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Faculty of Mathematics and Physics

DOCTORAL THESIS

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Isomorphic and isometric classification of spaces of continuous and Baire affine functions

Department of Mathematical Analysis

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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In Ostrava, 12th November 2013 Pavel Ludvík

Název práce: Izometrické a izomorfní klasifikace prostorů spojitých a baireovských afinních funkcí
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Abstrakt: Tato práce sestává z pěti odborných článků. V prvním dokazujeme, že za určitých podmínek plyne z existence isomorfismu mezi dvěma prostory spojitých afinních funkcí na kompaktních množinách existence homeomorfismu mezi množinami jejích extremálních bodů. Předmětem druhého je zkoumání přenosu deskriptivních vlastností prvků biduálů Banachových prostorů, které jsou chápány jako funkce na jednotkové duální kouli. Zabýváme se také vztahem mezi bairovskými a intrinsic bairovskými třídami L_1-preduálů. Ve třetím článku ztotožníme intrinsic bairovské třídy X s prostorem lichých, či homogenních bairovských funkcí na $\operatorname{ext} B_{X^*}$, kde X je separabilní reálný, či komplexní, L_1-preduál, jehož množina extremálních bodů duální jednotkové koule je typu F_σ. Poskytneme též příklad separabilní C^* algebry takové, že se druhá a druhá intrinsic bairovská třída jejího biduálu liší. Předmětem čtvrtého článku je zobecnění některých tvrzení článku předchozího pro reálné neseparabilní L_1-preduály. V pátém počítáme vzdálenost obecného zobrazení od třídy zobrazení první resolvable třídy pomocí kvantity frag a zkoumáme vlastnosti třídy zobrazení se spočetným oscilačním rankem.
Klíčová slova: spojité afinní funkce, borelovské a bairovské funkce, L_1 preduály, bairovské třídy a intrinsic bairovské třídy Banachových prostorů, oscilační rank

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Abstract: This thesis consists of five research papers. The first paper: We prove that under certain conditions, the existence of an isomorphism between spaces of continuous affine functions on the compact convex sets imposes homeomorphism between the sets of its extreme points. The second: We investigate a transfer of descriptive properties of elements of biduals of Banach spaces construed as functions on dual unit balls. We also prove results on the relation of Baire classes and intrinsic Baire classes of L_1-preduals. The third: We identify intrinsic Baire classes of X with the spaces of odd or homogeneous Baire functions on $\operatorname{ext} B_{X^*}$, provided X is a separable real or complex L_1-predual with the set of extreme points of its dual unit ball of type F_σ. We also provide an example of a separable C^*-algebra such that the second and second intrinsic Baire class of its bidual differ. The fourth: We generalize some of the above mentioned results for real non-separable L_1-preduals. The fifth: We compute the distance of a general mapping to the family of mappings of the first resolvable class via the quantity frag and we introduce and investigate a class of mappings of countable oscillation rank.
Keywords: continuous affine functions, Baire and Borel functions, L_1-preduals, Baire classes and intrinsic Baire classes of Banach spaces, oscillation rank

Contents

Introduction

The doctoral thesis consists of the five original research papers:

1. (with J. Spurný) Isomorphisms of spaces of continuous affine functions on compact convex sets with Lindelöf boundaries, *Proc. Amer. Math. Soc.*, 139(3):1099–1104, 2011.

2. (with J. Spurný) Descriptive properties of elements of biduals of Banach spaces, *Studia Math.*, 209(1):71–99, 2012.

3. (with J. Spurný) Baire classes of L_1-preduals and C^*-algebras, submitted.

4. (with J. Spurný) Baire classes of nonseparable L_1-preduals, submitted.

5. Distances to spaces of first H-class mappings, submitted.

Each paper corresponds to the one chapter of the thesis. The papers are presented in their original form (except for correcting a few typing errors). The first paper substantially comes out from my diploma thesis but the arguments are simplified and it is enriched by several counterexamples. The first and the second paper have been already published. The remaining papers have been submitted.

Let us provide a brief introduction to the topics investigated throughout the thesis. For the more extensive explanation of notions used in the Introduction we refer reader to the relevant subsequent chapter of the thesis.

In Chapter 1 we prove a generalization of the well-known Banach-Stone theorem which states that if X, Y are compact spaces, then X and Y are homeomorphic if and only if $C(X)$ (i.e. a Banach space of all continuous functions on a set X) is linearly isomorphic to $C(Y)$. The theorem can be carried into the context of compact convex sets and rephrased in a following form: *If X, Y are Bauer simplices then* $\operatorname{ext} X$ *and* $\operatorname{ext} Y$ *are homeomorphic if and only if* $\mathfrak{A}^c(X)$ *is linearly isomorphic to* $\mathfrak{A}^c(Y)$.

H. U. Hess, M. Cambern, C. H. Chu, H. B. Cohen and many others have managed to strengthen the result in several directions. The aim of Chapter 1 is to generalize and integrate two results of C. H. Chu, H. B. Cohen in a more general setting.

In Chapter 2 we study descriptive properties of strongly affine functions on compact convex spaces in general at first and then we apply the acquired results to the theory of Banach spaces.

To be more precise, if E is a (real or complex) Banach space, an element x^{**} of its bidual may be understood as a function on the dual space endowed with the weak* topology. Since the dual unit ball B_{E^*} is weak* compact, the set $\operatorname{ext} B_{E^*}$ of its extreme points is nonempty and its weak* closed convex hull is the whole unit ball. Hence one might expect that a behaviour of x^{**} on the set B_{E^*} might be in some sense predestined by the behaviour of x^{**} on $\operatorname{ext} B_{E^*}$. The aim of Chapter 2 is to support this expectation and provide several conditions on descriptive quality of $\operatorname{ext} B_{E^*}$ under which we get reasonable results. Our method lead to the generalization of some attainments of J. Saint Raymond and F. Jellett. In this chapter we also investigate the relation between Baire classes and intrinsic Baire classes of L_1-preduals which were introduced by S.A. Argyros, G. Godefroy and H.P. Rosenthal in [5, p. 1047]. We also present several examples indicating the natural limits of our positive results.

In Chapter 3 we continue with investigation of L_1-preduals but we restrict our attention to separable (real or complex) L_1-preduals. Let X be such an L_1-predual equipped with an additional descriptive property, namely, its dual unit ball B_{X^*} has the set $\operatorname{ext} B_{X^*}$ of its extreme points of type F_σ. Under this assumption we identify intrinsic Baire classes of X with the spaces of odd (in case X is a real Banach space) or homogeneous (in complex case) Baire functions on $\operatorname{ext} B_{X^*}$. In this chapter we generalize some results from the previous chapter and of J. Lindenstrauss and D. E. Wulbert.

Further, we answer a question of S. A. Argyros, G. Godefroy and H. P. Rosenthal by showing that there exists a separable C^*-algebra X for which the second intrinsic Baire class of X^{**} does not coincide with the second Baire class of X^{**}.

Chapter 4 is intended to be a sequel of Chapter 3 where we try to abandon an assumption of separability. We succeed for the class of real (not necessarily separable) L_1-preduals. We show that the intrinsic Baire classes of X can be identified with the spaces of odd Baire functions on the set $\operatorname{ext} B_{X^*}$ of the extreme points of the dual unit ball B_{X^*} if $\operatorname{ext} B_{X^*}$ satisfies some topological assumptions. Namely, if $\operatorname{ext} B_{X^*}$ is a Lindelöf H-set. Questioning whether analogous results for complex non-separable L_1-preduals hold true is a subject of our current research.

Chapter 5 has two leading motifs. At first, we study the mappings of the first resolvable class defined by G. Koumoullis in [47]. These mappings were intended to generalize functions of the first Baire class. We compute the distance of a general mapping to the family of mappings of the first resolvable class via the quantity frag and thus in part generalize a several results of C. Angosto, B. Cascales, and I. Namioka; B. Cascales, W. Marciszewski, and M. Raja; and J. Spurný.

In the second part of Chapter 5 we introduce a class of mappings with countable oscillation rank and relate its basic properties to the certain well known classes of mappings. This rank has been in a less general context considered by S. A. Argyros, R. Haydon and many others.

1. Isomorphisms of spaces of continuous affine functions on compact convex sets with Lindelöf boundaries

(joint work with Jiří Spurný)

1.1 Introduction

If X is a compact convex set in a real locally convex space, let $\mathfrak{A}^c(X)$ stand for the space of all continuous affine functions, $\mathfrak{A}^b(X)$ for the space of all bounded affine functions on X, and $\operatorname{ext} X$ for the set of extreme points.

We refer the reader to [15, pp. 72, 73, 75] for notions of the theory of compact convex sets. We just mention that X can be embedded to $(\mathfrak{A}^c(X))^*$ via the evaluation mapping $\phi\colon X \to (\mathfrak{A}^c(X))^*$ defined as $\phi(x)(f) = f(x)$, $f \in \mathfrak{A}^c(X)$, $x \in X$. The dual unit ball $B_{(\mathfrak{A}^c(X))^*}$ equals the convex hull $\operatorname{co}(X \cup -X)$, and $(\mathfrak{A}^c(X))^*$ coincides with $\operatorname{span} X$, the linear span of X. Further, any function $f \in \mathfrak{A}^b(X)$ has a unique extension to $\operatorname{span} X$, and this provides an identification of $(\mathfrak{A}^c(X))^{**}$ with $\mathfrak{A}^b(X)$.

For a set $F \subset X$, the complementary set F^{cs} is defined as the union of all faces of X disjoint from F. A face F of X is said to be a split face if its complementary set F^{cs} is convex (and hence a face; see [1, p. 132]) and every point in $X \setminus (F \cup F^{\mathrm{cs}})$ can be uniquely represented as a convex combination of a point in F and a point in F^{cs}.

We call $x \in \operatorname{ext} X$ a weak peak point if given $\varepsilon \in (0,1)$ and an open neighbourhood U of x, there exists $h \in \mathfrak{A}^c(X)$ such that $\|h\| \le 1$, $h(x) > 1-\varepsilon$ and $|h| < \varepsilon$ on $\operatorname{ext} X \setminus U$.

Let us also recall that any weak peak point of a compact convex set X is a split face and the converse holds if $\operatorname{ext} X$ is closed; see [15, Proposition 1].

The following results are proved in [15, Theorems 7 and 12] by C. H. Chu and H. B. Cohen:

Let X and Y be compact convex sets and let $T\colon \mathfrak{A}^c(X) \to \mathfrak{A}^c(Y)$ be an isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$. If

- *X and Y are metrizable and each point of $\operatorname{ext} X$ and $\operatorname{ext} Y$ is a weak peak point, or*
- *the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are closed and each extreme point of X and Y is a split face,*

then the sets $\operatorname{ext} X$ and $\operatorname{ext} Y$ are homeomorphic.

The aim of our paper is to show that the method of the proof of [15, Theorem 7] is applicable in a more general setting that covers both results mentioned above.

Theorem 1.1.1. *Let X, Y be compact convex sets such that every extreme point of X and Y is a weak peak point and both $\operatorname{ext} X$ and $\operatorname{ext} Y$ are Lindelöf spaces.*

Let $T : \mathfrak{A}^c(X) \to \mathfrak{A}^c(Y)$ be an isomorphism with $\|T\| \cdot \|T^{-1}\| < 2$. Then $\operatorname{ext} X$ *is homeomorphic to* $\operatorname{ext} Y$.

As in [15, Corollaries 13 and 14], this yields a corollary for function algebras: *Let $\mathcal{A}$ and $\mathcal{B}$ be function algebras with Lindelöf Choquet boundaries, and let $T\colon \operatorname{Re}\mathcal{A} \to \operatorname{Re}\mathcal{B}$ be an isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$. Then the Choquet boundaries of $\mathcal{A}$ and $\mathcal{B}$ are homeomorphic.*

We recall that the construction from [9, Section VII] (see also [1, Proposition I.4.15] or [6, Theorem 3.2.4]) yields an example of a non-metrizable simplex X such that $\operatorname{ext} X$ is a Lindelöf non-closed subset of X and every extreme point of X is a weak peak point. To see this, let $B \subset [0,1]$ be a Bernstein set (see [66, Theorem 5.3]) and let

$$K = ([0,1] \times \{0\}) \cup \bigcup_{x \in B} (\{x\} \times [0,1])$$

be endowed with the "porcupine" topology (see [9, Section VII]). Precisely, if $x \in B$ and $t \in (0,1]$, then a basis of neighborhoods of (x,t) consists of sets of the form $\{x\} \times U$, where $U \subset [0,1]$ is a neighborhood of t. If $x \in [0,1]$, then a basis of neighborhoods of $(x,0)$ consists of sets of the form

$$(U \times \{0\}) \cup \big((U \times [0,1]) \setminus \bigcup_{i=1}^{n} (\{x_i\} \times F_i)\big),$$

where $n \in \mathbb{N}$, $U \subset [0,1]$ is a neighborhood of x, $x_1, \dots, x_n$ are points in $B \cap U$ and $F_1, \dots, F_n$ are compact subsets of $(0,1]$.

If λ stands for Lebesgue measure on $[0,1]$, let

$$H = \{f \in \mathcal{C}(K)\colon f(x,0) = \int_{[0,1]} f(x,t)\, d\lambda(t), x \in B\}$$

and

$$X = \{s \in H^*\colon s \geq 0, s(1) = 1\}.$$

Then X endowed with the weak* topology is a simplex and $\operatorname{ext} X$ is homeomorphic to $(([0,1] \setminus B) \times \{0\}) \cup (K \setminus ([0,1] \times \{0\}))$. It is easy to see that $\operatorname{ext} X$ is a Lindelöf non-closed set and every extreme point of X is a weak peak point.

Example 1 on [15, p. 83] shows that Theorem 1.1.1 need not hold even for compact convex sets in finite dimensional spaces if we omit the assumption that extreme points are weak peak points. An example due to H. U. Hess (see [33]) shows that *for every $\varepsilon > 0$ there exist metrizable simplices X, Y and an isomorphism $T\colon \mathfrak{A}^c(X) \to \mathfrak{A}^c(Y)$ such that $\|T\| \cdot \|T^{-1}\| < 1 + \varepsilon$ and* $\operatorname{ext} X$ *is not homeomorphic to* $\operatorname{ext} Y$. Nevertheless, it is not clear whether Theorem 1.1.1 remains valid if we omit the topological assumption on the sets of extreme points.

Question 1.1.2. Let X, Y be compact convex sets such that every extreme point of X and Y is a weak peak point and let $T : \mathfrak{A}^c(X) \to \mathfrak{A}^c(Y)$ be an isomorphism with $\|T\| \cdot \|T^{-1}\| < 2$. Is it true that $\operatorname{ext} X$ is homeomorphic to $\operatorname{ext} Y$?

We need to recall several notions not explained in [15]. If X is a compact (Hausdorff) space, we write $\mathcal{C}(X)$ for the space of all continuous functions on

X and $\mathcal{M}^1(X)$ for the space of all probability Radon measures on X. (By a Radon measure we mean a complete measure that is inner regular with respect to compact sets and is defined on a σ-algebra including all Borel subsets of X. We refer the reader to [26, Section 416] for more information on Radon measures.) We always consider $\mathcal{M}^1(X)$ to be endowed with weak* topology. We say that a function $f : X \to \mathbb{R}$ is *universally measurable* if f is μ-measurable for every $\mu \in \mathcal{M}^1(X)$.

If X is a compact convex subset of a real locally convex space, any $\mu \in \mathcal{M}^1(X)$ has its unique barycenter $r(\mu) \in X$, i.e., the point $x \in X$ satisfying $f(x) = \mu(f)$ for any $f \in \mathfrak{A}^c(X)$. We sometimes say that μ *represents* x. A function $f : X \to \mathbb{R}$ is *strongly affine* (or *satisfies the barycentric formula*), if f is universally measurable, $\mu(f)$ exists and $f(r(\mu)) = \mu(f)$ for any $\mu \in \mathcal{M}^1(X)$. We write $\mathfrak{A}_{\mathrm{bf}}(X)$ for the space of all strongly affine functions on X and recall that it is easy to see that any strongly affine function is bounded (see the proof of [48, Satz 2.1(c)]). We also recall that any semicontinuous affine function on X is strongly affine; see [6, Theorem 1.6.1(ix)].

1.2 Proof of Theorem 1.1.1

The proof of the main theorem follows the idea of the proof of [15, Theorems 7 and 12]. Hence we recall the main steps of their proof and point out our modifications. We start the proof with a minimum principle which is crucial for us because then [78, Lemma 2.4] is applicable for functions $T^{**}f$, $f \in \mathfrak{A}_{\mathrm{bf}}(X)$.

Lemma 1.2.1. *Let X be a compact convex set such that $\operatorname{ext} X$ is Lindelöf. If $f \in \mathfrak{A}_{\mathrm{bf}}(X)$ satisfies $|f(x)| \le c$ for all $x \in \operatorname{ext} X$, then $|f(x)| \le c$ for all $x \in X$.*

Proof. Let $x \in X$ be given. We find a maximal measure $\mu \in \mathcal{M}^1(X)$ representing the point x (see [6, Theorem 1.6.4]) and define

$$A = \{y \in X \colon |f(y)| \le c\}.$$

Then A is a μ-measurable set and we claim that $\mu(A) = 1$.

Indeed, let $K \subset X$ be an arbitrary compact set disjoint from A. Since $A \supset \operatorname{ext} X$, for any $y \in \operatorname{ext} X$ we can find its closed neighborhood not intersecting K. The set $\operatorname{ext} X$ is Lindelöf, and thus we can select countably many closed sets $F_n \subset X$, $n \in \mathbb{N}$, such that

$$\operatorname{ext} X \subset \bigcup_{n=1}^{\infty} F_n \subset X \setminus K.$$

By [14, Theorem 27.11], $\mu(\bigcup_{n=1}^{\infty} F_n) = 1$, and hence $\mu(K) = 0$. By the regularity of μ, $\mu(X \setminus A) = 0$, and hence

$$|f(x)| = \left| \int_X f \, d\mu \right| \le \int_A |f| \, d\mu \le c.$$

This concludes the proof. $\square$

Proof of Theorem 1.1.1. Let $T : \mathfrak{A}^c(X) \to \mathfrak{A}^c(Y)$ be an isomorphism satisfying $\|T\| \cdot \|T^{-1}\| < 2$. We assume that there exists $c, c' \in \mathbb{R}$ such that $1 < c < c' < 2$ and $\|T\| < 2$ and $\|Tf\| \geq c'\|f\|$ for all $f \in \mathfrak{A}^c(X)$ (otherwise we would find $1 < c < c' < 2$ such that $\|T\| \cdot \|T^{-1}\| < \frac{2}{c'} < 2$ and consider the mapping $c'\|T^{-1}\|T$; see [15, p. 76]).

Claim 1. *For any $f \in \mathfrak{A}^b(X)$ and $g \in \mathfrak{A}^b(Y)$ non-zero, $\|T^{**}f\| > c\|f\|$ and $\|(T^{-1})^{**}g\| > \frac{1}{2}\|g\|$.*

Proof of Claim 1. The first inequality follows from

$$\|f\| = \|(T^{-1})^{**}T^{**}f\| \leq (c')^{-1}\|T^{**}f\| < c^{-1}\|T^{**}f\|,$$

the second one is analogous. □

If $x \in \operatorname{ext} X$, we recall that $(\mathfrak{A}^c(X))^* = \operatorname{span}\{x\} \oplus_{\ell^1} \operatorname{span}\{x\}^c$ because $\{x\}$ is a split face; see [15, p. 72]. Hence, given $y \in Y$, following [15, p. 76] we can write

$$T^*y = \lambda x + \mu \quad \text{for some } \lambda \in \mathbb{R} \text{ and } \mu \in \operatorname{span}\{x\}^{\mathrm{cs}}. \tag{1.1}$$

It is proved in [15, p. 77] that $\|\mu\| < 2 - c$ whenever $y \in Y$ satisfies $|\lambda| > c$.

We recall the construction of mappings $\rho\colon Y \to \operatorname{ext} X$ and $\tau\colon X \to \operatorname{ext} Y$. Given $x \in \operatorname{ext} X$, we denote by $\chi_{\{x\}}$ the characteristic function of the set $\{x\}$. The the upper envelope function $h_x = \hat{\chi}_{\{x\}}$, defined as

$$\hat{\chi}_{\{x\}}(z) = \inf\{h(z) : h \in \mathfrak{A}^c(X), h > \chi_{\{x\}}\} \quad \text{for } z \in X,$$

is upper semicontinuous and affine (see [15, p. 73]), and thus strongly affine (see [6, Theorem 1.6.1(ix)]). By [15, p. 77], for each $y \in \operatorname{ext} Y$ there is at most one point $x \in \operatorname{ext} X$ such that $|T^{**}h_x(y)| > c$. Let

$$Y' = \{y \in \operatorname{ext} Y : \text{there exists } x \in \operatorname{ext} X \text{ with } |T^{**}h_x(y)| > c\},$$

and let $\rho\colon Y' \to X$ be defined by the property that $\rho(y)$ equals the unique point $x \in \operatorname{ext} X$ satisfying $|T^{**}h_x(y)| > c$.

Analogously, if

$$X' = \{x \in \operatorname{ext} X : \text{there exists } y \in \operatorname{ext} Y \text{ with } |(T^{-1})^{**}h_y(x)| > \frac{1}{2}\},$$

then $\tau\colon X' \to \operatorname{ext} Y$ can be defined by the requirement that $\tau(x)$ is the unique $y \in \operatorname{ext} Y$ satisfying $|(T^{-1})^{**}h_y(x)| > \frac{1}{2}$.

Claim 2. *For any $x \in \operatorname{ext} X$, $T^{**}h_x \in \mathfrak{A}_{\mathrm{bf}}(Y)$.*

Proof of Claim 2. Since $T\colon \mathfrak{A}^c(X) \to \mathfrak{A}^c(Y)$, we have $T^*\colon \operatorname{span} Y \to \operatorname{span} X$. If $f \in \mathfrak{A}^b(X)$ and $\widetilde{f}$ is the linear extension of f to $\operatorname{span} X$, then $T^{**}f = \widetilde{f} \circ T^*$. Since $\|T\| < 2$,

$$T^*Y \subset 2B_{(\mathfrak{A}^c(X))^*} = \operatorname{co}(2X \cup -2X).$$

The sets $2X$ and $-2X$ are affinely homeomorphic to X, and hence $\widetilde{f}$ is strongly affine on both of them. By [78, Lemma 2.4(b)],

$$\widetilde{f} \in \mathfrak{A}_{\mathrm{bf}}(2B_{(\mathfrak{A}^c(X))^*}) = \mathfrak{A}_{\mathrm{bf}}(\operatorname{co}(2X \cup -2X)).$$

Since Y is affinely homeomorphic to T^*Y and $T^{**}f = \widetilde{f} \circ T^*$, we obtain that $T^{**}f \in \mathfrak{A}_{\mathrm{bf}}(Y)$. □

Claim 3. *The mappings $\rho\colon Y' \to \operatorname{ext} X$ and $\tau\colon X' \to \operatorname{ext} Y$ are surjective.*

Proof of Claim 3. Let $x \in \operatorname{ext} X$ be given and assume that $|T^{**}h_x(y)| \le c$ for all $y \in \operatorname{ext} Y$. By Claims 1, 2 and Lemma 1.2.1, $|T^{**}h_x| \le c$ on Y. Then

$$c \ge \|T^{**}h_x\| > c\|h_x\| = c$$

gives a contradiction. Hence ρ is surjective.

Analogously, using the second part of Claim 1 we obtain that τ is surjective. $\square$

The following claim is essentially Lemma 6 of [15]. However, we recall its proof since it uses Lemma 1.2.1.

Claim 4. *We have $X' = \operatorname{ext} X$ and $Y' = \operatorname{ext} Y$ and, for any $x \in \operatorname{ext} X$ and $y \in \operatorname{ext} Y$, $\rho(\tau(x)) = x$ and $\tau(\rho(y)) = y$.*

Proof of Claim 4. We show that, for any $y \in Y'$,

$$|(T^{-1})^{**}h_y(\rho(y))| > \frac{1}{2}. \tag{1.2}$$

Assuming $|(T^{-1})^{**}h_y(\rho(y))| \le \frac{1}{2}$, Claim 2 and Lemma 1.2.1 yield

$$d = \sup_{x \in \operatorname{ext} X} |(T^{-1})^{**}h_y(x)| = \sup_{x \in X} |(T^{-1})^{**}h_y(x)| = \|(T^{-1})^{**}h_y\|.$$

By Lemma 1.2.1 and Claim 1, $\frac{1}{2} < d$. Since $c > 1$, we have $d > \frac{d}{c}$. Let $x' \in \operatorname{ext} X$ be chosen such that

$$|(T^{-1})^{**}h_y(x')| > \max\{\frac{d}{c}, \frac{1}{2}\}.$$

By the assumption, $|(T^{-1})^{**}h_y(\rho(y))| \le \frac{1}{2}$, and thus $\rho(y) \ne x'$.

By Claim 3 we can select $y' \in Y'$ with $\rho(y') = x'$. Then $y' \in \{y\}^{\mathrm{cs}}$, and thus $h_y(y') = 0$. If $T^*y' = \lambda' x' + \mu'$, $\lambda' \in \mathbb{R}$ and $\mu' \in \operatorname{span}\{x'\}^{\mathrm{cs}}$ (see (1.1)), then

$$0 = h_y(y') = (T^{-1})^{**}h_y(T^*y') = (T^{-1})^{**}h_y(\lambda' x') + (T^{-1})^{**}h_y(\mu'). \tag{1.3}$$

Since $\lambda' = T^{**}h_{x'}(y')$, it follows from the definition of ρ that $|\lambda'| > c$.

Using this, (1.3) and (1.1) along with the subsequent remark, we obtain

$$\begin{aligned} d < |\lambda'|\frac{d}{c} &< |\lambda'||(T^{-1})^{**}h_y(x')| \\ &= |(T^{-1})^{**}h_y(\lambda' x')| \\ &= |(T^{-1})^{**}h_y(\mu')| \\ &\le d\|\mu'\| < d(2-c) < d. \end{aligned}$$

This contradiction yields the validity of (1.2).

Now, let $x \in \operatorname{ext} X$ be given. We find $y \in Y'$ with $\rho(y) = x$. It follows from (1.2) that $x \in X'$ and $\tau(x) = y$. Hence $X' = \operatorname{ext} X$ and $\tau(\rho(y)) = y$ for all $y \in Y'$.

If $y \in \operatorname{ext} Y$ is given, let $x \in \operatorname{ext} X$ be such that $\tau(x) = y$. If $y' \in Y'$ satisfies $\rho(y') = x$, from the previous argument we obtain

$$y = \tau(x) = \tau(\rho(y')) = y'.$$

Hence $Y' = \operatorname{ext} Y$ and it easily follows that $\rho(\tau(x)) = x$ for any $x \in \operatorname{ext} X$. $\square$

By the proof of Theorem 7 on p. 78 in [15], the mappings ρ and τ are continuous (we point out that this part of the argument is valid for arbitrary compact convex sets as mentioned in [15, p. 83]). This concludes the proof. $\square$

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2. Descriptive properties of elements of biduals of Banach spaces

(joint work with Jiří Spurný)

2.1 Introduction and main results

If E is a (real or complex) Banach space, an element x^{**} of its bidual may posses interesting descriptive properties if x^{**} is understood as a function on the dual space endowed with the weak* topology. Since the dual unit ball B_{E^*} is weak* compact, the set $\operatorname{ext} B_{E^*}$ of its extreme points is nonempty and its weak* closed convex hull is the whole unit ball. Hence one might expect that a behaviour of x^{**} on the set $\operatorname{ext} B_{E^*}$ in some sense determines the behaviour of x^{**} on B_{E^*}. The aim of our paper is to substantiate this general idea by presenting several results on transferring descriptive properties of $x^{**}|_{\operatorname{ext} B_{E^*}}$ to $x^{**}|_{B_{E^*}}$. To formulate our results precisely, we need to recall several notions.

Since the main results are mostly formulated for Banach spaces over real or complex field, we need to work with vector spaces over both real and complex numbers. So all the notions are considered, if not stated otherwise, with respect to the field of complex numbers. All topological spaces are considered to be Tychonoff (i.e., completely regular, see [20, p. 39]), in particular they are Hausdorff.

If K is a compact topological space, a *positive Radon measure* on K is a finite complete measure with values in $[0, \infty)$ defined at least on the σ-algebra of all Borel sets that is inner regular with respect to compact sets (see [26, Definition 411H]). A signed or complex measure μ on X is a Radon measure if its total variation $|\mu|$ is Radon. We often write $\mu(f)$ instead of $\int f \, d\mu$. We denote as $\mathcal{M}(K)$, $\mathcal{M}^+(K)$ and $\mathcal{M}^1(K)$ the set of all Radon measures, positive Radon measures and probability Radon measures, respectively. Using the Riesz representation theorem we view $\mathcal{M}(K)$ as the dual space to the space $\mathcal{C}(K)$ of all continuous functions on K. Unless stated otherwise, we consider the space $\mathcal{M}(K)$ endowed with the weak* topology. A function $f : K \to \mathbb{C}$ is *universally measurable* if f is μ-measurable for every $\mu \in \mathcal{M}(K)$. If $\mathcal{F}$ is a family of functions, we write $\mathcal{F}^b$ for the set of all bounded elements of $\mathcal{F}$.

Let X be a compact convex subset of a locally convex space. Then any measure $\mu \in \mathcal{M}^1(X)$ has its unique *barycenter* $x \in X$, i.e., the point $x \in X$ satisfying $\mu(f) = f(x)$ for each $f \in \mathfrak{A}^c(X)$ (here $\mathfrak{A}^c(X)$ stands for the space of all continuous affine functions on X). We write $\mathcal{M}_x(X)$ for the set of all probability measures with x as the barycenter. The mapping $r : \mathcal{M}^1(X) \to X$ assigning to every probability measure on X its barycenter is a continuous affine surjection, see [1, Proposition I.2.1] or [58, Proposition 2.38]. A function $f : X \to \mathbb{C}$ is called *strongly affine* (or a function satisfying the *barycentric formula*) if f is universally measurable and $\mu(f) = f(r(\mu))$ for every $\mu \in \mathcal{M}^1(X)$. It is easy to deduce that any strongly affine function is bounded (see e.g. [58, Lemma 4.5]).

If E is Banach space, B_{E^*} with the weak* topology is a compact convex set.

We call an element $f \in E^{**}$ *strongly affine* if its restriction to B_{E^*} is a strongly affine function. We also mention that a continuous affine function f on B_{E^*}, which satisfies $f(0) = 0$ and $f(ix^*) = if(x^*)$ for $x^* \in B_{E^*}$, is in fact an element of E, i.e., there exists $x \in E$ with $f(x^*) = x^*(x)$ for $x^* \in B_{E^*}$.

Further we need to recall descriptive classes of functions in topological spaces. We follow the notation of [79]. If X is a Tychonoff topological space, a *zero set* in X is an inverse image of a closed set in $\mathbb{R}$ under a continuous function $f : X \to \mathbb{R}$. The complement of a zero set is a *cozero set*. A countable union of closed sets is called an F_σ *set*, the complement of an F_σ set is a G_δ *set*. If X is normal, it follows from Tietze's theorem that a closed set is a zero set if and only if it is also a G_δ set. We recall that *Borel sets* are members of the σ–algebra generated by the family of all open subset of X and *Baire sets* are members of the σ–algebra generated by the family of all cozero sets in X. We write $\mathrm{Bos}(X)$ and $\mathrm{Bas}(X)$ for the algebras generated by open or cozero sets in X, respectively.

A set $A \subset X$ is *resolvable* (or an *H-set*) if for any nonempty $B \subset X$ (equivalently, for any nonempty closed $B \subset X$) there exists a relatively open $U \subset B$ such that either $U \subset A$ or $U \cap A = \emptyset$. It is easy to see that the family $\mathrm{Hs}(X)$ of all resolvable sets is an algebra, see e.g. [49, § 12, VI]. Let $\Sigma_2(\mathrm{Bas}(X))$, $\Sigma_2(\mathrm{Bos}(X))$ and $\Sigma_2(\mathrm{Hs}(X))$ denote countable unions of sets from the respective algebras.

Let $\mathrm{Baf}_1(X)$ denote the family of all $\Sigma_2(\mathrm{Bas}(X))$-measurable function on X, i.e., the functions $f : X \to \mathbb{C}$ satisfying $f^{-1}(U) \in \Sigma_2(\mathrm{Bas}(X))$ for all $U \subset \mathbb{R}$ open. Analogously we define families $\mathrm{Bof}_1(X)$ and $\mathrm{Hf}_1(X)$.

Now we use pointwise limits to create higher hierarchies of functions. More precisely, if Φ is a family of functions on X, we define $\Phi_0 = \Phi$ and, for each countable ordinal α, Φ_α consists of all pointwise limits of sequences from $\bigcup_{\beta<\alpha} \Phi_\beta$. Starting the procedure with $\mathrm{Baf}_1(X)$ and creating higher families $\mathrm{Baf}_\alpha(X)$ as pointwise limits of sequences contained in $\bigcup_{1\le\beta<\alpha} \mathrm{Baf}_\beta(X)$, we obtain the hierarchy of *Baire measurable* functions. Analogously we define, for $\alpha \in [1, \omega_1)$, families $\mathrm{Bof}_\alpha(X)$ and $\mathrm{Hf}_\alpha(X)$ of *Borel measurable* functions and *resolvably measurable* functions. (Theorem 5.2 in [79] explains the term "measurability" in these definitions.)

If we start the inductive process with the family $\Phi_0 = \Phi = \mathcal{C}(X)$, we obtain the families $\mathcal{C}_\alpha(X)$ of Baire-α functions on X, $\alpha < \omega_1$. Then the union $\bigcup_{\alpha<\omega_1} \mathcal{C}_\alpha(X)$ is the family of all *Baire* functions. It is easy to see that $\mathcal{C}_1(X) = \mathrm{Baf}_1(X)$ (see Proposition 2.2.3) and thus $\mathcal{C}_\alpha(X) = \mathrm{Baf}_\alpha(X)$ for any $\alpha \in [1, \omega_1)$.

Now we can state our first result concerning a preservation of descriptive properties. For separable Banach spaces and Baire functions, the results can be obtained from [73, Corollaire 8].

Theorem 2.1.1. *Let E be a (real or complex) Banach space and $f \in E^{**}$ be strongly affine. Then,*

- *for $\alpha \in [1, \omega_1)$, $f|_{\overline{\mathrm{ext}\, B_{E^*}}} \in \mathrm{Hf}_\alpha(\overline{\mathrm{ext}\, B_{E^*}})$ if and only if $f \in \mathrm{Hf}_\alpha(B_{E^*})$,*
- *for $\alpha \in [1, \omega_1)$, $f|_{\overline{\mathrm{ext}\, B_{E^*}}} \in \mathrm{Bof}_\alpha(\overline{\mathrm{ext}\, B_{E^*}})$ if and only if $f \in \mathrm{Bof}_\alpha(B_{E^*})$,*
- *for $\alpha \in [0, \omega_1)$, $f|_{\overline{\mathrm{ext}\, B_{E^*}}} \in \mathcal{C}_\alpha(\overline{\mathrm{ext}\, B_{E^*}})$ if and only if $f \in \mathcal{C}_\alpha(B_{E^*})$.*

We remark that the assumption of strong affinity is necessary because otherwise the transfer of properties fails spectacularly. An example witnessing this phenomenon can be constructed as follows. Consider the real Banach space $E = \mathcal{C}([0,1])$ and the function $f : \mathcal{M}([0,1]) \to \mathbb{R}$ assigning to each $\mu \in \mathcal{M}([0,1])$ its continuous part evaluated at function 1. Then f is a weak* discontinuous element of E^{**} contained in $\mathcal{C}_2(B_{\mathcal{M}([0,1])})$ that vanishes on $\operatorname{ext} B_{\mathcal{M}([0,1])}$. (Details can be found e.g. in [67, Chapter 14], [5, p. 1048] or [58, Proposition 2.63].)

The next theorem in a way extend results of F. Jellett in [40, Theorem].

Theorem 2.1.2. *Let E be a (real or complex) Banach space such that $\operatorname{ext} B_{E^*}$ is a Lindelöf set. Let $f \in E^{**}$ be a strongly affine element satisfying $f|_{\operatorname{ext} B_{E^*}} \in \mathcal{C}_\alpha(\operatorname{ext} B_{E^*})$ for some $\alpha \in [0, \omega_1)$. Then*

$$f \in \begin{cases} \mathcal{C}_{\alpha+1}(B_{E^*}), & \alpha \in [0, \omega_0), \\ \mathcal{C}_\alpha(B_{E^*}), & \alpha \in [\omega_0, \omega_1). \end{cases}$$

By assuming a stronger assumption on $\operatorname{ext} B_{E^*}$ we may ensure the preservation of all classes, including the finite ones.

Theorem 2.1.3. *Let E be a (real or complex) Banach space such that $\operatorname{ext} B_{E^*}$ is a resolvable Lindelöf set. Let $f \in E^{**}$ be a strongly affine element satisfying $f|_{\operatorname{ext} B_{E^*}} \in \mathcal{C}_\alpha(\operatorname{ext} B_{E^*})$ for some $\alpha \in [1, \omega_1)$. Then $f \in \mathcal{C}_\alpha(B_{E^*})$.*

We remark that the shift of classes may really occur without the assumption of resolvability as it is witnessed by Example 2.8.1. One may also ask whether results analogous to the ones of Theorems 2.1.2 and 2.1.3 remain true for functions from classes Bof_α and Hf_α. Examples 2.8.2 and 2.8.3 show that this is not the case.

Further we observe that, for a separable space E, the topological condition imposed on $\operatorname{ext} B_{E^*}$ in Theorem 2.1.3 is equivalent with the requirement that $\operatorname{ext} B_{E^*}$ is a set of type F_σ. This can be seen from the following two facts: a subset of a compact metrizable space is a resolvable set if and only if it is both of type F_σ and G_δ (use [49, § 30, X] and the Baire category theorem); the set of extreme points in a metrizable compact convex set is of type G_δ (see [1, Corollary I.4.4] or [58, Proposition 3.43]).

We also point out that the topological assumption in Theorem 2.1.3 is satisfied provided $\operatorname{ext} B_{E^*}$ is an F_σ set. To see this, we first notice that $\operatorname{ext} B_{E^*}$ is then a Lindelöf space. Second, we need to check that $\operatorname{ext} B_{E^*}$ is a resolvable set in B_{E^*}. To this end, assume that $F \subset B_{E^*}$ is a nonempty closed set such that both $F \cap \operatorname{ext} B_{E^*}$ and $F \setminus \operatorname{ext} B_{E^*}$ are dense in F. By [84, Théorème 2], we can write

$$\operatorname{ext} B_{E^*} = \bigcap_{n=1}^{\infty} (H_n \cup V_n),$$

where $H_n \subset B_{E^*}$ is closed and $V_n \subset B_{E^*}$ is open, $n \in \mathbb{N}$. Thus both $F \setminus \operatorname{ext} B_{E^*}$ and $F \cap \operatorname{ext} B_{E^*}$ are comeager disjoint sets in F, in contradiction with the Baire category theorem. Hence $\operatorname{ext} B_{E^*}$ is a resolvable set.

For a particular class of Banach spaces, namely the L_1-preduals, one can obtain some information on the affine class of a function from its descriptive class (we recall that a Banach space E is an L_1-*predual* if E^* is isometric to some space $L_1(\mu)$; see [41, p. 59], [51, Chapter 7] or [31, Section II.5]). Affine classes $\mathfrak{A}_\alpha(X)$, $\alpha < \omega_1$, of functions on a compact convex set X are created inductively from

$\mathfrak{A}_0(X) = \mathfrak{A}^c(X)$ (see [11] or [58, Definition 5.37]). We also remark that a pointwise convergent sequence of affine functions on X is uniformly bounded which easily follows from the uniform boundedness principle (see e.g. [58, Lemma 5.36]), and thus any function in $\bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(X)$ is strongly affine. If $X = B_{E^*}$ is the dual unit ball of a Banach space E, the affine classes are termed *intrinsic Baire classes* of E in [5, p. 1047] whereas strongly affine Baire functions on X creates hierarchy of *Baire classes* of E. Theorem 2.1.4 relates these classes for real L_1-preduals.

We recall that, given a compact convex set X in a real locally convex space, the real Banach space $\mathfrak{A}^c(X)$ is an L_1-predual if and only if X is a *simplex*, i.e., if for any $x \in X$ there exists a unique maximal measure $\delta_x \in \mathcal{M}^1(X)$ with $r(\delta_x) = x$ (see [25, Theorem 3.2 and Proposition 3.23]).

(A measure $\mu \in \mathcal{M}^+(X)$ is *maximal* if μ is maximal with respect to the Choquet ordering, i.e., μ fulfils the following condition: if a measure $\nu \in \mathcal{M}^+(X)$ satisfies $\mu(k) \leq \nu(k)$ for any convex continuous function k on X, then $\mu = \nu$. We refer the reader to [1, Chapter I, § 3] or [58, Section 3.6] for information on maximal measures.)

Theorem 2.1.4. *Let E be a real L_1-predual and $f \in E^{**}$ be a strongly affine function such that $f \in \mathcal{C}_\alpha(B_{E^*})$ for some $\alpha \in [2, \omega_1)$. Then*

$$f \in \begin{cases} \mathfrak{A}_{\alpha+1}(B_{E^*}), & \alpha \in [2, \omega_0), \\ \mathfrak{A}_\alpha(B_{E^*}), & \alpha \in [\omega_0, \omega_1). \end{cases}$$

If, moreover, $\operatorname{ext} B_{E^*}$ *is a Lindelöf resolvable set, then $f \in \mathfrak{A}_\alpha(B_{E^*})$.*

Let us point out that, for any Banach space E and a strongly affine function $f \in E^{**}$ satisfying $f \in \mathcal{C}_1(B_{E^*})$, we have $f \in \mathfrak{A}_1(B_{E^*})$. This follows from [69, Théorème 80] (see also [5, Theorem II.1.2] or [58, Theorem 4.24]). For higher Baire classes, there is a big gap between affine and Baire classes which is an assertion substantiated by M. Talagrand's example [85, Theorem] where he constructed a separable Banach space E and a strongly affine function $f \in E^{**}$ that is in $\mathcal{C}_2(B_{E^*})$ and not contained in $\bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(B_{E^*})$. Further, [78, Theorem 1.1] shows that the shift of classes in Theorem 2.1.4 for finite ordinals may occur even for separable L_1-preduals.

The strategy of the proofs of our main results is to reduce firstly the problem to the case of real Banach spaces and then to consider the dual unit ball with the weak* topology as a compact convex subset of a real locally convex space. Elements of the bidual are then bounded affine functions on the dual unit ball. The key results of Sections 2.3–2.6 are thus formulated for this setting. The proof of Theorem 2.1.4 is moreover based upon a result of W. Lusky stating that any real L_1-predual is complemented in a simplex space (i.e., a space of type $\mathfrak{A}^c(X)$ for a simplex X) and thus our above mentioned technique can be used only for real L_1-preduals. Since it is not clear whether Lusky's result remains true for complex L_1-preduals, *the validity of Theorem 2.1.4 for complex spaces remains open.*

The content of our paper is the following. The second section provides a more detailed information on descriptive classes of sets and functions. Then we prepare a proof of Theorem 2.1.1 in Section 2.3. Results necessary for dealing with Lindelöf sets of extreme points are collected in Section 2.4. They are used in Sections 2.5 and 2.6, which prepares ground for the proof of Theorems 2.1.2 and 2.1.3. All Sections 2.3–2.6 deal within the context of real spaces. Section 2.7 proves by means of prepared results the theorems stated in the introduction. The last Section 2.8 constructs spaces witnessing some natural bounds of our positive results.

When citing references, we try to include several sources to help the reader with finding relevant results.

2.2 Descriptive classes of sets and functions

We recall that, for a Tychonoff space X, $\mathrm{Bas}(X)$, $\mathrm{Bos}(X)$ and $\mathrm{Hs}(X)$ denote the algebras generated by cozero sets, open sets and resolvable sets in X, respectively. These algebras serve as a starting point of an inductive definition of descriptive classes of sets as was indicated in Section 2.1. More precisely, if $\mathcal{F}$ is any of the families above, $\Sigma_2(\mathcal{F})$ consists of all countable unions of sets from $\mathcal{F}$ and $\Pi_2(\mathcal{F})$ of all countable intersections of sets from $\mathcal{F}$. Proceeding inductively, for any $\alpha \in (2, \omega_1)$ we let $\Sigma_\alpha(\mathcal{F})$ to be made of all countable unions of sets from $\bigcup_{1\le\beta<\alpha} \Pi_\beta(\mathcal{F})$ and $\Pi_\alpha(\mathcal{F})$ is made of all countable intersections of sets from $\bigcup_{1\le\beta<\alpha} \Sigma_\beta(\mathcal{F})$. The family $\Pi_\alpha(\mathcal{F}) \cap \Sigma_\alpha(\mathcal{F})$ is denoted as $\Delta_\alpha(\mathcal{F})$. The union of all created additive (or multiplicative) classes is then the σ-algebra generated by $\mathcal{F}$.

(These classes and their analogues were studied by several authors, see e.g. [30], [68], [36] or [35]. We describe in [79, Remark 3.5] their relations to our descriptive classes. We refer the reader to [35] for a recent survey on descriptive set theory in nonseparable and nonmetrizable spaces.)

In case X is metrizable, all the resulting classes coincide (see [79, Proposition 3.4]). These classes characterize in terms of measurability the classes $\mathrm{Baf}_\alpha(X)$, $\mathrm{Bof}_\alpha(X)$ and $\mathrm{Hf}_\alpha(X)$ defined in the introduction. (We recall that a mapping $f : X \to \mathbb{C}$ is called *$\mathcal{F}$-measurable* if $f^{-1}(U) \in \mathcal{F}$ for every $U \subset \mathbb{C}$ open.) Precisely, it is proved in [79, Theorem 5.2] that *given a function $f : X \to \mathbb{C}$ on a Tychonoff space X and $\alpha \in [1, \omega_1)$, we have*

- *$f \in \mathrm{Baf}_\alpha(X)$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Bas}(X))$–measurable.*
- *$f \in \mathrm{Bof}_\alpha(X)$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Bos}(X))$–measurable.*
- *$f \in \mathrm{Hf}_\alpha(X)$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$–measurable.*

It follows easily from this characterization that all the classes $\mathrm{Baf}_\alpha(X)$, $\mathrm{Bof}_\alpha(X)$ and $\mathrm{Hf}_\alpha(X)$ are stable with respect to algebraic operations and uniform convergence (see [58, Theorem 5.10]). Also, a function f is measurable with respect to the σ-algebra generated by Hs if and only if f belongs to some class Hf_α. Analogous assertions hold true for the algebras Bos and Bas. Thus $\bigcup_{\alpha<\omega_1} \mathcal{C}_\alpha(X) = \bigcup_{\alpha<\omega_1} \mathrm{Baf}_\alpha(X)$ is the family of all functions measurable with respect to the σ-algebra of Baire sets.

The following characterization of functions from Hf_1 follows from the definition and results of G. Koumoullis in [47, Theorem 2.3].

Proposition 2.2.1. *For a function $f : K \to \mathbb{C}$ on a compact space K, the following assertions are equivalent:*

(i) $f \in \mathrm{Hf}_1(K)$,

(ii) $f|_F$ *has a point of continuity for every nonempty closed* $F \subset K$ *(i.e., f has the* point of continuity property*),*

(iii) for each $\varepsilon > 0$ *and nonempty* $F \subset K$ *there exists a relatively open nonempty set* $U \subset F$ *such that* $\operatorname{diam} f(U) < \varepsilon$ *(f is* fragmented*).*

Next we need to recall a characterization of resolvable sets that asserts that *a subset H of a topological space X is resolvable if and only if there exist an ordinal κ and an increasing sequence of open sets* $\emptyset = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_\gamma \subset \cdots \subset U_\kappa = X$ *and* $I \subset [0, \kappa)$ *such that, for a limit ordinal* $\gamma \in [0, \kappa]$*, we have* $\bigcup\{U_\lambda : \lambda < \gamma\} = U_\gamma$ *and* $H = \bigcup\{U_{\gamma+1} \setminus U_\gamma : \gamma \in I\}$ (see [37, Section 2] and references therein). We call such a transfinite sequence of open sets *regular* and such a description of a resolvable set a *regular representation* (this notion of regular representation is slightly more useful for us than the one used in [37, Section 2]).

A family $\mathcal{U}$ of subsets of a topological space X is *scattered* if it is disjoint and for each nonempty $\mathcal{V} \subset \mathcal{U}$ there is some $V \in \mathcal{V}$ relatively open in $\bigcup \mathcal{V}$. If $(U_\gamma)_{\gamma \le \kappa}$ is a regular sequence, then $\{U_{\gamma+1} \setminus U_\gamma : \gamma < \kappa\}$ is a scattered partition of X.

It is not difficult to deduce that a scattered union of resolvable sets is again a resolvable set. (Indeed, let $\{H_i : i \in I\}$ be a scattered family of resolvable sets. By [36, Fact 4], each H_i is a union of a scattered family $\mathcal{H}_i$ of sets in $\mathrm{Bos}(X)$. By [30, Lemma 2.2(c)], the family $\bigcup_{i \in I} \mathcal{H}_i$ is scattered, and thus again by [36, Fact 4], the set $\bigcup_{i \in I} H_i$ is resolvable.)

We will also need a fact that any resolvable subset of a compact space is universally measurable (see [47, Lemma 4.4]).

The following fact will be used in the proof of Theorem 2.6.4.

Proposition 2.2.2. *Let* $\alpha \in [2, \omega_1)$ *and* $(U_\gamma)_{\gamma \le \kappa}$ *be a regular sequence in a Tychonoff space X. Let* $A \subset X$ *be such that* $A \cap (U_{\gamma+1} \setminus U_\gamma) \in \Sigma_\alpha(\mathrm{Hs}(U_{\gamma+1} \setminus U_\gamma))$ *for each* $\gamma < \kappa$ *(or* $A \cap (U_{\gamma+1} \setminus U_\gamma) \in \Pi_\alpha(\mathrm{Hs}(U_{\gamma+1} \setminus U_\gamma))$, $\gamma < \kappa$*). Then* $A \in \Sigma_\alpha(\mathrm{Hs}(X))$ *(or* $A \in \Pi_\alpha(\mathrm{Hs}(X))$*).*

Proof. If $\alpha = 2$, the assertion for the additive class follows from the fact mentioned above that a scattered union of resolvable sets is again a resolvable sets. By taking complements we obtain the assertion for $\Pi_2(\mathrm{Hs})$. A straightforward transfinite induction then concludes the proof. □

For the sake of completeness, we include a proof of an easy observation mentioned in the introduction.

Proposition 2.2.3. *If X is a Tychonoff space,* $\mathcal{C}_1(X) = \mathrm{Baf}_1(X)$*.*

Proof. If $f \in \mathcal{C}_1(X)$, a straightforward reasoning gives $f \in \mathrm{Baf}_1(X)$. On the other hand, if $f \in \mathrm{Baf}_1(X)$, it is enough to assume that f is real-valued. If f is moreover bounded, a standard procedure (see e.g. [58, Lemma 5.7]) provides a uniform approximation by a sequence of simple functions, i.e., functions of the form $\sum_{i=1}^n c_i \chi_{A_i}$, where $c_1, \dots, c_n \in \mathbb{R}$ and $\{A_1, \dots, A_n\}$ is a disjoint cover of X such that each A_i is a countable unions of zero sets. A moment's reflection reveals that any such function is in $\mathcal{C}_1(X)$. Hence $f \in \mathcal{C}_1(X)$ as well.

If f is unbounded, we take a homeomorphism $\varphi : \mathbb{R} \to (0,1)$ and apply the procedure above to $\varphi \circ f \in \operatorname{Baf}_1(X)$ to infer $\varphi \circ f \in \mathcal{C}_1(X)$. We can then arrange an approximating sequence (f_n) of continuous functions on X in such a way that $0 < f_n < 1$, $n \in \mathbb{N}$. Then $\varphi^{-1} \circ f_n \to f$, and $f \in \mathcal{C}_1(X)$. $\square$

2.3 Transfer of descriptive properties from $\overline{\operatorname{ext} X}$ to X

Throughout this section we work with real spaces. The main result is Theorem 2.3.4 on transferring descriptive properties of strongly affine functions from the closure of the set of extreme points.

Lemma 2.3.1. *Let K be a compact space and H a universally measurable subset of K. Let $\widetilde{H}\colon \mathcal{M}^1(K) \to \mathbb{R}$ be defined as $\widetilde{H}(\mu) = \mu(H)$, $\mu \in \mathcal{M}^1(K)$. Then*

- $\widetilde{H} \in \operatorname{Hf}_1(\mathcal{M}^1(K))$ *if* $H \in \operatorname{Hs}(K)$,
- $\widetilde{H} \in \operatorname{Bof}_1(\mathcal{M}^1(K))$ *if* $H \in \operatorname{Bos}(K)$.

Proof. We first assume that H is a resolvable set. We select a regular sequence $(U_\gamma)_{\gamma \le \kappa}$ which provides a regular representation of H as mentioned in Section 2.2. We prove by transfinite induction that, *for every $\gamma \le \kappa$, the function $\mu \mapsto \mu(H \cap U_\gamma)$ is in* $\operatorname{Hf}_1(\mathcal{M}^1(K))$.

The statement holds trivially for $\gamma = 0$.

We suppose now that $\gamma \le \kappa$ is of the form $\gamma = \delta + 1$ and the claim is valid for δ. Then, for every $\mu \in \mathcal{M}^1(K)$, we have

$$\mu(H \cap U_\gamma) = \mu(H \cap U_\delta) + \mu(H \cap (U_{\delta+1} \setminus U_\delta)).$$

The second summand is either equal to 0 or $\mu(U_{\delta+1}) - \mu(U_\delta)$. Since the function $\mu \mapsto \mu(U)$ is lower semicontinuous on $\mathcal{M}^1(K)$ for every open set $U \subset K$, it follows e.g. from [47, Theorem 2.3] that the function $\mu \mapsto \mu(U_{\delta+1}) - \mu(U_\delta)$ is in $\operatorname{Hf}_1(\mathcal{M}^1(K))$.

The function $\mu \to \mu(H \cap U_\delta)$ is in $\operatorname{Hf}_1(\mathcal{M}^1(K))$ due to the induction hypothesis. Thus $\mu \mapsto \mu(H)$, as a sum of two functions in $\operatorname{Hf}_1(\mathcal{M}^1(K))$, is in $\operatorname{Hf}_1(\mathcal{M}^1(K))$ as well.

Assume now that $\gamma \le \kappa$ is a limit ordinal and the statement holds for each ordinal smaller than γ. Let $\widetilde{f}(\mu) = \mu(H \cap U_\gamma)$, $\mu \in \mathcal{M}^1(K)$. By Proposition 2.2.1, it is sufficient to show that $\widetilde{f}$ is fragmented. Let $M \subset \mathcal{M}^1(K)$ be nonempty and $\varepsilon > 0$. Let

$$s = \sup\{\mu(U_\gamma)\colon \mu \in M\}$$

and let $\mu_0 \in M$ be chosen such that $\mu_0(U_\gamma) > s - \frac{\varepsilon}{4}$. By the regularity of μ_0, there exists $\delta < \gamma$ with $\mu_0(U_\delta) > s - \frac{\varepsilon}{4}$. Then the set

$$V = \{\mu \in \mathcal{M}^1(K) : \mu(U_\delta) > s - \frac{\varepsilon}{4}\}$$

is an open neighborhood of μ_0.

Let $\widetilde{h} : \mathcal{M}^1(K) \to \mathbb{R}$ be defined as $\widetilde{h}(\mu) = \mu(H \cap U_\delta)$. Then for $\mu \in M \cap V$ we have

$$|\widetilde{h}(\mu) - \widetilde{f}(\mu)| = |\mu(H \cap U_\delta) - \mu(H \cap U_\gamma)| \leq |\mu(U_\gamma \setminus U_\delta)| \leq s - (s - \frac{\varepsilon}{4}) = \frac{\varepsilon}{4},$$

and, by the induction hypothesis, $\widetilde{h}$ is in $\mathrm{Hf}_1(\mathcal{M}^1(K))$ which means that $\widetilde{h}$ is fragmented.

Thus there exists an open set $W \subset \mathcal{M}^1(K)$ intersecting $M \cap V$ such that $\operatorname{diam} \widetilde{h}(M \cap V \cap W) < \frac{\varepsilon}{4}$. Then we have for $\mu_1, \mu_2 \in M \cap V \cap W$ estimate

$$|\widetilde{f}(\mu_1) - \widetilde{f}(\mu_2)| \leq |\widetilde{f}(\mu_1) - \widetilde{h}(\mu_1)| + |\widetilde{h}(\mu_1) - \widetilde{h}(\mu_2)| + |\widetilde{h}(\mu_2) - \widetilde{f}(\mu_2)| \leq \frac{3}{4}\varepsilon.$$

Hence $\operatorname{diam} \widetilde{f}(M \cap V \cap W) < \varepsilon$ and $\widetilde{f}$ is fragmented. This proves the claim as well as the proof of the first assertion (taking $\gamma = \kappa$).

Assume now that $H \in \mathrm{Bos}(K)$. Then H can be written as a finite disjoint union of differences of closed sets (see e.g. [58, Lemma 5.12]), i.e., $H = \bigcup_{i=1}^n E_i \setminus F_i$, where $F_i \subset E_i$ are closed and the family $\{E_1 \setminus F_1, \dots, E_n \setminus F_n\}$ is disjoint. Then the function $\mu \mapsto \mu(E_i \setminus F_i)$, as a difference of a couple of upper semicontinuous functions on $\mathcal{M}^1(K)$, is in $\mathrm{Bof}_1(\mathcal{M}^1(K))$ for each pair E_i, F_i.

Hence $\mu \mapsto \mu(H)$, $\mu \in \mathcal{M}^1(K)$, is a finite sum of functions contained in $\mathrm{Bof}_1(\mathcal{M}^1(K))$, and thus contained in $\mathrm{Bof}_1(\mathcal{M}^1(K))$. $\square$

Lemma 2.3.2. *Let K be a compact space, $f : K \to \mathbb{R}$ a bounded universally measurable function and let $\widetilde{f} \colon \mathcal{M}^1(K) \to \mathbb{R}$ be defined as $\widetilde{f}(\mu) = \mu(f)$, $\mu \in \mathcal{M}^1(K)$. Then*

- *$\widetilde{f} \in \mathrm{Hf}_1(\mathcal{M}^1(K))$ if $f \in \mathrm{Hf}_1(K)$,*
- *$\widetilde{f} \in \mathrm{Bof}_1(\mathcal{M}^1(K))$ if $f \in \mathrm{Bof}_1(K)$.*

Proof. We begin with the proof for $f \in \mathrm{Hf}_1(K)$. First, if $f = \chi_A$ is the characteristic function of a set $A \in \Delta_2(\mathrm{Hs}(K))$, we write $A = \bigcup_n A_n$, where $A_1 \subset A_2 \subset \cdots$ are sets in $\mathrm{Hs}(K)$. If $c \in \mathbb{R}$ is given, we have from Lemma 2.3.1 that

$$\{\mu \in \mathcal{M}^1(K) : \widetilde{f}(\mu) > c\} = \bigcup_{n=1}^{\infty} \{\mu \in \mathcal{M}^1(K) : \mu(A_n) > c\} \in \Sigma_2(\mathrm{Hs}(K)).$$

On the other hand, $K \setminus A \in \Sigma_2(\mathrm{Hs}(K))$ and hence it follows from the previous reasoning that

$$\{\mu \in \mathcal{M}^1(K) : \widetilde{f}(\mu) < c\} = \{\mu \in \mathcal{M}^1(K) : \mu(K \setminus A) > 1 - c\} \in \Sigma_2(\mathrm{Hs}(K)).$$

We conclude that a function $\widetilde{f}$ is $\Sigma_2(\mathrm{Hs}(\mathcal{M}^1(K)))$-measurable and hence $\widetilde{f} \in \mathrm{Hf}_1(\mathcal{M}^1(K))$.

If $f \in \mathrm{Hf}_1(K)$ is bounded, it can be uniformly approximated by simple functions in $\mathrm{Hf}_1(K)$, i.e., functions of the form $\sum_{i=1}^n c_i \chi_{A_i}$, provided $A_1, \dots, A_n \in \Delta_2(\mathrm{Hs}(K))$ are pairwise disjoint and $c_1, \dots, c_n \in \mathbb{R}$ (this standard procedure can be found e.g. in [58, Lemma 5.7]). Hence $\widetilde{f}$ can be uniformly approximated by functions in $\mathrm{Hf}_1(\mathcal{M}^1(K))$, and thus $\widetilde{f} \in \mathrm{Hf}_1(\mathcal{M}^1(K))$.

The proof for $f \in \mathrm{Bof}_1(K)$ would proceed in a similar fashion. $\square$

Lemma 2.3.3. *Let K be a compact space and $f : K \to \mathbb{R}$ be a bounded universally measurable function. Let $\widetilde{f}\colon \mathcal{M}^1(K) \to \mathbb{R}$ be defined as $\widetilde{f}(\mu) = \mu(f)$, $\mu \in \mathcal{M}^1(K)$. Then,*

(a) for $\alpha \in [1, \omega_1)$, $f \in \mathrm{Hf}_\alpha(K)$ if and only if $\widetilde{f} \in \mathrm{Hf}_\alpha(\mathcal{M}^1(K))$,

(b) for $\alpha \in [1, \omega_1)$, $f \in \mathrm{Bof}_\alpha(K)$ if and only if $\widetilde{f} \in \mathrm{Bof}_\alpha(\mathcal{M}^1(K))$,

(c) for $\alpha \in [0, \omega_1)$, $f \in \mathcal{C}_\alpha(K)$ if and only if $\widetilde{f} \in \mathcal{C}_\alpha(\mathcal{M}^1(K))$.

Proof. The "if" parts of the proof easily follows from the fact $f = \widetilde{f} \circ \phi$ where $\phi : K \to \mathcal{M}^1(K)$ sending a point $x \in K$ to the Dirac measure ε_x at x is a homeomorphic embedding.

The proof of "only if" part will be given by transfinite induction. If $\alpha = 1$ in (a) and (b), the assertion follows from Lemma 2.3.2, the case $\alpha = 0$ in (c) is obvious.

The assertions for higher ordinals α now follows by a straightforward induction. □

As we mentioned in the introduction, the following theorem is a generalization of [73, Corollaire 8].

Theorem 2.3.4. *Let X be a compact convex set and $f : X \to \mathbb{R}$ be a strongly affine function. Then,*

- *for $\alpha \in [1, \omega_1)$, $f|_{\overline{\mathrm{ext}\, X}} \in \mathrm{Hf}_\alpha(\overline{\mathrm{ext}\, X})$ if and only if $f \in \mathrm{Hf}_\alpha(X)$,*
- *for $\alpha \in [1, \omega_1)$, $f|_{\overline{\mathrm{ext}\, X}} \in \mathrm{Bof}_\alpha(\overline{\mathrm{ext}\, X})$ if and only if $f \in \mathrm{Bof}_\alpha(X)$,*
- *for $\alpha \in [0, \omega_1)$, $f|_{\overline{\mathrm{ext}\, X}} \in \mathcal{C}_\alpha(\overline{\mathrm{ext}\, X})$ if and only if $f \in \mathcal{C}_\alpha(X)$.*

Proof. It is easy to realize that all the families Hf_α, Bof_α and $\mathcal{C}_\alpha$ are preserved by making restrictions to subspaces of X. This observation gives the "if" parts of the proof.

For the proof of the "only if" parts, let $f : X \to \mathbb{R}$ be a strongly affine function with $f|_{\overline{\mathrm{ext}\, X}} \in \mathcal{F}(\overline{\mathrm{ext}\, X})$ where $\mathcal{F}$ is any of the classes Hf_α, Bof_α or $\mathcal{C}_\alpha$. Then the function $\widetilde{g} : \mathcal{M}^1(\overline{\mathrm{ext}\, X}) \to \mathbb{R}$ defined as

$$\widetilde{g}(\mu) = \mu(f), \quad \mu \in \mathcal{M}^1(\overline{\mathrm{ext}\, X}),$$

is in $\mathcal{F}(\mathcal{M}^1(\overline{\mathrm{ext}\, X}))$ by Lemma 2.3.3.

The mapping $r : \mathcal{M}^1(\overline{\mathrm{ext}\, X}) \to X$, which assigns $\mu \in \mathcal{M}^1(\overline{\mathrm{ext}\, X})$ its barycenter $r(\mu) \in X$, is a continuous surjection of the compact space $\mathcal{M}^1(\overline{\mathrm{ext}\, X})$ onto X (see [1, Proposition I.4.6 and Theorem I.4.8] or [58, Theorem 3.65 and Proposition 3.64]).

From the strong affinity of f we have $\widetilde{g} = f \circ r$. Now we use the fact that $\widetilde{g} \in \mathcal{F}(\mathcal{M}^1(\overline{\mathrm{ext}\, X}))$ if and only if $f \in \mathcal{F}(X)$. This fact can be found in [70, Theorem 5.9.13] and [58, Theorem 5.26] for classes $\mathcal{C}_\alpha$, and in [37, Theorems 4 and 10] for classes Bof_α and Hf_α (see also [58, Theorem 5.26]). Thus the function f is in $\mathcal{F}(X)$. □

2.4 Auxiliary result on compact convex sets with $\operatorname{ext} X$ being Lindelöf

Throughout this section we work with spaces over the field of real numbers. We aim for the proof of Proposition 2.4.8 which is a fact used both in Section 2.5 and 2.6. We recall that a topological space X is *K-analytic* if it is an image of a Polish space under an upper semicontinuous compact-valued map (see [70, Section 2.1]).

Lemma 2.4.1. *Let $\varphi\colon X \to Y$ be a continuous surjection of a K-analytic space X onto a K-analytic space Y and let $g : Y \to \mathbb{R}$. Then g is a Baire function on Y if and only if $g \circ \varphi$ is a Baire function on X.*

Proof. If g is a Baire function Y, then $g \circ \varphi$ is clearly a Baire function on X. Conversely, if $f = g \circ \varphi$ is a Baire function on X and $U \subset \mathbb{R}$ is an open set, then both $f^{-1}(U)$ and $f^{-1}(\mathbb{R} \setminus U)$ are Baire sets in X. Then they are K-analytic sets in X (see [70, Section 2]), and thus

$$g^{-1}(U) = \varphi(f^{-1}(U)), \quad g^{-1}(\mathbb{R} \setminus U) = \varphi(f^{-1}(\mathbb{R} \setminus U))$$

are K-analytic as well. It follows from the proof of the standard separation theorem (see [70, Theorem 3.3.1]) that they are Baire sets. Hence g is measurable with respect to the σ-algebra of Baire sets, and thus it is a Baire function. $\square$

Lemma 2.4.2. *Let B be a Lindelöf subset of a compact space X and f be a bounded continuous function on B. Then there exists a bounded Baire function on X extending f.*

Proof. Without loss of generality, let $0 \le f \le 1$. If

$$h(x) = \begin{cases} f(x), & x \in B, \\ \limsup_{y \to x, y \in B} f(y), & x \in \overline{B} \setminus B, \\ 0, & x \in X \setminus \overline{B}, \end{cases}$$

then h is an upper semicontinuous function on X. Hence

$$h = \inf\{a \in \mathcal{C}(X) : h \le a \le 1\}.$$

By the Lindelöf property of B and the continuity of the function f (see [58, Lemma A.54]), there exists a countable family $\{a_n : n \in \mathbb{N}\}$ of functions with $h \le a_n \le 1$, $n \in \mathbb{N}$, such that $f = \inf\{a_n : n \in \mathbb{N}\}$ on B. Then

$$g = \inf\{a_n : n \in \mathbb{N}\}$$

is a Baire function on X with values in $[0, 1]$ extending f. $\square$

Lemma 2.4.3. *Let $f\colon X \to \mathbb{R}$ be a strongly affine function on a compact convex set X for which there exists a Baire set $B \supset \operatorname{ext} X$ such that $f|_B$ is a Baire function. Then f is a Baire function on X.*

Proof. Let $B \supset \operatorname{ext} X$ and $f : X \to \mathbb{R}$ be as in the hypothesis. Let

$$M = \{\mu \in \mathcal{M}^1(X) \colon \mu(B) = 1\}.$$

Since the characteristic function of B is a Baire function, the function $\widetilde{B}(\mu) = \mu(B)$, $\mu \in \mathcal{M}^1(X)$, is a Baire function on $\mathcal{M}^1(X)$ as well by Lemma 2.3.3(c), and thus $M = \{\mu \in \mathcal{M}^1(X) \colon \widetilde{B}(\mu) = 1\}$ is a Baire and consequently K-analytic set in $\mathcal{M}^1(X)$.

Since $f|_B$ is a Baire function on B, it extends to a bounded Baire function g on X by Lemma 2.4.2 and transfinite induction. Then

$$\widetilde{g}(\mu) = \mu(g), \quad \mu \in \mathcal{M}^1(X),$$

is a Baire function on $\mathcal{M}^1(X)$ by Lemma 2.3.3(c).

Further, the function $\widetilde{f} : M \to \mathbb{R}$ defined as

$$\widetilde{f}(\mu) = \mu(f), \quad \mu \in M,$$

coincides on M with $\widetilde{g}$. Hence $\widetilde{f}$ is a Baire function on M.

Then $r : M \to X$ is a continuous surjective mapping satisfying $\widetilde{f} = f \circ r$ (see [1, Corollary I.4.12 and the subsequent remark] or [58, Theorem 3.79]). By Lemma 2.4.1, f is a Baire function. $\square$

Lemma 2.4.4. *Let X be a compact convex set with $\operatorname{ext} X$ Lindelöf, $\mu \in \mathcal{M}^1(X)$ be maximal and $B \supset \operatorname{ext} X$ be μ-measurable. Then $\mu(B) = 1$.*

Proof. Given $B \supset \operatorname{ext} X$ and maximal measure $\mu \in \mathcal{M}^1(X)$, by the regularity of μ it is enough to show that $\mu(K) = 0$ for every $K \subset X \setminus B$ compact. Given such a set K, for every $x \in \operatorname{ext} X$ we select a closed neighborhood U_x of x disjoint from K. By the Lindelöf property we choose a countable set $\{x_n \colon n \in \mathbb{N}\} \subset \operatorname{ext} X$ with $\operatorname{ext} X \subset \bigcup U_{x_n}$. By Corollary I.4.12 and the subsequent remark in [1] (see also [58, Theorem 3.79]), $\mu(\bigcup U_{x_n}) = 1$. Hence $\mu(K) = 0$, which concludes the proof. $\square$

Lemma 2.4.5. *Let X be a compact convex set with $\operatorname{ext} X$ Lindelöf and $f \in \mathcal{C}^b(\operatorname{ext} X)$. Then there exist a decreasing sequence (u_n) of continuous concave functions on X and an increasing sequence (l_n) of continuous convex functions on X such that*

$$\inf f(\operatorname{ext} X) \le \inf l_1(X), \quad \sup u_1(X) \le \sup f(\operatorname{ext} X),$$

and

$$u_n \searrow f,\ l_n \nearrow f \text{ on } \operatorname{ext} X.$$

Proof. Without loss of generality we may assume that

$$0 \le i = \inf f(\operatorname{ext} X) \le \sup f(\operatorname{ext} X) = s \le 1.$$

We construct a decreasing sequence (u_n) of continuous concave functions on X with values in $[0, 1]$ such that $u_n \searrow f$ on $\operatorname{ext} X$. To achieve this, we define $h \colon \overline{\operatorname{ext} X} \to [0, 1]$ as

$$h(x) = \begin{cases} f(x), & x \in \operatorname{ext} X, \\ \limsup_{y \to x, y \in \operatorname{ext} X} f(y), & x \in \overline{\operatorname{ext} X} \setminus \operatorname{ext} X. \end{cases}$$

Then h is upper semicontinuous on $\overline{\operatorname{ext} X}$ and the function

$$h^* = \inf\{a \in \mathfrak{A}^c(X) \colon a \geq f \text{ on } \operatorname{ext} X\}$$

satisfies $h = h^* = f$ on $\operatorname{ext} X$ by [1, Proposition I.4.1] (see also [58, Theorem 3.24]). Hence

$$f = \inf\{a \in \mathfrak{A}^c(X) \colon a \geq f \text{ on } \operatorname{ext} X\} \quad \text{ on } \operatorname{ext} X.$$

Since $\operatorname{ext} X$ is a Lindelöf space, there exists a countable family $\mathcal{H} = \{h_n \colon n \in \mathbb{N}\}$ of functions in $\mathfrak{A}^c(X)$ majorizing f on $\operatorname{ext} X$ such that $f = \inf \mathcal{H}$ on $\operatorname{ext} X$ (see [40, Lemma] or [58, Lemma A.54]). Then we obtain the desired sequence by setting

$$u_1 = s \wedge h_1,\ u_2 = s \wedge h_1 \wedge \cdots \wedge h_n,\ \ldots, \quad n \in \mathbb{N}.$$

Analogously we obtain an increasing sequence (l_n) of convex continuous functions converging to f on $\operatorname{ext} X$. ☐

Lemma 2.4.6. *Let X be a compact convex set with $\operatorname{ext} X$ Lindelöf and let $f \in \mathcal{C}_\alpha(\operatorname{ext} X)$ have values in $[0,1]$. Then there exist a Baire set $B \supset \operatorname{ext} X$ and a function $g \in \mathcal{C}_\alpha(B)$ such that*

- *$g = f$ on $\operatorname{ext} X$,*
- *$0 \leq g \leq 1$ on B, and*
- *$g(r(\mu)) = \mu(g)$ for any $\mu \in \mathcal{M}^1(X)$ satisfying $\mu(B) = 1$ and $r(\mu) \in B$.*

Proof. We proceed by transfinite induction on the class of a function f.

Assume first that f is continuous on $\operatorname{ext} X$. Using Lemma 2.4.5 we find relevant sequences (u_n) and (l_n), and define $u = \inf_{n\in\mathbb{N}} u_n$, $l = \sup_{n\in\mathbb{N}} l_n$. Then we observe that $l \leq u$ by the minimum principle (see [1, Theorem I.5.3] or [58, Theorem 3.16], both functions are Baire, u is upper semicontinuous concave and l is lower semicontinuous convex. Let

$$B = \{x \in X \colon u(x) = l(x)\} \quad \text{and} \qquad g(x) = u(x), \quad x \in B.$$

Then B is a Baire set containing $\operatorname{ext} X$ and, for $x \in B$ and $\mu \in \mathcal{M}_x(X)$ with $\mu(B) = 1$, we have by [58, Proposition 4.7]

$$g(x) = u(x) \geq \mu(u) = \mu(l) \geq l(x) = g(x).$$

Since g is continuous on B, the proof is finished for the case $\alpha = 0$.

Assume now that the claim holds true for all β smaller then some countable ordinal α. Given $f \in \mathcal{C}_\alpha(\operatorname{ext} X)$ with values in $[0,1]$, let (f_n) be a sequence of functions with $f_n \in \mathcal{C}_{\alpha_n}(\operatorname{ext} X)$ for some $\alpha_n < \alpha$, $n \in \mathbb{N}$, such that $f_n \to f$. Without loss of generality we may assume that all functions f_n have values in $[0,1]$. For each $n \in \mathbb{N}$, we use the induction hypothesis and find a Baire set $B_n \supset \operatorname{ext} X$ along with a function $g_n \in \mathcal{C}_{\alpha_n}(B_n)$ with values in $[0,1]$ that coincides with f_n on $\operatorname{ext} X$ and satisfies $g_n(r(\mu)) = \mu(g_n)$ for any $\mu \in \mathcal{M}^1(X)$ satisfying $\mu(B_n) = 1$ and $r(\mu) \in B_n$.

We set

$$B = \{x \in \bigcap_{n=1}^{\infty} B_n \colon (g_n(x)) \text{ converges}\} \quad \text{and} \quad g(x) = \lim_{n\to\infty} g_n(x),\ x \in B.$$

Then B is Baire set containing $\operatorname{ext} X$, $g \in \mathcal{C}_\alpha(B)$ with values in $[0,1]$,

$$g_n(x) = f_n(x) \to f(x) \quad \text{for every } x \in \operatorname{ext} X,$$

and, for $x \in B$ and $\mu \in \mathcal{M}_x(X)$ with $\mu(B) = 1$,

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \mu(g_n) = \mu(g).$$

This finishes the proof. □

Lemma 2.4.7. *Let X be a compact convex set with $\operatorname{ext} X$ Lindelöf and let $f \colon X \to \mathbb{R}$ be a strongly affine function such that $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X)$. Then there exists a Baire set $B \supset \operatorname{ext} X$ such that $f \in \mathcal{C}_\alpha(B)$.*

Proof. Given a function f as in the hypothesis, we assume without loss of generality that $0 \le f \le 1$. Using Lemma 2.4.6 we find a Baire set $B \supset \operatorname{ext} X$ together with a function $g \in \mathcal{C}_\alpha(B)$ with values in $[0,1]$ such that $g = f$ on $\operatorname{ext} X$ and $g(x) = \mu(g)$ for each $x \in B$ and $\mu \in \mathcal{M}_x(X)$ with $\mu(B) = 1$.

We claim that $f = g$ on B. To verify this, pick $x \in B$ and a maximal measure $\mu \in \mathcal{M}_x(X)$. Then μ is supported by B and $f = g$ μ-almost everywhere. (Indeed, the set $\{y \in X \colon f(y) = g(y)\}$ is μ-measurable and contains $\operatorname{ext} X$. The assertion thus follows from Lemma 2.4.4.) Hence

$$g(x) = \mu(g) = \mu(f) = f(x),$$

where the last equality follows from the strong affinity of f. This concludes the proof. □

Proposition 2.4.8. *Let X be a compact convex set with $\operatorname{ext} X$ Lindelöf and let $f \colon X \to \mathbb{R}$ be a strongly affine function such that $f|_{\operatorname{ext} X}$ is Baire. Then f is a Baire function on X.*

Proof. The assertion follows from Lemmas 2.4.7 and 2.4.3. □

2.5 Transfer of descriptive properties on compact convex sets with $\operatorname{ext} X$ being Lindelöf

The notions in this section are considered with respect to real numbers. The following key factorization result uses a method of a metrizable reduction available for Baire functions that can be found e.g. in [11], [70, Theorem 5.9.13], [86, Theorem 1], [7] or [58, Theorem 9.12]. The main result in this section, Theorem 2.5.2, is then consequences of a selection theorem by M. Talagrand (see [83]).

Lemma 2.5.1. *Let X be a compact convex set with $\operatorname{ext} X$ Lindelöf and let $f : X \to \mathbb{R}$ be strongly affine such that $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X)$ for some $\alpha \in [1, \omega_1)$. Then there exist a metrizable compact convex set Y, an affine surjection $\varphi : X \to Y$, a strongly affine Baire function $\widetilde{f} : Y \to \mathbb{R}$ and $\widetilde{g} \in \mathcal{C}^b_\alpha(\operatorname{ext} Y)$ such that*

$$\widetilde{g}(\varphi(x)) = f(x), \quad x \in \operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y),$$

and

$$f(x) = \widetilde{f}(\varphi(x)), \quad x \in X.$$

Proof. Given a function f as in the premise, we may assume without loss of generality that $0 \le f \le 1$. Let $\mathcal{F} = \{g_n \colon n \in \mathbb{N}\} \subset \mathcal{C}(\operatorname{ext} X)$ be a countable family of functions with values in $[0,1]$ satisfying $f \in \mathcal{F}_\alpha$.

For a fixed index $n \in \mathbb{N}$, using Lemma 2.4.5 we select finite families $\mathcal{U}_n^k$ and $\mathcal{L}_n^k$, $k \in \mathbb{N}$, of functions in $\mathfrak{A}^c(X)$ with values in $[0,1]$ such that, for

$$u_n^k = \inf \mathcal{U}_n^k, \quad l_n^k = \sup \mathcal{L}_n^k,$$

we have

- $\lim_{k\to\infty} l_n^k(x) = \lim_{k\to\infty} u_n^k = g_n(x)$ for each $x \in \operatorname{ext} X$,
- $(l_n^k)_{k=1}^\infty$ is increasing and $(u_n^k)_{k=1}^\infty$ is decreasing.

Further, by Proposition 2.4.8, f is a Baire function on X, say of class β. Let $\mathcal{F}' = \{h_n \colon n \in \mathbb{N}\} \subset \mathcal{C}(X)$ be a countable family satisfying $f \in (\mathcal{F}')_\beta$. For any $n, k \in \mathbb{N}$, by [1, Proposition I.1.1] (or [58, Proposition 3.11]) there exist finite families $\mathcal{V}_n^k, \mathcal{W}_n^k \subset \mathfrak{A}^c(X)$ such that, for $v_n^k = \inf \mathcal{V}_n^k$, $w_n^k = \sup \mathcal{W}_n^k$, we have

$$\|h_n - (v_n^k + w_n^k)\| < \frac{1}{k}.$$

By setting $\mathcal{G} = \{v_n^k, w_n^k \colon n, k \in \mathbb{N}\}$, we obtain a family satisfying $f \in \mathcal{G}_\beta$.

We set

$$\Phi = \bigcup_{n,k\in\mathbb{N}} (\mathcal{U}_n^k \cup \mathcal{L}_n^k \cup \mathcal{V}_n^k \cup \mathcal{W}_n^k)$$

and define $\varphi : X \to \mathbb{R}^{\mathbb{N}}$ as

$$\varphi(x) = (\phi(x))_{\phi\in\Phi}, \quad x \in X.$$

Then $Y = \varphi(X)$ is a metrizable compact convex set and, for each $\phi \in \Phi$, there exists $\widetilde{\phi} \in \mathfrak{A}^c(Y)$ with $\widetilde{\phi} \circ \varphi = \phi$.

For fixed $n, k \in \mathbb{N}$, let $\widetilde{\mathcal{U}}_n^k \subset \mathfrak{A}^c(Y)$ be such that

$$\mathcal{U}_n^k = \left\{\widetilde{u} \circ \varphi \colon \widetilde{u} \in \widetilde{\mathcal{U}}_n^k\right\}.$$

Analogously we pick $\widetilde{\mathcal{L}}_n^k$, $\widetilde{\mathcal{V}}_n^k$ and $\widetilde{\mathcal{W}}_n^k$ in $\mathfrak{A}^c(Y)$. Then

$$\widetilde{u}_n^k = \inf \widetilde{\mathcal{U}}_n^k,\ \widetilde{l}_n^k = \sup \widetilde{\mathcal{L}}_n^k,\ \widetilde{v}_n^k = \inf \widetilde{\mathcal{V}}_n^k \quad \text{and} \quad \widetilde{w}_n^k = \sup \widetilde{\mathcal{W}}_n^k$$

satisfy

$$\widetilde{u}_n^k \circ \varphi = u_n^k,\ \widetilde{l}_n^k \circ \varphi = l_n^k,\ \widetilde{v}_n^k \circ \varphi = v_n^k \quad \text{and} \quad \widetilde{w}_n^k \circ \varphi = w_n^k.$$

Given $y \in \operatorname{ext} Y$, we select $x \in \operatorname{ext} X \cap \varphi^{-1}(y)$. Then

$$\begin{aligned} \lim_{k\to\infty} \widetilde{u}_n^k(y) &= \lim_{k\to\infty} \widetilde{u}_n^k(\varphi(x)) = \lim_{k\to\infty} u_n^k(x) = g_n(x), \quad \text{and} \\ \lim_{k\to\infty} \widetilde{l}_n^k(y) &= \lim_{k\to\infty} \widetilde{l}_n^k(\varphi(x)) = \lim_{k\to\infty} l_n^k(x) = g_n(x). \end{aligned}$$

Thus $(\widetilde{u}_n^k)_{k=1}^\infty$ is a decreasing sequence on $\operatorname{ext} Y$, $(\widetilde{l}_n^k)_{k=1}^\infty$ is increasing on $\operatorname{ext} Y$and both converge to a common limit $\widetilde{g}_n : \operatorname{ext} Y \to \mathbb{R}$ given by

$$\widetilde{g}_n(y) = \lim_{k\to\infty} \widetilde{u}_n^k(y), \quad y \in \operatorname{ext} Y,$$

which is a continuous function on $\operatorname{ext} Y$ with values in $[0,1]$.

Thus, for every $n \in \mathbb{N}$, there exists a function $\widetilde{g}_n \in \mathcal{C}^b(\operatorname{ext} Y)$ satisfying $\widetilde{g}_n \circ \varphi = g_n$ on $\operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y)$. Let $\widetilde{\mathcal{F}} = \{\widetilde{g}_n \colon n \in \mathbb{N}\}$.

Now we claim that, for each $\gamma \in [0,\alpha]$ and $h \in \mathcal{F}_\gamma$, there exists $\widetilde{h} \in \widetilde{\mathcal{F}}_\gamma$ such that $h = \widetilde{h} \circ \varphi$ on $\operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y)$. To verify this, we proceed by transfinite induction. The claim is obvious for $\gamma = 0$. Assume that it holds for all $\gamma' < \gamma$ for some $\gamma \leq \alpha$ and that we are given $h \in \mathcal{F}_\gamma$. Let $\gamma_n < \gamma$ and $h_n \in \mathcal{F}_{\gamma_n}$, $n \in \mathbb{N}$, be such that $h = \lim h_n$. By the inductive assumption, there exist $\widetilde{h}_n \in \widetilde{\mathcal{F}}_{\gamma_n}$ satisfying $h_n = \widetilde{h}_n \circ \varphi$ on $\operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y)$. Then the sequence $(\widetilde{h}_n(y))$ converges for every point $y \in \operatorname{ext} Y$. Hence we may define a function $\widetilde{h} \in \widetilde{\mathcal{F}}_\gamma$ by

$$\widetilde{h}(y) = \lim_{n\to\infty} \widetilde{h}_n(y), \quad y \in \operatorname{ext} Y,$$

and then, for every $y \in \operatorname{ext} Y$ and $x \in \varphi^{-1}(y) \cap \operatorname{ext} X$,

$$\widetilde{h}(y) = \lim_{n\to\infty} \widetilde{h}_n(y) = \lim_{n\to\infty} h_n(x) = h(x).$$

This proves the claim.

It follows from the claim that there exists a function $\widetilde{g} \in \mathcal{C}_\alpha(\operatorname{ext} Y)$ such that

$$\widetilde{g}(\varphi(x)) = f(x), \quad x \in \operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y).$$

Analogously, let $\widetilde{\mathcal{G}}$ be the family satisfying

$$\mathcal{G} = \{\widetilde{z} \circ \varphi \colon \widetilde{z} \in \widetilde{\mathcal{G}}\}.$$

Then, for each $\gamma \in [0,\beta]$ and a function $h \in \mathcal{G}_\gamma$, it follows as above that there exists a function $\widetilde{h} \in \widetilde{\mathcal{G}}_\gamma$ satisfying $h = \widetilde{h} \circ \varphi$. Hence there exists a function $\widetilde{f} \in (\widetilde{\mathcal{G}})_\beta$ satisfying $f = \widetilde{f} \circ \varphi$. Obviously, $\widetilde{f}$ is a Baire function and, moreover, it is strongly affine by [75, Proposition 3.2] (see also [58, Proposition 5.29]). This concludes the proof. □

Theorem 2.5.2. *Let X be a compact convex set with $\operatorname{ext} X$ Lindelöf and $f : X \to \mathbb{R}$ be a strongly affine function. If $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X)$, then*

$$f \in \begin{cases} \mathcal{C}_{\alpha+1}(X), & \alpha \in [0,\omega_0), \\ \mathcal{C}_\alpha(X), & \alpha \in [\omega_0,\omega_1). \end{cases}$$

Proof. Let f be a strongly affine function f whose restriction to $\operatorname{ext} X$ is of Baire class α. If $\alpha = 0$, i.e., f is continuous and bounded on $\operatorname{ext} X$, Lemma 2.4.5 provides the relevant sequences (u_n) and (l_n). For $n \in \mathbb{N}$, $x \in X$ and $\mu_1, \mu_2 \in \mathcal{M}_x(X)$, we have

$$\mu_1(l_n) \leq \mu_1(f) = f(x) = \mu_2(f) \leq \mu_2(u_n).$$

If we denote

$$\begin{aligned}(l_n)^* &= \inf\{h \in \mathfrak{A}^c(X) : h \geq l_n \text{ on } X\} \quad \text{and}\\ (u_n)_* &= \sup\{h \in \mathfrak{A}^c(X) : h \leq u_n \text{ on } X\},\end{aligned}$$

then by [1, Corollary I.3.6] (see also [58, Lemma 3.21]),

$$(l_n)^* \leq f \leq (u_n)_*.$$

Using an argument based upon the Hahn-Banach theorem (see e.g. [58, Lemma 4.11]), there exists a sequence (h_n) of functions in $\mathfrak{A}^c(X)$ such that

$$(l_n)^* - \frac{1}{n} < h_n < (u_n)_* + \frac{1}{n}, \quad n \in \mathbb{N}.$$

Then $f \in \mathcal{C}_1(X)$ because $h_n \to f$ on $\operatorname{ext} X$, and thus on X. (Indeed, given $x \in X$, let $\mu \in \mathcal{M}_x(X)$ be maximal. Then the set

$$B = \{y \in X \colon h_n(y) \to f(y)\}$$

is μ-measurable and contains $\operatorname{ext} X$. By Lemma 2.4.4, $\mu(B) = 1$. Hence $f(x) = \mu(f) = \lim \mu(h_n) = h_n(x)$.)

Assume now that $\alpha \geq 1$. Then we use Lemma 2.5.1 to find a continuous affine surjection φ of X onto a metrizable compact convex set Y, $\widetilde{g} \in \mathcal{C}^b_\alpha(\operatorname{ext} Y)$ and a Baire function $\widetilde{f} : X \to \mathbb{R}$ such that

$$f = \widetilde{g} \circ \varphi \text{ on } \operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y) \quad \text{and} \quad f = \widetilde{f} \circ \varphi \text{ on } X. \tag{2.1}$$

Since $\operatorname{ext} Y$ is a G_δ set and $\alpha \geq 1$, we can extend $\widetilde{g}$ to the whole set Y (and denote it likewise) with preservation of class (see [49, § 35, VI, Théorème]). By [83, Théoréme 1] (see also [58, Theorem 11.41]), there exists a mapping $y \mapsto \nu_y$, $y \in Y$, such that

(a) ν_y is a maximal measure in $\mathcal{M}_y(Y)$,

(b) the function $y \mapsto \nu_y(h)$ is Baire-one on Y for every $h \in \mathcal{C}(Y)$.

Let

$$\widetilde{h}(y) = \nu_y(\widetilde{g}), \quad y \in Y.$$

Then

$$\widetilde{h} \in \begin{cases} \mathcal{C}_{\alpha+1}(Y), & \alpha \in [1, \omega_0),\\ \mathcal{C}_\alpha(Y), & \alpha \in [\omega_0, \omega_1).\end{cases}$$

Indeed, if $\alpha < \omega_0$, the claim follows from (b) by induction. If $\alpha = \omega_0$, let $(\widetilde{g}_n)$ be a bounded sequence of functions such that $\widetilde{g}_n \in \mathcal{C}_{\alpha_n}(Y)$ for some $\alpha_n < \omega_0$ and $\widetilde{g}_n \to \widetilde{g}$. Then the functions $\widetilde{h}_n(y) = \nu_y(\widetilde{g}_n)$ are in $\mathcal{C}_{\alpha_n+1}(Y)$ and converge to $\widetilde{h}$. Hence $\widetilde{h} \in \mathcal{C}_{\omega_0}(Y)$. For $\alpha > \omega_0$, the claim follows by transfinite induction.

Next we prove that $\widetilde{h} = \widetilde{f}$. To this end, let $y \in Y$ be fixed. Using [58, Proposition 7.49] we find a maximal measure $\mu \in \mathcal{M}^1(X)$ satisfying $\varphi_\sharp\mu = \nu_y$ (here $\varphi_\sharp : \mathcal{M}^1(X) \to \mathcal{M}^1(Y)$ denotes the mapping induced by $\varphi : X \to Y$, see [26, Theorem 418I]). Then it is easy to check (see e.g. the proof of Proposition 5.29 in [58]) that

$$\varphi(r(\mu)) = r(\varphi_\sharp\mu) = r(\nu_y) = y. \tag{2.2}$$

Further,

$$\mu(\varphi^{-1}(\operatorname{ext} Y)) = 1$$

and

$$\{x \in X\colon f(x) = \widetilde{g}(\varphi(x))\} \supset \operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y).$$

From these facts and Lemma 2.4.4 it follows that $f = \widetilde{g} \circ \varphi$ μ-almost everywhere. Thus we get from (2.2) and (2.1)

$$\begin{aligned}
\widetilde{h}(y) &= \int_{\operatorname{ext} Y} \widetilde{g}\, d\nu_y = \int_{\operatorname{ext} Y} \widetilde{g}\, d(\varphi_\sharp\mu) \\
&= \int_X \widetilde{g} \circ \varphi\, d\mu = \int_X f\, d\mu \\
&= f(r(\mu)) = \widetilde{f}(\varphi(r(\mu))) \\
&= \widetilde{f}(y).
\end{aligned}$$

Hence $\widetilde{f} = \widetilde{h}$ on Y.

By (2.1), f is of the same class as $\widetilde{f} = \widetilde{h}$. This concludes the proof. □

2.6 Transfer of decriptive properties on compact convex sets with $\operatorname{ext} X$ being a resolvable Lindelöf set

Again we point out that this section works within the context of real spaces. The first important ingredient is a result on separation of Lindelöf sets in Tychonoff spaces.

Lemma 2.6.1. *Let X_1 and X_2 be disjoint Lindelöf sets in a Tychonoff space X. Assume that there is no set $G \subset X$ satisfying $X_1 \subset G \subset X \setminus X_2$ which is a countable intersection of cozero sets. Then there exists a nonempty closed set $H \subset X$ with $\overline{H \cap X_1} = \overline{H \cap X_2} = H$.*

Proof. See [43, Proposition 11]. □

The following lemma is a kind of a selection result.

Lemma 2.6.2. *Let $\varphi : X \to Y$ be a continuous surjective mapping of a compact space X onto a compact space Y and let $f : X \to \mathbb{R}$ be a bounded $\Sigma_\alpha(\mathrm{Bos}(X))$-measurable function for some $\alpha \in [2, \omega_1)$. Then there exists a mapping $\phi : Y \to X$ such that*

- *$\varphi(\phi(y)) = y$, $y \in Y$,*

- *$f \circ \phi$ is a $\Sigma_\alpha(\mathrm{Bos}(Y))$-measurable function.*

Proof. Given a bounded $\Sigma_\alpha(\mathrm{Bos}(X))$-measurable function f on X, using a standard approximation technique and [79, Proposition 2.3(f)] (see also [58, Lemma 5.7]) we construct a bounded sequence (f_n) of $\Sigma_\alpha(\mathrm{Bos}(X))$-measurable simple functions uniformly converging to f. More precisely, each f_n is of the form

$$f_n = \sum_{k=1}^{k_n} c_{nk}\chi_{A_{nk}}, \quad c_{nk} \in \mathbb{R}, A_{nk} \in \Delta_\alpha(\mathrm{Bos}(X)) \text{ for } k = 1, \dots, k_n,$$

where the family $\{A_{nk} : k = 1, \dots, k_n\}$ is a disjoint cover of X. For every set A_{nk} we consider a countable family $\mathcal{A}_{nk} \subset \mathrm{Bos}(X)$ satisfying $A_{nk} \in \Sigma_\alpha(\mathcal{A}_{nk})$. We include all these families in a single family $\mathcal{A}$.

By [37, Lemma 8], there exists a mapping $\phi : Y \to X$ such that $\varphi(\phi(y)) = y$ for every $y \in Y$ and $\phi^{-1}(A) \in \mathrm{Bos}(Y)$ for every $A \in \mathcal{A}$. Then both $\phi^{-1}(A_{nk})$ and $\phi^{-1}(X \setminus A_{nk})$ are in $\Sigma_\alpha(\mathrm{Bos}(Y))$ for every set A_{nk}. Thus the functions $f_n \circ \phi$ are $\Sigma_\alpha(\mathrm{Bos}(Y))$-measurable and consequently, since they converge uniformly to $f \circ \phi$, the function $f \circ \phi$ is $\Sigma_\alpha(\mathrm{Bos}(Y))$-measurable as well. □

The next assertion provides an inductive step needed in the proof of Theorem 2.6.4.

Lemma 2.6.3. *Let X be a compact convex set with $\operatorname{ext} X$ being a resolvable Lindelöf set and $f : X \to \mathbb{R}$ be a strongly affine function such that $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X)$ for some $\alpha \in [1, \omega_0)$. Let $K \subset X$ be a nonempty compact set and $\varepsilon > 0$. Then there exists a nonempty open set U in K and a $\Sigma_{\alpha+1}(\mathrm{Hs}(U))$-measurable function g on U such that $|g - f| < \varepsilon$ on U.*

Proof. Without loss of generality we assume that $0 \le f \le 1$. Let K be a compact set in X and $\varepsilon > 0$. By Lemma 2.4.7, there exists a Baire set $B \supset \operatorname{ext} X$ such that $f \in \mathcal{C}_\alpha(B)$. We claim that there exists a G_δ set G with

$$X \setminus B \subset G \subset X \setminus \operatorname{ext} X. \tag{2.3}$$

Indeed, if there were no such set, Lemma 2.6.1 applied to $X_1 = X \setminus B$ and $X_2 = \operatorname{ext} X$ (observe that $X \setminus B$ is Lindelöf since it is a Baire set; see [70, Theorem 2.7.1]) would provide a nonempty closed set $H \subset X$ satisfying $\overline{H \cap (X \setminus B)} = \overline{H \cap \operatorname{ext} X} = H$. But this would contradict the fact that $\operatorname{ext} X$ is a resolvable set.

We pick a G_δ set G satisfying (2.3) and write $F = X \setminus G = \bigcup F_n$, where the sets $F_1 \subset F_2 \subset \cdots$ are closed in X. Then $\operatorname{ext} X \subset \bigcup F_n \subset B$.

For each $n \in \mathbb{N}$, we set

$$\begin{aligned} M_n &= \{\mu \in \mathcal{M}^1(X) \colon \mu(F_n) \ge 1 - \tfrac{\varepsilon}{2}\} \quad \text{and} \\ X_n &= \{x \in X \colon \text{there exists } \mu \in M_n \text{ such that } r(\mu) = x\} \ (= r(M_n)). \end{aligned}$$

Then each X_n is a closed set by the upper semicontinuity of the function $\mu \mapsto \mu(F_n)$ on $\mathcal{M}^1(X)$ and $X = \bigcup X_n$. Indeed, for any $x \in X$ there exists a maximal measure $\mu \in \mathcal{M}_x(X)$, which is carried by F (see [1, Corollary I.4.12 and the subsequent remark] or [58, Theorem 3.79]), and thus $\mu(F_n) \ge 1 - \frac{\varepsilon}{2}$ for $n \in \mathbb{N}$ large enough.

Since $K \subset \bigcup X_n$, by the Baire category theorem there exists $m \in \mathbb{N}$ such that $X_m \cap K$ has nonempty interior in K. Let U denote this interior. Since $f|_{F_m} \in \mathcal{C}_\alpha(F_m)$, we can extend $f|_{F_m}$ to a function $h \in \mathcal{C}_\alpha(X)$ satisfying $h(X) \subset \overline{\text{co}} f(F_m)$ (see [76, Corollary 3.5] or [58, Corollary 11.25]). Let the functions $\widetilde{h}, \widetilde{f} \colon \mathcal{M}^1(X) \to \mathbb{R}$ be defined as

$$\widetilde{h}(\mu) = \mu(h),\ \widetilde{f}(\mu) = \mu(f), \quad \mu \in \mathcal{M}^1(X).$$

Then

$$|\widetilde{f}(\mu) - \widetilde{h}(\mu)| < \varepsilon, \quad \mu \in M_m. \tag{2.4}$$

Indeed, for $\mu \in M_M$ we have

$$\begin{aligned} |\mu(f) - \mu(h)| &= \left| \int_{F_m} (f-h)\, d\mu + \int_{X \setminus F_m} (f-h)\, d\mu \right| \\ &\leq \int_{X \setminus F_m} |h - f|\, d\mu \leq \mu(X \setminus F_m) \leq \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

By Lemma 2.3.3(c), $\widetilde{h} \in \mathcal{C}_\alpha(\mathcal{M}^1(X))$, and thus it is $\Sigma_{\alpha+1}(\text{Bos}(\mathcal{M}^1(X)))$-measurable on $\mathcal{M}^1(X)$.

We consider the mapping $r : M_m \to r(M_m)$ and use Lemma 2.6.2 to find a selection $\phi : r(M_m) \to M_m$ such that

- $r(\phi(x)) = x$, $x \in r(M_m)$,
- $\widetilde{h} \circ \phi$ is $\Sigma_{\alpha+1}(\text{Bos}(r(M_m)))$-measurable on $r(M_m)$.

By setting $g = \widetilde{h} \circ \phi$ we obtain the desired function. Indeed, for a given point $x \in r(M_m)$, the measure $\phi(x)$ is contained in $\mathcal{M}_x(X) \cap M_m$, and hence by (2.4) and the strong affinity of f, we have

$$|g(x) - f(x)| = |\widetilde{h}(\phi(x)) - \widetilde{f}(\phi(x))| < \varepsilon.$$

Thus the function $g|_U$ is the required one because $\Sigma_{\alpha+1}(\text{Bos})$-measurability implies $\Sigma_{\alpha+1}(\text{Hs})$-measurability. □

Theorem 2.6.4. *Let X be a compact convex set with $\operatorname{ext} X$ being a resolvable Lindelöf set. Let $f : X \to \mathbb{R}$ be a strongly affine function such that $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X)$ for some $\alpha \in [1, \omega_1)$. Then $f \in \mathcal{C}_\alpha(X)$.*

Proof. Given such a function f, we assume that $0 \leq f \leq 1$. Also we may assume that $\alpha \in [1, \omega_0)$ since other cases are covered by Theorem 2.5.2. We claim that f is $\Sigma_{\alpha+1}(\text{Hs}(X))$-measurable.

To this end, let $\varepsilon > 0$ be arbitrary. We construct a regular sequence $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_\kappa = X$ and functions

$$g_\gamma \in \Sigma_{\alpha+1}(\text{Hs}(U_{\gamma+1} \setminus U_\gamma)), \quad \gamma < \kappa,$$

satisfying $|g - f| < \varepsilon$ on $U_{\gamma+1} \setminus U_\gamma$ as follows.

Let $U_0 = \emptyset$. Using Lemma 2.6.3 we select a nonempty open set U of X along with a $\Sigma_{\alpha+1}(\mathrm{Hs}(U))$-measurable function g on U with $|g - f| < \varepsilon$ on U. We set $U_1 = U$ and $g_0 = g$.

Assume now that U_δ and g_δ are chosen for all δ less then some γ. If γ is limit, we set $U_\gamma = \bigcup_{\delta<\gamma} U_\delta$.

Let $\gamma = \lambda + 1$. If $U_\lambda = X$, we set $\kappa = \lambda$ and stop the procedure. Otherwise we apply Lemma 2.6.3 to $K = X \setminus U_\lambda$ and obtain an open set $U \subset X$ intersecting K along with a $\Sigma_{\alpha+1}(\mathrm{Hs}(U \cap K))$-measurable function g on $U \cap K$ satisfying $|g - f| < \varepsilon$ on $U \cap K$. We set $U_\gamma = U_\lambda \cup U$ and $g_\lambda = g$. This finishes the construction.

Let $g : X \to \mathbb{R}$ be defined as $g = g_\gamma$ on $U_{\gamma+1} \setminus U_\gamma$, $\gamma < \kappa$. By Proposition 2.2.2, g is a $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$-measurable function.

By the procedure above we can approximate uniformly f by $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$-measurable functions which yields that f itself is $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$-measurable. But f is a Baire function by Proposition 2.4.8. Thus Theorem 5.2 and Corollary 5.5 in [79] imply $f \in \mathcal{C}_\alpha(X)$. This finishes the proof. □

2.7 Proofs of the main results

Before proving the main results we recall a simple observation.

Lemma 2.7.1. *Let E be a complex Banach space and let $f \in E^{**}$. Then f is strongly affine on B_{E^*} if and only if $\operatorname{Re} f$ is strongly affine on B_{E^*}.*

Proof. If f is strongly affine on B_{E^*} and $\mu \in \mathcal{M}^1(B_{E^*})$ has x^* as its barycenter, then
$$\operatorname{Re} f(x^*) + i \operatorname{Im} f(x^*) = f(x^*) = \mu(f) = \mu(\operatorname{Re} f) + i\mu(\operatorname{Im} f),$$
and thus $\mu(\operatorname{Re} f) = \operatorname{Re} f(x^*)$ and $\mu(\operatorname{Im} f) = \operatorname{Im} f(x^*)$.

Conversely, assuming that $\operatorname{Re} f$ is strongly affine on B_{E^*}, we infer that so is $\operatorname{Im} f$. To see this, consider the affine surjective homeomorphic mapping $\varphi : B_{E^*} \to B_{E^*}$ defined as
$$\varphi(y^*) = iy^*, \quad y^* \in B_{E^*}.$$
Since $\operatorname{Im} f(y^*) = -\operatorname{Re} f(iy^*)$ for $y^* \in E^*$, the function $\operatorname{Im} f$ is a composition of an affine homeomorphism and a strongly affine function, and hence it is strongly affine as well. Thus, for $\mu \in \mathcal{M}^1(B_{E^*})$ with the barycenter x^*,
$$\mu(f) = \mu(\operatorname{Re} f) + i\mu(\operatorname{Im} f) = \operatorname{Re} f(x^*) + i \operatorname{Im} f(x^*) = f(x^*),$$
and f is strongly affine. □

Proofs of Theorems 2.1.1, 2.1.2 and 2.1.3. We proceed to the proofs of Theorems 2.1.1, 2.1.2 and 2.1.3. Let E be a (real or complex) Banach space and f be an element of E^{**} whose restriction to B_{E^*} is strongly affine. By forgetting in E^* the multiplication by complex numbers, we can regard B_{E^*} to be a compact convex set in a real locally convex space. The function $\operatorname{Re} f$ is then a strongly affine function on a compact convex set B_{E^*} that inherits all descriptive properties from f. Thus if $f|_{\overline{\operatorname{ext} B_{E^*}}} \in \mathrm{Hf}_\alpha(\overline{\operatorname{ext} B_{E^*}})$, then $\operatorname{Re} f$ is a strongly affine real-valued function with $\operatorname{Re} f|_{\overline{\operatorname{ext} B_{E^*}}} \in \mathrm{Hf}_\alpha(\overline{\operatorname{ext} B_{E^*}})$. An application of Theorem 2.3.4 gives $\operatorname{Re} f \in \mathrm{Hf}_\alpha(B_{E^*})$. Then both $\operatorname{Re} f$ and $\operatorname{Im} f$ are in $\mathrm{Hf}_\alpha(B_{E^*})$, and thus $f = \operatorname{Re} f + i \operatorname{Im} f$ is in $\mathrm{Hf}_\alpha(B_{E^*})$. Similarly we prove the other assertions of Theorem 2.1.1.

Apparently, this procedure also verifies Theorems 2.1.2 and 2.1.3, which finishes their proof. □

Proof of Theorem 2.1.4. Now we prove Theorem 2.1.4. From now on we will be working with real spaces. We start with the following assertion which shows the required result for Banach spaces of continuous affine functions on simplices. The general result will be then obtained by means of a result of W. Lusky in [61].

Proposition 2.7.2. *Let $f : X \to \mathbb{R}$ be a strongly affine function on a simplex X such that $f \in \mathcal{C}_\alpha(X)$ for some $\alpha \geq 2$. Then*

$$f \in \begin{cases} \mathfrak{A}_{\alpha+1}(X), & \alpha \in [2, \omega_0), \\ \mathfrak{A}_\alpha(X), & \alpha \in [\omega_0, \omega_1). \end{cases}$$

If, moreover, $\operatorname{ext} X$ *is a Lindelöf resolvable set, then $f \in \mathfrak{A}_\alpha(X)$.*

Proof of Proposition 2.7.2. If X is a general simplex, the assertion for finite ordinals is proved in [11, Théorème 2], for infinite ordinals in [42, Theorem 1.2].

Assume now that X is a simplex with $\operatorname{ext} X$ being a Lindelöf resolvable set. For each $x \in X$, let δ_x denote the unique maximal measure in $\mathcal{M}_x(X)$. By [81, Theorem 1] and [58, Theorem 4.24], the function $Tg(x) = \delta_x(g)$, $x \in X$, is in $\mathfrak{A}_1(X)$ for any bounded $g \in \mathcal{C}_1(X)$. By induction, $Tg \in \mathfrak{A}_\beta(X)$ for any bounded function $g \in \mathcal{C}_\beta(X)$ and finite ordinal $\beta \in [2, \omega_0)$. Thus, for any $\alpha \in [2, \omega_0)$ and a strongly affine function $f \in \mathcal{C}_\alpha(X)$, $f = Tf \in \mathfrak{A}_\alpha(X)$. This finishes the proof. □

Let E be a real L_1-predual and $f \in E^{**}$ be a strongly affine function satisfying $f \in \mathcal{C}_\alpha(B_{E^*})$ for some $\alpha \in [2, \omega_1)$. By [61, Theorem], there exist a simplex X, an isometric embedding $j : E \to \mathfrak{A}^c(X)$ and a projection $P : \mathfrak{A}^c(X) \to j(E)$ of norm 1. Further, it is proved in [61, Corollary III] that there exists an affine continuous surjection $\varphi : X \to B_{E^*}$ such that

(1) $\varphi(\operatorname{ext} X) = \operatorname{ext} B_{E^*} \cup \{0\}$ and $\varphi^{-1}(\operatorname{ext} B_{E^*}) \subset \operatorname{ext} X$,

(2) $\varphi|_{\operatorname{ext} X}$ is injective,

(3) $\operatorname{ext} X \setminus \varphi^{-1}(\operatorname{ext} B_{E^*})$ is a singleton,

(4) $j(e)(x) = (e \circ \varphi)(x)$, $e \in E$, $x \in X$.

(In the notation of [61], the embedding j is denoted by T and φ is denoted by q. Conditions (1), (2) and (3) are explicitly stated in [61, Corollary III], condition (4) follows from the definitions of T on p. 175 and q on p. 176.)

The projection P provides for each $x \in X$ a measure $\mu_x \in B_{\mathcal{M}(X)}$ such that

$$Pg(x) = \mu_x(g), \quad g \in \mathfrak{A}^c(X). \tag{2.5}$$

Since P is identity on $j(E)$, we obtain from (4)

$$\mu_x(e \circ \varphi) = (e \circ \varphi)(x), \quad x \in X, e \in E.$$

We use equality (2.5) to extend the domain of P to any bounded universally measurable function on X.

We claim that

$$\mu_x(f \circ \varphi) = f(\varphi(x)), \quad x \in X. \tag{2.6}$$

To verify this, let $x \in X$ be given. We write

$$\mu_x = a_1\mu_1 - a_2\mu_2, \quad a_1, a_2 \geq 0 \text{ with } a_1 + a_2 \leq 1,\ \mu_1, \mu_2 \in \mathcal{M}^1(X),$$

and let $x_1, x_2 \in X$ be the barycenters of μ_1 and μ_2, respectively. Then

$$\varphi(x) = a_1\varphi(x_1) - a_2\varphi(x_2). \tag{2.7}$$

Indeed, let $e \in E$ be arbitrary. The we compute

$$\begin{aligned} e(\varphi(x)) &= \mu_x(e \circ \varphi) = a_1\mu_1(e \circ \varphi) - a_2\mu_2(e \circ \varphi) \\ &= a_1 e(\varphi(x_1)) - a_2 e(\varphi(x_2)) \\ &= e(a_1\varphi(x_1) - a_2\varphi(x_2)). \end{aligned}$$

Hence (2.7) holds.

Since $f \circ \varphi$ is strongly affine on X by [78, Lemma 2.3] (see also [58, Proposition 5.29]), we get from (2.7)

$$\begin{aligned} \mu_x(f \circ \varphi) &= a_1\mu_1(f \circ \varphi) - a_2\mu_2(f \circ \varphi) = a_1 f(\varphi(x_1)) - a_2 f(\varphi(x_2)) \\ &= f(a_1\varphi(x_1) - a_2\varphi(x_2)) = f(\varphi(x)). \end{aligned}$$

This verifies (2.6).

Now we prove by induction that $Pg \in (j(E))_\beta$ provided $g \in \mathfrak{A}_\beta(X)$ for some $\beta \geq 1$. First consider the case $\beta = 1$, i.e., there exists a bounded sequence (g_n) in $\mathfrak{A}^c(X)$ with $g_n \to g$. Then $Pg_n \in j(E)$ and, by the Lebesgue dominated convergence theorem, $Pg_n \to Pg$.

Assuming the validity of the assertion for all ordinals $\widetilde{\beta}$ smaller then some β, we consider $g \in \mathfrak{A}_\beta(X)$. Let (g_n) be a bounded sequence converging pointwise to g, where $g_n \in \mathfrak{A}_{\beta_n}(X)$ for some $\beta_n < \beta$. Then $Pg_n \in (j(E))_{\beta_n}$ and, as above, $Pg_n \to Pg$.

Now we get back to the function f. Since $f \circ \varphi \in \mathcal{C}_\alpha(X)$, Proposition 2.7.2 implies that the function $f \circ \varphi$ belongs to $\mathfrak{A}_\beta(X)$, where either $\beta = \alpha + 1$ if $\alpha < \omega_0$ or $\beta = \alpha$ otherwise. By the reasoning above and (2.6),

$$f \circ \varphi = P(f \circ \varphi) \in (j(E))_\beta.$$

Since $j(e) = e \circ \varphi$ for each $e \in E$, it follows that $f \in \mathfrak{A}_\beta(B_{E^*})$. This concludes the proof of the first part of the theorem.

If, moreover, we assume that $\operatorname{ext} B_{E^*}$ is a Lindelöf resolvable set, we observe that $\operatorname{ext} X$ is a Lindelöf resolvable set as well. To show this, we first notice that $\operatorname{ext} X$ differs from the resolvable set $\varphi^{-1}(\operatorname{ext} B_{E^*})$ by a singleton (see (1) and (3)), and thus it is a resolvable set. Second, let $F \subset X \setminus \operatorname{ext} X$ be a compact set. By (1), $\varphi(F)$ is disjoint from $\operatorname{ext} B_{E^*}$. Since $\operatorname{ext} B_{E^*}$ is Lindelöf, [81, Lemma 14] provides an F_σ set A with

$$\operatorname{ext} B_{E^*} \subset A \subset B_{E^*} \setminus \varphi(F).$$

If $x_0 \in X$ denotes the singleton $\operatorname{ext} X \setminus \varphi^{-1}(\operatorname{ext} B_{E^*})$, then $\varphi^{-1}(A)$ is an F_σ set in X satisfying

$$\operatorname{ext} X \subset \varphi^{-1}(A) \cup \{x_0\} \subset X \setminus F.$$

By [81, Lemma 15], $\operatorname{ext} X$ is a Lindelöf space.

Now we can conclude the proof as in the first part, the only difference is that we use the second part of Proposition 2.7.2. □

2.8 Examples

Banach spaces constructed in this section are real L_1-preduals and they are created using a notion of a *simplicial function space*. In order to illuminate the construction, we need to recall several definitions and facts.

If K is a compact topological space, $\mathcal{H} \subset \mathcal{C}(K)$ is a *function space* if $\mathcal{H}$ is a subspace of $\mathcal{C}(K)$, contains constant functions and separate points of K. For the sake of simplicity, we will construct real Banach spaces, and thus we will deal in this section only with real spaces $\mathcal{C}(K)$. For $x \in K$, we write $\mathcal{M}_x(\mathcal{H})$ for the set of all measures $\mu \in \mathcal{M}^1(K)$ with $\mu(h) = h(x)$ for all $h \in \mathcal{H}$. Let $\operatorname{Ch}_{\mathcal{H}}(K)$ be the *Choquet boundary* of $\mathcal{H}$, i.e., the set of those points $x \in K$ with $\mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}$. By defining $\mathcal{A}^c(\mathcal{H}) = \{f \in \mathcal{C}(K) \colon \mu(f) = f(x), \text{ for each } x \in K, \mu \in \mathcal{M}_x(\mathcal{H})\}$ we obtain a closed function space satisfying $\mathcal{H} \subset \mathcal{A}^c(\mathcal{H})$ (see [58, Definition 3.8]) and $\operatorname{Ch}_{\mathcal{H}}(K) = \operatorname{Ch}_{\mathcal{A}^c(\mathcal{H})}(K)$ (this follows easily from the definitions).

Let

$$\mathbf{S}(\mathcal{H}) = \{s \in \mathcal{H}^* \colon s \geq 0, \|s\| = 1\}$$

denote the *state space* of $\mathcal{H}$. Then $\mathbf{S}(\mathcal{H})$, endowed with the weak* topology, is a compact convex set and K is homeomorphically embedded in $\mathbf{S}(\mathcal{H})$ via the mapping $\phi : K \to \mathbf{S}(\mathcal{H})$ assigning to each $x \in K$ the point evaluation at x. Moreover, $\phi(\operatorname{Ch}_{\mathcal{H}}(K)) = \operatorname{ext} \mathbf{S}(\mathcal{H})$ (see [67, Proposition 6.2] or [58, Proposition 4.26]).

The function space $\mathcal{H}$ is called *simplicial* if $\mathbf{S}(\mathcal{A}^c(\mathcal{H}))$ is a simplex (see [58, Theorem 6.54]).

Further, let $\mathcal{H}^{\perp\perp}$ denote the space of all universally measurable functions $f : K \to \mathbb{R}$ satisfying $\mu(f) = 0$ for every $\mu \in \mathcal{H}^\perp \subset \mathcal{M}(K)$. It is proved in [78, Theorem 2.5] (see also [58, Corollary 5.41]) that for any function $f \in \mathcal{H}^{\perp\perp}$ there exists a strongly affine function $\widetilde{f} : \mathbf{S}(\mathcal{H}) \to \mathbb{R}$ with $f = \widetilde{f} \circ \phi$. Moreover, the function $\widetilde{f}$ inherits from f all descriptive properties considered in the paper, precisely, for any $\alpha \in [1, \omega_1)$ we have $f \in \mathcal{C}_\alpha(K)$, $f \in \operatorname{Bof}_\alpha(K)$ and $f \in \operatorname{Hf}_\alpha(K)$ if and only if $\widetilde{f} \in \mathcal{C}_\alpha(\mathbf{S}(\mathcal{H}))$, $\widetilde{f} \in \operatorname{Bof}_\alpha(\mathbf{S}(\mathcal{H}))$ and $\widetilde{f} \in \operatorname{Hf}_\alpha(\mathbf{S}(\mathcal{H}))$, respectively (the first two assertions are proved in [58, Corollary 5.41], the last one follows from Theorem 2.3.4).

A standard construction from [10, Section VII] of a simplicial function space $\mathcal{H}$ satisfying $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$ goes as follows. Take a compact space L, a subset B of L and define

$$K = (L \times \{0\}) \cup (B \times \{-1, 1\})$$

with the "porcupine topology", i.e., points of $K \setminus (L \times \{0\})$ are isolated and a point $(x, 0) \in K$ has a basis of neighborhoods consisting of sets of the form

$$K \cap (U \times \{-1, 0, 1\}) \setminus F,$$

where $U \subset L$ is a neighborhood of x and $F \subset K \setminus (L \times \{0\})$ is finite. Then K is a compact space and

$$\mathcal{H} = \{f \in \mathcal{C}(K)\colon f(x,0) = \frac{1}{2}(f(x,1) + f(x,-1)), x \in B\}$$

is a simplicial function space satisfying $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$ and

$$\mathrm{Ch}_{\mathcal{H}}(K) = K \setminus (B \times \{0\})$$

(for the verifications of these facts see [82] or [58, Definition 6.13 and Lemma 6.14]).

If $f : K \to \mathbb{R}$ is a bounded universally measurable function satisfying $f(x,0) = \frac{1}{2}(f(x,1) + f(x,-1))$ for each $x \in B$, it is easy to verify that $f \in \mathcal{H}^{\perp\perp}$ (see [58, Corollary 6.12]), and thus it induces a strongly affine function $\widetilde{f} : \mathbf{S}(\mathcal{H}) \to \mathbb{R}$ which satisfies $f = \widetilde{f} \circ \phi$ and shares with f all descriptive properties.

By this procedure we obtain a simplex $X = \mathbf{S}(\mathcal{H})$ and a strongly affine function on X with the desired descriptive properties. It is well known (see e.g. [58, Propositions 4.31 and 4.32]) that, given a compact convex set X, the dual space $(\mathfrak{A}^c(X))^*$ can be identified with $\operatorname{span} X$ and the dual unit ball with $\operatorname{co}(X \cup (-X))$, whereas the second dual $(\mathfrak{A}^c(X))^{**}$ equals to the space of all affine bounded functions on X. Hence the construction of a simplex X along with a strongly affine function f with the prescribed descriptive properties yields the resulting L_1-predual E: we set $E = \mathfrak{A}^c(X)$ and the element $x^{**} \in E^{**}$ is the function f.

This general construction is now used in the following examples.

Example 2.8.1. *There exist a separable L_1-predual E and a strongly affine function $f \in E^{**}$ such that $f|_{\operatorname{ext} B_{E^*}} \in \mathcal{C}_1(\operatorname{ext} B_{E^*})$ and $f \notin \mathcal{C}_1(B_{E^*})$.*

Proof. Let $L = [0,1]$ and B denote the set of all rational numbers in L. Let K, $\mathcal{H}$ and X be constructed as above. Then K is metrizable, and thus $E = \mathfrak{A}^c(X)$ is a separable space. Let $f : K \to \mathbb{R}$ be defined as

$$f(x,t) = \begin{cases} 1, & x \in B, \\ 0, & x \notin B, \end{cases} \qquad (x,t) \in K.$$

Then $f|_{\mathrm{Ch}_{\mathcal{H}}(K)} \in \mathcal{C}_1(\mathrm{Ch}_{\mathcal{H}}(K))$ since $f|_{\mathrm{Ch}_{\mathcal{H}}(K)}$ is the characteristic function of an open set in $\mathrm{Ch}_{\mathcal{H}}(K)$. On the other hand, f has no point of continuity on $L \times \{0\}$, and thus $f \notin \mathcal{C}_1(K)$. $\square$

Example 2.8.2. *There exist an L_1-predual E and a strongly affine function $f \in E^{**}$ such that $\operatorname{ext} B_{E^*}$ is an open set in $\overline{\operatorname{ext} B_{E^*}}$ (hence $\operatorname{ext} B_{E^*} \in \operatorname{Bos}(B_{E^*})$), $f|_{\operatorname{ext} B_{E^*}} \in \mathcal{C}(\operatorname{ext} B_{E^*})$ and f is not resolvably measurable on B_{E^*}.*

Proof. Let $L = B = [0,1]$ and A be an analytic non-Borel set in L (see [45, Theorem 14.2]) and let K, $\mathcal{H}$ and X be constructed as above. Then $\mathrm{Ch}_{\mathcal{H}}(K) = K \setminus (L \times \{0\})$ is an open set in $\overline{\mathrm{Ch}_{\mathcal{H}}(K)} = K$. Further, let $f : K \to \mathbb{R}$ be defined as

$$f(x,t) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases} \qquad (x,t) \in K.$$

Then $f|_{\mathrm{Ch}_{\mathcal{H}}(K)} \in \mathcal{C}(\mathrm{Ch}_{\mathcal{H}}(K))$ since $f|_{\mathrm{Ch}_{\mathcal{H}}(K)}$ is the characteristic function of a clopen set in $\mathrm{Ch}_{\mathcal{H}}(K)$. Since A is μ-measurable for any Radon measure μ on $[0,1]$, f is universally measurable on K (see [45, Theorem 21.10]). Obviously, $f|_{L\times\{0\}}$ is not Borel on $L \times \{0\}$. Since the σ-algebra of Borel sets in L coincides with the σ-algebra generated by resolvable sets in L (see [79, Proposition 3.4]), f is not measurable on K with respect to the σ-algebra generated by resolvable sets. □

Example 2.8.3. *Assuming (CH), there exist an L_1-predual E with $\operatorname{ext} B_{E^*}$ Lindelöf and a strongly affine function $f \in E^{**}$ such that f is not a resolvably measurable function and $f|_{\operatorname{ext} B_{E^*}} \in \mathrm{Bof}_1(\operatorname{ext} B_{E^*})$.*

Proof. Let $L = [0,1]$ and Q stand for the set of all rational numbers in L. Assuming the continuum hypothesis, by the method of the proof of [64, Proposition 4.9] we construct an uncountable set B disjoint from Q that concentrates around the set Q (i.e., the set $B \setminus U$ is countable for any open set $U \supset Q$). Let K, $\mathcal{H}$ and X be as above. Then $\mathrm{Ch}_{\mathcal{H}}(K) = K \setminus (B \times \{0\})$ is Lindelöf. Indeed, if $\mathcal{U}$ is an open cover of $\mathrm{Ch}_{\mathcal{H}}(K)$, we select a countable family $\mathcal{V} \subset \mathcal{U}$ satisfying

$$(L \times \{0\}) \setminus (B \times \{0\}) \subset V = \bigcup \{U \cap (L \times \{0\}) \colon U \in \mathcal{V}\}.$$

Then V is an open set in $L \times \{0\}$ containing $Q \times \{0\}$, and thus $B \setminus V$ is countable. Hence we may extract a countable family $\mathcal{W} \subset \mathcal{U}$ which covers that part of $\mathrm{Ch}_{\mathcal{H}}(K)$ not already contained in V. Thus $\mathcal{V} \cup \mathcal{W}$ is a countable subcover of $\mathrm{Ch}_{\mathcal{H}}(K)$.

Define a function $f : K \to \mathbb{R}$ by the formula

$$f(x,t) = \begin{cases} 1, & x \in B, \\ 0, & x \notin B, \end{cases} \qquad (x,t) \in K.$$

Then f is universally measurable on K. To see this, it is enough to verify that B is universally measurable. If $\mu \in \mathcal{M}^1([0,1])$ is a continuous measure (i.e., $\mu(\{x\}) = 0$ for each $x \in [0,1]$), let (U_n) be a sequence of open sets satisfying $\mu(U_n) < \frac{1}{n}$ and $U_n \supset Q$. Then $\mu(\bigcap U_n) = 0$ and $B \setminus \bigcap U_n$ is countable, and thus μ-measurable. Hence B is μ-measurable for every continuous measure. Obviously, B is μ-measurable for any discrete probability measure μ, and hence B is universally measurable.

On the other hand, B is not Borel, because otherwise, as an uncountable set, it would contain a copy of the Cantor set (see [45, Theorem 13.6]) which would contradict its concentration around Q.

Since f is the characteristic function of an open set in $\mathrm{Ch}_{\mathcal{H}}(K)$, we have $f|_{\mathrm{Ch}_{\mathcal{H}}(K)} \in \mathrm{Bof}_1(\mathrm{Ch}_{\mathcal{H}}(K))$. On the other hand, f is not Borel on $L \times \{0\}$ because the σ-algebra of Borel sets in L coincides with the σ-algebra generated by resolvable sets in L (see [79, Proposition 3.4]). Thus f is the required function. □

3. Baire classes of L_1-preduals and C^*-algebras

(joint work with Jiří Spurný)

3.1 Introduction

A real (or complex) Banach space X is called an *L_1-predual* (sometimes a *Lindenstrauss space*) if its dual X^* is isometric to a real (or complex) space $L^1(X, \mathcal{S}, \mu)$ for a measure space $(X, \mathcal{S}, \mu)$. Real L_1-preduals were in depth investigated in papers [17], [22], [23], [8], [52], [60], [24], [61], [27], [13] or [16]. The complex variant of L_1-preduals was studied e.g. in [38], [65], [54], [71], [19] or recently in [62]. It has turned out that a real Banach space V is an L_1-predual if and only if its dual unit ball B_{V^*} satisfies a "simplex-like" condition (see [53]). A complex version of this "simplex-like" characterization was provided by Effros in [18]. It is mentioned in this paper that "we have reason to believe that this result will make theory of complex Lindenstrauss spaces as accessible as that for real spaces".

The goal of our paper is to support this belief by results on real and complex L_1-predual spaces and their Baire classes. The significance of Effros's characterization becomes apparent especially from the comparision of Sections 3.2 and 3.3 of the paper in hand.

Let $\mathbb{F}$ denote the field $\mathbb{R}$ or $\mathbb{C}$.

For a topological space K, let $\mathcal{B}(K, \mathbb{F})$ be the space of all Borel functions with values in $\mathbb{F}$ and $\mathcal{B}^b(K, \mathbb{F})$ be the space of all bounded Borel functions on K with values in $\mathbb{F}$. For a compact (Hausdorff) topological space K, let $\mathcal{C}(K, \mathbb{F})$ stand for the space of all continuous functions on K with values in $\mathbb{F}$. In case K is compact, we write $\mathcal{M}(K, \mathbb{F})$ for the space of Radon measures on K and $\mathcal{M}^1(K)$ for Radon probability measures on K.

Let $\mathcal{H}$ be a subset of $\mathcal{C}(K, \mathbb{F})$. Then we set $\mathcal{B}^0(\mathcal{H}) = \mathcal{H}$ and, for $\alpha \in (0, \omega_1)$, let $\mathcal{B}^\alpha(\mathcal{H})$ consist of all pointwise limits of elements from $\bigcup_{\beta<\alpha} \mathcal{B}^\beta(\mathcal{H})$. Further, we denote by $\mathcal{B}^{\alpha,b}(\mathcal{H})$ the set of all bounded elements from $\mathcal{B}^\alpha(\mathcal{H})$. The symbol $\mathcal{B}^{\alpha,bb}(\mathcal{H})$ denotes the inductive families created by means of pointwise limits of bounded sequences of lower classes.

If we start the inductive procedure from the space of all continuous functions, we write simply $\mathcal{B}^\alpha(K, \mathbb{F})$ and $\mathcal{B}^{\alpha,b}(K, \mathbb{F})$ for the obtained spaces of Baire-α functions. Then we have $\mathcal{B}^{\alpha,b}(K, \mathbb{F}) = \mathcal{B}^{\alpha,bb}(K, \mathbb{F})$. Let us remind that for a metrizable K holds $\mathcal{B}^b(K, \mathbb{F}) = \bigcup_{\alpha<\omega_1} \mathcal{B}^{\alpha,b}(K, \mathbb{F})$. Having started with the space $\mathcal{A}(K, \mathbb{F})$ of all continuous affine functions on a compact convex set K in a locally convex space, we obtain spaces $\mathcal{A}^\alpha(K, \mathbb{F})$, $\mathcal{A}^{\alpha,b}(K, \mathbb{F})$ and $\mathcal{A}^{\alpha,bb}(K)$. As a consequence of the uniform boundedness principle we get $\mathcal{A}^{\alpha,bb}(K, \mathbb{F}) = \mathcal{A}^{\alpha,b}(K, \mathbb{F}) = \mathcal{A}^\alpha(K, \mathbb{F})$ (see e.g. [58, Lemma 5.36]) and elements of this set we call *functions of affine class α*.

If X is a Banach space over $\mathbb{F}$ and B_{X^*} is its dual unit ball endowed with the weak* topology, X is isometrically embedded in $\mathcal{C}(B_{X^*}, \mathbb{F})$ via the canonical embedding. We recall definitions of Baire classes of X^{**} from [5]. For $\alpha \in [0, \omega_1)$, we call $\mathcal{B}^\alpha(X)$ the *intrinsic α-Baire class of X^{**}*. Following [5, p. 1044] we denote the intrinsic α-th Baire class as X^{**}_α. Let us remark, that our definition is formally slightly different from the one introduced in [5]. While in our case elements of X^{**}_α are restrictions of the uniquely determined elements from X^{**} to the closed unit ball B_{X^*}, the functions considered in [5] are precisely these extensions. This is substantiated by Lemma 3.2.2.

Still considering X as a subspace of $\mathcal{C}(B_{X^*},\mathbb{F})$, *the α-th Baire class* of X^{**} is defined as

$$X^{**}_{\mathcal{B}_\alpha} = \{x^{**} \in X^{\perp\perp};\ x^{**}|_{B^*_X} \in \mathcal{B}^\alpha(B_{X^*},\mathbb{F})\}.$$

It can be verified that $x^{**} \in X^{**}_{\mathcal{B}_\alpha}$ if and only if $x^{**}|_{B_{X^*}} \in \mathcal{B}^\alpha(B_{X^*},\mathbb{F})$ and $x^{**}|_{B_{X^*}}$ satisfies the barycentric calculus, i.e.,

$$x^{**}\left(\int_{B_{X^*}} \mathrm{id}\ d\mu\right) = \int_{B_{X^*}} x^{**}\, d\mu$$

for every probability measure $\mu \in \mathcal{M}^1(B_{X^*})$. Where no confusion can arise, we do not distinguish between $X^{**}_{\mathcal{B}_\alpha}$ and $X^{**}_{\mathcal{B}_\alpha}|_{B_{X^*}}$.

Obviously, $X^{**}_\alpha \subset X^{**}_{\mathcal{B}_\alpha}$ but the converse need not hold by [85, Theorem] (for a detailed exposition on Baire classes of Banach spaces we refer the reader to [5, pp. 1043–1048]).

The first goal of our paper is the extension of the following result by Lindenstrauss and Wulbert proved in [55, Theorem 1]:

Let X be a real L_1-predual and T stand for the closure of extreme points $\operatorname{ext} B_{X^*}$ *of* B_{X^*}. *If $T = \operatorname{ext} B_{X^*}$, then $X = \mathcal{C}_\Sigma(T,\mathbb{R})$, where $\Sigma(x^*) = -x^*$, $x^* \in T$ and $\mathcal{C}_\Sigma(T,\mathbb{R})$ consists of real continuous functions on T satisfying $f(x^*) = -f(-x^*)$.*

We show in Theorem 3.2.10 that for a real L_1-predual X the space X^{**}_α can be identified with the space $\mathcal{B}^{\alpha,b}_\sigma(\operatorname{ext} B_{X^*},\mathbb{R})$ of all odd bounded Baire-α functions in case $\operatorname{ext} B_{X^*}$ is of type F_σ. An analogous result for complex L_1-predual is proved in Theorem 3.3.10.

The second goal of the paper is to extend to the complex setting the following result from [57] (see [57, Theorem 1.4]):

Let X be a real L_1-predual and let $x^{**} \in X^{**}$ satisfy $f = x^{**}|_{B_{X^*}} \in \mathcal{B}^\alpha(B_{X^*},\mathbb{R})$ for $\alpha \in [2,\omega_1)$. Then $f \in X^{**}_{\alpha+1}$ if $\alpha < \omega_0$ and $f \in X^{**}_\alpha$ if $\alpha \geq \omega_0$.

Proposition 3.3.6 fulfils our intentions at least for separable complex L_1-preduals.

Finally, a question posed in [5, p. 1048] asks *whether for a separable C^*-algebra X holds $X^{**}_{\mathcal{B}_\alpha} = X^{**}_\alpha$.* We answer this question in the negative, more precisely we prove using [62] and [78] that there is a separable C^*-algebra X satisfying $X^{**}_{\mathcal{B}_2} \neq X^{**}_2$.

Throughout the paper we work within separable Banach spaces, since our methods are based on the metrizability of their dual unit balls. The question of validity of the presented results for the case of nonseparable spaces is still open.

3.2 Real L_1-preduals

Let K be a compact convex set in a locally convex topological vector space. To a point $x \in K$, we can assign the set $\mathcal{M}^1_x(K)$ consisting of all probability measures on K satisfying $\int_K \mathrm{id}\ d\mu = x$ (equivalently, $\mu(h) = h(x)$ for any continuous affine function h on K). A function f on K is *strongly affine* if f is μ-measurable for each $\mu \in \mathcal{M}^1(K)$ and $f(x) = \mu(f)$ for any $x \in K$ and $\mu \in \mathcal{M}^1_x(K)$. Any strongly affine function is bounded (see e.g. [58, Lemma 4.5]).

The usual dilation order $\prec$ on $\mathcal{M}^1(K)$ is defined as $\mu \prec \nu$ if and only if $\mu(f) \leq \nu(f)$ for any convex continuous function f on K. We write $\mathcal{M}^{\max}(K)$ for the set of all probability measures on K which are maximal with respect to $\prec$. A measure $\mu \in \mathcal{M}(K, \mathbb{F})$ is *boundary* if either $\mu = 0$ or the probability measure $\frac{|\mu|}{\|\mu\|}$ is maximal.

For a function $f \in \mathcal{C}(K, \mathbb{R})$, let

$$\widehat{f}(x) = \sup\{\mu(f);\ \mu \in \mathcal{M}^1_x(K)\}.$$

By the Choquet representation theorem, for any $x \in K$ there exists $\mu \in \mathcal{M}^1_x(K) \cap \mathcal{M}^{\max}(K)$ (see [51, p. 192, Corollary]). The set K is termed *simplex* if this measure is uniquely determined for each $x \in K$ (see [51, § 20, Theorem 3]). In case K is metrizable, maximal measures are carried by the G_δ set $\operatorname{ext} K$ of extreme points of K (see [51, § 20, Theorem 5]).

Let X be a real Banach space. Then $\sigma(x^*) = -x^*$, $x^* \in B_{X^*}$, is a natural affine homeomorphism of B_{X^*} onto itself. A set $B \subset B_{X^*}$ is *symmetric* if $\sigma(B) = B$. An example of a symmetric set is the set $\operatorname{ext} B_{X^*}$. For a function f defined on a symmetric set $B \subset B_{X^*}$ we define

$$(\operatorname{odd} f)(x^*) = \frac{1}{2}(f(x^*) - f(-x^*)), \quad x^* \in B.$$

A function f defined on a symmetric subset is odd if $\operatorname{odd} f = f$.

For $\mu \in \mathcal{M}(B_{X^*}, \mathbb{R})$, let $\operatorname{odd} \mu \in \mathcal{M}(B_{X^*}, \mathbb{R})$ be defined as

$$(\operatorname{odd} \mu)(f) = \mu(\operatorname{odd} f), \quad f \in \mathcal{C}(B_{X^*}, \mathbb{R}).$$

The following characterization of L_1-preduals is due to Lazar (see [53, Theorem] or [51, § 21, Theorem 7])):

Let X be a Banach space. Then X is an L_1-predual if and only if $\operatorname{odd} \mu = \operatorname{odd} \nu$ *for each $x^* \in B_{X^*}$ and $\mu, \nu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$.*

Let X be a real separable L_1-predual and f be a bounded Borel function f defined on a Borel subset of B_{X^*} containing $\operatorname{ext} B_{X^*}$. We define

$$Tf(x^*) = (\operatorname{odd} \mu)(f), \quad x^* \in B_{X^*}, \mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*}). \tag{3.1}$$

Notice that Tf is well defined because of Lazar's characterization and because $\operatorname{odd} \mu$, as a boundary measure, is carried by the G_δ set $\operatorname{ext} B_{X^*}$.

The described mapping T is a natural generalization of a dilation mapping defined in the simplicial case e.g. in [58, Definition 6.7].

Proposition 3.2.1. *Let X be a real separable L_1-predual and T be defined as in* (3.1).

- *(a) If $f \in \mathcal{C}(B_{X^*}, \mathbb{R})$, then Tf is Baire*-1.
- *(b) If $f \in \mathcal{B}^b(\operatorname{ext} B_{X^*}, \mathbb{R})$, then Tf is an odd Borel strongly affine function on B_{X^*}.*

Proof. Since X is separable, B_{X^*} is a metrizable compact convex set (see [72, Theorems 3.15, 3.16]), and thus there exists a mapping $S : B_{X^*} \to \mathcal{M}^{\max}(B_{X^*})$ such that $Sx^* = \nu_{x^*} \in \mathcal{M}^1_{x^*}(B_{X^*})$ and the function $Sf : x^* \mapsto \nu_{x^*}(f)$ is a Baire-1 function on B_{X^*} for each continuous function f on B_{X^*} (see [83, Théoréme 1] or [58, Theorem 11.41]).

(a) Let $f \in \mathcal{C}(B_{X^*}, \mathbb{R})$ be given. Then $\operatorname{odd} f$ is a continuous function on B_{X^*} and, for a fixed $x^* \in B_{X^*}$, we have

$$Tf(x^*) = (\operatorname{odd} \nu_{x^*})(f) = \nu_{x^*}(\operatorname{odd} f) = S(\operatorname{odd} f)(x^*).$$

Thus $Tf = S(\operatorname{odd} f)$ is a Baire-1 function on B_{X^*}.

(b) Let now

$$\mathcal{F} = \{f \in \mathcal{B}^b(B_{X^*}, \mathbb{R});\ Tf \text{ is Borel}\}.$$

Then $\mathcal{F}$ is closed under the taking pointwise limits of bounded sequences by the Lebesgue dominated convergence theorem and contains $\mathcal{C}(B_{X^*}, \mathbb{R})$. Hence it contains any bounded Borel function on B_{X^*}.

Let f be a bounded Borel function on $\operatorname{ext} B_{X^*}$. Since $\operatorname{ext} B_{X^*}$ is a Borel set, we can consider f to be a bounded Borel function on B_{X^*}. Hence $f \in \mathcal{F}$ and Tf is Borel.

Let us show that Tf is strongly affine, i.e., that $\nu(Tf) = Tf(y^*)$ for each $y^* \in B_{X^*}$ and $\nu \in \mathcal{M}^1_{y^*}(B_{X^*})$. Given y^* and ν as above, let $\mu \in \mathcal{M}^1(B_{X^*})$ be defined as

$$\mu(g) = \int_{B_{X^*}} \nu_{x^*}(g)\, d\nu(x^*), \quad g \in \mathcal{C}(B_{X^*}, \mathbb{R}).$$

If g is a convex continuous function and $\widehat{g}$ is its upper envelope, due to Mokobodzki's maximality test (e.g. [51, § 20, Theorem 2]) we have $\nu_{x^*}(g) = \nu_{x^*}(\widehat{g})$, $x^* \in B_{X^*}$, and thus

$$\mu(\widehat{g}) = \int_{B_{X^*}} \nu_{x^*}(\widehat{g})\, d\nu(x^*) = \int_{B_{X^*}} \nu_{x^*}(g)\, d\nu(x^*) = \mu(g).$$

Hence μ is maximal. Further, for an affine continuous function h on B_{X^*} we have

$$\mu(h) = \int_{B_{X^*}} \nu_{x^*}(h)\, d\nu(x^*) = \int_{B_{X^*}} h(x^*)\, d\nu(x^*) = h(y^*),$$

and thus $\mu \in \mathcal{M}^1_{y^*}(B_{X^*})$. Hence $Tf(y^*) = (\operatorname{odd} \mu)(f)$ and it follows that

$$\begin{aligned}\nu(Tf) &= \int_{B_{X^*}} Tf(x^*)\, d\nu(x^*) = \int_{B_{X^*}} \nu_{x^*}(\operatorname{odd} f)\, d\nu(x^*) \\ &= \mu(\operatorname{odd} f) = (\operatorname{odd} \mu)(f) = Tf(y^*).\end{aligned}$$

Hence $\nu(Tf) = Tf(y^*)$ and Tf is strongly affine.

Finally we show that Tf is odd. Since Tf is affine, it is enough to show that $Tf(0) = 0$. Let x^* be an extreme point of B_{X^*}. Then the combination $\mu = \frac{1}{2}(\varepsilon_{x^*} + \varepsilon_{-x^*})$ of the Dirac measures ε_{x^*}, ε_{-x^*} is contained in $\mathcal{M}^1_0(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$. Because $\operatorname{odd} f$ is an odd function,

$$Tf(0) = (\operatorname{odd} \mu)(f) = \mu(\operatorname{odd} f) = \frac{1}{2}((\operatorname{odd} f)(x^*) + (\operatorname{odd} f)(-x^*)) = 0.$$

Hence Tf is odd. $\square$

Lemma 3.2.2. *Let X be a real Banach space and let f be an odd strongly affine function on the closed unit ball B_{X^*}. Then f is a restriction of a uniquely determined element of X^{**}.*

Proof. This simple observation is based on the fact that a strongly affine function f on B_{X^*} is bounded (e.g. [58, Lemma 4.5]). Thus the uniquely defined linear extension of f is an element of X^{**}. □

Proposition 3.2.3. *Let X be a real separable L_1-predual and $f \in \mathcal{B}^b(\operatorname{ext} B_{X^*}, \mathbb{R})$. If h is an odd strongly affine function on B_{X^*} extending f, then $h = Tf$.*

Proof. The function f, being extended by an odd function h, is odd as well.

Let $y^* \in B_{X^*}$ be given. We choose a maximal measure $\mu \in \mathcal{M}^1_{y^*}(B_{X^*})$ and compute

$$Tf(y^*) = (\operatorname{odd} \mu)(f) = \mu(\operatorname{odd} f) = \mu(f) = \int_{\operatorname{ext} B_{X^*}} h(x^*)\, d\mu(x^*) = h(y^*).$$

This concludes the proof. □

Proposition 3.2.4. *Let X be a real separable L_1-predual and assume that $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$.*

(a) If $\alpha \in [1, \omega_0)$, then $Tf \in X^{**}_{\alpha+1}$.

(b) If $\alpha \in [\omega_0, \omega_1)$, then $Tf \in X^{**}_{\alpha}$.

(c) If $\alpha \in [1, \omega_1)$ and $\operatorname{ext} B_{X^*}$ is of type F_σ, then $Tf \in X^{**}_{\alpha}$.

Proof. (a) If $\alpha = 1$, f can be extended to a bounded Baire-1 function on B_{X^*} ([1, Corollary I.4.4] and [49, § 35, VI, Theorem]). Let (f_n) be a bounded sequence in $\mathcal{C}(B_{X^*})$ converging to this extension on B_{X^*}. For a given $x^* \in B_{X^*}$, let $\mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$ be chosen. Then we have

$$Tf_n(x^*) = (\operatorname{odd} \mu)(f_n) \to (\operatorname{odd} \mu)(f) = Tf(x^*).$$

Since $Tf_n = \operatorname{odd} f_n$ on $\operatorname{ext} B_{X^*}$, each Tf_n is a continuous function on $\operatorname{ext} B_{X^*}$. By Proposition 3.2.1 and [57, Theorem 5.2], each Tf_n is an odd Baire-1 strongly affine function on B_{X^*}. By the Mokobodzki theorem ([5, Theorem II.1.2(a)]), $Tf_n \in X^{**}_1$. Hence $Tf \in X^{**}_2$.

The rest of the proof follows by induction.

(b) Let $\alpha = \omega_0$ and $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$. Let (f_n) be a bounded sequence of functions from $\mathcal{B}^{\alpha_n,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$, where $\alpha_n < \alpha$, converging to f. Then $Tf_n \to Tf$ and $Tf_n \in X^{**}_{\alpha_n+1}$ by (a). Hence $Tf \in X^{**}_{\alpha}$. For higher Baire classes the proof follows by transfinite induction.

(c) Let $\operatorname{ext} B_{X^*}$ is of type F_σ. If $f \in \mathcal{B}^{1,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$, then also $\operatorname{odd} f \in \mathcal{B}^{1,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$. Because a restriction of Tf to $\operatorname{ext} B_{X^*}$ is equal to $\operatorname{odd} f$, it is of Baire class 1. By Proposition 3.2.1 and [57, Theorem 1.3], Tf is a strongly affine function in $\mathcal{B}^{1,b}(B_{X^*}, \mathbb{R})$, and it is in X^{**}_1 by the Mokobodzki theorem (see [5, Theorem II.1.2(a)]).

For functions of higher Baire classes we proceed by transfinite induction.

□

We follow with a consequence of [52, Theorem 4.4] which could serve as a motivation for Proposition 3.2.6.

Corollary 3.2.5. *Let X be a real L_1-predual with* $\operatorname{ext} B_{X^*} \cup \{0\}$ *closed. If $\alpha \in [0, \omega_1)$ and f is an odd strongly affine function on B_{X^*} such that $f|_{\operatorname{ext} B_{X^*} \cup \{0\}}$ is a function of Baire class α, then $f \in X^{**}_\alpha$.*

Proof. If $\alpha = 0$, then using the proof of [52, Theorem 4.4] we deduce that $Tf \in X$ and $Tf|_{\operatorname{ext} B_{X^*} \cup \{0\}} = f|_{\operatorname{ext} B_{X^*} \cup \{0\}}$. Then due to Proposition 3.2.3 $f = Tf \in X = X^{**}_0$.

The proof for higher Baire classes follows by transfinite induction. □

Proposition 3.2.6. *Let X be a real separable L_1-predual, f an odd strongly affine function on B_{X^*} such that $f|_{\operatorname{ext} B_{X^*}}$ is of Baire class α on* $\operatorname{ext} B_{X^*}$.

(a) If $\alpha \in [0, \omega_0)$, then $f \in X^{**}_{\alpha+1}$.

(b) If $\alpha \in [\omega_0, \omega_1)$, then $f \in X^{**}_\alpha$.

(c) If $\alpha \in [1, \omega_1)$ and $\operatorname{ext} B_{X^*}$ *is of type F_σ, then $f \in X^{**}_\alpha$.*

Proof. (a) Let $\alpha \in [0, \omega_0)$ and f be an odd strongly affine function on B_{X^*} such that $f|_{\operatorname{ext} B_{X^*}}$ is of Baire class α. If $\alpha = 0$, i.e., f is continuous on $\operatorname{ext} B_{X^*}$, then f is Baire-1 on B_{X^*} by [57, Theorem 5.2]. As an odd strongly affine Baire-1 function, f is in X^{**}_1 by [5, Theorem II.1.2(a)]. If $\alpha \in [1, \omega_0)$, $f = Tf$ due to Proposition 3.2.3. By Proposition 3.2.4(a), $f \in X^{**}_{\alpha+1}$. This finishes the proof of (a).

(b) If $\alpha \in [\omega_0, \omega_1)$, f is an odd strongly affine function and $f|_{\operatorname{ext} B_{X^*}}$ is of Baire class α, then $f = Tf$ by Proposition 3.2.3. It follows from Proposition 3.2.4(b) that $f \in X^{**}_\alpha$.

(c) It suffices to use Propositions 3.2.3 and 3.2.4(c). □

Theorem 3.2.7. *Let X be a real separable L_1-predual and let f be an odd function in $\mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$.*

(a) If $\alpha \in [0, \omega_1)$ then there exists a function h such that $h = f$ on $\operatorname{ext} B_{X^*}$ *and $h \in X^{**}_{\alpha+1}$ in case $\alpha \in [0, \omega_0)$ and $h \in X^{**}_\alpha$ in case $\alpha \in [\omega_0, \omega_1)$.*

(b) If $\operatorname{ext} B_{X^*}$ *is of type F_σ, then for any $\alpha \in [1, \omega_1)$ and an odd function $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$ there exists a function $h \in X^{**}_\alpha$ such that $h = f$ on* $\operatorname{ext} B_{X^*}$.

Proof. (a) Let f be an odd bounded Borel function on $\operatorname{ext} B_{X^*}$. Thus by Proposition 3.2.1 the function Tf is an odd Borel strongly affine function on B_{X^*} satisfying

$$Tf(x^*) = (\operatorname{odd} \varepsilon_{x^*})(f) = \varepsilon_{x^*}(\operatorname{odd} f) = f(x^*), \quad x^* \in \operatorname{ext} B_{X^*}.$$

By Proposition 3.2.6(a),(b), the function $h = Tf$ is in $X^{**}_{\alpha+1}$ in case $\alpha \in [0, \omega_0)$ and $h \in X^{**}_\alpha$ in case $\alpha \in [\omega_0, \omega_1)$.

(b) We argue as above, only we use Proposition 3.2.6(c) instead.

□

Theorem 3.2.8. *Let X be a real separable L_1-predual. If the set $\operatorname{ext} B_{X^*}$ is not of type F_σ, then there exists an odd function $f \in \mathcal{B}^{1,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$ that is not extensible to a function from X_1^{**}.*

Proof. Assume that $\operatorname{ext} B_{X^*}$ is not of type F_σ. Since it is a G_δ subset of a compact metrizable space, by the Hurewicz theorem (see [45, Theorem 21.18]) there exists a closed set $A \subset B_{X^*}$ satisfying

$$\overline{A \cap \operatorname{ext} B_{X^*}} = \overline{A \setminus \operatorname{ext} B_{X^*}} = A$$

with $A \setminus \operatorname{ext} B_{X^*}$ countable. Let $\{x_n^*;\, n \in \mathbb{N}\}$ be an enumeration of $A \setminus \operatorname{ext} B_{X^*}$. For each $n \in \mathbb{N}$ we select a maximal measure $\mu_n \in \mathcal{M}^1_{x_n^*}(B_{X^*})$ and using the regularity of Radon measures we find a compact set $K_n \subset \operatorname{ext} B_{X^*}$ such that $\mu_n(K_n) > 1 - \frac{1}{n}$. Without loss of generality we may assume that $K_n = -K_n$. Since $\bigcup_n K_n$ is an F_σ set and $A \cap \operatorname{ext} B_{X^*} \setminus \bigcup K_n$ cannot be F_σ-separated from $A \setminus \operatorname{ext} B_{X^*}$ (otherwise $A \cap \operatorname{ext} B_{X^*}$ would be an F_σ set which is impossible), an application of [45, Theorem 21.22] provides a closed set $B \subset A \setminus \bigcup_n K_n$ such that

$$\overline{B \cap \operatorname{ext} B_{X^*}} = \overline{B \setminus \operatorname{ext} B_{X^*}} = B.$$

Let $b^* \in B$ be distinct from 0 and V be its closed neighborhood satisfying $V \cap -V = \emptyset$. Set $C = B \cap V$. Then

$$C \cap (-C) \subset V \cap (-V) = \emptyset.$$

Let

$$f(x^*) = \frac{1}{2}(\chi_C(x^*) - \chi_{-C}(x^*)), \quad x^* \in B_{X^*}.$$

Then f is a bounded odd Baire-1 function on B_{X^*}, and thus its restriction to $\operatorname{ext} B_{X^*}$ is also a bounded odd Baire-1 function on $\operatorname{ext} B_{X^*}$. We show that there is no odd Baire-1 strongly affine extension of $f|_{\operatorname{ext} B_{X^*}}$ to B_{X^*}.

Let h be such an extension. Then $h = Tf$ by Proposition 3.2.3. Let $n \in \mathbb{N}$ be such that

$$x_n^* \in C \setminus \operatorname{ext} B_{X^*} \subset A \setminus \operatorname{ext} B_{X^*} = \{x_k^*;\, k \in \mathbb{N}\}.$$

Since $K_n = -K_n$ and $C \cap K_n = \emptyset$, $(C \cup -C) \cap K_n = \emptyset$. Thus $\mu_n(C \cup -C) < \frac{1}{n}$ by the choice of the set K_n. Then we get

$$\begin{aligned} |Tf(x_n^*)| &= |(\operatorname{odd} \mu_n)(f)| = |\mu_n(f)| \\ &\leq \frac{1}{2}(\mu_n(C) + \mu_n(-C)) \leq \frac{1}{n}. \end{aligned}$$

On the other hand, if $x^* \in C \cap \operatorname{ext} B_{X^*}$, then $x^* \notin -C$ as $C \cap -C = \emptyset$. Hence it follows that

$$|Tf(x^*)| = |(\operatorname{odd} \varepsilon_{x^*})(f)| = |\varepsilon_{x^*}(f)| = \left|\frac{1}{2}(1-0)\right| = \frac{1}{2}.$$

Since both $C \cap \operatorname{ext} B_{X^*}$ and $C \setminus \operatorname{ext} B_{X^*}$ are dense in C, $h = Tf$ has no point of continuity on C. In particular, h is not a Baire-1 function on B_{X^*} by [45, Theorem 24.14], which concludes the proof. □

By a rephrasing a part of the previous results we get an analogue of [52, Theorem 4.4].

Corollary 3.2.9. *Let X be a separable real Banach space. Then the following statements are equivalent.*

(i) A space X is a real L_1-predual and $\operatorname{ext} B_{X^*}$ *is an F_σ set.*

(ii) Every odd function $f \in \mathcal{B}^{1,b}(\operatorname{ext} B_{X^*}, \mathbb{R})$ can be extended to a function in X_1^{**}.

Proof. (i) $\implies$ (ii). Due to Theorem 3.2.7(b).

(ii) $\implies$ (i). Assume $x^* \in X^*$ and let $\mu, \nu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$. For any $f \in \mathcal{C}(B_{X^*}, \mathbb{R})$ then there exists by (ii) a function $h \in X_1^{**}$ extending odd $f|_{\operatorname{ext} B_{X^*}}$. Maximal measures are carried by $\operatorname{ext} B_{X^*}$ and h is a strongly affine function, hence

$$(\operatorname{odd} \mu)(f) = \mu(\operatorname{odd} f) = \mu(h) = h(x^*) = \nu(h) = \nu(\operatorname{odd} f) = (\operatorname{odd} \nu)(f).$$

Thus $\operatorname{odd} \mu = \operatorname{odd} \nu$ and using Lazar's characterization of the real Lindenstrauss spaces (see [53, Theorem] or [51, § 21, Theorem 7]) we get that X is an L_1-predual.

Finally, due to Theorem 3.2.8, the set $\operatorname{ext} B_{X^*}$ is of type F_σ. □

For a symmetric set B and $\alpha \in [0, \omega_1)$ we denote a space of all bounded odd Baire-α function on B by $\mathcal{B}^{\alpha,b}_\sigma(B, \mathbb{R})$.

The following result extends [55, Theorem 1] of Lindenstrauss and Wulbert.

Theorem 3.2.10. *Let X be a real separable L_1-predual such that* $\operatorname{ext} B_{X^*}$ *is an F_σ set. Then for any $\alpha \in [1, \omega_1)$, the space X_α^{**} is isometric to the space* $\mathcal{B}^{\alpha,b}_\sigma(\operatorname{ext} B_{X^*}, \mathbb{R})$.

Proof. A function $f \in X_\alpha^{**}$ is bounded, Baire-α and strongly affine. The restriction mapping $r : X_\alpha^{**} \to \mathcal{B}^{\alpha,b}_\sigma(\operatorname{ext} B_{X^*}, \mathbb{R})$ is therefore an isometric isomorphism due to Theorem 3.2.7(b) and the minimum principle exposed by [58, Theorem 3.86]. □

Further, one can be tempted to investigate whether a topological quality of $\operatorname{ext} B_{X^*}$ can characterize possibility of extending Baire functions of higher classes. Theorem 3.2.11 below shows that this is not the case.

Let K be a compact convex set in a locally convex space and set $X = \mathcal{A}(K, \mathbb{R})$. Then we can make the natural identifications

$$\begin{aligned} B_{X^*} &= \operatorname{co}(K \cup -K), \\ \operatorname{ext} B_{X^*} &= \operatorname{ext} K \cup - \operatorname{ext} K \end{aligned} \tag{3.2}$$

using an affine homeomorphism $\varphi : \operatorname{co}(K \cup -K) \to B_{X^*}$ defined by the formula $\varphi(\lambda k_1 - (1-\lambda)k_2)(h) = \lambda h(k_1) - (1-\lambda)h(k_2)$, $\lambda \in [0,1]$, $k_1, k_2 \in K$ and $h \in X$.

Further, we need to establish a mapping I from the space $\mathcal{A}^\alpha(K, \mathbb{R})$ to a space of all affine functions on B_{X^*} by setting

$$If(s) = \mu(f), \quad \text{where } \mu \in B_{\mathcal{M}(K,\mathbb{R})} \text{ is any measure extending } s \in B_{X^*}.$$

For more detailed information concerning the mapping I consult e.g. [78, Theorem 2.5] or [58, Chapter 5.6].

Theorem 3.2.11. *There exist real separable L_1-preduals X, Y with the following properties.*

(a) The set $\operatorname{ext} B_{X^*}$ *is homeomorphic to* $\operatorname{ext} B_{Y^*}$.

(b) For any $\alpha \in [2, \omega_1)$ and any function $f \in \mathcal{B}^{\alpha,b}_{\sigma}(\operatorname{ext} B_{Y^*}, \mathbb{R})$ there exists a function $h \in Y^{**}_{\alpha}$ such that $h = f$ on $\operatorname{ext} B_{Y^*}$.

(c) There exists a function $f \in \mathcal{B}^{2,b}_{\sigma}(\operatorname{ext} B_{X^*}, \mathbb{R})$ not extensible to an element of X^{**}_2.

Proof. By [77, Theorem 1.1], there exists a couple of metrizable simplices K, L with the following properties:

- The set $\operatorname{ext} K$ is homeomorphic to $\operatorname{ext} L$.
- For $\alpha \in [2, \omega_1)$, any bounded Baire-α function on $\operatorname{ext} L$ can be extended to a function of affine class α on L.
- There exists a bounded function g on $\operatorname{ext} K$ of Baire-2 class that is not extensible to a function on K of affine class 2.

We set $X = \mathcal{A}(K, \mathbb{R})$ and $Y = \mathcal{A}(L, \mathbb{R})$. Then X and Y are separable L_1-preduals (see [51, § 19, Theorem 2]).

(a) The assertion follows from the identification (3.2).

(b) We claim that, for any $\alpha \in [2, \omega_1)$, every function $f \in \mathcal{B}^{\alpha,b}_{\sigma}(\operatorname{ext} B_{Y^*}, \mathbb{R})$ can be extended to a function $h \in Y^{**}_{\alpha}$.

Indeed, let $f \in \mathcal{B}^{\alpha,b}_{\sigma}(\operatorname{ext} B_{Y^*}, \mathbb{R})$. Using the identification (3.2) we may assert that $f|_{\operatorname{ext} L} \in \mathcal{B}^{\alpha,b}(\operatorname{ext} L, \mathbb{R})$ and set $g = f|_{\operatorname{ext} L}$. Due to the hypotheses there exists a function $\tilde{g} \in \mathcal{A}^{\alpha}(L, \mathbb{R})$ extending g. Then $I\tilde{g}|_L = \tilde{g}$ and applying [78, Theorem 2.5(f)] (see also [58, Theorem 5.40(f)]) we get that $I\tilde{g} \in Y^{**}_{\alpha}$. Hence $I\tilde{g} = f$ on $\operatorname{ext} B_{Y^*}$ and we may define $h = I\tilde{g}$ as the desired function.

(c) Let $g \in \mathcal{B}^{2,b}(\operatorname{ext} K, \mathbb{R})$ be a function not extensible to a function from $\mathcal{A}^2(K, \mathbb{R})$. The function g can be nevertheless naturally extended to an odd function $\tilde{g}$ defined on $\operatorname{ext} K \cup - \operatorname{ext} K$. Due to the identification (3.2) we may see $\tilde{g}$ as a function from $\mathcal{B}^{2,b}_{\sigma}(\operatorname{ext} B_{X^*}, \mathbb{R})$. We claim that the function $\tilde{g}$ cannot be extended to an element of X^{**}_2.

Suppose the contrary and let $\tilde{f} \in X^{**}_2$ be an extension of $\tilde{g}$. Due to [78, Theorem 2.5(f)] (see also [58, Theorem 5.40(f)]) there exists $f \in \mathcal{A}^2(K, \mathbb{R})$ such that $If = \tilde{f}$. The definition of I immediately provides that $f = g$ on $\operatorname{ext} K$ which gives us a contradiction with the properties of g.

□

3.3 Complex L_1-preduals

Main results, as well as their proofs, of this section are rather similar to those in Section 3.2. The principal technical inconvenience consists in the impossibility of using the notion of odd functions in the complex setting. The role of odd functions play homogeneous functions here.

Following notions are due to Effros (see [18]). Let $\mathbb{T}$ stand for the unit circle endowed with the unit Haar measure $d\alpha$. Let X be a complex Banach space. A set $B \subset B_{X^*}$ is called *homogeneous* if $\alpha B = B$ for each $\alpha \in \mathbb{T}$. An example of a homogeneous set is $\operatorname{ext} B_{X^*}$. A function f on a homogeneous set $B \subset B_{X^*}$ is called *homogeneous* (see e.g. [18, p. 53], [51, p. 240]) if

$$f(\alpha x^*) = \alpha f(x^*), \quad (\alpha, x^*) \in \mathbb{T} \times B.$$

If f is a Borel function defined on a homogeneous Borel set $B \subset B_{X^*}$, we set

$$(\hom f)(x^*) = \int_{\mathbb{T}} \alpha^{-1} f(\alpha x^*)\, d\alpha, \quad x^* \in B.$$

Then the function $\hom f$ is homogeneous on B and it is easy to see that it is continuous in case $f \in \mathcal{C}^b(B, \mathbb{C})$. By the Lebesgue dominated convergence theorem, $\hom f$ is well defined for each bounded Baire function on B and $\hom f$ is Baire-α whenever $f \in \mathcal{B}^{\alpha,b}(B, \mathbb{C})$. A function f is homogeneous if and only if $\hom f = f$.

The mapping hom provides a mapping on $\mathcal{M}(K, \mathbb{C})$ defined as

$$(\hom \mu)(f) = \mu(\hom f), \quad f \in \mathcal{C}(K, \mathbb{C}), \mu \in \mathcal{M}(K, \mathbb{C}).$$

For $x^* \in B_{X^*}$, let $\mathcal{M}^1_{x^*}(B_{X^*})$ be defined as in Section 3.2. Similarly, symbols $\prec$ and $\mathcal{M}^{\max}(B_{X^*})$ are defined as above.

If X is a complex Banach space, then we have the following analogue of [53, Theorem] due to Effros:

The Banach space X is an complex L_1-predual if and only if, for any $x^* \in B_{X^*}$ and measures $\mu, \nu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$, it holds $\hom \mu = \hom \nu$ (see [18, Theorem 4.3] or [51, § 23, Theorem 5]).

This theorem permits to define a mapping T analogously as in the real case (see Section 3.2). Namely, for a separable complex L_1-predual X and a bounded Borel function f defined at least on $\operatorname{ext} B_{X^*}$ we set

$$Tf(x^*) = (\hom \mu)(f), \quad \mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*}). \tag{3.3}$$

Since $\hom \mu$ is a boundary measure if μ is maximal (see [18, Lemma 4.2] or [51, § 23, Lemma 10]), the mapping T is well defined.

The first result is an extension of [5, Theorem II.1.2(a)] to the context of complex Banach spaces.

Proposition 3.3.1. *Let X be a complex Banach space and f a Baire-1 affine homogeneous function on B_{X^*}. Then $f \in X^{**}_1$.*

Proof. We first notice that for a homogeneous function $\phi : Y \to \mathbb{C}$, where Y is a complex Banach space, holds the identity

$$\operatorname{Im} \phi(y) = -\operatorname{Re} \phi(iy), \quad y^* \in Y. \tag{3.4}$$

Let $f = \operatorname{Re} f + i \operatorname{Im} f$. Then both $\operatorname{Re} f$ and $\operatorname{Im} f$ are Baire-1 and affine. Since f is homogeneous, $f(0) = \operatorname{Re} f(0) = \operatorname{Im} f(0) = 0$.

Let $X_{\mathbb{R}}$ be the real version of X (we forget multiplication by complex numbers) and $\pi : X^* \to (X_{\mathbb{R}})^*$ be defined as

$$\pi(x^*) = \operatorname{Re} x^*, \quad x^* \in X^*.$$

Then π is a weak*-weak* homeomorphic isometric isomorphism of X^* and $(X_{\mathbb{R}})^*$ (due to (3.4)). Thus the function

$$(\operatorname{Re} f) \circ \pi^{-1} : B_{(X_{\mathbb{R}})^*} \to \mathbb{R}$$

is a Baire-1 affine odd function on $B_{(X_{\mathbb{R}})^*}$. By [5, Theorem II.1.2(a)], there exists a sequence (x_n) in X such that

$$(\operatorname{Re} f) \circ \pi^{-1}(x^*) = \lim x_n(x^*), \quad x^* \in B_{(X_{\mathbb{R}})^*}.$$

Then we have for $x^* \in B_{X^*}$

$$\begin{aligned} f(x^*) &= \operatorname{Re} f(x^*) - i \operatorname{Re} f(ix^*) = (\operatorname{Re} f \circ \pi^{-1})(\pi(x^*)) - i(\operatorname{Re} f \circ \pi^{-1})(\pi(ix^*)) \\ &= \lim x_n(\pi(x^*)) - i \lim x_n(\pi(ix^*)) = \lim(\pi(x^*) - i\pi(ix^*))(x_n) \\ &\overset{(3.4)}{=} \lim(\operatorname{Re} x^* + i \operatorname{Im} x^*)(x_n) = \lim x_n(x^*). \end{aligned}$$

Hence $f \in X_1^{**}$. □

Proposition 3.3.2. *Let X be a complex separable L_1-predual and T be the mapping defined by* (3.3)*.*

*(a) If $f \in \mathcal{C}(B_{X^*}, \mathbb{C})$, then Tf is Baire-*1*.*

(b) If $f \in \mathcal{B}^b(\operatorname{ext} B_{X^*}, \mathbb{C})$, then Tf is a homogeneous Borel strongly affine function on B_{X^*}.

Proof. Since X is separable, B_{X^*} is a metrizable compact convex set (see [72, Theorems 3.15, 3.16]), and thus there exists a mapping $S : B_{X^*} \to \mathcal{M}^{\max}(B_{X^*})$ such that $Sx^* = \nu_{x^*} \in \mathcal{M}^1_{x^*}(B_{X^*})$ and the function $Sf : x^* \mapsto \nu_{x^*}(f)$ is a Baire-1 function on B_{X^*} for each continuous function f on B_{X^*} (see [83, Théoréme 1] or [58, Theorem 11.41]).

(a) Let f be a continuous function on B_{X^*}. Then the function

$$g(\alpha, z^*) = \alpha^{-1} f(\alpha z^*), \quad (\alpha, z^*) \in \mathbb{T} \times B_{X^*}$$

is continuous on $\mathbb{T} \times B_{X^*}$, and thus the function

$$h(z^*) = \int_{\mathbb{T}} g(\alpha, z^*) \, d\alpha, \quad z^* \in B_{X^*},$$

is continuous on B_{X^*}.

Then, for each $x^* \in B_{X^*}$,

$$Tf(x^*) = (\hom \nu_{x^*})(f) = \nu_{x^*}(\hom f) = \nu_{x^*}(h) = Sh(x^*).$$

Thus $Tf = Sh$ is a Baire-1 function on B_{X^*}.

(b) Let now

$$\mathcal{F} = \{f \in \mathcal{B}^b(B_{X^*}, \mathbb{C});\ Tf \text{ is Borel}\}.$$

Then $\mathcal{F}$ is closed under the taking pointwise limits of bounded sequences by the Lebesgue dominated convergence theorem and contains $\mathcal{C}(B_{X^*}, \mathbb{C})$. Hence it contains any bounded Borel function on B_{X^*}.

Let f be a bounded Borel function on $\operatorname{ext} B_{X^*}$. Since $\operatorname{ext} B_{X^*}$ is a Borel set, we can consider f to be a Borel function on B_{X^*}. Hence $f \in \mathcal{F}$ and Tf is Borel.

Further, Tf is homogeneous. Indeed, let $\beta \in \mathbb{T}$ be given and let an affine homeomorphism $\sigma_\beta : B_{X^*} \to B_{X^*}$ be defined as $\sigma_\beta(x^*) = \beta x^*$. Given $y^* \in B_{X^*}$ and a maximal measure $\mu \in \mathcal{M}^1_{y^*}(B_{X^*})$, the measure $\sigma_\beta\mu \in \mathcal{M}^1_{\beta y^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$. Thus

$$\begin{aligned} Tf(\beta y^*) &= (\hom \sigma_\beta\mu)(f) = (\sigma_\beta\mu)(\hom f) = \int_{\operatorname{ext} B_{X^*}} (\hom f)(\beta x^*)\, d\mu(x^*) \\ &= \beta\mu(\hom f) = \beta(\hom \mu)(f) = \beta Tf(y^*). \end{aligned}$$

Hence Tf is homogeneous.

Let us show that Tf is strongly affine, i.e., that $\nu(Tf) = Tf(y^*)$ for each $y^* \in B_{X^*}$ and $\nu \in \mathcal{M}^1_{y^*}(B_{X^*})$. Given y^* and ν as above, let

$$\mu(g) = \int_{B_{X^*}} \nu_{x^*}(g)\, d\nu(x^*), \quad g \in \mathcal{C}(B_{X^*}, \mathbb{C}).$$

As in the proof of Proposition 3.2.1 we obtain that μ is maximal and contained in $\mathcal{M}^1_{y^*}(B_{X^*})$. Thus $Tf(y^*) = (\hom \mu)(f)$ and it follows that

$$\begin{aligned} \int_{B_{X^*}} Tf(x^*)\, d\nu(x^*) &= \int_{B_{X^*}} \nu_{x^*}(\hom f)\, d\nu(x^*) \\ &= \mu(\hom f) = (\hom \mu)(f) = Tf(y^*). \end{aligned}$$

Hence $\nu(Tf) = Tf(y^*)$ and Tf is strongly affine. □

Proposition 3.3.3. *Let X be a complex separable L_1-predual and f be a Borel bounded function on $\operatorname{ext} B_{X^*}$. If h is a homogeneous strongly affine function on B_{X^*} extending f, then $h = Tf$.*

Proof. Since f is extended by a homogeneous function h, it is homogeneous on $\operatorname{ext} B_{X^*}$. Let $y^* \in B_{X^*}$ be given. We pick a maximal measure $\mu \in \mathcal{M}^1_{y^*}(B_{X^*})$ and obtain

$$Tf(y^*) = (\hom \mu)(f) = \mu(f) = \int_{\operatorname{ext} B_{X^*}} h(x^*)\, d\mu(x^*) = h(y^*).$$

□

Proposition 3.3.4. *Let X be a complex separable L_1-predual and assume that $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$.*

(a) If $\alpha \in [1, \omega_0)$, then $Tf \in X^{**}_{\alpha+1}$.

(b) If $\alpha \in [\omega_0, \omega_1)$, then $Tf \in X^{**}_{\alpha}$.

(c) If $\alpha \in [1, \omega_1)$ and $\operatorname{ext} B_{X^*}$ is F_σ, then $Tf \in X^{**}_{\alpha}$.

Proof. (a) If $\alpha = 1$, f can be extended to a Baire-1 function on B_{X^*} by [1, Corollary I.4.4] and [49, § 35, VI, Theorem]. Let (f_n) be a bounded sequence of continuous functions on B_{X^*} converging to f on $\operatorname{ext} B_{X^*}$.

For a given $x^* \in B_{X^*}$, let $\mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$ be chosen. Then

$$Tf_n(x^*) = (\hom \mu)(f_n) \to (\hom \mu)(f) = Tf(x^*).$$

By Proposition 3.3.2, each Tf_n is a Baire-1 homogeneous strongly affine function on B_{X^*}. By Proposition 3.3.1, $Tf_n \in X_1^{**}$. Hence $Tf \in X_2^{**}$.

The rest of the proof follows by induction.

(b) Let $\alpha = \omega_0$ and $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$. Let (f_n) be a bounded sequence of functions from $\mathcal{B}^{\alpha_n,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$, $\alpha_n < \alpha$, converging to f. Then $Tf_n \to Tf$ and $Tf_n \in X^{**}_{\alpha_n+1}$ by (a). Hence $Tf \in X^{**}_\alpha$. For higher Baire classes the proof follows by transfinite induction.

(c) Let $\operatorname{ext} B_{X^*}$ be of type F_σ and $f \in \mathcal{B}^{1,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$. We have $\hom f \in \mathcal{B}^{1,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$. Then Tf is a homogeneous strongly affine function on B_{X^*} whose restriction to $\operatorname{ext} B_{X^*}$ equals $\hom f$, and thus it is of Baire class 1 on $\operatorname{ext} B_{X^*}$. By [57, Theorem 1.3], Tf is of Baire class 1 on B_{X^*}, and thus it is in X_1^{**} by Proposition 3.3.1.

For functions of higher classes we proceed by transfinite induction. ☐

As a motivation for Proposition 3.3.6 we offer the following consequence of [65, Theorem 9].

Corollary 3.3.5. *Let X be a complex separable L_1-predual with $\operatorname{ext} B_{X^*} \cup \{0\}$ closed. If $\alpha \in [0, \omega_1)$ and f is a homogeneous strongly affine function on B_{X^*} such that $f|_{\operatorname{ext} B_{X^*}}$ is of Baire class α on $\operatorname{ext} B_{X^*}$, then $f \in X^{**}_\alpha$.*

Proof. If $\alpha = 0$, then using the proof of [65, Theorem 9] we get $Tf \in X$ and $Tf|_{\operatorname{ext} B_{X^*}} = f|_{\operatorname{ext} B_{X^*}}$. Then due to Proposition 3.3.3 $f = Tf \in X = X_0^{**}$.

The proof for higher Baire classes follows by transfinite induction. ☐

The subsequent result is also an improvement of [57, Theorem 1.4] for the case of separable complex spaces.

Proposition 3.3.6. *Let X be a complex separable L_1-predual and f be a homogeneous strongly affine function on B_{X^*} such that $f|_{\operatorname{ext} B_{X^*}}$ is of Baire class α on $\operatorname{ext} B_{X^*}$.*

(a) If $\alpha \in [0, \omega_0)$, then $f \in X^{**}_{\alpha+1}$.

(b) If $\alpha \in [\omega_0, \omega_1)$, then $f \in X^{**}_\alpha$.

(c) If $\operatorname{ext} B_{X^*}$ is of type F_σ, $\alpha \in [1, \omega_1)$, then $f \in X^{**}_\alpha$.

Proof. (a) Let $\alpha \in [0, \omega_0)$ and f be a homogeneous strongly affine function on B_{X^*} such that $f|_{\operatorname{ext} B_{X^*}}$ is of Baire class α. If $\alpha = 0$, i.e., f is continuous on $\operatorname{ext} B_{X^*}$, then f is Baire-1 on B_{X^*} by [57, Theorem 5.2]. As a homogeneous strongly affine Baire-1 function, $f \in X_1^{**}$ by Proposition 3.3.1.

If $\alpha \in [1, \omega_0)$, $f = Tf$ due to Proposition 3.3.3. By Proposition 3.3.4(a), $f = Tf \in X^{**}_{\alpha+1}$. This finishes the proof of (a).

(b) If $\alpha \in [\omega_0, \omega_1)$ and f is homogeneous strongly affine such that $f|_{\operatorname{ext} B_{X^*}}$ is of Baire class α, then $f = Tf$ by Proposition 3.3.3 and it follows from Proposition 3.3.4(b) that $f = Tf \in X^{**}_\alpha$.

(c) It is enough to use Propositions 3.3.3 and 3.3.4(c). ☐

Theorem 3.3.7. *Let X be a complex separable L_1-predual and let f be an homogeneous function in $\mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$.*

(a) If $\alpha \in [0, \omega_1)$ then there exists a function h such that $h = f$ on $\operatorname{ext} B_{X^*}$ and $h \in X^{**}_{\alpha+1}$ in case $\alpha \in [0, \omega_0)$ and $h \in X^{**}_{\alpha}$ in case $\alpha \in [\omega_0, \omega_1)$.

(b) If $\operatorname{ext} B_{X^*}$ is of type F_σ and $\alpha \in [1, \omega_1)$ then there exists a function $h \in X^{**}_{\alpha}$ such that $h = f$ on $\operatorname{ext} B_{X^*}$.

Proof. (a) Let f be an homogeneous bounded Borel function on $\operatorname{ext} B_{X^*}$. Thus by Proposition 3.3.2 the function Tf is a homogeneous Borel strongly affine function on B_{X^*} satisfying

$$Tf(x^*) = (\hom \varepsilon_{x^*})(f) = \varepsilon_{x^*}(\hom f) = f(x^*), \quad x^* \in \operatorname{ext} B_{X^*}.$$

By Proposition 3.3.6(a),(b), the function $h = Tf$ is in $X^{**}_{\alpha+1}$ in case $\alpha \in [0, \omega_0)$ and $h \in X^{**}_{\alpha}$ in case $\alpha \in [\omega_0, \omega_1)$.

(b) We argue as above, only we use Proposition 3.3.6(c) instead. □

Theorem 3.3.8. *Let X be a separable complex L_1-predual whose set of extreme points is not of type F_σ. Then there exists a homogeneous bounded Baire-1 function on $\operatorname{ext} B_{X^*}$ that is not extensible to a function from X^{**}_1.*

Proof. We start the reasoning as in the proof of Theorem 3.2.8. Let A be a nonempty closed set in B_{X^*} with $A \setminus \operatorname{ext} B_{X^*}$ countable enjoying the property $\overline{A \cap \operatorname{ext} B_{X^*}} = \overline{A \setminus \operatorname{ext} B_{X^*}} = A$. Let $\{x^*_n;\, n \in \mathbb{N}\}$ be an enumeration of $A \setminus \operatorname{ext} B_{X^*}$ and let μ_n be chosen from $\mathcal{M}^1_{x_n^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$. Let K_n, $n \in \mathbb{N}$, be compact sets in $\operatorname{ext} B_{X^*}$ satisfying $\mu_n(K_n) \geq 1 - \frac{1}{n}$. Without loss of generality we may assume that K_n are homogeneous. The F_σ-separation argument provides a nonempty closed set $B \subset A \setminus \bigcup_n K_n$ such that $\overline{B \cap \operatorname{ext} B_{X^*}} = \overline{B \setminus \operatorname{ext} B_{X^*}} = B$.

Fix $\delta \in (0, \frac{\pi}{2})$. We pick a nonzero element $b^* \in B$ and choose its closed neighborhood V such that

$$\forall t_1 \in [-\frac{\delta}{2}, \frac{\delta}{2}]\ \forall t_2 \in [-\pi, -\delta] \cup [\delta, \pi] : e^{it_1}V \cap e^{it_2}V = \emptyset.$$

Let

$$C = B \cap V, \quad D = \{e^{it}c^*;\, t \in [-\frac{\delta}{2}, \frac{\delta}{2}], c^* \in C\} \quad \text{and} \quad f = \hom \chi_D.$$

We claim that f has no extension in X^{**}_1. Assume that h is such an extension. Then $h = Tf$ by Proposition 3.3.3. Let $n \in \mathbb{N}$ satisfy $x^*_n \in C$. Then

$$\forall \alpha \in \mathbb{T}\ \forall x^* \in K_n : \alpha x^* \notin D. \tag{3.5}$$

Indeed, if αx^* were in D for some $\alpha \in \mathbb{T}$ and $x^* \in K_n$, then there would exist $t \in [-\frac{\delta}{2}, \frac{\delta}{2}]$ and $c^* \in C$ such that $\alpha x^* = e^{it}c^*$. Then $e^{-it}\alpha x^* \in K_n \cap C$, a contradiction.

Thus we have from (3.5)

$$\begin{aligned}
|Tf(x^*_n)| = |(\hom \mu_n)(f)| &= \left| \int_{B_{X^*}} \int_{\mathbb{T}} \alpha^{-1} \chi_D(\alpha x^*)\, d\alpha\, d\mu_n(x^*) \right| \\
&= \left| \int_{K_n} \int_{\mathbb{T}} \alpha^{-1} \chi_D(\alpha x^*)\, d\alpha\, d\mu_n(x^*) + \int_{\operatorname{ext} B_{X^*} \setminus K_n} f d\mu_n \right| \\
&= \left| \int_{\operatorname{ext} B_{X^*} \setminus K_n} f d\mu_n \right| \leq \frac{1}{n}.
\end{aligned}$$

On the other hand, let $x^* \in C \cap \operatorname{ext} B_{X^*}$ be given. Then we have

$$\forall t \in [-\pi, -\delta] \cup [\delta, \pi] : \chi_D(e^{it}x^*) = 0. \tag{3.6}$$

Indeed, if $e^{it_1}x^* \in D$ for some $t_1 \in [-\pi, -\delta] \cup [\delta, \pi]$, then there exists $t_2 \in [-\frac{\delta}{2}, \frac{\delta}{2}]$ and $c^* \in C$ such that $e^{it_1}x^* = e^{it_2}c^*$. Then

$$e^{it_1}x^* \in e^{it_1}V \cap e^{it_2}V = \emptyset,$$

a contradiction. Thus (3.6) holds.

Further, we have

$$\forall t \in [-\frac{\delta}{2}, \frac{\delta}{2}] : \chi_D(e^{it}x^*) = 1. \tag{3.7}$$

Using (3.6) and (3.7) we obtain

$$\begin{aligned}
|Tf(x^*)| &= |(\operatorname{hom} \varepsilon_{x^*})(x^*)| = |\varepsilon_{x^*}(f)| = |f(x^*)| \\
&= \left| \int_{\operatorname{ext} B_{X^*}} \alpha^{-1}\chi_D(\alpha x^*)\, d\alpha \right| = \left| \int_{\alpha \in \{e^{it};\, t \in [-\delta,\delta]\}} \alpha^{-1}\chi_D(\alpha x^*)\, d\alpha \right| \\
&\geq \left| \operatorname{Re} \int_{\alpha \in \{e^{it};\, t \in [-\delta,\delta]\}} \alpha^{-1}\chi_D(\alpha x^*)\, d\alpha \right| \\
&= \left| \int_{\alpha \in \{e^{it};\, t \in [-\delta,\delta]\}} (\operatorname{Re} \alpha^{-1})\chi_D(\alpha x^*)\, d\alpha \right| \\
&\geq (\cos\delta) \int_{\alpha \in \{e^{it};\, t \in [-\frac{\delta}{2},\frac{\delta}{2}]\}} \chi_D(\alpha x^*)\, d\alpha \\
&= (\cos\delta) \int_{\alpha \in \{e^{it};\, t \in [-\frac{\delta}{2},\frac{\delta}{2}]\}} 1\, d\alpha \\
&= (\cos\delta)\delta.
\end{aligned}$$

Thus $h = Tf$ has no point of continuity in the set C, a contradiction with $h \in X_1^{**}$. This concludes the proof. □

Corollary 3.3.9. *Let X be a separable complex Banach space. Then the following assertions are equivalent.*

(i) The space X is an L_1-predual and $\operatorname{ext} B_{X^*}$ is an F_σ-set.

(ii) Every homogeneous function $f \in \mathcal{B}^{1,b}(\operatorname{ext} B_{X^*}, \mathbb{C})$ can be extended to a function in X_1^{**}.

Proof. The implication (i) $\implies$ (ii) follows from Theorem 3.3.7(b).

(ii) $\implies$ (i). Let $x^* \in B_{X^*}$ and $\mu, \nu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$ be given. Pick an arbitrary $f \in \mathcal{C}(B_{X^*}, \mathbb{C})$. Then $\operatorname{hom} f|_{\operatorname{ext} B_{X^*}}$ is a homogeneous function on $\operatorname{ext} B_{X^*}$, and thus there exists its extension $h \in X_1^{**}$. Since μ, ν are carried by $\operatorname{ext} B_{X^*}$ and h is strongly affine, we obtain

$$(\operatorname{hom} \mu)(f) = \mu(\operatorname{hom} f) = \mu(h) = h(x) = \nu(h) = \nu(\operatorname{hom} f) = (\operatorname{hom} \nu)(f).$$

Thus $\operatorname{hom} \mu = \operatorname{hom} \nu$ and X is an L_1-predual by the omnipresent result of Effros (see [18, Theorem 4.3] or [51, § 23, Theorem 5]). The fact that $\operatorname{ext} B_{X^*}$ is of type F_σ then follows from Theorem 3.3.8. □

Theorem 3.3.10. *Let X be a complex separable L_1-predual such that $\operatorname{ext} B_{X^*}$ is of type F_σ. Let $\alpha \in [1, \omega_1)$. Then the space X^{**}_α is isometrically isomorphic to the space $\mathcal{B}^{\alpha,b}_{\text{hom}}(\operatorname{ext} B_{X^*}, \mathbb{C})$ of homogeneous bounded Baire-α functions on $\operatorname{ext} B_{X^*}$.*

Proof. Obviously, for any $f \in X^{**}_\alpha$ the restriction $f|_{\operatorname{ext} B_{X^*}}$ is clearly in $\mathcal{B}^{\alpha,b}_{\text{hom}}(\operatorname{ext} B_{X^*}, \mathbb{C})$ and it also preserves a norm. Indeed, consider a homogeneous set $B \subset B_{X^*}$ and a bounded complex function g on B. Then

$$\sup\{|g(x^*)|; x^* \in B\} = \sup\{|\operatorname{Re} g(x^*)|; x^* \in B\}.$$

Moreover, for any $g \in X^{**}_\alpha$ the function $\operatorname{Re} g$ is strongly affine. Thus due to the previous observations and [58, Theorem 3.86] we get

$$\|f\| = \|\operatorname{Re} f\| = \|(\operatorname{Re} f)|_{\operatorname{ext} B_{X^*}}\| = \|f|_{\operatorname{ext} B_{X^*}}\|.$$

On the other hand, by Proposition 3.3.4(c), any function in $\mathcal{B}^{\alpha,b}_{\text{hom}}(\operatorname{ext} B_{X^*}, \mathbb{C})$ is extended by Tf to the element of X^{**}_α. The restriction mapping is thus the required isometric isomorphism. □

3.4 C^*-algebras

The main result of this section answers a question from [5, p. 1048].

In order to prove it we need to recall a notion of a *function space* which is a linear subspace of $\mathcal{C}(K, \mathbb{F})$ containing constants and separating points of K. If $\mathcal{H} \subset \mathcal{C}(K, \mathbb{F})$ is a function space, we write $\mathcal{H}^{\perp\perp}$ for the set of all bounded Borel functions on K satisfying $\mu(f) = 0$ for each $\mu \in \mathcal{H}^\perp$.

Proposition 3.4.1. *Let K be a metrizable compact space and $f \in \mathcal{B}^b(K, \mathbb{C})$. Then the function $F : B_{\mathcal{M}(K,\mathbb{C})} \to \mathbb{C}$ defined as $F(\mu) = \mu(f)$, $\mu \in B_{\mathcal{M}(K,\mathbb{C})}$, is strongly affine on $B_{\mathcal{M}(K,\mathbb{C})}$.*

Proof. If $f \in \mathcal{C}(K, \mathbb{C})$, F is strongly affine on $B_{\mathcal{M}(K,\mathbb{C})}$ by the definition. If (f_n) is a bounded sequence of Borel functions pointwise converging to f such that the relevant functions F_n are strongly affine on $B_{\mathcal{M}(K,\mathbb{C})}$, (F_n) converges pointwise to F by the Lebesgue dominated convergence theorem. Since F_n are strongly affine, F is strongly affine as well again due to the Lebesgue dominated convergence theorem.

Hence the family of all Borel functions f, for which F is strongly affine, is closed under the taking pointwise limits of bounded sequences and contains continuous functions. Hence it contains any bounded Borel function. □

Next we recall a result which is essentially from [75].

Proposition 3.4.2. *Let $\pi : K \to L$ be a continuous affine surjection of a compact convex set K onto a compact convex set L. Let $g : L \to \mathbb{C}$ be a bounded function. Then g is strongly affine on L if and only if $g \circ \pi$ is strongly affine on K.*

Proof. We notice that a function $g : L \to \mathbb{C}$ is strongly affine if and only if both $\operatorname{Re} g$ and $\operatorname{Im} g$ are strongly affine. Then use [75, Proposition 3.2] (see also [58, Proposition 5.29]). □

Proposition 3.4.3. *Let K be a metrizable compact space, $\mathcal{A} \subset \mathcal{C}(K, \mathbb{C})$ be a function space and let $\pi : B_{\mathcal{M}(K,\mathbb{C})} \to B_{\mathcal{A}^*}$ be the restriction mapping. If $f \in \mathcal{B}^{\alpha,b}(K, \mathbb{C}) \cap \mathcal{A}^{\perp\perp}$, then the function $F : B_{\mathcal{A}^*} \to \mathbb{C}$ defined as*

$$F(a^*) = \mu(f), \quad \mu \in B_{\mathcal{M}(K,\mathbb{C})}, \pi(\mu) = a^*,$$

is a well defined homogeneous strongly affine function on $B_{\mathcal{A}^*}$ of Baire class α.

Proof. Let $f \in \mathcal{B}^{\alpha,b}(K, \mathbb{C}) \cap \mathcal{A}^{\perp\perp}$. First we notice that F is well defined. Indeed, if $a^* \in B_{\mathcal{A}^*}$, let $\mu \in B_{\mathcal{M}(K,\mathbb{C})}$ be extending a^*. If $\nu \in B_{\mathcal{M}(K,\mathbb{C})}$ is another extension, then $\mu - \nu \in \mathcal{A}^{\perp}$, and thus $\mu(f) = \nu(f)$.

Let $\alpha \in \mathbb{T}$, $a^* \in B_{\mathcal{A}^*}$ and $\mu \in B_{\mathcal{M}(K,\mathbb{C})}$ such that $\pi(\mu) = a^*$. Then $\pi(\alpha\mu) = \alpha a^*$ and $F(\alpha a^*) = \alpha F(a^*)$, thus F is homogeneous. For the verification of the strong affinity of F we use Proposition 3.4.2. Let $G : B_{\mathcal{M}(K,\mathbb{C})} \to \mathbb{C}$ be defined as $G(\mu) = \mu(f)$, $\mu \in B_{\mathcal{M}(K,\mathbb{C})}$. Then

$$G = F \circ \pi.$$

Since π is a continuous affine surjection of the compact convex set $B_{\mathcal{M}(K,\mathbb{C})}$ onto the compact convex set $B_{\mathcal{A}^*}$, the strong affinity of F follows from Propositions 3.4.1 and 3.4.2. If f is of Baire class α, G is of class α as well by the Lebesgue dominated convergence theorem. Hence F is of class α by [73] (see also [58, Theorem 5.16]). □

Theorem 3.4.4. *There exists a separable C^*-algebra X such that $X^{**}_{\mathcal{B}_2} \neq X^{**}_2$.*

Proof. Let $\mathcal{H} \subset \mathcal{C}(K, \mathbb{R})$ be the real function space constructed in [78, Section 5]. By the construction, $\mathcal{H}$ is closed in $\mathcal{C}(K, \mathbb{C})$ (see [78, p. 1674]) and K is metrizable (see [78, p. 1673]). Further, $\mathcal{H}$ is a real L_1-predual (see [78, Lemma 6.1(a)] and [58, Theorem 6.25]) and

$$\mathcal{B}^{2,bb}(\mathcal{H}) \subsetneq \mathcal{B}^{2,b}(K, \mathbb{R}) \cap \mathcal{H}^{\perp\perp}$$

by [78, Lemmas 6.5, 6.6]. Let

$$\mathcal{A} = \{g \in \mathcal{C}(K, \mathbb{C});\ \operatorname{Re} g, \operatorname{Im} g \in \mathcal{H}\}.$$

Then $\mathcal{A}$ is selfadjoint and $\operatorname{Re} \mathcal{A} = \mathcal{H}$ is a real L_1-predual. Thus $\mathcal{A}$ is a complex L_1-predual by [34, Theorem 2] (see also [51, § 23, Theorem 6]). We claim that $\mathcal{A}^{**}_2 \subsetneq \mathcal{A}^{**}_{\mathcal{B}_2}$. Indeed, pick

$$f \in (\mathcal{B}^{2,b}(K, \mathbb{R}) \cap \mathcal{H}^{\perp\perp}) \setminus \mathcal{B}^{2,bb}(\mathcal{H}).$$

Since $f \in \mathcal{H}^{\perp\perp}$, clearly $f \in \mathcal{A}^{\perp\perp}$ as well. Due to Proposition 3.4.3 we are able to define $F : B_{\mathcal{A}^*} \to \mathbb{C}$ as

$$F(a^*) = \mu(f), \quad \mu \in B_{\mathcal{M}(K,\mathbb{C})}, \quad \pi(\mu) = a^*,$$

such that F is a homogeneous strongly affine function on $B_{\mathcal{A}^*}$ of Baire class 2.

On the other hand, $F \notin \mathcal{A}^{**}_2$. Indeed, assume that $F \in \mathcal{A}^{**}_2$. Let

$$S = \{\phi(k);\ k \in K\} \subset B_{\mathcal{A}^*},$$

where $\phi(k)(a) = a(k)$, $a \in \mathcal{A}$. Then $\phi : K \to S$ is a homeomorphic embedding and $f = F \circ \phi$. Since $F \in \mathcal{A}_2^{**}$, also $f = F \circ \phi \in \mathcal{B}^{2,bb}(\mathcal{A})$. So let

$$\{a_{nk};\, n, k \in \mathbb{N}\}$$

be a family in $\mathcal{A}$ such that

$$f = \lim_{n\to\infty} \lim_{k\to\infty} a_{nk},$$

where $(\lim_{k\to\infty} a_{nk})_{n\in\mathbb{N}}$ is a bounded sequence as well as every sequence $(a_{nk})_{k\in\mathbb{N}}$ for any given $n \in \mathbb{N}$. Since f is real,

$$f = \operatorname{Re} f = \lim_{n\to\infty} \lim_{k\to\infty} \operatorname{Re} a_{nk},$$

and thus $f \in \mathcal{B}^{2,bb}(\mathcal{H})$, which is not the case. Thus $F \notin \mathcal{A}_2^{**}$.

Now we use [62, Theorem] asserting that $\mathcal{A}$ is a 1-complemented subspace of a separable C^*-algebra X. We claim that $X_{\mathcal{B}_2}^{**} \neq X_2^{**}$. Indeed, recall that $F \in \mathcal{A}_{\mathcal{B}_2}^{**} \setminus \mathcal{A}_2^{**}$. Let $P : X \to \mathcal{A}$ be a projection of norm 1 and $\pi : B_{X^*} \to B_{\mathcal{A}^*}$ be the restriction mapping. Then

$$(\pi \circ P^*)(a^*) = a^*, \quad a^* \in B_{\mathcal{A}^*}.$$

Let

$$G = F \circ \pi.$$

By Proposition 3.4.2, $G \in X_{\mathcal{B}_2}^{**}$. Suppose $G \in X_2^{**}$ and let $(x_{nk})_{n,k\in\mathbb{N}}$ witness that $G \in X_2^{**}$. Then $(Px_{nk})_{n,k\in\mathbb{N}}$ witness that $F \in \mathcal{A}_2^{**}$, because

$$\begin{aligned} F(a^*) &= F(\pi(P^*a^*)) = G(P^*a^*) = \lim_{n\to\infty} \lim_{k\to\infty} x_{nk}(P^*a^*) \\ &= \lim_{n\to\infty} \lim_{k\to\infty} Px_{nk}(a^*), \quad a^* \in B_{\mathcal{A}^*}. \end{aligned}$$

But this contradicts our choice of F. $\square$

4. Baire classes of nonseparable L_1-preduals

(joint work with Jiří Spurný)

4.1 Introduction

A Banach space X is called an *L_1-predual* (or a *Lindenstrauss space*) if its dual X^* is isometric to a space $L^1(X, \mathcal{S}, \mu)$ for a measure space $(X, \mathcal{S}, \mu)$. Real L_1-preduals were in depth investigated in papers [17], [22], [23], [8], [52], [60], [24], [61], [16], [27], [13] or [57]. It has turned out that a real Banach space X is an L_1-predual if and only if its dual unit ball B_{X^*} satisfies a "simplex-like" condition (see [53]). This result indicates that methods developed in the theory of simplices might be useful in the study of L_1-preduals.

The paper is intended to follow the line of the cited papers and present a more detailed knowledge of the structure of L_1-preduals, namely, we generalize some results of Lindenstrauss and Wulbert in [55], Jellett in [40] and of ours in [56].

We work within the context of the field of real numbers and all topological spaces are considered to be Hausdorff.

For a topological space K, let $\mathcal{C}(K)$ be the space of all continuous functions on K, $\mathcal{B}(K)$ be the space of all Borel functions on K and $\mathcal{B}^b(K)$ be the space of all bounded Borel functions on K. In case K is compact, we write $\mathcal{M}(K)$ for the space of Radon measures on K and $\mathcal{M}^1(K)$ for the set of all Radon probability measures on K. For a point $x \in K$, ε_x stands for the Dirac measure at x. If $B \subset K$ is a Borel subset of K, we write $\mathcal{M}^1(B)$ for the subset of $\mathcal{M}^1(K)$ containing measures μ satisfying $\mu(B) = 1$.

Let K be a topological space and $\mathcal{H}$ be a subset of $\mathcal{C}(K)$. We set $\mathcal{B}^0(\mathcal{H}) = \mathcal{H}$ and, for $\alpha \in (0, \omega_1)$, let $\mathcal{B}^\alpha(\mathcal{H})$ consist of all pointwise limits of elements from $\bigcup_{\beta<\alpha} \mathcal{B}^\beta(\mathcal{H})$. Further we denote by $\mathcal{B}^{\alpha,b}(\mathcal{H})$ the set of all bounded elements from $\mathcal{B}^\alpha(\mathcal{H})$. The symbol $\mathcal{B}^{\alpha,bb}(\mathcal{H})$ denotes the inductive families created by means of pointwise limits of bounded sequences of lower classes.

If we start the inductive procedure from the space of all continuous functions, we write simply $\mathcal{B}^\alpha(K)$ and $\mathcal{B}^{\alpha,b}(K)$ for the spaces of Baire-α functions. Then we have $\mathcal{B}^{\alpha,b}(K) = \mathcal{B}^{\alpha,bb}(K)$. Let us remind that for a metrizable space K holds $\mathcal{B}^b(K) = \bigcup_{\alpha<\omega_1} \mathcal{B}^{\alpha,b}(K)$. Having started with the space $\mathcal{A}(K)$ of all continuous affine functions on a compact convex set K in a locally convex space, we obtain spaces $\mathcal{A}^\alpha(K)$, $\mathcal{A}^{\alpha,b}(K)$ and $\mathcal{A}^{\alpha,bb}(K)$. As a consequence of the uniform boundedness principle we get $\mathcal{A}^{\alpha,bb}(K) = \mathcal{A}^{\alpha,b}(K) = \mathcal{A}^\alpha(K)$ (see e.g. [58, Lemma 5.36]) and elements of this set we call *functions of affine class α*.

If X is a Banach space and B_{X^*} is its dual unit ball endowed with the weak* topology, X is isometrically embedded in $\mathcal{C}(B_{X^*})$ via canonical embedding. We recall definitions of Baire classes of X^{**} from [5]. For $\alpha \in [0, \omega_1)$, we call $\mathcal{B}^\alpha(X)$ the *intrinsic α-Baire class of* X^{**}. Following [5, p. 1044], we denote the intrinsic α-th Baire class as X^{**}_α. Let us remark that our definition is formally slightly different from the one in [5]. While in our case elements of X^{**}_α are restrictions of uniquely determined elements from X^{**} to the closed unit ball B_{X^*}, the functions considered in [5] are precisely these extensions.

Still considering X as a subspace of $\mathcal{C}(B_{X^*})$, *the α-th Baire class* of X^{**} is defined as

$$X^{**}_{\mathcal{B}_\alpha} = \{x^{**} \in X^{\perp\perp} : x^{**}|_{B^*_X} \in \mathcal{B}^\alpha(B_{X^*})\}.$$

Given an element $x^{**} \in X^{**}$, it can be verified that $x^{**} \in X^{**}_{\mathcal{B}_\alpha}$ if and only if $x^{**}|_{B_{X^*}} \in \mathcal{B}^\alpha(B_{X^*})$ and $x^{**}|_{B_{X^*}}$ satisfies the barycentric calculus, i.e.,

$$x^{**}\left(\int_{B_{X^*}} \mathrm{id}\; d\mu\right) = \int_{B_{X^*}} x^{**}\, d\mu$$

for every probability measure $\mu \in \mathcal{M}^1(B_{X^*})$. Where no confusion can arise, we do not distinguish between $X^{**}_{\mathcal{B}_\alpha}$ and $X^{**}_{\mathcal{B}_\alpha}|_{B_{X^*}}$.

Obviously, $X^{**}_\alpha \subset X^{**}_{\mathcal{B}_\alpha}$ but the converse need not hold by [85, Theorem] (we refer the reader for a detailed exposition on Baire classes of Banach spaces to [5, pp. 1043–1048]).

In [56] we have proven [56, Theorem 2.7]: *Let X be a separable L_1-predual.*

(a) Let $\alpha \in [0, \omega_1)$ and f be an odd function in $\mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*})$. Then there exists a function h such that $h = f$ on $\operatorname{ext} B_{X^*}$ and $h \in X^{**}_{\alpha+1}$ in case $\alpha \in [0, \omega_0)$ and $h \in X^{**}_\alpha$ in case $\alpha \in [\omega_0, \omega_1)$.

(b) If $\operatorname{ext} B_{X^*}$ is of type F_σ, then for any $\alpha \in [1, \omega_1)$ and an odd function $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*})$ there exists a function $h \in X^{**}_\alpha$ such that $h = f$ on $\operatorname{ext} B_{X^*}$.

The first goal of the paper is to give an argument that the previous assertion can be generalized to the nonseparable setting. This is accomplished by Theorems 4.2.14 and 4.2.15.

The second goal of our paper is to extend [56, Theorem 2.10] which states: *Let X be a separable L_1-predual such that $\operatorname{ext} B_{X^*}$ is an F_σ set. Then for any $\alpha \in [1, \omega_1)$, the space X^{**}_α is isometric to the space of all bounded odd Baire-α functions on $\operatorname{ext} B_{X^*}$.*

Corollary 4.2.16 carries the result to the context of nonseparable L_1-preduals. It generalizes a result by Lindenstrauss and Wulbert proved in [55, Theorem 1].

It is worth pointing out that for a separable Banach space X, the set of extreme points $\operatorname{ext} B_{X^*}$ is an F_σ set if and only if it is a Lindelöf H-set. In the nonseparable case only one implication remains valid in general: $\operatorname{ext} B_{X^*}$ is a Lindelöf H-set provided it is of type F_σ. For a detailed argument consult e.g. [57, p. 4].

4.2 Results

Let K be a compact convex set in a locally convex topological vector space. For a point $x \in K$, we can assign the set $\mathcal{M}^1_x(K)$ consisting of all probability measures on K satisfying $\int_K \mathrm{id}\; d\mu = x$ (equivalently, $\mu(h) = h(x)$ for any continuous affine function h on K). Given a measure $\mu \in \mathcal{M}^1(K)$, we write $r(\mu)$ for the unique point $x \in K$ satisfying $x = \int_K \mathrm{id}\; d\mu$ (see [1, Proposition I.2.1] or [51, Chapter 7, § 20]).

A function f on K is *strongly affine* if f is μ-measurable for each $\mu \in \mathcal{M}^1(K)$ and $f(x) = \mu(f)$ for any $x \in K$ and $\mu \in \mathcal{M}^1_x(K)$.

The usual dilation order $\prec$ on $\mathcal{M}^1(K)$ is defined as $\mu \prec \nu$ if and only if $\mu(f) \le \nu(f)$ for any convex continuous function f on K. We write $\mathcal{M}^{\max}(K)$ for the set of all probability measures on K which are maximal with respect to $\prec$. A measure $\mu \in \mathcal{M}(K)$ is *boundary* if either $\mu = 0$ or the probability measure $\frac{|\mu|}{\|\mu\|}$ is maximal. The symbol $\mathcal{M}^{\mathrm{bnd}}(K)$ denotes the space of all boundary measures on K.

For a function $f \in \mathcal{C}(K)$, let

$$\widehat{f}(x) = \sup\{\mu(f);\ \mu \in \mathcal{M}^1_x(K)\}, \quad x \in K.$$

By the Choquet representation theorem, for any $x \in K$ there exists $\mu \in \mathcal{M}^1_x(K) \cap \mathcal{M}^{\max}(K)$ (see [51, p. 192, Corollary]). The set K is termed *simplex* if this measure is uniquely determined for each $x \in K$ (see [51, § 20, Theorem 3]). In case K is metrizable, maximal measures are carried by the G_δ set $\operatorname{ext} K$ of extreme points of K (see [51, § 20, Theorem 5]). If K is a simplex, the space $\mathcal{A}(K)$ is an example of an L_1-predual (see [25, Proposition 3.23]).

We recall that a topological space X is *K-analytic* if it is the image of a Polish space under an upper semicontinuous compact-valued map (see [70, Section 2.1]).

If K is a topological space, a *zero set* in K is the inverse image of a closed set in $\mathbb{R}$ under a continuous function $f : K \to \mathbb{R}$. The complement of a zero set is a *cozero set*. A countable union of closed sets is called an *F_σ set*, the complement of an F_σ set is a *G_δ set*. If K is normal, it follows from Tietze's theorem that a closed set is a zero set if and only if it is also a G_δ set. We recall that *Borel sets* are members of the σ–algebra generated by the family of all open subset of K and *Baire sets* are members of the σ–algebra generated by the family of all cozero sets in K.

A set $A \subset K$ is an *H-set* (or a *resolvable set*) if for any nonempty $B \subset K$ (equivalently, for any nonempty closed $B \subset K$) there exists a relatively open $U \subset B$ such that either $U \subset A$ or $U \cap A = \emptyset$. It is easy to see that the family of all H-sets is an algebra (see e.g. [49, § 12, VI]).

We say that a function $f : K \to \mathbb{R}$ from a topological space K is a *Baire function* if it is measurable with respect to the σ-algebra of Baire sets (i.e., $f^{-1}(U)$ is a Baire set for every open set $U \subset \mathbb{R}$). It is well known that any Baire function belongs to some $\mathcal{B}^\alpha(K)$ for a suitable ordinal $\alpha \in [0, \omega_1)$.

Lemma 4.2.1. *Let K, L be K-analytic topological spaces and $r : K \to L$ be a continuous surjection. Let $g : L \to \mathbb{R}$. Then g is a Baire function on L if and only if $g \circ r$ is a Baire function on K.*

Proof. If g is a Baire function L, then $g \circ r$ is clearly a Baire function on K. Conversely, if $f = g \circ r$ is a Baire function on K and $U \subset \mathbb{R}$ is an open set, then both $f^{-1}(U)$ and $f^{-1}(\mathbb{R} \setminus U)$ are Baire sets in K. Then they are K-analytic sets in K (see [70, Section 2]), and thus

$$g^{-1}(U) = r(f^{-1}(U)), \quad g^{-1}(\mathbb{R} \setminus U) = r(f^{-1}(\mathbb{R} \setminus U))$$

are K-analytic as well. It follows from the proof of the standard separation theorem (see [70, Theorem 3.3.1]) that they are Baire sets. Hence g is measurable with respect to the σ-algebra of Baire sets, and thus it is a Baire function. $\square$

Lemma 4.2.2. *Let K be a compact convex set with $\operatorname{ext} K$ Lindelöf and let $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} K)$. Then there exist a Baire set $B \supset \operatorname{ext} K$ and a function $g \in \mathcal{B}^{\alpha,b}(B)$ such that*

- *$g = f$ on $\operatorname{ext} K$,*
- *$\mu(g) = \nu(g)$ for any $\mu, \nu \in \mathcal{M}^1(B)$ with $\mu \prec \nu$.*

Proof. We proceed by transfinite induction on the class of a function f.

Assume first that f is a bounded continuous function on $\operatorname{ext} K$. Using [57, Lemma 4.5] we find sequences (u_n) and (l_n) such that

- the functions u_n are continuous concave on K, l_n are continuous convex on K,
- $\inf f(\operatorname{ext} K) \leq \inf l_1(K)$, $\sup u_1(K) \leq \sup f(\operatorname{ext} K)$,
- $u_n \searrow f$, $l_n \nearrow f$ on $\operatorname{ext} K$.

We define $u = \inf_{n\in\mathbb{N}} u_n$, $l = \sup_{n\in\mathbb{N}} l_n$. Then we observe that $l \leq u$ by the minimum principle (see [1, Theorem I.5.3] or [58, Theorem 3.16]), both functions are Baire, u is upper semicontinuous concave and l is lower semicontinuous convex. Let

$$B = \{x \in K \colon u(x) = l(x)\} \quad \text{and} \qquad g(x) = u(x), \quad x \in B.$$

Then B is a Baire set containing $\operatorname{ext} K$ and, for $\mu, \nu \in \mathcal{M}^1(B)$ with $\mu \prec \nu$, we have by [58, Proposition 3.56]

$$\begin{aligned} \mu(g) &= \int_B g\, d\mu = \int_B u\, d\mu \geq \int_B u\, d\nu = \int_B g\, d\nu \\ &= \int_B l\, d\nu \geq \int_B l\, d\mu = \int_B g\, d\mu = \mu(g). \end{aligned}$$

Hence $\mu(g) = \nu(g)$. Finally, since g is continuous on B, the proof is finished for the case $\alpha = 0$.

Assume now that the claim holds true for all β smaller then some countable ordinal α. Given $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} K)$, let (f_n) be a bounded sequence of functions with $f_n \in \mathcal{B}^{\alpha_n,b}(\operatorname{ext} K)$ for some $\alpha_n < \alpha$, $n \in \mathbb{N}$, such that $f_n \to f$. For each $n \in \mathbb{N}$, we use the induction hypothesis and find a Baire set $B_n \supset \operatorname{ext} K$ along with a function $g_n \in \mathcal{B}^{\alpha_n,b}(B_n)$ that coincides with f_n on $\operatorname{ext} K$ and satisfies $\mu(g_n) = \nu(g_n)$ for any $\mu, \nu \in \mathcal{M}^1(B_n)$ with $\mu \prec \nu$.

We set

$$B = \{x \in \bigcap_{n=1}^{\infty} B_n \colon (g_n(x)) \text{ converges}\} \quad \text{and} \quad g(x) = \lim_{n\to\infty} g_n(x),\ x \in B.$$

Then B is a Baire set containing $\operatorname{ext} K$, $g \in \mathcal{B}^{\alpha,b}(B)$,

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x) = f(x), \quad x \in \operatorname{ext} K,$$

and, for $\mu, \nu \in \mathcal{M}^1(B)$ satisfying $\mu \prec \nu$,

$$\mu(g) = \int_B (\lim g_n)\, d\mu = \lim \mu(g_n) = \lim \nu(g_n) = \int_B (\lim g_n)\, d\nu = \nu(g).$$

This finishes the proof. $\square$

Lemma 4.2.3. *Let K be a compact convex set, $B \supset \operatorname{ext} K$ be a Baire set and $f\colon B \to \mathbb{R}$ be a function such that*

(a) f is bounded and Baire,

(b) $\mu(f) = \nu(f)$ for every $\mu, \nu \in \mathcal{M}^1(B)$ with $\mu \prec \nu$,

(c) $\mu(f) = 0$ for every $\mu \in \mathcal{M}^{\text{bnd}}(K) \cap \mathcal{A}(K)^{\perp}$.

Then there exists an affine bounded Baire function $h : K \to \mathbb{R}$ such that

(d) $h = f$ on B,

(e) $\mu(h) = h(r(\mu))$ for any $\mu \in \mathcal{M}^{\max}(K) \cap \mathcal{M}^1(K)$.

Proof. Let $B \supset \operatorname{ext} K$ and $f : B \to \mathbb{R}$ be as in the hypothesis.

We set

$$h(x) = \nu(f), \quad \nu \in \mathcal{M}^1_x(K) \cap \mathcal{M}^{\max}(K), x \in K.$$

Then h is correctly defined because of (c) and the fact that any maximal measure is carried by B (see [1, Remark, p. 38] or [58, Theorem 3.79(a)]).

Further, h is affine. Indeed, let $\alpha x + (1-\alpha)y$ be a convex combination of points $x, y \in K$. Pick $\nu_x \in \mathcal{M}^1_x(K) \cap \mathcal{M}^{\max}(K)$ and $\nu_y \in \mathcal{M}^1_y(K) \cap \mathcal{M}^{\max}(K)$. Since the set of maximal measures is a convex cone and the mapping r is affine,

$$\alpha\nu_x + (1-\alpha)\nu_y \in \mathcal{M}^1_{\alpha x+(1-\alpha)y}(K) \cap \mathcal{M}^{\max}(K).$$

Thus

$$h(\alpha x+(1-\alpha)y) = (\alpha\nu_x+(1-\alpha)\nu_y)(f) = \alpha\nu_x(f)+(1-\alpha)\nu_y(f) = \alpha h(x)+(1-\alpha)h(y),$$

and h is affine.

Obviously, due to (b), the fact that any maximal measure is carried by B and the definition of h we have

$$\begin{aligned} h(x) = \nu(f) = \varepsilon_x(f) = f(x), &\quad \nu \in \mathcal{M}^1_x(K) \cap \mathcal{M}^{\max}(K), x \in B,\\ h(r(\mu)) = \mu(f) = \mu(h), &\quad \mu \in \mathcal{M}^1(K) \cap \mathcal{M}^{\max}(K). \end{aligned}$$

Thus (d) and (e) hold.

Finally we show that h is Baire. Since the characteristic function of B is a Baire function, the function $\widetilde{B}(\mu) = \mu(B)$, $\mu \in \mathcal{M}^1(K)$, is a Baire function on $\mathcal{M}^1(K)$ as well. Thus

$$\mathcal{M}^1(B) = \{\mu \in \mathcal{M}^1(K)\colon \widetilde{B}(\mu) = 1\}$$

is a convex Baire set. Consequently, it is a K-analytic set in $\mathcal{M}^1(K)$ (see e.g. [70, Section 2.5]). Since f is a bounded Baire function on B, the function $\widetilde{f}$: $\mathcal{M}^1(B) \to \mathbb{R}$ defined as

$$\widetilde{f}(\mu) = \int_B f\, d\mu, \quad \mu \in \mathcal{M}^1(B),$$

is a well defined Baire function on $\mathcal{M}^1(B)$. The mapping $r : \mathcal{M}^1(B) \to K$ is an affine continuous surjection (this follows from [1, p. 12] or [58, Proposition 2.38]) and $\widetilde{f} = h \circ r$.

Indeed, let $\mu \in \mathcal{M}^1(B)$. We pick a maximal measure $\nu \in \mathcal{M}^{\max}(K)$ with $\mu \prec \nu$. Then $\nu \in \mathcal{M}^1(B)$ and $r(\mu) = r(\nu)$, thus due to (b)

$$\widetilde{f}(\mu) = \mu(f) = \nu(f) = h(r(\nu)) = h(r(\mu)) = (h \circ r)(\mu).$$

By Lemma 4.2.1, h is a Baire function on K. □

Lemma 4.2.4. *Let K be a topological space, $\mathcal{H} \subset \mathcal{C}(K)$, $\alpha \in [0, \omega_1)$ and $f \in \mathcal{B}^\alpha(\mathcal{H})$. Then there exists a countable set $\mathcal{F} \subset \mathcal{H}$ such that $f \in \mathcal{B}^\alpha(\mathcal{F})$.*

Proof. The assertion follows by transfinite induction. □

Lemma 4.2.5. *Let K be a compact convex set and $f : K \to \mathbb{R}$ be a bounded Baire affine function such that $\mu(f) = f(r(\mu))$ for every $\mu \in \mathcal{M}^{\max}(K)$. Then f is strongly affine.*

Proof. Since f is a Baire function, there exists an ordinal $\alpha \in [0, \omega_1)$ and a countable family $\mathcal{F} \subset \mathcal{C}(K)$ such that $f \in \mathcal{B}^\alpha(\mathcal{F})$ (see Lemma 4.2.4). Then each function in $\mathcal{F}$ can be uniformly approximated by a sequence of functions of the form $p_1 - p_2$, where p_1, p_2 are continuous convex functions (see [1, Proposition I.1.1]). We denote the set of all functions p_1, p_2 needed in the approximation of elements in $\mathcal{F}$ by $\mathcal{P}$. Next, every continuous convex function in $\mathcal{P}$ can be uniformly approximated by a sequence of functions of the form $\sup\{a_1, \dots, a_n\}$, where $n \in \mathbb{N}$ and $a_1, \dots, a_n$ are continuous affine functions (see [1, Corollary I.1.3]). Let us denote the countable set of all such continuous affine functions involved in the approximation of functions in $\mathcal{P}$ by $\{a_n : n \in \mathbb{N}\}$.

We define an affine continuous mapping

$$\begin{aligned} \varphi\colon K &\to \mathbb{R}^{\mathbb{N}} \\ x &\mapsto \{a_n(x)\}_{n\in\mathbb{N}}, \quad x \in K, \end{aligned}$$

which maps X onto a metrizable compact convex set $L = \varphi(K)$. Further, if $\varphi(x_1) = \varphi(x_2)$ for $x_1, x_2 \in K$, then clearly $f(x_1) = f(x_2)$. Thus we can define a function

$$\begin{aligned} g\colon L &\to \mathbb{R} \\ y &\mapsto f(x), \quad x \in \varphi^{-1}(y), y \in Y. \end{aligned}$$

Then $f = g \circ \varphi$.

By Lemma 4.2.1, the function g is Baire and it is easy to see that it is bounded and affine. Further, for any $\nu \in \mathcal{M}^{\max}(L)$ we have $\nu(g) = g(r(\nu))$.

Indeed, let $\nu \in \mathcal{M}^{\max}(L)$ be given. We find a measure $\mu \in \mathcal{M}^{\max}(K)$ with $\varphi\mu = \nu$ (see [58, Proposition 7.49]). Since

$$h(\varphi(r(\mu))) = \mu(h \circ \varphi) = (\varphi\mu)(h) = \nu(h) = h(r(\nu)), \quad h \in \mathcal{A}(L),$$

we obtain $\varphi(r(\mu)) = r(\nu)$. Thus

$$\nu(g) = \mu(g \circ \varphi) = \mu(f) = f(r(\mu)) = (g \circ \varphi)(r(\mu)) = g(r(\nu)),$$

and the claim is proved.

Since L is metrizable, by [85, Théorème 1] (see also [58, Theorem 11.41]) there exists a Borel mapping $S : y \mapsto \nu_y$ from L to $\mathcal{M}^1(L)$ such that $\nu_y \in$

$\mathcal{M}^1_y(L) \cap \mathcal{M}^{\max}(L)$ for each $y \in L$. Now we prove that g is strongly affine. Let $\nu \in \mathcal{M}^1(L)$ be given. We define a measure $\omega \in \mathcal{M}^1(L)$ as

$$\omega(c) = \int_L \nu_y(c)\, d\nu(y), \quad c \in \mathcal{C}(L).$$

Since, for any $c \in \mathcal{C}(L)$,

$$\omega(\widehat{c}) = \int_L \nu_y(\widehat{c})\, d\nu(y) = \int_L \nu_y(c)\, d\nu(y) = \omega(c),$$

ω is maximal due to [58, Theorem 3.58]. Further,

$$\omega(h) = \int_L \nu_y(h)\, d\nu(y) = \nu(h) = h(r(\nu)), \quad h \in \mathcal{A}(L),$$

and thus $\omega \in \mathcal{M}^1_{r(\nu)}(L)$.

Hence

$$g(r(\nu)) = \omega(g) = \int_L \nu_y(g)\, d\nu(y) = \int_L g(y)\, d\nu(y) = \nu(g),$$

and g is strongly affine.

Now it suffices to use [75, Proposition 3.2] (see also [58, Proposition 5.29]) to conclude that f is strongly affine. □

Definition 4.2.6. Let X be a Banach space. Then $\sigma(x^*) = -x^*$, $x^* \in B_{X^*}$, is a natural affine homeomorphism of B_{X^*} onto itself. A set $B \subset B_{X^*}$ is *symmetric* if $\sigma(B) = B$. An example of a symmetric set is the set $\operatorname{ext} B_{X^*}$. For a function f defined on a symmetric set $B \subset B_{X^*}$ we define

$$(\operatorname{odd} f)(x^*) = \frac{1}{2}(f(x^*) - f(-x^*)), \quad x^* \in B.$$

A function f defined on a symmetric subset of B_{X^*} is odd if $\operatorname{odd} f = f$.

For $\mu \in \mathcal{M}(B_{X^*})$, let $\operatorname{odd} \mu \in \mathcal{M}(B_{X^*})$ be defined as

$$(\operatorname{odd} \mu)(f) = \mu(\operatorname{odd} f), \quad f \in \mathcal{C}(B_{X^*}).$$

The following characterization of L_1-preduals is due to Lazar (see [53, Theorem] or [51, § 21, Theorem 7])):

Let X be a Banach space. Then X is an L_1-predual if and only if $\operatorname{odd} \mu = \operatorname{odd} \nu$ *for each $x^* \in B_{X^*}$ and $\mu, \nu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$.*

Lemma 4.2.7. *Let X be an L_1-predual such that* $\operatorname{ext} B_{X^*}$ *is Lindelöf. Then for every bounded odd Baire function on* $\operatorname{ext} B_{X^*}$ *there exist its odd Baire strongly affine extension on B_{X^*}.*

Proof. Let f be an odd bounded Baire function on $\operatorname{ext} B_{X^*}$. By Lemma 4.2.2 there exist a Baire set $\widetilde{B} \supset \operatorname{ext} B_{X^*}$ and a bounded Baire function $\widetilde{h} : \widetilde{B} \to \mathbb{R}$ such that

- $\widetilde{h} = f$ on $\operatorname{ext} B_{X^*}$,

- $\mu(\widetilde{h}) = \nu(\widetilde{h})$ for any $\mu, \nu \in \mathcal{M}^1(\widetilde{B})$ with $\mu \prec \nu$.

We set

$$B = \widetilde{B} \cap -\widetilde{B} \quad \text{and} \quad h = \operatorname{odd} \widetilde{h} \text{ on } B.$$

Then h is an odd Baire function on a symmetric Baire set $B \supset \operatorname{ext} B_{X^*}$ extending f and satisfying $\mu(h) = \nu(h)$ for any $\mu, \nu \in \mathcal{M}^1(B)$ with $\mu \prec \nu$.

Indeed, let $\mu \prec \nu$ with $\mu, \nu \in \mathcal{M}^1(B)$ be given. Then $\mu, \nu \in \mathcal{M}^1(\widetilde{B})$ as well as $\sigma\mu, \sigma\nu \in \mathcal{M}^1(\widetilde{B})$. Further, $\sigma\mu \prec \sigma\nu$. Thus $\sigma\mu(\widetilde{h}) = \sigma\nu(\widetilde{h})$ and

$$\begin{aligned}\mu(h) = \mu(\operatorname{odd} \widetilde{h}) &= \int_B \frac{1}{2}(\widetilde{h}(x^*) - \widetilde{h}(-x^*))\, d\mu(x^*) \\ &= \frac{1}{2}(\mu(\widetilde{h}) - \sigma\mu(\widetilde{h})) = \frac{1}{2}(\nu(\widetilde{h}) - \sigma\nu(\widetilde{h})) = \nu(h).\end{aligned}$$

Let

$$\omega \in \mathcal{M}^{\mathrm{bnd}}(B_{X^*}) \cap \mathcal{A}(B_{X^*})^{\perp}$$

be given. Without loss of generality we may assume that $\omega = \mu - \nu$, where $\mu, \nu \in \mathcal{M}^{\max}(B_{X^*})$. By Lazar's theorem [53, Theorem] (see also [51, § 21, Theorem 7]),

$$\mu(h) = \mu(\operatorname{odd} h) = (\operatorname{odd} \mu)(h) = (\operatorname{odd} \nu)(h) = \nu(\operatorname{odd} h) = \nu(h).$$

Hence $\omega(h) = 0$. By Lemma 4.2.3 there exists an affine bounded Baire extension g of h satisfying $\mu(g) = g(r(\mu))$ for each $\mu \in \mathcal{M}^{\max}(B_{X^*})$. By Lemma 4.2.5, the extension g is strongly affine.

To show that g is odd, it is enough to verify that $g(0) = 0$. But this is obvious since, for a point $x^* \in \operatorname{ext} B_{X^*}$,

$$g(0) = \frac{1}{2}(\varepsilon_{x^*} + \varepsilon_{-x^*})(g) = \frac{1}{2}(\varepsilon_{x^*} + \varepsilon_{-x^*})(h) = 0.$$

This concludes the proof. □

Lemma 4.2.8. *Let K be a compact convex set with* $\operatorname{ext} K$ *being Lindelöf. Then any bounded Baire function on* $\operatorname{ext} K$ *can be extended to a bounded Baire function on K.*

Proof. Let $\mathcal{F}$ be the family of all bounded Baire functions on $\operatorname{ext} K$ that are extendable to a bounded Baire function on K.

By [43, Theorem 30], $\mathcal{F}$ contains any bounded Baire-1 function on $\operatorname{ext} K$. Let (f_n) be a bounded sequence of functions from $\mathcal{F}$ converging to a function f on $\operatorname{ext} K$. Let $\widetilde{f_n}$ be a bounded Baire extension of f_n, $n \in \mathbb{N}$. Without loss of generality we may assume that the functions $\widetilde{f_n}$ are bounded by the same constant. By setting

$$\widetilde{f} = \limsup \widetilde{f_n},$$

we obtain a bounded Baire function extending f. Hence $\mathcal{F}$ is closed with respect to taking pointwise limits of bounded sequences. Thus $\mathcal{F}$ contains every bounded Baire function on $\operatorname{ext} K$. □

Definition 4.2.9. Let X be an L_1-predual with $\operatorname{ext} B_{X^*}$ Lindelöf. For any bounded Baire function f on $\operatorname{ext} B_{X^*}$ we define

$$Tf(x^*) = (\operatorname{odd}\mu)(\widetilde{f}), \quad \mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*}), x^* \in B_{X^*},$$

where $\widetilde{f}$ is an arbitrary bounded Baire function on B_{X^*} extending f.

We hasten to add that Tf is well defined since

- $\operatorname{odd}\mu = \operatorname{odd}\nu$ for any $\mu, \nu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$ and $x^* \in B_{X^*}$ by the mentioned Lazar theorem,
- f has a bounded Baire extension on B_{X^*} (see Lemma 4.2.8),
- given two bounded Baire extensions $\widetilde{f}_1$, $\widetilde{f}_2$ of f, they coincide on a Baire set containing $\operatorname{ext} B_{X^*}$, and thus $(\operatorname{odd}\mu)(\widetilde{f}_1) = (\operatorname{odd}\mu)(\widetilde{f}_2)$ for any $\mu \in \mathcal{M}^{\max}(B_{X^*})$.

The described mapping T is a natural generalization of the dilation mapping defined in the simplicial case e.g. in [58, Definition 6.7].

Lemma 4.2.10. *Let X be an L_1-predual with $\operatorname{ext} B_{X^*}$ Lindelöf. Let f be a bounded Baire function on $\operatorname{ext} B_{X^*}$. Then Tf is a bounded odd Baire strongly affine function on B_{X^*} such that $Tf = \operatorname{odd} f$ on $\operatorname{ext} B_{X^*}$.*

Proof. Let $\widetilde{f}$ be a bounded Baire function on B_{X^*} extending f (see Lemma 4.2.8). Since $\operatorname{odd}\widetilde{f}$ is an odd bounded Baire function on B_{X^*}, by Lemma 4.2.7 there exists an odd Baire strongly affine function h on B_{X^*} satisfying $h = \operatorname{odd}\widetilde{f}$ on $\operatorname{ext} B_{X^*}$. Let $x^* \in B_{X^*}$ be given and let $\mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$. Since $\operatorname{odd}\mu$ is boundary and $h = \operatorname{odd}\widetilde{f}$ on a Baire set containing $\operatorname{ext} B_{X^*}$, we obtain

$$Tf(x^*) = (\operatorname{odd}\mu)(\widetilde{f}) = \mu(\operatorname{odd}\widetilde{f}) = \mu(h) = h(x^*).$$

Thus Tf is an odd Baire strongly affine function on B_{X^*}.

Finally, for a point $x^* \in \operatorname{ext} B_{X^*}$ we have

$$Tf(x^*) = h(x^*) = (\operatorname{odd}\widetilde{f})(x^*) = (\operatorname{odd} f)(x^*).$$

The proof is finished. □

Lemma 4.2.11. *Let X be an L_1-predual with $\operatorname{ext} B_{X^*}$ Lindelöf. Let (f_n) be a bounded sequence of Baire functions on $\operatorname{ext} B_{X^*}$ converging pointwise to f on $\operatorname{ext} B_{X^*}$. Then $Tf_n \to Tf$.*

Proof. Let $\widetilde{f}_n$ be bounded Baire extensions of the functions (f_n) (see Lemma 4.2.8), obviously we may assume that they are bounded by the same constant. Then

$$\widetilde{f} = \limsup \widetilde{f}_n$$

is a bounded Baire function extending f. The set

$$B = \{x^* \in B_{X^*} \colon \widetilde{f}(x^*) = \lim_{n\to\infty} \widetilde{f}_n(x^*)\}$$

is a Baire set containing $\operatorname{ext} B_{X^*}$. Thus, for $x^* \in B_{X^*}$ and $\mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^{\max}(B_{X^*})$, we have

$$Tf_n(x^*) = (\operatorname{odd}\mu)(\widetilde{f}_n) = \int_B \widetilde{f}_n \, d(\operatorname{odd}\mu) \to \int_B \widetilde{f} \, d(\operatorname{odd}\mu) = (\operatorname{odd}\mu)(\widetilde{f}) = Tf(x^*).$$

The proof is finished. □

Lemma 4.2.12. *Let X be an L_1-predual with* $\operatorname{ext} B_{X^*}$ *Lindelöf and $\alpha \in [0, \omega_1)$. Let $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*})$. Then*

- *$Tf \in X^{**}_{\alpha+1}$ if $\alpha \in [0, \omega_0)$,*
- *$Tf \in X^{**}_{\alpha}$ if $\alpha \in [\omega_0, \omega_1)$.*

Proof. If $\alpha = 0$, then Tf is a strongly affine function whose restriction to $\operatorname{ext} B_{X^*}$ is equal to a continuous function $\operatorname{odd} f$ (see Lemma 4.2.10). By [57, Theorem 5.2], $Tf \in \mathcal{B}^{1,b}(B_{X^*})$. Thus $Tf \in X^{**}_1$ by [5, Theorem II.1.2(a)].

For $\alpha < \omega_0$ now the proof follows by induction using Lemma 4.2.11.

If $\alpha = \omega_0$, let $f_n \in \mathcal{B}^{\alpha_n,b}(\operatorname{ext} B_{X^*})$, $\alpha_n < \alpha$, form a bounded sequence converging to $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*})$. By Lemma 4.2.11, $Tf_n \to Tf$. By the first part of the proof, $Tf \in X^{**}_{\alpha}$.

For higher Baire classes we use again transfinite induction. □

Lemma 4.2.13. *Let X be an L_1-predual with* $\operatorname{ext} B_{X^*}$ *being a Lindelöf H-set and $\alpha \in [1, \omega_1)$. Let $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*})$. Then $Tf \in X^{**}_{\alpha}$.*

Proof. The proof is analogous to the proof of Lemma 4.2.12, we only use instead of [57, Theorem 5.2] as the starting point of transfinite induction the following fact from [57, Theorem 6.4]: *If* $\operatorname{ext} B_{X^*}$ *is a Lindelöf H-set and h is a strongly affine function on B_{X^*} whose restriction on* $\operatorname{ext} B_{X^*}$ *is Baire-*1*, then h is Baire-*1 *on B_{X^*}.* Any such function is then in X^{**}_1 by [5, Theorem II.1.2(a)]. □

Theorem 4.2.14. *Let X be an L_1-predual with* $\operatorname{ext} B_{X^*}$ *Lindelöf and $\alpha \in [0, \omega_1)$. Then for every odd function $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*})$ there exists a function h on B_{X^*} extending f such that*

- *$h \in X^{**}_{\alpha+1}$ if $\alpha \in [0, \omega_0)$,*
- *$h \in X^{**}_{\alpha}$ if $\alpha \in [\omega_0, \omega_1)$.*

Proof. By Lemma 4.2.12, if $\alpha \in [0, \omega_0)$ then the function Tf is in $X^{**}_{\alpha+1}$, and if $\alpha \in [\omega_0, \omega_1)$ then $Tf \in X^{**}_{\alpha}$. Since $Tf = \operatorname{odd} f = f$ on $\operatorname{ext} B_{X^*}$, the proof is finished. □

Theorem 4.2.15. *Let X be an L_1-predual such that* $\operatorname{ext} B_{X^*}$ *is a Lindelöf H-set. Let $\alpha \in [1, \omega_1)$. Then for every odd function $f \in \mathcal{B}^{\alpha,b}(\operatorname{ext} B_{X^*})$ there exists a function $h \in X^{**}_{\alpha}$ extending f.*

Proof. The proof is analogous to the proof of Theorem 4.2.14, only we use Lemma 4.2.13 instead of Lemma 4.2.12. □

Corollary 4.2.16. *Let X be an L_1-predual such that* $\operatorname{ext} B_{X^*}$ *is a Lindelöf H-set. Let $\alpha \in [1, \omega_1)$. Then the space X^{**}_{α} is isometric to the space of all bounded odd Baire-α functions on* $\operatorname{ext} B_{X^*}$.

Proof. A function $f \in X^{**}_{\alpha}$ is bounded, Baire-α and strongly affine. The restriction mapping $f \in X^{**}_{\alpha} \mapsto f|_{\operatorname{ext} B_{X^*}}$ is therefore an isometric isomorphism onto the space of all bounded odd Baire-α functions on $\operatorname{ext} B_{X^*}$ due to Theorem 4.2.15 and the minimum principle [58, Theorem 3.86]. □

4.3 Questions

Let X be an L_1-predual. Theorems 4.2.14 and 4.2.15 show that under some topological assumptions on $\operatorname{ext} B_{X^*}$ it is possible to extend odd bounded Baire functions from $\operatorname{ext} B_{X^*}$ to odd strongly affine Baire functions on B_{X^*}. The question is whether the topological assumptions are not only sufficient but also necessary. To be more precise, we propose the following questions.

Question 4.3.1. Let X be an L_1-predual such that every continuous odd bounded function on $\operatorname{ext} B_{X^*}$ can be extended to an element from X_1^{**}. Is the set $\operatorname{ext} B_{X^*}$ Lindelöf?

This question stems from the result of Jellett in [40] stating that any bounded continuous function on the set of all extreme points of a simplex K can be extended to an affine Baire-1 function provided the set $\operatorname{ext} K$ is Lindelöf.

Question 4.3.2. Let X be an L_1-predual such that every odd bounded Baire-1 function on $\operatorname{ext} B_{X^*}$ can be extended to an element from X_1^{**}. Is the set $\operatorname{ext} B_{X^*}$ a Lindelöf H-set?

The answer to Question 4.3.2 is known to be positive in case X is the space $\mathcal{A}(K)$ for a simplex K (see [81]).

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5. Distances to spaces of first H-class mappings

5.1 Introduction

We open the section with an important notion of fragmented mapping introduced in [47]. Let X be a topological space and E a metric space. Then a mapping $f : X \to E$ is called *ε-fragmented* if for every nonempty (equivalently, nonempty closed) set $F \subset X$ there exists an open set $V \subset X$ such that $V \cap F \neq \emptyset$ and $\operatorname{diam} f(V \cap F) \leq \varepsilon$. We write briefly

$$\operatorname{frag}(f) = \inf\{\varepsilon > 0 : f \text{ is } \varepsilon\text{-fragmented}\}$$

and term f to be *fragmented* if $\operatorname{frag}(f) = 0$. We designate the set of all fragmented mapping $f : X \to E$ by $\operatorname{Frag}(X, E)$.

A set H in X is *resolvable* if its characteristic function χ_H is fragmented, that is, for any nonempty (equivalently nonempty closed) $A \subset X$ there exists a nonempty relatively open set $U \subset A$ such that either $U \subset H$ or $U \subset X \setminus H$. We refer the reader to [49, § 12, V-VI] or [47, Section 2] for elementary properties of resolvable sets. We just recall that the family $\operatorname{Hs}(X)$ of all resolvable sets forms an algebra containing all open sets.

A useful characterization of resolvable sets says that a set $H \subset X$ is resolvable if and only if there exist an ordinal κ, an ordinal interval $I \subset [0, \kappa)$ and an increasing sequence of open sets $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_\kappa = X$ satisfying

- $U_\gamma = \bigcup\{U_\lambda : \lambda < \gamma\}$ for a limit ordinal $\gamma \leq \kappa$,

- $H = \bigcup\{U_{\gamma+1} \setminus U_\gamma : \gamma \in I\}$.

A system of sets $\{U_{\alpha+1} \setminus U_\alpha : \alpha < \kappa\}$ with such properties is called a *resolvable partition* of X whereas a system of open sets $\{U_\gamma : \gamma \leq \kappa\}$ is termed a *regular sequence* of open sets.

Let $\mathcal{F}$ be a family of subsets of X. Then f is *ε-σ-fragmented by sets of $\mathcal{F}$* if there is a system $\{X_n : n \in \mathbb{N}\} \subset \mathcal{F}$ covering X such that $f|_{X_n}$ is ε-fragmented for each $n \in \mathbb{N}$.

We set

$$\sigma\text{-}\operatorname{frag}_{\mathcal{F}}(f) = \inf\{\varepsilon > 0 : f \text{ is } \varepsilon\text{-}\sigma\text{-fragmented by sets of } \mathcal{F}\}$$

and say that f is *σ-fragmented by sets of $\mathcal{F}$* if $\sigma\text{-}\operatorname{frag}_{\mathcal{F}}(f) = 0$. For the sake of brevity we write just σ-frag instead of $\sigma\text{-}\operatorname{frag}_{\operatorname{Hs}(X)}$ if $\mathcal{F} = \operatorname{Hs}(X)$ (cf. [39]).

Following Koumoullis in [47], we consider a certain generalization of functions of Baire class 1 (i.e. functions expressible as pointwise limits of sequences of continuous functions) from the metric spaces within the context of topological spaces. Let $\operatorname{Bos}(X)$ stand for the algebra generated by open sets and $\operatorname{Hs}(X)$ for the algebra of all resolvable sets. Let $\Sigma_2(\operatorname{Bos}(X))$ stand for the family of all countable unions of sets from $\operatorname{Bos}(X)$, analogously we define $\Sigma_2(\operatorname{Hs}(X))$. We say that $f : X \to E$ is a mapping of the *first Borel class* if f is $\Sigma_2(\mathrm{Bos}(X))$-measurable, that is, $f^{-1}(U) \in \Sigma_2(\mathrm{Bos}(X))$ for every open set $U \subset X$. The family of such mappings is designated by $\mathrm{Bof}_1(X, E)$. Similarly we define the family of all mappings of the *first resolvable class* and denote them by $\mathrm{Hf}_1(X, E)$. If X is metrizable then $\mathrm{Hf}_1(X, E)$ and $\mathrm{Bof}_1(X, E)$ coincide.

It is easy to check that any function of Baire class 1 is also of the first Borel class (see, e.g., [59, Exercise 3.A.1]) and that any function of the first Borel class is also of the first resolvable class.

The aim of our paper is twofold. In Section 5.3 of the paper we extend some results of [12], [4] and [80] by computing the distance of a general mapping to the family of mappings of the first resolvable class via the quantity frag. In Section 5.4 we provide results analogous to those in [3], [4] concerning quantitative difference between countable compactness and compactness in $\mathrm{Hf}_1(X, E)$. Kindred results may be also found in [2].

Second, in Section 5.5 we study a class of mappings with a countable oscillation rank and relate its basic properties to the aforementioned classes of mappings. This rank has been in a less general context considered by many authors (see, e.g., [32]). It also possess a connection to the Szlenk index (e.g., [21, Definition 8.5]) in a following way. Assume X is an infinite-dimensional Banach space and K a w^*-compact subset of X^*. Let f be an identity mapping of (K, w^*) onto $(K, \|\cdot\|)$. Then a Szlenk index of K is equal to the oscillation rank of f.

An investigation of a class of mappings with a countable oscillation rank is also motivated by the following well known characterization of functions of Baire class 1: Let E be a compact metrizable space. Then a function $f : E \to \mathbb{R}$ is of Baire class 1 if and only if its oscillation rank is countable ([46, Proposition 2]).

5.2 Preliminaries

This section contains auxiliary results dealing particularly with an approximation of fragmented mappings which serves us well in the arguments appearing in Section 5.3.

Having a metric space E, the function diam of making a diameter is always meant with respect to the metric of the space E. Having a subset A of a linear space X we denote its convex hull (i.e., intersection of all convex sets of X containing A) by $\operatorname{co} A$. Throughout the paper we also adopt a convention that $\inf \emptyset = \infty$.

Lemma 5.2.1. *Let X be a hereditarily Baire topological space, E a metric space. Then for a mapping $f : X \to E$ holds σ-$\mathrm{frag}(f) = \mathrm{frag}(f)$.*

Proof. An inequality σ-$\mathrm{frag}(f) \leq \mathrm{frag}(f)$ follows immediately from the definitions.

If σ-$\mathrm{frag}(f) = \infty$ then also the remaining inequality holds. Provided σ-$\mathrm{frag}(f) \neq \infty$, let us assume that σ-$\mathrm{frag}(f) < \varepsilon$ for some $\varepsilon \in \mathbb{R}$. Then there exists a sequence $\{H_n : n \in \mathbb{N}\} \subset \mathrm{Hs}(X)$ covering X such that $f|_{H_n}$ is ε-fragmented for each $n \in \mathbb{N}$. Given a closed subset F of X let us define resolvable sets $E_n = H_n \cap F$, $n \in \mathbb{N}$. From the proof of [47, Proposition 2.1(iv)] follows that for each $n \in \mathbb{N}$ there exist sets U_n, N_n respectively open and nowhere dense in F

satisfying $E_n = U_n \cup N_n$. Since F is a Baire space there exists $k \in \mathbb{N}$ such that $U_k \neq \emptyset$.

Employing ε-fragmentability of $f|_{H_k}$ we find an open subset V of X such that $V \cap U_k \cap F \neq \emptyset$ and $\operatorname{diam} f(V \cap U_k \cap F) < \varepsilon$. Hence, f is ε-fragmented and $\operatorname{frag}(f) \leq \sigma\text{-frag}(f)$.

□

Lemma 5.2.2. *Let X be a topological space, (E, ρ) a metric space and $f : X \to E$ an ε-fragmented mapping for some $\varepsilon > 0$. Then there exists a mapping $g : X \to E$ which is constant on each set of some resolvable partition of X and $\rho(f(x), g(x)) < \varepsilon$ for every $x \in X$.*

Moreover, if $E = \mathbb{R}$ then such g can be found that $\rho(f(x), g(x)) < \frac{\varepsilon}{2}$ for every $x \in X$.

Proof. We find an ordinal Γ and construct sets G_α, F_α for $\alpha < \Gamma$ by transfinite induction.

Set $G_0 = F_0 = \emptyset$. Let us assume that sets G_ξ, F_ξ are constructed for every $\xi < \gamma$. If $\bigcup_{\xi<\gamma} G_\xi = X$ we set $\Gamma := \gamma$ and stop the construction. Otherwise, due to ε-fragmentability of f there exists an open set G_γ satisfying

$$\operatorname{diam} f(G_\gamma \setminus \bigcup_{\xi<\gamma} G_\xi) < \varepsilon.$$

We set $F_\gamma = G_\gamma \setminus \bigcup_{\xi<\gamma} G_\xi$.

Then $\{F_\gamma : \gamma < \Gamma\}$ is a resolvable partition of X and we can define a mapping $g : X \to E$ as follows: Given $\gamma < \Gamma$ and $x \in F_\gamma$ we set $g(x) = t_\gamma$ where in the general case $t_\gamma \in f(F_\gamma)$ is chosen arbitrarily whereas in case $E = \mathbb{R}$ we take t_γ as the center of $\operatorname{co}(f(F_\gamma))$.

A moment of reflection shows that $g \in \mathrm{Hf}_1(X, E)$. Moreover, in the general case the inequality $\rho(f(x), g(x)) < \varepsilon$ holds for every $x \in E$ and if $E = \mathbb{R}$ then even $\rho(f(x), g(x)) < \frac{\varepsilon}{2}$ for every $x \in E$ holds. □

Remark 5.2.3. Realize that a mapping $g : X \to E$ from a topological space X to a metric space E, which is constant on each set of some resolvable partition $\{G_{\alpha+1} \setminus G_\alpha : \alpha < \kappa\}$ of X, is fragmented. Indeed, let $\varepsilon > 0$ and F be a closed subset of X and set $\delta = \inf\{\alpha \leq \kappa : G_\alpha \cap F \neq \emptyset\}$. It can be easily observed that δ is a successive ordinal, i.e., there exists $\gamma < \kappa$ such that $\delta = \gamma + 1$. Hence, the mapping g is constant on the set

$$F \cap G_{\gamma+1} = F \cap (G_{\gamma+1} \setminus G_\gamma)$$

and therefore $\operatorname{diam} g(F \cap G_{\gamma+1}) = 0$. The function g is thus ε-fragmented for every $\varepsilon > 0$ and hence fragmented.

Corollary 5.2.4. *Let X be a topological space and (E, ρ) a metric space. Then $f : X \to E$ is a fragmented mapping if and only if there exists a sequence of mappings $(f_n)_n$ such that $f_n \rightrightarrows f$ and every f_n is constant on each set of some resolvable partition of X.*

Proof. "Only if part" is a straightforward consequence of Lemma 5.2.2.

For the proof of the "if part" let $\varepsilon > 0$. Then there exists $k \in \mathbb{N}$ such that $\rho(f(x), f_k(x)) < \frac{\varepsilon}{3}$ for every $x \in X$. Then, due to Remark 5.2.3 for every closed set

F in X there is an open set U in X satisfying $U \cap F \neq \emptyset$ and $\operatorname{diam} f_k(U \cap F) < \frac{\varepsilon}{3}$. Hence, $\operatorname{diam} f(U \cap F) < \varepsilon$ which concludes the proof.

□

5.3 Distances to spaces of first resolvable class mappings

The aim of this section is to estimate a distance between a general mapping from a topological space X to a metric space E and the family $\mathrm{Hf}_1(X, E)$ via the quantity frag. However, as is demonstrated later in this chapter, these two quantities are not related in general and to get reasonable results we were forced to restrict ourselves to the particular pairs of spaces X and E.

Let X be a topological space and (E, ρ) a metric space. We define a distance of a pair of mappings $f, g : X \to E$ by

$$d(f, g) = \sup\{\rho(f(x), g(x)) : x \in X\}.$$

If $\mathcal{F}$ is a system of mappings from X to E, we denote

$$d(f, \mathcal{F}) = \inf\{d(f, g) : g \in \mathcal{F}\}.$$

Proposition 5.3.1. *If X is a topological space, (E, ρ) a metric space, then for a mapping $f : X \to E$ holds*

$$d(f, \mathrm{Hf}_1(X, E)) \leq \sigma\text{-frag}(f). \tag{5.1}$$

Moreover, if $E = \mathbb{R}$, then $d(f, \mathrm{Hf}_1(X, \mathbb{R})) \leq \frac{1}{2}\, \sigma\text{-frag}(f)$.

Proof. If σ-frag $= \infty$ then (5.1) clearly holds.

Suppose otherwise that $\sigma\text{-frag}(f) < \varepsilon$ for some $\varepsilon \in \mathbb{R}$. Then X can be covered by a system $\{X_n : n \in \mathbb{N}\}$ of resolvable sets such that $f|_{X_n}$ is ε-fragmented for every $n \in \mathbb{N}$.

If we construct a partition $\{Y_n : n \in \mathbb{N}\}$ of X in a usual way

$$Y_1 = X_1 \text{ and } Y_n = X_n \setminus \bigcup_{m=1}^{n-1} X_m,$$

then $f|_{Y_n}$ is ε-fragmented for every $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ we apply Lemma 5.2.2 and obtain a mapping $g_n : Y_n \to E$ constant on each set of some resolvable partition of Y_n and satisfying $\rho(g_n(x), f(x)) < \varepsilon$, $x \in Y_n$. Now we define a mapping $g : X \to E$ by setting $g(x) = g_n(x)$ for $x \in Y_n$, $n \in \mathbb{N}$.

For every $x \in X$ holds $\rho(f(x), g(x)) < \varepsilon$ and it can be easily checked that $g \in \mathrm{Hf}_1(X, E)$. Hence $\rho(f, \mathrm{Hf}_1(X, E)) \leq \sigma\text{-frag}(f)$.

The special case for $E = \mathbb{R}$ follows from Lemma 5.2.2 analogously. □

We would like to use the quantity σ-frag (or frag) to get a lower estimate of $d(f, \mathrm{Hf}_1(X, E))$. We proceed in two steps. First, we show that a certain qualitative property of the space $\in \mathrm{Hf}_1(X, E)$ already guaranties a quantitative estimate. Second, we bring out examples of families of spaces X, E with $\mathrm{Hf}_1(X, E)$ having the mentioned qualitative property.

Proposition 5.3.2. *Let X be a topological space and (E,ρ) a metric space such that every $h \in \mathrm{Hf}_1(X,E)$ satisfies σ-frag$(h)=0$.*

Then for every mapping $f : X \to E$ holds

$$\frac{1}{2}\,\sigma\text{-frag}(f) \le d(f, \mathrm{Hf}_1(X,E)). \tag{5.2}$$

Proof. If $d(f,\mathrm{Hf}_1(X,E)) = \infty$ then the statement is clearly valid.

Otherwise, suppose $d(f,\mathrm{Hf}_1(X,E)) < \alpha$ for some $\alpha \in \mathbb{R}$. Then there exists $h \in \mathrm{Hf}_1(X,E)$ such that $\rho(f(x),h(x)) < \alpha$ for every $x \in X$.

Due to the hypothesis σ-frag$(h) = 0$ holds and hence, for a given $\varepsilon > 0$ there exists a system $\{H_n : n \in \mathbb{N}\}$ of resolvable sets covering X such that $h|_{H_n}$ is ε-fragmented for each $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$ and let F be a closed subset of H_n. Then there exists an open subset U of X such that $U \cap F \ne \emptyset$ and $\operatorname{diam} h(U \cap F) < \varepsilon$. From

$$\begin{aligned}\operatorname{diam} f(U\cap F) &= \sup_{a,b\in U\cap F} \rho(f(a),f(b))\\ &\le \sup_{a,b\in U\cap F} \rho(f(a),h(a)) + \rho(h(a),h(b)) + \rho(h(b),f(b)) \le 2\alpha + \varepsilon,\end{aligned}$$

we conclude that σ-frag$(f) \le 2\alpha$ and this finally yields

$$\frac{1}{2}\,\sigma\text{-frag}(f) \le d(f, \mathrm{Hf}_1(X,E)).$$

$\square$

The following examples indicate that the assumption of Proposition 5.3.2 need not be satisfied even in case X is a metric space or a hereditarily Baire topological space.

Example 5.3.3. Assuming Martin's Axiom and the negation of the continuum hypothesis, [63, p. 162] assures the existence of an uncountable set Z in $\mathbb{R}$ whose every subset is an F_σ set in Z. Let E be the set Z endowed with the Euclidean metric and D be Z with the discrete metric. Consider the identity mapping $h : E \to D$.

Since E is a metric space, the family of sets $\Sigma_2(\mathrm{Hs}(E))$ corresponds to the family of F_σ subsets of E (see [79, Proposition 3.4(d2)]). Hence $h \in \mathrm{Hf}_1(E,D)$.

On the other hand, σ-frag$(h) = 1$. Indeed, given $0 < \varepsilon < 1$ and a system $\{E_n : n \in \mathbb{N}\}$ of resolvable sets covering E, the mapping $h|_{E_n}$ cannot be ε-fragmented on each E_n. If that was the case, we would select $k \in \mathbb{N}$ with E_k uncountable. Observe that every subset of E_k would have an isolated point due to the ε-fragmentability of $h|_{E_k}$. Since E_k has a countable base, E_k would be a countable set, which would contradict the cardinality of the selected E_k. Since clearly σ-frag$(h) \le 1$, the desired conclusion follows.

Example 5.3.4. In [47, Examples 2.4(2)] there is a construction carried out of a hereditarily Baire space X, a metric space E and a mapping $h \in \mathrm{Hf}_1(X,E)$ such that h has no point of continuity. Such a mapping cannot satisfy σ-frag$(h) = 0$. Indeed, assume it does. Then Lemma 5.2.1 implies σ-frag$(h) = \text{frag}(h) = 0$. Hence, h is fragmented and thus its set of points of continuity is comeager in X by [47, Theorem 2.3], in particular, it is nonempty. But h has no point of continuity.

However, the following result documents that if we consider the target space to be a separable metric space, the property needed in Proposition 5.3.2 is satisfied.

Proposition 5.3.5. *Let X be a topological and (E, ρ) a separable metric space. If $f \in \mathrm{Hf}_1(X, E)$, then σ-frag$(f) = 0$.*

Proof. Given $\varepsilon > 0$, we find a set of points $\{y_i \in E : i \in \mathbb{N}\}$ such that the system of open balls $\{B(y_i, \frac{\varepsilon}{2}) : i \in \mathbb{N}\}$ covers E. Then $f^{-1}(B(y_i, \frac{\varepsilon}{2})) \in \Sigma_2(\mathrm{Hs}(X))$, and thus there exists a family of resolvable sets $\{H_{i,n}\colon i, n \in \mathbb{N}\}$ satisfying

$$f^{-1}(B(y_i, \frac{\varepsilon}{2})) = \bigcup_{n=1}^{\infty} H_{i,n}, \quad i \in \mathbb{N}.$$

Then $\{H_{i,n} : i, n \in \mathbb{N}\}$ is a countable cover of X such that $\operatorname{diam} f(H_{i,n}) < \varepsilon$, for every $i, n \in \mathbb{N}$. We may conclude that σ-frag$(f) = 0$. □

Let us recall some topological notions needed in the rest of the chapter. A subset of a topological space X is said to be *Suslin* if it arises as a result of the Suslin operation applied to a system of closed sets in X.

A topological space X is an *absolute Suslin space* if it is homeomorphic to a Suslin set in a complete metric space.

Let $\mathcal{V}$ be a family of sets in a topological space X. Then it is *Suslin-additive* if $\bigcup \mathcal{U}$ is Suslin for every subfamily $\mathcal{U} \subset \mathcal{V}$.

Let us also remind that a family $\mathcal{V}$ is called *discrete* if every point $x \in X$ has a neighborhood that intersects at most one set of $\mathcal{V}$. A family $\{A_\alpha : \alpha \in I\}$ is said to be *σ-discretely decomposable* if there exists a family $\{A_{\alpha,n} : \alpha \in I, n \in \mathbb{N}\}$ of sets in X such that $\{A_{\alpha,n} : \alpha \in I\}$ is discrete for every $n \in \mathbb{N}$ and $A_\alpha = \bigcup_{n\in\mathbb{N}} A_{\alpha,n}$.

For more details regarding these concepts consult, e.g., [28], [29]. Introducing the notion of an absolute Suslin space allows us to find another relation between mapping of the first H-class and the quantity σ-frag.

Proposition 5.3.6. *Let X be an absolute Suslin and E a metric space. If $h \in \mathrm{Hf}_1(X, E)$, then σ-frag$(h) = 0$.*

Proof. Let $\varepsilon > 0$. Stone's theorem (see [20, Theorem 4.4.1]) provides an open cover $\mathcal{V}$ of E consisting of open sets of diameter smaller then ε such that $\mathcal{V} = \bigcup \mathcal{V}_n$, where a family $\mathcal{V}_n$ is discrete for every $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. Then a family $\{h^{-1}(V) : V \in \mathcal{V}_n\}$ is disjoint and since any resolvable set in a metric space is Borel (see [49, § 30, X, Theorem 5]) it is also Suslin-additive.

Due to [44, Theorem 1], the above-mentioned family is σ-discretely decomposable, namely, there exists a family $\{H_{V,k} : V \in \mathcal{V}_n, k \in \mathbb{N}\}$ such that the family $\{H_{V,k} : V \in \mathcal{V}_n\}$ is discrete for every $k \in \mathbb{N}$ and $h^{-1}(V) = \bigcup_{k\in\mathbb{N}} H_{V,k}$ for every $V \in \mathcal{V}_n$. Without loss of generality we may assume that $\{H_{V,k} : V \in \mathcal{V}_n, k \in \mathbb{N}\} \subset \Sigma_2(\mathrm{Hs}(X))$ (otherwise we would consider a family $\{\overline{H_{V,k}} \cap h^{-1}(V) : V \in \mathcal{V}_n\, k \in \mathbb{N}\}$ instead).

Then for every $H_{V,k}$, where $V \in \mathcal{V}_n$, $k \in \mathbb{N}$, there exists its partition $\{H_{V,k,m} : m \in \mathbb{N}\}$ comprised of resolvable sets. We define sets

$$H_{k,m,n} = \bigcup_{V\in\mathcal{V}_n} H_{V,k,m}, \quad \text{where } k, m, n \in \mathbb{N},$$

which are unions of discrete families of resolvable sets and hence resolvable as well.

A straightforward verification suffices to realize that $X = \bigcup_{k,m,n} H_{k,m,n}$ and the mapping $h|_{H_{m,n,k}}$ is ε-fragmentable for each $k, m, n \in \mathbb{N}$. Hence σ-frag$(h) < \varepsilon$ and σ-frag$(h) = 0$.

$\square$

Now we can summarize the results of this section in the following theorem.

Theorem 5.3.7. *If one of the following conditions is satisfied*

- *X is a topological space and E a separable metric space,*
- *X is an absolute Suslin space and E a metric space,*

then for every $f : X \to E$ holds

$$\frac{1}{2}\,\sigma\text{-frag}(f) \leq d(f, \mathrm{Hf}_1(X, E)) \leq \sigma\text{-frag}(f).$$

Moreover, if X is a topological space and $E = \mathbb{R}$, then

$$\sigma\text{-frag}(f) = d(f, \mathrm{Hf}_1(X, \mathbb{R})).$$

Proof. A consequence of Propositions 5.3.1, 5.3.2, 5.3.5 and 5.3.6. $\square$

5.4 Quantitative difference between compactness and countable compactness in $\mathrm{Hf}_1(X, E)$

We follow the line of reasoning which has appeared in [3] and [4] where the quantitative differences between compactness and countable compactness in $C(X, E)$ (see [3, Theorem 2.3]) and $B_1(X, E)$ (see [4, Corollary 3.2]) have been studied. The goal of this chapter is to provide analogous results for $\mathrm{Hf}_1(X, E)$.

For this purposes we adopt the following notions. Let X be a topological space. Given a subset $A \subset X$, we denote the set of all sequences in A by $A^{\mathbb{N}}$ and the set of all cluster points of a sequence $\varphi \in A^{\mathbb{N}}$ in X by $\mathrm{clust}(\varphi)$.

Let A, B be nonempty subsets of a metric space (E, d). We employ a notion of a *usual distance* between A and B defined by

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

and the *Hausdorff non-symmetrical distance* from A to B defined by

$$\hat{d}(A, B) = \sup\{d(a, B) : a \in A\}.$$

The following theorem is proved as [4, Proposition 3.1] provided X is a separable metric space, nevertheless the same arguments serve equally well in a little more general setting:

Theorem 5.4.1. *Let X be a second countable topological space, E a metrizable space and H a pointwise relatively compact subset of (E^X, τ_p). Then*

$$\sup_{f \in \overline{H}^{\tau_p}} \mathrm{frag}(f) = \sup_{\varphi \in H^{\mathbb{N}}} \inf\{\mathrm{frag}\, f : f \in \mathrm{clust}(\varphi)\}.$$

Let X be a topological space, (E,d) a metric space and H a relatively compact subset of the space (E^X, τ_p). Then we may define the quantity

$$\operatorname{ck}(H) = \sup_{\varphi \in H^{\mathbb{N}}} d(\operatorname{clust}(\varphi), \operatorname{Hf}_1(X,E)).$$

Remark 5.4.2. Note that for a general topological space X, a metric space E and a relatively compact set $H \subset (E^X, \tau_p)$ trivially holds

$$\operatorname{ck}(H) \leq \hat{d}(\overline{H}^{\tau_p}, \operatorname{Hf}_1(X,E)).$$

If H is moreover a countably compact subset of $(\operatorname{Hf}_1(X,E), \tau_p)$, then $\operatorname{ck}(H) = 0$ (use, e.g., [20, Theorem 3.10.3]).

Now, with the aid of the previous results we deduce a few corollaries.

Corollary 5.4.3. *Let X be a second countable hereditarily Baire topological space, E be a metric space and H be τ_p-relatively compact subset of E^X. If moreover one of the following conditions is satisfied*

(i) X is an absolute Suslin topological space,

(ii) E is a separable metric space,

then

$$\hat{d}(\overline{H}^{\tau_p}, \operatorname{Hf}_1(X,E)) \leq 2\operatorname{ck}(H).$$

Proof. Taking the definition of $\hat{d}$ into consideration and following Theorem 5.3.7 together with Lemma 5.2.1 we infer that

$$\hat{d}(\overline{H}^{\tau_p}, \operatorname{Hf}_1(X,E)) = \sup_{f \in \overline{H}^{\tau_p}} d(f, \operatorname{Hf}_1(X,E)) \leq \sup_{f \in \overline{H}^{\tau_p}} \sigma\text{-frag}(f) = \sup_{f \in \overline{H}^{\tau_p}} \operatorname{frag}(f) = (*).$$

We continue by employing Theorem 5.4.1 and a closing argument is again due to Theorem 5.3.7 and Lemma 5.2.1

$$\begin{aligned}(*) &= \sup_{\varphi \in H^{\mathbb{N}}} \inf\{\operatorname{frag} f : f \in \operatorname{clust}(\varphi)\} \\ &\leq \sup_{\varphi \in H^{\mathbb{N}}} \inf\{2d(f, \operatorname{Hf}_1(X,E)) : f \in \operatorname{clust}(\varphi)\} = 2\operatorname{ck}(H).\end{aligned}$$

□

Corollary 5.4.4. *Let X be a separable completely metrizable space and H be a τ_p-relatively compact subset of $\mathbb{R}^X$. Then*

$$\hat{d}(\overline{H}^{\tau_p}, \operatorname{Hf}_1(X)) = \operatorname{ck}(H).$$

Proof. An argument makes use of Theorems 5.4.1, 5.3.7, Lemma 5.2.1 and Remark 5.4.2 and goes along a similar pattern as a proof of Corollary 5.4.3. □

5.5 Oscillation rank

In this section we recall a definition of the oscillation rank which has been adopted by many authors, including S. Argyros, R. Haydon, A. S. Kechris and A. Louveau (see, e.g., [32], [46]). However, this rank has been so far investigated for the functions defined on metrizable compact spaces. The main purpose of this section is to provide a view on general situation considering the oscillation rank of the mappings from topological spaces to metric spaces.

We adhere to a standard convention that $\inf\emptyset = \infty$ and ∞ is greater then any ordinal.

Definition 5.5.1. Let X be a topological space, (E, ρ) a metric space and $\varepsilon > 0$. For a given mapping $f : X \to E$ and an ordinal α we construct the open set U_α. We proceed by transfinite induction.

Let $U_0 = \emptyset$. Assume that U_γ is constructed for every ordinal $\gamma < \alpha$. If $\alpha = \gamma' + 1$ is a successive ordinal then we set

$$U_\alpha = U_{\gamma'} \cup \{x \in X \setminus U_{\gamma'} : \exists \text{ open set } U \ni x \text{ such that } \operatorname{diam} f(U \setminus U_{\gamma'}) < \varepsilon\}.$$

If α is a limit ordinal we set $U_\alpha = \bigcup_{\gamma<\alpha} U_\gamma$.

We define $\beta(f, \varepsilon)$ as the first ordinal α satisfying $U_\alpha = X$ and if such an ordinal does not exist we set $\beta(f, \varepsilon) = \infty$.

Then we define the *oscillation rank of a mapping* f as

$$\beta(f) = \sup_{\varepsilon>0} \beta(f, \varepsilon).$$

Further, we define

$$\beta^*(f, \varepsilon) = \inf\{\kappa : \text{there exists a transfinite sequence } (V_\alpha)_{\alpha\le\kappa} \text{ of sets in } X \\ \text{with a property } *(f, \varepsilon)\},$$

where we say that a transfinite sequence $(V_\alpha)_{\alpha\le\kappa}$ of sets in X has the *property* $*(f, \varepsilon)$ if

(i) A transfinite sequence $(V_\alpha)_{\alpha\le\kappa}$ is nondecreasing, composed of open sets in X and such that $V_0 = \emptyset$ and $V_\kappa = X$.

(ii) For a limit ordinal $\gamma \le \kappa$ holds $V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$.

(iii) For each ordinal $\alpha < \kappa$ holds: Let $x \in V_{\alpha+1} \setminus V_\alpha$. Then there exists an open neighborhood V of x such that $\operatorname{diam} f(V \setminus V_\alpha) < \varepsilon$.

If there is no such a transfinite sequence we set $\beta^*(f, \varepsilon) = \infty$.

We define

$$\beta^*(f) = \sup_{\varepsilon>0} \beta^*(f, \varepsilon).$$

Remark 5.5.2. In some cases it is more convenient to use a slightly modified definition of $\beta(f, \varepsilon)$ where instead of constructing open subsets $U_\alpha \subset X$ for ordinals α we construct its complements F_α. Then $F_0 = X$, for a limit ordinal α holds $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$ and for a successive ordinal $\alpha = \beta' + 1$ we set

$$
\begin{aligned}
F_\alpha &= \{x \in F_{\beta'} : \forall \text{ open set } U \ni x \text{ holds } \operatorname{diam} f(U \cap F_{\beta'}) \geq \varepsilon\} \\
&= F_{\beta'} \setminus \bigcup \{U \subset X : U \text{ is open, } \operatorname{diam} f(U \setminus U_{\beta'}) < \varepsilon\}.
\end{aligned}
$$

Then $\beta(f, \varepsilon)$ is the first ordinal α such that $F_\alpha = \emptyset$.

Remark 5.5.3. We have adopted a notation of [32], though our definition of the oscillation rank is formally different. A glimpse of these two definitions shows, however, that they provide identical concepts.

The following lemma proves that concepts β and β^* actually coincide.

Lemma 5.5.4. *Let X be a topological space, (E, ρ) a metric space and $\varepsilon > 0$. Then for a mapping $f : X \to E$ holds $\beta(f, \varepsilon) = \beta^*(f, \varepsilon)$.*

Proof. An inequallity $\beta^*(f, \varepsilon) \leq \beta(f, \varepsilon)$ follows immediately from the definitions. If $\beta^*(f, \varepsilon) = \infty$ then clearly $\beta(f, \varepsilon) = \infty$.

Let $\beta^*(f, \varepsilon) = \kappa_0$ take any $\infty \neq \kappa_1 > \kappa_0$. Then there exists an ordinal $\kappa_0 \leq \kappa < \kappa_1$ and a corresponding transfinite sequence $(V_\alpha)_{\alpha \leq \kappa}$ of sets in X with $*(f, \varepsilon)$. We find a transfinite sequence $(U_\alpha)_{\alpha \leq \kappa}$ provided by the construction in the definition of $\beta(f, \varepsilon)$. Now we observe that for every $\alpha \leq \kappa$ holds $V_\alpha \subset U_\alpha$.

Indeed, for $\alpha = 0$ the statement follows immediately from the definitions of indices. Assume that $\gamma \leq \kappa$ is an ordinal and the statement is valid for every $\alpha < \gamma$. If γ is a limit ordinal then $V_\gamma = \bigcup_{\alpha < \gamma} V_\alpha \subset \bigcup_{\alpha < \gamma} U_\alpha = U_\gamma$. If γ is a successive ordinal then $\gamma = \alpha + 1$ for some ordinal α and

$$
U_{\alpha+1} = U_\alpha \cup \{x \in X \setminus U_\alpha : \exists \text{ open set } U \ni x \text{ such that } \operatorname{diam} f(U \setminus U_\alpha) < \varepsilon\}.
$$

Let $x \in V_{\alpha+1}$. Then either $x \in U_\alpha \subset U_{\alpha+1}$, or $x \in V_{\alpha+1} \setminus U_\alpha \subset V_{\alpha+1} \setminus V_\alpha$ and due to $*(f, \varepsilon)$ there exists an open neighborhood V of x such that $\operatorname{diam} f(V \setminus V_\alpha) < \varepsilon$ and thus also $x \in U_{\alpha+1}$.

It follows that $V_\kappa \subset U_\kappa$ and hence $\beta(f, \varepsilon) \leq \kappa$ for every selected $\kappa_1 > \kappa_0$. Consequently, $\beta(f, \varepsilon) \leq \kappa_0$ which concludes the proof. $\square$

We may ask whether a transfinite process described in a definition of the oscillation rank must always stop. A quite natural answer is given by a following lemma.

Lemma 5.5.5. *Let X be a topological space, (E, ρ) a metric space and $f : X \to E$. Then $\beta(f) \neq \infty$ if and only if f is fragmented.*

Proof. If f is fragmented then a moment of reflection shows that $\beta(f) \neq \infty$. Specifically, $\beta(f) \leq \operatorname{card}(X)$, where $\operatorname{card}(X)$ is a cardinality of X.

Suppose f is not fragmented. Then there exists a nonempty closed set $F \subset X$ and $\varepsilon > 0$ such that for every open set $U \subset X$ intersecting F holds $\operatorname{diam} f(F \cap U) > \varepsilon$. We prove that then $\beta(f, \varepsilon) = \infty$. Suppose the contrary, let $\beta(f, \varepsilon) = \kappa \neq \infty$ and let a transfinite sequence $(U_\alpha)_{\alpha \leq \kappa}$ be the one constructed in the definition of $\beta(f, \varepsilon)$.

Let $\delta = \min\{\alpha : U_\alpha \cap F \neq \emptyset\}$. It is easy to see that $\delta \leq \kappa$ is a successive ordinal, hence there exists an ordinal γ such that $\delta = \gamma + 1$.

We choose a point $x \in F \cap U_{\gamma+1} = (F \cap U_{\gamma+1}) \setminus U_\gamma$. Therefore, we can find an open neighborhood U of x such that $\operatorname{diam} f(U \setminus U_\gamma) < \varepsilon$. Then

$$\emptyset \neq U \cap F \cap U_{\gamma+1} = U \cap F \cap (U_{\gamma+1} \setminus U_\gamma) \subset U \setminus U_\gamma$$

and from the assumption follows that $\operatorname{diam} f(U \cap F \cap U_{\gamma+1}) < \varepsilon$ which contradicts the properties of the set F. Hence, for every $\alpha \leq \kappa$ holds $U_\alpha \cap F = \emptyset$. We conclude that $\beta(f, \varepsilon) = \infty$. □

Let X be a topological space and E a metric space. We denote the set of the mappings $f : X \to E$ satisfying $\beta(f) < \omega_1$ by the symbol ω_1-$\beta(X, E)$.

In the following statements we examine a stability of ω_1-$\beta(X, E)$ under making uniform limits of nets (Lemma 5.5.6) and composing with a uniformly continuous mappings (Lemma 5.5.7).

Lemma 5.5.6. *Let X be a topological space and (E, ρ) a metric space. If $(f_\gamma)_{\gamma\in\Gamma}$ is a net of mappings in ω_1-$\beta(X, E)$ which converges uniformly to f, then $f \in \omega_1$-$\beta(X, E)$.*

Proof. Let $\varepsilon > 0$. Then there exists $\gamma \in \Gamma$ such that $\rho(f(x), f_\gamma(x)) \leq \frac{\varepsilon}{3}$ for every $x \in X$. We set $\kappa = \beta(f, \frac{\varepsilon}{3}) = \beta^*(f, \frac{\varepsilon}{3}) < \omega_1$ (see Lemma 5.5.4) and find a family $(U_\alpha)_{\alpha\leq\kappa}$ with $*(f_\gamma, \frac{\varepsilon}{3})$. Then for each $\alpha < \kappa$ and $x \in U_{\alpha+1} \setminus U_\alpha$ there exists a neighborhood U of x such that

$$\operatorname{diam} f(U \setminus U_\alpha) \leq \operatorname{diam} f_\gamma(U \setminus U_\alpha) + \frac{2\varepsilon}{3} \leq \varepsilon.$$

Hence, $\beta^*(f, \varepsilon) \leq \kappa$ and due to Lemma 5.5.4 also $\beta(f) \leq \kappa$ and $f \in \omega_1$-$\beta(X, E)$. □

Lemma 5.5.7. *Let X be a topological space, (E, ρ), (F, σ) metric spaces and κ an ordinal. If $f : X \to E$ is a mapping satisfying $\beta(f) \leq \kappa$ and $h : E \to F$ is an uniformly continuous mapping, then $\beta(h \circ f) \leq \kappa$.*

Proof. Let $\varepsilon > 0$. We find $\delta > 0$ such that for any $x, y \in E$ satisfying $\rho(x, y) < \delta$ holds $\sigma(h(x), h(y)) < \varepsilon$. Lemma 5.5.4 allows us to find a family $\{U_\alpha : \alpha \leq \kappa\}$ with $*(f, \delta)$.

For each $\alpha < \kappa$ and $x \in U_{\alpha+1} \setminus U_\alpha$ there exists a neighborhood U of x such that $\operatorname{diam} f(U \setminus U_\alpha) < \delta$. Hence

$$\operatorname{diam}(h \circ f)(U \setminus U_\alpha) \leq \varepsilon,$$

which finalizes the proof of the statement. □

The subsequent example indicates that continuous mappings do not generally preserve the oscillation rank.

Example 5.5.8. *Let $f : \mathbb{Q}^+ \to \mathbb{R}^+$ be defined as $f(\frac{p}{q}) = \frac{1}{q}$ for $\frac{p}{q} \in \mathbb{Q}^+$ where $p, q \in \mathbb{N}$ are coprime numbers. Further, we define a function $h : \mathbb{R}^+ \to \mathbb{R}^+$ by setting $h(x) = \frac{1}{x}$ for $x \in \mathbb{R}^+$. Notice that h is a continuous function and $\beta(f) < \omega_1$. Yet, the composition $h \circ f : \mathbb{Q}^+ \to \mathbb{R}^+$ is not fragmented and hence $\beta(h \circ g) = \infty$.*

Lemmas 5.5.10, 5.5.11 and 5.5.12 provide us an information about stability of ω_1-$\beta(X, E)$ on making various algebraic operations. First, we prove the following auxiliary result.

Lemma 5.5.9. *Let X be a topological space, (E, ρ) a metric space, κ_1, κ_2 ordinals and $f, g : X \to E$ mappings satisfying $\beta(f) \leq \kappa_1$, $\beta(g) \leq \kappa_2$, where $\kappa_1, \kappa_2 > 0$. Then for a diagonal mapping $f\Delta g : X \to (E \times E, \rho_{\max})$ defined by $(f\Delta g)(x) = (f(x), g(x))$, $x \in X$, holds $\beta(f\Delta g) \leq \kappa_1\kappa_2$ ($\rho_{\max}$ is a maximal metric).*

Proof. Let $\varepsilon > 0$. Due to Lemma 5.5.4 we may find $(U_\alpha)_{\alpha\leq\kappa_1}$ in X with $*(f, \varepsilon)$ and $(V_\alpha)_{\alpha\leq\kappa_2}$ in X with $*(g, \varepsilon)$.

We remind that considering $\gamma < \kappa_1\kappa_2$ there exists a uniquely determined pair of ordinals α, β such that $\gamma = \kappa_1\beta + \alpha$ and $\alpha < \kappa_1$, $\beta < \kappa_2$ (e.g., [50, Theorem VII.5.4]). Hence, we may define

$$G_\gamma = (U_\alpha \cap V_{\beta+1}) \cup V_\beta, \gamma < \kappa_1\kappa_2, \text{ and } G_{\kappa_1\kappa_2} = X.$$

In the remainder of the proof we verify that the transfinite sequence $(G_\alpha)_{\alpha\leq\kappa_1\kappa_2}$ satisfies $*(f\Delta g, \varepsilon)$. First, G_γ is an open set in X for every $\gamma \leq \kappa_1\kappa_2$, $G_0 = \emptyset$ and $G_{\kappa_1\kappa_2} = X$. Let $\gamma, \gamma' < \kappa_1\kappa_2$ be ordinals satisfying $\gamma < \gamma'$. Then $\gamma = \kappa_1\beta + \alpha$ and $\gamma' = \kappa_1\beta' + \alpha'$ where either $\beta < \beta'$, or $\beta = \beta'$ together with $\alpha < \alpha'$ holds. In case $\beta < \beta'$ holds

$$G_\gamma = (U_\alpha \cap V_{\beta+1}) \cup V_\beta \subset V_{\beta'} \subset G_{\gamma'} \subset X$$

and if $\beta = \beta'$ with $\alpha < \alpha'$ is the case then

$$G_\gamma = (U_\alpha \cap V_{\beta+1}) \cup V_\beta \subset (U_{\alpha'} \cap V_{\beta+1}) \cup V_\beta = G_{\gamma'} \subset X.$$

Second, let $\gamma \leq \kappa_1\kappa_2$ be a limit ordinal with the unique decomposition $\gamma = \kappa_1\beta + \alpha$, $\alpha < \kappa_1$, $\beta \leq \kappa_2$. Then either α is a limit ordinal as well:

$$G_\gamma = (U_\alpha \cap V_{\beta+1}) \cup V_\beta = \bigcup_{\alpha'<\alpha} \big((U_{\alpha'} \cap V_{\beta+1}) \cup V_\beta\big) = \bigcup_{\gamma'<\gamma} G_{\gamma'},$$

or $\alpha = 0$ and κ_1 is a limit ordinal:

$$G_\gamma = V_\beta = \bigcup_{\beta'<\beta} (V_{\beta'+1} \cup V_{\beta'}) = \bigcup_{\beta'<\beta} \bigcup_{\alpha'<\kappa_1} (U_{\alpha'} \cap V_{\beta'+1}) \cup V_{\beta'} = \bigcup_{\gamma'<\gamma} G_{\gamma'},$$

or $\alpha = 0$ and β is a limit ordinal:

$$G_\gamma = V_\beta = \bigcup_{\alpha<\kappa_1} V_\beta = \bigcup_{\alpha<\kappa_1} \bigcup_{\beta'<\beta} (U_\alpha \cap V_{\beta'+1}) \cup V_{\beta'} = \bigcup_{\gamma'<\gamma} G_{\gamma'}.$$

Third, let $x \in G_{\gamma+1} \setminus G_\gamma$, where $\gamma = \kappa_1\beta + \alpha < \kappa_1\kappa_2$. A decomposition of $\gamma + 1$ is then either $\kappa_1\beta + (\alpha + 1)$ or $\kappa_1(\beta + 1)$ (when $V_{\alpha+1} = X$). Both cases imply

$$G_{\gamma+1} \setminus G_\gamma = (U_{\alpha+1} \setminus U_\alpha) \cap (V_{\beta+1} \setminus V_\beta)$$

and we can find open sets $U, V \subset G_{\gamma+1}$ enjoying the following properties: $x \in U$, $x \in V$, $\operatorname{diam} f(U\setminus U_\alpha) < \varepsilon$ and $\operatorname{diam} g(V\setminus V_\beta) < \varepsilon$. It follows that $\operatorname{diam}(f\Delta g)((U\cap V) \setminus G_\gamma) < \varepsilon$.

Hence, $\beta^*(f\Delta g, \varepsilon) \leq \kappa_1\kappa_2$ and employing Lemma 5.5.4 completes the proof.

□

Lemma 5.5.10. *Let X be a topological space and E a Banach space. If $f, g \in \omega_1\text{-}\beta(X, E)$ then $f + g \in \omega_1\text{-}\beta(X, E)$.*

Proof. Let $f, g \in \omega_1\text{-}\beta(X, E)$. We define $h : E \times E \to E$ as $h(x, y) = x + y$ for every $x, y \in E$. It can be verified straightforwardly that a mapping h is uniformly continuous and that $f + g = h \circ (f\Delta g)$. Applying Lemmas 5.5.9 and 5.5.7 we get a required inequality $\beta(f + g) < \omega_1$. $\square$

Lemma 5.5.11. *Let X be a topological space and E a Banach lattice. If $f, g \in \omega_1\text{-}\beta(X, E)$ then* $\inf(f, g), \sup(f, g) \in \omega_1\text{-}\beta(X, E)$.

Proof. First, we realize that the lattice mappings $i : E \times E \to E$ and $j : E \times E \to E$, defined as $i(x, y) = \inf(x, y)$ and $j(x) = \sup(x, y)$ for $x, y \in E$, are uniformly continuous (cf. [74, Proposition 5.2]). For any $f, g \in \omega_1\text{-}\beta(X, E)$ and $x \in X$ the following identities clearly hold:

$$\begin{aligned} \sup(f, g)(x, y) &= \sup(f(x), g(x)) = (i \circ f)(x, y), \\ \inf(f, g)(x, y) &= \inf(f(x), g(x)) = (j \circ f)(x, y). \end{aligned}$$

Thanks to Lemma 5.5.7 holds $\sup(f, g), \inf(f, g) \in \omega_1\text{-}\beta(X, E)$. $\square$

Concerning stability under making products we were able to achieve only a partial result.

Lemma 5.5.12. *Let X be a topological space, E a commutative Banach algebra and $f, g \in \omega_1\text{-}\beta(X, E)$ bounded mappings. Then $fg \in \omega_1\text{-}\beta(X, E)$.*

Proof. Since $fg = \frac{1}{2}\left((f + g)^2 - f^2 - g^2\right)$ it suffices to prove that for every bounded $f : X \to E$ such that $f \in \omega_1\text{-}\beta(X, E)$ holds $f \in \omega_1\text{-}\beta(X, E)$.

We may write f^2 as a composition $f^2 = \varphi \circ f$ where $\varphi : E \to E$ is defined as $\varphi(x) = x^2$ for each $x \in E$. Since f is a bounded mapping, $\varphi|_{\mathrm{range}\, f}$ is uniformly continuous and employing Lemma 5.5.7 finishes the proof. $\square$

However, the general problem remains unsolved.

Question 5.5.13. Let X be a topological space, E a commutative Banach algebra and $f, g \in \omega_1\text{-}\beta(X, E)$. Is it then true that $fg \in \omega_1\text{-}\beta(X, E)$?

We summarize the previous stability results in the following proposition.

Proposition 5.5.14. *Let X be a topological space.*

(a) *If E is a Banach space then the space of all bounded elements of $\omega_1\text{-}\beta(X, E)$ endowed with a supremum norm (i.e., $\|f\| = \sup\{\|f(x)\| : x \in X\}$) is a Banach space.*

(b) *If E is a Banach lattice then the space of all bounded elements of $\omega_1\text{-}\beta(X, E)$ is a Banach lattice in a supremum norm and a pointwise ordering.*

(c) *If E is a commutative Banach algebra then the space of all bounded elements of $\omega_1\text{-}\beta(X, E)$ a commutative Banach algebra.*

Proof. A straightforward verification using Lemmas 5.5.10, 5.5.11 and 5.5.12. $\square$

The following lemma shows that the oscillation rank β may attain a value of arbitrary ordinal.

Lemma 5.5.15. *For every $\alpha \leq \omega_1$ there exists a hereditarily Baire metrizable topological space X_α and a function $f_\alpha : X_\alpha \to \{0,1\}$ satisfying $\beta(f_\alpha) = \alpha$ and $f_\alpha \in \mathrm{Bof}_1(X, E)$.*

Proof. Let $\alpha \leq \omega_1$. Then we find a hereditarily Baire metrizable topological space X_α and its subset $A_\alpha \subset X_\alpha$ such that for its characteristic function χ_{A_α} holds $\beta(\chi_{A_\alpha}) = \alpha$.

The construction proceeds by transfinite induction on $\alpha < \omega_1$. Let $X_0 = A_0 = \emptyset$. Then for every $0 < \varepsilon < 1$ holds $\beta(\chi_{A_0}) = \beta(\chi_{A_0}, \varepsilon) = 0$.

Assume that $\alpha < \omega_1$ is a limit ordinal and for every $\beta < \alpha$ we have constructed a hereditarily Baire metrizable topological space X_β and its subset $A_\beta \subset X_\beta$ such that $\beta(\chi_{A_\beta}) = \beta$. We define topological spaces

$$X = \bigcup_{\beta<\alpha} X_\beta \times \{\beta\} \quad \text{and} \quad A = \bigcup_{\beta<\alpha} A_\beta \times \{\beta\}$$

equipped with the disjoint union topology (e.g., [20, Section 2.2]). A space X is clearly hereditarily Baire and due to [20, Theorem 4.2.1] also metrizable.

Now, let us fix $0 < \varepsilon < 1$. For $\beta < \alpha$ let $\{U_\beta^\gamma : \gamma < \omega_1\}$ be the open sets emerging in the definition of $\beta(\chi_{A_\beta}, \varepsilon)$. The system of sets

$$\Big\{ \bigcup_{\beta<\alpha} U_\beta^\gamma \times \{\beta\} : \gamma < \omega_1 \Big\}$$

is then precisely that one constructed in the definition of $\beta(\chi_A, \varepsilon)$ and $\gamma = \alpha$ is clearly the first ordinal satisfying $\bigcup_{\beta<\alpha} U_\beta^\gamma \times \{\beta\} = X$. Hence, $\beta(\chi_A) = \alpha$ and we may set $X_\alpha = X$ and $A_\alpha = A$.

Let us assume that for an ordinal $\alpha < \omega_1$ we have a corresponding set A_α and a topological space X_α constructed. We define topological spaces

$$X = \{p\} \cup \Big(\bigcup_{n\in\mathbb{N}} X_\alpha \times \{n\}\Big) \quad \text{and} \quad A = \{p\} \cup \Big(\bigcup_{n\in\mathbb{N}} A_\alpha \times \{2n\}\Big) \cup \big((X_\alpha \setminus A_\alpha) \times \{2n-1\}\big)$$

with a following topology: points of the form $(x, n) \in X_\alpha \times \{n\}$, where $n \in \mathbb{N}$, has a basis of neighborhoods consisting of sets $U \times \{n\}$ where $U \subset X_\alpha$ is a neighborhood of $x \in X_\alpha$ and the point $p \in X$ has a basis of neighborhoods consisting of sets of the form

$$\{p\} \cup \bigcup_{n\geq n_0} X_\alpha \times \{n\},$$

where $n_0 \in \mathbb{N}$. A routine verification shows that the described topological space X is metrizable and hereditarily Baire.

Following the inductive process in the definition of $\beta(f, \varepsilon)$ (considering Remark 5.5.2) we construct closed sets $\{F_\beta : \beta \leq \beta(f, \varepsilon)\}$. It is easy to realize that if $f : X \to \mathbb{R}$ is a characteristic function of a set $A \subset X$ then for every $\beta < \omega_1$ holds

$$F_{\beta+1} = \overline{F_\beta \cap A} \cap \overline{F_\beta \setminus A} \tag{5.3}$$

Our aim is to prove that $F_\alpha = \{p\}$. Once this assertion is justified it follows that $\beta(\chi_A) = \beta(\chi_A, \varepsilon) = \alpha + 1$ and we may set $X_{\alpha+1} = X$ and $A_{\alpha+1} = A$.

First, we take a system $\{H_\beta : \beta \le \alpha\}$ of closed sets $\{H_\beta : \beta \le \alpha\}$ from the definition of $\beta(\chi_{A_\alpha}, \varepsilon)$ (considering Remark 5.5.2). We prove by transfinite induction on β a claim that $F_\beta = \{p\} \cup \bigcup_{n\in\mathbb{N}} H_\beta \times \{n\}$ for every $\beta \le \alpha$.

A little reflection on (5.3) makes it clear that it suffices to prove that for every $\beta \le \alpha$ holds $p \in F_\beta$. The statement is obvious for $\beta = 0$. Assuming $\beta \le \alpha$ is a limit ordinal and $p \in F_\gamma$ for every $\gamma < \beta$ we have $p \in F_\beta = \bigcap_{\gamma<\beta} F_\gamma$. Now suppose $\beta < \alpha$ and $p \in F_\beta = \{p\} \cup \bigcup_{n\in\mathbb{N}} H_\beta \times \{n\}$.

Then $H_\beta \neq \emptyset$, because $\beta < \alpha$. Hence at least one of the sets $H_\beta \cap A_\alpha$, $H_\beta \setminus A_\alpha$ is nonempty. A glance at the definitions of the set A and its topology shows that both cases guarantee $p \in \overline{F_\beta \setminus A}$. Since also trivially $p \in \overline{F_\beta \cap A}$, we may infer that $p \in F_{\beta+1}$ and the claim is proved.

The claim implies that

$$F_\alpha = \{p\} \cup \bigcup_{n\in\mathbb{N}} H_\alpha \times \{n\} = \{p\},$$

which was to prove.

□

Remark 5.5.16. According to [32, Proposition 2.8] for all $\alpha < \omega_1$ there exists a quasireflexive (of order 1) Banach space Q_α such that $Q_\alpha^{**} = Q_\alpha \oplus \langle f_\alpha \rangle$ where $\beta(f_\alpha) > \alpha$.

Now we clarify the relations between ω_1-$\beta(X, E)$ and the following classes of mappings: $\mathrm{Bof}_1(X, E)$, $\mathrm{Hf}_1(X, E)$, $\mathrm{Frag}(X, E)$, mappings with the point of continuity property and mappings of Baire class 1. The positive results of this kind are summarized in Theorem 5.5.18.

Lemma 5.5.17. *Let X be a topological space, E a metric space and $f : X \to E$ with $\beta(f) < \omega_1$. Then $f \in \mathrm{Bof}_1(X, E)$.*

Proof. For every $n \in \mathbb{N}$ we set $\alpha_n = \beta(f, \frac{1}{n}) < \omega_1$ and find open sets $\{U_\alpha^n : \alpha \le \alpha_n\}$ from the definition of the oscillation rank. Then, we define the families

$$\mathcal{D}_n^\alpha = \{G \subset X : G \text{ is open, } \operatorname{diam} f(G \setminus U_\alpha^n) < \frac{1}{n}\}, \quad n \in \mathbb{N}, \alpha \le \alpha_n,$$

and set $\mathcal{D} = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n^\alpha$.

Now, let $U \subset E$ be an open set. Then clearly

$$f^{-1}(U) = \bigcup_{n\in\mathbb{N}} \bigcup_{\alpha\le\alpha_n} \{G \setminus U_\alpha^n : G \in \mathcal{D}_n^\alpha, f(G \setminus U_\alpha^n) \subset U\} = \bigcup_{n\in\mathbb{N}} \bigcup_{\alpha\le\alpha_n} G_n^\alpha \setminus U_\alpha^n,$$

where $G_\alpha^n = \bigcup\{G \in \mathcal{D}_n^\alpha : f(G \setminus U_\alpha^n) \subset U\}$ for every $n \in \mathbb{N}$ and $\alpha \le \alpha_n$.

Since $\alpha_n < \omega_1$ for every $n \in \mathbb{N}$, it follows that $f \in \mathrm{Bof}_1(X, E)$. □

Theorem 5.5.18. *Let X be a topological space and E a metric space. Then the following schema holds true*

$$\begin{array}{ccc} \mathrm{Hf}_1(X,E) & \supset & \mathrm{Bof}_1(X,E) \\ \cup & & \cup \\ \mathrm{Frag}(X,E) & \supset & \omega_1\text{-}\beta(X,E) \end{array}$$

and no other nontrivial inclusion is valid generally.

Proof. Inclusion $\mathrm{Frag}(X,E) \subset \mathrm{Hf}_1(X,E)$ was established by [47, Theorem 2.3].

Inclusion $\mathrm{Bof}_1(X,E) \subset \mathrm{Hf}_1(X,E)$ is a direct consequence of definitions. Assuming Martin's Axiom and the negation of the continuum hypothesis there is a function from $\mathrm{Bof}_1(X,E)$ constructed in [47, Example 2.4(3)] which is not fragmented. Therefore $\mathrm{Bof}_1(X,E) \not\subseteq \mathrm{Frag}(X,E)$ and hence also $\mathrm{Hf}_1(X,E) \not\subseteq \mathrm{Frag}(X,E)$.

Inclusion ω_1-$\beta(X,E) \subset \mathrm{Frag}(X,E)$ is the content of Lemma 5.5.5 and inclusion ω_1-$\beta(X,E) \subset \mathrm{Bof}_1(X,E)$ is exactly Lemma 5.5.17. The preceding two inclusions generally cannot be reversed due to Lemma 5.5.15.

In [79, Remark 3.3] there is a space constructed containing a resolvable non-Borel set A. A characteristic function χ_A of a set A is clearly fragmented and $\chi_A \notin \mathrm{Bof}_1(X,E)$. Hence, $\mathrm{Frag}(X,E) \not\subseteq \mathrm{Bof}_1(X,E)$ and consequently also $\mathrm{Bof}_1(X,E) \not\subseteq \omega_1$-$\beta(X,E)$, $\mathrm{Hf}_1(X,E) \not\subseteq \mathrm{Bof}_1(X,E)$ and $\mathrm{Frag}(X,E) \not\subseteq \omega_1$-$\beta(X,E)$ in general.

□

Remark 5.5.19. It is worth mentioning that a mapping $f : X \to E$, where X is a topological space, E a metric space and $\beta(f) < \omega_1$, need not have a separable range. A simple counterexample provides the identity mapping on $[0,1]$ endowed with a discrete topology. Such a mapping has $\beta(f) = 1$ despite having a non-separable range.

From the viewpoint of Theorem 5.5.18 and [47, Theorem 2.3] it seems reasonable to investigate a relation between the mappings with the point of continuity property and the mappings with a countable oscillation rank. Let us remind that a mapping f from a topological space X to a metric space E has the *point of continuity property* provided for every nonempty closed $F \subset X$, the restriction $f|F$ of f to a set F has a point of continuity.

Example 5.5.20. *There exists a function $f : \mathbb{Q} \to \mathbb{R}$ lacking the point of continuity property such that $f \in \omega_1$-$\beta(\mathbb{Q},\mathbb{R})$.*

Proof. A function f from [47, Examples 2.4(4)] does the job. □

Example 5.5.21. *There exists a topological space X, a metric space E and a mapping $f : X \to E$ with the point of continuity property and satisfying $\beta(f) = \omega_1$.*

Proof. Lemma 5.5.15 provides a function $f : X \to \{0,1\}$ such that $\beta(f) = \omega_1$. A glance at Lemma 5.5.5 shows that f is fragmented and hence, as f is a characteristic function of a resolvable set, it clearly has the point of continuity property. □

Let K be a compact metrizable space. Then, according to [46, Proposition 2], a function $f : K \to \mathbb{R}$ is of Baire class 1 if and only if $\beta(f) < \omega_1$. A question arises whether there is a chance for an analogous proposition to hold in a more general setting. Alas, there is no relation between functions of Baire class 1 and class ω_1-$\beta(X,\mathbb{R})$ for a general topological space X.

Example 5.5.22. *There exists a function $g : \mathbb{Q} \to \mathbb{R}$ of Baire class* 1 *which is not fragmented.*

Proof. The function g from [47, Examples 2.4(4)] has the desired properties. □

Example 5.5.23. *There exists a topological space* X *and a function* $f : X \to \mathbb{R}$ *such that* $f \in \omega_1\text{-}\beta(X, \mathbb{R})$ *but which is not of Baire class* 1*.*

Proof. It suffices to consider a nonmetrizable topological X and a characteristic function of an open set in X which is not F_σ. □

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