}(R) \setminus \{Q\}\}$ is non-empty. Suppose for a contradiction that there is $\mathcal{J} \subseteq \text{m-Spec}(R) \setminus \{Q\}$ such that $Q \subseteq \bigcup \mathcal{J}$. Because $R$ is noetherian, $Q$ is finitely generated, and thus there is a finite subset $\mathcal{F}$ of $\mathcal{J}$ such that $Q \subseteq \bigcup \mathcal{F}$. Applying [AM69, Proposition 1.11], we get that there is $P \in \mathcal{F}$ such that $Q \subseteq P$, a contradiction to $Q$ being a maximal ideal. $\square_{\text{Claim 3.16}}$

Put $\varkappa = \text{card}(\text{m-Spec}(R))$. It follows from Claim 3.16 that

$$\sum_{P \in \text{m-Spec}(R)} H/PH = \varkappa.$$

We claim that $\text{gen}(H) > \varkappa$. Since $R$ is noetherian, it is enough to find a submodule of $H$ which cannot be generated by $\varkappa$ elements. Let $\mathcal{A}$ be an almost disjoint system of subsets of $\varkappa$ such that $\text{card}(\mathcal{A}) > \varkappa$, which exists by Lemma 3.14. To each $A \in \mathcal{A}$ we assign an element $\mathbf{x}^A \in H$ by setting $x_P^A = 1 + P$ for $P \in A$ and $x_P^A = 0 + P$ for $P \notin A$. Put $X = \{\mathbf{x}^A \mid A \in \mathcal{A}\}$ and $G = \text{Span}(X) \subseteq H$. It can be easily seen that $X$ is a weak basis of $G$. Lemma 3.8 shows that $\text{gen}(G) = \text{card}(X) = \text{card}(\mathcal{A}) > \varkappa$. We proved that $\text{gen}(H) > \varkappa = \text{card}(\text{m-Spec}(R))$.

Thus we can apply Corollary 3.13 to infer that $H$ is not weakly based. $\square$

The following lemma shows that a projection of a module onto a large enough semisimple module of a certain form can be used to construct a weak basis, providing a kind of converse to Lemma 3.11.

**Lemma 3.17.** *Let $R$ be a ring, let $M$ be an $R$-module and let $Y$ a subset of $M$. Suppose that there is a projection of $M$ onto a semisimple module $S = \bigoplus_{\alpha<\sigma}(R/P_\alpha)^{(\lambda_\alpha)}$ where $\sigma$ is an even ordinal, $(P_\alpha \mid \alpha < \sigma)$ a sequence of pairwise distinct left maximal ideals of $R$ and $(\lambda_\alpha \mid \alpha < \sigma)$ a sequence of cardinal numbers such that $\lambda_\alpha \leq \lambda_{\alpha+1}$ for each $\alpha < \sigma$ even and $\text{card}(Y) \leq \sum_{\alpha<\sigma \text{ even}} \lambda_\alpha$. Then there is a subset $X$ of $M$ lifting a weak basis of $S$ such that $Y \subseteq \text{Span}(X)$.*

*Proof.* Let $\pi : M \to S$ be the given projection, let $K$ stand for $\operatorname{Ker} \pi$ and put $\varkappa = \text{card}(Y)$. For each $\alpha < \sigma$ denote by $S_\alpha$ the given submodule of $S$ of form $(R/P_\alpha)^{(\lambda_\alpha)}$ (in particular, $S = \bigoplus_{\alpha<\sigma} S_\alpha$). Let $W_\alpha \subseteq S_\alpha$ denote the weak basis of $S_\alpha$ consisting of projections of 1 to each $R/P_\alpha$ in the decomposition $S_\alpha =$

$(R/P_\alpha)^{(\lambda_\alpha)}$. Note that the cardinality of $W_\alpha$ is $\lambda_\alpha$. Let $U_\alpha = \{u_{(\alpha,\gamma)} \mid \gamma < \lambda_\alpha\}$ be (a well-ordering of) a subset of $M$ lifting $W_\alpha$ via $\pi$ for each $\alpha < \sigma$. By the choice of $W_\alpha$, the annihilator of the image of $u_{(\alpha,\gamma)}$ in $S$ is $P_\alpha$ for each $\alpha < \sigma$ and $\gamma < \lambda_\alpha$. For brevity we put $u_{(\alpha,\gamma)} = 0$ for every $\alpha < \sigma$ and $\gamma \geq \lambda_\alpha$. Since for each even $\alpha < \sigma$ the maximal ideals $P_\alpha$ and $P_{\alpha+1}$ are distinct, there are $p_{\alpha+i} \in P_{\alpha+i}$ for each $i \in 2$ such that $p_\alpha + p_{\alpha+1} = 1$. Now for each $\alpha < \sigma$ even and $\gamma < \lambda_{\alpha+1}$ we define $z_{(\alpha,\gamma)} = u_{(\alpha,\gamma)} + u_{(\alpha+1,\gamma)}$ and for each $\alpha < \sigma$ even put $Z_\alpha = \{z_{(\alpha,\gamma)} \mid \gamma < \lambda_{\alpha+1}\}$. Note that $\pi(p_{\alpha+1-i}z_{(\alpha,\gamma)}) = \pi(p_{\alpha+1-i}u_{(\alpha+i,\gamma)}) \neq 0$ for each $\alpha < \sigma$ even, $i \in 2$, and $\gamma < \lambda_\alpha$. Thus $\{p_{\alpha+1-i}z_{(\alpha,\gamma)} \mid \gamma < \lambda_{\alpha+i}\}$ lifts a weak basis of $S_{\alpha+i}$ via $\pi$ for each $\alpha < \sigma$ even and $i \in 2$. It follows that both $Z_\alpha$ and the set $\{p_\alpha z_{(\alpha,\gamma)}, p_{\alpha+1}z_{(\alpha,\gamma)} \mid \gamma < \lambda_\alpha\} \cup \{z_{(\alpha,\gamma)} \mid \lambda_\alpha \leq \gamma < \lambda_{\alpha+1}\}$ lift a weak basis of $S_\alpha \oplus S_{\alpha+1}$ via $\pi$ for each $\alpha < \sigma$ even.

Put $Z = \bigcup_{\alpha<\sigma \text{ even}} Z_\alpha$. Then $Z$ lifts a weak basis of $S$ via $\pi$. Let $Y'$ be a subset of $K$ of cardinality $\varkappa$ such that $Y \subseteq \mathrm{Span}(Y') + \mathrm{Span}(Z)$. By the assumption, there is a system of maps $\{f_\alpha : \lambda_\alpha \to Y' \mid \alpha < \sigma \text{ even}\}$ such that $Y' \subseteq \bigcup_{\alpha<\sigma \text{ even}} f_\alpha[\lambda_\alpha]$. For each even $\alpha < \sigma$ let us define $X_\alpha = \{p_{\alpha+1}z_{(\alpha,\gamma)} + f_\alpha(\gamma) \mid \gamma < \lambda_\alpha\}$ and $X_{\alpha+1} = \{p_\alpha z_{(\alpha,\gamma)} - f_\alpha(\gamma) \mid \gamma < \lambda_\alpha\} \cup \{z_{(\alpha,\gamma)} \mid \lambda_\alpha \leq \gamma < \lambda_{\alpha+1}\}$. Now we set $X = \bigcup_{\alpha<\sigma \text{ even}} X_\alpha$. Since $f_\alpha(\gamma)$ is an element of $K$ for each $\alpha < \sigma$ even and $\gamma < \lambda_\alpha$, we have that $X$ lifts a weak basis of $S$ via $\pi$. Also, because $z_{(\alpha,\gamma)} = (p_{\alpha+1}z_{(\alpha,\gamma)} + f_\alpha(\gamma)) + (p_\alpha z_{(\alpha,\gamma)} - f_\alpha(\gamma))$ for each even $\alpha < \sigma$ and $\gamma < \lambda_\alpha$ and $z_{(\alpha,\gamma)} \in X_{\alpha+1}$ for each $\lambda_\alpha \leq \gamma < \lambda_{\alpha+1}$, we have that $Z \subseteq \mathrm{Span}(X)$. Thus also $f_\alpha[\lambda_\alpha] \subseteq \mathrm{Span}(X)$ for each even $\alpha < \sigma$, and so $Y' \subseteq \mathrm{Span}(X)$. Therefore, since $Y \subseteq \mathrm{Span}(Y') + \mathrm{Span}(Z)$, also $Y \subseteq \mathrm{Span}(X)$. □

**Corollary 3.18.** *Let $R$ be a ring and $M$ an $R$-module such that there are two distinct left maximal ideals $P_0, P_1$ of $R$ and a projection of $M$ onto a module isomorphic to $(R/P_0)^{(\varkappa)} \oplus (R/P_1)^{(\varkappa)}$ where $\varkappa = \mathrm{gen}(M)$. Then $M$ is weakly based.*

*In particular, if $R$ is commutative, then $M$ is weakly based provided that* $\mathrm{card}(\mathrm{Spec}(M,=)) \geq 2$.

*Proof.* Let $Y$ be a set of generators of $M$ of cardinality $\varkappa$. Then, putting $\sigma = 2$ and $S = (R/P_0)^{(\varkappa)} \oplus (R/P_1)^{(\varkappa)}$, Lemma 3.17 gives us a subset $X$ of $M$ which lifts a weak basis of $S$ via the given projection such that $Y \subseteq \mathrm{Span}(X)$. It follows that $X$ is a weak basis of $M$.

For the last part of this corollary, it is enough to discuss the definition of $\mathrm{Spec}(M,=)$ and Remark 3.12. □

**Example 3.19.** Let $R$ be a countable non-local ring and $\varkappa$ a cardinal. Then $R^\varkappa$ is weakly based.

*Proof.* If $\varkappa$ is finite, then $R^\varkappa$ is finitely generated, and thus weakly based. Suppose further that $\varkappa \geq \aleph_0$. Since $R$ is non-local, there are two distinct left maximal ideals $P_0, P_1$. Because $P_0 + P_1 = R$, we have a projection $R \to R/(P_0 \cap P_1) \simeq R/P_0 \oplus R/P_1$. It follows that there is a projection

$$R^\varkappa \to (R/P_0 \oplus R/P_1)^\varkappa \simeq (R/P_0)^\varkappa \oplus (R/P_1)^\varkappa.$$

Now $(R/P_i)^\varkappa$ is a semisimple $R$-module, with all simple submodules isomorphic to $R/P_i$, for each $i \in 2$. Therefore, it is enough to show that $\mathrm{len}((R/P_i)^\varkappa) = \mathrm{gen}(R^\varkappa)$

for both $i \in 2$ and use Corollary 3.18. Indeed, we have, using [Jec06, Lemma 5.6] and our hypothesis that $\operatorname{card}(R) \leq \aleph_0$, that

$$\operatorname{len}((R/P_i)^{\varkappa}) = \operatorname{card}((R/P_i)^{\varkappa}) \geq 2^{\varkappa} = \operatorname{card}(R)^{\varkappa} = \operatorname{gen}(R^{\varkappa})$$

and we are done. □

**Corollary 3.20.** *Let $R$ be a non-local ring, $M$ an $R$-module and $F$ a free $R$-module on set of cardinality at least* $\operatorname{gen}(M)$*. Then $M \oplus F$ is weakly based.*

*Proof.* If $M$ is finitely generated, then $M$ is weakly based and the result follows. Put $\varkappa = \operatorname{gen}(M)$ and suppose further that $\varkappa \geq \aleph_0$. We can, without loss of generality, assume that $F = R^{(\varkappa)}$. Since $R$ is non-local, there are two distinct left maximal ideals $P, Q$ of $R$. Because $P + Q = 1$, we have

$$R/(P \cap Q) \simeq R/P \oplus R/Q,$$

and thus there is a sequence of projections

$$M \oplus F \to F = R^{(\varkappa)} \to (R/P)^{(\varkappa)} \oplus (R/Q)^{(\varkappa)}.$$

From infinitude of $\varkappa$ it follows that $\operatorname{gen}(M \oplus F) = \varkappa$, and we can apply Corollary 3.18 to conclude that $M \oplus F$ is weakly based. □

Corollary 3.20 does not in general hold for local rings (see Example 4.30).

**Example 3.21.** There is a module which is not weakly based such that its infinite direct power is weakly based.

*Proof.* The abelian group $\mathbb{Q} \oplus \mathbb{Z}$ is not weakly based by Corollary 3.13. If $\varkappa$ is an infinite cardinal, then Corollary 3.20 shows that

$$(\mathbb{Q} \oplus \mathbb{Z})^{(\varkappa)} \simeq \mathbb{Q}^{(\varkappa)} \oplus \mathbb{Z}^{(\varkappa)}$$

is weakly based. □

**Corollary 3.22.** *Let $R$ be a ring and $M$ be an $R$-module such that there is a projection onto a semisimple module $S$ isomorphic to $\bigoplus_{\alpha<\sigma}(R/P_\alpha)^{(\lambda_\alpha)}$ where $\lambda_\alpha <$* $\operatorname{gen}(M)$ *for each $\alpha < \sigma$ and $\sum_{\alpha<\sigma} \lambda_\sigma =$* $\operatorname{gen}(M)$*. Then $M$ is weakly based.*

*Proof.* Put $\varkappa = \operatorname{gen}(M)$. If $\varkappa < \aleph_0$, then the statement is obviously true. Suppose that $\varkappa$ is an infinite cardinal. Then necessarily $\sigma$ is infinite and we can suppose that $\sigma$ is a limit ordinal and that the sequence $\lambda_\alpha, \alpha < \sigma$ is non-decreasing. Let $Y$ be a generating set of $M$ of cardinality $\varkappa$. Note that since $\sigma$ is limit ordinal, $\sup_{\alpha<\sigma \text{ even}} \lambda_\alpha = \varkappa$. Then we are in a situation where Lemma 3.17 applies and provides us with a subset $X$ of $M$ lifting a basis of $S$ via the given projection such that $Y \subseteq \operatorname{Span}(X)$. Then $X$ is a weak basis of $M$. □

We can readily characterize weakly based modules over a class of rings including simple rings and endomorphism rings of vector spaces.

**Theorem 3.23.** *Let $R$ be a ring such that no quotient of $R$ is a division ring and let $M$ be an $R$-module. Then $M$ is weakly based if and only if there is a projection of $M$ onto a semisimple $R$-module $S$ with* $\operatorname{len}(S) = \operatorname{gen}(M)$*.*

*Proof.* (⇒) Follows directly from Lemma 3.11.

(⇐) If $M$ is finitely generated, then this implication is trivially true; further assume that $\varkappa = \mathrm{gen}(M) \geq \aleph_0$. Suppose first that there is a left maximal ideal $P$ of $R$ such that there is a submodule of $S$ isomorphic to $(R/P)^{(\varkappa)}$. Since $R$ has no quotient which is a division ring, $P$ is not a two sided ideal, and necessarily $\mathrm{Ann}(R/P) \neq P$, and therefore there is $x \in R/P$ with $\mathrm{Ann}(x) \neq P$. It follows that there is a left maximal ideal $Q$ such that $Q \neq P$ and $R/Q \simeq R/P$. Because $\varkappa$ is infinite, we have that $(R/P)^{(\varkappa)} \simeq (R/P)^{(\varkappa)} \oplus (R/Q)^{(\varkappa)}$ and Corollary 3.18 applies.

Now suppose otherwise, that is for each left maximal ideal $P$ there is no submodule of $S$ isomorphic to $(R/P)^{(\varkappa)}$. Since $\mathrm{len}(S) = \varkappa$, it follows that the assumptions of Corollary 3.22 are fulfilled for the semisimple module $S$ and again, we have that $M$ is weakly based. □

Let us summarize the criteria we proved for modules over general rings:

**Proposition 3.24.** *Let $R$ be a ring and $M$ a (left) $R$-module with $\varkappa = \mathrm{gen}(M)$. If $M$ is weakly based, then there is a projection of $M$ onto a semisimple $R$-module $S$ which is isomorphic to one of the following:*

1. $(R/P)^{(\varkappa)} \oplus (R/Q)^{(\varkappa)}$ *for two distinct left maximal ideals $P, Q$;*
2. $\bigoplus_{\alpha<\sigma}(R/P_\alpha)^{(\lambda_\alpha)}$ *for pairwise distinct left maximal ideals $P_\alpha, \alpha < \sigma$ and cardinals $\lambda_\alpha < \varkappa$ with $\sum_{\alpha<\sigma} \lambda_\alpha = \varkappa$;*
3. $(R/P)^{(\varkappa)}$ *for some left maximal ideal $P$.*

*The module $S$ being of form as stated in 1 or 2 is a sufficient condition for $M$ to be weakly based.*

If $M$ is infinitely generated and has a projection onto a module described in condition 3, but there is no projection onto a module described in conditions 1 or 2 of Proposition 3.24, then necessarily the ideal $P$ is two-sided. Condition 3 is in general not sufficient for $M$ to be weakly based. We will see in the next Chapter that over a Dedekind domain, condition 3 is sufficient for torsion modules but in general insufficient for torsion-free modules.

**Remark 3.25.** If $R$ is commutative, then conditions 1 and 3 can be reformulated as follows:

1. $\dim(M/PM) = \dim(M/QM) = \mathrm{gen}(M)$ for some $P, Q \in \text{m-Spec}(R)$ distinct.

3. $\dim(M/PM) = \mathrm{gen}(M)$ for some $P \in \text{m-Spec}(R)$.

The following Proposition shows that if $R$ is commutative and has only countably many maximal ideals, then condition 2 can be simplified in a similar fashion as the other two conditions are in Remark 3.25. Proposition 3.26 cannot be generalized to rings with uncountable spectra, as shown in Example 4.45.

**Proposition 3.26.** *Let $R$ be a commutative ring and $M$ an infinitely generated $R$-module. Suppose that* $\mathrm{card}(\text{m-Spec}(R)) = \aleph_0$. *Then the following properties are equivalent:*

1. *There is a projection of $M$ onto a module $\bigoplus_{P\in\mathrm{Spec}(M,<)} V_P$ where $V_P$ is a vector space over $R/P$ for each $P\in\mathrm{Spec}(M,<)$ and $\sum_{P\in\mathrm{Spec}(M,<)}\dim(V_P) = \mathrm{gen}(M)$.*

2. $\sum_{P\in\mathrm{Spec}(M,<)}\dim(M/(PM+N)) = \mathrm{gen}(M)$ *for each submodule $N$ of $M$ with* $\mathrm{gen}(N) < \mathrm{gen}(M)$.

*Proof.* ($1\Rightarrow 2$) Easy to see.

($2\Rightarrow 1$) Put $\varkappa = \mathrm{gen}\, M$ and let $Y = \{y_\alpha \mid \alpha < \varkappa\}$ be some generating set of $M$. Write $Y = \bigcup_{n\in\omega} Y_n$ as follows: if $\varkappa = \aleph_0$, put $Y_n = \{y_i \mid i < n\}$, while if $\varkappa > \aleph_0$, (but note that the cofinality of $\varkappa$ must be countable) put $Y_n = \{y_\alpha \mid \alpha < \lambda_n\}$ where $0 = \lambda_0 < \lambda_1 < \cdots$ is some increasing countable sequence of infinite cardinals (except $\lambda_0 = 0$) whose supremum is $\varkappa$. If $\varkappa = \aleph_0$, then put $\lambda_n = 1$ for all $n\in\omega$.

We will construct inductively a sequence $\{P_n \mid n\in\omega\}$ of maximal ideals from $\mathrm{Spec}(M,<)$ and a sequence $\{X_n \mid n\in\omega\}$ of subsets of $M$ such that, defining recursively $Q_0 = R$, $Q_n = P_{n-1}Q_{n-1}$ and, similarly, $N_0 = 0$, $N_n = N_{n-1} + \mathrm{Span}(X_{n-1})$, for all $n\in\omega$, the following properties are satisfied:

(a) $X_n \subseteq Q_nM$;

(b) $M/N_n$ is $Q_n$-divisible;

(c) $\mathrm{card}(X_n) < \varkappa$;

(d) $Y_n \subseteq N_{n+1}$;

(e) $\dim(M/(P_nM + N_n)) \geq \lambda_n$.

Initially, let $P_0$ be an arbitrary maximal ideal from $\mathrm{Spec}(M,<)$ and $Z_0$ a subset of $M$ which lifts a basis of $M/P_0M$ over $P_0M$. Put $X_0 = Z_0\cup Y_0$. In this case, properties (a) and (b) are vacuous, while (c), (d), and (e) are easily verified.

Let $0 < n$ and suppose that we have picked $P_0,\dots,P_{n-1}\in\mathrm{Spec}(M,<)$ and $X_0,\dots,X_{n-1}$ so that the properties (a-e) hold. By (c) we have that $\mathrm{gen}(N_n) < \varkappa$, and thus by (2) there is a maximal ideal $P_n$ from $\mathrm{Spec}(M,<)$ such that $\dim M/(P_nM+N_n) \geq \lambda_n$. Let $Z_n$ be a subset of $Q_nM$ which lifts a basis of $M/(P_nM+N_n)$. Since $M/N_n$ is $Q_n$-divisible by (b), there is a subset $Y'_n$ of $Q_nM$ with $\mathrm{card}(Y'_n) = \mathrm{card}(Y_n)$ such that $Y_n \subseteq Y'_n + N_n$. Put $X_n = Z_n\cup Y'_n$. It is straightforward to verify (a–e).

Put $X = \bigcup_{n<\omega} X_n$ and note that, since $Y\subseteq\mathrm{Span}(X)$ by (d), $X$ is a generating set of $M$. Let $\phi : M \to \prod_{n\in\omega} M/(P_nM+N_n)$ be the product of the canonical projections $\phi_n : M\to M/(P_nM+N_n)$, $n<\omega$. It follows from the construction that if $n<m$, then $X_n\subseteq N_m$, while if $n>m$, then $X_n\subseteq P_mM$ by (a). We conclude that $\phi_m[X_n] = 0$ for each $m\neq n$. Since $\phi[X]$ generates $\phi[M]$, we have that $\phi[M] = \bigoplus_{n<\omega} M/(P_nM+N_n)$. By (e), $\sum_{n<\omega}\dim(M/(P_nM+N_n)) = \varkappa$, and hence the implication holds. □

The unimportance of finitely generated modules in investigation of weakly based modules is further illustrated by the following lemma.

**Lemma 3.27.** [Rů10, Lemma 5.1] *Let $R$ be a ring, $M$ an $R$-module and $N$ a finitely generated submodule of $M$. Then $M$ is weakly based if and only if $M/N$ is weakly based.*

*Proof.* We make a heavy use of Lemma 2.4 in this proof.

($\Rightarrow$) Let $X$ be a weak basis of $M$. Since $N$ is finitely generated, there is a finite subset $X_0$ of $X$ such that $N \subseteq \operatorname{Span}(X_0)$. Put $X_1 = X \setminus X_0$. Since $X_1$ lifts a weakly independent set over $X_0$, we have that the image of $X_1$ in $M/N$ is weakly independent. Denote by $\bar{X}_i$ the image of $X_i$ in $M/N$ for each $i \in 2$. We have that $M/(N + \operatorname{Span}(X_1))$ is a finitely generated module (generated by the image of $\bar{X}_0$), thus we can pick its weak basis $\bar{Y}$. Pick a subset $Y$ of $M/N$ lifting $\bar{Y}$ over $\operatorname{Span}(\bar{X}_1)$ such that $Y$ is contained in $\operatorname{Span}(\bar{X}_0) \subseteq M/N$. We have that $Y$ lifts a weakly independent set over $\operatorname{Span}(\bar{X}_1)$ and $\bar{X}_1$ lifts a weakly independent set over $\operatorname{Span}(\bar{X}_0) \supseteq Y$. Since $M/N = \operatorname{Span}(\bar{X}_1) + \operatorname{Span}(Y)$, we have that $\bar{X}_1 \cup Y$ is a weak basis of $M/N$.

($\Leftarrow$) Let $\bar{X}$ be a weak basis of $M/N$ and pick a subset $X$ of $M$ lifting $\bar{X}$ over $N$. Then $M/\operatorname{Span}(X) \simeq N/N \cap \operatorname{Span}(X)$ is a finitely generated module, and hence we can find its weak basis $\bar{Y}$. Therefore, there is a subset $Y$ of $N \subseteq M$ lifting $\bar{Y}$ over $\operatorname{Span}(X)$. Again, we have that $X$ lifts a weakly independent set over $N \supseteq Y$ and $Y$ lifts a weakly independent set over $\operatorname{Span}(X)$. It follows that $X \cup Y$ is a weak basis of $M$. $\square$

# Chapter 4

# Dedekind domains

In this chapter we restrict our study of weakly based modules to the case of modules over a Dedekind domain. Because Dedekind domains are the central topic of this thesis, let us give for convenience several equivalent definitions of them which we will use the most.

**Definition.** We say that a ring $R$ is a *discrete valuation ring (DVR)* if $R$ is a local principal ideal domain which is not a field. By a prime $p$ of $R$ we mean a generator of the maximal ideal of $R$.

Let $R$ be an integral domain which is not a field. The following conditions for $R$ are equivalent (see [Bou72, VII §2.1, Theorem 1]):

- The ring $R$ is hereditary (i.e., any submodule of a projective $R$-module is projective).
- Every non-zero ideal $I$ of $R$ factors into a product of prime ideals. This factorization is unique up to the order of factors.
- Every fractional ideal of $R$ is invertible.
- $R$ is noetherian and the localization of $R$ at each maximal ideal is a discrete valuation ring.
- $R$ is integrally closed, noetherian, and has Krull dimension one (i.e., every non-zero prime ideal is maximal).

If $R$ is a non-field integral domain fulfilling any of those conditions, we say that $R$ is a *Dedekind domain*.

Example of a Dedekind domain is any principal ideal domain (in particular, $\mathbb{Z}$ and rings of polynomials over a field) and the ring of algebraic integers of any number field (see [Rib01, p. 128]).

Let $R$ be a Dedekind domain. We denote by $\operatorname{Spec}(R)$ the set of non-zero prime ideals of $R$. Note that since $R$ is not a field and non-zero prime ideals coincide with maximal ideals of $R$, we have that $\text{m-Spec}(R) = \operatorname{Spec}(R)$ and we will prefer the latter symbol. Note that an $R$-module $M$ is divisible if and only if it is $P$-divisible for any $P \in \operatorname{Spec}(R)$.

We begin with a result concerning direct sums of modules, a stronger version of Corollary 3.20 for Dedekind domains. First we prove the following auxiliary lemma.

**Lemma 4.1.** *Let $R$ be a ring and let $M$ and $N$ be $R$-modules. Let $\varphi, \psi : M \to N$ be two homomorphisms such that* $\mathrm{Ker}(\varphi) + \mathrm{Ker}(\psi) = M$. *Then* $(\varphi + \psi)[M] = \varphi[M] + \psi[M]$.

*Proof.* This lemma is proved by the following computation:

$(\varphi{+}\psi)[M] = (\varphi{+}\psi)(\mathrm{Ker}(\varphi){+}\mathrm{Ker}(\psi)) = (\varphi{+}\psi)(\mathrm{Ker}(\varphi)){+}(\varphi{+}\psi)(\mathrm{Ker}(\psi)) = \psi(\mathrm{Ker}(\varphi)) + \varphi(\mathrm{Ker}(\psi)) = \psi(\mathrm{Ker}(\varphi) + \mathrm{Ker}(\psi)) + \varphi(\mathrm{Ker}(\varphi) + \mathrm{Ker}(\psi)) = \psi[M] + \varphi[M]$. □

**Lemma 4.2.** ([Rů10, Proposition 1.5]) *Let $R$ be a Dedekind domain and let $M, N$ be $R$-modules such that $M$ is weakly based,* $\mathrm{gen}(N) \leq \mathrm{gen}(M)$ *and* $\mathrm{Ext}^1_{\mathrm{R}}(M, N) = 0$. *Then:*

1. *If $R$ is non-local, then $M \oplus N$ is weakly based.*
2. *If $M$ is not finitely generated and it does not have a free direct summand of rank* $\mathrm{gen}(M)$, *then $M \oplus N$ has a weak basis which lifts a weak basis of $M$ over $N$.*

*Proof.* The statement is obvious if $M$ is finitely generated; we further assume that $\mathrm{gen}(M) \geq \aleph_0$. Let $X$ be a weak basis of $M$, put $F = R^{(X)}$ and let $\pi : F \to M$ be the projection extending identity on $X$. Denote by $K$ the kernel of $\pi$. Suppose first that $M$ has a free direct summand of rank $\mathrm{gen}(M)$. If $R$ is non-local, then $M \oplus N$ is weakly based by Corollary 3.20. Now suppose that $M$ does not have a free direct summand of rank $\mathrm{gen}(M)$, thus $\mathrm{gen}(K) = \mathrm{gen}(M)$. Since $R$ is Dedekind, $K$ is an infinitely generated projective module. By [Pas04, Theorem 7.7], $K$ is a free module of rank $\mathrm{gen}(M)$, so there is a projection $\varphi$ of $K$ onto $N$. Since $\mathrm{Ext}^1_{\mathrm{R}}(M, N) = 0$, we can extend $\varphi : K \to N$ to a projection $\psi : F \to N$.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\subseteq"] \arrow[dr, "\varphi"] & F \arrow[r, "\pi"] \arrow[d, "\psi"] & N \arrow[r] & 0 \\ & & N & & \end{tikzcd}$$

Figure 4.1

We have that $\psi[K] = \psi[F]$, and hence $F = K + \mathrm{Ker}\,\psi = \mathrm{Ker}\,\pi + \mathrm{Ker}\,\psi$. By Lemma 4.1, it follows that $(\pi + \psi)[F] = \pi[F] + \psi[F] = M \oplus N$. Since $X$ is the free basis of $F$, we have that $(\pi + \psi)[X]$ generates $M \oplus N$. Because it lifts $X$ in $M$ over $N$, the set $(\pi + \psi)[X]$ is weakly independent, and therefore it is a weak basis of $M \oplus N$. □

**Remark 4.3.** The typical application of Lemma 4.2 will be in a situation where $M$ is a torsion $R$-module and $N$ is a divisible $R$-module. Since $R$ is Dedekind domain, divisible and injectives modules coincide (see [Lam99, Corollary 3.24]), and thus $\mathrm{Ext}^1_{\mathrm{R}}(M, D) = 0$. The module $M$ being torsion ensures that $M$ does not contain any non-trivial free direct summand, and we can thus apply Lemma 4.2 to conclude that $M \oplus D$ has a weak basis whenever $\mathrm{gen}(M) \geq \mathrm{gen}(D)$.

**Example 4.4.** The abelian group $\bigoplus_{n>0} \mathbb{Z}_{p^n} \oplus \mathbb{Z}_{p^\infty}$ is weakly based.

Lemma 4.2 indicates a kind of dichotomy between the local Dedekind domains (i.e., discrete valuation rings) and non-local ones. That is indeed the case, and therefore we will treat discrete valuation rings separately. We start with the characterization of torsion weakly based modules where the general Dedekind domain case can actually be easily derived from the special case of modules over a DVR via localization.

## 4.1 Torsion modules over a discrete valuation ring

In this section, let $R$ always denote a discrete valuation ring and let us fix a prime $p$ (a generator of the maximal ideal of $R$). Lemma 3.6 translates immediately as:

**Lemma 4.5.** *Let $M$ be an $R$-module and suppose that $X$ is a weak basis of $M$. Then $X$ lifts a basis of $M/pM$ (viewed as a vector space over $R/pR$) over $pM$. In particular,* $\dim(M/pM) = \operatorname{gen}(M)$.

The aim of this section is to show that for torsion $R$-modules the condition $\dim(M/pM) = \operatorname{gen}(M)$ actually characterizes weakly based modules. It turns out that, at least in our approach, it is needed to treat countably generated modules separately.

### 4.1.1 Countably generated modules

As in the case of abelian groups in [Rů10], we have to treat the case of countably generated modules over discrete valuation rings on their own, using the knowledge of their structure. The proof of [Rů10, Lemma 4.3] made use of the classical classification of countable abelian p-groups by their *Ulm invariants* (see [Kur60, Chapter 7]). Similar classification exists for a class of reduced (a module is said to be *reduced* if its only divisible submodule is zero) torsion $R$-modules called *totally projective* modules (see [TK08, pp. 243-244], [Wal73]). This class contains all countably generated reduced torsion modules and can be characterized by the following property.

**Definition.** We say that an $R$-module $M$ is *simply presented* if there is a presentation of $M$ with defining relations only of form $px = 0$ or $px = y$. To be more explicit, we require that there is a free $R$-module $F$ on set $X$ such that $M \simeq F/K$, where $K$ is a submodule of $F$ generated by some elements of form $px$ or $px - y$ for distinct $x, y \in X$.

**Remark 4.6.** It is easy to see that a simply presented module can be equivalently defined as a module having a presentation such that all defining relations use at most two variables. On the other hand, any $R$-module has a presentation where any defining relation use at most three variables. Indeed, we can take $M$ as a set of generators, and defining relation will consist of all equations of form $x = y + z$ valid for some $x, y, z \in M$.

**Fact 4.7.** ([TK08, Theorem 30.4]) *The classes of totally projective and reduced torsion simply presented $R$-modules coincide.*

**Lemma 4.8.** *A reduced simply presented $R$-module is weakly based.*

*Proof.* Let $M$ be a reduced simply presented $R$-module. By definition, there is a set $X$ of generators of $M$ such that the kernel $K$ of the projection $R^{(X)} \to M$ extending the identity on $X$ is generated by elements of form $px$ or $y - px$ where $x, y \in X$. We define a binary relation $\leq$ on $X$ by setting

$$x \leq y \Leftrightarrow p^n y = x \text{ for some positive integer } n$$

for all $x, y \in X$. Observe that $\leq$ is a partial order. Indeed, its reflexivity and transitivity is clear and if $p^n x = y$ and $p^m y = x$ for some $x, y \in X$ and some positive integers $m, n$, then $p^{m+n} x = x$, and so $(1 - p^{m+n})x = 0$. Observe that either $m = n = 0$, or $1 - p^{m+n}$ is invertible (recall that $p$ is a prime of a discrete valuation ring $R$, in particular $p$ is quasi-regular). Then either $x = y$ or $x = y = 0$, proving that $\leq$ is antisymmetric, and therefore an order.

Since the module $M$ is reduced, there is not any infinite strictly increasing chain with respect to the above defined partial order $\leq$ in $X$. (Otherwise the elements of a strictly increasing chain would generate a non-trivial divisible submodule of $M$.)

We claim that the set $Y$ of maximal elements (with respect to $\leq$) of $X$ forms a weak basis of $M$. Suppose that there is $y \in Y$ such that $y \in \operatorname{Span}(Y \setminus \{y\})$. Then the projection of $K$ onto the $y$-th (recall that $K \subseteq R^{(X)}$) coordinate necessarily contains $y$. Because $K$ can be generated just by elements of form $px$ and $y - px$, there is $x \in X$ such that $y - px \in K$. This is a contradiction to $y$ being a maximal element with respect to $\leq$, therefore $Y$ is weakly independent. Since $X$ does not contain an infinite strictly increasing chain (with respect to $\leq$), for every $x \in X$ there is some $y \in Y$ such that $x \leq y$, that is, $x = p^n y$ for some positive integer $n$. It follows that $\operatorname{Span}(Y) = M$. □

**Corollary 4.9.** *A countably generated reduced torsion $R$-module is weakly based.*

**Remark 4.10.** A classical example of a reduced torsion abelian group which is not totally projective is the torsion subgroup of $\prod_{n \in \omega} \mathbb{Z}_{p^n}$. In Example 4.23, we show that this abelian group is not weakly based.

**Lemma 4.11.** *Let $M$ be a torsion $R$-module with $\operatorname{gen}(M) = \aleph_0$. Then $M$ is weakly based if and only if $\dim(M/pM) = \aleph_0$.*

*Proof.* ($\Leftarrow$) Since a divisible $R$-module is injective (see [Lam99, Corollary 3.24]), we can write $M$ as $D \oplus S$ where $D$ is divisible and $S$ reduced. Observe that

$$\operatorname{gen}(S) \geq \dim(S/pS) = \dim(M/pM) = \aleph_0.$$

By Corollary 4.9, the module $S$ has a weak basis of cardinality $\aleph_0$. Then $D \oplus S$ is weakly based by Lemma 4.2.

($\Rightarrow$) Follows from Lemma 4.5. □

### 4.1.2 Uncountably generated and general torsion modules

Before proving a version of Lemma 4.11 for uncountably generated $R$-modules, we need the following auxiliary set-theoretic lemmata.

**Lemma 4.12.** *Let $\lambda \leq \varkappa$ be infinite cardinals and let $X$ be a subset of $\varkappa$ of cardinality $\varkappa$. Then there is a map $h : X \to \varkappa$ such that $h(\alpha) \leq \alpha$ and* $\operatorname{card}(h^{-1}[\{\alpha\}]) = \lambda$ *for each $\alpha \in X$.*

*Proof.* Let $X = \bigcup_{\alpha<\varkappa} X_\alpha$ be a partition of $X$ such that $\operatorname{card}(X_\alpha) = \lambda$ for each $\alpha < \varkappa$. We define a bijection $f : \varkappa \to \varkappa$ by induction on $\varkappa$. For the first step, put $f(0) = 0$. Suppose that we have already defined $f(\alpha)$ for all $\alpha < \beta$ for some $\beta < \varkappa$. Suppose first that $\beta \in X$ and let $\gamma < \varkappa$ be such that $\beta \in X_\gamma$. If $\gamma \notin f[\beta]$, we put $f(\beta) = \gamma$. Otherwise, or if $\beta \notin X$, we put $f(\beta) = \min\{\delta < \varkappa \mid \delta \notin f[\beta]\}$. Note that $f$ is indeed a bijection $\varkappa \to \varkappa$. We claim that for any $\alpha < \varkappa$ we have that $\alpha \leq \beta$ for any $\beta \in X_{f(\alpha)}$. Towards a contradiction, suppose that there is $\alpha < \varkappa$ and $\beta \in X_{f(\alpha)}$ such that $\alpha > \beta$. By the construction, we have $f(\alpha) \notin f[\beta]$. But $\beta \in X_{f(\alpha)}$, and thus $f(\beta) = f(\alpha)$, a contradiction with $f$ being a bijection.

The desired function $h$ is obtained by setting $h[X_\alpha] = f^{-1}(\alpha)$ for each $\alpha < \varkappa$. □

**Lemma 4.13.** *Let $\lambda < \varkappa$ be infinite cardinals, let $Y$ be a set of cardinality $\varkappa$, and let $f : Y \to \varkappa$ be a map such that* $\operatorname{card}(f^{-1}[\{\alpha\}]) \leq \lambda$ *for every $\alpha < \varkappa$. Then there is a map $g : Y \to \varkappa$ such that $g \leq f$ and* $\operatorname{card}(g^{-1}[\{\alpha\}]) = \lambda$ *for each $\alpha < \varkappa$.*

*Proof.* Set $X = f[Y] \subseteq \varkappa$. We have that $Y = \bigcup_{\alpha\in X} f^{-1}[\{\alpha\}]$, and therefore $\varkappa \leq \operatorname{card}(X) \cdot \lambda$. Since $\lambda < \varkappa$, we have that $\operatorname{card}(X) = \varkappa$. Let $h : X \to \varkappa$ be a map given by Lemma 4.12. We then obtain the desired map $g$ by putting $g = h \circ f$. □

The following lemma generalizes and corrects [Rů10, Lemma 4.4], which in general does not hold for torsion-free abelian groups. The idea of the proof is basically the same, the important ingredients being decomposing of divisible modules to a direct sum of countable divisible modules and Lemma 4.2. The reason why the proof of Lemma 4.14 does not work for countably generated modules is the unremovable hypothesis "$\lambda < \varkappa$" in the wording of Lemma 4.13.

**Lemma 4.14.** *Let $\varkappa$ be an uncountable cardinal and let $M$ be a $\varkappa$-generated torsion $R$-module. If* $\dim(M/pM) = \varkappa$, *then $M$ is weakly based.*

*Proof.* Let $X$ be a subset of $M$ lifting a basis of $M/pM$ over $pM$. By our hypothesis, $\operatorname{card}(X) = \varkappa$; fix an enumeration $X = \{x_\alpha \mid \alpha < \varkappa\}$. Put $N = \operatorname{Span}(X)$. Since $R$ is a DVR, we have that $D = M/N$ is a divisible module, and by [TK08, Theorem 6.3], there is a decomposition $D \simeq \bigoplus_{\gamma<\varkappa} D_\gamma$ such that $D_\gamma$ is an at most countably generated divisible module for each $\gamma < \varkappa$. There are at most countably generated submodules $C_\gamma$, $\gamma < \varkappa$, of $M$ such that $D_\gamma = (C_\gamma + N)/N$. Since the ring $R$ is noetherian, there are for all $\gamma < \varkappa$ at most countable $J_\gamma \subseteq \varkappa$ such that $C_\gamma \cap N \subseteq \operatorname{Span}(\{x_\alpha \mid \alpha \in J_\gamma\})$.

**Claim 4.15.** *There is a map $g\colon \varkappa \to \varkappa$ such that* $\operatorname{card}(g^{-1}[\{\gamma\}]) = \aleph_0$ *and $J_\gamma \subseteq g^{-1}[\gamma + 1]$ for all $\gamma < \varkappa$.*

*Proof of Claim 4.15.* Define a map $f : \varkappa \to \varkappa$ by putting

$$f(\gamma) = \begin{cases} \min\{\alpha \mid \gamma \in J_\alpha\} & \text{if } \gamma \in J_\alpha \text{ for some } \alpha < \varkappa, \\ \gamma & \text{otherwise.} \end{cases}$$

Note that $f^{-1}[\{\gamma\}] \subseteq J_\gamma \cup \{\gamma\}$, hence $f^{-1}[\{\gamma\}]$ has cardinality at most $\aleph_0$ for each $\gamma < \varkappa$. Since $\varkappa > \aleph_0$, we can use Lemma 4.13 in order to obtain a map $g\colon \varkappa \to \varkappa$ such that $g(\gamma) \leq f(\gamma)$ for each $\gamma < \varkappa$, and $\operatorname{card}(g^{-1}[\{\gamma\}]) = \aleph_0$ for all $\gamma < \varkappa$. Observe that whenever $\gamma \in J_\alpha$, then $f(\gamma) \leq \alpha$. Thus $J_\gamma \subseteq g^{-1}[\gamma+1]$ for all $\gamma < \varkappa$. $\square_{\text{Claim 4.15}}$

For every $\gamma < \varkappa$ put $X_\gamma = \{x_\alpha \mid \alpha \in g^{-1}[\{\gamma\}]\}$ and $X_{<\gamma} = \{x_\alpha \mid \alpha \in g^{-1}[\gamma]\}$. Further, set $N_\gamma = \operatorname{Span}(X_\gamma)$, $N_{<\gamma} = \operatorname{Span}(X_{<\gamma})$, and define $B_\gamma = N_{<\gamma+1}/N_{<\gamma}$. Since $J_\gamma \subseteq g^{-1}[\gamma+1]$ by Claim 4.15, we have that $\{x_\alpha \mid \alpha \in J_\gamma\} \subseteq X_{<\gamma+1}$, hence $C_\gamma \cap N \subseteq N_{<\gamma+1}$ for all $\gamma < \varkappa$. It follows that

$$D_\gamma = (C_\gamma + N)/N \simeq C_\gamma/(C_\gamma \cap N) C_\gamma/(C_\gamma \cap N_{<\gamma+1}) \simeq (C_\gamma + N_{<\gamma+1})/N_{<\gamma+1}$$

and, consequently,

$$D_\gamma \oplus B_{\gamma+1} \simeq ((C_\gamma + N_{<\gamma+1})/N_{<\gamma+1}) \oplus (N_{<\gamma+2}/N_{<\gamma+1}) \simeq (C_\gamma + N_{<\gamma+2})/N_{<\gamma+1}$$

for all $\gamma < \varkappa$.

Since $X$ lifts a basis of $M/pM$ over $pM$, it is weakly independent. It readily follows that $X_{\gamma+1}$ lifts a weakly independent subset, say $V_{\gamma+1}$, of $B_{\gamma+1}$ over $N_{<\gamma+1}$ for all $\gamma < \varkappa$. Since $\operatorname{card}(X_{\gamma+1}) = \operatorname{card}(g^{-1}[\{\gamma+1\}]) = \aleph_0$, we infer that $\operatorname{gen}(B_{\gamma+1}) = \aleph_0$. Applying Lemma 4.2, we get that the factor module $D_\gamma \oplus B_{\gamma+1} \simeq (C_\gamma + N_{<\gamma+2})/N_{<\gamma+1}$ has a weak basis, say $W_{\gamma+1}$, which lifts $V_{\gamma+1}$ over $D_\gamma \simeq (C_\gamma + N_{<\gamma+1})/N_{<\gamma+1}$. Since $D_\gamma$ is divisible, there are elements $c_\alpha \in C_\gamma$, $\alpha \in g^{-1}[\{\gamma+1\}]$ such that $W_{\gamma+1} = \{x_\alpha + pc_\alpha + N_{<\gamma+1} \mid \alpha \in g^{-1}[\{\gamma+1\}]\}$ for all $\gamma < \varkappa$.

For each $\gamma < \varkappa$ and every $\alpha \in g^{-1}[\{\gamma\}]$, put

$$y_\alpha = \begin{cases} x_\alpha & \text{if } \gamma \text{ is a limit ordinal,} \\ x_\alpha + pc_\alpha & \text{otherwise,} \end{cases}$$

and set $Y = \{y_\alpha \mid \alpha < \varkappa\}$. We claim that $Y$ is a weak basis of $M$.

It is obvious that $Y$ lifts a basis of $M/pM$ over $pM$. Thus $Y$ is a weakly independent subset of $M$. It remains to verify that $M = \operatorname{Span}(Y)$. To do so, put $Y_\gamma = \{y_\alpha \mid \alpha \in g^{-1}[\{\gamma\}]\}$ and $Y_{<\gamma} = \{y_\alpha \mid \alpha \in g^{-1}[\gamma]\}$, and set $M_\gamma = \operatorname{Span}(Y_\gamma)$ and $M_{<\gamma} = \operatorname{Span}(Y_{<\gamma})$ for each $\gamma < \varkappa$. We will prove by induction that $N_{<\gamma} \subseteq M_{<\gamma}$ for each $\gamma < \varkappa$ and $C_\alpha \subseteq M_{<\gamma}$ whenever $\gamma = \alpha + 2 < \varkappa$.

For the initial step, observe that $N_{<0} = M_{<0} = 0$. Let $0 < \gamma < \varkappa$ and suppose that the claim holds for all $\beta < \gamma$. First assume that $\gamma$ is a limit ordinal. Then, by the induction hypothesis, $N_{<\gamma} = \bigcup_{\beta<\gamma} N_\beta \subseteq \bigcup_{\beta<\gamma} M_\beta = M_{<\gamma}$. If $\gamma = \alpha + 1$, where $\alpha < \varkappa$, then $N_{<\alpha} \subseteq M_{<\alpha}$ by the induction hypothesis. Suppose that $\alpha$ is a limit ordinal. Then $Y_\alpha = X_\alpha$ by definition, hence $M_\alpha = N_\alpha$, whence $N_{<\gamma} = N_{<\alpha} + N_\alpha \subseteq M_{<\alpha} + M_\alpha = M_{<\gamma}$. Finally, suppose that $\gamma = \alpha + 2$ for some $\alpha < \varkappa$. Then $N_{<\alpha+1} \subseteq M_{<\alpha+1}$ by the induction hypothesis, and $Y_{\alpha+1}$ lifts $W_{\alpha+1}$ over $N_{<\alpha+1}$ by definition. Since $W_{\alpha+1}$ is a weak basis of $(C_\alpha + N_{<\alpha+2})/N_{<\alpha+1}$, we conclude that $C_\alpha + N_{<\alpha+2} \subseteq M_{<\alpha+2}$. $\square$

Now we are ready to characterize torsion weakly bases $R$-modules.

**Theorem 4.16.** *Let $R$ be a discrete valuation ring with prime $p$. Let $\varkappa$ be a cardinal and $M$ a torsion $R$-module such that* $\operatorname{gen}(M) = \varkappa$. *Then $M$ is weakly based if and only if* $\dim(M/pM) = \varkappa$.

*Proof.* ($\Rightarrow$) Follows from Lemma 4.5.

($\Leftarrow$) It is obvious for $M$ finitely generated. The rest follows from Lemma 4.14 if $M$ is uncountably generated and from Lemma 4.11 if $M$ is countably generated. $\square$

## 4.2 Torsion modules over a Dedekind domain

In this section we extend the characterization of weakly based torsion modules to all Dedekind domains. It is well known that the standard decomposition of torsion abelian groups into a direct sum of their $p$-primary components generalizes to torsion modules over Dedekind domains. Explicitly, each torsion module $M$ over a Dedekind domain can be written as a direct sum of its $P$-primary components $M_P$, $P \in \operatorname{Spec}(R)$, where $M_P$ is a submodule of $M$ which can be naturally identified with $M \otimes_R R_P$, where $R_P$ is the localization of $R$ at $P$. We show that for $P$-primary modules, localization preserves weakly independent sets (which is not the case for torsion-free modules; for example the localization of $\mathbb{Z}$ at $p$ is not a weakly based abelian group by Proposition 4.40, but it is of course a weakly based $\mathbb{Z}_{(p)}$-module).

Following [Mat89], we say that a prime ideal $P$ of a commutative ring $R$ is an *associated prime ideal* of an $R$-module $M$ provided that $P = \operatorname{Ann}_R(m)$ for some $m \in M$. An $R$-module $M$ is said to be *$P$-primary* if $P$ is the only associated ideal of $M$. If $R$ is a Dedekind domain, then a torsion $R$-module $M$ is $P$-primary (for a prime ideal $P$) if and only if for every non-zero $m \in M$ there is a positive integer $k$ such that $\operatorname{Ann}_R(m) = P^k$. Let $M$ be a torsion module over a Dedekind domain $R$. We denote by $M_P$ the $P$-primary component of $M$, i.e. the submodule of $M$ consisting of elements $m \in M$ such that $\operatorname{Ann}_R(m) = P^k$ for some positive integer $k$. Then $M$ decomposes into a direct sum of all of its $P$-primary components.

Up to the end of this section let $R$ be a Dedekind domain.

**Lemma 4.17.** *Let $P$ be a prime ideal of $R$ and let $M$ be a $P$-primary $R$-module. Then for every $m \in M$ and every $s \in R \setminus P$ there is $r \in R$ such that $m = rsm$.*

*Proof.* If $m = 0$ then it is obviously true, hence suppose that $m$ is non-zero. Since $R$ is a Dedekind domain and $M$ a $P$-primary $R$-module, we have that $\operatorname{Ann}(m) = P^k$ for some $k > 0$, i.e. $Rm \simeq R/P^k$. Since $s \notin P$, and $P$ is prime (hence a maximal ideal of $R$), we have that $R = Rs + P$. It follows that $R = R^k = (Rs + P)^k \subseteq Rs + P^k$, therefore $1 - rs \in P^k = \operatorname{Ann}(m)$ for some $r \in R$. We showed that $m = rsm$. $\square$

It follows readily from Lemma 4.17 that $M \simeq M \otimes_R R_P$ for a $P$-primary module $M$. This allows us to view the module $M$ as a module over the ring $R_P$.

Given a torsion module $M$ over a Dedekind domain $R$, and a prime ideal $P$ of $R$, a $P$-primary component of $M$ correspond naturally with the localization $M_P \simeq M \otimes_R R_P$ of the module $M$. It justifies our use of the notation $M_P$ for the $P$-primary component of the module $M$.

**Lemma 4.18.** *Let $P$ be a prime ideal of $R$ and let $M$ be a $P$-primary $R$-module. Given a subset $X$ of $M$:*

1. *If $X$ generates $M$ as an $R_P$-module, then $X$ generates $M$ as an $R$-module.*

2. *If $X$ is weakly independent in $M$ as an $R$-module, then $X$ is a weakly independent subset of $M$ as an $R_P$-module.*

*In particular, $X$ is a weak basis of $M$ as an $R$-module if and only if it is a weak basis of $M$ as an $R_P$-module.*

*Proof.*

1. Let $m \in M$. Then $m = \sum_{i=0}^{n} t_i x_i$ where $t_0, \dots, t_n \in R_P$ and $x_0, \dots, x_n \in X$. Then there is $s \in R \setminus P$ such that $st_i \in R$ for all $i \in n$. By Lemma 4.17, there is $r \in R$ be such that $rsm = m$. Thus $m = \sum_{i=0}^{n} rst_i x_i$, and so $X$ generates $M$ as an $R$-module.

2. Suppose for a contradiction that $x_0 = \sum_{i=1}^{n} t_i x_i$ for $t_1, \dots, t_n \in R_P$ and pairwise distinct $x_0, \dots, x_n \in X$. Let $s \in R \setminus P$ satisfy $st_i \in R$ for each $i = 1, \dots, n$. By Lemma 4.17, there is $r \in R$ such that $rsx_0 = x_0$. Then $x_0 = \sum_{i=1}^{n} rst_i x_i$, and thus $X$ is not a weakly independent subset of an $R$-module $M$. This is a contradiction.

□

**Lemma 4.19.** *Let $\varkappa$ be an infinite cardinal and let $M$ be a $P$-primary $R$-module with* $\mathrm{gen}(M) = \varkappa$. *Then $M$ is weakly based if and only if* $\dim_{R/P}(M/PM) = \varkappa$. *Furthermore, any weak basis of $M$ lifts a basis of $M/PM$ over $PM$.*

*Proof.* According to Lemma 4.18, the $R$-module $M$ is weakly based if and only if $M$ is weakly based as an $R_P$-module. Applying [Mat89, Theorem 11.4], we get that the localization $R_P$ is a discrete valuation ring. By Theorem 4.16, the $R_P$-module $M$ is weakly based if and only if $\varkappa = \dim_{R_P/P_P}(M/P_P M)$. Now the statement of the lemma follows from $R/P \simeq R_P/P_P$ and the following: $\dim_{R_P/PR_P}(M/P_P M) = \dim_{R/P}(M/PM)$. □

**Lemma 4.20.** *Let $\varkappa$ be an infinite cardinal and $M$ be a $\varkappa$-generated torsion $R$-module. If*

$$\sum_{P \in \mathrm{Spec}(M,<)} \dim(M/PM) = \varkappa, \tag{4.1}$$

*then $M$ is weakly based.*

*Proof.* For each $P \in \mathrm{Spec}(R)$ we denote by $\psi_P : M \to M/PM$ the canonical projection. Define $\psi : M \to \prod_{P \in \mathrm{Spec}(R)} M/PM$ the product of maps $\psi_P$ over $\mathrm{Spec}(R)$. Observe that since $M$ is torsion, $\psi[M]$ is also torsion, and thus $\psi[M] \subseteq \tau(\prod_{P \in \mathrm{Spec}(R)} M/PM) = \bigoplus_{P \in \mathrm{Spec}(R)} M/PM$. Then, by the definition of $\psi$, $\psi[M] \simeq \bigoplus_{P \in \mathrm{Spec}(R)} M/PM \simeq \bigoplus_{P \in \mathrm{Spec}(R)} (R/P)^{(\dim(M/PM))}$, and we can use Corollary 3.22 to conclude that $M$ is weakly based. □

**Lemma 4.21.** *Let $M$ be a torsion $R$-module and suppose that there is $P \in$* $\mathrm{Spec}(R)$ *such that* $\dim(M/PM) = \mathrm{gen}(M)$. *Then $M$ has a weak basis lifting a basis of $M/PM$ over $PM$.*

*Proof.* Since the module $M$ is torsion and $R$ is Dedekind, it is an easy observation that $\mathrm{Ext}^1_{\mathrm{R}}(M_P, M/M_P) = 0$. By Lemma 4.19, $M_P$ has a weak basis, say $X$. The set $X$ lifts a basis of $M_P/PM_P$ over $PM_P$ by Lemma 4.11 and Lemma 4.18. Applying Lemma 4.2, we obtain a weak basis of $M \simeq M_P \oplus (M/M_P)$ which lifts $X$ over $M/M_P$. □

Let $M$ be a $P$-primary $R$-module. We say that a submodule $B$ of $M$ is a *basic submodule* if $B$ is a pure submodule, it is isomorphic to a direct sum of cyclic modules, and the quotient module $M/B$ is divisible. Since all these properties hold for $M$ viewed as an $R$-module if and only if they hold if $M$ is viewed as an $R_P$-module, and since $R_P$ is a discrete valuation ring, we have by [TK08, Theorem 9.4] that every $P$-primary $R$-module has a basic submodule, unique up to isomorphism. Let $M$ be a torsion $R$-module, then we say that submodule $B$ of $M$ is a basic submodule if $B_P$ is a basic submodule of $M_P$ for each prime ideal $P$.

**Theorem 4.22.** *Let $R$ be a Dedekind domain. Let $\varkappa$ be an infinite cardinal and let $M$ be a torsion $R$-module with $\operatorname{gen}(M) = \varkappa$. Then the following conditions are equivalent:*

1. *$M$ is weakly based;*
2. *There is a projection of $M$ onto a semisimple $R$-module $S$ with $\operatorname{gen}(S) = \varkappa$;*
3. *Any basic submodule $B$ of $M$ has $\operatorname{gen}(B) = \varkappa$;*
4. *$\sum_{P\in\operatorname{Spec}(R)} \dim(M/PM) = \varkappa$.*

*Proof.* ($1 \Rightarrow 2$) See Lemma 3.11.

($2 \Rightarrow 3$) Let $B$ be a basic submodule of $M$. Suppose that $\operatorname{gen}(B) < \varkappa$ and denote by $\bar{B}$ the image of $B$ in the given projection of $M$ onto $S$. Since $M/B$ is divisible, we have that $S/\bar{B}$ is a divisible semisimple module. It follows that $S/\bar{B}$ is zero. But this is a contradiction since $\operatorname{gen}(\bar{B}) \leq \operatorname{gen}(B) < \varkappa = \operatorname{gen}(S)$.

($3 \Rightarrow 4$) Let $B$ be a basic submodule of $M$. Since $B$ is weakly based, we have that $\sum_{P\in\operatorname{Spec}(R)} B/PB = \varkappa$ by Corollary 3.13. Since $B$ is a pure submodule of $M$, $\dim(B/PB) \leq \dim(M/PM)$ for all $P \in \operatorname{Spec}(R)$.

($4 \Rightarrow 1$) If there is $P \in \operatorname{Spec}(R)$ such that $\dim(M/PM) = \operatorname{gen}(M)$, we apply Lemma 4.21. If $\dim(M/PM) < \varkappa$ for every $P \in \operatorname{Spec}(R)$, then the implication follows from Lemma 4.20. □

**Example 4.23.** Let $R$ be a Dedekind domain, $P$ be a prime ideal and $H = \prod_{n>0} R/P^n$. Then the module $T = \tau H$ is not weakly based.

*Proof.* View $B = \bigoplus_{n>0} R/p^nR$ as a submodule of $T$. We claim that $B$ is a basic submodule of $T$. It is easy to see that $B$ is a pure submodule of $T$ and, obviously, it is a direct sum of cyclic $R$-modules. We need to show that $T/B$ is divisible; since $T$ is $P$-primary, it is enough to show that $T/B$ is $P$-divisible. Pick $t \in T$ and let $n \in \omega$ be such that $\operatorname{Ann}(t) = P^n$. Let $t'$ be an element of $T$ given by zeroing the first $n+1$ coordinates of $t$ (viewed as an $\mathbb{N}$-sequence). It follows that $t' \in PT$ and that $t - t' \in B$. Hence, $t + B \in P(T/B)$ and $T/B$ is divisible.

Since $\operatorname{gen}(B) = \aleph_0 < \operatorname{gen}(T)$, Theorem 4.22 shows that $T$ is not weakly based. □

**Example 4.24.** Let $R$ be a Dedekind domain and $M$ an infinitely generated torsion-free $R$-module of finite rank. Then $M$ is weakly based if and only if there is a projection of $M$ onto a semisimple $R$-module $S$ with $\operatorname{len}(S) = \operatorname{gen}(M)$.

*Proof.* Let $B$ be a maximal linearly independent subset of $M$. Since $M$ has finite rank, we have that $\mathrm{card}(B) < \aleph_0$. Put $F = \mathrm{Span}(B)$ and $T = M/F$. Then $T$ is a torsion $R$-module. Since $F$ is finitely generated, Lemma 3.27 shows that $M$ is weakly based if and only if $T$ is weakly based. By Theorem 4.22, $T$ is weakly based if and only if there is a projection of $T$ onto a semisimple module $S$ with $\mathrm{len}(S) = \mathrm{gen}(T) = \mathrm{gen}(M)$. □

## 4.3 General modules over discrete valuation rings

Within this section let $R$ stand for a discrete valuation ring with a prime $p$. In the local case, the situation of torsion-free modules is very simple:

**Lemma 4.25.** *A torsion-free $R$-module $M$ is weakly based if and only if $M$ is free.*

*Proof.* ($\Leftarrow$) Obvious.

($\Rightarrow$) Let $X$ be a weak basis of $M$. We claim that $X$ is $R$-linearly independent. Let $x_0, \cdots, x_n \in X$ be pairwise distinct and suppose that there are non-zero $r_0, \cdots, r_n \in R$ such that $\sum_{i=0}^{n} r_i x_i = 0$. Then there are units $s_0, \cdots, s_n \in R$ and non-negative integers $k_0, \cdots, k_n$ such that $r_i = p^{k_i} s_i$. Put $k = \min\{k_0, \cdots, k_n\}$. Since $M$ is torsion-free, we have that $\sum_{i=0}^{n} p^{k_i - k} s_i x_i = 0$. Since at least one of the coefficients $p^{k_i-k} s_i$ is a unit, we get a contradiction with $X$ being a weak basis. We conclude that $X$ is $R$-linearly independent, and thus a free basis of $M$. □

**Example 4.26.** Let $R$ be a discrete valuation ring and $\varkappa$ an infinite cardinal. The $R$-module $R^{\varkappa}$ is not weakly based.

*Proof.* By Lemma 4.25, it is enough to show that $R^{\varkappa}$ is not free. Let us quickly prove this well-known fact. Suppose for a contradiction that $R^{\varkappa}$ is free. Since $R^{\omega}$ is naturally a submodule of $R^{\varkappa}$ and $R$ being a principal ideal domain ensures that $R^{\omega}$ is itself free, it is enough to show that this leads to a contradiction in the special case of $\varkappa = \omega$. Put $H = R^{\omega}$ and view $B = R^{(\omega)}$ as a submodule of $H$ in the obvious way. Since $\mathrm{gen}(B) = \aleph_0$, there is a free direct summand $F$ of $H$ of rank $\aleph_0$ such that $B \subseteq F$. Denote by $C$ some complement of $F$ in $H$. Note that $H$ is not countably generated (for example, by finding an uncountably generated submodule by virtue of the proof of Example 3.15 and using the noetherian property of $R$). It is then obvious that $C$ is a free $R$-module of rank $\mathrm{gen}(H) > \aleph_0$. Let $p$ be a prime of $R$ and put $G = \prod_{n\in\omega} p^n R \subseteq \prod_{n\in\omega} R = H$, again viewed as a submodule of $H$ in the natural way. Also, we once more have that $G$ is not countably generated.

Put $E = (F + G)/F$. Since $H = F \oplus C$, we can consider $E$ as a submodule of $C$. Now $F$ is countably generated, while $G$ is not, whence $E$ is a non-zero module. We claim that $E \subseteq p^k C$ for any $k \in \omega$. Indeed, this follows quickly from the definition of $G$ and the fact that $B \subseteq F$. But $C$ being free ensures that $\bigcap_{k\in\omega} p^k C = 0$, a contradiction with $E$ being non-zero.

□

**Remark 4.27.** The proof of [EM02, Theorem 2.8] shows that $R^{\omega}$ is an example of an *almost free* $R$-module which is not weakly based. By almost free $R$-module in this context we mean that each countably generated submodule is free.

**Lemma 4.28.** *Let $N$ be a torsion-free $R$-module and $M$ be an $R$-module such that* $\dim(M/pM) < \operatorname{gen}(M)$. *Then no extension of $M$ by $N$ has a weak basis.*

*Proof.* Let $C$ be an extension of $M$ by $N$ and let $\pi\colon C \to N$ be the projection coming with this extension. Suppose that $C$ has a weak basis, say $X$. Put $F = R^{(X)}$, and let $\varphi\colon F \to C$ be the projection extending the identity on $X$. Define a projection $\psi\colon F \to N$ by setting $\psi = \pi\varphi$, and let $K = \operatorname{Ker}\psi$. Since $R$ is hereditary, $K$ is a projective module, and hence $K$ is free by [Pas04, Theorem 10.8]. Observe that $K = \varphi^{-1}[M]$ and since $\varphi$ is a projection, it follows that $\varphi[K] = M$. (See Figure 4.2 below.)

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & M \arrow[u] \arrow[r, "\subseteq"] & C \arrow[u] \arrow[r, "\pi"] & N \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[u, "\varphi\restriction K"'] \arrow[r, "\subseteq_*"] & F \arrow[u, "\varphi"'] \arrow[r, "\psi"] & N \arrow[u, equal] \arrow[r] & 0
\end{tikzcd}$$

Figure 4.2

**Claim 4.29.** *We claim that $\varphi^{-1}[pM] = pK$.*

*Proof of Claim 4.29.* Clearly, $pK \subseteq \varphi^{-1}[pM]$. Assume that there is $k \in K \setminus pK$ with $\varphi(k) \in pM$. Since $F/K \simeq N$ is torsion-free, $K$ is a pure submodule of $F$, i.e. $pK = K \cap pF$. It follows that $k \in F \setminus pF$, whence there are pairwise distinct $x_0, x_1, \cdots, x_n \in X$ such that $k = ux_0 + \sum_{i=1}^n r_i x_i$ for some $r_1, \cdots, r_n \in R$ and a unit $u \in R$. Hence $ux_0 + \sum_{i=1}^n r_i x_i = \varphi(k) \in pM \subseteq pC$ which is a contradiction with weak independence of $X$ in $C$ by Lemma 4.5. $\square_{\text{Claim 4.29}}$

It follows from Claim 4.29 that $\varphi$ induces an injection of vector spaces $K/pK \to M/pM$; in particular, $\dim(K/pK) \leq \dim(M/pM)$. Then we have that

$$\operatorname{gen}(K) = \dim(K/pK) \leq \dim(M/pM) < \operatorname{gen}(M) \leq \operatorname{gen}(K).$$

The last inequality follows from the fact that $\varphi_{\restriction K}$ is a projection onto $M$. This establishes the contradiction. $\square$

**Example 4.30.** In particular, Lemma 4.28 shows that if $D$ is a non-zero divisible $R$-module then the module $D \oplus F$ is not weakly based for any free $R$-module $F$, showing that Corollary 3.20 and Lemma 4.2 do in general fail for local rings.

**Lemma 4.31.** *Let $M$ be an $R$-module with a weak basis $X$. For any subset $X_0$ of $X$, the span of $X_0$ is a pure submodule of $M$.*

*Proof.* Put $N = \operatorname{Span}(X_0)$ and let $y \in N \cap pM$. Let $x_0, \dots, x_n \in X_0$ be pairwise distinct such that $y = \sum_{i=0}^n r_i x_i$ for some $r_0, \dots, r_n \in R$. Since $y \in pM$, we have that $r_0, \dots, r_n \in pR$. Indeed, otherwise $X$ would not lift a basis of $M/pM$ over $pM$ which would contradict Lemma 4.5. Hence, for each $i \in \{0, \dots, n\}$ we have that $r_i = p^{k_i} s_i$ for some $s_i \in R$ and a positive integer $k_i$. We put $y' = \sum_{i=0}^n p^{k_i - 1} s_i x_i$, thus $py' = y$ and $y' \in \operatorname{Span}(X_0) = N$, hence $y \in pN$. It follows that $N$ is a pure submodule of $M$. $\square$

**Lemma 4.32.** *Let $M$ be an $R$-module such that* $\mathrm{gen}(\tau M) < \mathrm{gen}(M)$. *If $M$ is weakly based, then $M \simeq F \oplus N$ where $F$ is a free module and $N$ is a weakly based $R$-module such that* $\mathrm{gen}(N) = \mathrm{gen}(\tau M)$.

*Proof.* Suppose that $M$ has a weak basis, say $X$. If $\tau M$ is finitely generated, then $M \simeq \tau M \oplus \phi M$ by [TK08, Theorem 7.2], and the statement holds by Lemma 4.25 and Lemma 3.27. Further, suppose that $\tau M$ is infinitely generated. There is a subset $X_0$ of $X$ such that $\mathrm{card}(X_0) = \mathrm{gen}(\tau M)$ and $\tau M \subseteq \mathrm{Span}(X_0)$. Put $N = \mathrm{Span}(X_0)$ and observe that, by Lemma 4.31, $N$ is a pure submodule of $M$. Then, since $\tau M \subseteq N$, we get that $M/N$ is torsion-free. Because the image of $X \setminus X_0$ in $M/N$ is a weak basis of $M/N$, we have by Lemma 4.25 that $M/N$ is free. Hence $M \simeq F \oplus N$, where $F$ is a free module of rank $\mathrm{gen}(M)$ and $N$ is a module containing $\tau M$ with a weak basis $X_0$. □

Before proceeding further we need to make a little detour. We would like to know how $\mathrm{Ext}^1_{\mathrm{R}}(R/P, -)$ can be computed given a prime ideal $P$ of a Dedekind domain $R$. In the case of abelian groups (or more generally, modules over principal ideal domains), it is easy to show that there is a natural isomorphism $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \simeq A/nA$ for any non-zero $n \in \mathbb{Z}$ and any abelian group $A$ (see [Rot08, Theorem 7.17]). This allows us to use Lemma 4.2 for modules of form $M \oplus N$ where $M$ is a vector space over $R/P$ for some $P \in \mathrm{Spec}(R)$, and $N$ is $P$-divisible with $\mathrm{gen}(N) \leq \mathrm{gen}(M)$ whenever $R$ is a principal ideal domain. Proving that $\mathrm{Ext}^1_{\mathrm{R}}(R/I, M) \simeq M/IM$ for a general Dedekind domain $R$, for any $R$-module $M$, and a non-zero ideal $I$ is a bit more tricky. In what follows we present some needed terminology and prove this fact in a quite elementary way (also, while we are confident that this is a well-known result, we failed to find a suitable reference).

Let $R$ be an integral domain and let $Q$ be its field of fractions. An $R$-submodule of $Q$, say $A$, is called a *fractional ideal* of $R$ if there is non-zero $r \in R$ such that $rA \subseteq R$. For any non-zero fractional ideal $A$ of $R$ we put $A^{-1} = \{q \in Q \mid qA \subseteq R\}$ which is indeed again a fractional ideal. Note that $R \subseteq I^{-1}$ for every non-zero ideal $I$ of $R$. Every fractional ideal $A$ of $R$ is in fact isomorphic to some ideal of $R$. Indeed, let $r \in R$ be a non-zero element such that $rA \subseteq R$. Then $A \subseteq r^{-1}R \simeq R$.

It is well known that every ideal $I$ of a Dedekind domain $R$ is *strongly 2-generated*, i.e. for every non-zero cyclic submodule $C$ of $I$ the quotient $I/C$ is also cyclic. Since every fractional ideal is as an $R$-module isomorphic to an ideal of $R$, this strong 2-generator property holds for all fractional ideals of Dedekind domains.

**Lemma 4.33.** *Let $R$ be a Dedekind domain and $I$ a non-zero ideal. Then $I^{-1}/R \simeq R/I$.*

*Proof.* By the discussion above, $R \subseteq I^{-1}$ and $I^{-1}$ is strongly 2-generated; it follows that there is $q \in I^{-1}$ such that $I^{-1} = \mathrm{Span}(\{1, q\})$. Let us define $\varphi : R \to I^{-1}/R$ by putting $\varphi(r) = qr + R$ for each $r \in R$. Note that $\varphi$ is surjective. It is well known that if $R$ is Dedekind domain, then $(I^{-1})^{-1} = I$. Thus $rI^{-1} \subseteq R$ if and only if $r \in I$ for all $r \in R$. It follows that $\{r \in R \mid qr \in R\} = I$, and so $\mathrm{Ker}\,\varphi = I$. We have proved that $\varphi$ induces the desired isomorphism. □

**Lemma 4.34.** *Let $R$ be a Dedekind domain. Then*

$$\mathrm{Ext}^1_{\mathrm{R}}(R/I, M) \simeq M/IM$$

*for every non-zero ideal $I$ of $R$ and every $R$-module $M$.*

*Proof.* Let us first prove the lemma in the special case when $M = R$.

**Claim 4.35.** *Let $R$ be a Dedekind domain. Then* $\mathrm{Ext}^1_{\mathrm{R}}(R/I, R) \simeq R/I$ *for any non-zero ideal $I$.*

*Proof of Claim 4.35.* Since $\mathrm{Hom}_{\mathrm{R}}(R/I, R) = 0$, application of the contravariant functor $\mathrm{Hom}_{\mathrm{R}}(-, R)$ on exact sequence

$$0 \to I \xrightarrow{i} R \to R/I \to 0 \tag{4.2}$$

(where $i$ represents the inclusion map) yields the following exact sequence:

$$0 \to \mathrm{Hom}_{\mathrm{R}}(R, R) \xrightarrow{\mathrm{Hom}_{\mathrm{R}}(i,R)} \mathrm{Hom}_{\mathrm{R}}(I, R) \to \mathrm{Ext}^1_{\mathrm{R}}(R/I, R) \to 0.$$

By [Pas04, Lemma 7.1], for each non-zero ideal $J$ of $R$ there is an isomorphism $J^{-1} \simeq \mathrm{Hom}_{\mathrm{R}}(J, R)$ given by assigning to each $q \in J^{-1}$ the element of $\mathrm{Hom}_{\mathrm{R}}(J, R)$ corresponding to a multiplication by $q$. Hence $\mathrm{Hom}_{\mathrm{R}}(R, R) \simeq R$ and $\mathrm{Hom}_{\mathrm{R}}(I, R) \simeq I^{-1}$. The homomorphism $\mathrm{Hom}_{\mathrm{R}}(i, R)$ sends $1_R \in \mathrm{Hom}_{\mathrm{R}}(R, R)$ to $i \in \mathrm{Hom}_{\mathrm{R}}(I, R)$. Since both of these homomorphisms correspond to a multiplication by 1, under the above described isomorphisms, we have that $\mathrm{Hom}_{\mathrm{R}}(i, R)$ is in fact the inclusion of $R$ into $I^{-1}$ and $\mathrm{Ext}^1_{\mathrm{R}}(R/I, R) \simeq I^{-1}/R$. Then the claim holds by Lemma 4.33. $\square_{\text{Claim 4.35}}$

Now we will prove the general case using Claim 4.35. Applying $\mathrm{Hom}_{\mathrm{R}}(-, M)$ to exact sequence (4.2) yields an exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}_{\mathrm{R}}(R/I, M) \to \mathrm{Hom}_{\mathrm{R}}(R, M) \xrightarrow{\mathrm{Hom}_{\mathrm{R}}(i,M)} \mathrm{Hom}_{\mathrm{R}}(I, M) \to \\ \to \mathrm{Ext}^1_{\mathrm{R}}(R/I, M) \to 0,\end{aligned} \tag{4.3}$$

hence

$$\mathrm{Ext}^1_{\mathrm{R}}(R/I, M) \simeq \mathrm{Hom}_{\mathrm{R}}(I, M)/\mathrm{Hom}_{\mathrm{R}}(i, M)[\mathrm{Hom}_{\mathrm{R}}(R, M)]. \tag{4.4}$$

By [AF93, Proposition 20.10] and the fact that $I$ is finitely generated and projective, the natural transformations

$$\mathrm{Hom}_{\mathrm{R}}(I, R) \otimes M \to \mathrm{Hom}_{\mathrm{R}}(I, M)$$

and

$$\mathrm{Hom}_{\mathrm{R}}(R, R) \otimes M \to \mathrm{Hom}_{\mathrm{R}}(R, M)$$

are isomorphisms. By their naturality, the following diagram commutes:

$$\begin{array}{ccc}\mathrm{Hom}_{\mathrm{R}}(I, R) \otimes M & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{R}}(I, M) \\ \uparrow{\scriptstyle \mathrm{Hom}_{\mathrm{R}}(i,R)\otimes 1_M} & & \uparrow{\scriptstyle \mathrm{Hom}_{\mathrm{R}}(i,M)} \\ \mathrm{Hom}_{\mathrm{R}}(R, R) \otimes M & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{R}}(R, M)\end{array}$$

Together with (4.4) and the right exactness of the tensor functor $-\otimes_R M$, we get that $\mathrm{Ext}^1_{\mathrm{R}}(R/I,R)\otimes M \simeq \mathrm{Ext}^1_{\mathrm{R}}(R/I,M)$. The result follows by Claim 4.35 since

$$\mathrm{Ext}^1_{\mathrm{R}}(R/I,R)\otimes M \simeq R/I\otimes M \simeq M/IM.$$

$\square$

The following is the last lemma we need to characterize weakly based modules over a DVR. Using Lemma 4.34, we are able to prove this result for a general Dedekind domain, which will be essential in finishing the characterization for all Dedekind domains in the next section.

**Lemma 4.36.** *Let $R$ be a Dedekind domain and let $M$ be an $R$-module. If there is $P \in \mathrm{Spec}(R)$ with $\dim(\tau M/P\tau M) = \mathrm{gen}(M)$, then the module $M$ is weakly based.*

*Proof.* The statement is obvious if $M$ is finitely generated. Suppose that $M$ is infinitely generated and put $\varkappa = \mathrm{gen}(M)$, $T = \tau M$, $N = \phi M \simeq M/T$ and $N' = M/PT$. Since $\tau N' = T/PT$ is a bounded module (meaning that $\mathrm{Ann}(N') \subsetneq R$), it is easy to see from [Kap84, Theorem 5] that $N' \simeq \tau N' \oplus N$. Pick a subset $Z'$ of $N$ lifting a basis of $N/PN$ over $PN$ and put $N'' = N/\mathrm{Span}(Z')$. By our initial assumption, $\dim(T/PT) = \varkappa$, and so we can pick an $R/P$-linearly independent subset $Y''$ of $T/PT$ such that $\dim(\mathrm{Span}(Y'')) = \mathrm{codim}(\mathrm{Span}(Y'')) = \varkappa$. Since $Y'' \subseteq \tau N'$, the modules $\mathrm{Span}(Y'')$ and $N''$, viewed as submodules of $\tau N' \oplus N'' \simeq N'/\mathrm{Span}(Z')$, have a trivial intersection. Put $M'' = \mathrm{Span}(Y'') \oplus N''$. Note that since $Z'$ lifts a basis of $N/PN$ over $PN$, we have that $N'' = PN''$. By [Rot08, Proposition 7.21] and Lemma 4.34, we have that

$$\mathrm{Ext}^1_{\mathrm{R}}(\mathrm{Span}(Y''),N'') \simeq \mathrm{Ext}^1_{\mathrm{R}}((R/P)^{(\varkappa)},N'') \simeq \prod_{\alpha<\varkappa}\mathrm{Ext}^1_{\mathrm{R}}(R/P,N'') \simeq$$
$$\simeq \prod_{\alpha<\varkappa} N''/PN'' = \prod_{\alpha<\varkappa} 0 = 0.$$

It follows from Lemma 4.2 that $M''$ has a weak basis, say $Y'$, lifting $Y''$ over $N''$. View $Z'$, defined as a subset of $N$, as a subset of $N' = M/PT \simeq \tau N' \oplus N$ and let $Z$ be a subset of $M$ lifting $Z'$ over $PT$. Now pick $Y \subseteq M$ lifting $Y'$ over $\mathrm{Span}(Z) + PT$ and put $C = \mathrm{Span}(Y \cup Z)$. Observe that $M = C + T$, in particular, $T' = M/C$ is a torsion module. Since $\dim(T'/PT') = \mathrm{codim}(Y') = \varkappa$, we have by Theorem 4.21 that $T'$ has a weak basis $X'$ which lifts a basis of $T'/PT'$ over $PT'$. Let $X$ be a subset of $M$ lifting $X'$ over $C$. Since $Y$ lifts a linearly independent subset of $M/(PM + \mathrm{Span}(Z))$ over $(PM + \mathrm{Span}(Z))$, and $X$ lifts a linearly independent subset of $M/(PM + \mathrm{Span}(Y \cup Z))$, it follows that $X \cup Y \cup Z$ lifts a linearly independent subset of $M/PM$. Therefore, $X \cup Y \cup Z$ is weakly independent subset of $M$. Since $M = \mathrm{Span}(X) + C$, we have that $X \cup Y \cup Z$ is a weak basis of $M$. $\square$

Now we are ready to give a full characterization of weakly based modules over discrete valuation rings.

**Theorem 4.37.** *Let $R$ be a discrete valuation ring with a prime $p$. Let $M$ be an $R$-module. Then $M$ is weakly based if and only if $M \simeq F \oplus N$ where $F$ is free and $\dim(\tau N/p\tau N) = \mathrm{gen}(N)$.*

*Proof.* ($\Leftarrow$) By Lemma 4.36, $N$ is weakly based. It readily follows that $F \oplus N$ is weakly based.

($\Rightarrow$) Suppose that the module $M$ is weakly based. It follows from Lemma 4.28 that $\dim(\tau M/p\tau M) = \mathrm{gen}(\tau M)$, thus if $\mathrm{gen}(\tau M) = \mathrm{gen}(M)$, the implication follows from Lemma 4.36. If $\mathrm{gen}(\tau M) < \mathrm{gen}(M)$, then $M \simeq F \oplus N$ where $F$ is free and $\mathrm{gen}(N) = \mathrm{gen}(\tau M)$ by Lemma 4.31. We can pick $N$ to be a submodule of $M$ and, observing that $\tau N = \tau M$, we get that

$$\dim(\tau N/p\tau N) = \dim(\tau M/p\tau M) = \mathrm{gen}(\tau M) = \mathrm{gen}(N).$$

This establishes the result. □

## 4.4 Torsion-free modules over non-local Dedekind domains

Within this section let $R$ be a Dedekind domain.

**Lemma 4.38.** *Let $P \in \mathrm{Spec}(R)$, let $M$ be a torsion-free $R$-module and let $X$ be a subset of $M$ which lifts a linearly independent subset of $M/PM$ over $PM$. Then the set $X$ is $R$-linearly independent in $M$.*

*Proof.* Since $M$ is a flat $R$-module([GT06, Theorem 4.4.9]), the natural mapping $M \to M_P$ taking $m$ to $\frac{m}{1}$ is an injection. Therefore, we can, without loss of generality, assume that $M = M_P$ and naturally view $M$ as an $R_P$-module. Suppose for a contradiction that there are pairwise distinct $x_0, \cdots, x_n \in X$ and non-zero $r_0, \cdots, r_n \in R$ such that $\sum_{i=0}^n r_i x_i = 0$. Since $R$ is assumed to be a Dedekind domain, $R_P$ is a discrete valuation ring ([Mat89, Theorem 11.4]). Denote by $p$ some prime of $R_P$. Then there are units $u_0, \cdots, u_n \in R_P$ and non-negative integers $k_0, \cdots, k_n$ such that $r_i = u_i p^{k_i}$ for all $i = 0, \cdots, n$. Put $k = \min\{k_0, \cdots, k_n\}$ and set $s_i = u_i p_i^{k_i - k}$ for all $i = 0, \cdots, n$. Since the module $M$ is torsion-free (both as $R$-module and $R_P$-module), we have that $\sum_{i=0}^n s_i x_i = 0$. There is a unit $t \in R_P$ such that $ts_i \in R$ for all $i = 0, \cdots, n$ (take a least common multiple of denominators of $s_0, \ldots, s_n$ for $t$). It follows that $\sum_{i=0}^n (ts_i)x_i = 0$ in $R$. This a contradiction with $X$ lifting a linearly independent subset of $M/PM$ over $PM$ because there is $i \in \{1, \cdots, n\}$ such that $s_i$ is a unit in $R_P$, and, consequently, $ts_i \notin P$. □

Recall that having fixed a weak basis $X$ of an $R$-module $M$, we denoted by $\mathrm{Spec}(M, x)$ the set of all prime ideals $P$ of $R$ such that $M/(\mathrm{Span}(X \setminus \{x\}))$ is not $P$-divisible.

**Lemma 4.39.** *Suppose that $R$ is non-local. Let $M$ be an infinitely generated $R$-module such that $\mathrm{Spec}(M, =) = \{P\}$ and $\dim(\tau M/P\tau M) < \mathrm{gen}(M)$. If $X$ is a weak basis of $M$, then there is a subset $X_0$ such that $\mathrm{card}(X_0) = \mathrm{card}(X)$, and $\mathrm{Spec}(M, x)$ contains at least one prime ideal different from $P$ for each $x \in X_0$.*

*Proof.* Towards a contradiction, suppose that there is a subset $Y$ of $X$ with $\mathrm{card}(Y) < \mathrm{card}(X)$ such that $\mathrm{Spec}(M, x) = \{P\}$ for each $x \in X' = X \setminus Y$. We claim that $X'$ lifts a linearly independent set in $M/PM$ over $PM$. Indeed, otherwise there would exist $x \in X'$ such that $x \in \operatorname{Span}(X \setminus \{x\}) + PM$, and thus $C_x = PC_x$ where $C_x = M/\operatorname{Span}(X \setminus \{x\})$, a contradiction to $P \in \operatorname{Spec}(M, x)$. Since $\dim(\tau M/P\tau M) < \operatorname{gen}(M)$, there is a submodule $N$ of $M$ with $\operatorname{gen}(N) < \operatorname{gen}(M)$ such that $\tau M \subseteq N + PM$. Let $Y' \subseteq X$ be such that $Y, N \subseteq \operatorname{Span}(Y')$ and $\operatorname{card}(Y') < \operatorname{card}(X)$, and put $X'' = X \setminus Y'$. Denote by $Z$ the image of $X''$ in the projection of $M$ onto $\phi M$. Then $Z$ lifts a linearly independent subset of $\phi M/P\phi M$, and thus, by Lemma 4.38, $Z$ is linearly independent in $\phi M$. Since $\phi M = \operatorname{Span}(Z) + L$ for some submodule $L$ of $\phi M$ such that $\operatorname{gen}(L) < \operatorname{gen}(\phi M) = \operatorname{gen}(M)$, we have that $\phi M$ necessarily contains a free direct summand of rank $\operatorname{gen}(M)$. Then also $M$ has a free direct summand of rank $\operatorname{gen}(M)$, and thus $\operatorname{Spec}(M, =) = \operatorname{Spec}(R)$. Since $R$ is non-local, this is a contradiction to $\operatorname{Spec}(M, =)$ having just one element. □

The next proposition shows, that weakly based torsion-free modules are characterized exactly by properties 1 and 2 from Proposition 3.24.

**Proposition 4.40.** *Let $R$ be a non-local Dedekind domain and $M$ be an infinitely generated torsion-free $R$-module. Then $M$ is weakly based if and only if either*

1. $\operatorname{card}(\operatorname{Spec}(M, =)) \geq 2$;
2. *There is a projection of $M$ onto $\bigoplus_{P \in \operatorname{Spec}(M,<)} V_P$ where $V_P$ is a vector space over $R/P$ for each $P \in \operatorname{Spec}(M, <)$ and $\sum_{P \in \operatorname{Spec}(M,<)} \dim(V_P) = \operatorname{gen}(M)$.*

*Proof.* ($\Leftarrow$) Follows from Lemmas 3.18 and 3.22.

($\Rightarrow$) If $\operatorname{Spec}(M, =) = \emptyset$, then (2) holds by Lemma 3.11. Suppose that $\operatorname{Spec}(M, =) = \{P\}$ for some $P \in \operatorname{Spec}(R)$ and that $X$ is a weak basis of $M$. Then since $M$ is torsion-free, we have by Lemma 4.39 that there is a subset $X_0$ of $X$ with $\operatorname{card}(X_0) = \operatorname{card}(X)$ such that for each $x \in X_0$ we have that $P_x \in \operatorname{Spec}(M, x)$ for some $P_x \in \operatorname{Spec}(R) \setminus \{P\}$. Then property (2) follows from the second part of Lemma 3.11. □

Note that an $R$-module $M$ is isomorphic to $M_P$ under the natural map if and only if $M = QM$ for all $Q \in \operatorname{Spec}(R) \setminus \{P\}$ if and only if $M$ is an $R_P$-module in the natural way. In this case, the module $M$ is sometimes called *$P$-local*.

**Corollary 4.41.** *Let $R$ be a non-local Dedekind domain and $M$ a torsion-free $R$-module. If $M$ is $P$-local for some $P \in \operatorname{Spec}(R)$, then $M$ is not weakly based.*

**Example 4.42.** The abelian group of $p$-adic integers is not weakly based.

*Proof.* Let us recall that $p$-adic integers can be constructed by taking the inverse limit of the following chain of abelian groups:

$$\cdots \to \mathbb{Z}_{p^{n+1}} \to \mathbb{Z}_{p^n} \to \cdots \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$$

where each arrow is the projection such that its kernel is a simple group. Let us denote this inverse limit by $A$. It follows that $A$ is a torsion-free subgroup of $\prod_{n \in \mathbb{N}} \mathbb{Z}_{p^n}$. It is enough to show that $A = qA$ for any prime $q$ distinct from $p$. Indeed, then Corollary 4.41 shows that $A$ is not weakly based.

Let $q$ be a prime distinct from $p$ and let $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ be an element of $A$. By Lemma 4.17, there is $y_n \in \mathbb{Z}_{p^n}$ such that $qy_n = x_n$ for each $n \in \mathbb{N}$. Put $\mathbf{y} =$

$(y_n)_{n\in\mathbb{N}}$. We want to show that $\mathbf{y} \in A$. Since $\mathbf{x} \in A$, we have that $(x_n - x_m) \equiv 0 \pmod{p^n}$ for any $n \leq m$. Therefore, $q(y_n - y_m) \equiv 0 \pmod{p^n}$. Since $q$ is a unit of the ring $\mathbb{Z}_{p^n}$, it follows that $(y_n - y_m) \equiv 0 \pmod{p^n}$ for any $n \leq m$, and thus $\mathbf{y} \in A$. □

## 4.5 General modules over (non-local) Dedekind domains

Properties characterizing the existence of a weak basis in modules over a Dedekind domain $R$ depend fairly on the fact whether the domain $R$ is local. Indeed, in the local case, we cannot apply Lemma 3.20. Therefore, we shall treat the local and non-local case separately. The local case was solved by Theorem 4.37. The characterization in the non-local case follows:

**Theorem 4.43.** *Let $R$ be a non-local Dedekind domain and let $M$ be an infinitely generated $R$-module. Then $M$ is weakly based if and only if at least one of the following properties is satisfied:*

1. $\operatorname{card}(\operatorname{Spec}(M, =)) \geq 2$*;*
2. $\dim(\tau M/P\tau M) = \operatorname{gen}(M)$ *for some* $P \in \operatorname{Spec}(R)$*;*
3. *There is a projection of $M$ onto a module $\bigoplus_{P\in\operatorname{Spec}(M,<)} V_P$ where $V_P$ is a vector space over $R/P$ for each $P \in \operatorname{Spec}(R)$ and $\sum_{P\in\operatorname{Spec}(M,<)} \dim(V_P) = \operatorname{gen}(M)$.*

*Proof.* ($\Leftarrow$) If (1) holds, then the implication follows from Lemma 3.18. If (3) holds, then we use Lemma 3.22. If (2) holds than the implication follows from Lemma 4.36.

($\Rightarrow$) Let $X$ be a weak basis of $M$ and suppose that (1) and (2) are false. Then by Lemma 4.38 there is a subset $X_0$ of $X$ with $\operatorname{card}(X_0) = \operatorname{card}(X)$ such that for each $x \in X_0$ there is a prime ideal $P_x \in \operatorname{Spec}(M, x)$ such that $P_x \notin \operatorname{Spec}(M, =)$. Applying the second part of Lemma 3.11, we infer that there is a projection of $M$ onto $\bigoplus_{x\in X_0} R/P_x$ and thus (3) holds. □

Even though the next statement is quite clumsy, we combine the local and non-local case to formulate the characterization of weakly based modules over Dedekind domains. We also add several supplements which simplify the conditions in some special cases.

**Theorem 4.44.** *Let $R$ be a Dedekind domain and let $M$ be an $R$-module with* $\operatorname{gen}(M) = \varkappa$ *for some infinite cardinal $\varkappa$. Then $M$ is weakly based if and only if at least one of the following conditions is satisfied:*

1. *There are two distinct prime ideals $P, Q$ of $R$ such that* $\dim_{R/P}(M/PM) = \dim_{R/Q}(M/QM) = \varkappa$*;*
2. *There is a prime ideal $P$ of $R$ and a decomposition $M \simeq F \oplus N$ where $F$ is a free module and* $\dim_{R/P}(\tau N/P\tau N) = \operatorname{gen}(N)$*;*

3. *There is a projection of $M$ onto an $R$-module $\bigoplus_{P\in\mathrm{Spec}(R)} V_P$ where $V_P$ is a vector space over $R/P$ with $\dim_{R/P}(V_P) < \varkappa$ for each $P \in \mathrm{Spec}(R)$ and $\sum_{P\in\mathrm{Spec}(R)} \dim_{R/P}(V_P) = \varkappa$.*

*Moreover, the following should be noted:*

- *If $\mathrm{cf}(\mathrm{gen}(M)) > \mathrm{card}(\mathrm{Spec}(R))$, then $M$ is weakly based if and only if at least one of properties* (1) *and* (2) *holds;*
- *If the ring $R$ is local, the $M$ is weakly based if and only if property* (2) *holds;*
- *If the ring $R$ is non-local, then property* (2) *can be simplified to*

*(2')* $\dim_{R/P}(\tau M/P\tau M) = \mathrm{gen}(M)$ *for some prime ideal $P$ of $R$;*

- *If the spectrum of $R$ is countable, then we can simplify condition* (3) *to*

*(3')* $\sum_{P\in\mathrm{Spec}(M,<)} \dim(M/(PM+N)) = \mathrm{gen}(M)$ *for each submodule $N$ of $M$ with* $\mathrm{gen}(N) < \mathrm{gen}(M)$*;*

- *If the spectrum of $R$ is countable and furthermore $M$ is uncountably generated, then condition* $(3')$ *is simplified to*

*(3")* $\sum_{P\in\mathrm{Spec}(M,<)} \dim(M/PM) = \mathrm{gen}(M)$.

*Proof.* Combine Theorems 4.37 and 4.43. If the spectrum of $R$ is countable, then the simplification of (3) to $(3')$ is provided by Proposition 3.26. Furthermore, if $M$ is uncountably generated, then the sum of cardinals in condition $(3')$ can be rearranged to form a non-decreasing countable sequence of cardinals smaller, then $\mathrm{gen}(M)$ with limit $\mathrm{gen}(M)$. Then any submodule $N$ of $M$ with $\mathrm{gen}(N) < \mathrm{gen}(M)$ fits somewhere in this sequence of cardinals, and it follows that $(3'')$ implies $(3')$ in this case. □

The following example, apart from serving as a non-trivial application of Theorem 4.44, shows that Proposition 3.26 does not generalize to commutative rings with uncountable spectra, and that condition $(3')$ in Theorem 4.44 is not sufficient in general.

**Example 4.45.** Let $R$ be a Dedekind domain with $\mathrm{card}(R) = \mathrm{card}(\mathrm{Spec}(R)) = 2^{\aleph_0}$[1]. Put

$$H = \prod_{P\in\mathrm{Spec}(R)} R/P$$

and let $M$ be a submodule of $H$ consisting of all sequences which are zero for all but countably many prime ideals $P$. Then condition $(3')$ of Theorem 4.44 does hold for $M$, but condition (3) does not. Furthermore, $M$ is not weakly based.

[1]Given any uncountable cardinal $\varkappa$, there is a PID $R$ of cardinality $\varkappa$ such that its spectrum has cardinality $\varkappa$. It is well known that there is an algebraically closed field $k$ of any given uncountable cardinality (and any such two fields of some fixed characteristics $p$ are isomorphic; see [CK90, Chapter 7]). Setting $R = k[x]$, we get the desired PID. Indeed, $\mathrm{Spec}(R) = \{(x-a) \mid a \in k\}$.

*Proof.* Using the same argumentation as in the proof of Example 3.15, we infer that $\dim(M/PM) = 1$ for any $P \in \mathrm{Spec}(R)$ (alternatively, this can be obtained by observing that $M$ is a pure submodule of $H$ and using the corresponding part of the proof of Example 3.15 directly). Also, we have that $\mathrm{gen}(M) = 2^{\aleph_0}$. Indeed, it is obvious that $\mathrm{gen}(M) \geq 2^{\aleph_0}$, and the reversed inequality follows from the fact that $\mathrm{card}(R) \leq 2^{\aleph_0}$ implies $\mathrm{card}(M) = 2^{\aleph_0}$. Therefore,

$$\sum_{P \in \mathrm{Spec}(R)} \dim(M/PM) = \mathrm{card}(\mathrm{Spec}(R)) = 2^{\aleph_0} = \mathrm{gen}(M).$$

We claim that there is no projection from $M$ onto a direct sum of $2^{\aleph_0}$ simple $R$-modules. Supposing otherwise yields that there is a subset $\mathcal{J}$ of $\mathrm{Spec}(R)$ of cardinality $2^{\aleph_0}$ and a projection $\pi : M \to \bigoplus_{P \in \mathcal{J}} R/P$. Observe that the kernel of any projection of $M$ onto $R/P$ is exactly $PM$ for any $P \in \mathrm{Spec}(R)$. It follows that $\mathrm{Ker}(\pi) = \bigcap_{P \in \mathcal{J}} PM$. But it is easy to see, using again a similar argumentation as in the proof of Example 3.15, that the module $N = M/(\bigcap_{P \in \mathcal{J}} PM)$ is isomorphic to the submodule consisting of all sequences with countable support in the module $\prod_{P \in \mathcal{J}} R/P$. This is a contradiction, since $N$ is manifestly not semisimple.

The fact that $M$ is not weakly based follows by applying Theorem 4.44. $\square$

We provide an example of a module such that existence of its weak basis is independent on ZFC (for brevity we confine ourselves to the abelian group case).

**Example 4.46.** Let $\varkappa$ be an infinite cardinal and let us define an abelian group $A = \prod_{p \in \mathbb{P}} \mathbb{Z}_p^{(\varkappa)}$. Then $A$ has a weak basis if and only if $\varkappa = \varkappa^{\aleph_0}$.

*Proof.* First we observe that $\dim(A/pA) = \varkappa$ for all $p \in \mathbb{P}$. It follows that $\sum_{p \in \mathbb{P}} \dim(A/pA) = \aleph_0 \cdot \varkappa = \varkappa$. Since $\mathrm{card}(\mathbb{Z}) = \aleph_0$, we have

$$\mathrm{gen}(A) = \mathrm{card}(A) = \varkappa^{\aleph_0}.$$

Using Theorem 4.44, we concur that $A$ is weakly based if and only if $\varkappa = \varkappa^{\aleph_0}$. $\square$

Putting $\varkappa = \aleph_1$ in Example 4.46, we obtain an abelian group $A$ such that $A$ is weakly based if and only if $\aleph_1 = \aleph_1^{\aleph_0}$. By [Jec06, Formula 5.22] and [Jec06, Lemma 5.6], we have that $\aleph_1^{\aleph_0} = \aleph_0^{\aleph_0} \cdot \aleph_1 = 2^{\aleph_0}$. Therefore, the abelian group $A$ is weakly based if and only if the continuum hypothesis holds.

## 4.6 Closure properties

Let us sum up the already presented (counter)examples and make use of the characterization obtained in the previous section to make several judgements about the closure properties of the class of weakly based $R$-modules over a Dedekind domain $R$.

**Lemma 4.47.** *Let $R$ be a Dedekind domain. There is an extension of weakly based abelian groups which is not weakly based.*

*Proof.* We prove the Lemma separately for local and non-local Dedekind domains. First, suppose that $R$ is non-local. Let $P$ be a prime ideal and $\iota : R \to R_P$ an injection of $R$ into the localization of $R$ at $P$ given by sending $1 \in R$ to $p$, a prime in the DVR $R_P$. Since the torsion-free rank of $R_P$ is one, we have that $R_P/\iota[R]$ is a torsion $R$-module, hence it has a primary decomposition. It is not hard to see that the $P$-primary component is isomorphic to $R/P$ and the $Q$-primary component is divisible for all prime ideals $Q$ other than $P$. It follows that $\iota$ induces the following short exact sequence:

$$0 \to R \xrightarrow{\iota} R_P \to (R/P) \oplus D \to 0, \tag{4.5}$$

where $D$ is a divisible $R$-module. Note that

$$\mathrm{gen}(D) \leq \mathrm{gen}(R_P) \leq \max(\aleph_0, \mathrm{card}(\mathrm{Spec}(R))).$$

Denote the latter cardinal by $\varkappa$. Taking a direct sum of $\varkappa$ copies of 4.5, we obtain a short exact sequence

$$0 \to R^{(\varkappa)} \xrightarrow{\iota^{(\varkappa)}} R_P^{(\varkappa)} \to (R/P)^{(\varkappa)} \oplus D^{(\varkappa)} \to 0.$$

The leftmost module is free, and thus weakly based, while the weak basedness of $(R/P)^{(\varkappa)} \oplus D^{(\varkappa)}$ follows from Lemma 4.2. But the middle module is non-zero torsion-free and divisible by all prime ideals except for $P$, and therefore it is not weakly based by Corollary 4.41.

Suppose now that $R$ is a local Dedekind domain (i.e. a DVR) and denote by $p$ a prime of $R$. Denote by $Q_R$ the ring of quotients of $R$. Take into consideration the short exact sequence

$$0 \to R \to R \oplus Q_R \to (R/pR) \oplus Q_R \to 0 \tag{4.6}$$

where the inclusion is given by sending 1 to $p$ in the first direct summand $R$ of the middle module. Taking a direct sum of $\aleph_0$ copies of 4.6, we obtain

$$0 \to R^{(\aleph_0)} \to R^{(\aleph_0)} \oplus Q_R^{(\aleph_0)} \to (R/pR)^{(\aleph_0)} \oplus Q_R^{(\aleph_0)} \to 0. \tag{4.7}$$

Again, the leftmost module is free, and thus weakly based, while the weak basedness of the rightmost module follows from Lemma 4.2. But the middle module is not weakly based by Lemma 4.28. □

**Lemma 4.48.** *Let $A, B$ be weakly based modules over a Dedekind domain $R$ and let $C$ be some extension of $A$ over $B$. Then $C$ is weakly based provided that $A$ and $B$ are torsion or that the extension is pure.*

*Proof.* Let

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0 \tag{4.8}$$

be a short exact sequence in Mod-$R$ with $A$ and $C$ being weakly based. If both $A$ and $C$ are finitely generated, then so is $B$ and we are done. Hence we can assume that $\mathrm{gen}(B) = \max(\mathrm{gen}(A), \mathrm{gen}(C)) \geq \aleph_0$. By Theorem 4.44, both $A$ and $C$ fulfill at least one of the conditions 1, 2, or 3 of Theorem 4.44. We will refer to those conditions throughout the proof. Let us identify $A$ with the submodule $\iota[A]$ of $B$. We will make another assumption about $C$ if condition 2 holds for it. In this case, $C \simeq F \oplus C'$, where $C'$ has $\dim(\tau C'/P\tau C') = \operatorname{gen}(C')$ (for some $P \in \operatorname{Spec}(R)$) and $F$ is free. Then we can remove the direct summand $F$ from both $C$ and $B$. We can thus already assume that $C = C'$, that is, that $C$ satisfies $\operatorname{gen}(C) = \operatorname{gen}(\tau C)$.

**Case I.** $\operatorname{gen}(A) \leq \operatorname{gen}(C)$

*Proof of Case I.* If $C \to S$ is a projection of $C$ onto a semisimple module witnessing the validity of condition 1 or 3 for module $C$, then we prove that $B$ is weakly based simply using the projection $B \xrightarrow{\pi} C \to S$ and the fact that $\operatorname{gen}(B) = \operatorname{gen}(C)$. Suppose that $C$ fulfills condition 2 of Theorem 4.44, which by our additional assumption means that $\dim(\tau C/P(\tau C)) = \operatorname{gen}(C)$ for some $P \in \operatorname{Spec}(R)$. Pick a subset $\bar{Y}$ of $\tau C$ lifting a basis of $\tau C/P(\tau C)$ over $P(\tau C)$. It is enough to show that there is a subset $Y$ of $\tau B$ lifting $\bar{Y}$ over $A$. If $A$ and $C$ are torsion, then so is $B$ and we are done in this case. Suppose that $\iota$ is a pure injection. Pick some subset $Y'$ of $B$ lifting $\bar{Y}$ over $A$. For any $y \in Y'$ we have that $y + A$ is torsion in $C$, and thus there is $r \in R$ non-zero with $ry \in A$. By the purity of $\iota$, there is $a_y \in A$ with $ry = ra_y$. Then $r(y - a_y) = 0$ and thus $(y - a_y)$ is an element of $\tau B$. Putting $Y = \{y - a_y \mid y \in Y'\}$, we obtain the desired subset of $\tau B$ lifting $\bar{Y}$ over $A$. $\square_{\text{Case}}$

**Case II.** $\operatorname{gen}(A) > \operatorname{gen}(C)$

*Proof of Case II.* Choose a subset $Y$ of $B$ lifting a generating subset of $C$ lifting with $\operatorname{card}(Y) < \operatorname{gen}(A)$. Suppose that $A$ fulfills condition 1 or 3. Then there is a projection of $A$ onto a semisimple $R$-module $S$ with $\operatorname{gen}(S) = \operatorname{gen}(A)$ witnessing the validity of one of these conditions. Since $B = A + \operatorname{Span}(Y)$, we have a series of projections

$$\begin{aligned} B \to B/\operatorname{Span}(Y) = (A + \operatorname{Span}(Y))/\operatorname{Span}(Y) \simeq \\ A/(A \cap \operatorname{Span}(Y)) \to S/(S \cap \operatorname{Span}(\bar{Y}) = S' \quad (4.9) \end{aligned}$$

where $\bar{Y}$ is the image of $Y$ in $S$. Then $S'$ is a semisimple $R$-module with $\operatorname{gen}(B) = \operatorname{gen}(S)$, since $\operatorname{card}(Y) < \operatorname{gen}(A) = \operatorname{gen}(S)$. If $S$ witnesses the validity of condition 1 or 3 of Theorem 4.44 for module $A$, then so does $S'$ for module $B$. Suppose now that $A$ meets condition 2, i.e., $A \simeq F \oplus A'$, where $\dim(\tau A'/P(\tau A')) = \operatorname{gen}(A')$ for some $P \in \operatorname{Spec}(R)$. We would like to assume that $F = 0$. If $\operatorname{rank}(F) > \operatorname{gen}(A')$, then $B$ has a direct summand of rank $\operatorname{rank}(F)$ because we assumed that $\operatorname{gen}(C) < \operatorname{gen}(A)$. Therefore, the exact sequence 4.8 splits into a direct sum of trivial free exact sequence and exact sequence

$$0 \to A' \xrightarrow{\iota} B' \xrightarrow{\pi} C \to 0 \tag{4.10}$$

where $B'$ is a direct complement of the free direct summand in $B$. Note that if the short exact sequence 4.8 is pure, then so is 4.10. If $\operatorname{gen}(A') \leq \operatorname{gen}(C)$, then $B'$ is weakly based by Case I, and therefore $B$ is weakly based. If $\operatorname{gen}(A') > \operatorname{gen}(C)$, we redenote $A = A'$ and continue. We have a series of projections

$$\begin{aligned} \tau B \to \tau(B)/(\operatorname{Span}(Y) \cap \tau B) = (\tau A + \operatorname{Span}(Y) \cap \tau B)/(\operatorname{Span}(Y) \cap \tau B) \simeq \\ \tau A/(\tau A \cap \operatorname{Span}(Y)) \to V/(\operatorname{Span}(\bar{Y}) = V' \quad (4.11) \end{aligned}$$

where $V$ is the vector space over $R/P$ of dimension $\mathrm{gen}(A)$ together with the projection $A \to V$ and $\bar{Y}$ is the image of $Y$ in $V$. It follows that $\dim(\tau B/P(\tau B)) = \mathrm{gen}(A) = \mathrm{gen}(B)$ and $B$ meets condition 2 of Theorem 4.44. $\square_{\mathrm{Case}}$

This concludes the proof. $\square$

**Lemma 4.49.** *Let $R$ be a non-local Dedekind domain. Then there are modules $M, N$ which are not weakly based such that $M \oplus N$ is weakly based.*

*Proof.* Let $P, Q$ be two distinct prime ideals of $R$. Let $\varkappa$ stand for the maximum of cardinals $\aleph_0$ and $\mathrm{card}(\mathrm{Spec}(R))$. Put $M = R_P^{(\varkappa)}$ and $N = R_Q^{(\varkappa)}$. Note that $M, N$ are both torsion-free $R$-modules and $\mathrm{gen}(M) = \mathrm{gen}(N) = \varkappa$. Also, $\dim(M/LM) = 0$ for each $L \in \mathrm{Spec}(R) \setminus \{P\}$ and $\dim(M/PM) = \varkappa$ by Lemma 4.38 (the torsion-free rank of $M$ is $\varkappa$). The same statement holds *mutatis mutandis* for $N$ and $Q$.

It follows from Corollary 4.41 that $M$ and $N$ are not weakly based. On the other hand, since $\dim((M \oplus N)/P(M \oplus N)) = \dim((M \oplus N)/Q(M \oplus N)) = \varkappa = \mathrm{gen}(M \oplus N)$, we have by Proposition 4.40 that $M \oplus N$ is weakly based. $\square$

**Lemma 4.50.** *Let $R$ be a DVR and $M, N$ two $R$-modules which are not weakly based. Then $M \oplus N$ is also not weakly based.*

*Proof.* Let $R$ be a DVR with a prime $p$ and let $M, N$ be $R$-module such that $M \oplus N$ is weakly based. It follows from Theorem 4.37 that $M \oplus N \simeq F \oplus A$ where $F$ is free $R$-module and $\dim(\tau A/p(\tau A)) = \mathrm{gen}(A)$. If $A$ is finitely generated, then $M, N$ are obviously weakly based; we further assume that $\mathrm{gen}(A) \geq \aleph_0$. Since $\tau A = \tau M \oplus \tau N$, it follows that we can, without loss of generality, assume that $\dim(\tau M/p(\tau M)) = \mathrm{gen}(A)$. We have

$$\dim(\tau M/p(\tau M)) = \mathrm{gen}(A) = \mathrm{gen}(\tau A) = \mathrm{gen}(\tau M).$$

If $\mathrm{gen}(M) = \mathrm{gen}(\tau M)$, then $M$ is weakly based by Theorem 4.37. Suppose that $\mathrm{gen}(M) > \mathrm{gen}(\tau M)$. Since $M \subseteq A \oplus F$ and $\mathrm{gen}(M) > \mathrm{gen}(A)$, we have that $G = M/A \cap M$ is isomorphic to a submodule of $F$. Since $R$ is a PID, the module $G$ is free, and thus $M \simeq G \oplus (A \cap M)$. Since $\tau M \subseteq A \cap M$ and $\mathrm{gen}(A \cap M) = \tau M$, we have that $M$ is weakly based by Theorem 4.37. $\square$

**Example 4.51.** Let $R$ be a Dedekind domain and $P \in \mathrm{Spec}(R)$. Put $H = \prod_{n \in \mathbb{N}} R/P^n$. Then $H$ is not weakly based.

*Proof.* We showed in Example 4.23 that $\tau H$ is not weakly based. If $R$ is local, then $H$ is not weakly based by Lemma 4.28. If $R$ is non-local then, since $H = QH$ for any $Q \in \mathrm{Spec}(R) \setminus \{P\}$, it follows from Theorem 4.43 that $H$ is not weakly based. $\square$

**Theorem 4.52.** *Let $R$ be a Dedekind domain and denote respectively by $\mathcal{W}$ and $\mathcal{TW}$ the class of all weakly based and all torsion weakly based $R$-modules. Then:*

- *$\mathcal{W}$ is not closed under direct summands, direct products, extensions, direct limits, submodules, or quotients (for any Dedekind domain $R$),*
- *$\mathcal{TW}$ is closed under extensions,*

- $\mathcal{W}$ *is closed under arbitrary direct sums and pure extensions,*
- Mod-$R \setminus \mathcal{W}$ *is closed under finite direct sums if and only if $R$ is local.*

*Proof.* Let $R$ be a Dedekind domain. For any torsion module $T$ and non-zero divisible module $D$ with $\mathrm{gen}(T) \geq \mathrm{gen}(D)$, we have by Lemma 4.2 that $T \oplus D$ is weakly based, but $D$ is not weakly based by Corollary 3.2. The non-closedness under extensions is shown by Lemma 4.47. By [GT06, Lemma 1.2.3], any module is a direct limit of finitely presented, and thus weakly based modules. In particular, a non-trivial divisible module is a direct limit of weakly based modules.

The counterexample to $\mathcal{W}$ being closed under direct products for any Dedekind domain $R$ is shown in Example 4.51. We note that if $R$ is local, then even the direct product of infinite number of copies of $R$ is not weakly based by Example 4.26. That $\mathcal{W}$ is not closed under submodules or quotients is clear.

Let $R$ be a Dedekind domain. That $\mathcal{W}$ is closed under direct sums, is demonstrated by Lemma 3.4. The closedness of $\mathcal{W}$ under pure extensions and of $\mathcal{TW}$ under extensions is shown by Lemma 4.48.

The fact that for a non-local Dedekind domain a direct sum of two modules which are not weakly based can indeed be weakly based is demonstrated in Lemma 4.49, while the closedness of Mod-$R \setminus \mathcal{W}$ under finite direct sums in the local case is proved in Lemma 4.50. $\square$

# Chapter 5

# Weak basis properties of rings

In this chapter we are going to investigate rings $R$ such that all (left) modules over $R$ are weakly based, and an even more rare scenario in which any generating set of any $R$-module contains a weak basis. The latter property of a module is stronger than being just weakly based; we also give some attention to modules over PIDs with this property and characterize them over a DVR.

**Definition.** Let $R$ be a ring.

- We say that an $R$-module $M$ is *strongly weakly based*[1] if for any generating set $X$ of $M$ there is a subset $Y$ of $X$ which is a weak basis of $M$.
- If any (left) $R$-module is (strongly) weakly based, then we say that $R$ has the (left) *(strong) weak basis property.*

Of course, any finitely generated module is strongly weakly based, and any division ring has the strong weak basis property (Lemma 5.20).

It may be worth noting that in the literature both the class of rings with the (left) weak basis property and with the strong weak basis property can be found identified with the class of left perfect rings. A counterexample to the claim that commutative rings with the weak basis property are perfect, found in [Ran77], was shown in [NN91a]; we cite this example and show a variety of other ones in the next section. On the other hand, the authors of [NN91a] proved that any ring with the strong weak basis property is perfect, and asked whether the reversed inclusion holds. A positive answer is given in [Zho95], but the proof is flawed (as shown in the erratum of this paper) and the question remains open as far as we know.

**Remark 5.1.** Neither the strong weak basis property nor the weak basis property are left-right symmetric. Let us get slightly ahead of ourselves and show a counterexample right away. A classical example of Bass (see [Lam01, Example 23.22]) of a left perfect ring which is not right perfect is local, and therefore has the left strong weak basis property by Lemma 5.20. By the theorem of Bass, the radical of this ring cannot be right $T$-nilpotent, and thus we can use Lemma 5.3 to infer that this ring does not have the right weak basis property.

Continuing in our convention of all modules being left, whenever referring to a ring property, we have its left version in mind (unless said otherwise).

[1] The author would like to apologize for not being able to devise a less quaint term, albeit it might evoke a fairy tale connotation.

## 5.1 Weak basis property

First we give a very simple lower and upper bound for a class of rings with the weak basis property.

**Definition.** Let $R$ be a ring.

- We say that $R$ is *semisimple*[2] if it is semisimple as an $R$-module.
- We call $R$ a (left) *max* ring provided that any non-zero (left) $R$-module has a maximal submodule.

**Lemma 5.2.** *Any semisimple ring has the weak basis property. All rings with the weak basis property are max rings.*

*Proof.* The first part follows from Lemma 3.4, and the second part is an immediate consequence of Lemma 3.1. □

In what follows we show that in order to study the class of rings with the weak basis property it is enough to concern ourselves with rings with zero radical (i.e. the *semiprimitive* rings).

**Definition.** Let $R$ be a ring and $I$ a left ideal of $R$. Following [Lam01, Theorem 23.16], we say that $I$ is (left) $T$*-nilpotent* provided that for any left $R$-module $M$, $IM = M$ implies $M = 0$.

We say that $R$ is a (left) *perfect* ring if the radical $J$ of $R$ is left $T$-nilpotent and the quotient ring $R/J$ is semisimple.

**Lemma 5.3.** *Let $R$ be a ring and $J$ its radical. Then $R$ has the weak basis property if and only if $R/J$ has the weak basis property and $J$ is $T$-nilpotent.*

*Proof.* ($\Rightarrow$) Suppose that $R$ has the weak basis property. By Lemma 3.6, any weak basis lifts over the radical, and therefore the ring $R/J$ has the weak basis property. Let $M$ be an $R$-module and suppose that $JM = M$. Since $R$ has the weak basis property, there is a weak basis $X$ of $M$. Again, Lemma 3.6 shows that $X$ lifts a weak basis of $M/JM = 0$, and thus $M = 0$. Hence, by definition, $J$ is $T$-nilpotent.

($\Leftarrow$) Let $M$ be an $R$-module. The factor module $M/JM$ is naturally an $R/J$-module. By hypothesis, the ring $R/J$ has the weak basis property, and thus there is a weak basis $\bar{X}$ of the $R/J$-module $M/JM$, which is of course also a weak basis of the $R$-module $M/JM$. Denote by $X$ some subset of $M$ lifting $\bar{X}$ over $JM$. We claim that $X$ is a weak basis of $M$. Clearly, $X$ is weakly independent. Furthermore, we have

$$M = JM + \operatorname{Span}(X).$$

It follows that $M/\operatorname{Span}(X) = J(M/\operatorname{Span}(X))$. By the definition of $T$-nilpotence, $M/\operatorname{Span}(X) = 0$, and therefore $X$ is a weak basis of $M$. Hence, $R$ has the weak basis property. □

**Definition.** The following conditions for the ring $R$ are equivalent (see [Goo79, Theorem 1.1]):

[2]This is a left-right symmetric property of a ring - see [Lam01, Corollary 3.7].

1. For any $x \in R$ there is $y \in R$ such that $xyx = x$.
2. Any (left) principal ideal of $R$ is generated by an idempotent.
3. Any (left) finitely generated ideal of $R$ is generated by an idempotent.

If $R$ fulfills any of those conditions, we say that $R$ is *(Von Neumann) regular*. Furthermore, $R$ is an *abelian regular*[3] ring if all its left ideals of $R$ are two-sided.

**Fact 5.4.** *The following statements are true:*

*1.* ([Goo79, Corollary 1.2]) *Any regular ring is semiprimitive.*

*2.* ([Fai95, Kaplansky's Theorem, First Max Theorem]) *For a commutative semiprimitive ring, the notions of regular and max rings coincide.*

Denote by $\mathcal{W}$ the class of rings with the (left) weak basis property, and by $\mathcal{W}'$ the subclass of $\mathcal{W}$ consisting of commutative and semiprimitive rings. Putting Lemmas 5.2, 5.3, and Fact 5.4 together we obtain following bounds:

$$\text{Perfect} \subseteq \mathcal{W} \subseteq \text{Max},$$

$$\text{Semisimple} \subseteq \mathcal{W}' \subseteq \text{Regular},$$

Recall that by Lemma 5.3 it is enough to concern ourselves with semiprimitive rings. Therefore, in order to investigate commutative rings with the weak basis property, it is enough to treat the class $\mathcal{W}'$.

In the rest of this section we provide a variety of examples, proving in particular that all the inclusions above are proper.

### 5.1.1 Semiartinian rings

The following is an example of a non-perfect ring with the weak basis property by Nashier and Nichols:

**Example 5.5.** ([NN91a]) Let $k$ be a field, $\varkappa$ an infinite cardinal and $R = k^{\varkappa}$. Put $V = k^{(\varkappa)}$, viewed as a $k$-subspace of $R$. Let $S$ be a $k$-span of $V \cup \{1_R\}$. Then $S$ is a non-perfect commutative ring with the weak basis property.

*Proof.* It is easy to see that $S$ is a $k$-subalgebra of $R$ and that the ring $S$ is regular. In particular, $S$ has zero radical. From $\varkappa$ being infinite it follows that $S$ is not artinian, and hence $S$ is not semisimple. Therefore, $S$ is a non-perfect commutative ring. The weak basis property of $S$ follows from Lemma 5.6. $\square$

We generalize the example of Nashier and Nichols in the following, fairly straightforward, way.

**Definition.** Let $R$ be a ring. We define the (left) socle of $R$, denoted $\operatorname{Soc}(R)$, to be the left ideal of $R$ generated by all the simple (minimal) left ideals of $R$. We note that $\operatorname{Soc}(R)$ is always a two-sided ideal.

**Lemma 5.6.** *Let $R$ be a ring. If $R/\operatorname{Soc}(R)$ has the weak basis property, then so does $R$.*

[3] Also referred to as *strongly regular* rings in literature.

*Proof.* Let $M$ be an $R$-module and put $S = \operatorname{Soc}(R)$. Then $M/SM$ is naturally an $R/S$-module, and thus there is a weak basis $\bar{X}$ of the $R$-module $M/SM$. Denote by $X$ some subset of $M$ lifting $\bar{X}$ over $SM$ and note that $X$ is weakly independent. Put $C = \operatorname{Span}(X) \cap SM$.

Because $SM$ is an epimorphic image of some direct power of $S$ and $S$ is a semisimple module, we have that $SM$ is a semisimple module. Thus $C$ is a direct summand of $SM$ and $SM = C \oplus D$ for some complement $D$. It follows that

$$M = SM + \operatorname{Span}(X) = (C \oplus D) + \operatorname{Span}(X) = D \oplus \operatorname{Span}(X).$$

Hence, $M$ is a direct sum of two weakly based modules, and whence $M$ is weakly based. □

**Definition.** Let $R$ be a ring. We put $\operatorname{Soc}_0(R) = 0$ and inductively define:

- If $\alpha$ is a successor ordinal, then $\operatorname{Soc}_\alpha(R)$ is the ideal of $R$ such that the quotient $\operatorname{Soc}_\alpha(R)/\operatorname{Soc}_{\alpha-1}(R)$ is the socle of $R/\operatorname{Soc}_{\alpha-1}(R)$.
- If $\alpha$ is a limit ordinal, then $\operatorname{Soc}_\alpha(R) = \bigcup_{\beta<\alpha} \operatorname{Soc}_\beta(R)$.

We say that $R$ is a left *semiartinian* (or *Loewy*) ring if there is an ordinal $\sigma$ such that $\operatorname{Soc}_\sigma(R) = R$. In this case we say that $\sigma$ is the (left) *Loewy length* of $R$, denoted by $L(R) = \sigma$. Note that since $R$ is a finitely generated $R$-module, $L(R)$ is always a successor ordinal.

**Corollary 5.7.** *A semiartinian ring of finite Loewy length has the weak basis property.*

*Proof.* Follows by a finite induction from Lemma 5.6. □

**Lemma 5.8.** *Let $M$ be an $R$-module and $X$ a subset of $M$. If there are pairwise distinct elements $x_n \in X, n \in \omega$ and elements $y_0, \dots, y_n \in X$ such that $\{x_n \mid n \in \omega\} \subseteq \operatorname{Span}(\{y_0, \dots, y_n\})$, then $X$ is not weakly independent.*

*Proof.* Since $\operatorname{card}(\{x_n \mid n \in \omega\}) = \aleph_0$, there is $n \in \omega$ such that $x_n \notin \{y_0, \dots, y_n\}$. Hence $x_n \in \operatorname{Span}(X \setminus \{x_n\})$ and $X$ is not weakly independent. □

As shown in the following example, Corollary 5.7 does not hold for general semiartinian rings.

**Example 5.9.** There is a commutative semiartinian ring which does not have the weak basis property.

*Proof.* Inductively, we construct partitions $\mathcal{P}_\alpha, \alpha < \omega_1$ of $\aleph_1$ such that:

1. For each $\alpha < \omega_1$ the partition $\mathcal{P}_\alpha$ is a refinement of $\mathcal{P}_{\alpha+1}$, but no set from $\mathcal{P}_{\alpha+1}$ is a finite union of sets from $\mathcal{P}_\alpha$.
2. For each $\alpha < \omega_1$ we have $\operatorname{card}(\mathcal{P}_\alpha) = \aleph_1$ and for each $P \in \mathcal{P}_\alpha$ we have $\operatorname{card}(P) \leq \aleph_0$.
3. There is $P_\alpha^0 \in \mathcal{P}_\alpha$ for $\alpha < \omega_1$ such that $P_\alpha^0 \subseteq P_{\alpha+1}^0$ for each $\alpha < \omega_1$ and $\bigcup_{\alpha<\omega_1} P_\alpha^0 = \aleph_1$.

For the first step, we just let $\mathcal{P}_0$ to be the set of all singletons from $\aleph_1$. Suppose that we already constructed $\mathcal{P}_\beta$ for all $\beta < \alpha$ for some $\alpha < \omega_1$. For each $\beta < \alpha$ we fix enumerations $\mathcal{P}_\beta = \{P_\beta^\gamma \mid \gamma < \omega_1\}$. First, let $\alpha$ be a successor. We put $P_\alpha^0 = \bigcup_{\gamma<\max(\omega,\alpha)} P_{\alpha-1}^\gamma$. The rest of sets in $\mathcal{P}_{\alpha-1}$ can easily be divided to $\aleph_1$ disjoint subsets of cardinality $\aleph_0$, and their unions will be enumerated by $P_\alpha^\gamma, 0 < \gamma < \omega_1$. Putting $\mathcal{P}_\alpha = \{P_\alpha^\gamma \mid \gamma < \omega_1\}$, we fulfilled conditions (1) and (2).

Suppose now that $\alpha$ is limit. For each $a \in \aleph_1$, we define a subset $P_\alpha^a$ of $\aleph_1$ by putting $P_\alpha^a = \bigcup_{\beta<\alpha}\{P \in \mathcal{P}_\beta | a \in P\}$. We set $\mathcal{P}_\alpha = \{P_\alpha^a \mid a \in \aleph_1\}$. This is easily verified to be a partition of $\aleph_1$. Since $\alpha < \omega_1$, it follows from (1) and (2) that $\operatorname{card}(P_\alpha^a) \le \aleph_0$, and thus also $\operatorname{card}(\mathcal{P}_\alpha) = \aleph_1$. Note that $P_\alpha^0 = \bigcup_{\beta<\alpha} P_\beta^0$. Finally, since $\alpha \subseteq P_\alpha^0$ for each $\alpha < \omega_1$, it follows that $\bigcup_{\alpha<\omega_1} P_\alpha^0 = \aleph_1$. We let $\mathcal{P} = \bigcup_{\alpha<\omega_1} \mathcal{P}_\alpha$.

We construct a subring of $\mathbb{Z}_2{}^{\omega_1}$ by taking the $\mathbb{Z}_2$-linear span of $\mathcal{P} = \bigcup_{\alpha<\omega_1} \mathcal{P}_\alpha$ united with $\{1\}$, identifying subsets of $\aleph_1$ with elements of $\mathbb{Z}_2{}^{\omega_1}$ in the obvious way. This is easily seen to be a commutative regular ring. Also, we naturally get a socle sequence for $R$ of length $\omega_1 + 1$ (the $(\alpha+1)$-th socle is generated by $\mathcal{P}_\alpha$ for each $\alpha < \omega_1$), hence $R$ is semiartinian[4]. Take the $\omega_1$-th socle $S = S_{\omega_1} = \operatorname{Span}_{\mathbb{Z}_2}(\mathcal{P})$. We claim that $S$ does not have a weak basis. Suppose that $X$ is a weak basis of $S$. Since $S$ is obviously not finitely generated, there are $x_n, n \in \omega$ pairwise different elements in $X$. Then there are sets $P_n \in \mathcal{P}, n \in \omega$ such that $\{x_n \mid n \in \omega\} \subseteq \operatorname{Span}(\{P_n \mid n \in \omega\})$. Since each $P_n$ is a countable subset of $\aleph_1$, there is by (3) and (1) a set $P \in \mathcal{P}$ such that $\bigcup_{n<\omega} P_n \subseteq P$. But $P \in S = \operatorname{Span}(X)$ and using Lemma 5.8, we conclude from $\{x_n\} \subseteq \operatorname{Span}(\{P\})$ that $X$ is not weakly independent. $\square$

The ring constructed in Example 5.9 is regular (this can be seen either directly from the construction or noting that the ring has zero radical and using [NP68, Théorème 3.1.]), and hence max by Fact 5.4. Therefore, we have already found an example of a (commutative, semiprimitive) max ring without the weak basis property. In the next subsection we provide a wide class of non-semiartinian regular rings without the weak basis property.

**Lemma 5.10.** *Let $R$ be a regular ring and $Y = \{y_n \mid n \in \omega\}$ a countable subset of $R$. Then there is a countable set $\{x_n \in n \in \omega\}$ of orthogonal idempotents of $R$ such that $\operatorname{Span}(Y) = \bigoplus_{n\in\omega} Rx_n$. Furthermore, we can require that $y_n \in \operatorname{Span}(\{y_0, \dots, y_{n-1}, x_n\})$ for each $n \in \omega$.*

*Proof.* Let $I_n$ be a left ideal of $R$ generated by $\{y_0, \dots, y_n\}$ for each $n \in \omega$. We will inductively construct the desired elements $x_n, n \in \omega$. For the first step, let $x_0$ be an idempotent such that $Rx_0 = I_0$. Suppose that we have already found orthogonal idempotents $x_0, \dots, x_n$ such that $I_n = \bigoplus_{i=0}^n Rx_i$. Put $f_n = x_0 + \dots + x_n$; by the orthogonality of $x_0, \dots, x_n$ we have that $f_n$ is an idempotent with $I_n = Rf_n$. Now $I_{n+1} = Rf_n \oplus I_{n+1}(1 - f_n)$. Let $x_{n+1}$ be an idempotent such that $Rx_{n+1} = I_{n+1}(1 - f_n)$. It follows easily that $x_{n+1}f_n = f_n x_{n+1} = 0$, and therefore $x_0, \dots, x_{n+1}$ are orthogonal. Also, we can see that $I_{n+1} = \bigoplus_{i=0}^{n+1} Rx_i$.

Together we have that $\operatorname{Span}(Y) = \bigoplus_{n\in\omega} Rx_n$. The furthermore part follows from the fact that $I_n = I_{n-1} + Rx_n$ for each $n > 0$. $\square$

[4] For more details we refer the reader to [CF74], where the authors used similar construction to show that there is a semiartinian ring of length $\alpha + 1$ for any ordinal $\alpha$.

**Corollary 5.11.** *Let $R$ be a regular ring. Then any countably generated ideal of $R$ is weakly based.*

So far all examples of rings with the weak basis property were semiartinian of finite Loewy length. We show that there are examples of infinite Loewy length as well.

**Lemma 5.12.** *Let $R$ be a commutative semiartinian ring such that any ideal of $R$ decomposes into a direct sum of principal ideals. Then $R$ has the weak basis property.*

*Proof.* By [NP68, Théorème 3.1.] and Lemma 5.3, we can assume that $R$ has zero radical.

Let $L(R) = \lambda$ and denote $S_\alpha = \mathrm{Soc}_\alpha(R)$ for each $\alpha \leq \lambda$. Observe that given any $R$-module $M$, the socle sequence of $R$ induces a natural filtration of $M$ by putting $M_\alpha = S_\alpha M$ for each $\alpha \leq \lambda$. In particular, $M_{\alpha+1}/M_\alpha$ is semisimple for each $\alpha < \lambda$ because it is a homomorphic image of a direct sum of $S_{\alpha+1}/S_\alpha$. We define $\ell(M)$ to be the smallest ordinal $\alpha \leq \lambda$ such that $M = M_\alpha$ (i.e., $M = S_\alpha M$).

Now we show by induction on $\ell(M)$ that each $R$-module $M$ is weakly based. If $\ell(M) \leq 1$, then $M$ is semisimple, and thus weakly based. Suppose that all $R$-modules with $\ell(M) < \mu \leq \lambda$ are weakly based. First we assume that $\mu$ is a successor. Since $M_\mu/M_{\mu-1}$ is a semisimple module, there is a subset $X$ of $M = M_\mu$ lifting a weak basis of $M/M_{\mu-1}$. The module $M/\operatorname{Span}(X)$ is then divisible by $S_{\mu-1}$, and thus $\ell(M/\operatorname{Span}(X)) < \mu$. Therefore, induction applies and there is a weak basis $\bar{Y}$ of $M/\operatorname{Span}(X)$. Since $M/\operatorname{Span}(X) = S_{\mu-1}(M/\operatorname{Span}(X))$, we can choose a subset $Y$ of $M_{\mu-1}$ lifting $\bar{Y}$ over $\operatorname{Span}(X)$. Since $X$ lifts a weakly independent set over $M_{\mu-1} \supseteq \operatorname{Span}(Y)$ and $Y$ lifts a weakly independent set over $\operatorname{Span}(X)$, it follows that $X \cup Y$ is weakly independent by Lemma 2.4, and thus is the desired weak basis of $M$.

Suppose now that $\mu$ is a limit ordinal. By hypothesis, $S_\mu$ is a direct sum of principal ideals. Since $R$ has zero radical, we have by [NP68, Théorème 3.1.] that $R$ is regular, and thus we can safely assume that $S_\mu = \bigoplus_{i\in I} e_i R$ where $e_i$ is an idempotent of $R$ for each $i$ from some index set $I$. Because $M = S_\mu M$ and $R$ is commutative, we have $M = \sum_{i\in I} e_i M$. If $i, j \in I$ are distinct and $m \in e_i M \cap e_j M$, we have by idempotency of $e_i, e_j$ and the commutativity of $R$ that $m = e_i m = e_j e_i m = 0m = 0$. Therefore $e_i M \cap e_j M = 0$, and it follows that $M = \bigoplus_{i\in I} e_i M$. Hence it is enough to show that $e_i M$ is weakly based for each $i \in I$. But $e_i$ is an element of $S_\mu = \bigcup_{\alpha<\mu} S_\alpha$, and thus there is $\alpha < \mu$ such that $e_i \in S_\alpha$. We have $S_\alpha(e_i M) = e_i M$ which implies that $\ell(e_i M) \leq \alpha < \mu$, and the induction hypothesis shows that $e_i M$ is indeed weakly based. This concludes the proof. □

**Definition.** Let $\varkappa$ be a cardinal number. We say that a ring $R$ is $\varkappa$-noetherian if any ideal of $R$ is at most $\varkappa$-generated.

**Corollary 5.13.** *An $\aleph_0$-noetherian regular semiartinian commutative ring has the weak basis property.*

*Proof.* By Lemma 5.10, any countably generated ideal of $R$ is a direct sum of principal ideals. The rest is an application of Lemma 5.12. □

**Example 5.14.**
- There is an $\aleph_0$-noetherian regular semiartinian commutative ring with zero radical of infinite Loewy length. In particular, there is by Corollary 5.13 a semiartinian ring with the weak basis property of infinite Loewy length.
- There is a commutative semiartinian ring with the weak basis property with an ideal which does not decompose into a direct sum of principal ideals.

*Proof.*
- It is easy to construct a sequence $\mathcal{P}_n, n \in \omega$ of subsequently refining partitions of $\omega$ such that for each $n > 0$ there is no $P \in \mathcal{P}_n$ which can be written as a finite union of sets from $\mathcal{P}_{n-1}$. We view subsets of $\omega$ as elements of $\mathbb{Z}_2^\omega$ and define $R$ to be the $\mathbb{Z}_2$-span of the set $\bigcup_{n\in\omega} \mathcal{P}_n \cup \{1\}$. It is easy to check that $R$ is a subalgebra of $\mathbb{Z}_2^\omega$ and that $R$ is a semiartinian ring with $L(R) = \omega + 1$ (the $(n+1)$-th socle of $R$ is generated by $\mathcal{P}_n$ for every $n \in \omega$). Also, $R$ is obviously a regular ring and the $\aleph_0$-noetherian property follows from the fact that $R$ is countable.
- We construct ring $R$ with the desired properties in a similar fashion as above, as a subring of $\mathbb{Z}^\omega$. Let $\mathcal{A}$ be a uncountable almost disjoint system of subsets of $\omega$ (see Lemma 3.14 and the definition preceding it).

  Defining $R$ as a $\mathbb{Z}_2$-span of $\{\{n\} \mid n \in \omega\} \cup \mathcal{A} \cup \{1\}$ in the $\mathbb{Z}_2$-algebra $\mathbb{Z}^\omega$, we again obtain a ring. It is easy to check that $R$ is semiartinian with $S_1$ generated by the singletons, $S_2$ is generated by $\mathcal{A}$ and $S_3 = R$. Thus $R$ is of finite Loewy length and Corollary 5.7 shows that $R$ has the weak basis property. On the other hand, $S_2$ does not decompose into a direct sum of principal ideals. Indeed, that would induce a partition of $\omega$ into uncountable amount of non-empty disjoint sets, which is not possible.

$\square$

### 5.1.2 Baer regular rings

**Definition.** A ring $R$ is a *Baer* ring[5] provided that for any subset $X$ of $R$ the left annihilator $\{r \in R \mid rX = 0\}$ of $X$ is generated by an idempotent (as a left ideal of $R$).

**Fact 5.15.** [Ber88, Corollary 1.22] *Let $R$ be a regular ring. Then $R$ is Baer if and only if the lattice of left principal ideals of $R$ is complete.*

**Lemma 5.16.** *Let $R$ be a regular Baer ring. Let $A$ be a set and $\{I_\alpha \mid \alpha \in A\}$ a collection of pairwise disjoint left principal ideals of $R$. Then for any subset $B$ of $A$, there are left principal ideals $J_0, J_1$ of $R$ such that $J_0 \cap J_1 = 0$, $I_\alpha \subseteq J_0$ for each $\alpha \in B$ and $I_\alpha \subseteq J_1$ for each $\alpha \in A \setminus B$.*

*Proof.* Put $B_0 = B$ and $B_1 = A \setminus B$. Let $J_i$ be the intersection of all principal ideals of $R$ which contain all ideals $I_\alpha, \alpha \in B_i$ for each $i \in 2$. By Fact 5.15, ideal $J_i$ is principal for each $i \in 2$. It suffices to show that $J_0 \cap J_1 = 0$. Towards a contradiction, suppose that $j \in J_0 \cap J_1$ is non-zero. Since $j \in J_0$, it follows from the definition of $J_0$ that there is $\alpha \in B_0$ such that $Rj \cap I_\alpha \neq 0$. Then $J_1 \cap I_\alpha \neq 0$. Because $R$ is regular and $I_\alpha$ is principal, $I_\alpha$ is a direct summand of $R$. It follows

[5] This is a left-right symmetric property - see [Lam99, Proposition 7.46].

that there is an ideal $C$ such that $I_\alpha \oplus C = R$ and $I_\beta \subseteq C$ for all $\beta \in B_1$. We have by the definition of $J_1$ that $J_1 \subseteq C$. This is a contradiction with $J_1 \cap I_\alpha \neq 0$. $\square$

**Lemma 5.17.** *Let $R$ be a regular ring. Then* $\mathrm{Soc}(R)$ *is exactly the intersection of left non-principal maximal ideals of $R$.*

*Proof.* Let $V$ be a left minimal ideal. Then $R = V \oplus M$, where $M$ is a left maximal ideal such that $V \simeq R/M$, hence $M$ is principal. It is easily seen that $V$ lies in the intersection of all left non-principal maximal ideals.

On the other hand, let $m$ be an element contained in each left non-principal maximal ideal. We can suppose that $m$ is an idempotent. Then $(1-m)$ is not contained in any left non-principal maximal ideal. Note that any left maximal ideal containing $\mathrm{Soc}(R)$ is necessarily non-principal. Thus $R(1-m)+\mathrm{Soc}(R)) = R$, in particular, $m \in \mathrm{Soc}(R)$. $\square$

**Proposition 5.18.** *Let $R$ be a Baer regular ring. Then all the non-principal left maximal ideals of $R$ are not weakly based. In particular, $R$ has the weak basis property if and only if $R$ is semisimple.*

*Proof.* Let $I$ be a left maximal ideal of $R$ such that $I$ is not principal. Then $I$ is infinitely generated. Towards a contradiction, let $Y$ be a weak basis of $I$. Pick pairwise distinct elements $y_n, n \in \omega$ from $Y$. Let $x_n, n \in \omega$ be orthogonal idempotents as in Lemma 5.10, in particular $\mathrm{Span}(\{y_n \mid n \in \omega\}) = \bigoplus_{n\in\omega} \mathrm{Span}(\{x_n\})$. We claim that $X = (Y \setminus \{y_n \mid n \in \omega\}) \cup \{x_n \mid n \in \omega\}$ is a weak basis of $Y$. Of course $I = \mathrm{Span}(X)$. Suppose that $X$ is not weakly independent, that is, there is $x \in X$ such that $x \in \mathrm{Span}(X \setminus \{x\})$. If $x \neq x_m$ for any $m \in \omega$, then $x \in \mathrm{Span}(Y \setminus \{y\})$, because $\mathrm{Span}(\{y_n \mid n \in \omega\}) = \mathrm{Span}(\{x_n \mid n \in \omega\})$, a contradiction with weak independence of $Y$. Suppose that $x = x_m$ for some $m \in \omega$. Because $x_n, n \in \omega$ are orthogonal idempotents, we infer that $x_m \in \mathrm{Span}(Y \setminus \{y_n \mid n \in \omega\})$. But then, by Lemma 5.10, we have that $y_m \in \mathrm{Span}(\{y_0, \dots, y_{m-1}, x_m\}) \subseteq \mathrm{Span}(Y \setminus \{y_n \mid n \geq m\})$, a contradiction with weak independence of $Y$.

Let $\omega = A_0 \cup A_1$ be a partition of $\omega$ into two infinite sets and put $M_i = \mathrm{Span}(\{x_n \mid n \in A_i\})$ for each $i \in 2$. Since $R$ is Baer regular, we can by Lemma 5.16 find left principal ideals $J_i \subseteq R, i \in 2$ such that $J_i$ contains $M_i$ for each $i \in 2$ and $J_0 \cap J_1 = 0$. By [Goo79, Proposition 2.11], there are idempotents $a_i, i \in 2$ of $R$ such that $J_i = Ra_i$, and $a_1 a_0 = 0$.

Since $X$ was assumed to be weakly independent, Lemma 5.8 shows that $a_i \notin I$ for each $i \in 2$. Therefore, there are $z_0, \dots, z_k \in X$ such that $1 \in \mathrm{Span}(\{z_0, \dots, z_k, a_0\})$. Hence, $a_1 \in \mathrm{Span}(\{z_0, \dots, z_k\}) \subseteq \mathrm{Span}(X)$ which is a contradiction.

To demonstrate the validity of the last claim of the Proposition, we refer to Lemma 5.17, which shows that a regular ring has a non-principal maximal left ideal if and only if it is semisimple. $\square$

Baer regular rings include products of division rings, endomorphism rings of vector spaces, self-injective regular rings ([Lam99, Corollary 7.53]), and $C^*$-algebras.

## 5.2 Strong weak basis property

An analogy of Lemma 5.3 holds for the strong weak basis property. The proof is basically the same.

**Lemma 5.19.** *Let $R$ be a ring and $J$ its radical. Then $R$ has the strong weak basis property if and only if $J$ is $T$-nilpotent and $R/J$ has the strong weak basis property.*

*Proof.* ($\Rightarrow$) Because $R$ has *a fortiori* the weak basis property, Lemma 5.3 shows that $J$ is $T$-nilpotent. The strong weak basis property of $R/J$ is obvious.

($\Leftarrow$) Let $M$ be an $R$-module and $X$ its generating set. Let $\bar{X}$ denote the image of $X$ in the canonical projection of $M$ onto $M/JM$. Since $M/JM$ is naturally an $R/J$-module and $R/J$ has the strong weak basis property, there is a subset $\bar{Y}$ of $\bar{X}$ which is a weak basis of $M/JM$. Let $Y$ be a subset of $X$ lifting $\bar{Y}$ over $JM$. We claim that $Y$ is a weak basis of $M/JM$. The weak independence of $Y$ is clear. Note that $M = JM + \operatorname{Span}(Y)$. It follows that $M/\operatorname{Span}(Y) = J(M/\operatorname{Span}(Y))$, and therefore, by the $T$-nilpotency of $J$, we have that $M = \operatorname{Span}(Y)$. Hence, $R$ has the strong weak basis property. $\square$

**Lemma 5.20.** *([NN91b, Theorem 1.1]) Any local perfect ring has the strong weak basis property.*

*Proof.* Let $R$ be a local perfect ring. By Lemma 5.19, we can, without loss of generality, assume that $R$ has zero radical. In this case $R$ is a division ring. Let $M$ be an $R$-module. It is easy to see that the set of all weakly independent subsets of $M$ is inductive. Therefore, we can apply Zorn's Lemma in order to obtain a maximal weakly independent subset $X$ of $M$. We claim that $X$ is a weak basis of $M$. Suppose that there is $m \in M$ such that $m \notin \operatorname{Span}(X)$. By maximality of $X$, there are $x_1, \dots, x_n \in X$ and $r_0, \dots, r_n \in R$ such that

$$r_0 m + r_1 x_1 + \cdots + r_n x_n = 0$$

where at least one of the scalars $r_i$ is invertible in $R$ (i.e., $r_i$ is non-zero). Since $X$ is weakly independent, $r_0$ is non-zero, and thus invertible in $R$. It follows that $m \in \operatorname{Span}(X)$, a contradiction. $\square$

In [NN91a], the authors proved that any ring with the left strong weak basis property is left perfect. We present a simpler proof of this fact.

**Lemma 5.21.** *Let $R$ be a ring with the left strong weak basis property. Then $R$ is left perfect.*

*Proof.* By Lemma 5.19, the radical $J$ of $R$ is left $T$-nilpotent and $R/J$ has the weak basis property. We can thus assume that $R$ is already semiprimitive and the goal is to show that $R$ is semisimple. Suppose that $R$ is not semisimple. Then $R$ contains an infinite amount of distinct left maximal ideals (otherwise $R$ would be semisimple by Chinese remainder theorem, because the radical is zero). Let $\{M_n \mid n \in \omega\}$ be a collection of pairwise distinct left maximal ideals and put $S = \bigoplus_{n\in\omega} R/M_n$. Denote by $e_n$ the projection of 1 to $R/M_n$ for all $n \in \omega$. Define a subset $X$ of $S$ as follows: $X = \{e_0 + \cdots + e_n \mid n \in \omega\}$. It is easy to see that $X$ generates $S$. Because $\operatorname{Ann}(e_n) = M_n$ for any $n \in \omega$, we have that

$\mathrm{Span}(\{e_0 + \cdots + e_n\}) = \bigoplus_{i \leq n} R/M_i$. Then any pair of elements of $X$ is weakly dependent and it follows that $X$ does not contain any weak basis of $S$. This is a contradiction with $R$ having the strong weak basis property. $\square$

We alter the original proof of Lemma 5.21 by Nashier and Nichols slightly to show that the module witnessing that a non-perfect ring does not possess the strong weak basis property can actually be always chosen as a free module of an infinite rank.

**Proposition 5.22.** *Let $R$ be a (left) non-perfect ring. Then any free $R$-module of an infinite rank is not strongly weakly based.*

*Proof.* Let $R$ be a left non-perfect ring. By [NN91a, Proposition 1 and Theorem 2], there is a sequence $r_n \in R, n \in \omega$ such that for each $i \in \omega$ and $n > i$ we have that $Rr_{i+1} \dots a_n \neq Rr_i r_{i+1} \dots r_n$. We can safely assume that $r_n$ is a non-unit in $R$ for any $n \in \omega$. Let $F = R^{(\omega)}$ be a free $R$-module of rank $\omega$ and let $B = \{b_n \mid n \in \omega\}$ be a free basis of $F$. Define for each $n \in \omega$:

$$x_n = b_n + r_n b_{n+1};$$

$$y_n = r_n b_{n+1}$$

and put $Z = \{x_n, y_n \mid n \in \omega\}$, $X = \{x_n \mid n \in \omega\}$. It is easy to see that $B \subseteq \mathrm{Span}(Z)$, and thus $Z$ generates $F$. We will show that $Z$ does not contain any weak basis of $F$. Suppose towards a contradiction that $Z' \subseteq Z$ is a weak basis of $F$. Since $r_n$ is a non-unit for each $n \in \omega$, we conclude that necessarily $X \subseteq Z'$. Put $X_k = X \cup \{y_k\}$ for each $k \in \omega$. We claim that $b_{k+1} \notin \mathrm{Span}(X_k)$ for any $k \in \omega$. Suppose towards a contradiction that $b_{k+1} = sy_k + \sum_{n \in \omega} s_n x_n$ for some $s, s_n \in R$, $n \in \omega$ such that $s_n = 0$ for all $n > m$ for some $m \in \omega$. Comparing the coefficients in basis $B$, we obtain:

$$s_0 = 0, s_1 = 0, \dots, s_k = 0;$$

$$1 = sr_k + s_{k+1}, 0 = s_{k+1} r_{k+1} + s_{k+2}, \dots, 0 = s_{m-1} r_{m-1} + s_m, 0 = s_m r_m.$$

Together we get (modulo a sign)

$$(1 - sr_k) r_{k+1} \dots r_m = 0,$$

and therefore

$$r_{k+1} \dots r_m = sr_k r_{k+1} \dots r_m.$$

But then $Rr_{k+1} \dots r_m = Rr_k r_{k+1} \dots r_m$, a contradiction with our choice of elements $r_n, n \in \omega$. On the other hand, it can be easily seen that $b_n \in \mathrm{Span}(X_k)$ for any $n < k$. It follows that there are $k < l \in \omega$ (and in fact there is an infinite number of such indices) such that $X_k, X_l \subseteq Z'$. But since $y_k \in \mathrm{Span}(X_l)$ and $y_k \notin X_l$, this contradicts $Z'$ being weakly independent. $\square$

### 5.2.1 Strongly weakly based modules over Dedekind domains

At this point we address strongly weakly based torsion modules over Dedekind domains. We do not provide a complete characterization. In fact, the gap left in the following proposition can serve as a motivation to the main problem of Nashier and Nichols presented in this section - do non-local perfect rings have the strong weak basis property ?

**Lemma 5.23.** *Let $M$ be a strongly weakly based module over a ring $R$. Then any direct summand of $M$ is strongly weakly based.*

*Proof.* Let $N$ be a submodule of $M$ such that $M = N \oplus C$ for some submodule $C$. Let $X$ be a generating set of $N$ and $Y$ be a generating set of $C$. Since $M$ is strongly weakly based, there is a subset $Z$ of $X \cup Y$ which is a weak basis of $M$. Then $X \cap Z$ is a weak basis of $N$. Therefore, $N$ is strongly weakly based. $\square$

**Lemma 5.24.** *Let $R$ be a Dedekind domain, and $M$ an $R$-module. Suppose that either $R$ is local or $M$ is $P$-primary for some $P \in \mathrm{Spec}(R)$. If there is a projection of $M$ onto a non-zero divisible $R$-module, then $M$ is not strongly weakly based.*

*Proof.* Suppose that there is a projection $\pi : M \to D$ where $D$ is a non-zero divisible module. Since $D$ is divisible, there is a subset $X$ of $PM$ (if $R$ is local let $P$ be the radical of $R$) lifting some generating set of $D$ via $\pi$. Let $Y$ be some generating set of $\mathrm{Ker}(\pi)$. We have $M = \mathrm{Span}(X \cup Y)$. Suppose that there is a subset $Z$ of $X \cup Y$ which is a weak basis of $M$. By Lemma 4.5 or Lemma 4.19, the set $Z$ lift a basis of $M/PM$ over $PM$. Since $X \subseteq PM$, it follows that $Z \subseteq Y \subseteq \mathrm{Ker}(\pi)$, a contradiction to $M = \mathrm{Span}(Z)$. $\square$

We say that an $R$-module $M$ over a Dedekind domain $R$ is *bounded* if $\mathrm{Ann}(M) \subsetneq R$.

**Proposition 5.25.** *Let $R$ be a Dedekind domain and let $M$ be a torsion $R$-module. Then:*

1. *If $M$ is strongly weakly based, then $M$ is bounded.*
2. *Suppose that $M$ is $P$-primary for some $P \in \mathrm{Spec}(R)$. Then $M$ is strongly weakly based if and only if $M$ is bounded.*

*Proof.* 1. Suppose that $M$ is an unbounded torsion $R$-module and recall that given $P \in \mathrm{Spec}(R)$ we denoted by $M_P$ the $P$-primary component of $M$. Then either there is a $P \in \mathrm{Spec}(R)$ such that $M_P$ is unbounded or there is an infinite number of distinct prime ideals $Q$ such that $M_Q$ is non-zero. Let us treat those two cases separately.

Suppose that $M_P$ is unbounded and let us show that $M$ is not strongly weakly based. Since $M_P$ is a direct summand of $A$, we can by Lemma 5.23 assume that already $M = M_P$. We claim that there is a projection of $M$ onto a non-zero divisible $R$-module. Let $B$ be a basic submodule of $M$ (for the definition see Chapter 4). If $B$ is a proper submodule of $M$, then we are done by the definition of basic submodule. If $M = B$, then $M$

is a direct sum of cyclic $P$-primary modules of unbounded order, and the standard construction of the Prüfer $P$-module as a direct limit of cyclic $P$-modules provides us with the desired projection. In both cases, we have a projection of $M$ onto a non-zero divisible $R$-module. By Lemma 5.24, $M$ is not strongly weakly based.

Suppose that there is an infinite subset $\Phi$ of $\mathrm{Spec}(R)$ such that $M_Q$ is non-zero for each $Q \in \Phi$. By [Kap54, Theorem 9], there is either a divisible or cyclic direct summand $B_Q$ of $M_Q$ for each $Q \in \Phi$. If $B_Q$ is divisible, then $M$ is not strongly weakly based by Lemma 5.24 and Lemma 5.23. Suppose that $B_Q$ is cyclic for all $Q \in \Phi$. Put $B = \bigoplus_{Q\in\Phi} B_Q$. Since $B$ is a direct summand of $M$, it is by Lemma 5.23 enough to show that $B$ is not strongly weakly based. Let $x_Q$ be a generator of $B_Q$ for each $Q \in \Phi$. Enumerate $\Phi = \{Q_n \mid n \in \omega\}$ and put $y_n = \sum_{i=0}^{n} x_{Q_i}$. Observe that $B = \mathrm{Span}(\{y_n \mid n \in \omega\})$. On the other hand, for each $n > 0$ we have $y_0, \dots, y_{n-1} \in \mathrm{Span}(\{y_n\})$. This readily shows that we cannot choose a subset of $\{y_n \mid n \in \omega\}$ which is a weak basis of $B$.

2. It suffices to show that a bounded $P$-primary $R$-module $M$ is strongly weakly based. Let $n \in \omega$ be such that $P^n M = 0$. Then $M$ can be viewed naturally as an $R/P^n$-module. But $R/P^n$ is a local perfect ring, and therefore, by Lemma 5.20, a ring with the strong weak basis property. Hence, $M$ is strongly weakly based.

□

**Example 5.26.** There is a semisimple abelian group which is not strongly weakly based.

*Proof.* Put $A = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p$. Then $A$ is a semisimple abelian group. Since $A$ is not bounded, it is not strongly weakly based by Proposition 5.25. □

**Theorem 5.27.** *Let $R$ be a DVR and $M$ an $R$-module. Then $M$ is strongly weakly based if and only if $M \simeq G \oplus T$ where $G$ is a free module of finite rank and $T$ is a bounded torsion module.*

*Proof.* ($\Leftarrow$) Let $X$ be a generating set of $M = G \oplus T$ where $G$ is free of finite rank and $T$ is bounded torsion. Since $G$ is finitely generated module, there is a finite subset $X_0$ of $X$ such that $G \subseteq \mathrm{Span}(X_0)$. Then $T' = M/\mathrm{Span}(X_0)$ is a bounded torsion module. Denote by $\bar{X}$ the image of $X$ in $T'$. By virtue of Proposition 5.25, there is a weak basis $\bar{X}_1 \subseteq \bar{X}$ of $T'$. Let $X_1$ be a subset of $X$ lifting $\bar{X}_1$ over $\mathrm{Span}(X_0)$. Put $N = M/\mathrm{Span}(X_1)$. Then $N$ is generated by the image of $X_0$. In particular, $N$ is finitely generated, and therefore strongly weakly based. It follows that there is a subset $X_2$ of $X_0$ lifting a weak basis of $N$ over $\mathrm{Span}(X_1)$.

Put $Y = X_1 \cup X_2$. We have that $M = \mathrm{Span}(Y)$. Also, by Lemma 2.4, $Y$ is weakly independent. We found a weak basis in $X$, showing that $M$ is indeed strongly weakly based.

($\Rightarrow$) Let $p$ be a prime of $R$. Since $M$ is *a fortiori* weakly based, we have by Theorem 4.16 that $M \simeq F \oplus N$ where $F$ is a free module and $\dim(\tau N/p\tau N) = \mathrm{gen}(N)$. By Proposition 5.22, a free $R$-module is strongly weakly based if and only if it has a finite rank. Therefore, $F$ is necessarily of finite rank. Now it is enough to show that $\tau N$ is bounded. Indeed, then (by [Kap84, Theorem 5]) we have $N \simeq \tau N \oplus \phi N$, implying by Lemma 5.23 that $\phi N$ is strongly weakly based, from which follows by Lemma 4.25 and Proposition 5.22 that $\phi N$ is free of finite rank.

Suppose that $\tau N$ is not bounded. We want to show that then $N$ is not strongly weakly based. By Lemma 5.24, it is enough to find a projection of $N$ onto a non-zero divisible $R$-module. Since $\tau N$ is not bounded, there is a projection of $\tau N$ onto a $p$-primary divisible $R$-module $D$ (by the same argument as in the proof of Proposition 5.25). Let $K$ be the kernel of this projection. Then $N/K$ contains an isomorphic copy of $D$. Because $D$ is injective (see [Lam99, Corollary 3.24]), we have that $D$ is a direct summand of $N/K$, and we are done. □

### 5.2.2 Perfect rings

The gap in Proposition 5.25 boils down to determining whether abelian groups which are naturally modules over $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$ have the strong weak basis property. This question generalizes to the question found in [NN91a] which asks if the rings with the strong weak basis property are exactly the perfect rings. By virtue of Lemma 5.19, this is equivalent to all semisimple rings having the strong weak basis property. As far as we know, this is an open problem. The difficulty behind this question is that "hungry" arguments using Zorn's lemma in a simple way do not work in this setting, even for finitely generated modules. This is demonstrated in the following very simple example.

**Example 5.28.** Let $p, q$ be two distinct primes and let us consider an abelian group $A = \mathbb{Z}_p^2 \oplus \mathbb{Z}_q^2$. Let us define a generating set of $A$ using the following diagram.

$$\mathbb{Z}_p \overset{x_1}{\longleftrightarrow} \mathbb{Z}_q \overset{x_2}{\longleftrightarrow} \mathbb{Z}_p \overset{x_3}{\longleftrightarrow} \mathbb{Z}_q$$

The vertices in this diagram are the simple subgroups of $A$ forming some chosen decomposition to simple groups. Each edge represent an element of $A$ chosen so that it generates the simple subgroups it connects. We put $X = \{x_1, x_2, x_3\}$; it is easy to observe that $A = \mathrm{Span}(X)$. Then $\{x_1, x_2\}$ is a maximal weakly independent subset of $X$, but it does not generate $A$.

Since our effort to devise some more intricate way to choose weak bases came to no avail, we tried to prove this or to find a counterexample in a special setting. A natural thing to consider is the smallest non-division ring type - a semisimple ring $R = V_1 \oplus V_2$ where $V_i$ are left simple $R$-modules. The second restriction we imposed was on the properties of the generating set - we examined only those sets of elements which can be represented by a similar diagram as used in Example 5.28. We show that in this special setting there is no counterexample.

**Definition.** Let $M$ be a module and $I$ a set. For each $i \in I$ let us denote by $M^{(i)}$ the $i$-th copy of $M$ as a submodule of $M^{(I)}$.

Let $R$ be a semisimple ring such that $R = V_1 \oplus V_2$ where $V_1, V_2$ are simple $R$-modules and $M$ an $R$-module. We say that a subset $X$ of $M$ is *lucid* provided that there is a decomposition $M = V_1^{(I)} \oplus V_2^{(J)}$ such that for each $x \in X$ there is $i \in I$ and $j \in J$ such that $\mathrm{Span}(\{x\}) \in \{V_1^{(i)}, V_2^{(j)}, V_1^{(i)} \oplus V_2^{(j)}\}$.

Let $M$ be a module over $R = V_1 \oplus V_2$ and let $X$ be a lucid generating set of $M$. Then we can describe this situation satisfactorily by a graph $G$ in the following way. The vertices $V$ of $G$ will be the copies of simple modules $V_1, V_2$ forming the decomposition of $M$ witnessing the lucidity of $X$. The (non-oriented) edges $E$ of $G$ will correspond to elements $x \in X$; the edge corresponding to $x$ will connect exactly the simple modules from $V$ it generates. If $x \in X$ generates only one simple module, we represent it as a loop. In this way, we defined a non-oriented graph $G$ with the possibility of multiple edges and loops.

The situation can be further simplified. Defining a pre-order $\leq$ on $X$ by setting $x \leq y$ if and only if $\mathrm{Span}(\{x\}) \subseteq \mathrm{Span}(\{y\})$ for $x, y \in X$ and taking the set $X'$ of representives of maximal elements in the factor-order, we get again a generating set (because the lattice of cyclic submodules of $M$ has bounded height). Taking $X'$ instead of $X$, the graph $G$ now does not contain multiple edges. We can also get rid of the loops. Any element $x \in X'$ generating a simple module now fulfills $\mathrm{Span}(\{x\}) \cap \mathrm{Span}(X' \setminus \{x\}) = 0$. It follows that we can, without loss of generality, assume that all elements $x \in X$ generate a module isomorphic to $V_1 \oplus V_2$ and that $G$ is a simple non-oriented graph without loops.

For any subset $Y \subseteq X$, let $G(Y)$ be the subgraph of $G$ consisting from all vertices $V$ and a subset of edges $E$ corresponding to elements from $Y$. It is easy to see that $Y$ generates $M$ if and only if each vertex of $G(Y)$ has degree of at least 1. On the other hand, $Y$ is weakly independent if and only if $G(Y)$ does not contain three distinct consecutive edges.

**Lemma 5.29.** *Let $G = (V, E)$ be a (simple, non-oriented) graph such that each vertex has a degree of at least 1. Then there is a subset $E'$ of $E$ such that $G' = (V, E')$ still has all vertices of a degree of at least 1 and such that $G'$ does not contain three distinct consecutive edges.*

*Proof.* We can, without loss of generality, suppose that $G$ is connected. By Zorn's lemma, we can, again without loss of generality, suppose that $G$ is already a tree (by taking the spanning tree). Choose a root of $G$ and let $V_n$ be the set of vertices on the $n$-th level for each $n \in \omega$ ($V_0$ is a singleton containing the root). Let $E_n$ be the set of edges connecting vertices from $V_n$ to vertices from $V_{n+1}$. Now let us find $E'$ by working out down the levels of the tree.

Begin with $E'$ empty. For the 0-th level, do the following: If there are leaves (i.e., vertices of degree one) in $V_1$, put all the edges connecting the root with those vertices to $E'$ and continue to next level. If there are no leaves in $V_1$, pick one vertex from $V_1$ and put the single edge connecting the root with this vertex to $E'$, then continue to next level. Now suppose we have worked out levels $0, \dots, n-1$. For each vertex from $v \in V_n$ do the following: if $v$ is not connected to any vertex from $V_{n-1}$, apply the procedure used for the root vertex. If $v$ is connected to some vertex in $V_{n-1}$, do the following: if there are leaves in $V_n$ under $v$, put all the edges connecting them with $v$ to $E'$, otherwise do nothing.

This procedure gives the desired set $E'$ of edges. □

**Corollary 5.30.** *Let $R$ be a semisimple ring which has a decomposition to two simple $R$-modules and let $M$ be an $R$-module. If $X$ is a lucid generating set of $M$, then there is a weak basis of $M$ contained in $X$.*

We conclude this Chapter with a certain generalization of the classical Steinitz exchange lemma from linear algebra. Let $R$ be a semisimple ring. Then there are orthogonal idempotents $e_0, \ldots, e_{n-1} \in R$ such that $R = Re_0 \oplus \cdots \oplus Re_{n-1}$, and each $Re_i$ is a simple left $R$-module (see [Goo79, Proposition 2.11]). It follows that for each $r \in R$ there is a unique subset $A \subseteq n$ such that $r = u \sum_{i \in A} e_i$ for some unit $u \in R$.

**Lemma 5.31.** *Let $M$ be a left $R$-module and $X \subseteq M$ a weakly independent subset. Let $m$ be an element of $M$ such that $m \notin \mathrm{Span}(X)$. Then either $X \cup \{m\}$ is weakly independent or there is a finite subset $F \subseteq X$ such that $X' = (X \setminus F) \cup \{m\}$ is weakly independent and $\mathrm{Span}(X) \subseteq \mathrm{Span}(X')$. Furthermore, $\mathrm{card}(F) \leq 2^{n-1} - 1$.*

*Proof.* Suppose that $X \cup \{m\}$ is not weakly independent. For each $A \subseteq n$ put

$$X_A = \{x \in X \mid x = \sum_{i=1}^{k} r_i x_i + u \sum_{j \in A} e_j m\},$$

where $k$ range over integers, $r_i$ over elements of $R$ and $u$ over units of $R$. Note that since $m \notin \mathrm{Span}(X)$, there is $j \in n$ such that $X_A = \emptyset$ whenever $j \in A$.

Let $\Phi = \{A \subseteq n \mid X_A \neq \emptyset\}$. Now pick $x_A \in X_A$ for each $A \in \Phi$. We claim that $X'' = (X \setminus \{x_A \mid A \in \Phi\}) \cup \{m\}$ is weakly independent. Suppose that there is $x \in X''$ such that $x \in \mathrm{Span}(X'' \setminus \{x\})$. Thus,

$$x = \sum_{i=1}^{k} r_i x_i + u \sum_{j \in A} e_j m$$

for some $x_i \in X'' \setminus \{x\}$, $r_i \in R$, $k \in \omega$ and unit $u$ of $R$. Since $X_A$ was non-empty, we had

$$x_A = \sum_{i=1}^{k'} r'_i x'_i + u' \sum_{j \in A} e_j m,$$

But noting that $x_A \notin \{x, x_1, \ldots, x_k\}$ and that $u, u'$ are units, we quickly obtain a contradiction with weak independence of $X$.

Since there are at most finitely many elements $x_A$ (indexed by subsets of $n-1$), which can possibly miss in $\mathrm{Span}(X'')$, we can easily remove the elements $x_A$ from $X \cup \{m\}$ iteratively until we arrive to the desired weakly independent set $X'$ such that $\mathrm{Span}(X) \subseteq \mathrm{Span}(X')$.

The bound $\mathrm{card}(F) \leq 2^{n-1} - 1$ comes from the number of non-empty subsets of $n - 1$. □

We tried, alas unsuccessfully, to employ this lemma in choosing weak bases from generating sets of modules over a semisimple ring. One would like to use some kind of "hungry" argument to the set of weakly independent subsets of some generating set $X$ ordered by inclusions of spans instead of the standard inclusion. Lemma 5.31 shows that any maximal element of this order is necessarily a weak basis. The problem is then to find a weakly independent upper bound of infinite chains of form

$$\mathrm{Span}(X_0) \subseteq \mathrm{Span}(X_1) \subseteq \mathrm{Span}(X_2) \subseteq \ldots \subseteq \mathrm{Span}(X_\alpha) \subseteq \ldots$$

where $X_\alpha$ are weakly independent.

# Chapter 6

# Open problems

In this chapter we gather several open problems concerning weak bases of modules.

**Problem 6.1.** Describe rings with the (left) weak basis property. In particular:

- Which commutative semiartinian rings have the weak basis property?
- Are all rings with the weak basis property semiartinian?

**Problem 6.2.** [NN91a] Do semisimple rings enjoy the strong weak basis property? Particular problems:

- Let $p, q$ be distinct prime numbers. Is the abelian group $\mathbb{Z}_p^{(\aleph_0)} \oplus \mathbb{Z}_q^{(\aleph_0)}$ strongly weakly based?
- Let $V$ be an infinite dimensional vector space over some field. Let $V_i, i \in I$ be a system of subspaces of $V$ such that $\dim(V_i) \leq n$ for some $n > 1$ for each $i \in I$ such that $V = \sum_{i \in I} V_i$. Is there an inclusion-minimal subset $J \subseteq I$ such that $\sum_{j \in J} V_j = V$?

**Problem 6.3.** Describe torsion-free modules over non-local Dedekind domains (torsion-free abelian groups in particular) which are strongly weakly based. Are they exactly the finitely generated ones ?

**Problem 6.4.** *(Pavel Příhoda)* For which rings is every projective module weakly based?

Example 3.7 shows that there are projective modules which are not weakly based.

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