Charles University in Prague<br>Faculty of Mathematics and Physics

## MASTER THESIS



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# Relation between Forces Induced by Fluid Flow and Dissipative Processes in Boundary Layers 

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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In on

Název práce: Souvislost mezi silami působícími na obtékaná tělesa a disipativními procesy v mezních vrstvách

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Abstrakt: Práce se zabývá vlivem disipativních procesů v mezních vrstvách, konkrétně vlivem viskozity tekutiny a dále vlivem difúze vodní páry ve vzduchu na vznik sil působících na obtékaná tělesa. Z Croccovy věty plyne, že gradient entropie má svůj podíl na vzniku cirkulace a důsledkem toho i na vznik vztlakové síly. Práce zkoumá vznik vztlakové síly při obtékání rotujícího válce, v softwaru FEniCS modeluje tuto situaci, porovnává s modelem nevazké nestlačitelné tekutiny a následně is experimentem provedeným Ing. Zdeňkem Trávníčkem a Ing. Zuzanou Broučkovou. Taktéž zkoumá vznik vztlakové síly při vypařování na horní straně válce a srovnává s případem rotujícího válce.

Klíčová slova: disipace, vztlak, rotující válec, difúze

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Abstract: The thesis concerns with the influence of dissipative processes in boundary layers, particularly with the influence of viscosity of the fluid and further with the influence of diffusion of water vapor in air on induction of forces acting on the flown-around bodies. Crocco's theorem implies that gradient of entropy contributes to induction of circulation, and in consequence to induction of lift. The thesis studies the generation of lift acting on the rotating cylinder; it models this situation using FEniCS software. Results are then compared with model of potential flow and with visualization of the experiment performed by Ing. Zdeněk Trávníček and Ing. Zuzana Broučková. The thesis also studies induction of lift acting on the cylinder, which has water vapor released on its upper part.

Keywords: dissipation, lift, rotating cylinder, diffusion

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## Introduction

As bachelor thesis was concerned with induction of lift acting on flown-around bodies, master thesis continues in this effort and studies what influence dissipative processes have on the acting forces. During these processes, in the course of change of one form of energy to another, dissipation (loss) of energy happens. Examples of dissipative processes include: friction, motion of viscous fluid, transfer of heat from a warmer body to a colder one, electrical current flowing through a resistance, and diffusion. Dissipative processes are real, irreversible, the system does not return to its original state without interaction with outer environment, the same as when tea particles infused in water do not organize back into tea leaves. Entropy production is positive during these processes. Crocco's theorem connects gradient of entropy with vorticity of velocity field. Vorticity results in circulation. Circulation creates lift.

In this thesis, we will first and foremost study dissipation of energy during viscous flow and, furthermore, diffusion of water vapor in air. Using mathematical modelling, we will create a program in FEniCS software, we will obtain visualized results and also numerically compute lift. These results can be compared with the results in the bachelor thesis, where potential flow of inviscid fluid was inquired into. We also have great opportunity to compare numerical results with visualization of the experiment performed by Ing. Zdeněk Trávníček and Ing. Zuzana Broučková from Institute of Thermomechanics, Academy of Sciences CR.

Will the influence of dissipation be perceptible? Is it nonsense to devise a plane with wings emitting water vapor on one side?

## Chapter 1

## Definitions of fundamental terms and relations

Before we start deriving and computing, let us make clear some fundamental terms used for mathematical description of reality, motion and moving objects. We will also define essential quantities and express laws of balance, from which we will proceed in further derivation. Special attention will be paid to terms and forms of relations connected with fluids. It is relevant to note that while describing reality and creating model situation we neglect and simplify certain facts, which leads to inaccurate results.

### 1.1 Material point, velocity, vorticity

For the sake of simplicity, let us have tree-dimensional vector space including matter. In reality, matter consists of atoms, molecules and so on, however we will consider it as a continuum, continuously filled with mass. On this space, we will define functions - physical quantities - which will also be time-dependent. Body is understood as a domain $V \subset \mathbb{R}^{3}$, which, in time $t_{0}$, occupies domain $V_{0}$ in the reference frame.

Motion will be described in two different ways: Lagrangian specification of the flow field (material), and Eulerian specification of the flow field (spatial).

Definition. Material point is the point in continuum, which has in initial time $t_{0}$, or in reference frame, location $\boldsymbol{X}=\left(X_{1}, X_{2}, X_{3}\right)$. In time $t$, it occurs at the point of space $\boldsymbol{x}=\left(x_{1}, x_{2}, x_{3}\right)=\boldsymbol{x}(\boldsymbol{X}, t)$, so it draws a curve in space, which is called trajectory.

Jacobian matrix of the mapping $\mathbf{x}=\mathbf{x}(\mathbf{X}, t)$ we will call deformation gradient and denote it $\mathbf{F}$. We will assume that the mapping has non-zero Jacobian (that is the determinant of Jacobian matrix) in every point $(\mathbf{X}, t) \in V_{0}$. So

$$
j=\left|\operatorname{det}\left(\begin{array}{ccc}
\frac{\partial x_{1}}{\partial X_{1}} & \frac{\partial x_{1}}{\partial X_{2}} & \frac{\partial x_{1}}{\partial X_{3}}  \tag{1.1}\\
\frac{\partial x_{2}}{\partial X_{1}} & \frac{\partial x_{2}}{\partial X_{2}} & \frac{\partial x_{2}}{\partial X_{3}} \\
\frac{\partial x_{3}}{\partial X_{1}} & \frac{\partial x_{3}}{\partial X_{2}} & \frac{\partial x_{3}}{\partial X_{3}}
\end{array}\right)\right| \neq 0 .
$$

Moreover, we assume that the mapping is injective and is an element of $C^{1}$ (continuous first derivatives). These are conditions for the existence of unique
inverse mapping $\mathbf{x}=\mathbf{x}(\mathbf{X}, t)$

$$
\begin{equation*}
\mathbf{X}=\mathbf{X}(\mathbf{x}, t) . \tag{1.2}
\end{equation*}
$$

These requirements mean that any two given different material points do not occupy the same spot at any given time and any material point occurs at one spot exactly.

In Lagrangian specification of the flow field, all functions (quantities depending on space) have $\mathbf{X}$ as variables, therefore Lagrangian specification is called 'material specification'. In Eulerian specification, the variables are points in space $\mathbf{x}$, and, at the same time, functions of variables $\mathbf{X}$ and $t$. This is the reason why material derivative is more difficult in this specification. It is derived by chain rule, see theorem 3 .

Definition. Velocity of material point is defined as derivative

$$
\tilde{\boldsymbol{v}}(\boldsymbol{X}, t)=\frac{\partial \boldsymbol{x}(\boldsymbol{X}, t)}{\partial t}
$$

or for $i$-th component, $i=1,2,3$

$$
\tilde{v}_{i}(\boldsymbol{X}, t)=\frac{\partial x_{i}(\boldsymbol{X}, t)}{\partial t} .
$$

We formulate velocity in spatial specification

$$
\tilde{\boldsymbol{v}}(\boldsymbol{X}, t)=\tilde{\boldsymbol{v}}(\boldsymbol{X}(\boldsymbol{x}, t), t)=\boldsymbol{v}(\boldsymbol{x}, t),
$$

where $\boldsymbol{X}(\boldsymbol{x}, t)$ is the inverse mapping above.
For the sake of comprehensiveness, we write acceleration in both different specifications.

Material specification:

$$
\tilde{\mathbf{a}}(\mathbf{X}, t)=\frac{\partial \tilde{\mathbf{v}}(\mathbf{X}, t)}{\partial t}=\frac{\partial^{2} \mathbf{x}(\mathbf{X}, t)}{\partial t^{2}} .
$$

Spatial specification:

$$
\begin{aligned}
\mathbf{a}(\mathbf{x}, t) & =\frac{d \mathbf{v}(\mathbf{x}, t)}{d t}=\frac{d \mathbf{v}(\mathbf{x}(\mathbf{X}, t), t)}{d t}=\frac{\partial \tilde{\mathbf{v}}(\mathbf{X}, t)}{\partial t} \\
& =\frac{\partial \mathbf{v}(\mathbf{x}(\mathbf{X}, t), t)}{\partial t}+\sum_{i=1}^{3} \frac{\partial \mathbf{v}(\mathbf{x}(\mathbf{X}, t), t)}{\partial x_{i}} \frac{\partial x_{i}(\mathbf{X}, t)}{\partial t} \\
& =\frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t}+\sum_{i=1}^{3} v_{i}(\mathbf{x}, t) \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial x_{i}} .
\end{aligned}
$$

This kind of derivative is called material derivative. From now on, we will not distinguish $\tilde{\mathbf{v}}$ and $\mathbf{v}$.

Definition. Streamline is a curve in $V$, such that the vector of velocity is, at any given time, tangent to it at all (spatial) points.

It is clear from the definition that if $d x_{1}, d x_{2}, d x_{3}$ are the components of elementary curve at given point, and $v_{1}, v_{2}, v_{3}$ are the components of velocity at that point, then $d x_{1}: d x_{2}: d x_{3}=v_{1}(\mathbf{x}, t): v_{2}(\mathbf{x}, t): v_{3}(\mathbf{x}, t)$, or

$$
\frac{d x_{1}}{v_{1}}=\frac{d x_{2}}{v_{2}}=\frac{d x_{3}}{v_{3}} .
$$

If $\mathbf{v}(\mathbf{x}, t) \neq \mathbf{0}$ and $\mathbf{v}$ has continuous first derivatives with respect to coordinates, then one and only one streamline runs through every point. Points where $\mathbf{v}(\mathbf{x}, t)=\mathbf{0}$ are called stagnation points and streamlines are not uniquely defined there.

In case of steady flow (flow which does not change in time), streamlines coincide with trajectories.

Definition. Let $\underline{s}$ be closed curve in space. If there runs one and only one streamline through every point of this curve and if this closed curve does not coincide with any streamline, then the set of all streamlines running through curve $\underline{s}$ is called a stream tube.

Fluid flowing through a stream tube acts as if the surface of the tube is impenetrable. The vector of velocity is tangent to the streamlines, thus its normal component is zero.

Definition. Gradient of scalar function $\varphi: V \subset \mathbb{R}^{3} \rightarrow \mathbb{R}$ is

$$
\nabla \varphi(\boldsymbol{b})=\left(\frac{\partial \varphi}{\partial x_{1}}, \frac{\partial \varphi}{\partial x_{2}}, \frac{\partial \varphi}{\partial x_{3}}\right)(\boldsymbol{b}) .
$$

Definition. Divergence of vector field $\boldsymbol{v}: V \subset \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ is defined as

$$
\begin{equation*}
\operatorname{div} \boldsymbol{v}(\boldsymbol{x})=\operatorname{tr} \nabla \boldsymbol{v}(\boldsymbol{x}) \tag{1.3}
\end{equation*}
$$

where tr denotes trace of matrix.
Definition. Let $P$ be a point in space, $r$ a plane that runs through this point, and o a normal vector to the plane at point $P$. Let $s$ be a closed curve around $P$, so that index of point $P$ to curve $s$ is equal to 1, in other words the curve is oriented counterclockwise. Circulation of vector $v$ is

$$
\begin{equation*}
\gamma=\oint_{s} \boldsymbol{v} \cdot d \boldsymbol{s}=\oint_{s} v_{s} d s \tag{1.4}
\end{equation*}
$$

Curl of vector $\boldsymbol{v}$ with respect to axis o is

$$
\operatorname{curl}_{\boldsymbol{o}} \boldsymbol{v}=\lim _{\Delta S \rightarrow 0} \frac{1}{\Delta S} \oint_{s} \boldsymbol{v} \cdot d \boldsymbol{s}
$$

Expression of curl in Cartesian coordinates follows from its definition (see [2]), i -th component is expressed as (by definition, the axis in the equation is the $x_{i}$ axis)

$$
\operatorname{curl}_{i} \mathbf{v}=\sum_{j, k=1}^{3} \epsilon_{i j k} \frac{\partial v_{k}}{\partial x_{j}},
$$

where $\epsilon_{i j k}$ is Levi-Civita symbol, defined as

$$
\epsilon_{i j k}= \begin{cases}1 & \text { if }(i, j, k) \text { is odd permutation } \\ -1 & \text { if }(i, j, k) \text { is even permutation } \\ 0 & \text { otherwise }\end{cases}
$$

Then full notation of all components of curl of vector $\mathbf{u}$ in Cartesian coordinates is

$$
\operatorname{curl} \mathbf{v}=\left(\frac{\partial v_{3}}{\partial x_{2}}-\frac{\partial v_{2}}{\partial x_{3}}, \frac{\partial v_{1}}{\partial x_{3}}-\frac{\partial v_{3}}{\partial x_{1}}, \frac{\partial v_{2}}{\partial x_{1}}-\frac{\partial v_{1}}{\partial x_{2}}\right) .
$$

Definition. Vorticity $\boldsymbol{w}$ is defined as $\boldsymbol{w}=\operatorname{curl} \boldsymbol{v}$, where $\boldsymbol{v}$ is velocity. Curves, defined analogically as streamlines, namely curves to which vector of vorticity is tangent in every fixed time in every point, are called vortex lines. Thus

$$
d x_{1}: d x_{2}: d x_{3}=w_{1}(\boldsymbol{x}, t): w_{2}(\boldsymbol{x}, t): w_{3}(\boldsymbol{x}, t),
$$

where $d x_{i}$ for $i=1,2,3$ are elementary components of a curve of a vortex line, and $w_{i}$ are components of vorticity $\boldsymbol{w}$.

Vortex tube is a set of all vortex lines passing trough a closed curve, such that every point of this vortex tube lies on a different vortex line. Let us consider a cross-section of the tube with a surface $S$, then vortex flux is defined as

$$
\mu=\int_{S} \boldsymbol{w} \cdot \boldsymbol{\nu} d S=\int_{S} \boldsymbol{w}_{\boldsymbol{\nu}} d S,
$$

where $\boldsymbol{\nu}$ is unit normal vector to the cross-section of the tube. The expression above represents flux of a vector field of vorticity through a cross-section of the vortex tube.

If there exists a point such that $\mathbf{w} \neq \mathbf{0}$, then motion of the fluid is rotational, otherwise it is irrotational and therefore there exists a velocity potential $\varphi$ which is a scalar function of variables $(\mathbf{x}, t)$, is an element of $C^{2}(V)$, and fulfills

$$
\begin{equation*}
\nabla \varphi=\left(\frac{\partial \varphi}{\partial x_{1}}, \frac{\partial \varphi}{\partial x_{2}}, \frac{\partial \varphi}{\partial x_{3}}\right)=\mathbf{v}, \forall(\mathbf{x}, t), \mathbf{x} \in V, t \geq t_{0} \tag{1.5}
\end{equation*}
$$

(Proof is to be found in e.g. [6].)
Reverse implication holds as well: if there exists a potential for velocity, then curl $\mathbf{v}=\mathbf{0}$, which can easily be demonstrated by symmetry of second derivatives of function $\varphi$.

Vortex flux does not depend on the choice of the cross-section of the tube. (Proof in [2].)

Definition. Let us have a vector $\boldsymbol{v}(\boldsymbol{x}, t)$, then Laplace operator is defined as

$$
\Delta \boldsymbol{v}=\operatorname{div}(\nabla \boldsymbol{v})=\sum_{i=1}^{3} \frac{\partial^{2} v_{i}}{\partial x_{i}^{2}}
$$

### 1.2 Balance laws

In the following sub-chapter, we will express balance laws of mass, linear momentum, angular momentum, energy and entropy. Let us assume that there are no surfaces of discontinuity in our system. (Balance laws for system with surfaces of discontinuity is to be found in [10]). All functions, unless stated otherwise, have variables ( $\mathbf{x}, t$ ). Balance laws are written in Eulerian specification.

### 1.2.1 Mass balance

The mass of the system does not change in time, which is mathematically expressed as

$$
\frac{d}{d t} m(t)=\int_{V}\left(\frac{\partial \rho}{\partial t}+\sum_{i=1}^{3} \frac{\partial\left(\rho v_{i}\right)}{\partial x_{i}}\right) d v=0
$$

where $\rho$ is density. Differential form of mass balance is

$$
\begin{align*}
\frac{\partial \rho}{\partial t}+\sum_{i=1}^{3} \frac{\partial\left(\rho v_{i}\right)}{\partial x_{i}} & =0 \\
& \text { or }  \tag{1.6}\\
\left.\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{v})\right) & =0 .
\end{align*}
$$

This equation is called continuity equation.
If we consider incompressible fluid, continuity equation is substantially simplified, for $\rho=\rho_{0} \equiv c, c \in \mathbb{R}^{+}$. Consequently, we can write

$$
\sum_{i=1}^{3} \frac{\partial v_{i}}{\partial x_{i}}=\operatorname{div} \mathbf{v}=0
$$

For the important case of steady flow of compressible fluid, the continuity equation takes the form

$$
\sum_{i=1}^{3} \frac{\partial\left(\rho v_{i}\right)}{\partial x_{i}}=\operatorname{div}(\rho \mathbf{v})=0
$$

### 1.2.2 Linear momentum balance and angular momentum balance

Change of linear momentum of the system in time is equal to the sum of surface forces and volume forces acting on the system. Linear momentum (denoted M) is defined as

$$
\mathbf{M}(t)=\int_{V} \rho \mathbf{v} d v
$$

Note that linear momentum is a vector quantity, and therefore linear momentum balance provides three equations.

Now we are interested in which forces are acting on the system. Volume forces are forces that act on every material point, and are long-range. For example
gravity, electromagnetic forces (we will not consider these), or fictitious forces, inertial forces in non-inertial reference frame (in this thesis, however, we are only concerned with inertia reference frame). Volume forces are expressed as

$$
\mathbf{F}_{v o l}(t)=\int_{V} \rho \mathbf{f} d v
$$

where $\mathbf{f}$ is the density of volume forces.
Surface forces are short-range. They are expressed as

$$
\mathbf{F}_{s u r}(t)=\int_{\partial V} \mathbf{T}(\mathbf{x}, t, \boldsymbol{\nu}(\mathbf{x})) d s
$$

where $\mathbf{T}$ is stress vector, and $\boldsymbol{\nu}(\boldsymbol{x})$ is unit outward normal vector to surface $\partial V$.
Let us now turn to angular momentum balance.

$$
\frac{d}{d t} \int_{V} \mathbf{r} \times(\rho \mathbf{v}) d v=\int_{V} \mathbf{r} \times(\rho \mathbf{f})+\rho \mathbf{A} d v+\int_{\partial V} \mathbf{r} \times \mathbf{T}(\mathbf{x}, t, \nu(\mathbf{x})) d s
$$

where $\mathbf{r}(\mathbf{x})=\mathbf{x}-\mathbf{x}_{0}$ is position vector and $\rho \mathbf{A}$ is density of inner production of angular momentum (this term is non-zero for so called polar materials). Angular momentum balance law says that change of angular momentum in time is equal to moment (torque) of forces acting on the system. It is obvious that angular momentum depends on the choice of point $\mathbf{x}_{0}$, thus angular momentum is not an objective quantity and whenever we change coordinate system it is necessary to add fictitious forces accordingly.

From Cauchy theorem (see Theorem 22), the relation between stress vector and stress tensor will be derived. It is sufficient to substitute $\boldsymbol{\nu}$ with elements of orthonormal basis $e_{i}, i=1,2,3$, in (1), so

$$
\boldsymbol{\tau} \cdot e_{i}=\sum_{j=1}^{3} \tau_{i j}=\mathbf{T}\left(\mathbf{x}, t, e_{i}\right),
$$

thus $\tau_{i j}=T_{j}\left(\mathbf{x}, t, e_{i}\right)$. And for $\boldsymbol{\nu}$

$$
\sum_{i=1}^{3} \nu_{i}(\mathbf{x}) \tau_{i j}(\mathbf{x}, t)=T_{j}(\mathbf{x}, t, \boldsymbol{\nu}(\mathbf{x}))
$$

(For details see [2].)
Therefore equation of motion is to be written in the following form, using divergence theorem (see Theorem 4),

$$
\begin{equation*}
\frac{\partial\left(\rho v_{j}\right)}{\partial t}+\operatorname{div}\left(\rho v_{j} \mathbf{v}\right)=\sum_{i=1}^{3} \frac{\partial \tau_{i j}}{\partial x_{i}}+\rho f_{j}, \quad j=1,2,3 . \tag{1.7}
\end{equation*}
$$

Elements on diagonal of tensor $\boldsymbol{\tau}$, i.e. $\tau_{i i}, i=1,2,3$, are called normal stresses, the other elements are called shear stresses.

In our case, we want to examine flow of a viscous fluid (viscosity is a property of a fluid which expresses the resistance of the fluid to deformation) around a rigid body. We express stress tensor in terms of rate-of-strain tensor. Let us
denote velocity gradient $\mathbf{L}=\nabla \mathbf{v}$. Let us make a constitutive assumption on the form of the stress tensor:

$$
\boldsymbol{\tau}=-p_{0} \mathbf{I}+\mathbf{C}(\mathbf{L})
$$

where $p_{0}$ is the pressure of the fluid at rest. In case that $\mathbf{C}$ is linear function, fluid is called Newtonian. Let us consider a Newtonian fluid in conjunction with incompressible flow, i.e. $\operatorname{tr} \mathbf{L}=\operatorname{div} \mathbf{v}=0$.

Assumption that $\mathbf{C}$ is linear implies that if fluid is at rest then $\mathbf{L}=\mathbf{0}, \mathbf{C}(\mathbf{0})=$ $\mathbf{0}$. Cauchy stress has the form $\boldsymbol{\tau}=-p_{0} \mathbf{I}$, and the fluid behaves like an ideal, inviscous one.

Now we will select scalar function of $\mathbf{L}$ as

$$
\alpha(\mathbf{L})=-\frac{1}{3} \operatorname{tr} \mathbf{C}(\mathbf{L}),
$$

and Cauchy stress is written as

$$
\begin{aligned}
\boldsymbol{\tau} & =-\left(p_{0}+\alpha(\mathbf{L})\right) \mathbf{I}+(\mathbf{C}(\mathbf{L})+\alpha(\mathbf{L}) \mathbf{I}) \\
& =\left(-p_{0}+\frac{1}{3} \operatorname{tr} \mathbf{C}(\mathbf{L})\right) \mathbf{I}+\left(\mathbf{C}(\mathbf{L})-\frac{1}{3} \operatorname{tr} \mathbf{C}(\mathbf{L}) \mathbf{I}\right) .
\end{aligned}
$$

Let us denote

$$
\begin{gathered}
p=p_{0}-\frac{1}{3} \operatorname{tr} \mathbf{C}(\mathbf{L})=-\frac{1}{3} \operatorname{tr} \boldsymbol{\tau}, \\
\mathbf{B}(\mathbf{L})=\mathbf{C}(\mathbf{L})-\frac{1}{3} \operatorname{tr} \mathbf{C}(\mathbf{L}) \mathbf{I} .
\end{gathered}
$$

So $\boldsymbol{\tau}=-p \mathbf{I}+\mathbf{B}(\mathbf{L})$.
Theorem 1. Response of Newtonian incompressible fluid is independent of the observer, i.e. it fulfills objectivity principle if and only if the response function $\boldsymbol{B}(\boldsymbol{L})$ has the form

$$
\boldsymbol{B}(\boldsymbol{L})=2 \mu \boldsymbol{D}, \forall \boldsymbol{L}: \operatorname{tr} \boldsymbol{L}=0
$$

where $\mu \in \mathbb{R}_{0}^{+}$is dynamical viscosity of the fluid,

$$
\begin{aligned}
\boldsymbol{D} & =\frac{1}{2}\left(\boldsymbol{L}+\boldsymbol{L}^{T}\right), \\
\text { so }\left(d_{i j}\right)_{i, j=1}^{3} & =\frac{1}{2}\left(\frac{\partial v_{i}}{\partial x_{j}}+\frac{\partial v_{j}}{\partial x_{i}}\right) .
\end{aligned}
$$

$\boldsymbol{D}$ being the above-mentioned rate-of-strain tensor.
See proof in 6].
Now we are able to express Cauchy stress of Newtonian incompressible fluid $\boldsymbol{\tau}$ in this special form. Using the form in motion equation (1.7) we obtain

$$
\rho \frac{\partial v_{j}}{\partial t}+\rho \operatorname{div}\left(v_{j} \mathbf{v}\right)=-\frac{\partial p}{\partial x_{j}}+\mu \sum_{i=1}^{3} \frac{\partial}{\partial x_{i}}\left(\frac{\partial v_{i}}{\partial x_{j}}+\frac{\partial v_{j}}{\partial x_{i}}\right)+\rho f_{j}, \quad j=1,2,3
$$

It is obvious that term

$$
\sum_{i=1}^{3} \frac{\partial^{2} v_{i}}{\partial x_{i} \partial x_{j}}=\frac{\partial}{\partial x_{j}} \operatorname{div} \mathbf{v}=0
$$

Furthermore, motion equation written using Laplace operator is

$$
\rho \frac{\partial v_{j}}{\partial t}+\rho \operatorname{div}\left(v_{j} \mathbf{v}\right)=-\frac{\partial p}{\partial x_{j}}+\mu \Delta v_{j}+\rho f_{j}, \quad j=1,2,3 .
$$

Let us denote

$$
\nu=\frac{\mu}{\rho},
$$

and call $\nu$ kinematic viscosity. If gravity is the only volume force considered, then $\rho \mathbf{f}=\rho \mathbf{g}$. Finally we obtain Navier-Stokes equations

$$
\begin{equation*}
\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v} \cdot \nabla) \mathbf{v}=-\frac{\nabla p}{\rho}+\nu \Delta \mathbf{v}+\mathbf{g} \tag{1.8}
\end{equation*}
$$

I-th component of forces acting on the body is

$$
F_{i, s u r}(t)+F_{i, o b j}(t)=-\int_{V} \frac{\partial p}{\partial x_{i}} d v+\int_{V} \rho f_{i} d v
$$

Euler equations for inviscid fluid

$$
\frac{\partial\left(\rho v_{i}\right)}{\partial t}+\sum_{k=1}^{3} \frac{\partial\left(\rho v_{i} \mathbf{v}\right)}{\partial x_{k}}=-\frac{\partial p}{\partial x_{i}}+\rho f_{i}
$$

where $\mathbf{x} \in V, t \geq t_{0}, i=1,2,3$. This form is then:

$$
\rho \frac{\partial v_{i}}{\partial t}+v_{i} \frac{\partial \rho}{\partial t}+v_{i} \operatorname{div}(\rho \mathbf{v})+(\rho \mathbf{v}) \cdot \nabla v_{i}=-\frac{\partial p}{\partial x_{i}}+\rho f_{i} .
$$

Notice that continuity equation (1.6) applies to terms $v_{i}\left(\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{v})\right)$, therefore these terms vanish. Then

$$
\rho \frac{\partial v_{i}}{\partial t}+\rho \sum_{k=1}^{3} v_{k} \frac{\partial v_{i}}{\partial x_{k}}=-\frac{\partial p}{\partial x_{i}}+\rho f_{i}, \quad i=1,2,3
$$

If the fluid is incompressible ( $\rho \equiv c, c \in \mathbb{R}^{+}$), we obtain equations

$$
\frac{\partial v_{i}}{\partial t}+\sum_{k=1}^{3} v_{k} \frac{\partial v_{i}}{\partial x_{k}}=-\frac{1}{c} \frac{\partial p}{\partial x_{i}}+f_{i}, \quad i=1,2,3
$$

### 1.2.3 Energy balance

Change of total energy (internal energy and kinetic energy) of the system in time is equal to heat added to the system plus work performed on the system.

$$
\begin{gathered}
E=\int_{V}\left(\rho u+\frac{\rho \mathbf{v} \cdot \mathbf{v}}{2}\right) d v \\
\frac{d}{d t} \int_{V}\left(\rho u+\frac{\rho \mathbf{v} \cdot \mathbf{v}}{2}\right) d v=\int_{\partial V} \sum_{k=1}^{3}\left(\sum_{i=1}^{3} \tau_{k i} v_{i}+q_{k}\right) d s_{k}+\int_{V}\left(\sum_{i=1}^{3} \rho f_{i} v_{i}+\tilde{q}\right) d v
\end{gathered}
$$

where $\rho u$ is density of internal energy, $\frac{\rho \mathbf{v} \cdot \mathbf{v}}{2}$ is density of kinetic energy, term that includes $\tau_{k i} v_{i}$ stands for work of surface forces, $\rho f_{i} v_{i}$ stands for work of volume
forces, and $\tilde{q}$ is density of exterior energetic influences (e.g. radiance). Let us convert the energy balance law to its proper form:

$$
\begin{align*}
\frac{\partial}{\partial t}\left(\rho u+\frac{\rho \mathbf{v} \cdot \mathbf{v}}{2}\right)+\sum_{k=1}^{3}\left[\frac{\partial}{\partial x_{k}}\left(\rho u+\frac{\rho \mathbf{v} \cdot \mathbf{v}}{2}\right) v_{k}\right. & \left.-\frac{\partial}{\partial x_{k}}\left(\sum_{i=1}^{3} \tau_{k i} v_{i}+q_{k}\right)\right]  \tag{1.9}\\
& -\left(\sum_{i=1}^{3} \rho f_{i} v_{i}+\tilde{q}\right)=0
\end{align*}
$$

Let us formulate internal energy balance which is equivalent with the first law of thermodynamics: Change of internal energy is equal to added heat minus work performed by the system, or $d U=d Q-d W, U$ is internal energy, $Q$ heat, $W$ work. To express $d Q$ in terms of thermodynamic temperature $T: d Q=c d T$, where $c$ is the specific heat capacity of given examined material. Internal energy balance law says:

$$
\begin{equation*}
\frac{\partial(\rho u)}{\partial t}+\sum_{k=1}^{3} \frac{\partial}{\partial x_{k}}\left(\left(\rho v_{k} u\right)+q_{k}+\sum_{i=1}^{3} \tau_{k i} v_{i}\right)+\tilde{q}=0 . \tag{1.10}
\end{equation*}
$$

Some quantities which feature in total energy balance are not objective; it is necessary to keep this in mind when changing coordinate system to a moving one.

### 1.2.4 Entropy balance

Quantity $S$ called entropy will be established unusually, with the help of following Clausius inequality:

$$
\frac{d Q}{T} \leq d S
$$

Using entropy, we are able to express the second law of thermodynamics: The entropy of thermodynamically isolated system does not decrease. Using the expression of $d Q$ from the first law of thermodynamics $(d U=d Q-d W)$, we obtain fundamental thermodynamic inequality

$$
\begin{gathered}
T d S-d W-d U \geq 0 . \\
S=\int_{V} \rho s d v,
\end{gathered}
$$

where $\rho s$ stands for density of entropy. Clausius inequality can be formulated

$$
\frac{d S}{d t}=H(S)+P(S) \geq \frac{d Q}{T d t}=-\int_{\partial V} \sum_{k=1}^{3} \frac{q_{k}}{T} d s_{k}+\int_{V} \frac{\tilde{q}}{T} d v
$$

where $H(S)$ is flux of etropy through the surface of the system, $P(S)$ is the production of entropy in time unit. General balance law (see [10]) implies that

$$
P(S)=\frac{d S}{d t}-H(S)
$$

Using Clausius inequality:

$$
P(S) \geq-\int_{\partial V} \sum_{k=1}^{3} \frac{q_{k}}{T} d s_{k}+\int_{V} \frac{\tilde{q}}{T} d v-H(S)
$$

Flux of entropy is defined:

$$
H(S)=-\int_{\partial V} \frac{q_{k}}{T} d s_{k}+\int_{V} \frac{\tilde{q}}{T} d v
$$

Thus production of entropy fulfills:

$$
P(S)=\int_{V} \boldsymbol{\sigma}(S) d v=\int_{V}\left[\rho\left(\frac{\partial s}{\partial t}+\sum_{k=1}^{3} \frac{\partial s}{\partial x_{k}} v_{k}\right)+\sum_{k=1}^{3} \frac{\partial}{\partial x_{k}}\left(\frac{q_{k}}{T}\right)-\frac{\tilde{q}}{T}\right] d v \geq 0
$$

No surfaces of discontinuity were considered in the system, therefore

$$
\boldsymbol{\sigma}(S)=\rho\left(\frac{\partial s}{\partial t}+\sum_{k=1}^{3} \frac{\partial s}{\partial x_{k}} v_{k}\right)+\sum_{k=1}^{3} \frac{\partial}{\partial x_{k}}\left(\frac{q_{k}}{T}\right)-\frac{\tilde{q}}{T} \geq 0, \forall \mathbf{x} \in V
$$

because Clausius inequality and entropy balance hold for every system we consider, and because all functions are continuous. This can be easily verified by contradiction.

Internal energy balance (see (1.10)) implies that

$$
\sum_{k=1}^{3} \frac{\partial}{\partial x_{k}}\left(\frac{q_{k}}{T}\right)-\frac{\tilde{q}}{T}=\sum_{k=1}^{3}\left(q_{k} \frac{\partial}{\partial x_{k}}\left(\frac{1}{T}\right)+\frac{1}{T} \sum_{i=1}^{3} \tau_{k i} \frac{\partial v_{i}}{\partial x_{k}}\right)-\frac{\rho}{T} \frac{d u}{d t} .
$$

Thus the density of the production of entropy is expressed
$\rho\left(\frac{\partial s}{\partial t}+\frac{1}{T} \frac{\partial u}{\partial t}\right)+\sum_{k=1}^{3}\left[\rho\left(\frac{\partial s}{\partial x_{k}}+\frac{1}{T} \frac{\partial u}{\partial x_{k}}\right) v_{k}+q_{k} \frac{\partial}{\partial x_{k}}\left(\frac{1}{T}\right)+\frac{1}{T} \sum_{i=1}^{3} \tau_{k i} \frac{\partial v_{i}}{\partial x_{k}}\right] \geq 0$.

### 1.2.5 Constitutive assumptions for thermoviscoelastic fluid

Considering thermoviscoelastic fluid, we are able to use following constitutive assumptions

$$
q_{k}=-\lambda(\rho, T) \frac{\partial T}{\partial x_{k}}, \quad k=1,2,3
$$

where $\lambda$ is heat conductivity. Further, for Cauchy stress,

$$
\begin{equation*}
\tau_{k i}=-p(\rho, T) \delta_{k i}+\tau_{k i}^{d i s}, \quad k, i=1,2,3 \tag{1.11}
\end{equation*}
$$

where

$$
\tau_{k i}^{d i s}=\mu_{V}(\rho, T) \operatorname{div} \mathbf{v} \delta_{k i}+2 \mu(\rho, T)\left(d_{k i}-\frac{1}{3} \operatorname{div} \mathbf{v} \delta_{k i}\right)
$$

$\mu_{V}$ is volume viscosity, $\mu$ is shear viscosity, $\mathbf{D}$ is rate-of-strain tensor, $\delta_{k i}$ is Kronecker delta. Thus

$$
\boldsymbol{\tau}=\left(-p+\operatorname{div} \mathbf{v}\left(\mu_{V}-\frac{2 \mu}{3}\right)\right) \mathbf{I}+2 \mu \mathbf{D}
$$

Remember that $\operatorname{div} \mathbf{v}=\operatorname{tr} \mathbf{D}$. Derivation of these assumptions is to be found in 10.

## Chapter 2

## Alternative formulation of linear momentum balance and energy balance

### 2.1 Energy balance using stagnation enthalpy

In the previous considerations, we discussed balance laws in the usual order, i.e. mass, linear momentum, angular momentum, energy, entropy. Now new quantity which occurs in energy balance (1.9) will be used to formulate linear momentum balance. Let us assume that fluid is incompressible and that there exists potential $-\nabla \Phi(\mathbf{x})=\mathbf{f}$, and assume that $\tilde{q}=0$. The equation of total energy is to be expressed in form

$$
\begin{aligned}
\frac{d}{d t}\left(\rho u+\rho \frac{\mathbf{v} \cdot \mathbf{v}}{2}\right)-\sum_{k=1}^{3} \frac{\partial}{\partial x_{k}}\left(\sum_{i=1}^{3} \tau_{k i} v_{i}-q_{k}\right)-\rho \frac{d \Phi}{d t} & =0, \\
\rho \frac{d}{d t}\left(u+\frac{\mathbf{v} \cdot \mathbf{v}}{2}+\Phi\right)+\sum_{k=1}^{3} \frac{\partial}{\partial x_{k}}\left(p v_{k}-\mu \sum_{i=1}^{3} d_{k i} v_{i}+q_{k}\right) & =0, \\
\rho \frac{d}{d t}\left(u+\frac{\mathbf{v} \cdot \mathbf{v}}{2}+\Phi+\frac{p}{\rho}\right)-\frac{\partial p}{\partial t}+\sum_{k=1}^{3} \frac{\partial}{\partial x_{k}}\left(-\mu \sum_{i=1}^{3} d_{k i} v_{i}+q_{k}\right) & =0,
\end{aligned}
$$

using continuity equation (1.6) and incompressibility.

$$
\rho \frac{d}{d t}\left(u+\frac{\mathbf{v} \cdot \mathbf{v}}{2}+\Phi+\frac{p}{\rho}\right)=\frac{\partial p}{\partial t}+\sum_{k=1}^{3} \frac{\partial}{\partial x_{k}}\left(\mu \sum_{i=1}^{3} d_{k i} v_{i}-q_{k}\right)
$$

The following form of energy balance is ideal for analysis of properties of velocity fields, especially for finding a connection between generation of vorticity and change of entropy.

$$
\begin{array}{r}
\frac{d}{d t}\left(u+\frac{\mathbf{v} \cdot \mathbf{v}}{2}+\Phi+\frac{p}{\rho}\right)=\frac{\partial h_{c}}{\partial t}+\sum_{l=1}^{3} v_{l} \frac{\partial h_{c}}{\partial x_{l}}= \\
\frac{1}{\rho}\left[\frac{\partial p}{\partial t}+\sum_{k=1}^{3}\left(-\frac{\partial q_{k}}{\partial x_{k}}+\mu \frac{\partial}{\partial x_{k}} \sum_{i=1}^{3} v_{i}\left(\frac{\partial v_{i}}{\partial x_{k}}+\frac{\partial v_{k}}{\partial x_{i}}\right)\right)\right] \tag{2.1}
\end{array}
$$

It appears convenient to define a new quantity:

$$
\begin{equation*}
h_{c}=u+\frac{\mathbf{v} \cdot \mathbf{v}}{2}+\Phi+\frac{p}{\rho}, \tag{2.2}
\end{equation*}
$$

This new quantity is called total enthalpy. Its constituents are total energy, including internal and kinetic energy, pressure energy generated by surface forces and potential energy of corresponding volume forces. In case of incompressible, inviscid and heat non-conducting flow, it holds that quantity $h_{c}$ is constant along trajectories (see [1]). In case of steady flow, trajectories and streamlines coincide, thus $h_{c}$ is constant along streamlines. To express this result in a condensed form: Total enthalpy (2.2) is preserved in case of steady flow, neglecting heat conduct and viscosity.

### 2.2 Linear momentum balance - Crocco's theorem

Using the first law of thermodynamics in form for material point

$$
\begin{gather*}
T \frac{d S}{d t}=\frac{d u}{d t}+p \frac{d}{d t}\left(\frac{1}{\rho}\right)=T\left(\frac{\partial S}{\partial t}+\mathbf{v} \cdot \nabla S\right)= \\
\frac{\partial u}{\partial t}+\mathbf{v} \cdot \nabla u+p\left(\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right)+\mathbf{v} \cdot \nabla\left(\frac{1}{\rho}\right)\right) \tag{2.3}
\end{gather*}
$$

an alternative expression of linear momentum balance will be provided, using the concept of total enthalpy. In case of steady flow, the first law of thermodynamics (2.3) implies

$$
\begin{equation*}
\mathbf{v} \cdot\left[\nabla u+p \nabla\left(\frac{1}{\rho}\right)-T \nabla S\right]=\mathbf{v} \cdot\left[\nabla\left(u+\frac{p}{\rho}\right)-T \nabla S-\frac{1}{\rho} \cdot \nabla p\right]=0 \tag{2.4}
\end{equation*}
$$

if $\mathbf{v} \neq 0$. We are able to compute gradient of pressure $\nabla p$ and use vector identity

$$
\begin{equation*}
\mathbf{v} \cdot \nabla \mathbf{v}=\nabla\left(\frac{\mathbf{v} \cdot \mathbf{v}}{2}\right)-\mathbf{v} \times \operatorname{curl} \mathbf{v} \tag{2.5}
\end{equation*}
$$

to convert convective term. We will now use equations (2.4) and (2.5) in the equation of linear momentum balance for steady flow of incompressible viscous Newtonian fluid

$$
\begin{equation*}
(\mathbf{v} \cdot \nabla) \mathbf{v}=-\frac{\nabla p}{\rho}+\nu \Delta \mathbf{v}-\nabla \Phi \tag{2.6}
\end{equation*}
$$

We modify this equation to obtain Crocco's theorem:

$$
\begin{equation*}
\nabla h_{c}-\mathbf{v} \times \operatorname{curl} \mathbf{v}=T \nabla S+\nu \Delta \mathbf{v} \tag{2.7}
\end{equation*}
$$

Crocco's theorem provides an alternative expression of stationary linear momentum balance $(\sqrt{1.8})$, or $(\sqrt{2.6})$. In case that total enthalpy $(2.2)$ is constant, i.e. for flow of inviscid heat non-conducting fluid, it holds that

$$
\begin{equation*}
-\mathbf{v} \times \operatorname{curl} \mathbf{v}=T \nabla S \tag{2.8}
\end{equation*}
$$

Therefore vorticity $\mathbf{w}=\nabla \times \mathbf{v}$ can be generated merely by change of entropy, e.g. by heat of condensation. In case of constant entropy, the flow is potential.

Note: Crocco's theorem (2.7) holds only for stationary process. The main reason is using the first law of thermodynamics (2.3). This law holds only for material point. Flow sets material points in motion, and in the case of nonstationary flow, the pressure gradient (which appears in momentum balance (2.6)) is not to be easily excluded from equation (2.3).

## Chapter 3

## Numerical simulation of flow around rotating cylinder

The bachelor thesis (see [11]) studied, among other things, potential flow around cylinder. Potential flow model describes motion of inviscous incompressible fluid. In this thesis, we would like to compare the experiment performed by Ing. Zdeněk Trávníček, CSc., and Ing. Zuzana Broučková (see e. g. [13]) with numerical computation, using either potential flow model or stationary Navier-Stokes equations ((1.8)), obtained by finite element method. These results can later be compared with flow around cylinder which emits water vapor.

### 3.1 Flow around cylinder using complex potential

Potential flow is irrotational and, in two dimensions, can be described by complex potential. Complex potential is a complex holomorphic function. Derivative of complex potential is called complex velocity and it holds that velocity in the direction of the x -axis is equal to real part of complex velocity and velocity in the direction of the $y$-axis is equal to minus imaginary part of the complex velocity. For more information about potential flow see [2], [9] or [11]. Complex potential describing the flow around cylinder has this form:

$$
u=v_{\infty}\left(z+\frac{a^{2}}{z}\right)+\frac{\gamma}{2 \pi i} \ln z
$$

where $v_{\infty}$ is the velocity of incoming fluid, $a$ is radius of the cylider, $\gamma$ is circulation. The experiment (Trávníček and Broučková) consisted in placing a rotating cylinder ( $a=0.0125 \mathrm{~m}$ ) into wind tunnel, $v_{\infty}=0.5 \mathrm{~m} / \mathrm{s}$, rotational speed $\omega$ was, in sequence, $0,5,12.5,20 \mathrm{~Hz}$ in clockwise direction. Water mist was used for visualization of the flow. Let us compute circulation of velocity field in all cases. Circulation is, by definition (1.4):

$$
\gamma=-\int_{0}^{2 \pi} 2 \pi a^{2} \omega d \theta=-4 \pi a^{2} \omega .
$$

For resulting values, see table 3.1. Velocity field is visualized by Wolfram Mathematica. See figure 3.1. The lift acting on the 1 m long cylinder can be computed, table 3.1 shows computed values (for derivation of the formula see [2]):

$$
R_{y}=-\Gamma v_{\infty} \rho,
$$

where density $\rho=1.2041 \mathrm{~kg} / \mathrm{m}^{3}$.


Figure 3.1: Flow around rotating cylinder, rotational speed in sequence $0,5,12.5$ and 20 Hz .

### 3.2 Flow around rotating cylinder using N-S equations and their numerical solution

The flow around rotating cylinder can be described better - by stationary NavierStokes equations. The flow is incompressible, the same as in potential flow model, but it is also viscous. The equations of steady viscous incompressible flow (no volume forces are considered) are

$$
\begin{align*}
\operatorname{div} \mathbf{v} & =0 \operatorname{in} \Omega \\
\mathbf{v} \cdot \nabla \mathbf{v} & =\frac{1}{\rho} \operatorname{div} \boldsymbol{\tau} \text { in } \Omega \tag{3.1}
\end{align*}
$$

where $\rho=$ const is density, $\mathbf{v}$ is velocity field, $\tau=-p \mathbf{I}+2 \mu \mathbf{D}$ is Cauchy stress, $p$ is pressure, $\mu$ is dynamic viscosity, $\mathbf{D}=\frac{1}{2}\left(\nabla \mathbf{v}+(\nabla \mathbf{v})^{T}\right)$ is rate-ofstrain tensor. The domain is depicted in figure 3.2. Boundary conditions are following:

| $\omega[\mathrm{Hz}]$ | $\gamma\left[\mathrm{m}^{2} / \mathrm{s}\right]$ | $R_{y}[\mathrm{~N}]$ |
| :---: | :---: | :---: |
| 0 | 0 | 0 |
| 5 | -0.03084 | 0.01857 |
| 12.5 | -0.07711 | 0.04642 |
| 20 | -0.1234 | 0.0743 |

Table 3.1: Circulation depending on rotational speed of the cylinder and computed lift.

$$
\begin{aligned}
\mathbf{v} & =\mathbf{v}_{\text {in }} \text { on } \Gamma_{\text {in }} \\
\mathbf{v} & =\mathbf{0} \text { on } \Gamma_{1} \cup \Gamma_{2} \\
\mathbf{v} & =\mathbf{v}_{\theta} \text { on } \Gamma_{c} \\
\boldsymbol{\tau} \cdot \mathbf{n} & =-p_{\text {out }} \mathbf{n}+\mu(\nabla \mathbf{v})^{T} \cdot \mathbf{n} \text { on } \Gamma_{\text {out }},
\end{aligned}
$$

where $\mathbf{v}_{i n}$ is velocity of incoming fluid, $\Gamma_{1} \cup \Gamma_{2}$ are solid walls and the flow is viscous, therefore fluid is sticking to the walls, $\mathbf{v}_{\theta}=2 \pi \omega a(\sin \theta,-\cos \theta)$ is velocity of the surface of the rotating cylinder (using polar coordinates), $\omega$ is rotational speed of the cylinder, $a$ is radius of the cylinder, $\theta$ is the angle, $p_{\text {out }}$ is pressure on $\Gamma_{\text {out }}$.


Figure 3.2: Domain where the equations of flow around rotating cylinder are to be solved.

### 3.2.1 Weak formulation of equations of flow around rotating cylinder

Let $\hat{\mathbf{v}} \in W^{1,2}(\Omega)$ be such that

$$
\begin{aligned}
\operatorname{Tr} \hat{\mathbf{v}} & =\mathbf{v}_{i n} \text { on } \Gamma_{i n} \\
& =\mathbf{v}_{\theta} \text { on } \Gamma_{c},
\end{aligned}
$$

where $W^{1,2}(\Omega)$ is a Sobolev space and Tr is a trace operator. Let us define spaces

$$
\begin{aligned}
& \tilde{V}=\left\{\mathbf{u} \in W^{1,2}(\Omega), \mathbf{u}=\mathbf{0} \text { on } \Gamma_{i n} \cup \Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{c}\right\} \\
& \tilde{P}=\left\{q \in L^{2}(\Omega)\right\} .
\end{aligned}
$$

Weak formulation of the problem above is:
We search for the functions $\tilde{\mathbf{v}} \in \tilde{V}$ and $p \in \tilde{P}$ such that

$$
\begin{array}{r}
\int_{\Omega} q \operatorname{div}(\tilde{\mathbf{v}}+\hat{\mathbf{v}}) d \mathbf{x}=0 \forall q \in \tilde{P} \\
\int_{\Omega}(\tilde{\mathbf{v}}+\hat{\mathbf{v}}) \cdot \nabla(\tilde{\mathbf{v}}+\hat{\mathbf{v}}) \cdot \mathbf{u} d \mathbf{x} \\
+\frac{1}{\rho} \int_{\Gamma_{\text {out }}}\left(p_{\text {out }}-\mu \nabla(\tilde{\mathbf{v}}+\hat{\mathbf{v}})^{T}\right) \mathbf{n} \cdot \mathbf{u} d s \\
+\frac{1}{\rho} \int_{\Omega}\left(-p \mathbf{I}+\mu\left(\nabla(\tilde{\mathbf{v}}+\hat{\mathbf{v}})+(\nabla(\tilde{\mathbf{v}}+\hat{\mathbf{v}}))^{T}\right)\right): \nabla \mathbf{u} d \mathbf{x}=0 \forall \mathbf{u} \in \tilde{V} .
\end{array}
$$

The solution will then be $\mathbf{v}=\tilde{\mathbf{v}}+\hat{\mathbf{v}}$ and $p$.


Figure 3.3: Discretisation of the domain $\Omega$, mesh A.


Figure 3.4: Discretisation of the domain $\Omega$, mesh B.

| mesh | number of vertices | number of cells | dim |
| :---: | :---: | :---: | :---: |
| A | 2951 | 5730 | 26215 |
| B | 11632 | 22920 | 104000 |
| C | 46184 | 91680 | 414280 |

Table 3.2: Number of vertices, cells and dimension of space $\tilde{V}_{h} \times \tilde{P}_{h}$

### 3.2.2 Discretisation

Discretisation of the domain $\Omega$ was created using Gmsh and FEniCS software. Three meshes were used. Mesh A is the roughest, mesh B is refined mesh A, mesh C is refined mesh B. See figures 3.3, 3.4 and 3.5. In the process of refinement, each triangle was divided into four, thus mesh C has sixteen times more cells than mesh A. $\Omega_{h}$ denotes discretisation of the domain $\Omega$.

Spaces of functions $\tilde{V}$ and $\tilde{P}$ are approximated by their finite-dimensional subspaces $\tilde{V}_{h}$ and $\tilde{P}_{h}$, where

$$
\begin{aligned}
& \tilde{V}_{h}=\left\{\mathbf{v}_{h} \in C\left(\Omega_{h}\right),\left.\mathbf{v}_{h}\right|_{E} \in P^{2}(E) \forall E \in A, \mathbf{v}_{h}=\mathbf{0} \text { on } \Gamma_{i n} \cup \Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{c}\right\}, \\
& \tilde{P}_{h}=\left\{p_{h} \in C\left(\Omega_{h}\right),\left.p_{h}\right|_{E} \in P^{1}(E) \forall E \in A\right\},
\end{aligned}
$$

where $E$ is an element in triangulation $A$. Discretisations of function spaces using meshes $B$ or $C$ have analogic definitions. For total dimension of space $\tilde{V}_{h} \times \tilde{P}_{h}$ see table 3.2. FEniCS used Newton's method to solve algebraic system obtained by finite element method (for more information see [15]).

Following computation used these data:


Figure 3.5: Discretisation of the domain $\Omega$, mesh C.
density $\rho=1.2041 \mathrm{~kg} / \mathrm{m}^{3}$
dynamic viscosity $\mu=18 \cdot 10^{-5} \mathrm{~Pa} \cdot \mathrm{~s}$
$\Omega$ was a rectangle $0.4 m \times 0.3 \mathrm{~m}$ minus circle, its center located at [0.1, 0.15], its radius $a=0.0125 \mathrm{~m}$
$\mathbf{v}_{i n}=\left(\frac{0.5 \cdot 4 \cdot \cdot \cdot(0.3-y)}{0.3^{2}}, 0\right) \mathrm{m} / \mathrm{s}$
$\mathbf{v}_{\theta}=2 \pi \omega a(\sin \theta,-\cos \theta)$, where $\omega$ is rotational speed of cylinder
$p_{\text {out }}=101325$ Pa
$\omega$ was, in sequence, $5 / \mathrm{s}, 12.5 / \mathrm{s}, 20 / \mathrm{s}$
Computed solution is visualized in Paraview. Lift and drag acting on the cylinder (on the cylinder 1 m long) was computed for all meshes and all rotational speeds, see table 3.3.

| $\omega[H z]$ | mesh | drag $[N]$ | lift $[N]$ |
| :---: | :---: | :---: | :---: |
| 0 | A | $4.298214 \cdot 10^{-3}$ | $1.263989 \cdot 10^{-5}$ |
| 0 | B | $4.646994 \cdot 10^{-3}$ | $3.613350 \cdot 10^{-6}$ |
| 0 | C | $4.658672 \cdot 10^{-3}$ | $7.840737 \cdot 10^{-7}$ |
| 5 | A | $4.328917 \cdot 10^{-3}$ | $7.093461 \cdot 10^{-3}$ |
| 5 | B | $4.379473 \cdot 10^{-3}$ | $7.087945 \cdot 10^{-3}$ |
| 5 | C | $4.390504 \cdot 10^{-3}$ | $7.079696 \cdot 10^{-3}$ |
| 12.5 | A | $4.012619 \cdot 10^{-3}$ | $2.108079 \cdot 10^{-2}$ |
| 12.5 | B | $4.054664 \cdot 10^{-3}$ | $2.108019 \cdot 10^{-2}$ |
| 12.5 | C | $4.071689 \cdot 10^{-3}$ | $2.105339 \cdot 10^{-2}$ |
| 20 | A | $6.318441 \cdot 10^{-3}$ | $4.161933 \cdot 10^{-2}$ |
| 20 | B | $6.353315 \cdot 10^{-3}$ | $4.167484 \cdot 10^{-2}$ |
| 20 | C | $6.378742 \cdot 10^{-3}$ | $4.161071 \cdot 10^{-2}$ |

Table 3.3: Drag and lift acting on the rotating cylinder, using meshes A, B and C.

As we can see, computed solution is more precise when refined mesh is used. However, mesh A was created to be finer in the neighbourhood of the cylinder. The table 3.3 shows that computed force acting on the rotating cylinder differs, except for steady cylinder, less than $1 \%$ whether computed using mesh A or using refined meshes. In the case of steady cylinder, the lift should be zero, because the cylinder is placed symmetrically in the wind tunnel. Computed values of lift are therefore subject to numerical error, which has order $10^{-5}$ for mesh A, $10^{-6}$ for mesh B and $10^{-7}$ for mesh C. This all indicates that for computation of the force acting on the rotating cylinder mesh A was chosen well.

Furthermore, figures visualizing velocity field with streamlines of flow around cylinder rotating at higher speed, e. g. figure 3.16, show that the fluid returns back into the domain on the boundary $\Gamma_{\text {out }}$. Does it mean that the boundary condition (do-nothing condition) on the outflow was not right? This leads us to another problem: What if the domain was longer, what would the velocity field look like? Let $\Omega_{\text {long }}=1 m \times 0.3 m$ minus circle, its center located at [0.1, 0.15], its radius $a=0.0125 \mathrm{~m}$. Again, discretisation was created using Gmsh and FEniCS. Rotational speed is 20 Hz . See figures of used mesh and velocity field with streamlines 3.18 and 3.19).

The fluid is rotating near upper wall. The restricted velocity field is similar to the one computed on the smaller domain. Boundary condition is therefore right. It seems that the fluid is flung at the lower wall of the domain by the rotating cylider, and the fluid near the upper wall has almost zero velocity. The flow in the lower part causes the slow rotation of the fluid near upper wall.

This part of the thesis studied flow around rotating cylinder. The problem was defined, numericaly computed and the solution will be compared with the experimental data.


Figure 3.6: Computed velocity field for flow around cylinder without rotation on mesh A, mesh B and mesh C.


Figure 3.7: Streamlines for flow around cylinder without rotation, mesh A, mesh $B$ and mesh C.


Figure 3.8: Relative pressure, i. e. $p-101325 \mathrm{~Pa}$, for flow around cylinder without rotation, mesh A , mesh B and mesh C .


Figure 3.9: Computed velocity field for cylinder rotating 5 times per second, mesh A, mesh B and mesh C.


Figure 3.10: Streamlines for cylinder rotating 5 times per second, mesh A, mesh B and mesh C.


Figure 3.11: Relative pressure, i. e. $p-101325 P a$, for cylinder rotating 5 times per second, mesh A, mesh B and mesh C.


Figure 3.12: Computed velocity field for cylinder rotating 12.5 times per second, mesh A, mesh B and mesh C.


Figure 3.13: Streamlines for cylinder rotating 12.5 times per second, mesh A, mesh B and mesh C.


Figure 3.14: Relative pressure, i. e. $p-101325 P a$, for cylinder rotating 12.5 times per second, mesh A, mesh B and mesh C.


Figure 3.15: Computed velocity field for cylinder rotating 20 times per second, mesh A, mesh B and mesh C.


Figure 3.16: Streamlines for cylinder rotating 20 times per second, mesh A, mesh B and mesh C.


Figure 3.17: Relative pressure, i. e. $p-101325 P a$, for cylinder rotating 20 times per second, mesh A, mesh B and mesh C.


Figure 3.18: Discretisation of the domain $\Omega_{\text {long }}$, mesh D.


Figure 3.19: Top: computed velocity field with streamlines for cylinder rotating 20 times per second on the longer domain, mesh D. Bottom: velocity field with streamlines for cylinder rotating 20 times per second on the original domain, computed on mesh C.

## Chapter 4

## Numerical simulation of flow around cylinder emitting vapor

In the previous part of the thesis, the rotating cylinder problem was solved, its weak formulation was derived and its numerical solution was visualized. Now we would like to compare this solution with the solution of flow around cylinder emitting vapor problem. The task is to show the similarity of effects of dissipative processes on flow in both cases. Flow will be considered stationary, viscous and incompressible, but density of air will change depending on concentration of water vapor in air.

### 4.1 Humid air

Humid air is a mixture of dry air and water vapor. Humid air can also contain small drops of water (e. g. fog) or small pieces of ice (e. g. snowflakes), which are visible, unlike water vapor. To simplify the problem, we will omit the possibility that water in both liquid and solid state is present in the air. Dry air is a mixture of gases, like nitrogen, oxygen, argon, carbon dioxide and other gases. Its composition is internationally standardized (see [4]).

Molar mass of dry air (denoted $M_{a}$ ) is $28.964 \mathrm{~kg} \cdot \mathrm{kmol}^{-1}$. Molar mass of water (denoted $M_{v}$ ) is $18.01534 \mathrm{~kg} \cdot \mathrm{kmol}^{-1}$. Dry air, under common atmospherical conditions, can be considered ideal gas. Water vapor at low pressure can also be considered ideal gas. Therefore, the equation for ideal gas can be used for humid air consisting of dry air and water vapor at low partial pressure:

$$
\begin{equation*}
\frac{p}{\rho}=\frac{R T}{M}, \tag{4.1}
\end{equation*}
$$

where $p$ is pressure, $\rho$ is density, $T$ is temperature, $M$ is molar mass and $R=$ $8314.41 \mathrm{~J} \cdot \mathrm{kmol}^{-1} \cdot \mathrm{~K}^{-1}$ is ideal gas constant.

Let $p_{a}$ be partial pressure of dry air, $p_{v}$ be partial pressure of water vapor and $p$ be total pressure of humid air. It holds that

$$
p=p_{a}+p_{v}
$$

Let $\rho_{a}$ be density of dry air, $\rho_{v}$ be density of water vapor and $\rho$ be density of humid air. Then

$$
\rho=\rho_{a}+\rho_{v} .
$$

Using equation for ideal gas (see 4.1),

$$
\begin{equation*}
\rho=\frac{p_{a} M_{a}}{R T}+\frac{p_{v} M_{v}}{R T}=\frac{M_{a} p}{R T}\left(\frac{p_{a}+\frac{p_{v} M_{v}}{M_{a}}}{p_{a}+p_{v}}\right)=\frac{M_{a} p}{R T}\left(\frac{1+\frac{p_{v} M_{v}}{p_{a} M_{a}}}{1+\frac{p_{v}}{p_{a}}}\right) . \tag{4.2}
\end{equation*}
$$

Expression $\frac{p_{v} M_{v}}{p_{a} M_{a}}$ is specific humidity. Concentration of water vapor (denoted $w_{v}$ ) is defined:

$$
w_{v}=\frac{\rho_{v}}{\rho_{a}+\rho_{v}} .
$$

Let us derive, using (4.2):

$$
\begin{aligned}
\rho_{a} & =\rho-\rho_{v}=\frac{M_{a} p}{R T}\left(\frac{1+\frac{p_{v} M_{v}}{p_{a} M_{a}}}{1+\frac{p_{v}}{p_{a}}}\right)-\rho_{v}=\frac{M_{a} p}{R T}\left(\frac{p_{a} M_{a}+p_{v} M_{v}-\rho_{v} R T}{p_{a} M_{a}+p_{v} M_{a}}\right) \\
& =\frac{M_{a} p}{R T}\left(\frac{p_{a} M_{a}+p_{v} M_{v}-p_{v} M_{v}}{p_{a} M_{a}+p_{v} M_{a}}\right)=\frac{M_{a} p}{R T}\left(\frac{p_{a} M_{a}}{p_{a} M_{a}+p_{v} M_{a}}\right) \\
& =\frac{M_{a} p}{R T}\left(1-\frac{p_{v} M_{a}}{p_{a} M_{a}+p_{v} M_{a}}\right)=\frac{M_{a} p}{R T}-\frac{\rho_{v} M_{a}}{M_{v}} .
\end{aligned}
$$

Thus,

$$
\begin{equation*}
\rho_{a}=\frac{M_{a} p}{R T}-\frac{\rho_{v} M_{a}}{M_{v}} . \tag{4.3}
\end{equation*}
$$

### 4.2 Equations of flow around cylinder emitting water vapor

The equations are similar to equations describing flow around rotating cylinder (see (3.1)), but the density in motion equation is not constant. It depends on the concentration of water vapor in air. Partial density of dry air fulfills equation (4.3). Moreover, we will use equation of diffusion. If we assume that diffusion coefficient $D$ is constant, steady diffusion equation is

$$
\mathbf{v} \cdot \nabla w_{v}=D \Delta w_{v} .
$$

(See [8].) Thus, all equations of flow around cylinder emitting water vapor are

$$
\begin{aligned}
\operatorname{div} \mathbf{v} & =0 \text { in } \Omega \\
\mathbf{v} \cdot \nabla \mathbf{v} & =\frac{1}{\rho_{a}+\rho_{v}} \operatorname{div} \boldsymbol{\tau} \text { in } \Omega \\
\mathbf{v} \cdot \nabla\left(\frac{\rho_{v}}{\rho_{v}+\rho_{a}}\right) & =D \Delta\left(\frac{\rho_{v}}{\rho_{v}+\rho_{a}}\right) \text { in } \Omega,
\end{aligned}
$$

where $\mathbf{v}$ is velocity field, $\tau=-p \mathbf{I}+2 \mu \mathbf{D}$ is Cauchy stress, $\mu$ is dynamic viscosity of dry air, $\mathbf{D}=\frac{1}{2}\left(\nabla \mathbf{v}+(\nabla \mathbf{v})^{T}\right)$ is rate-of-strain tensor, $D$ is diffusivity coefficient of water vapor in dry air. Dynamic viscosity in the real world is not constant, since concentration of water vapor changes in $\Omega$ and the properties of humid air vary depending on humidity. This time, change of dynamic viscosity is neglected. Also, the flow is considered to be isothermic: the value of temperature influences partial density of water vapor in saturated humid air, as can be seen from ideal gas equation (see (4.1)). Density of dry air that has no contact with vapor is $\rho_{a, d r y}=$ const. Boundary conditions are following:

$$
\begin{aligned}
\mathbf{v} & =\mathbf{v}_{\text {in }} \text { on } \Gamma_{\text {in }} \\
\mathbf{v} & =\mathbf{0} \text { on } \Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{c, \text { up }} \cup \Gamma_{c, \text { low }} \\
\boldsymbol{\tau} \cdot \mathbf{n} & =-p_{\text {out }} \mathbf{n}+\mu(\nabla \mathbf{v})^{T} \cdot \mathbf{n} \text { on } \Gamma_{\text {out }} \\
\rho_{v} & =0 \text { on } \Gamma_{\text {in }} \cup \Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{c, \text { low }} \\
\rho_{v} & =\rho_{v, \text { sat }} \text { on } \Gamma_{c, \text { up }} \\
\frac{\partial \rho_{v}}{\partial \mathbf{n}} & =0 \text { on } \Gamma_{\text {out }},
\end{aligned}
$$

where $\mathbf{v}_{\text {in }}$ is velocity of incoming fluid, $\Gamma_{1} \cup \Gamma_{2}$ are solid walls, $p_{\text {out }}$ is pressure on $\Gamma_{o u t}, \rho_{v, \text { sat }}$ is partial density of water vapor on the surface of the cylinder, where the air is saturated. It can be computed using Antoine equation, which provides formula for partial pressure of water vapor in saturated humid air:

$$
\begin{equation*}
p_{v, \text { sat }}(T)=10^{5} e^{\left(11.964-\frac{3984.95}{T-39.724}\right)} \tag{4.4}
\end{equation*}
$$

Then, partial density of water vapor on the upper part of the surface of the cylinder is equal to

$$
\rho_{v, \text { sat }}(T)=\frac{p_{v, s a t}(T) M_{v}}{R T}
$$

See figure of the domain $\Omega$ (4.1).
Numeric solution of this problem will be obtained as the limit of sequence of numeric solutions of these problems: Let $k=0, \ldots, \infty, \rho_{a, 0}=\rho_{a, d r y}, \rho_{v, 0}=0$. The first two equations have the unknowns $\mathbf{v}_{k}$ and $p_{k}, \rho_{v, k+1}$ is the unknown in the last equation. First, continuity and motion equations are solved; $\rho_{a, k}$ and $\rho_{v, k}$ are functions which were obtained in the previous step. Second, diffusion equation is solved, the unknown function is $\rho_{v, k+1}$.

$$
\begin{align*}
\operatorname{div} \mathbf{v}_{k} & =0 \text { in } \Omega \\
\mathbf{v} \cdot \nabla \mathbf{v}_{k} & =\frac{1}{\rho_{a, k}+\rho_{v, k}} \operatorname{div}\left(-p_{k} \mathbf{I}+\mu\left(\nabla \mathbf{v}_{k}+\left(\nabla \mathbf{v}_{k}\right)^{T}\right)\right) \text { in } \Omega \\
\mathbf{v}_{k} \cdot \nabla\left(\frac{\rho_{v, k+1}}{\rho_{v, k+1}+\rho_{a, k}}\right) & =D \Delta\left(\frac{\rho_{v, k+1}}{\rho_{v, k+1}+\rho_{a, k}}\right) \text { in } \Omega \tag{4.5}
\end{align*}
$$



Figure 4.1: Domain $\Omega$, where equations of flow around cylinder emitting water vapor are to be solved.

Boudary conditions for problem (4.5) $\forall k=0, \ldots, \infty$ are:

$$
\begin{aligned}
\mathbf{v}_{k} & =\mathbf{v}_{\text {in }} \text { on } \Gamma_{\text {in }} \\
\mathbf{v}_{k} & =\mathbf{0} \text { on } \Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{c, \text { up }} \cup \Gamma_{c, \text { low }} \\
\tau_{k} \cdot \mathbf{n} & =-p_{\text {out }} \mathbf{n}+\mu\left(\nabla \mathbf{v}_{k}\right)^{T} \cdot \mathbf{n} \text { on } \Gamma_{\text {out }} \\
\rho_{v, k} & =0 \text { on } \Gamma_{\text {in }} \cup \Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{c, \text { low }} \\
\rho_{v, k} & =\rho_{v, \text { sat }} \text { on } \Gamma_{c, \text { up }} \\
\frac{\partial \rho_{v, k}}{\partial \mathbf{n}} & =0 \text { on } \Gamma_{\text {out }},
\end{aligned}
$$

Assuming that ideal gas equation (see (4.1)) holds, (4.3) also holds and we can compute $\rho_{a, k+1}$ for the next step:

$$
\begin{equation*}
\rho_{a, k+1}=\frac{M_{a} p_{k}}{R T}-\frac{\rho_{v, k+1} M_{a}}{M_{v}} . \tag{4.6}
\end{equation*}
$$

The same problem in different notation using concentration of water vapor $\left(w_{v, k}=\frac{\rho_{v, k+1}}{\rho_{a, k}+\rho_{v, k+1}}\right):$

$$
\begin{align*}
\operatorname{div} \mathbf{v}_{k} & =0 \text { in } \Omega \\
\mathbf{v} \cdot \nabla \mathbf{v}_{k} & =\frac{1}{\rho_{a, k}+\rho_{v, k}} \operatorname{div}\left(-p_{k} \mathbf{I}+\mu\left(\nabla \mathbf{v}_{k}+\left(\nabla \mathbf{v}_{k}\right)^{T}\right)\right) \text { in } \Omega  \tag{4.7}\\
\mathbf{v}_{k} \cdot \nabla w_{v, k} & =D \Delta w_{v, k} \text { in } \Omega .
\end{align*}
$$

Boundary conditions:

$$
\begin{align*}
\mathbf{v}_{k} & =\mathbf{v}_{\text {in }} \text { on } \Gamma_{\text {in }} \\
\mathbf{v}_{k} & =\mathbf{0} \text { on } \Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{c, \text { up }} \cup \Gamma_{c, \text { low }} \\
\tau_{k} \cdot \mathbf{n} & =-p_{\text {out }} \mathbf{n}+\mu\left(\nabla \mathbf{v}_{k}\right)^{T} \cdot \mathbf{n} \text { on } \Gamma_{\text {out }} \\
w_{v, k} & =0 \text { on } \Gamma_{\text {in }} \cup \Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{c, \text { low }}  \tag{4.8}\\
w_{v, k} & =\frac{\rho_{v, \text { sat }}}{\rho_{v, \text { sat }}+\rho_{a, k}} \text { on } \Gamma_{c, \text { up }} \\
\frac{\partial w_{v, k}}{\partial \mathbf{n}} & =0 \text { on } \Gamma_{\text {out }},
\end{align*}
$$

### 4.2.1 Weak formulation of equations of flow around cylinder emitting vapor

Let $\hat{\mathbf{v}} \in W^{1,2}(\Omega)$ and $\hat{w}_{v, k} \in W^{1,2}(\Omega), \forall k=0, \ldots, \infty$, be such that

$$
\begin{aligned}
\operatorname{Tr} \hat{\mathbf{v}} & =\mathbf{v}_{i n} \text { on } \Gamma_{i n} \\
\operatorname{Tr} \hat{w}_{v, k} & =\frac{\rho_{v, \text { sat }}}{\rho_{v, s a t}+\rho_{a, k}} \text { on } \Gamma_{c, u p} .
\end{aligned}
$$

Let us define spaces

$$
\begin{aligned}
\tilde{V} & =\left\{\mathbf{u} \in W^{1,2}(\Omega), \mathbf{u}=\mathbf{0} \text { on } \Gamma_{i n} \cup \Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{c, \text { up }} \cup \Gamma_{c, \text { low }}\right\} \\
\tilde{P} & =\left\{q \in L^{2}(\Omega)\right\} \\
\tilde{W}_{v} & =\left\{w_{v, \text { test }} \in W^{1,2}(\Omega), w_{v, \text { test }}=0 \text { on } \Gamma_{i n} \cup \Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{c, \text { up }} \cup \Gamma_{c, \text { low }}\right\} .
\end{aligned}
$$

We search for the functions $\tilde{\mathbf{v}_{k}} \in \tilde{V}, p_{k} \in \tilde{P}$ such that

$$
\begin{gather*}
\int_{\Omega} q \operatorname{div}\left(\hat{\mathbf{v}}+\tilde{\mathbf{v}_{k}}\right)=0 \forall q \in \tilde{P} \\
\int_{\Omega}\left(\rho_{a, k}+\rho_{v, k}\right)\left(\hat{\mathbf{v}}+\tilde{\mathbf{v}_{k}}\right) \cdot \nabla\left(\hat{\mathbf{v}}+\tilde{\mathbf{v}_{k}}\right) \cdot \mathbf{u} d \mathbf{x} \\
+\int_{\Omega}\left(-p \mathbf{I}+\mu\left(\nabla\left(\hat{\mathbf{v}}+\tilde{\mathbf{v}_{k}}\right)+\left(\nabla\left(\hat{\mathbf{v}}+\tilde{\mathbf{v}_{k}}\right)\right)^{T}\right)\right): \nabla \mathbf{u} d \mathbf{x}  \tag{4.9}\\
+\int_{\Gamma_{\text {out }}}\left(p_{\text {out }}-\mu\left(\nabla\left(\hat{\mathbf{v}}+\tilde{\mathbf{v}_{k}}\right)\right)^{T}\right) \cdot \mathbf{n} \cdot \mathbf{u} d s=0 \forall \mathbf{u} \in \tilde{V} .
\end{gather*}
$$

The solution will be $\mathbf{v}_{k}=\hat{\mathbf{v}}+\tilde{\mathbf{v}_{k}}$ and $p_{k}$. Having obtained $\mathbf{v}_{k}$ and $p_{k}$, we search for the function $\tilde{v}_{v, k} \in \tilde{W}_{v}$ such that

$$
\begin{align*}
& \int_{\Omega} \mathbf{v}_{k} \cdot \nabla\left(\hat{w}_{v, k}+\tilde{w_{v, k}}\right) \cdot w_{v, \text { test }} d \mathbf{x} \\
+ & D \int_{\Omega} \nabla\left(\hat{w}_{v, k}+\tilde{w_{v, k}}\right) \cdot \nabla w_{v, \text { test }} d \mathbf{x}=0 \forall w_{v, \text { test }} \in \tilde{W}_{v} . \tag{4.10}
\end{align*}
$$

The solution will be $w_{v, k}=\hat{w}_{v, k}+w_{v, k}$.

| mesh | number of vertices | number of cells | $\operatorname{dim}$ of $\tilde{V}_{h} \times \tilde{P}_{h}$ | $\operatorname{dim}$ of $\tilde{W}_{v, h}$ |
| :---: | :---: | :---: | :---: | :---: |
| A | 2951 | 5730 | 26215 | 11632 |
| B | 11632 | 22920 | 104000 | 46184 |
| C | 46184 | 91680 | 414280 | 184048 |

Table 4.1: Number of vertices, cells and dimension of spaces $\tilde{V}_{h} \times \tilde{P}_{h}$ and $\tilde{W}_{v, h}$.

| temperature $[K]$ | mesh | drag $[N]$ | lift $[N]$ | power $[W]$ |
| :---: | :---: | :---: | :---: | :---: |
| 293.15 | A | $4.598289 \cdot 10^{-3}$ | $1.151051 \cdot 10^{-5}$ | $3.32 \cdot 10^{-5}$ |
| 293.15 | B | $4.647143 \cdot 10^{-3}$ | $2.497831 \cdot 10^{-6}$ | $1.21 \cdot 10^{-5}$ |
| 293.15 | C | $4.658793 \cdot 10^{-3}$ | $-3.135890 \cdot 10^{-7}$ | $2.75 \cdot 10^{-6}$ |
| 313.15 | C | $4.46993 \cdot 10^{-3}$ | $-2.465866 \cdot 10^{-6}$ | $1.5 \cdot 10^{-4}$ |

Table 4.2: Drag and lift acting on the cylinder emitting vapor computed using meshes A, B and C. Power needed for vaporization.

### 4.2.2 Discretisation

We will use the same discretisation of the domain $\Omega$ as in the problem of the rotating cylinder (see mesh A, B and C). $\Omega_{h}$ denotes discretisation of the domain. Function spaces $\tilde{V}, \tilde{P}$ and $\tilde{W}_{v}$ were approximated by their finite-dimensional subspaces $\tilde{V}_{h}, \tilde{P}_{h}$ and $\tilde{W}_{v, h}$, where

$$
\begin{aligned}
\tilde{V}_{h} & =\left\{\mathbf{v}_{h} \in C\left(\Omega_{h}\right),\left.\mathbf{v}_{h}\right|_{E} \in P^{2}(E) \forall E \in A, \mathbf{v}_{h}=\mathbf{0} \text { on } \Gamma\right\}, \\
\tilde{P}_{h} & =\left\{p_{h} \in C\left(\Omega_{h}\right),\left.p_{h}\right|_{E} \in P^{1}(E) \forall E \in A\right\}, \\
\tilde{W}_{v, h} & =\left\{w_{v, h} \in C\left(\Omega_{h}\right),\left.w_{v, h}\right|_{E} \in P^{2}(E) \forall E \in A, w_{v, h}=0 \text { on } \Gamma\right\}
\end{aligned}
$$

where $E$ is a cell in triangulation $A, \Gamma=\Gamma_{i n} \cup \Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{c, u p} \cup \Gamma_{c, \text { low }}$. Discretisations of function spaces using meshes $B$ or $C$ have analogic definitions. For total dimensions of space $\tilde{V}_{h} \times \tilde{P}_{h}$ and $\tilde{W}_{v, h}$, see table 4.1. FEniCS used Newton's method, and algebraic system obtained by FEM from diffusion equation was solved by direct solver (Krylov solver).

The following computation used these data:
$\rho_{a, d r y}=1.2041 \mathrm{~kg} / \mathrm{m}^{3}$
dynamic viscosity $\mu=18 \cdot 10^{-5} \mathrm{~Pa} \cdot \mathrm{~s}$
$\Omega$ was a rectangle $0.4 \mathrm{~m} \times 0.3 \mathrm{~m}$ without circle, center [0.1, 0.15], radius $a=$ 0.0125 m
$\mathbf{v}_{\text {in }}=\left(\frac{0.5 \cdot 4 \cdot \cdot \cdot(0.3-y)}{0.3^{2}}, 0\right) \mathrm{m} / \mathrm{s}$
$p_{\text {out }}=101325 \mathrm{~Pa}$
$T=293.15 \mathrm{~K}$
$D=282 \cdot 10^{-7} \mathrm{~m}^{2} / \mathrm{s}$
Numerical solution of the problem consists in solving a sequence of problems. The difference in $L^{2}$-norm between new and previous solution $\rho_{v, k}$ is watched and this process runs until the difference is reasonably small (in this case, lower than $10^{-8}$, which was achieved by 4 iterations, or, at higher temperature by 5 iterations).

Drag and lift was computed for the problem of the flow around cylinder emitting vapor, see table 4.2. These values can be compared with computed force acting on the rotating cylinder. It seems that vapor on the upper part of the cylinder has some influence on the lift acting on the cylinder. The lift, compared with the lift acting on steady cylinder (see table 3.3), has lower value. Thus, the influence of vapor is that the lift acts downwards. In case the temperature is higher ( 313.15 K ), the partial density of saturated air is higher and the value of lift is one order higher than error. See also figure 4.7. Power needed for vaporization of water in quantity corresponding to the model, was computed using formula

$$
P=-h_{v a p} D \int_{\Gamma_{c, u p}} \nabla w_{v} \cdot d \mathbf{s},
$$

where enthalpy of vaporization $h_{\text {vap }}=2.5 \cdot 10^{6} \mathrm{~J} / \mathrm{kg}$. See table 3.3. Computed values of power are very low and change a lot when they are computed on meshes A, B and C. Further refinement of mesh would be needed to obtain more precise value.

This part of the thesis studied the flow around cylider with evaporation, and it was shown that there is some influence of nonconstant density on the lift; the influence on the velocity field is not perceptible in figures. Also when the temperature was higher, the influence of water vapor on the lift was more apparent. We cannot forget that there were some restrictions, as the viscosity and the temperature were constant. In the real world, if there was a phase transition on the cylinder, there would probably be some change in temperature and in entropy. As we saw in Crocco's theorem (see (2.7)), change in entropy causes nonzero vorticity and lift is induced. The effect of evaporization on lift could be stronger than our results would suggest.


Figure 4.2: Computed velocity field for flow around cylinder emitting water vapor on mesh A , mesh B and mesh C .


Figure 4.3: Streamlines for flow around cylinder emitting water vapor, mesh A, mesh B and mesh C.


Figure 4.4: Partial density of water vapor for flow around cylinder emitting water vapor, mesh $A$, mesh $B$ and mesh $C$.


Figure 4.5: Partial density of dry air for flow around cylinder emitting water vapor, mesh $A$, mesh $B$ and mesh $C$.


Figure 4.6: Relative pressure, i. e. $p-101325 \mathrm{~Pa}$, for flow around cylinder emitting water vapor, mesh $A$, mesh $B$ and mesh $C$.


Figure 4.7: Computed partial density of water vapor for flow around cylinder emitting water vapor, $T=313.15 \mathrm{~K}$, mesh C.

## Chapter 5

## Comparison of numerical solution with experiment

In previous chapters, we visualized potential flow around cylider and numerical solution of steady Navier-Stokes equations. The figures will be compared with visualization of the experiment performed by Ing. Z. Trávníček, CSc., and Ing. Z. Broučková from the Institute of Thermomechanics AS CR (see [13]). The experiment was already explained in chapter 3. The visualization of the experiment focused on the neighbourhood of the cylinder. There are groups of figures, each obtained by a different method, describing the same situation, or at least similar situation.

### 5.1 Steady cylinder

The first figure in this chapter (see 5.1) shows flow around steady cylinder: using potential flow method, the visualization of experiment, numerical solution of Navier-Stokes equation using mesh C and numerical solution of flow around cylinder emitting vapor problem using mesh C . The potential flow describes motion of inviscous incompressible fluid, meaning it differs a little from other pictures in that the fluid does not create circular streamlines behind the cylinder. It can be seen that numerical solution of Navier-Stokes equations corresponds to reality. In addition to, we computed the lift (see table 3.3), meaning that if the experiment included measurement of lift, the values could be compared. Flow around cylinder emitting vapor has almost the same velocity field as the flow around steady cylinder, but lift is slightly different (compare tables 3.3 and 4.2).

### 5.2 Rotating cylinder

More comparative figures follow, the comparison always includes potential flow, the visualization of the experiment and numerical solution of N-S equations. See figures 5.2, 5.3 and 5.4 . It can be seen that the higher circulation is, the more potential flow model differs from the visualization of the experiment. On the other hand, numerical solution of N-S equations is similar to the experiment.


Figure 5.1: Flow around steady cylinder. Bottom: cylinder with evaporation.


Figure 5.2: Flow around rotating cylinder, $\omega=5 \mathrm{~Hz}$.


Figure 5.3: Flow around rotating cylinder, $\omega=12.5 \mathrm{~Hz}$.



Figure 5.4: Flow around rotating cylinder, $\omega=20 \mathrm{~Hz}$.

## Conclusion

The thesis was concerned with the influnce of dissipative processes in boundary layers on forces acting on flown-around body. It studied the influence of viscosity of the fluid and the influence of diffusion of water vapor in air. Both cases were modelled, numerically computed using FEM in FEniCS software, visualized in Paraview and compared with the experiment performed by Ing. Z. Trávníček, CSc., and Ing. Z. Broučková.

The conclusion is that the influence of viscosity is apparent unlike the influence of diffusion. The results demonstrate, that rotating cylinder is better described by Navier-Stokes equations than by model of potential flow. Lift was computed. Model of cylinder emitting vapor showed that diffusion has very little effect on both velocity field and lift.

We have to bear in mind that in case of real flow around cylinder with evaporation, the gradient of entropy would be nonzero, as a result of phase transition. There would probably be temperature change; depending on concentration of water vapor, change in viscosity of humid air would also play a part. This would have to be demonstrated by further experiment with cylinder placed in wind tunnel, with some moistened porous material on its upper part.

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## Used theorems

Theorem 2. Cauchy theorem, existence of stress
Let the same volume forces and surface forces as above act on moving system composed of non-polar materials. Then linear momentum balance and angular momentum balance hold if and only if there exists tensor $\boldsymbol{\tau}=\left(\tau_{i j}\right)_{i, j=1}^{3}$, called Cauchy stress, such that:
(1) $\forall \boldsymbol{\nu} \boldsymbol{T}(x, t, \boldsymbol{\nu})=\boldsymbol{\tau} \cdot \boldsymbol{\nu}$;
(2) $\boldsymbol{\tau}$ is symmetric ( $\left.\tau_{i j}=\tau_{j i}, \forall i, j=1,2,3\right)$;
(3) $\boldsymbol{\tau}$ satisfies equation of motion

$$
\operatorname{div} \boldsymbol{\tau}+\rho \boldsymbol{f}=\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}+\sum_{i=1}^{3} \frac{\partial \boldsymbol{v}}{\partial x_{i}} v_{i}\right) .
$$

Proof is to be found in [6].

## Theorem 3. Derivative of composed mapping

Let $\varphi_{1}, \ldots, \varphi_{r}$ be real functions of $s$ variables which have differentials at point $\boldsymbol{a}=\left(a_{1}, \ldots, a_{s}\right), b_{j}=\varphi_{j}(\boldsymbol{a}), j=1, \ldots, r$. Let $f$ be a function of $r$ variables, and let $f$ have differential at point $\boldsymbol{b}=\left(b_{1}, \ldots, b_{r}\right)$. Then function $F(\boldsymbol{t})=$ $f\left(\varphi_{1}(\boldsymbol{t}), \ldots, \varphi_{r}(\boldsymbol{t})\right)$ has differential at point $\boldsymbol{a}$, which is expressed:

$$
L(\boldsymbol{h})=\sum_{j=1}^{r} \frac{\partial f(\boldsymbol{b})}{\partial x_{j}} \sum_{k=1}^{s} \frac{\partial \varphi_{j}(\boldsymbol{a})}{\partial t_{k}} h_{k},
$$

and for $k=1, \ldots, s$ it holds

$$
\frac{\partial F(\boldsymbol{a})}{\partial t_{k}}=\sum_{j=1}^{r} \frac{\partial f(\boldsymbol{b})}{\partial x_{j}} \frac{\partial \varphi_{j}(\boldsymbol{a})}{\partial t_{k}} .
$$

Proof is to be found in [7].

## Theorem 4. Divergence theorem

Let $V$ be bounded regular domain in $\mathbb{R}^{3}, \partial V$ is boundary of $V$, let $\boldsymbol{v}: V \rightarrow \mathbb{R}$ be smooth vector field, then

$$
\int_{\partial V} \boldsymbol{v} \cdot \boldsymbol{\nu} d S=\int_{V} \operatorname{div} \boldsymbol{v} d v
$$

where $\boldsymbol{\nu}$ is unit outward normal to $\partial V$.

See [6].
Theorem 5. Stokes' theorem Let $\boldsymbol{v}$ be smooth vector field in $V \subset \mathbb{R}^{3}$. Let $S$ be a disc, c be a curve encircling $S$ such that $\boldsymbol{\nu}$ unit normal to $S$ fulfills righ-had rule. Then

$$
\int_{S} \operatorname{curl} \boldsymbol{v} \cdot \boldsymbol{\nu} d S=\int_{c} \boldsymbol{v} \cdot d \boldsymbol{s}
$$

Proof is to be found in [6].

## Theorem 6. Cauchy-Riemann equations

Let $f$ is complex function of complex variable. Let us denote $\tilde{f}=\left(\tilde{f}_{1}, \tilde{f}_{2}\right)$ function of two real variables with values in $\mathbb{R}^{2}$ corresponding with $f$ by matching $\mathbb{C}$ and $\mathbb{R}^{2}$, i.e. such that $f(x+i y)=\tilde{f}_{1}(x, y)+i \tilde{f}_{2}(x, y), x+i y$ in domain of $f$. $z=a+i b$, where $a, b \in \mathbb{R}$. Then $f$ has at point $z$ derivative with respect to complex variable if and only if $\tilde{f}$ has at point $(a, b)$ differential and it hold that

$$
\begin{aligned}
& \frac{\partial \tilde{f}_{1}}{\partial x}(a, b)=\frac{\partial \tilde{f}_{2}}{\partial y}(a, b) \\
& \frac{\partial \tilde{f}_{1}}{\partial y}(a, b)=-\frac{\partial \tilde{f}_{2}}{\partial x}(a, b) .
\end{aligned}
$$

Proof in [3].

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