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MASTER THESIS

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Extremal Polyominoes

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I would like to thank my family and my employer for the support while I was writing the thesis. I also dedicate this thesis to my supervisor Doc.R̃NDr.P̃avel Valtr,D̃r., who helped me to choose an interesting topic, led inspirational conversations and gave me the hope when I stuck in an unsolvable problem.

I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Extremal Polyominoes

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Abstrakt: Práce se zabývá tématem polymin a dalších rovinných obrazců, které se skládají z pravidelných mnohoúhelníků, konkrétně polyiamondů a polyhexů. Zaměřili jsme se na základní geometrické vlastnosti: obvod, konvexní obal a ohraničující čtverec/šestiúhelník. Tyto parametry minimalizujeme nebo maximalizujeme pro pevně danou velikost polymina, kterou značíme jako n. Vzhledem k n odvozujeme vzorec pro maximální a minimální hodnoty zvoleného parametru a také se snažíme vyjmenovat všechna polymina, která tohoto maxima dosahují. Některé problémy už byly vyřešeny dříve jinými autory a my přinášíme shrnutí jejich výsledků. Jiné jsme vyřešili my, jmenovitě problém maximálního ohraničujícího čtverce/šestiúhelníku a maximálního konvexního obalu pro polyiamondy. Některé otázky zůstávají i nadále otevřeny a my nabízíme alespoň pozorování, která mohou posloužit v dalším výzkumu.

Klíčová slova: Polymino, konvexní obal, extremální otázky, rovina

Title: Extremal Polyominoes

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Abstract: The thesis is focused on polyominoes and other planar figures consisting of regular polygons, namely polyiamonds and polyhexes. We study the basic geometrical properties: the perimeter, the convex hull and the bounding rectangle/hexagon. We maximise and minimise these parameters and for the fixed size of the polyomino, denoted by n. We compute the extremal values of a chosen parameter and then we try to enumerate all polyominoes of the size n, which has the extremal property. Some of the problems were solved by other authors. We summarise their results. Some of the problems were solved by us, namely the maximal bounding rectangle/hexagon and maximal convex hull of polyiamonds. There are still several topics which remain open. We summarise the literature and offer our observations for the following scientists.

Keywords: Polyomino, convex hull, extremal questions, plane

Contents

Introduction

The aim of our work are polyominoes and other figures consisting of regular polygons which are edge-to-edge connected. They were introduced at the beginning of 20th century, but they became much more popular after the work of Solomon Golomb from the beginning of 50s and his book [6] is still the bible of this area of combinatorics.

The formal definition of a polyomino says it is a planar figure consisting of $n \in \mathbb{N}$ interior disjoint unit squares connected edge-to-edge. An easy generalisation into d-dimensional object is possible considering unit d-cubes instead of unit squares. We can also substitute unit squares by any other planar polygon. Literature usually mentions polyiamonds (unit regular triangles), polyhexes (unit regular hexagons), polyabolos (unit right isosceles triangles). The general term for all kinds of such planar figures is *animals*. All the mentioned shapes can tile a plane completely, so the interesting question always was, which animals can also tile a plane. Later there were introduced generalized polyominoes (authors use the term polyomino, not animal) of any other shapes, which sometimes even cannot tile the plane, but it was interesting to study how many different shapes they can make. In our thesis we are focus on the planar figures only. We study squares, regular triangles and hexagons. The squares/triangles/hexagons are called *cells* and if a figure has n cells it is said it has size (or area) n or it is n-animal, e. g. n-polyomino.

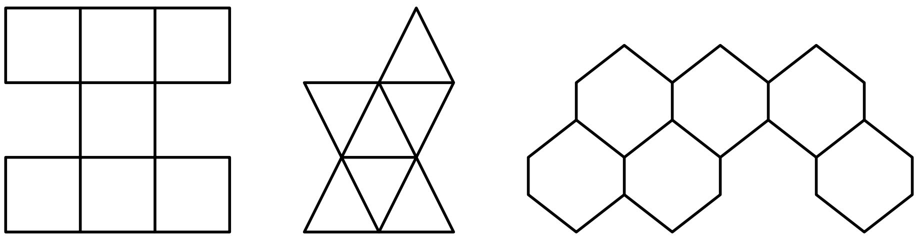


Figure 1: Animals: 7-polyomino, 7-polyiamond and 6-polyhex

As already mentioned, many questions arise. The two basic categories of the problems with polyominoes are plane-tiling and enumeration. The first problem is, which figures can tile a plane or rectangle or parts of plane as half-plane, stripe and so on. Golomb divided the shapes into several classes according their ability to tile a part of a plane. Some questions from his paper [5] are open till today, such as if there is a shape which can tile a half of plane, can it always tile the quarter of the plane, too? Or are the classes of tiling always in proper inclusion? Also the number of distinct tilings was studied. The variation with coloured tiles and plane was studied, too, especially the black-and-white case.

The second category, which contains our work, too, contains the problems how to enumerate all polyominoes with a given property. The original problem is, how many figures can n unit squares form. It is still an open question, but the mathematicians tried to restrict on more special classes and so they were able to answer the question partially. In our thesis we use restriction, which maximise and minimise some basic geometrical parameters, namely the perimeter and the area of the convex hull. As the boundaries of the animals consist of segments in only two or three directions, the natural criterion is the bounding rectangle or hexagon with boundary segments in the same directions.

The thesis is divided into three chapters, each focused on one extremal criterion: the bounding rectangle/hexagon, the perimeter and the convex hull. We try to characterise figures which are minimal or maximal in one of the criteria and count them for fixed n.

As we enumerate distinct polyominoes we need a definition of the equivalence of polyominoes. There are three basic function which can identify two polyominoes: translation, rotation and reflection. Equivalence using combination of all these three function is the one, which we use in the thesis and such polyominoes are usually called $free$.

The first chapter is about the bounding rectangle for polyominoes and about the bounding hexagon for polyiamonds and polyhexes. For minimisation and maximisation we use the length of the perimeter of the bounding rectangle/hexagon. For the polyominoes we give the estimate on the size of the bounding rectangle.

Theorem 1. *The perimeter of a minimal bounding rectangle for polyominoes of size n is equal to $2ab$ where $|a-b| \leq 1$ and $ab \geq n$ is minimal possible.*

Unfortunately we are unable to enumerate all free polyominoes with the maximal bounding rectangle. We think the problem is close to the problem of enumeration all polyominoes for fixed n, because $k = ab - n$ "empty" squares can form any polyomino which can be placed inside the bounding rectangle to its border. This process can probably be used for the enumeration of all polyominoes, but as we said above, the question of enumeration of all free polyominoes for fixed n is still open. The siuation is similar with the polyiamonds and polyhexes, too.

The problem of maximal bounding rectangle and hexagon is much easier but surprisingly we did not find any literature about this topic. Therefore we bring out the theorem about the size of the maximal bounding rectangle for the polyominoes and the bounding hexagon for polyimonds and polyhexes.

Theorem 2. *The maximal possible perimeter of a bounding rectangle of an n-polyomino is $2(n+1)$.*

Theorem 3. *The maximal perimeter of a bounding hexagon of an n-polyiamond is $n+2$.*

Theorem 4. *The maximal perimeter of a bounding rectangle of an n-polyhex is $4n+2$.*

For the polyominoes the figures with the maximal bounding rectangle look like stairs which can be extended by piles under the first and on the last step. We give the algorithm to enumerate all free polyominoes with the maximal bounding rectangle.

The same description of the figures with the maximal bounding hexagon fits for the polyiamonds, too. We omit the exact algorithm as we think there is nothing surprising in it. The polyhexes are a bit different and we prove the only optimal figure is a line.

The second chapter is about the perimeter. The perimeter is defined as if the animal is a polygon. The problem of the minimal perimeter of polyominoes was solved by F. Harary and H. Harborth in [7], but the enumeration was found almost thirty years later by S. Kurz in [8].

Theorem 5 (Harary, Harborth). *The perimeter of a polyomino of size n with the minimal perimeter is $p(n) = 2\lceil 2\sqrt{n}\rceil$.*

Theorem 6 (Kurz). *Any polyomino with minimal perimeter $p(n)$ can be made from any rectangular poloymino consisting of at least n squares and perimeter $p(n)$ by removing the squares from corners.*

For the exact numbers of the polyominoes of the size n see the theorem 17.

The minimal perimeter for polyiamonds was computed by W. C. Yang and R. R. Meyer in [11], but they did not give the enumeration of all of them. One of the authors published similar result for polyhexes in [10]. The question of enumeration remains open.

Theorem 7 (Yang, Meyer). *The minimal perimeter over all polyiamonds of the size n is $\lceil\sqrt{6n}\rceil$ or $\lceil\sqrt{6n}\rceil + 1$ and it has the same parity as n.*

Theorem 8 (Yang). *The minimal perimeter over all polyhexes of the size n is $2\lceil\sqrt{12n-3}\rceil$*

The problem of maximal perimeter was not studied that much, we found only one paper [7] about it. The authors proved the theorem about the maximal perimeter for polyominoes, polyiamonds and polyhexes.

Theorem 9 (Harary, Harborth). *For triangular ($a = 3$), square ($a = 4$) and hexagonal ($a = 6$) animals*

$$\max p = an - 2n + 2.$$

An animal attains this maximum perimeter p if and only if its skeleton is a tree (for skeleton see figure 2.4).

The problem of enumeration is still open and it seems it would be useful to use some techniques from graph theory. We can define the dual of an animal as a graph, where the cells are vertices and the vertices are connected by an edge if their cells share and edge. It is an interesting question how such a graph class looks like, if we consider the animals with the maximal perimeter. This class is subclass of trees. The class of polyiamonds is a subclass of the polyominoes and their class is a subclass of polyhexes. But the characterisation of this classes remains open.

The last chapter is about the convex hull. We minimise and maximise its area while the cells have unit area. The minimal convex hull has in case of polyiamonds and polyominoes the size equal to n. The number of the figures for a fixed n can be characterised by the number of solution of equations based on the formulae for volume of rectangle and hexagon, respectively.

The problem of the maximum convex hull was solved by S. Kurz in [9]. He gave the bound for the maximal convex hull.

Theorem 10 (Kurz). *The area of the convex hull of a planar n-polyomino is at most $n + \lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor$.*

He enumerated all the maximal figures for fixed n. The maximal figures consist of one line and at most one orthogonal line on each side.

Theorem 11 (Kurz). *Denote the number of different n-polyominoes with the maximal area of the convex hull by $c(n)$. Then we have*

$$c(n) = \begin{cases} \frac{n^3-2n^2+4n}{16} & \text{if } n \equiv 0 \bmod 4 \\ \frac{n^3-2n^2+13n+20}{32} & \text{if } n \equiv 1 \bmod 4 \\ \frac{n^3-2n^2+4n+8}{16} & \text{if } n \equiv 2 \bmod 4 \\ \frac{n^3-2n^2+5n+8}{32} & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

Inspired by this paper we proved the exactly same theorem for the polyiamonds.

Theorem 12. *The area of the convex hull of any polyiamond consisting of n triangles is at most*

$$\frac{n^2+10n+1}{12}.$$

Theorem 13. *The polyiamonds with the maximum area of convex hull are three pointed starts and washtub (see figure 3.10).*

We tried to prove it for polyhexes, too, but we did not succeed. We give at least some basic observations, which can be useful for the following research.

1. Extremal Bounding Rectangle And Hexagon

The first chapter is focused on the bounding rectangle (and bounding hexagon for polyiamonds and polyhexes). The bounding rectangle of a polyomino is defined as minimal rectangle which contains the whole polyomino and its edges are parallel to the edges of the squares of the polyomino. The definition of the bounding hexagon is analogous. See figure 1.1.

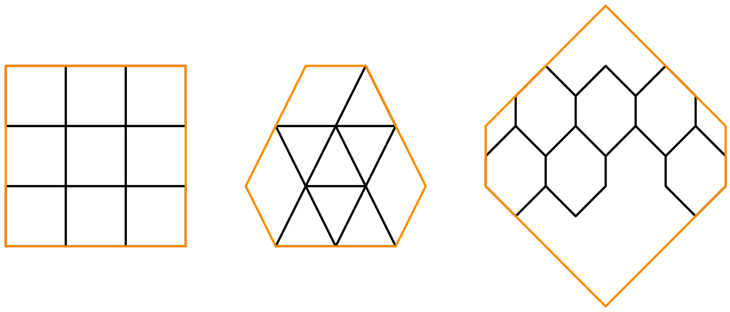


Figure 1.1: Bounding rectangle and hexagons

We are interested in the minimal (resp. maximal) bounding rectangle/hexagon of an n-animal. Here minimising and maximising are measured by the perimeter of the bounding rectangle/hexagon. The cells have the edges of length one.

Our aim is to compute the minimal and maximal perimeter of the bounding rectangle/hexagon and to describe, how the rectangle/hexagon looks like. Then we try to enumerate all the possible free n-polyominoes which have the minimal/maximal bounding rectangle/hexagon.

1.1 Minimal Bounding Rectangle and Hexagon

1.1.1 Polyominoes

First we look at the polyominoes. We were inspired by a compilation [2] published on Art of Problem Solving Website, where authors describe how many polyominoes exist for a chosen size of the bounding rectangle. They don't fix n, the number of used squares. This question is very hard to answer for a general polyomino, so they chose several classes of polyominoes such as so-called skyline, vertically convex or directed convex polyominoes.

Our approach is a bit different as it was announced in the introduction of this chapter. We fix the size n of the polyomino and then we want to compute the perimeter of the minimal (and next the maximal) possible bounding square. When the perimeter of the bounding rectangle is computed, it is fixed and we want to enumerate all polyominoes consisting of n squares and having bounding rectangle with this perimeter.

Theorem 14. *Suppose n is the size of a polyomino. Then there is a minimal bounding rectangle, which has edges a, b so that $|a - b| \leq 1$ and $ab \geq n$ minimal possible.*

Proof. The latter condition from the theorem is necessary. If $ab < n$ then the polyomino cannot fit into the rectangle because the rectangle is too small. The other condition follows from the observation that if the difference is bigger, then there exist a_1, b_1 which sum equals to $a + b$ (so the perimeter is the same) but the area of the bounding rectangle is bigger, so n squares still fit into it. □

Note that because of the minimality the bounding rectangle is determined uniquely. It can be computed in the following way. Take a square root of n. If $\sqrt{n} \in \mathbb{N}$ then $a = b = \sqrt{n}$. Otherwise consider three possible combinations:

1. $a = b = \lfloor\sqrt{n}\rfloor$
2. $a = b = \lceil\sqrt{n}\rceil$
3. $a = \lfloor\sqrt{n}\rfloor, b = \lceil\sqrt{n}\rceil$

Choose the combination which minimise $a + b$ and which satisfies the condition $ab \geq n$.

As a corollary recall the theorem from the introduction.

Theorem 1. *The perimeter of a minimal bounding rectangle for polyominoes of size n is equal to $2ab$ where $|a - b| \leq 1$ and $ab \geq n$ is minimal possible.*

There is an important fact, that the minimal bounding rectangle is not uniquely determined. For example, let $n = 15$. Then the optimal rectangle can be 4×4 (from the theorem), so we know $a + b = 8$. But 3×5 is also possible. Of course using induction we can enumerate all optimal rectangles in time O(n).

Enumeration of all polyominoes with minimal bounding rectangle seems to be very hard. Let n be the size of the polyomino and let $A = ab$ be the area of the bounding rectangle. Then we define $k = A - n$ as a number of "empty" squares, that means number of squares which are not occupied by any square of the polyomino. From the theorem we know the size of the bounding rectangle grows with the size of polyomino, so k can be arbitrarily large, because it can equal $a-1$. Now suppose we can enumerate all polyominoes of size k. Almost each polyominoes can be placed into the bounding rectangle so the "nonempty" squares remain connected and the empty squares do not create a hole(see figure 1.2). So the problem of the enumeration of all polyominoes with minimal bounding rectangle seems to be very close to the problem of enumeration all polyominoes of specified size. And this question has not been solved yet.

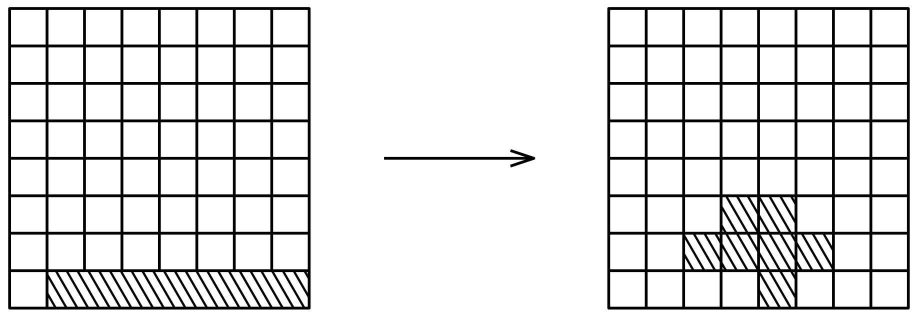


Figure 1.2: Empty squares can form almost any polyomino inside.

Note: there are very specific polyominoes which are not convex on any side, so it is not possible to place them into the bounding rectangle. But we think it is a very specific class of polyominoes therefore it does not have a big influence on the problem hardness.

1.1.2 Polyiamonds and Polyhexes

The problem about the minimal bounding hexagon is the same as for the squares. To enumerate them one would need an enumeration of all figures of size n. We did not find any literature focused on this topic.

1.2 Maximal Bounding Rectangle

1.2.1 Polyominoes

The problem of the maximal bounding rectangle is much easier but it was not studied that much. We found only the paper [1] in which authors enumerate all polyominoes of minimal size which can be inscribed into a specified rectangle. This theorem is used to count n-polyominoes with maximal bounding rectangle. But they omit the symmetries, so in the view of standard polyomino enumeration their results are not that interesting. This subsection presents our own results.

Before we start we want to recall that we maximise the perimeter of the bounding rectangle. We can choose another parameter, for example the area of the bounding rectangle. In that case the bounding rectangle will have the maximal perimeter too, but the size of edges would differ by at most one. Therefore we decided to consider the perimeter as it is more general and our algorithm can be easily restricted to the maximal area.

Recall the theorem from the introduction.

Theorem 2. *The maximal possible perimeter of a bounding rectangle of an n-polyomino is* $2(n + 1)$.

Proof. We prove the theorem by induction on n. One square has the bounding rectangle with the perimeter $4 = 2(1 + 1)$. Any other square can enlarge the bounding rectangle only in one direction (to enlarge it in both direction it has to be connected only by a corner and it is not allowed). It adds 2 to the bounding rectangle's perimeter. That proves the upper bound. We can create a line of squares of length n and it has the perimeter $2n + 2 = 2(n + 1)$. □

As the proof of the previous theorem shows any figure whose construction enlarge the bounding rectangle in every step is optimal. The example of line in the proof is only one of many possible figures. The other are polyominoes look like crosses, stairs or other similar variations. It is not easy to see how to enumerate all of them with the respect to symmetries.

We give an algorithm which describe how all the optimal figures look like and how to enumerate all of them. We omit the exact formula for the number of all different figures, because it is really complicated to express. Using the description one can make a program, which generate the numbers easily.

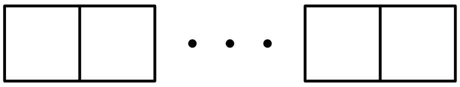


Figure 1.3: A square line

Definition. S*quare line is a polyomino of the line shape. See figure 1.3.*

S*tairs are a polyomino consisting of one or more square lines* $l_1, l_2, \ldots l_k$ *where* l_i, l_{i+1} *for* $i \in [1 \ldots k-1]$ *are connected by one edge, for the lower line it is the top right edge, for the higher line it is the bottom left edge. One square line is called layer. The stairs with* k *layers are denoted as* k*-stairs. The top and bottom layer consists of at least two squares. Any other layer can consist of one square only. See figure 1.4 for some examples.*

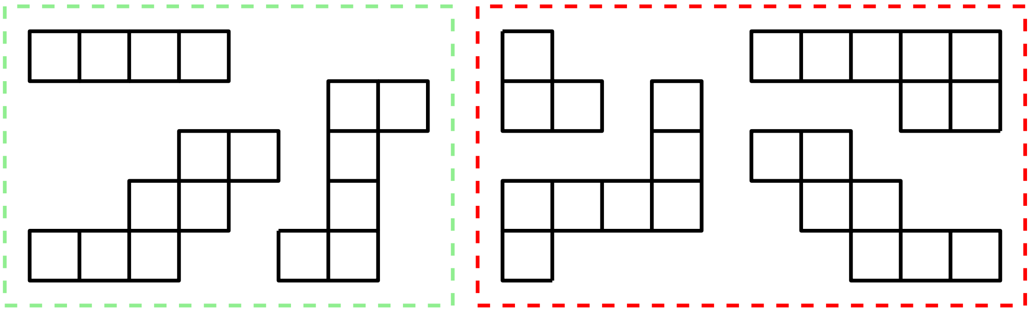


Figure 1.4: The stairs in green, invalid shapes in red (one square in last layer, two connected squares between two lines, from right to left)

The stairs go up from the left to the right. We could define stairs in the other orientation, but it is not necessary as we want to enumerate all shapes with the respect to all symmetries, we can get the right-left stair using the mirror symmetry. The condition for two squares in the outer layers is described in the algorithm and it also helps to count a figure only once.

Theorem 15. *The polyominoes with the maximal bounding rectangle are stairs with a pile of squares on the top layer and other one under the bottom layer. One or both piles can be missing. See Figure 1.5.*

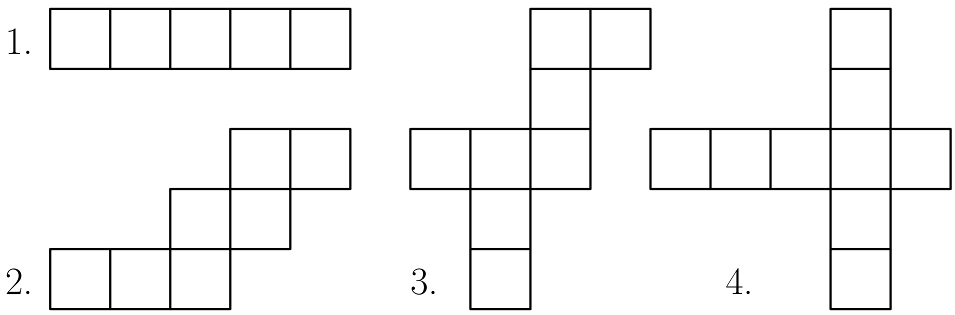


Figure 1.5: 1. 1-stairs, 2. 3-stairs, 3. 3-stairs with a 2-pile under the bottom layer, 4. 1-stairs with two 2-piles.

Proof. Consider the construction square-by-square. We can form without loss of generality a horizontal line of squares and it has the maximal bounding rectangle (from the previous theorem). If we add a square to a side of the line, say to the upper side, we cannot add any other square to this side, because it will not enlarge the bounding square. The only place, where it can be added is to the first square. We can always add it on the last square, so we build a pile, or we can add it to left or right side, but only if there are no squares in the same vertical coordinates already. Consider this construction restrictions the only possible shapes are stairs with piles at the ends. Because of the mirror symmetry it is enough to build stairs from the left to the right. □

The theorem describes, which shapes are optimal, but it does not specified, how to enumerate them such that each shape is counted only once with the respect to all symmetries. The algorithm is described in the following paragraphs.

Enumeration algorithm We enumerate the figures so that the following property is fulfilled. The bounding rectangle is of the form $a \times b$ where $a \geq b$.

The input of the algorithm is n, the size of the polyomino. From the previous condition it is obvious that $b \leq \lfloor \frac{n}{2} \rfloor$.

The algorithm constructs the polyominoes according the number of the layers of the stairs. We denote the number of layers by $i \in [1, \dots \lfloor \frac{n}{2} \rfloor]$.

$\boldsymbol{i = 1}$: If there is only one layer it is kind of a special case. There is 1 square line and then there are other "cross-like" polyominoes formed by one layer and a pile on one or both sides. There are $\lceil \frac{n+1}{2} \rceil \dots n$ squares in the layer. Define k as the number of squares which are not in the layer, they can be used in the piles, and $n_l = n - k$, the number of squares in the layer. Divide k squares into two groups of size k_1, k_2 where $k_1 \in [0, \dots k]$ and $k_2 = k - k_1$. If $k_1 \neq 0$ and $k_2 \neq 0$ then there are n_l^2 figures. If $k_1 = k_2 = 0$ then there is 1 figure. If $k_1 = 0$ and $k_2 \neq 0$ then there are n_l figures and the same number is in the last case. In total there are $kn_l^2 + 2n_l$ figures for a fixed k, but we must consider the symmetries.

If n_l is odd, we add 2 to the total number of figures, so each figure where $k_1 = 0$ or $k_2 = 0$ appears four times (the figure with a pile in the middle appears only twice), because of vertical and horizontal mirror symmetries.

If k is even, we add n_l^2 to the total number of figures, so each figure where $k_1 = k_2$ appears four times. All other figures appears four times naturally.

In total we have

$$1 + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(k-1)n_l^2 + even(k)n_l^2 + 2n_l + 2odd(n_l)}{4} \tag{1.1}$$

distinct figures with one layer. We used two functions $even : \mathbb{N} \to \{0, 1\}$, which returns 1 if the argument is even and returns 0 otherwise, and $odd : \mathbb{N} \to \{0, 1\}$, which returns 1 if the argument is odd and returns 0 otherwise.

$\boldsymbol{i \in [2, \dots \lfloor \frac{n}{2} \rfloor]}$: If we have more then one layer we must keep at least two squares in the top layer and in the bottom layer, too. If we allow to have only one square in one of these layers, we cannot distinguish the case, when it is a layer and when it is a part of a pile.

In the following we consider two cases: figures with no piles and figures with nonzero piles.

Without piles: We need to distribute all squares into the layers. There are 4 squares fixed in the top layer and in the bottom layer and 1 square in each layer, so all the layers are nonempty. There are $l = n - i - 2$ squares left. They are distributed into all i layers. So according the formula for combination with repetition we get

$$\frac{\binom{i+l-1}{l}}{2} + \frac{1}{2}(even(i)even(l)\binom{\frac{i}{2}+\frac{l}{2}-1}{\frac{l}{2}}) + odd(i)\sum_{c=0}^{l} even(l-c)\binom{\frac{i-1}{2}+\frac{l-c}{2}-1}{\frac{l-c}{2}}. \tag{1.2}$$

The division by 2 is necessary because each distribution can be turned by 180° and it gives us an already counted figure. The only exception are symmetric distributions, for example $(3, 1, 3)$ for odd number of layers or $(3, 1, 1, 3)$ for even number of layers, which are counted only once. As we divide the original number by 2 we have to add the lost half of the symmetric figures again. The number of symmetric figures depends on the parity of the number of layer and of the parity of l. If all parity condition are fulfilled then we distribute the half of the l squares into the upper half layers and the rest of the distribution is given. The formula explains all the parity conditions and gives the number of symmetric distribution in all cases. Note that if the number of layers is odd, then we choose number of squares in the central layer (the number is denoted by c) and then distribute the rest into the layers above the central layer.

With piles: In the last case we combine the previous computations. We denote the number of squares which are used in piles by k (same as in case $i = 1$). It can be chosen from $k \in [1, \dots min(l, \lfloor \frac{n}{2} \rfloor - i)]$, because we can use only squares which are not fixed in a layer and we cannot use more than $\lfloor \frac{n}{2} \rfloor - i$ squares, because each square in pile add 1 to the b size of the bounding rectangle and the size of this side is at most $\lfloor \frac{n}{2} \rfloor$, where i was already used by layers. In addition we use n_l for the number of squares in layers, therefore $n = n_l + k$ as in the case of $i = 1$.

The exact formulae start to be really complicated, so we describe here, how to get them but for the exact values would be better to use a computer.

Let the previous case generate all possible figures for all possible n_l. For each n_l group the figures according the number of the squares in the top layer p_1 and bottom layer p_2. So we have groups for each pair (p_1, p_2) of size s. We use analogous formula as (1.1) to get the total number of the figures of specified parameters:

$$s((k-1)p_1p_2 + p_1 + p_2).$$

Note there is no need to add "fake" figures, because each figure is already generated only once, because the original figures are generated with respect to all symmetries already.

The algorithm is finished and from its description it is obvious it generate all figures and it generates each of them only once.

1.2.2 Polyiamonds

The polyiamonds behave a bit differently. The first triangle has the perimeter of the bounding hexagon equal to 3. If we construct a line, each triangle adds one to the perimeter. If we add a triangle in any other direction, it enlarges the bounding rectangles by 1 on both side edges, but it shortened the edge parallel to the one which is the connection between the triangle and the rest of the polyiamond. See figure 1.6. As a corollary of this observation we get the theorem about the size of the maximal bounding hexagon.

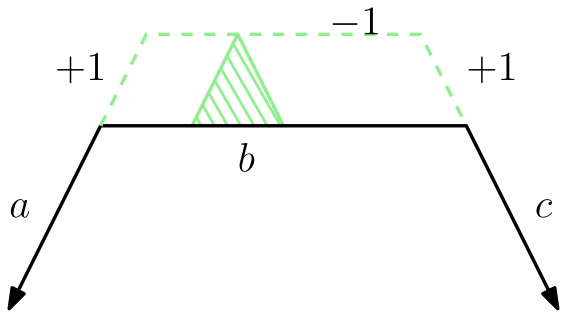


Figure 1.6: The added triangle enlarged a, c by 1, but shortened b by 1

Recall the theorem from the introduction.

Theorem 3. *The maximal perimeter of a bounding hexagon of an n-polyiamond is $n + 2$.*

Therefore the shapes of the maximal figures are similar to the stairs figures for polyominoes. We think the algorithm for polyominoes can be easily modified to get the same enumeration for the polyiamonds. We omit this generalisation, because we think there is nothing tricky or new in it.

1.2.3 Polyhexes

The hexagons behaves in absolute different way. The only optimal figure is a line. Forming a line, each hexagon adds one to four sides of the bounding hexagon (see figure 1.7). It not possible that one hexagon enlarge more than four sides. But the thing is, one hexagon can enlarge one side by more than one, by at most two exactly.

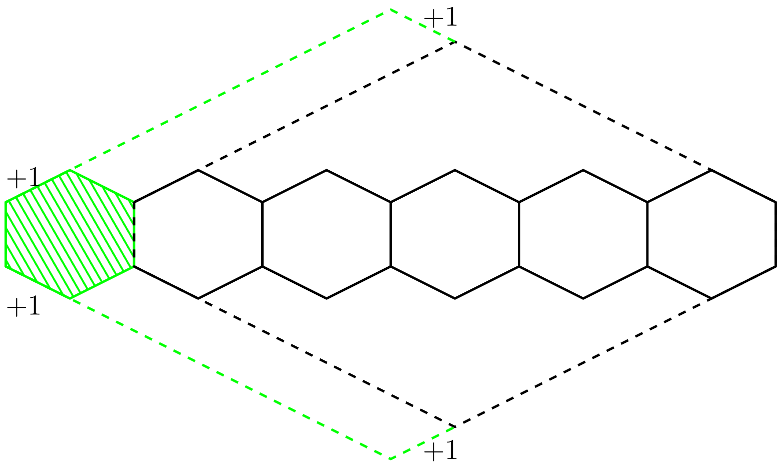


Figure 1.7: Hexagon line, the original bounding hexagon has the perimeter 22, the new one $22 + 4 = 26$

Let us describe the situation, when a side of the bounding rectangle is enlarged by 2. We denote three side of the bounding hexagon by a, b, c. The new hexagon is added to b. Then a, c is enlarged by at most 2, but b is shortened by 2. Therefore the perimeter of the bounding rectangle was enlarged by 2 and it is definitely worse then in the case of line (see figure 1.8). It can happen one side is enlarged by two and the other only by 1 and then b is shortened only by 1. But again the perimeter is enlarged by 2 only.

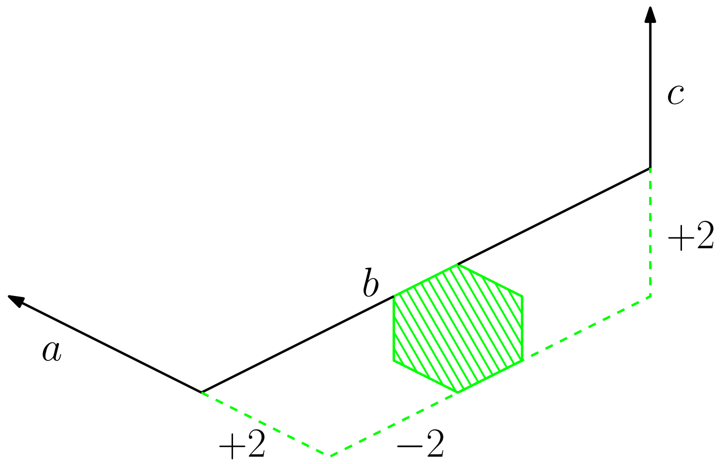


Figure 1.8: The added hexagon enlarged a, c by 2, but shortened b by 2

From the argument above it is obvious any other figure must contains a step when the perimeter was enlarged by 2 or less and hence the line is the only optimal polyhex.

As corollary recall the theorem from the introduction and formulate the theorem about the optimal polyhex figure.

Theorem 4. *The maximal perimeter of a bounding rectangle of an n-polyhex is $4n + 2$.*

Theorem 16. *The only polyhex with the maximal bounding hexagon is a line of hexagons.*

2. Extremal Perimeter

The perimeter is one of the most studied parameter of polyominoes. The general problem of enumaration all polyominoes for a fixed perimeter have not been solved in past years. So again the scientist try to restrict the enumeration on special classes of polyominoes. See [4] for examples of column-convex or staircase polyominoes.

Another nice class polyominoes are the equable polyominoes. The definition says that they are polyominoes whose size equals their perimeter. It is a simple observation you cannot make an equable polyomino of size less than 15 and all equable polyominoes has even size.

In our thesis we consider only standard perimeter (sum of external edges), but one can often find a term site-perimeter. This kind of perimeter counts the cells adjacent to the polyomino by an edge. Surprisingly this parameter seems to be easier to describe and was fully characterized for polyominoes and enumeration of minimal polyiamonds and polyhexes.

2.1 Minimal Perimeter

2.1.1 Polyominoes

The answer about minimal perimeter was completely solved by Sascha Kurz in [8], but before[7] published the exact formula for the minimal perimeter. The description of the optimal polyomino of size n is quite simple. Let take a rectangle with area at least n and minimal perimeter. Then remove from corners k squares, where k is the difference between n and rectangle area. The sketch of proof is described below.

Recall the theorem from the introduction.

Theorem 5 (Harary, Harborth)**.** *The perimeter of a polyomino of size n with the minimal perimeter is $p(n) = 2\lceil 2\sqrt{n}\rceil$.*

The proof of this theorem was published in [7] and it is based on the spiral construction. See Figure 2.1.

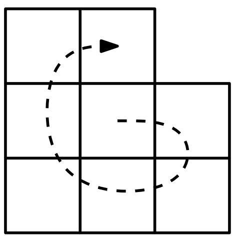


Figure 2.1: Spiral construction

Theorem 17. *The number $e(n)$ of polyominoes with n squares and minimum perimeter $p(n)$ is given by following cases:*

$$e(n) = \begin{cases} 1 & \text{if } n = s^2 \\ \sum_{c=0}^{\lfloor -\frac{1}{2}+\frac{1}{2}\sqrt{1+4s-4t}\rfloor} r_{s-c-c^2-t} & \text{if } n = s^2 + t \quad (0 < t < s) \\ 1 & \text{if } n = s^2 + s \\ q_{s+1-t} + \sum_{c=1}^{\lfloor\sqrt{s+1-t}\rfloor} r_{s+1-c^2-t} & \text{if } n = s + s + t \quad (0 < t \leq s) \end{cases}$$

where $s = \lfloor\sqrt{n}\rfloor$ and with r_k, q_k being the coefficient of x^k in the following generating function $r(x)$ and $q(x)$, respectively. The two generating functions

$$s(x) = 1 + \sum_{k=1}^{\infty} x^{k^2} \prod_{j=1}^{k} \frac{1}{1 - x^{2j}}$$

and

$$a(x) = \prod_{j=1}^{\infty} \frac{1}{1 - x^j}$$

are used in the definition of

$$r(x) = \frac{1}{4}\big(a(x)^4 + 3a(x^2)^2\big)$$

and

$$q(x) = \frac{1}{8}\big(a(x)^4 + 3a(x^2)^2 + 2s(x)^2 a(x^2) + 2a(x^4)\big).$$

We can see from the function definition that when the squares are able to make a square or rectangle which sides differ only by one, there is only one optimal polyomino. Local maximum is reach for $n = s^2 + 1$ and $n = s^2 + s + 1$. The exact values for small cases are presented in [8]

It is obvious that optimal polyomino must be connected and without holes.

Observation 1. *Denote the number of common edges in polyomino by $B(n)$. Then we have*

$$B(n) = \frac{4n - p(n)}{2} = 2n - \lceil 2\sqrt{n} \rceil$$

The proof of the theorem require the definition of bounding walk.

Definition. The bounding walk H *is a closed walk trough all edge-to-edge neighbouring squares of the perimeter (squares which are not disjoint with the perimeter, even by a vertex only).*

If we consider the squares as vertices of a graph, which are connected by an edge if they are adjacent. Then we can define $|H|$ as the length of the "cycle" defined by it (note that the vertices can repeat on the cycle). Let furthermore h_i denote the number of squares of degree i. See

$$|H| = h_1 + h_2 + h_3 + h_4.$$

If $h_1 > 0$ then $B(n) - B(n-1) = 1$, because if the square is removed the number of common is decreased by its degree. Therefore if $h_1 > 0$ then $n = s^2 + 1$ or $s^2 + s + 1$, because of the original equation for $B(n) = 2n - 2\lceil\sqrt{n}\rceil$. In general we can assume $h_1 = 0$

Following lemmas express technical properties and we omit their proof.

Lemma 1. *If* $h_1 = 0 then h_2 = h_4 + 4$.

Lemma 2. *If* $h_1 = 0$ *then the number* m *of common edges of squares of the polyomino is*

$$m = 2n - \frac{|H|}{2} - 2.$$

Lemma 3. *For the maximum area* $A(|H|)$ *of a polyomino with boundary* H *and* $h_1 = 0$ *we have*

$$A(|H|) = \begin{cases} (\frac{|H|+4}{4})^2 & \textit{if } |H| \equiv 0 \bmod 4 \\ \\ (\frac{|H|+4}{4})^2 - \frac{1}{4} & \textit{if } |H| \equiv 2 \bmod 4 \end{cases}$$

Lemma 4. *For a polyomino with* $h_1 = 0$ *with minimum perimeter* $p(n)$ *we have* $|H| = 2\lceil 2\sqrt{n}\rceil - 4$.

From these technical lemmas we can derive the theorem describing how to create a polyomino with minimal perimeter. Let us recall the theorem from the introduction.

Theorem 6 (Kurz)**.** *Any polyomino with minimal perimeter* $p(n)$ *can be made from any rectangular poloymino consisting of at least* n *squares and perimeter* $p(n)$ *by removing the squares from corners.*

Proof. First consider $h_1 = 0$. Denote the bounding rectangle by R. If the cardinality of the boundary of R is smaller then $|H|$ then due to Lemma 2 and the fact $B(n)$ is increasing. Thus $|H|$ equals the cardinality of R and the polyomino can be obtained from R removing the squares. Only the squares of degree 2 can be removed, because they don't change the perimeter. Such squares are only in the corners.

For $h_1 > 0$ we know $n = s^2 + 1$ or $s^2 + s + 1$ and can be created by removing $s - 1$ squares from $s \times s + 1$ from the shorter side, from $s^2 + 1 \times s^2 + 1$ or from $s^2 \times s^2 + 2$ respectively. □

The rest of the proof describes how to get the exact formula for the number of different polyominoes of the size n and minimal perimeter $p(n)$. The idea is based on the observation, that the shape of a removed corner can be associated with a Ferrer's diagram, so we can use the generating function for Ferrer's diagram to describe the situation in one corner. The combination of the function for all four corners and usage of Cauchy - Frobenius's Lemma completes the proof.

2.1.2 Polyiamonds

The question about the minimal perimeter for polyiamonds was solved by Winston C. Yang and Robert R. Meyer in [11]. they proved the exact formula for the minimal perimeter over all polyiamonds of the size n. Their construction and the idea of its proof is presented below.

Recall the theorem from the introduction.

Theorem 7 (Yang, Meyer)**.** *The minimal perimeter over all polyiamonds of the size n is $\lceil\sqrt{6n}\rceil$ or $\lceil\sqrt{6n}\rceil + 1$ and it has the same parity as n.*

To prove the theorem above the authors use *maximal polyiamonds* which are figures for the fixed perimeter p containing maximum number of triangles. They state the theorem about number of the triangles, then describe an algorithm for construction of the maximum polyiamond and then they derive the figure of the same perimeter but with minimal number of triangles. The result follows.

The proofs are based on the structure of the *polyiamond slices*. Let have a polyiamond. Triangles on one line in one of three direction (horizontal, antidiagonal, diagonal) are a slice. The slice consist of subslices, a subslice is a group of connected triangles in one slice. See Figure 2.2 for an example. We sum all subslices in horizontal direction and denote the number by h. Do the same with antidiagonal (a) and diagonal (d) direction. We get (h, a, d) the description of the polyiamond. If the polyiamond has (h, a, d) subslices we say it has HAD (h, a, d).

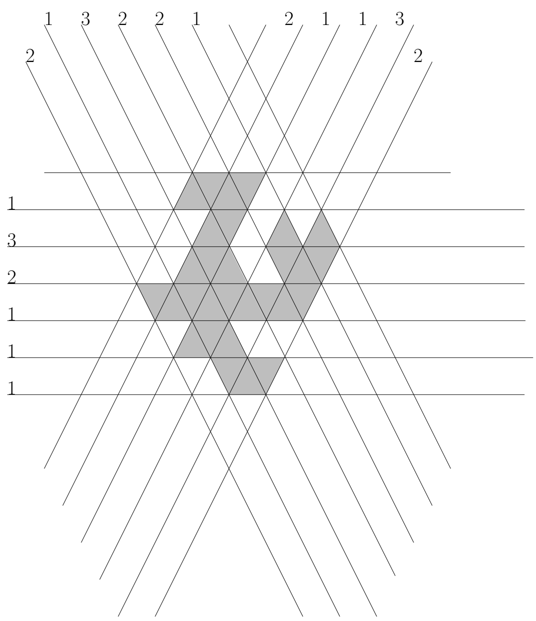


Figure 2.2: Polyiamond with HAD $(9, 9, 11)$

Theorem 18. *The maximum number of triangles in a polyiamond with perimeter p is*

$$round(\frac{p^2}{6}) - \begin{cases} 0 & if\ p \equiv 0 \bmod 6 \\ 1 & else \end{cases}$$

Proof. Sketch: They design an linear integer program to get the meximum number of triangles based on several observation about HAD. The most important observation is, that $h + a + d = p$. □

Theorem 19. *The algorithm for the maximal polyiamond with perimeter p. Choose* (h, a, d) *so that* $p = h + a + d$ *and* $h \leq a \leq d$. *Draw parallelogram with HAD (a+d, a, d). Pick up the horizontal lines with the most triangles.*

The authors gives another alternative algorithm using spiral construction. It is very similar to the spiral construction for the polyominoes.

They use the spiral construction for construction of the polyiamonds with minimal perimeter. The theorems above are used for the proof such a figure is minimal.

The problem of the enumeration seems to be much more difficult than for the polyominoes. We cannot use the argument of minimal bounding rectangle/hexagon, because removing one triangle never preserve the perimeter (the perimeter has always the same parity as n). The extension can be done be removing diamonds from corners (diamond = two adjacent triangles). But still there is a problem we cannot achieve all optimal figures. For example $n = 5$ has the optimal perimeter $p = 7$ and one optimal figure is regular hexagon with edge of the size 1 with one missing triangle. We may construct the hexagon polyiamonds with perimeter $p - 1$ and start the removing process by one triangle which will improve the perimeter. Then we can proceed by removing diamonds. But this algorithm would need more careful analysis.

2.1.3 Polyhexes

Winston C Yang, who was on e of the authors of the paper about minimal perimeter of polyiamonds, is the author of the paper about the polyhexes, too. In [10] he use absolutely same approach, which he used in the previous paper. The enumeration of all possible minimal polyhexes with minimal perimeter is missing.

Recall the theorem from the introduction.

Theorem 8 (Yang)**.** *The minimal perimeter over all polyhexes of the size* n *is* $2\lceil\sqrt{12n-3}\rceil$

2.2 Maximal Perimeter

Compare to the minimal perimeter, the topic of the maximal perimeter was not studied that much. The only information which we found is in the paper [7], where the authors proved the following theorem for all *a*nimals, that means for polyiamonds, polyominoes and polyhexes. We call a *c*ell one triangle, square or hexagon of an animal.

Definition. E*dge of an animal is an edge of a cell of the animal. If two cells share an edge, this edge is taken as one edge. We denote by* q *the number of all edges in an animal. See figure 2.3.*

S*keleton of an animal is defined as its dual, that is a graph where vertices are cells and there is an edge between two vertices if their cells share an edge. See figure 2.4*

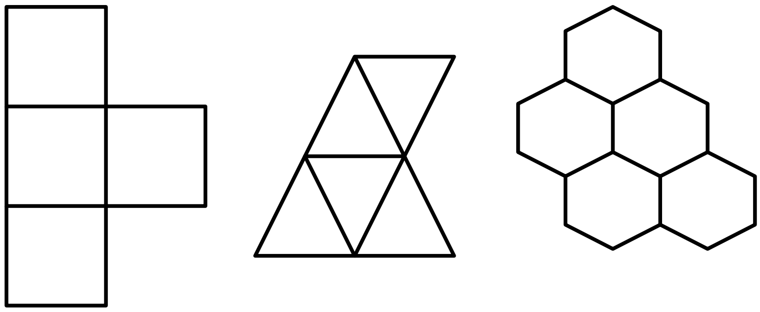


Figure 2.3: Number of edges from left to rigth: q = 12, q = 11, q = 23

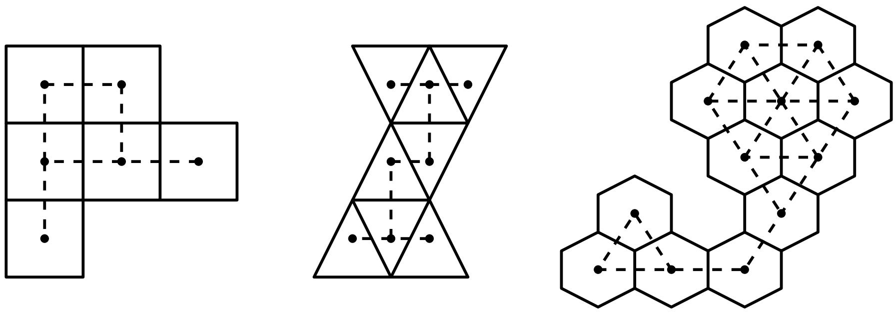


Figure 2.4: Skeletons - vertices are inside the cells and edges a dashed.

Theorem 20. *For triangular ($a = 3$), square ($a = 4$) and hexagonal ($a = 6$) animals*

$$\max q = (a - 1)n + 1.$$

An animal attains this maximum number of edges if and only if its skeleton is a tree.

Proof. Let q_1 and q_2 denote the number of lines which belong to one and two cells, respectively, so that $q = q_1 + q_2$. With respect to the number and kind of cells we get $na = q_1 + 2q_2 = q + q_2$ and from which we derive

$$q = na - q_2. \tag{2.1}$$

We observe the skeleton $G(V, E)$ has $|V| = n$ and $|E| = q_2$ therefore it is obvious that $\max q$ means $\min q_2 = n - 1$ and so the skeleton is a tree. From (2.1) we get $\max q = (a - 1)n + 1$ and by the fact that such an animal exists (e. g. path) the theorem is proven. □

As a corollary of this theorem we recall the theorem from the introduction.

Theorem 9 (Harary, Harborth)**.** *For triangular ($a = 3$), square ($a = 4$) and hexagonal ($a = 6$) animals*

$$\max p = an - 2n + 2.$$

An animal attains this maximum perimeter p if and only if its skeleton is a tree (for skeleton see figure 2.4).

This theorem says what the size of the perimeter is and how does the animal looks like. However the question about the enumeration all shapes remains open.

One can consider all trees of degree at most three for triangles and for hexagons, four for squares. But we cannot use it, not only because of symmetries, but also because there exist trees which has the correct degree, but does not represent any animal.

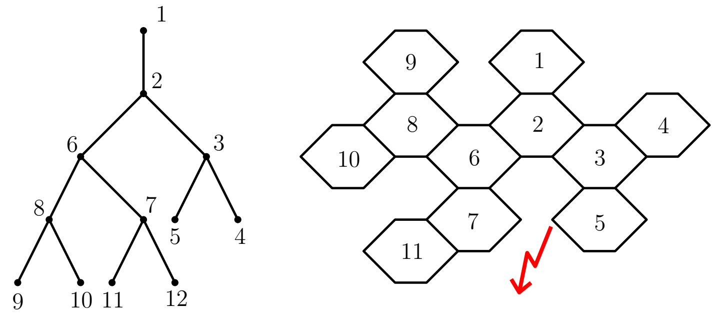


Figure 2.5: The tree has degree 3, but it is not a skeleton of any animal. On the left you see the analysis for polyhexes (for polyiamonds and polyominoes it is obvious).

The problem of enumeration can be translated onto enumeration of trees of unit length edges and edges in only few possible direction (two orthogonal for squares and three for triangles and hexagons). It looks as a very interesting class of trees, but we did not find any literature with at least the definition of such a class.

3. Extremal Convex Hull

The last chapter is about the convex hull. As the problem of the minimal convex hull is usually pretty simple we focus on the maximal convex hull. For all animals we consider the cells of the area equal to one. For polyominoes it means the unit squares, but for the triangles the size of its edges is $2/\sqrt[4]{3}$. For hexagon it is $2/6\sqrt[4]{3}$.

3.1 Minimal Convex Hull

3.1.1 Polyominoes

It is obvious that the best way how to arrange the unit squares so they have the minimal convex hull is to place them into a rectangle so they fill it completely. The number of ways how to do it is the same how many ways we can write n as product of two integers. Each partition $n = n_1 n_2$ then corresponds to a rectangle $n_1 \times n_2$.

3.1.2 Polyiamonds

The situation of the minimal convex hull of a polyiamond is similar to the situation of polyominoes. We can always create a long strip consisting of unit triangles, so the area of its convex hull equals the area of the triangles, so it is minimal.

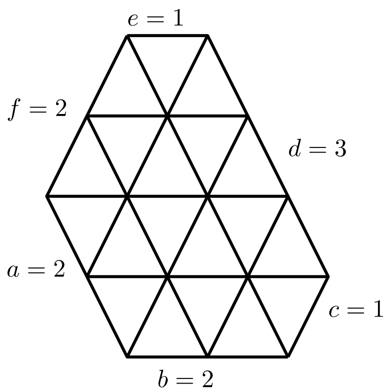


Figure 3.1: Convex figure example with the labels of the edges

As squares can form rectangles which are convex, so they have minimal convex hull, triangles can form hexagons with the same property. The area of a hexagon can be computed as

$$n = 2(ac + bc + bd) - (a - d)^2 \tag{3.1}$$

where a, b, c, d are sizes of four consecutive edges of the polyiamond. Four segments determine the hexagon, because the sizes of the other two can be expressed as

$$e = a + b - d$$
$$f = c + d - a$$

Setting some of the variables to 0 we get another shapes, which are illustrated on the picture 3.2.

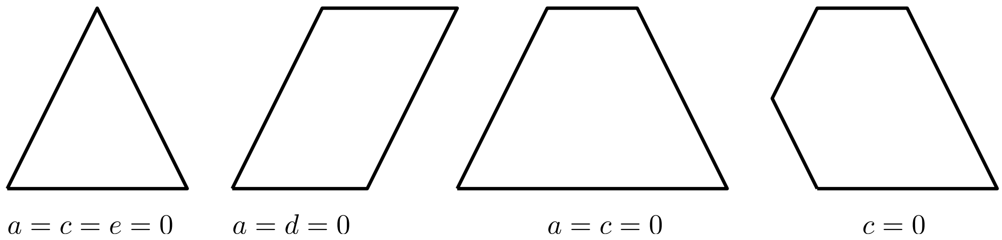


Figure 3.2: Possible convex polyiamonds beside hexagon

So the number of the distinct figures is the number of integral solutions of 3.1 with the respect to the twelve symmetries (identity, rotation, reflection through a vertex, reflection through an edge).

3.1.3 Polyhexes

We wanted to solve the problem of the minimal convex hull for the polyhexes, too, but it turned out it is much more difficult then for the other two cases. Here we give some observations and conjectures we were unable to prove.

The complication of the minimal convex hull is, that we cannot create a convex figure. Even two hexagons are not convex, so their convex hull must be strictly larger than two, it is $2\frac{1}{3}$

There is no obvious way how to create minimal shapes, but it is possible to make them such that each hexagon creates at most 1/6 area, that is the case of the line of hexagons.

Theorem 21. *The minimal area of the convex hull of an n-polyhex is at most $n + n/6$.*

3.2 Maximal Convex Hull

3.2.1 Polyominoes

The problem of polyominoes with the maximal convex hull is much more interesting. It was firstly introduced in [3] in 1994. They conjectured that the volume of the maximal convex hull of a polyomino in d-dimensional space equals

$$\sum_{I \subseteq \{1,\ldots,d\}} \frac{1}{|I|!} \prod_{i \in I} \lfloor \frac{n-2+i}{d} \rfloor$$

and proved it for $d = 2$.

Ten years later Sascha Kurz proved in [9] their conjecture for general dimension and enumerate all planar polyominoes. His proof is presented bellow and our own result for polyiamonds follows the same idea.

Recall the 2-dimensional case from the introduction.

Theorem 10 (Kurz)**.** *The area of the convex hull of a planar n-polyomino is at most $n + \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor$.*

Before we start the proof we need to define a function, which we use to bound the area of convex hull.

Definition. The description *of a planar polyomino p is the* 4-*tuple* (l_1, l_2, v_1, v_2) *such that for a single square it is* $(1, 1, 0, 0)$ *and for a bigger polyomino we consider the following process of its construction square-by-square. If the square enlarges the actual bounding rectangle in width then l_1 increase by one. If the square changes the height of the bounding rectangle then l_2 increase by one. Otherwise v_1 (resp. v_2) increase by one if the square is adjacent to the rest in horizontal (resp. vertical) direction. If the square is adjacent in both direction we can choose either v_1 or v_2 arbitrary and let it increase by one. See figure 3.3.*

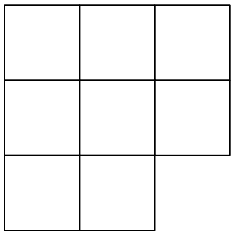


Figure 3.3: Possible descriptions: $(3, 3, 2, 1)$, $(3, 3, 1, 2)$, $(3, 3, 3, 0)$, $(3, 3, 0, 3)$

Observation 2. *The following equation expresses the relationship between polyomino size and its description*

$$n = l_1 + l_2 - 1 + v_1 + v_2.$$

Definition. *The bounding function is a function $f : \mathbb{N}^4 \to \mathbb{R}$ defined by the following expression:*

$$f(l_1, l_2, v_1, v_2) = l_1+l_2-1+\frac{(l_1 - 1)(l_2 - 1)}{2}+v_1+v_2+\frac{v_1(l_2 - 1)}{2}+\frac{v_2(l_1 - 1)}{2}+\frac{v_1v_2}{2}$$

Proof. First we prove that the area of the convex hull of any polyomino is at most the value of its bounding function. We prove it by induction on n.

For $n = 1$ we have $f(1, 1, 0, 0) = 1$ so the statement holds. Now we assume it is true for any description such that $n - 1 = l_1 + l_2 - 1 + v_1 + v_2$. Because of symmetry we can omit the cases of increasing l_2 and v_2.

Suppose first l_1 be increased by one and the new square has a neighbour on the left side. We can see on the picture 3.4 that the square enlarge the convex hull at most by 1 (square itself), and by $\frac{l_2-1}{2}$ (two triangles). While we have

$$f(l_1 + 1, l_2, v_1, v_2) - f(l_1, l_2, v_1, v_2) = 1 + \frac{l_2 - 1}{2} + \frac{v_2}{2}$$

the statement is satisfied and the first part of the induction step is done.

It remains to prove it for v_1 and its increase by 1. Again we suppose there is a neighbour on the left side. On the figure 3.5 we can see two triangles, which are beside the square itself the biggest possible contribution of the new square to the convex hull. From the similar computation we get

$$f(l_1, l_2, v_1 + 1, v_2) - f(l_1, l_2, v_1, v_2) = 1 + \frac{l_2 - 1}{2} + \frac{v_2}{2}$$

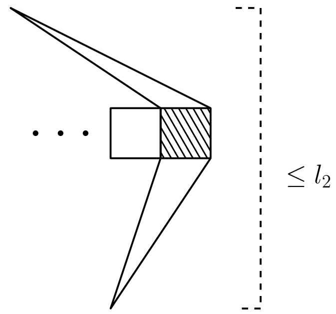


Figure 3.4: l_1 increase

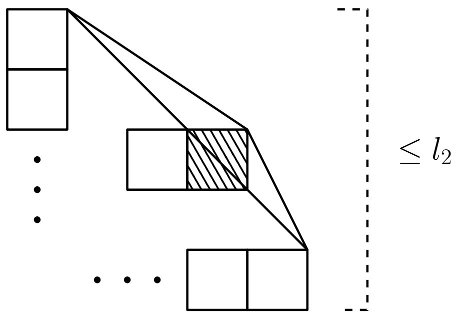


Figure 3.5: v_1 increase

so the proof, that the bounding function bounds the area of the convex hull is finished.

To finish the proof of the theorem we need to find the maximum of the function and compare it with the given estimate.

First we observe $v_1 = v_2 = 0$. It follows from the symmetry and $f(l_1 + 1, l_2, v_1 - 1, v_2) - f(l_1, l_2, v_1, v_2) = 0$. Without loss of generality we can assume $l_1 \leq l_2$. Now from

$$f(l_1 + 1, l_2 - 1, 0, 0) - f(l_1, l_2, 0, 0) = \frac{l_2 - l_1 - 1}{2} > 0$$

we observe $l_2 - l_1 \leq 1$. Because of $n = l_1 + l_2 - 1$ we derive $l_1 = \lfloor \frac{n+1}{2} \rfloor$ and $l_2 = \lfloor \frac{n+2}{2} \rfloor$. Finally substitution into the bounding function proves the theorem. $\square$

Beside the theorem about the maximal size of the convex hull Sasha Kurz shows in his paper all possible shapes of the polyominoes. The shape is described as a linear strip with at most two orthogonal strips on each side. Using Cauchy-Frobenius lemma he calculated the exact number of extremal polynominoes for fixed n. Let us recall this theorem from the introduction.

Theorem 11 (Kurz)**.** *Denote the number of different n-polyominoes with the* *maximal area of the convex hull by c(n). Then we have*

$$c(n) = \begin{cases} \frac{n^3-2n^2+4n}{16} & \textit{if } n \equiv 0 \bmod 4 \\ \frac{n^3-2n^2+13n+20}{32} & \textit{if } n \equiv 1 \bmod 4 \\ \frac{n^3-2n^2+4n+8}{16} & \textit{if } n \equiv 2 \bmod 4 \\ \frac{n^3-2n^2+5n+8}{32} & \textit{if } n \equiv 3 \bmod 4 \end{cases}$$

Using the same idea and generalized version of the bounding function he proved the estimate for maximal convex hull of polyominoes in d-dimensional space. The question of enumeration all shapes in general dimension remains open, but the conjecture is that the polyomioes consist of several orthogonal arms.

3.2.2 Polyiamonds

In this section we give estimate for the maximal convex hull of a polyiamond and enumerate all possible figures for given n. The idea of the proof of the estimate follows the proof of Kurz for polyominoes.

Recall the theorem from the introduction.

Theorem 12. *The area of the convex hull of any polyiamond consisting of n triangles is at most*

$$\frac{n^2+10n+1}{12}.$$

Before we start with the proof we need to define description for polyiamonds and bounding function for them.

Definition. The description *of a polyiamond p is the* 6*-tuple* $(l_1, l_2, l_3, v_1, v_2, v_3)$ *such that for a single triangle it is* $(1, 1, 1, 0, 0, 0)$ *and for a bigger polyiamond we consider the following process of its construction triangle-by-triangle. If the triangle enlarge the actual bounding hexagon* l_1*,* l_2 *or* l_3 *increase by one depends on the direction in which the hexagon changed (see on the picture 3.6). Otherwise either* v_1*,* v_2 *or* v_3 *increase by one depends on the side which is adjacent to the rest of the polyiamond. If it is adjacent in more directions it can be chosen arbitrary. See figure 3.7.*

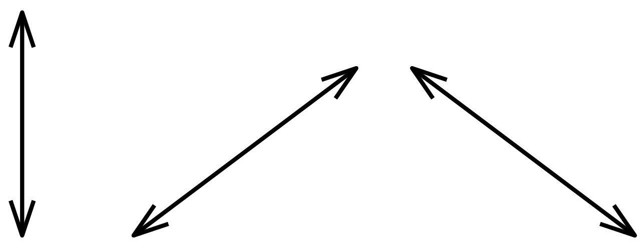


Figure 3.6: Directions from left to right: 1, 2, 3

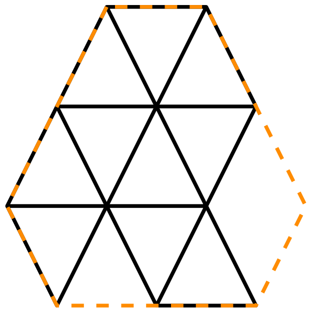


Figure 3.7: Possible descriptions: $(3, 3, 3, 1, 1, 1)$, $(3, 3, 3, 2, 1, 1)$

Observation 3. *Following equation express the relationship between the size of the a polyiamond and its description.*

$$n = l_1 + l_2 + l_3 - 2 + v_1 + v_2 + v_3$$

Definition. *Let* $f : \mathbb{N}^6 \to \mathbb{R}$ b*ounding function be defined by following expression:*

$$\begin{aligned} f(l_1, l_2, l_3, v_1, v_2, v_3) = {} & l_1 + l_2 + l_3 - 2 + \frac{(l_1 - 1)(l_2 - 1)}{4} + \frac{(l_1 - 1)(l_3 - 1)}{4} \\ & + \frac{(l_2 - 1)(l_3 - 1)}{4} + v_1 + v_2 + v_3 + \frac{v_1(l_2 - 1)}{4} + \frac{v_1(l_3 - 1)}{4} \\ & + \frac{v_2(l_1 - 1)}{4} + \frac{v_2(l_3 - 1)}{4} + \frac{v_3(l_1 - 1)}{4} + \frac{v_3(l_2 - 1)}{4} \\ & + \frac{v_1 v_2}{4} + \frac{v_1 v_3}{4} + \frac{v_2 v_3}{4} \end{aligned}$$

Proof. At first we prove the area of the convex hull is at most the value of the bounding function.

We prove it by induction on n. For $n = 1$ we have $f(1, 1, 1, 0, 0, 0) = 1$ so the statement holds. Now we assume it is true for any description such that $n - 1 = l_1 + l_2 + l_3 - 2 + v_1 + v_2 + v_3$. Because of symmetry we can omit the cases of increasing l_2, l_3 and v_2, v_3.

Let increase l_1 by one and the new triangle has neighbour at the bottom. We can see on the figure 3.8 the triangle enlarge the convex hull by 1 (triangle itself) and by at most $\frac{l_2-1}{4} + \frac{l_3-1}{4}$. While we have

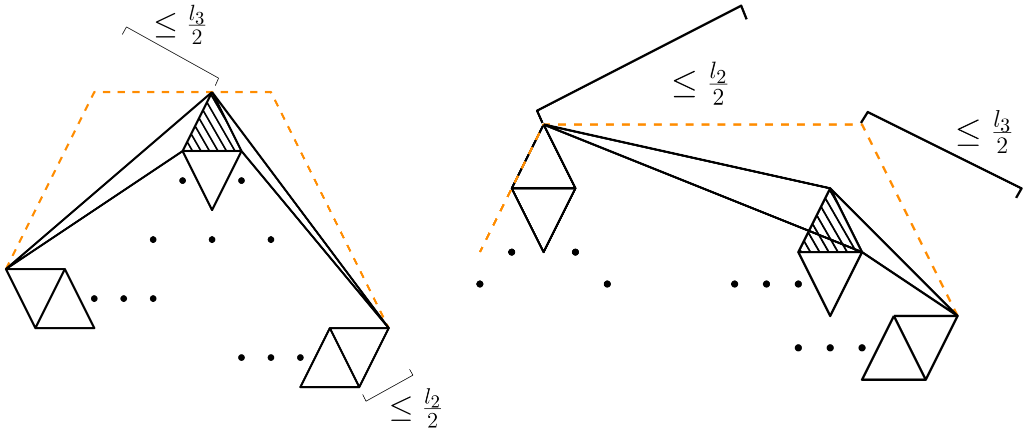
$$f(l_1 + 1, l_2, l_3, v_1, v_2, v_3) - f(l_1, l_2, l_3, v_1, v_2, v_3) = 1 + \frac{l_2 - 1}{4} + \frac{l_3 - 1}{4} + \frac{v_2}{4} + \frac{v_3}{4}$$

the statement holds and the first step of the induction is completed.

Let increase v_1 by one and the new triangle has neighbour at the bottom. We can see on the picture the new triangle enlarge the convex hull by 1 or less (triangle itself) and by at most $\frac{l_2-1}{4} + \frac{l_3-1}{4}$. While we have

$$f(l_1, l_2, l_3, v_1 + 1, v_2, v_3) - f(l_1, l_2, l_3, v_1, v_2, v_3) = 1 + \frac{l_2 - 1}{4} + \frac{l_3 - 1}{4} + \frac{v_2}{4} + \frac{v_3}{4}$$

the statement holds and the proof, that the bounding function bounds the area of the convex hull, is finished.



l_1 increase $\qquad$ v_1 increase

Figure 3.8: Induction step illustration

To finish the proof of the theorem we need to find the maximum of the bounding function and compute the estimate from it.

At first we observe $v_1 = v_2 = v_3 = 0$. It follows from the symmetry and $f(l_1+1, l_2, l_3, v_1, v_2, v_3) - f(l_1, l_2, l_3, v_1+1, v_2, v_3) = 0$. Without loss of generality we assume $l_1 \leq l_2 \leq l_3$. Now from

$$f(l_1+1, l_2-1, l_3, v_1, v_2, v_3) - f(l_1, l_2, l_3, v_1, v_2, v_3) = \frac{l_2 - l_1 - 1}{2} > 0$$

we observe $l_2 - l_1 \leq 1$ and similarly $l_3 - l_1 \leq 1$, $l_3 - l_2 \leq 1$. Because of $n = l_1 + l_2 + l_3 - 2$ we get

$$l_1 = \lfloor \frac{n+2}{3} \rfloor \qquad l_2 = \lfloor \frac{n+3}{3} \rfloor \qquad l_3 = \lfloor \frac{n+4}{3} \rfloor$$

Finally the substitution into the bounding function prove the theorem. $\square$

Enumeration of optimal polyiamonds is a bit more complicated than it was in the case of polyominoes, because we cannot omit all figures with non-zero v_i.

Recall the theorem from the introduction.

Theorem 13. *The polyiamonds with the maximum area of convex hull are three pointed starts and washtub (see figure 3.10).*

Proof. Firstly we observe polyiamond cannot contain subfigure on picture 3.9, because its area of convex hull is strictly less than area of a star or a washtub. We also know $|l_1 - l_2| \leq 1$, $|l_1 - l_3| \leq 1$ and $|l_2 - l_3| \leq 1$, therefore the only

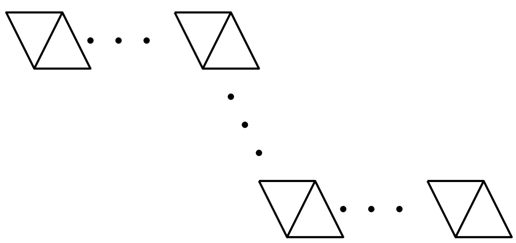


Figure 3.9: Forbidden subfigure

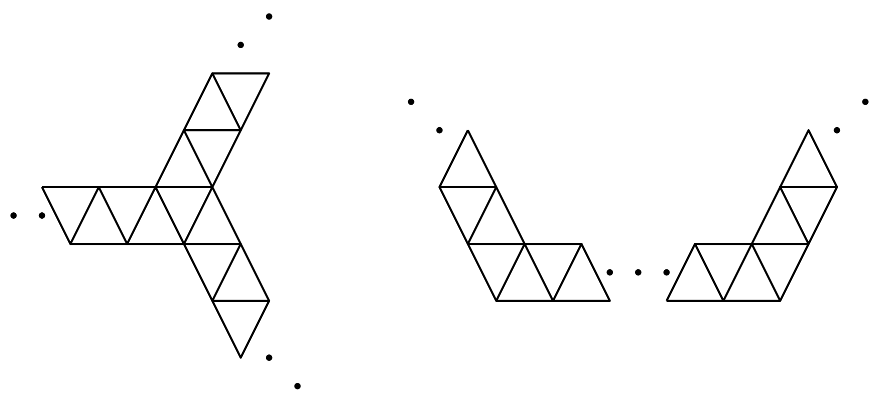


Figure 3.10: Optimal figures

possible shape is star and washtub for $n \geq 7$. For $n \leq 6$ we need to analyse the shape one by one and the result of our computation is on 3.11.

Analysis of the star gives us the result that the length of star arms should differ by at most one. For the washtub we observe similar result. So for each $n \geq 7$ there is one star and for odd n one washtub, too. $\square$

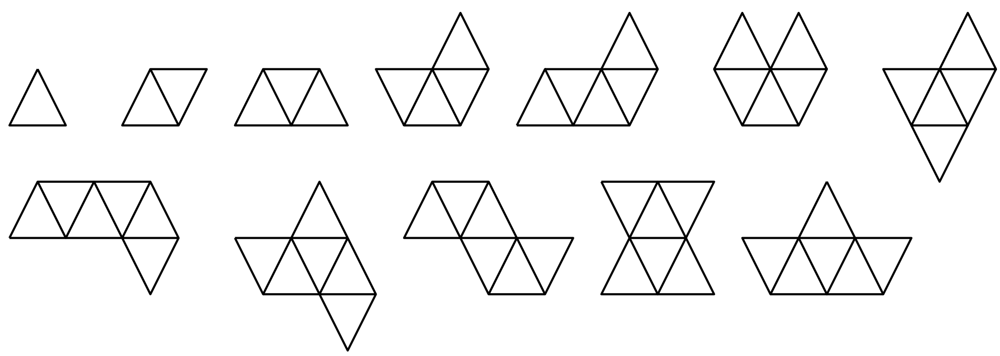


Figure 3.11: Optimal figures for $n \leq 6$

3.2.3 Polyhexes

Sasha Kurz conjectured in his paper [9] that the maximal area convex hull of a polyhex is

$$\frac{1}{6}\lfloor n^2 + \frac{14}{3}n + 1 \rfloor$$

and he thought it is possible to prove it in the same way as the similar theorem for squares. We tried to prove it, but it seems much more difficult, because for squares there are only two ways how a square can enlarge the area of the convex hull, but the hexagon shape is more complicated. We were unable to design the bounding function for the hexagons which would converge to the estimate for the area of the convex hull.

Anyway, our analysis of the shapes discovered interesting unexpected fact. For the polyhexes the optimal shape is probably only the washtub, because the star has strictly less value. For example for $n = 10$ the area of the tub is 24.5, but for star it is only 23.5 (see figure 3.12).

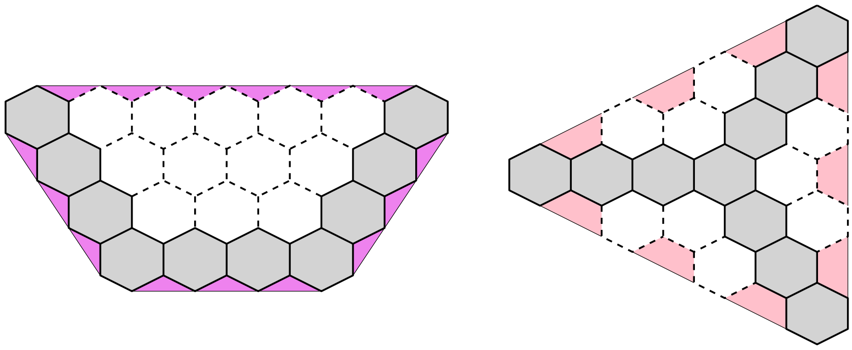


Figure 3.12: Difference between star and washtub for $n = 10$, violet areas equal to 1/6 and pink areas to 1/2

We derive the formulae for the area of the convex hull for the washtub and the star. The area of the convex hull of the washtub equals to the Kurz's estimate. The area of the convex hull of the star equals to $\frac{1}{6}(n^2 + 4n + 1)$ for $n \equiv 1 \bmod 3$ and $\frac{1}{6}(n^2 + 4n)$ for others. That proves the washtub has always (for $n \leq 4$, for smaller n the shapes are same) a larger convex hull than the star of the same size.

Conclusion

We gave an exhaustive report about the topic of the extremal animals. We considered the following characteristics: the bounding rectangle/hexagon, the perimeter and the convex hull. We minimised and maximised all off them and looked through the literature to find any remark about the chosen categories.

We solved two problems, the first one is about enumeration all polyominoes with the maximal bounding rectangle, the second one is about enumeration all polyiamonds with the maximal convex hull. The solution for the first problem can be probably applied on polyiamonds, too.

Our work can serve as a source for further research, because there are still several topics, which we did not solve. At least we found the reference literature connected to these problems and tried to give some observation, how one can try to solve the problem and which problems are probably more difficult than they seem to be.

We hope we can continue in the research and finish the algorithm for the maximum bounding hexagon for the polyiamonds and the maximum convex hull of the polyhexes. We really want to see the progress in the problem of the maximum perimeter, because it seems challenging, but probably using tools from graph drawing it can happen to be solved easily. The minimum bounding rectangle is even more difficult and so far we have no idea how it can be solved. For this parameter it would be nice to see the solution for at least some restricted classes of polyominoes.

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