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MASTER THESIS

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Lipschitz mappings in the plane

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Abstrakt: V této práci se zabýváme otevřenou Feigeho otázkou, která se ptá, jestli existuje konstantně lipschitzovská bijekce každé n^2-bodové podmnožiny $\mathbb{Z}^2$ na pravidelnou mřížku $[n] \times [n]$ pro každé $n \in \mathbb{N}$. Dáme tuto otázku do vztahu s již vyřešeným problémem existence hustoty v $\mathbb{R}^2$, která není jakobiánem žádného bilipschitzovského zobrazení. Tento problém byl vyřešen Buragem a Kleinerem [1] a nezávisle McMullenem [12]. Předvedeme práci Buraga a Kleinera, zanalyzujeme její vztah k Feigeho problému a navrhneme spojitou formulaci Feigeho otázky ve speciálním případě. Poté předvedeme konstrukci hustoty Buraga a Kleinera, uděláme několik pozorování ohledně vlastností této hustoty a následně zkonstruujeme hustotu, která je všude nerealizovatelná jako jakobián bilipschitzovského zobrazení. Dále se zabýváme naší spojitou formulací Feigeho otázky, uděláme několik pozorování o této otázce a nakonec se pokusíme tuto otázku zodpovědět s použitím dříve zkonstruované všude nerealizovatelné hustoty. Nicméně tento poslední úkol zůstává stále nesplněn.

Klíčová slova: lipschitzovské zobrazení, bilipschitzovské zobrazení, nerealizovatelná hustota, jakobián, Feigeho problém, Burago–Kleinerova hustota

Title: Lipschitz mappings in the plane

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Abstract: In this thesis we consider an open question of Feige that asks whether there always exists a constantly Lipschitz bijection of an n^2-point subset of $\mathbb{Z}^2$ onto a regular grid $[n] \times [n]$ for every $n \in \mathbb{N}$. We relate this question to an already resolved problem of the existence of a bounded positive measurable density in $\mathbb{R}^2$ that is not the Jacobian of any bilipschitz map. This problem was resolved by Burago and Kleiner [1], and independently, by McMullen [12]. We present the work of Burago and Kleiner, analyze its relation to Feige's problem and suggest a continuous formulation of Feige's question in a special case. Then we present the Burago–Kleiner density, make several observation about the properties of this density, and after that we construct a density that is everywhere nonrealizable as the Jacobian of a bilipschitz map. Subsequently, we discuss our continuous variant of Feige's question, provide several observation concerning it, and finally, we try to use the everywhere nonrealizable density constructed before to answer our continuous variant of Feige's question. However, this last task still remains incomplete.

Keywords: Lipschitz map, bilipschitz map, nonrealizable density, Jacobian determinant, Feige's problem, Burago–Kleiner density

Contents

List of Symbols and Abbreviations

$A \setminus B$	The relative complement of a set B to a set A.
$B(x, \gamma)$	The closed ball of radius γ centered at x.
$Df(x)$	The Jacobian matrix of a map f at a point x.
R_N	The rectangle of the form $[0,1] \times [0, 1/N]$.
$[n]$	The set of integers $\{1, \ldots, n\}$.
$\lvert S\rvert$	The cardinality of a set S.
$\lvert r\rvert$	The absolute value of a real number r.
$\lceil r \rceil$	The smallest integer not less than a real number r.
$\mathrm{diam}(S)$	The diameter of a set S.
$\overline{\mathbf{ab}}$	The line segment between points $\mathbf{a}$ and $\mathbf{b}$.
$\lfloor r \rfloor$	The integer part of a real number r.
$\inf \rho$	The infimum of a real-valued function ρ.
$\mathrm{int}\, A$	The interior of a set A.
$\mathrm{Jac}(f)(x)$	The Jacobian of a map f at a point x.
λ_n or just λ	The n-dimensional Lebesgue measure.
$\mathbf{0}$	The point with the coordinates $(0, 0)$.
$\mathbf{1}$	The point with the coordinates $(1, 0)$.
$\lVert\mathbf{a} - \mathbf{b}\rVert$	The Euclidean distance of points $\mathbf{a}$ and $\mathbf{b}$.
ν_h	The measure with a density h with respect to the Lebesgue measure.
$f\rvert_S$	The restriction of a map f to a set S.
f^{-1}	The inverse map to a map f.
$f_*(\nu)$	The image measure (the pushforward measure) of a measure ν induced by a map f.
h'	The derivative of a real function h of one real variable.
$:=$	Definitional equality.
a.e.	Almost everywhere.

Introduction

What is a Lipschitz map? Informally speaking, it is a map that stretches all distances at most by a constant factor. Formally, we can define it in the following general way:

Definition: *Let $f\colon (X_1, d_1) \to (X_2, d_2)$ be a map between metric spaces (X_1, d_1) and (X_2, d_2). Let $L > 0$ be a real number. Then we call the map f **L-Lipschitz** if it satisfies the following condition for every pair of points $x, y \in (X_1, d_1)$:*

$$d_2(f(x), f(y)) \leq L \cdot d_1(x, y).$$

*We call the map f **Lipschitz** if there exists $L > 0$ such that f is L-Lipschitz. Finally, we call such a number L a **Lipschitz constant** of f.*

The definition of a Lipschitz map is clearly dependent on the metrics defined on the domain and the range of a given map. However, we work in $\mathbb{R}^2$ or, more generally, in $\mathbb{R}^n$, and so we use, throughout the whole thesis, the Euclidean distance ℓ_2.

Lipschitz maps have been a topic of an intensive research for decades, especially in the context of real and functional analysis, theory of differential equations or geometric measure theory. Our motivation to study these maps comes from the field of discrete mathematics, namely from a beautiful question raised by an Israeli mathematician Uriel Feige [10, Problem 2.12], which still remains unresolved. Feige's question is stated as follows:

Feige's question. Is there a constant $C > 0$ such that for every $n \in \mathbb{N}$ and $S \subset \mathbb{Z}^2$, $|S| = n^2$, there exists a C-Lipschitz bijection $f\colon S \to [n] \times [n]$? We denote by $[n]$ the set of integers $\{1, \ldots, n\}$.

To the best of our knowledge, there has been almost no progress towards answering Feige's question so far. The only works concerning this problem have been an unpublished work of Sinha [15] from 2010, in which Feige's question was answered positively for sets S in the shape of the long and thin rectangle, and the author's bachelor thesis [9] from 2012, in which the result of Sinha [15] was reproved in a way allowing a slight extension.

In the aforementioned bachelor thesis [9] there was also presented another problem which is loosely related to that of Feige. It is a question of whether every two separated nets in Euclidean space are bilipschitz equivalent. This question was first posed, according to Burago and Kleiner [1], by Furstenberg in the 1960s. Later, Gromov raised this question in general geometric setting [4, Page 16].

To explain this problem more precisely we have to introduce two definitions:

Definition: *A **separated net** in a metric space (X, d) is a set $S \subset X$ for which there are two constants $r, R > 0$ fulfilling the following two conditions:*

- *For every $x \in X$ there is some $a \in S$ such that $d(x, a) \leq R$, i.e., the set S is an **R-net** in (X, d).*
- *For every $a, b \in S$, $a \neq b$, we have $d(a, b) > r$, i.e., the set S is **r-separated** in (X, d).*

Definition: *A map* $f\colon (X_1, d_1) \to (X_2, d_2)$ *is called* ***L-bilipschitz*** *if both* f *and* f^{-1} *are L-Lipschitz, or equivalently, for every* $x, y \in X_1$ *we have*

$$\frac{1}{L} \cdot d_1(x,y) \leq d_2(f(x), f(y)) \leq L \cdot d_1(x,y).$$

We call the map f ***bilipschitz*** *if there exists some* $L \geq 1$ *such that* f *is L-bilipschitz.*

It is clear from the above definition that a bilipschitz map is always injective. Moreover, it is even a *homeomorphism* of the domain and its image, i.e., an isomorphism of the respective topological spaces. Another apparent property of bilipschitz maps is that a composition of two bilipschitz maps is again bilipschitz.

We call two separated nets S and P in the metric space (X, d) *bilipschitz equivalent* if there is a bilipschitz bijection $f\colon S \to P$. Now we can state the problem mentioned above properly:

The problem of anomalous separated nets. For any given separated net $S \subset \mathbb{R}^2$ is there always a bilipschitz bijective map $f\colon S \to \mathbb{Z}^2$?

In 1998, Burago and Kleiner [1] constructed a separated net in $\mathbb{R}^2$ that is not bilipschitz equivalent to the integer lattice $\mathbb{Z}^2$, and thus answered this question negatively.[1] For convenience, let us abbreviate the term "separated net in $\mathbb{R}^2$ that is not bilipschitz equivalent to the integer lattice $\mathbb{Z}^2$" as "*anomalous separated net*".

Feige's problem seems to be a hard problem. There are no apparent techniques that can be used to solve it. That is why we try to relate it to the problem of anomalous separated nets. This problem looks similar enough to that of Feige that one can hope to use the work of Burago and Kleiner to answer Feige's question.

In Feige's question we look for a Lipschitz map, on the other hand, in the problem of anomalous separated nets in $\mathbb{R}^2$ we look for a bilipschitz map. This is, quite evidently, a substantial difference. To see it we fix some $n \in \mathbb{N}$ and take a sequence of n^2-point sets $P_i := \{(0, 1^i), (0, 2^i), \ldots, (0, n^i)\}$. For the diameters of these sets we have $\operatorname{diam}(P_i) \to \infty$. Thus, it is easily seen that such sets cannot be mapped by a constantly-bilipschitz map onto the grid $[n] \times [n]$. However, it is not difficult to obtain a constantly Lipschitz bijection onto $[n] \times [n]$ for these sets.

The illustrated difference between bilipschitz and Lipschitz maps in the context of Feige's question seems to be rather artificial. Nevertheless, there is another difference, maybe crucial, which we are going to explain below.

An example of folding. In what follows, we present our attempt to adapt the technique of Burago and Kleiner to a form applicable to Feige's problem. We start with the following simple observation concerning Feige's problem in a very specific case.

[1] In their article [1] Burago and Kleiner claim that their construction also works generally in $\mathbb{R}^n$.

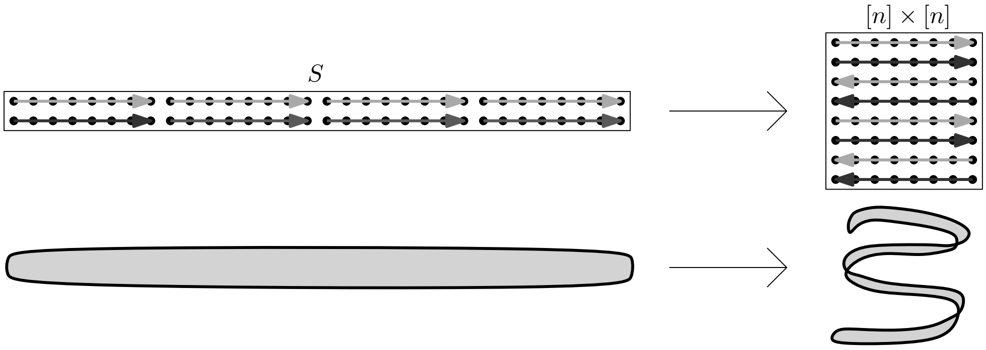

Figure 1: The idea of folding.

We take a set $S \subset \mathbb{Z}^2$ consisting of n^2 points arranged in the shape of thin and long rectangle, as illustrated in Figure 1. Now we try to find a constantly-Lipschitz bijection that maps the points of S onto $[n] \times [n]$. By the constantly-Lipschitz map we mean here a map with Lipschitz constant not dependent on n, but possibly dependent on the height of S, i.e., the number of the rows of points in S.

After a while, one can try to take this long and thin rectangle S as if it were a strip of paper and try to fold it several times in order to form the shape of square. Such an idea can indeed be imitated producing a d-Lipschitz bijection, in the case of S with height at most d, onto the grid $[n] \times [n]$. Both the idea and its realization are also depicted in Figure 1.

We do not need to discuss this example in detail, it only serves us as an illustration of a more general idea. For further treatment of Feige's question in such special cases, mainly how to get rid of a dependency of the Lipschitz constant on the number of the rows d in S, see author's bachelor thesis [9].[2]

Informally speaking, a bilipschitz map cannot realize such folds, because some pairs of points that were far away in the domain become close to each other. This is even more apparent when looking at some kind of "continuous analogs" as was the idea of folding the strip of paper above. The construction of such a "continuous analog" from the map between discrete sets is exactly what is happening in the proof of correctness of the construction of the anomalous separated net of Burago and Kleiner [1].

However, further examining the previous example we can observe that the Lipschitz map produced above is in fact bilipschitz on each part of the "strip" between the folds.

Before we proceed to an explanation of the Burago–Kleiner construction, we have to notice that we are going to use the basics of measure theory and Lebesgue integration. We expect that the reader is familiar with the basics of this theory, especially with the notions of a measure and its density, the Lebesgue measure in $\mathbb{R}^n$, the Lebesgue integral, a null (negligible) set, a measurable or a Borel set, a measurable or a Borel function, etc. There are plenty of materials covering this

[2]In that work the ℓ_1 metric has been used instead of the Euclidean distance ℓ_2. However, all ℓ_p metrics on $\mathbb{R}^n$ are *strongly equivalent*, which means that every distance in one specific ℓ_p metric is bounded above and below by a constant multiple of the same distance in another ℓ_p metric. This implies that the answer to Feige's question is independent of the chosen ℓ_p metric.

topic. Besides many others, the reader can see a book by Rudin [14] or more exhaustive monograph by Fremlin [3] and [2], for example. We denote the n-dimensional Lebesgue measure by λ_n. If the dimension is understood, then we sometimes denote it simply by λ. We implicitly use the Lebesgue measure if not stated otherwise.

Consider a map $\varphi\colon A \subset \mathbb{R}^n \to \mathbb{R}^n$, for some $n \in \mathbb{N}$, $\varphi = (\varphi^1, \varphi^2, \ldots, \varphi^n)$, where φ^i is the i-th coordinate map of φ. We remind that the matrix $D\varphi(x) := \left(\frac{\partial \varphi^i}{\partial x_j}(x)\right)_{i,j=1}^n$ consisting of all partial derivatives of φ at a point x, if they exist, is called the *Jacobian matrix* of φ at x. If the map φ is *strongly differentiable* (or *Fréchet differentiable*)[3] at x, then the Jacobian matrix of φ exists at x and it is the matrix of the unique linear map approximating φ at x, whose existence is guaranteed by strong differentiability.

The determinant of the Jacobian matrix at x is called the *Jacobian determinant* or simply the *Jacobian* of φ at x and denoted by $\operatorname{Jac}(\varphi)(x)$. The absolute value of $\operatorname{Jac}(\varphi)(x)$ determines the factor by which φ changes volume near x. We also remind that if the sign of $\operatorname{Jac}(\varphi)(x)$ is negative, then φ reverses orientation near x; if it is positive, then it preserves orientation near x. We know that bilipschitz maps are homeomorphisms. Therefore, the Jacobian of a bilipschitz map on a sufficiently nice domain, e.g., $[0,1]^2$, cannot change the sign.

In this thesis we look at Jacobians of bilipschitz maps in order to examine the change of area (or volume) caused by these maps. For that reason, we assume, without loss of generality, that the bilipschitz maps considered in this thesis have nonnegative Jacobians.

An anomalous density. The Burago–Kleiner anomalous separated net is based on a construction of a bounded measurable density[4] function $\rho\colon [0,1]^2 \to \mathbb{R}$ with $\inf \rho > 0$ such that there is no bilipschitz map $f\colon [0,1]^2 \to \mathbb{R}^2$ for which the equation $\operatorname{Jac}(f) = \rho$ holds almost everywhere (abbreviated a.e.).

The existence of such a density function in $\mathbb{R}^2$ had been an open question since the 1960s, firstly asked by Moser and Reimann. Burago and Kleiner [1], and independently McMullen [12], answered this question positively in 1998.

The problem with McMullen's construction is that McMullen has not provided a proof of its correctness in his paper. He just noticed that the proof can be obtained in a way similar to that of Burago and Kleiner. Although I have been able to provide the proof of correctness of McMullen's construction [9], this proof implies weaker properties than does the Burago–Kleiner proof on their construction. That is the reason why we will not describe nor use McMullen's construction anymore in this thesis.

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[3]This notion is a standard part of elementary courses on real analysis. See [2], for example.

[4]We should clarify using the notion of *density* in this context. The question of whether a given positive integrable function $\rho\colon A \subset \mathbb{R}^n \to \mathbb{R}$ is the Jacobian of a bilipschitz homeomorphism f can be viewed equivalently in the following way. We take a measure ν_ρ defined on A with a density ρ with respect to the Lebesgue measure λ, i.e., $\nu_\rho(B) := \int_B \rho \, d\lambda$ for every Lebesgue measurable subset $B \subseteq A$. Then we look at the *image measure* $f_*(\nu_\rho)$ (sometimes also called the *pushforward measure*), which is defined by $f_*(\nu_\rho)(C) := \nu_\rho(f^{-1}(C))$. According to the change of variables formula for the Lebesgue integral we obtain that the density of $f_*(\nu_\rho)$ with respect to the Lebesgue measure is the function $(\rho/\operatorname{Jac}(f)) \circ f^{-1}$ a.e., and hence $\rho = \operatorname{Jac}(f)$ a.e. if and only if $f_*(\nu_\rho) = \lambda$.

Again, for convenience, let us abbreviate the term "bounded measurable density function defined on $[0,1]^2$ with a positive infimum not realizable as the Jacobian of a bilipschitz homeomorphism in $\mathbb{R}^2$" as "*anomalous density*". We will describe the construction of the anomalous density of Burago and Kleiner in detail at the beginning of Chapter 1. Now we sketch how they use their anomalous density to produce an anomalous separated net in $\mathbb{R}^2$.

Transforming a density function into a separated net. Let $\rho\colon [0,1]^2 \to \mathbb{R}$ be a bounded measurable function with $\inf \rho > 0$. Take a sequence of pairwise disjoint squares $S_k \subset \mathbb{R}^2$ with vertices in $\mathbb{Z}^2$ and sidelengths l_k such that $l_k \to \infty$. We also take similitudes $\varphi_k\colon [0,1]^2 \to S_k$. Thus, we can "implant" ρ into each S_k using φ_k and looking at $\rho \circ \varphi_k$. Now, we subdivide each S_k into m_k^2 identical squares $\{T_k^i\}_{i=1}^{m_k^2}$. We choose the quantities m_k so that the sidelengths l_k/m_k of the squares T_k^i go to the infinity with k and so that the quantities m_k go to the infinity as well.

Finally, we produce a separated net $X \subset \mathbb{R}^2$ in two steps. First, we place one point at the center of each square with integer vertices not contained in $\bigcup_{k=1}^{\infty} S_k$ and sidelength equal to one. Second, we place $\lfloor \int_{T_k^i} \rho \circ \varphi_k \, d\lambda_2 \rfloor$ points regularly into each T_k^i, for every $i \in [m_k^2]$ and every $k \in \mathbb{N}$. The latter step of the construction is depicted in Figure 2.

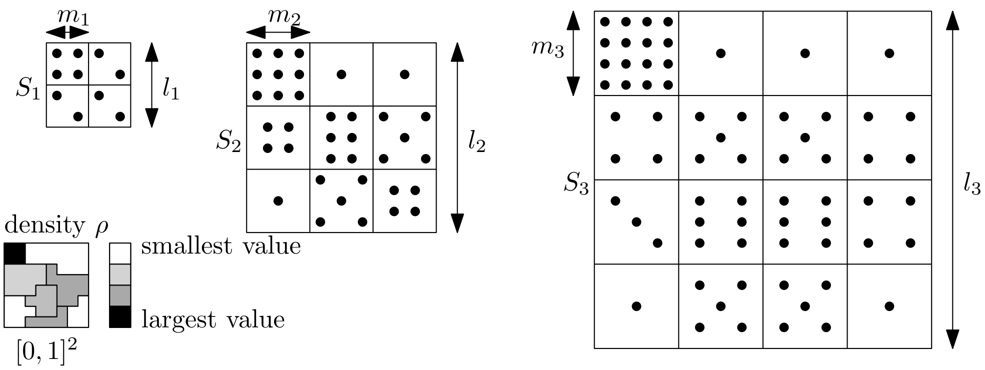

Figure 2: The construction of the separated net X from the density ρ inside of each square S_k.

The boundedness of ρ implies that the set X is indeed a separated net. Moreover, the construction of X inside of each S_k "encodes" the structure of ρ more and more precisely with growing k. This is done by the choice of values of m_k's. Since we have $m_k \to \infty$, each T_k^i contains smaller and smaller part of ρ. In addition, the property $l_k/m_k \to \infty$ allows one to catch smaller and smaller details of ρ inside of each T_k^i by the values of $\int_{T_k^i} \rho \circ \varphi_k \, d\lambda_2$. Consequently, every detail of ρ is reflected by X in every S_k starting at some k_0.

Burago and Kleiner then prove that, if we apply this construction to their anomalous density function ρ, we obtain an anomalous separated net. This is done arguing that a bilipschitz bijection $g\colon X \to \mathbb{Z}^2$ can be used to construct a bilipschitz homeomorphism $f\colon [0,1]^2 \to \mathbb{R}^2$ with $\mathrm{Jac}(f) = \rho$ a.e.

As in the simple example of folding of a strip of paper (see Figure 1), such f can be viewed as a "continuous analog" of the maps $g|_{S_k}$, where $g|_{S_k}$ denotes the restriction of g to the set S_k. We do not describe this part of the Burago–

Kleiner construction [1] in detail for two reasons. First, it is a measure theoretic technicality. Second, it is sufficient for us to use it as a black box and stick to the idea of a "continuous analog".

Nevertheless, it is important to note that we do not need the points of X outside $\bigcup_{k=1}^{\infty} S_k$ here. Likewise, we do not need g to be defined on these points to produce f. These points are defined only in order to create a net of the whole $\mathbb{R}^2$. The property that makes X bilipschitz nonequivalent to $\mathbb{Z}^2$ is fully contained in the finite sets S_k.

For completeness, we add that McMullen [12] also proved the opposite implication, i.e., that an anomalous separated net can be used to produce an anomalous density. Consequently, we can regard these two problems, i.e., the problem of the existence of an anomalous separated net and the problem of the existence of an anomalous density, as two versions of only one problem - a discrete version and a continuous one.

A continuous version of Feige's question? After the exposition presented above one can reasonably ask as follows: Is there a continuous formulation of Feige's question? What does such a formulation look like? Unfortunately, we cannot provide satisfactory answers to these questions. However, we present here some progress in this direction.

The first guess might be that one tries to look for a density not realizable as the Jacobian of any Lipschitz map.[5] In Chapter 1 we will present a modification of the Burago–Kleiner construction possessing this property at least for a subclass of Lipschitz maps wider than the class of bilipschitz maps. Nevertheless, this is not the right question with respect to Feige's problem.

The problem is that Feige's question cannot be easily transformed to a question about the Jacobians of maps from a certain class. The reason for this is illustrated on the simple example of folding of a rectangle presented above (see Figure 1).

In that example we transform one density (represented by the points inside of a long and thin rectangle), let us call it ρ, to a density equal to 1 (represented by the points of $[n] \times [n]$) by a map realizing some sort of folding. Looking at a "continuous analog" of this map (the folded strip of paper in Figure 1), call it f for now, we observe that the density on the image is not a simple one-to-one transformation of the density on the domain. Some points from the image have two preimages under f, and thus a density on the image around these points is rather a sum of transformed densities around the respective preimages. Hence, the density on the image cannot be related to the density on the domain only by a simple Jacobian transformation as in the Burago–Kleiner construction, i.e., by the equation $\rho/\operatorname{Jac}(f) = 1$ a.e.

Towards a continuous version of the example with folding. In connection with the discussion presented above we ask the following question, which can

[5] In this case, i.e., given $\rho\colon [0,1]^2 \to \mathbb{R}$ a density with $\inf \rho > 0$ and a Lipschitz map $f\colon [0,1]^2 \to \mathbb{R}^2$, it is more appropriate to ask whether $\rho = |\operatorname{Jac}(f)|$ holds a.e. than just whether $\rho = \operatorname{Jac}(f)$ holds a.e. In contrast to bilipschitz maps on $[0,1]^2$, the Jacobian of a Lipschitz map can change a sign.

be regarded as a generalization of the problem of the existence of an anomalous density:

Question 1: *Let I_1 and I_2 be two disjoint copies of the square $[0,1]^2$. For $i \in \{1,2\}$ are there bounded measurable densities $\rho_i\colon I_i \to \mathbb{R}$ with $\inf \rho_i > 0$ such that there are no bilipschitz maps $f_i\colon I_i \to \mathbb{R}^2$ verifying the equation*

$$\left(\frac{\rho_1}{\mathrm{Jac}(f_1)} \circ f_1^{-1}\right)(x) + \left(\frac{\rho_2}{\mathrm{Jac}(f_2)} \circ f_2^{-1}\right)(x) = 1$$

for almost every $x \in f_1(I_1) \cup f_2(I_2)$? (If either of the formulas in the equation above is not well defined, we treat it as a zero.)

If we had such densities ρ_1 and ρ_2, we explain briefly how they could be used with respect to Feige's problem. Each ρ_i can be transformed to a sequence of finite separated sets $\{S_k^i\}_{k=1}^{\infty}$ as in the Burago–Kleiner construction. There is some $r > 0$ such that every S_k^i is r-separated for $i \in \{1,2\}$ and $k \in \mathbb{N}$. Therefore, all these sets can be scaled by the factor $2/r$, and thus become 2-separated. Every point in such a 2-separated set can be rounded to the nearest integer point in $\mathbb{Z}^2$ obtaining a set $\widetilde{S}_k^i$. Any two points from the same set S_k^i cannot be rounded to the same integer point in $\widetilde{S}_k^i$, because these sets are 2-separated.

Suppose now that there are constantly-bilipschitz transformations of each $\widetilde{S}_k^i$ into a subset of $\mathbb{Z}^2$ such that the images of each pair $\widetilde{S}_k^1$, $\widetilde{S}_k^2$ sums to the uniform density of some "connected" bounded piece of $\mathbb{Z}^2$ (we would say to $[n] \times [n]$, but we want to avoid number theoretic assumptions concerning cardinality of each S_k^i here). Reversing the rounding procedure, scaling, and using the machinery of the Burago–Kleiner technique these maps would yield bilipschitz maps f_1 and f_2 contradicting the properties of ρ_1 and ρ_2 from Question 1.

In Chapter 1 we construct a candidate density expected to have the properties required in Question 1. Chapter 2 contains my attempt to prove that this candidate density is the right one, but this proof still remains incomplete.

Obviously, the preceding question can be further generalized to any finite number of squares I_i and densities ρ_i. It might seem that answering such a question would answer Feige's question only in a very special case. Nonetheless, there is a result by Jones [8] from 1988 suggesting that these special cases are not far from a general case.

Jones' result roughly says that any Lipschitz map on a bounded domain can be decomposed into bilipschitz maps everywhere except for a set of arbitrarily small measure. Formally, it is stated as follows:[6]

Theorem 0.1 (Jones [8]): *Let $f\colon [0,1]^n \to \mathbb{R}^n$ be 1-Lipschitz. Then for every $\delta > 0$ there is $M(\delta) < \infty$ and there are compact sets $K_1, \dots K_M \subset [0,1]^n$, for some $M \leq M(\delta)$, such that $|f(x) - f(y)| \geq \delta\,|x-y|/2$ for every $x, y \in K_j$ and every $j \in [M]$, and such that*

$$\lambda_n\left(f\left([0,1]^n \setminus \bigcup_{j=1}^{M} K_j\right)\right) < \delta.$$

[6] We state it here in a less general form, because we are interested only in the case when the dimension of a domain equals the dimension of a range.

We remark that the assumption that f be 1-Lipschitz is not restrictive, because every Lipschitz map can be made 1-Lipschitz by an appropriate scaling.

The result of Jones says that every Lipschitz map (on a reasonable domain) is bilipschitz on a finite number of subsets of the domain whose images occupy arbitrarily large portion of the whole image of the domain. Hence, loosely speaking, a majority of the image under a Lipschitz map looks like a result of a finite sum of bilipschitz maps, i.e., like something very similar to the conditions contained in our Question 1.

1. Nonrealizable densities

1.1 The Burago–Kleiner anomalous density

Before we proceed to the description of the Burago–Kleiner construction we state two classical and well known theorems about Lipschitz maps together with one easy observation, which will be needed later.

Theorem 1.1 (The area formula for bilipschitz maps): *Let $f\colon A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a bilipschitz map. Then for every measurable set E, $E \subseteq A$, the set $f(E)$ is measurable and we have*

$$\int_E |\mathrm{Jac}(f)| \; d\lambda_n = \lambda_n(f(E)).$$

This variant of the area formula can be found in Fremlin's monograph [2, 263F], for example. We now state Rademacher's theorem; see [5] for a reference.

Theorem 1.2 (Rademacher): *Let $A \subseteq \mathbb{R}^n$ be open and $f\colon A \to \mathbb{R}^n$ be Lipschitz. Then f is Fréchet differentiable a.e. in A, i.e., $Df(x)$ exists at almost every point $x \in A$.*

The following observation tells us that the Jacobian of an L-bilipschitz homeomorphism in $\mathbb{R}^2$ is bounded above and below by L^2 and $1/L^2$, respectively.

Observation 1.3: *Let $f\colon A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be an L-bilipschitz map. Then for every $x \in A$ such that $Df(x)$ exists we have $1/L^2 \leq |\mathrm{Jac}(f)(x)| \leq L^2$.*

Proof. We pick $x \in A$ such that $Df(x)$ exists. Then for every vector $h \in \mathbb{R}^2$ and every $t \in \mathbb{R}$ we have $\|th\|/L \leq \|f(x+th) - f(x)\| \leq L\|th\|$. Thus, for the directional derivative of f at x in the direction h we have

$$\frac{1}{L}\|h\| \leq \left\| \lim_{t\to 0} \frac{f(x+th) - f(x)}{t} \right\| = \|Df(x)h\| \leq L\|h\|.$$

This in turn implies that $1/L^2 \leq |\mathrm{Jac}(f)(x)| \leq L^2$. **Q.E.D.**

Describing the Burago–Kleiner construction properly involves definitions of several terms and introduction of some parameters, whose values will be either set at the end or we will do just with proving their existence. The following definitions and notation will be used throughout the whole chapter.

For a given $N \in \mathbb{N}$ we define the rectangle $R_N := [0,1] \times [0,1/N]$ in $\mathbb{R}^2$. This rectangle can be partitioned, in an obvious manner, to N squares of the form $[\frac{i-1}{N}, \frac{i}{N}] \times [0, 1/N]$. We denote these squares by S_i, $i \in [N]$. We use the simpler sign λ instead of λ_2 in the rest of this thesis. Furthermore, we denote the line segment between points $\mathbf{a}$ and $\mathbf{b}$ by $\overline{\mathbf{ab}}$.

Definition 1.4: *Let $d > c > 0$ be real constants. For a given $N \in \mathbb{N}$ we define a function $\rho_N\colon [0,1] \times [0,1/N] \to \{c,d\} \subset \mathbb{R}$ so that it is constant on the interior of each square S_i, $i \in [N]$, attains the value c on the S_i's with i odd, and the value d on the S_i's with i even.*

Although the definition of ρ_N depends on the specific values of c and d, we do not explicitly specify these values in the symbol ρ_N in order to simplify the notation. We always assume that these values are understood within a context. The reader may also think of c and d as of constants fixed throughout the whole chapter.

We denote by $\mathbf{0}$ and $\mathbf{1}$ the points in the plane with the coordinates $(0,0)$ and $(1,0)$, respectively. We now describe the main idea of the Burago–Kleiner construction.

The construction of the Burago–Kleiner anomalous density is based on the construction of a density function that is not realizable as the Jacobian of any L-bilipschitz homeomorphism. A core of such a density is the function ρ_N, where the value of N is chosen with respect to the value of L. In Lemma 1.5 we observe that there is $\epsilon = \epsilon(N, L) > 0$ with the following property. If f is an L-bilipschitz homeomorphism whose Jacobian equals ρ_N everywhere except for a set of measure less than ϵ, then we can find a finite set of nonintersecting segments such that f has to stretch some of these segments "much" more than it stretches the distance between $\mathbf{0}$ and $\mathbf{1}$. The collection of these segments is formed by the segments between the vertices of some regular grid inside R_N.

In the second step we iterate the preceding step as follows. We cover each segment from the collection mentioned in the first step by a sufficiently small copy of $R_{N'}$ so that these copies do not intersect each other. Then we modify ρ_N on these rectangles "implanting" $\rho_{N'}$ into each copy of $R_{N'}$. We choose the value of N' large enough in order to keep the total area of these copies of $R_{N'}$ smaller than ϵ. Hence, we preserve the properties of ρ_N obtained in the first step. At the same time, we get a new finite collection of nonintersecting segments from which at least one is stretched "much" more under L-bilipschitz homeomorphism than are all the segments from the collection in the first step.

Thus, if we iterate this process several times, we obtain a density that cannot be the Jacobian of any L-bilipschitz homeomorphism. If it were not true, then the distance between some pair of points in R_N would be stretched under an L-bilipschitz homeomorphism with this prescribed Jacobian more than it is allowed by the Lipschitz constant L of a given map.

Now we are going to describe the process sketched above precisely. Our presentation follows the exposition of Burago and Kleiner in their article [1].[1]

Lemma 1.5: *For every* $L \geq 1$ *and* $c, d \in \mathbb{R}$, $0 < c < d$, *there are* $N_0 \in \mathbb{N}$, $M \in \mathbb{N}$, *and* $k > 0$ *with the property that for every* $N \in \mathbb{N}$, $N \geq N_0$, *there is* $\epsilon > 0$ *such that every* L*-bilipschitz homeomorphism* f *whose Jacobian* $\mathrm{Jac}(f)$ *equals* ρ_N *everywhere except for a set of measure less than* ϵ *has to stretch the distance between some pair of points of the form* $(\frac{p}{NM} + \frac{i-1}{N}, \frac{s}{NM}) \in R_N$ *and* $(\frac{p+1}{NM} + \frac{i-1}{N}, \frac{s}{NM}) \in R_N$ *at least* $(1+k)$ *times more than it stretches the distance between the points* $\mathbf{0}$ *and* $\mathbf{1}$.

Proof. We consider the values of L, c, and d, $0 < c < d$, fixed until the end of the proof. Let us choose an integer M, whose value will be set at the end of

[1] In fact, we make one small and purely technical change to the original Burago–Kleiner [1] construction. Burago and Kleiner present their construction with $c = 1$. These change is made to fit more the way we use this construction later.

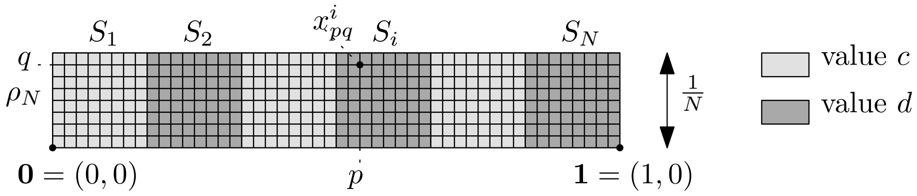

Figure 1.1: The function ρ_N with marked vertices x^i_{pq}.

the argument. We now subdivide each S_i into M^2 smaller squares, we call their vertices the *marked vertices* or the *marked points* and denote them by $x^i_{pq} := (\frac{p}{NM} + \frac{i-1}{N}, \frac{q}{NM})$, where $i \in [N]$ and $p, q \in \{0, \dots, M\}$. So x^i_{pq} lies in the square S_i and its "coordinates" inside of S_i are p and q. Later in the proof we will look at the pairs of marked points and examine how much they are stretched under bilipschitz maps with the Jacobian ρ_N.

For a given marked vertex x^i_{pq} the marked vertex x^{i+1}_{pq} is called the *corresponding* marked vertex to x^i_{pq}. It is the vertex in the neighbor square S_{i+1} with the same "coordinates" inside of this square. Sometimes we call a pair of marked vertices (x^i_{pq}, x^{i+1}_{pq}) a *marked pair.*

We now fix four constants $N_0 \in \mathbb{N}$, $k = k(N_0) > 0$, $N \geq N_0$, and $\epsilon = \epsilon(N) > 0$, whose values will be specified later, and continue proving this lemma by contradiction.

Let us assume that $f \colon R_N \to \mathbb{R}^2$ is an L-bilipschitz map whose Jacobian is equal to ρ_N everywhere except for a set of measure less than ϵ. We denote the stretching of the distance between $\mathbf{0}$ and $\mathbf{1}$ by $A := \|f(\mathbf{1}) - f(\mathbf{0})\| \,/\, \|\mathbf{1} - \mathbf{0}\|$. Moreover, we assume that the distance between every pair of marked points of the form $(x^i_{pq}, x^i_{p+1,q})$ is stretched under f by a factor at most $(1+k)A$.

We further denote by $W^i_{pq} := f(x^{i+1}_{pq}) - f(x^i_{pq})$ the vector between the images of a marked pair (x^{i+1}_{pq}, x^i_{pq}) under the map f. Using the hypothesis that every pair of marked points $(x^i_{pq}, x^i_{p+1,q})$ is stretched at most by the factor $(1+k)A$, we deduce that the length of each W^i_{pq} is at most $(1+k)A/N$.

We assume, without loss of generality, that $f(\mathbf{0}) = \mathbf{0}$ and $f(\mathbf{1}) = (A, 0)$. The main idea of the proof is to show that under the assumptions made before there must be some $i \in [N]$ such that all the vectors W^i_{pq}, for $p, q \in \{0, \dots, M\}$, lie close to the vector $W := (A/N, 0)$. This allows us to bound above the difference between areas of the sets $f(S_i)$ and $f(S_{i+1})$. On the other hand, the knowledge of $\mathrm{Jac}(f)$ permits us to infer a lower bound on the preceding quantity, and hence obtain a contradiction.

To find such a square S_i, we must first define what it does exactly mean that some W^i_{pq} is close to W. Let us assume we have some t, $0 < t < 1$. We say that the vector W^i_{pq} is *t-regular* (or just *regular*) if its projection to the x-axis is longer than $\frac{(1-t)A}{N}$. A square S_i is called *t-regular* (or again just *regular*) if all the W^i_{pq}'s are t-regular, $p, q \in \{0, \dots, M\}$.

Observation: *There exist $k_1 = k_1(t, M) > 0$ and $N_0 = N_0(t, M)$ such that for every $k \leq k_1$ and every $N \geq N_0$ there is a t-regular square S_i.*

Proof. We will reason by contradiction. The fact that there is no t-regular square implies that there must be a lot of irregular vectors W^i_{pq}, and hence, for N big enough, we can find using the pigeonhole principle a row s with the property that there are at least $\frac{N}{M+1}$ irregular vectors W^i_{ps}, each in a different square S_i. Thus we can find sequences $\{p_j\}_{j=1}^J$ and $\{i_j\}_{j=1}^J$, where $J \geq \frac{N}{2(M+1)}$, such that $\{i_j\}_{j=1}^J$ is an increasing sequence of fixed parity (all i_j's are even or all of them are odd), and such that the corresponding vectors $W^{i_j}_{p_j s}$ are irregular. The situation is depicted in Figure 1.2.

In other words, we look at one row s in the rectangle R_N and find at least $\frac{N}{2(M+1)}$ non-overlapping segments $\overline{x^{i_j}_{p_j s} x^{i_j+1}_{p_j s}}$ such that the corresponding vectors $W^{i_j}_{p_j s}$ are irregular. This means that their projection to the x-axis is short.

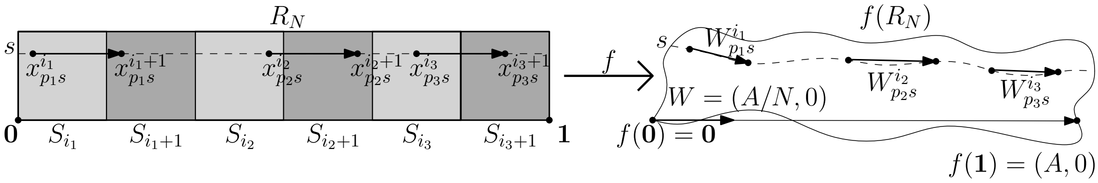

Figure 1.2: The choice of $J \geq \frac{N}{2(M+1)}$ irregular vectors $W^{i_j}_{p_j s}$ in the row s.

We see that the polygonal curve with the vertices $\mathbf{0}, (0, \frac{s}{NM}), (1, \frac{s}{NM}), \mathbf{1}$ connects its endpoints $\mathbf{0}$ and $\mathbf{1}$. It means that the image of this curve connects the points $f(\mathbf{0})$ and $f(\mathbf{1})$, and thus the projection of this curve to the x-axis has to be at least A long. On the other hand, we can bound above the length of this projection in the following way:

First, we use the upper bound on the length of irregular vectors W^i_{pq} and the lower bound on J. Then we know that there are $M(N-J)$ segments of the form $\overline{x^i_{ps} x^i_{p+1,s}}$ not contained in the segments $\overline{x^{i_j}_{p_j s} x^{i_j+1}_{p_j s}}$. Also, we can use the assumption that these $M(N-J)$ segments cannot be stretched more than by the factor $(1+k)A$ under f. Next, we use the fact that f is L-Lipschitz. Therefore, the vertical segments $\overline{\mathbf{0}, (0, \frac{s}{NM})}$ and $\overline{(1, \frac{s}{NM}), \mathbf{1}}$ cannot be stretched more than by the factor L.

Finally, combining these two bounds we obtain the following inequality:

$$\left(\frac{N}{2(M+1)}\right)\left(\frac{(1-t)A}{N}\right) + \left(NM - \frac{NM}{2(M+1)}\right)\left(\frac{(1+k)A}{NM}\right) + 2\frac{L}{N} \geq A.$$

It can easily be seen that this inequality is false when k is sufficiently small and N is sufficiently large (depending on L, M, and t, of course). Since f is L-bilipschitz, we have $A \geq 1/L$.

Q.E.D.

We now make a simple observation which tells us that by a sufficiently small choice of t we can make every t-regular vector W^i_{pq} as close to the vector W as we want:

Observation: *For every $m > 0$ there is a value of $t_0 = t_0(m) > 0$ such that whenever $k \leq t \leq t_0$ we have $\left\|W - W^i_{pq}\right\| \leq m/N$ for every t-regular vector W^i_{pq}.*

Proof. It is a straightforward application of Pythagoras' theorem. Let W^i_{pq} be a regular vector with the coordinates (X, Y). Because $W = (A/N, 0)$, we have $W - W^i_{pq} = (A/N - X, -Y)$. By the t-regularity of W^i_{pq} we have $X \geq \frac{(1-t)A}{N}$.

As we have assumed to obtain a contradiction, $\left\|W_{pq}^i\right\| \leq \frac{(1+k)A}{N}$, so we also have $X \leq \frac{(1+k)A}{N}$. Thus, we can write $A/N - X \leq \frac{tA}{N}$ and $A/N - X \geq -\frac{kA}{N}$. Combining these two bounds we infer that the difference of the x-coordinates of W and W_{pq}^i is bounded by $\frac{(t+k)A}{N} \leq \frac{2tA}{N}$.

The length of W_{pq}^i is equal to $X^2 + Y^2$, which is no greater than $\left(\frac{(1+k)A}{N}\right)^2$. To get an upper bound on Y^2 we substitute the smallest possible value $\frac{(1-t)A}{N}$ for X, and hence we obtain $Y^2 \leq \frac{A^2(2(k+t)+k^2-t^2)}{N^2} \leq \frac{4tA^2}{N^2}$.

Finally, we conclude that

$$\left\|W - W_{pq}^i\right\| \leq \sqrt{\left(\frac{2tA}{N}\right)^2 + \frac{4tA^2}{N^2}} \leq \frac{2A}{N}\sqrt{t^2+t} \leq \frac{2L}{N}\sqrt{t^2+t}.$$

Q.E.D.

The last observation tells us that by a suitable choice of constants m and M we can make the difference of the areas of the images of some regular square S_i and its neighbor square S_{i+1} arbitrarily close to zero.

Observation: *For any real value $\alpha > 0$ there are $m_0 > 0$ and $M_0 \in \mathbb{N}$ with the following property. If we pick $m \leq m_0$ and $M \geq M_0$ such that there exists an index $i \in [N]$ with the property that for every p, q we have $\left\|W - W_{pq}^i\right\| \leq m/N$, then*

$$|\lambda(f(S_i)) - \lambda(f(S_{i+1}))| < \frac{\alpha}{N^2}.$$

The values of m_0 and M_0 do not depend on N.

Proof. We choose $i \in [N]$ for which the hypothesis of this observation holds. Let us write $Q := f(S_i)$ and $R := f(S_{i+1})$. We look at the translation of the set Q by the vector W; we denote the resulting set by $T := Q + W$. Since every W_{pq}^i lies very close to W, the whole set T has to lie very close to R. This idea can be made more precise considering the images of the marked vertices x_{pq}^i and x_{pq}^{i+1} lying on the boundaries of Q and R, respectively.

We recall that f is a bilipschitz map, and thus a homeomorphism, which means that it maps boundaries onto boundaries. The marked points form the $1/NM$-nets on the boundaries of S_i and S_{i+1}. Therefore, the images of the marked points under f form the L/NM-nets on the boundaries of Q and R. We assume $\left\|W - W_{pq}^i\right\| \leq m/N$; consequently, the image of the marked point x_{pq}^i on the boundary of T lies within the m/N-neighborhood of the image of the corresponding marked point x_{pq}^{i+1} on the boundary of R. Hence, we get that the boundary of T lies within the $m/N + 2L/NM$-neighborhood of the boundary of R. Because f is L-Lipschitz, the length of the boundary of R is at most $4L/N$.

The area of an r-neighborhood of a curve of the length $\gamma < \infty$ can be bounded above[2] by $2r\gamma + \pi r^2$. This implies that the difference of the areas of the sets Q

[2]This bound, also called the area of the so called *tubular neighborhood* of a curve, was first given by Hotelling [7].

and R can be bounded above by

$$|\lambda(R) - \lambda(Q)| \leq \frac{8L}{N}\left(\frac{m}{N} + \frac{2L}{NM}\right) + \pi\left(\frac{m}{N} + \frac{2L}{NM}\right)^2$$
$$\leq \frac{1}{N^2}\left(8Lm + \frac{16L^2}{M} + \pi\left(m + \frac{2L}{M}\right)^2\right).$$

This quantity can clearly be made smaller than α/N^2 by a suitable choice of the parameters m and M (larger values of M implies larger values of N, nevertheless, this do not affect the validity of the desired inequality, because the choice of m and M depends only on α). **Q.E.D.**

We assume now that ρ_N takes the value c on S_i, while its value on S_{i+1} is d (the other case can be treated in the same way). Since $\operatorname{Jac}(f)$ is equal to ρ_N everywhere except for a set of measure less than ϵ, we can use it to make another bounds on the quantities $\lambda(f(S_i))$ and $\lambda(f(S_{i+1}))$. Using Observation 1.3 and the area formula from Theorem 1.1 we get that $\lambda(f(S_i)) \leq c(1/N^2 - \epsilon) + \epsilon L^2$ and $\lambda(f(S_{i+1})) \geq d(1/N^2 - \epsilon)$. We conclude that

$$\lambda(f(S_{i+1})) - \lambda(f(S_i)) \geq (d - c)(1/N^2 - \epsilon) - \epsilon L^2.$$

The right hand side of the previous inequality is positive provided $\epsilon < \frac{d-c}{N^2(d-c+L^2)}$. Thus, if we set $\alpha := (d-c)/2$ and $\epsilon := \frac{\alpha}{N^2(d-c+L^2)}$, we deduce that

$$\lambda(f(S_{i+1})) - \lambda(f(S_i)) \geq \frac{d-c}{N^2} - \frac{\alpha}{N^2} = \frac{d-c}{2N^2}.$$

On the other hand, combining all the observations above we deduce that, by a choice of $M \geq M_0(\alpha)$ and $m \leq m_0(\alpha)$, picking some $t \leq t_0(m)$, $k \leq \min(t, k_1(t, M))$, and $N \geq N_0(t, M)$, there is an index $i \in [N]$ such that

$$|\lambda(f(S_{i+1})) - \lambda(f(S_i))| < \frac{\alpha}{N^2} = \frac{d-c}{2N^2}.$$

This is a contradiction proving Lemma 1.5.

Q.E.D.

A majority of the rest of this chapter is devoted to the examination of the proof of Lemma 1.5. Therefore, we reserve the names of the parameters from this proof until the end of this thesis. Especially, the parameters N and ϵ will be referred to frequently.

The construction above is a basis on which Burago and Kleiner built up a density that is L-bilipschitz nonrealizable. The idea is to iterate the previous construction on smaller and smaller scales in the neighborhood of marked points. In such a way we can prove that, after the j-th iteration, there is a pair of points $\mathbf{a}$ and $\mathbf{b}$ whose distance has to be stretched under an L-bilipschitz map by the factor at least $(1+k)^j A \geq (1+k)^j/L$.

We observe that all the properties of the above construction are preserved under a proportional scaling of the domain. This enables us to iterate the preceding construction on smaller and smaller scales. We remind that by the appropriate choice of $N \geq N_0$ in the proof above we can push $\lambda(R_N)$ arbitrarily close to zero.

Corollary 1.6 (Construction of an L-bilipschitz nonrealizable density): *For every $L \geq 1$ and $d > c > 0$ there exist $N \in \mathbb{N}$, a density $\rho^L_{c,d}\colon R_N \to \{c,d\} \subset \mathbb{R}$, and $\bar{\epsilon} > 0$ such that there is no L-bilipschitz map f with $\mathrm{Jac}(f) = \rho^L_{c,d}$ everywhere except for a set of measure less than $\bar{\epsilon}$.*

Proof. We choose $M = M(L,c,d)$ as in the above proof of Lemma 1.5 first. Then we take ρ_N with the values $d > c > 0$, where $N \geq N_0(M,L,c,d)$. Next, we choose $N' \geq N$ and cover each segment $\overline{x^i_{pq}x^i_{p+1,q}}$ between the marked points in R_N with small nonintersecting, but possibly touching, copies of $R_{N'}$ with total area less then ϵ (ϵ is also as in the proof of Lemma 1.5).

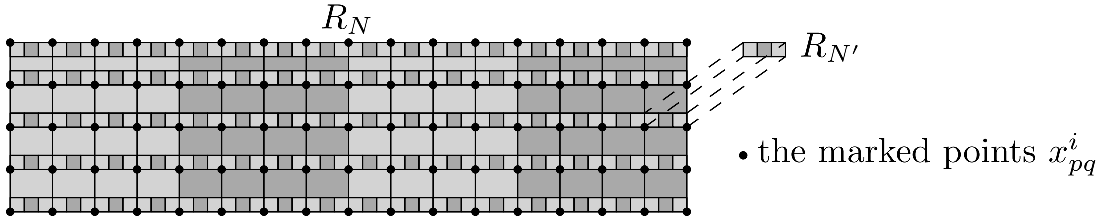

Figure 1.3: The first two steps in the construction of $\rho^L_{c,d}$.

In the following step, we implant $\rho_{N'}$ into each of these smaller rectangles, and thus redefine the density there. This step is illustrated in Figure 1.3. Finally, we inductively repeat this step inside of each of these smaller rectangles, and eventually we obtain the desired density $\rho^L_{c,d}$. The number of inductive steps needed to destroy the L-bilipschitz realizability is at most $\lceil \log_k(L^2) \rceil$, where k is as in the above proof of Lemma 1.5. **Q.E.D.**

At last, we can use the densities $\rho^L_{c,d}$ to construct an anomalous density:[3]

Corollary 1.7 (Construction of an anomalous density): *For every $d > c > 0$ there exists a density $\rho_{c,d}\colon [0,1]^2 \to \{c,d\} \subset \mathbb{R}$ such that there is no bilipschitz map f with $\mathrm{Jac}(f) = \rho_{c,d}$ a.e.*

Proof. We take a sequence of disjoint rectangles $Q_L \subset [0,1]^2$, where $L \in \mathbb{N}$. We now define the density $\rho_{c,d}$. For every $L \in \mathbb{N}$ we implant $\rho^L_{c,d}$ into Q_L. Finally, we define $\rho_{c,d}$ to be equal to a constant, e.g., to c, on the rest of $[0,1]^2$.

Of course, $\rho_{c,d}$ is an anomalous density. If there were a bilipschitz map $f\colon [0,1]^2 \to \mathbb{R}^2$ with $\mathrm{Jac}(f) = \rho_{c,d}$ a.e., then this f would be $\bar{L}$-bilipschitz for a certain value of $\bar{L} \in \mathbb{N}$. Consequently, for every $L \geq \bar{L}$, the map $f|_{Q_L}$ would also be L-bilipschitz with $\mathrm{Jac}(f|_{Q_L}) = \rho^L_{c,d}$ a.e. This would be a contradiction. **Q.E.D.**

1.2 Modifying the Burago–Kleiner anomalous density

In this section we further investigate the properties of the Burago–Kleiner densities and present a modification of their construction. For a number $z \in \mathbb{R}$

[3] In their article [1] Burago and Kleiner produced an anomalous density that is even continuous. We do not need this additional property. However, it is only a technicality. A continuity can be obtained by an appropriate *mollification* of each $\rho^L_{c,d}$.

and a function $\varphi\colon A \subset \mathbb{R}^2 \to \mathbb{R}$ we denote by $\varphi + z$ a function that satisfies $(\varphi + z)(x) := \varphi(x) + z$ for every $x \in A$.

We now come back to the proof of Lemma 1.5. If we look at this proof we observe that, in fact, the whole construction does not depend directly on the specific values of c and d. It rather depends on the value of their difference, i.e., on $d - c$. Valid choices of the parameters in Lemma 1.5 for particular values of c and d remain also valid for the values $c + z$ and $d + z$.

Hence, this implies that all the properties of the Burago–Kleiner densities discussed in Section 1.1 are preserved as well for the densities $\rho^L_{c,d} + z$ or $\rho_{c,d} + z$ for an arbitrary value[4] of $z \in (-c, \infty)$. This leads us to the following definition:

Definition 1.8: *We call a bounded measurable density $\rho\colon [0,1]^2 \to \mathbb{R}$ with $\inf \rho > 0$* ***strongly anomalous*** *if for every $z \in (-\inf \rho, \infty)$ the function $\rho + z$ is an anomalous density.*

Observation 1.9: *The density $\rho_{c,d}$ of Burago and Kleiner is strongly anomalous.*

It is evident that every strongly anomalous density is also an anomalous density. What can be said about the opposite implication? As we have already said earlier, McMullen also constructed an anomalous density in his article [12]. We also noticed that he did not provide a full proof of its correctness. Even though I have been able to provide one in [9], the problem is that this proof does not imply that McMullen's density is also strongly anomalous. That is why we do not use his construction here.

We state the following question:

Question 2: *Is every anomalous density also strongly anomalous?*

Although it seems plausible that the answer to Question 2 is positive, we have not been able to prove it. However, this question is very important to us, because it may be closely related to Feige's problem. In Chapter 2 we will prove that a positive answer to this question already implies a positive answer to Question 1, which was stated in the Introduction.

We now proceed to the second modification of the Burago–Kleiner densities. So far, in order to preserve the L-bilipschitz nonrealizability of $\rho^L_{c,d}$, we are allowed to modify $\rho^L_{c,d}$ in two ways. First, we know that there is $\epsilon > 0$ such that we can change the values of $\rho^L_{c,d}$ arbitrarily on a set of measure less than ϵ. Second, we have observed that we can change $\rho^L_{c,d}$ on the whole domain adding a constant to it. Next, we would like to gain a possibility to modify $\rho^L_{c,d}$ not just by adding a constant to it, but also by adding a sufficiently bounded function to it.

In the proof of Lemma 1.5 we have assumed the existence of an L-bilipschitz homeomorphism f with $\mathrm{Jac}(f) = \rho_N$ everywhere except for a set of measure less than ϵ. We observe that the only place in that proof where we have worked with $\mathrm{Jac}(f)$ has been its very end. There we have picked an index $i \in [N]$ and looked at S_i and S_{i+1}. We have also assumed that ρ_N takes the value c on S_i and the value d on S_{i+1}. Then we have used Observation 1.3 and the area formula

[4] In fact, summing the density $\rho^L_{c,d}$ with a constant $z > L^2 - d$ is not very useful, because the resulting density is not L-bilipschitz realizable for trivial reasons. This can be seen from Observation 1.3. The same holds for the values of z smaller than $1/L^2 - c$.

from Theorem 1.1 to infer that $\lambda(f(S_i)) \leq c(1/N^2 - \epsilon) + \epsilon L^2$ and $\lambda(f(S_{i+1})) \geq d(1/N^2 - \epsilon)$.

Actually, in order to compute the quantities $\lambda(f(S_i))$ and $\lambda(f(S_{i+1}))$ we are not interested in exact values of $\mathrm{Jac}(f)$ on the sets S_i and S_{i+1}. For this purpose, determining are the average values of $\mathrm{Jac}(f)$ over these sets. This leads us to an introduction of a new parameter $\omega > 0$. His role is exposed in the following definition and thereafter:

Definition 1.10 (AVG(ϵ, ω)-property): *Let $f\colon R_N \to \mathbb{R}^2$ be an L-bilipschitz map. For real numbers $\epsilon > 0$ and $\omega > 0$ we define that f has an **AVG**(ϵ, ω)-**property** as follows:*

> *For every S_i in R_N there is a measurable set $E_i \subset S_i$, $\lambda(S_i \setminus E_i) < \epsilon$, such that the average value of $\mathrm{Jac}(f)$ over E_i differs from the value of $\rho_N|_{S_i}$ by at most ω. In other words,*
>
> $$\frac{\int_{E_i} \mathrm{Jac}(f)\ d\lambda}{\lambda(E_i)} \in [c - \omega, c + \omega] \qquad \text{for } \rho_N|_{S_i} = c,$$
> $$\frac{\int_{E_i} \mathrm{Jac}(f)\ d\lambda}{\lambda(E_i)} \in [d - \omega, d + \omega] \qquad \text{for } \rho_N|_{S_i} = d.$$

Hence, all the properties described in Lemma 1.5 are not possessed just by the L-bilipschitz maps with $\mathrm{Jac}(f) = \rho_N$ everywhere except for a set of measure less than ϵ, but the same properties also possess the L-bilipschitz maps with an AVG(ϵ, ω)-property (for appropriate values of ϵ and ω):

Lemma 1.11 (A strengthening of Lemma 1.5): *For every $L \geq 1$ and c, d, $0 < c < d$, there are $N_0 \in \mathbb{N}$ and $k > 0$ with the property that for every $N \in \mathbb{N}$, $N \geq N_0$, there are $\epsilon > 0$ and $\omega > 0$ such that every L-bilipschitz homeomorphism f having the AVG(ϵ, ω)-property has to stretch the distance between some pair of points in R_N at least $(1 + k)$ times more than it stretches the distance between the points $\mathbf{0}$ and $\mathbf{1}$.*

Proof. We choose $\omega := (d - c)/3$. Then we imitate the whole proof of Lemma 1.5. The only difference is that the choices of all of the parameters during that construction are done with respect to the values $c + \omega$ and $d - \omega$ instead of c and d.

This is correct, since Lemma 1.5 works for arbitrary values of c, d such that $d > c > 0$. Therefore, it also works for values $c + \omega$ and $d - \omega$. The proof is finished by observation that if particular choices of the values of the parameters from the proof of Lemma 1.5 work for some values $d > c > 0$, then the same choices also work for any values c', d' such that $d' \geq d > c \geq c'$. **Q.E.D.**

Let us assume, once and for all, that ω takes the value $\omega = \omega(c, d) := (d - c)/3$. It is easy to see that this strengthening of the nonrealizability properties of ρ_N transfers to $\rho^L_{c,d}$ and $\rho_{c,d}$, too. We denote these modified densities by $\hat{\rho}^L_{c,d}$ and $\hat{\rho}_{c,d}$, respectively. That is, we assume that the constructions of the densities $\hat{\rho}^L_{c,d}$ and $\hat{\rho}_{c,d}$ are carried out in the same way as were the constructions of the original Burago–Kleiner densities $\rho^L_{c,d}$ and $\rho_{c,d}$, with the modification as in the preceding proof. Thus, these constructions take into account the parameter $\omega = (d - c)/3$, i.e., the choices of all of the parameters in these constructions are made with respect to the values $c+\omega$ and $d-\omega$ instead of c and d.

However, we emphasize that the values of $\hat{\rho}^L_{c,d}$ and $\hat{\rho}_{c,d}$ still remain c and d, but the other parameters, e.g., N, M, ϵ or k, may have changed. This means, for example, that the number of iterative steps needed in the construction of $\hat{\rho}^L_{c,d}$ may have been slightly increased in comparison with $\rho^L_{c,d}$. Sometimes, we will refer to this modification as the *ω-modification.*

Corollary 1.12 (A strengthening of the properties of $\rho^L_{c,d}$): *For every $L \geq 1$ and $d > c > 0$ there exists $\bar{\epsilon} > 0$ such that there is no L-bilipschitz map f possessing the $AVG(\bar{\epsilon}, \omega)$-property with respect to all of the copies of ρ_N (for appropriate values of $N \in \mathbb{N}$) in all iterative steps of the construction of $\hat{\rho}^L_{c,d}$.*

In a similar manner, this ω-modification also works for $\hat{\rho}_{c,d}$, since this density is built upon the densities $\hat{\rho}^L_{c,d}$, for every $L \in \mathbb{N}$. We do not state it formally, because it would be rather confusing than explanatory. However, we believe that the idea is perfectly clear to the reader.

Putting this corollary and the notion of a strongly anomalous density together we easily obtain the following corollary, which concludes this section:

Corollary 1.13 (Properties of the ω-modification): *For every $z \in (-c+\omega, \infty)$ and every function $h\colon [0,1]^2 \to [z-\omega, z+\omega]$ the density $\hat{\rho}_{c,d}+h$ is strongly anomalous.*

Proof. Let h be as in the hypothesis. It suffices to observe that, for an arbitrary value of $\epsilon > 0$, the density $\rho_N + h - z$ has the AVG(ϵ, ω)-property.

In the proof of Lemma 1.11 we have observed that the properties of ρ_N guaranteed by Lemma 1.5 are also preserved for the density $\rho_N + h - z$. As we have noticed in Corollary 1.12, these properties are also transfered to $\hat{\rho}^L_{c,d} + h - z$. Therefore, for every L there is $\bar{\epsilon} > 0$ such that there is no L-bilipschitz map whose Jacobian is equal to $\hat{\rho}^L_{c,d} + h - z$ except for a set of measure less than $\bar{\epsilon}$. Consequently, $\hat{\rho}_{c,d} + h$ is a strongly anomalous density. **Q.E.D.**

1.3 An everywhere strongly anomalous density

We finish this chapter by a construction of a density that is everywhere bilipschitz nonrealizable. It roughly means that arbitrarily small piece of such a density cannot be itself the Jacobian of a bilipschitz map.

We denote by $B(x,\gamma)$ the closed ball in $\mathbb{R}^n$ of radius γ centered at x. Moreover, we denote by $\operatorname{int} A$ the interior of a set $A \subseteq \mathbb{R}^n$.

We also extend the notions of an anomalous density and a strongly anomalous density to the densities defined on a closed ball, not only on $[0,1]^2$. That is, for any $x \in \mathbb{R}^2$ and $\gamma > 0$ we say that a bounded density $\rho\colon B(x,\gamma) \to \mathbb{R}$ is an *anomalous density* if there is no bilipschitz map $f\colon B(x,\gamma) \to \mathbb{R}^2$ with $\operatorname{Jac}(f) = \rho$ a.e. Similarly, we do it in the case of a *strongly anomalous density.*[5]

[5]It is quite clear that we restrict ourselves to these domains, i.e. $[0,1]^2$ and closed balls, only for our convenience. The definitions of an anomalous density or a strongly anomalous density can be extended, without changing the meaning, to any sufficiently connected domain.

Definition 1.14: *Let $\rho\colon [0,1]^2 \to \mathbb{R}$ be a bounded measurable density with* $\inf \rho > 0$. *We say that ρ is **everywhere anomalous** if for every $x \in [0,1]^2$ and every $\gamma > 0$ the function $\rho|_{B(x,\gamma)}$ is an anomalous density. We say that ρ is **everywhere strongly anomalous** if for every $x \in [0,1]^2$ and every $\gamma > 0$ the function $\rho|_{B(x,\gamma)}$ is a strongly anomalous density.*

We have the anomalous density $\rho_{c,d}$, which is based on the densities $\rho^L_{c,d}$. Therefore, if we want to produce an everywhere anomalous density we may try to form a density such that for every $L \in \mathbb{N}$, $x \in [0,1]^2$, and $\gamma > 0$ every ball $B(x,\gamma)$ contains a rectangle on which our density has the properties of $\rho^L_{c,d}$ (appropriately scaled). This idea leads to a construction similar in spirit to Cantor-like sets.[6] Such an idea can be realized in several ways, each resulting in a density with slightly different properties than the others. We will show the one that is the simplest to describe:

Theorem 1.15: *For every $c, d \in \mathbb{R}$ such that $d > c > 0$ there is an everywhere strongly anomalous density $\pi_{c,d}\colon [0,1]^2 \to \{c,d\}$ having the properties of the ω-modification.*

Proof. We take $\{q_i\}_{i=1}^\infty$ an enumeration of the rational points in $\operatorname{int}[0,1]^2$, i.e., $\{q_i\}_{i=1}^\infty = \mathbb{Q}^2 \cap (0,1)^2$. We start with $\pi_{c,d}\colon [0,1]^2 \to \{c,d\}$ equal to the constant c everywhere. Our construction than proceeds inductively for $i = 1, 2, \ldots$

In the i-th step, $i \in \mathbb{N}$, we look at the density $\hat{\rho}^i_{c,d}$. Its domain is a rectangle R_{N_i} for an appropriate value of $N_i \in \mathbb{N}$. We define a rectangle $Q_i \subset (0,1)^2$ such that q_i lies at the center of Q_i. We consider only the rectangles with sides parallel to the coordinate axes. We also assume that the ratio of the sidelengths of Q_i is equal to N_i and that the horizontal side of Q_i is longer than its vertical side. That is, Q_i is a rescaled copy of R_{N_i}. We denote the length of the horizontal side of Q_i by $\ell(Q_i)$. We define $Q_0 := [0,1]^2$.

Then we take a similitude $\varphi_i\colon Q_i \to R_{N_i}$ and redefine $\pi_{c,d}$ on Q_i setting $\pi_{c,d}|_{Q_i} := \hat{\rho}^i_{c,d} \circ \varphi_i$. We leave the values of $\pi_{c,d}$ on the set $[0,1]^2 \setminus Q_i$ unchanged during the i-th iteration. We denote by $\mathcal{Q}_i := \{Q_j \mid j < i \;\&\; q_i \in \operatorname{int} Q_j\}$ the collection of all of the rectangles Q_j such that $j < i$ and q_i lies in the interior of Q_j. The idea of this construction is depicted in Figure 1.4.

Finally, it suffices to describe how to choose $\ell(Q_i)$ in the i-th iteration. We always choose $\ell(Q_i)$ irrational; this ensures that q_i does not lie on the boundary of any Q_j. Thus, for every j either $q_i \in \operatorname{int} Q_j$ or $q_i \notin Q_j$. Therefore, we can choose $\ell(Q_i)$ so that $Q_i \subset \operatorname{int} \bigcap \mathcal{Q}_i$ and $Q_i \cap \bigcup\{Q_j \mid j < i \;\&\; Q_j \notin \mathcal{Q}_i\} = \emptyset$.

Let us define $\epsilon_0 := 1$. For every $j \in \mathbb{N}$, $j < i$, there is an $\epsilon_j > 0$ such that redefining the strongly anomalous density $\hat{\rho}^j_{c,d} \circ \varphi_j$ on a set of measure less than ϵ_j does not destroy its property of being strongly anomalous. We need to preserve the properties of the copy of $\hat{\rho}^j_{c,d}$ on Q_j for every $j < i$. For that reason, we also require $\lambda(Q_i) < \min \left\{ \frac{\epsilon_j}{2^i} \middle| \, j < i \;\&\; Q_j \in \mathcal{Q}_i \right\}$. That is, since $\lambda(Q_i) = \ell(Q_i)^2/N_i$ according to the condition set up above, we choose $\ell(Q_i)$ so that $\ell(Q_i) < \min \left\{ \sqrt{\frac{\epsilon_j N_i}{2^i}} \middle| \, j < i \;\&\; Q_j \in \mathcal{Q}_i \right\}$.

The last condition ensures that, at the end of the construction, the nonrealizability properties of $\hat{\rho}^j_{c,d}$ "implanted" in Q_j are preserved for every $j \in \mathbb{N}$. If we

[6] The description of a Cantor set and related constructions can be found in Mattila's book [11], for example.

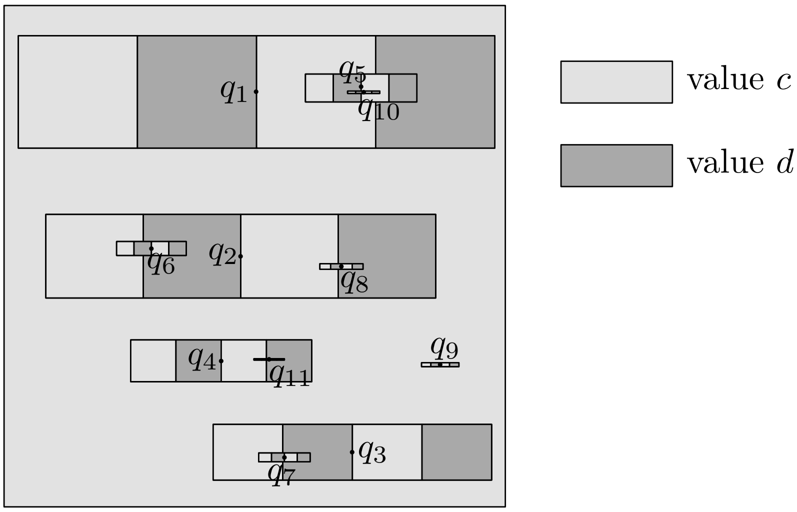

Figure 1.4: The idea of the construction of $\pi_{c,d}$.

now pick any $x \in [0,1]^2$, $\gamma > 0$, and any $L \geq 1$ we can find $j \in \mathbb{N}$ such that $j \geq L$ and $Q_j \subset B(x, \gamma)$. Therefore, $\pi_{c,d}|_{Q_j}$ cannot be the Jacobian of any L-bilipschitz map a.e. This implies that $\pi_{c,d}|_{B(x,\gamma)}$ is a strongly anomalous density.

Moreover, we observe that we have not used nor changed the properties of each $\hat{\rho}^j_{c,d}$ ensured by the parameter ω throughout the construction. Consequently, $\pi_{c,d}$ possesses these properties as well.

We conclude this proof claiming that $\pi_{c,d}$ is well defined a.e. The density $\pi_{c,d}$ is not defined only in the points that lie in the intersection of infinitely many different Q_j's. Since $Q_0 \in \mathcal{Q}_j$ for every $j \in \mathbb{N}$, we have $\lambda(Q_j) < \epsilon_0/2^j = 1/2^j$ according to the condition stated above. This implies that, after the j-th step of the construction, $\pi_{c,d}$ can be modified during the remaining steps only on a set of measure less than $\sum_{i=j+1}^{\infty} \frac{1}{2^i} = 1/2^j$. Hence, the set of all points lying in an infinite number of different Q_j's has measure zero. The claim follows. **Q.E.D.**

2. Towards Question 1

2.1 Transforming the problem of Question 1

In this section we will focus on the problem stated in the Introduction in Question 1, which asked for two densities defined on two disjoint squares that cannot be summed, after a bilipschitz transformation, to the uniform density equal to 1. Employing our everywhere strongly anomalous density $\pi_{c,d}$ we will show that a positive answer to Question 2 implies a positive answer to Question 1. For reader's convenience we restate Question 1 here:

Question (1): *Let I_1 and I_2 be two disjoint copies of the square $[0,1]^2$. For $i \in \{1,2\}$ are there bounded measurable densities $\rho_i \colon I_i \to \mathbb{R}$ with $\inf \rho_i > 0$ such that there are no bilipschitz maps $f_i \colon I_i \to \mathbb{R}^2$ verifying the equation*

$$\left(\frac{\rho_1}{\mathrm{Jac}(f_1)} \circ f_1^{-1}\right)(x) + \left(\frac{\rho_2}{\mathrm{Jac}(f_2)} \circ f_2^{-1}\right)(x) = 1$$

for almost every $x \in f_1(I_1) \cup f_2(I_2)$? (If either of the formulas in the equation above is not well defined, we treat it as a zero.)

At first, we observe that using everywhere anomalous densities in the problem from Question 1 we can simplify the relation between $f_1(I_1)$ and $f_2(I_2)$.

Observation 2.1: *If ρ_1 is an everywhere anomalous density, we can assume that $f_1(I_1)$ is a subset of $f_2(I_2)$ in Question 1. If both ρ_1 and ρ_2 are everywhere anomalous densities, we can assume that $f_1(I_1) = f_2(I_2)$.*

Proof. Indeed, if there were some $x \in f_1(I_1) \setminus f_2(I_2)$, then its distance to $f_2(I_2)$ would be positive, since $f_2(I_2)$ is compact. Therefore, there would be some $\gamma > 0$ such that $B(x,\gamma) \cap f_2(I_2) = \emptyset$. Moreover, the interior of the set $f_1^{-1}(B(x,\gamma))$ would be nonempty. Consequently, we would obtain that $\mathrm{Jac}(f_1) = \rho_1$ a.e. in $\mathrm{int}\, f_1^{-1}(B(x,\gamma))$. This would contradict the everywhere nonrealizability property of ρ_1.

The second part of the observation follows by symmetry. **Q.E.D.**

Before we proceed we state the following strengthening of the change of variables formula for bilipschitz maps, which is based on Theorem 1.1 and can be found in Fremlin's monograph [2, 263F]:

Theorem 2.2 (The change of variables formula for bilipschitz maps): *Let $f \colon A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a bilipschitz map. Let $h \colon f(A) \to \mathbb{R}$ be a real function. Then for every measurable set E, $E \subseteq A$, the set $f(E)$ is measurable and we have*

$$\int_E |\mathrm{Jac}(f)| \cdot (h \circ f)\ d\lambda_n = \int_{f(E)} h\ d\lambda_n$$

if either integral is defined.

To simplify the notation we introduce the following definition:

Definition 2.3: *Let* $h\colon A \subseteq \mathbb{R}^2 \to [0,\infty)$ *be a measurable function. Then we denote by* ν_h *the measure with the density* h*, i.e.,* $\nu_h(B) := \int_B h\ d\lambda$ *for every measurable set* $B \subseteq A$*.*

We remind that for a bilipschitz map f and a measure ν_h the pushforward measure $f_*(\nu_h)$ is defined by $f_*(\nu_h)(B) := \nu_h(f^{-1}(B))$ for every measurable set B. As we have noticed in the Introduction, we assume, without loss of generality, that the bilipschitz maps used here have positive Jacobians.

Now, we present the following simple observation, which tells us that a bilipschitz transformation of an anomalous density is still not realizable as the Jacobian of a bilipschitz map.

Observation 2.4 (A bilipschitz transformation of an anomalous density is bilipschitz nonrealizable): *Let* $\rho\colon [0,1]^2 \to \mathbb{R}$ *be an anomalous density and* $f\colon [0,1]^2 \to \mathbb{R}^2$ *be a bilipschitz map. Then the density of* $f_*(\nu_\rho)$ *cannot be the Jacobian of any bilipschitz map a.e. If* ρ *is an everywhere anomalous density, then so is the density of* $f_*(\nu_\rho)$*.*

Proof. Let us denote by σ the density of $f_*(\nu_\rho)$.[1] We will reason by contradiction. Let us suppose we have $\chi\colon f([0,1]^2) \to \mathbb{R}^2$ a bilipschitz map with $\mathrm{Jac}(\chi) = \sigma$ a.e. For every measurable set $B \subseteq [0,1]^2$, using the area formula from Theorem 1.1, we have

$$\int_{f(B)} \sigma\ d\lambda = \int_{f(B)} \mathrm{Jac}(\chi)\ d\lambda = \lambda((\chi \circ f)(B)) = \int_B \mathrm{Jac}(\chi \circ f)\ d\lambda.$$

On the other hand, since σ is the density of the measure $f_*(\nu_\rho)$, we have that

$$\int_{f(B)} \sigma\ d\lambda = f_*(\nu_\rho)(f(B)) = \nu_\rho(f^{-1}(f(B)) = \int_B \rho\ d\lambda.$$

Thus, we have $\int_B \rho\ d\lambda = \int_B \mathrm{Jac}(\chi \circ f)\ d\lambda$ for every measurable set $B \subseteq [0,1]^2$. This implies that $\mathrm{Jac}(\chi \circ f) = \rho$ a.e., a contradiction.

If the last implication were not correct, then the set $B_+ := \{x |\ \mathrm{Jac}(\chi \circ f)(x) > \rho(x)\}$ would have a positive measure. Since this set is measurable, because ρ and $\mathrm{Jac}(\chi \circ f)$ are measurable, we would obtain a contradiction to $\int_{B_+} \rho\ d\lambda = \int_{B_+} \mathrm{Jac}(\chi \circ f)\ d\lambda$.

If ρ is an everywhere anomalous density, we can apply the argumentation above to each $\rho|_{B(x,\gamma)}$ for every $x \in [0,1]^2$ and every $\gamma > 0$. **Q.E.D.**

Let us come back to Question 1. We now choose ρ_1 to be our everywhere strongly anomalous density $\pi_{c,d}$ from Theorem 1.15 and ρ_2 to be a constant density equal to some $z \in (0,\infty)$. Thus, we can state the following special form of Question 1:

[1] The existence of a density of this measure is a corollary of the change of variables formula here. In general, the Radon-Nikodým theorem implies that every σ-finite measure absolutely continuous with respect to λ admits a density with respect to λ. See Mattila's book [11] or Fremlin's monograph [2], for example.

Question 3 (A special case of Question 1): *Let z be a positive real number. Is it true that there are no bilipschitz maps $f_1\colon I_1 \to \mathbb{R}^2$ and $f_2\colon I_2 \to \mathbb{R}^2$ satisfying the equation*

$$\left(\frac{\pi_{c,d}}{\mathrm{Jac}(f_1)} \circ f_1^{-1}\right)(x) + \left(\frac{z}{\mathrm{Jac}(f_2)} \circ f_2^{-1}\right)(x) = 1$$

for almost every $x \in f_1(I_1) \cup f_2(I_2)$? (If either of the formulas in the equation above is not well defined, we treat it as a zero.)

We are now ready to present our reduction of Question 1 to Question 2:

Theorem 2.5: *A positive answer to Question 2 yields a positive answer to Question 3, and thus, a positive answer to Question 1.*

Proof. We will reason by contradiction. Let us assume that there are bilipschitz maps f_1 and f_2 verifying the equation from Question 3. According to Observation 2.1, we can suppose that $f_1(I_1)$ is a subset of $f_2(I_2)$. This implies that the map $g := f_2^{-1} \circ f_1$ is well defined. It is also bilipschitz and maps I_1 into I_2.

The density $\pi_{c,d}$ induces the measure $\nu_{\pi_{c,d}}$ on I_1. We look at the measure $g_*(\nu_{\pi_{c,d}})$ and denote its density by σ. From the change of variables formula we obtain that $\int_B \sigma \; d\lambda = \int_B (\pi_{c,d} \circ g^{-1}) \cdot \mathrm{Jac}(g^{-1}) \; d\lambda$ for every measurable set $B \subseteq g(I_1)$. This in turn implies, as we have already justified in the proof of Observation 2.4, that $\sigma = (\pi_{c,d} \circ g^{-1}) \cdot \mathrm{Jac}(g^{-1})$ a.e.

Using the formulas for Jacobians of a composite mapping and an inverse map,[2] one can compute:

$$\mathrm{Jac}(g^{-1})(x) = \mathrm{Jac}(f_1^{-1} \circ f_2)(x) = \left(\frac{1}{\mathrm{Jac}(f_1)} \circ f_1^{-1} \circ f_2\right)(x) \cdot \mathrm{Jac}(f_2)(x).$$

Therefore, we can write

$$\frac{\sigma}{\mathrm{Jac}(f_2)}(x) = \left(\frac{\pi_{c,d}}{\mathrm{Jac}(f_1)} \circ g^{-1}\right)(x)$$

for a.e. $x \in g(I_1)$.

Next, we take the formula from Question 3,

$$\left(\frac{\pi_{c,d}}{\mathrm{Jac}(f_1)} \circ f_1^{-1}\right)(y) + \left(\frac{z}{\mathrm{Jac}(f_2)} \circ f_2^{-1}\right)(y) = 1,$$

and we substitute for $y = f_2(x)$. Then we substitute the quantity obtained in the last step for the summand on the left. We obtain that

$$\frac{\sigma}{\mathrm{Jac}(f_2)}(x) + \frac{z}{\mathrm{Jac}(f_2)}(x) = 1$$

for a.e. $x \in g(I_1)$. Finally, we have

$$(\sigma + z)(x) = \mathrm{Jac}(f_2)(x)$$

for a.e. $x \in g(I_1)$.

According to Observation 2.4, σ is an everywhere anomalous density. On the other hand, we have deduced that $\sigma + z$, where z is a real number from Question 3, is the Jacobian of the bilipschitz map f_2 a.e. in $g(I_1)$. Consequently, if every anomalous density were also a strongly anomalous density, as it is asked in Question 2, we would have a contradiction. **Q.E.D.**

[2] These formulas can be deduced, e.g., from the change of variables formula from Theorem 2.2. They are also a standard part of elementary courses on real analysis.

2.2 An attempt to answer Question 1

In this section I present briefly my attempt to prove that the answer to Question 3 is positive, which would imply that the answer to Question 1 is also positive. However, my attempt has failed. The reason for this will be explained at the end of this Section.

We start by a presentation of a simple definition slightly generalizing the classical notion of continuity:

Definition 2.6: *Let $f\colon A \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a map. We call f* ***essentially continuous*** *at $x \in A$ if for every $\gamma > 0$ there is $\delta > 0$ such that for a.e. $y \in B(x, \delta) \cap A$ we have $\|f(x) - f(y)\| < \gamma$. Such a point x is called the point of* ***essential continuity*** *of f.*

The difference between usual continuity and essential continuity is in words "a.e." in the definition above. For example, Dirichlet's function (a characteristic function of the set of rational numbers) is nowhere continuous, however, it is essentially continuous in every irrational point.

The basis of my attempt was the following humble observation:

Observation 2.7: *Let $h\colon [0,1]^2 \to (0, \infty)$ be a function. Moreover, we suppose that h is essentially continuous at a point $x_0 \in [0,1]^2$. Then $\pi_{c,d} + h$ is an anomalous density.*

Proof. From Theorem 1.15 we know that $\pi_{c,d}$ is an everywhere strongly anomalous density with the properties of the ω-modification. We define $z := h(x_0)$.

From the essential continuity of h at x_0 we find $\delta > 0$ such that for a.e. $y \in B(x_0, \delta) \cap [0,1]^2$ we have $|z - h(y)| < \omega$. We know that $\pi_{c,d} + z$ is an everywhere anomalous density having the properties of the ω-modification. Consequently, the function $(\pi_{c,d} + h)|_{B(x_0,\delta)}$ is an anomalous density on $B(x_0, \delta)$, and thus, $\pi_{c,d} + h$ itself is an anomalous density. **Q.E.D.**

The problem from Question 3 can be transformed in a way analogous to the transformation performed in Theorem 2.5, but in the opposite direction, i.e., using the map $g^{-1}|_{g(I_1)} = (f_1^{-1} \circ f_2)|_{g(I_1)}$ instead of $g = f_2^{-1} \circ f_1$.

The constant density equal to $z \in \mathbb{R}$ on I_2 induces the measure that is the Lebesgue measure λ multiplied by z, and hence, we denote this measure by $z\lambda$.

Moreover, we denote by ζ the density of the measure $(g^{-1}|_{g(I_1)})_*(z\lambda)$. This ζ is a bilipschitz transformation of the constant density z. By the change of variables formula we have that $\zeta = z \cdot \mathrm{Jac}(g)$ a.e., thus, it is a constant multiple of the Jacobian of a bilipschitz map. In a way analogous to that of Theorem 2.5 we obtain

$$(\pi_{c,d} + \zeta)(x) = \mathrm{Jac}(f_1)(x)$$

for a.e. $x \in [0,1]^2$.

The advantage of this transformation is that we know exactly how does $\pi_{c,d}$ look. Therefore, we can try to employ its specific properties. Combining the transformation above and Observation 2.7, we see that if the Jacobian of every bilipschitz map $[0,1]^2 \to \mathbb{R}^2$ were essentially continuous in at least one point of $[0,1]^2$, we would have a positive answer to Question 3.

I was not able to find any result, positive nor negative, concerning the continuity of Jacobians of bilipschitz maps in literature. For that reason, I tried to prove that Jacobians of bilipschitz maps enjoy the desired property.

The Jacobian of an everywhere Fréchet differentiable map $[0,1]^2 \to \mathbb{R}^2$ is continuous in the points of a dense subset of type G_δ (that is, a countable intersection of open sets). This result can be deduced by the Baire category theorem; see Munkres' monograph [13, Theorem 48.5] for reference.

This can be even strengthened in the following manner. The Jacobian of every map that is Fréchet differentiable everywhere outside a set that is a subset of a set of measure zero and of type F_σ (that is, a countable union of closed sets) is essentially continuous in the points of a dense subset of $[0,1]^2$. For example, this implies that the Jacobian of a map $[0,1]^2 \to \mathbb{R}^2$ that is Fréchet differentiable everywhere outside of a countable set is essentially continuous in the points of a dense subset of $[0,1]^2$.

By Rademacher's theorem we know that every bilipschitz map $[0,1]^2 \to \mathbb{R}^2$ is Fréchet differentiable a.e. in $[0,1]^2$. This shows that my belief was not so unlikely.

However, after many unsuccessful attempts to resolve this problem, positively or negatively, I have consulted it with doc. Hencl, a specialist in this field from Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University in Prague. He was able to provide an example of a real function of one real variable that is bilipschitz and its derivative is nowhere essentially continuous.

Now, we present his example here. We will need the following well known theorem of Lebesgue (see Mattila's book [11], for example):

Theorem 2.8 (Lebesgue - one dimensional version): *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an integrable function. Then the function $F(x) := \int_{-\infty}^{x} f(s)\, ds$ is differentiable a.e. and $F' = f$ at the points of differentiability of F.*

Theorem 2.9 (Hencl [6]): *There is a bilipschitz function $h\colon [0,1] \to \mathbb{R}$ such that its derivative h' is nowhere essentially continuous.*

Proof. We do not construct h directly; rather we construct its derivative h'. More precisely, we construct a function $v\colon [0,1] \to \mathbb{R}$ and then we define $h(x) := \int_0^x v(s)\, ds$. The function v takes only two values $d > c > 0$. The idea is to construct v so that the sets $C := \{x|\, v(x) = c\}$ and $D := \{x|\, v(x) = d\}$ are both dense in $[0,1]$ and that the measures of both of these sets in every interval inside $[0,1]$ are positive.

We choose $\{q_i\}_{i=1}^{\infty}$ an enumeration of the rational numbers in $(0,1)$. We start with $v = c$ everywhere on $[0,1]$. The construction then proceeds inductively for $i = 1, 2, \dots$

In the i-th step we choose a quantity r_i and redefine v on the interval $(q_i - r_i, q_i + r_i)$. We always choose r_i to be irrational and so as all the endpoints of all the intervals $(q_i - r_i, q_i + r_i)$ are unique, i.e., $q_i + r_i, q_i - r_i \notin \{q_j + r_j|\, j < i\} \cup \{q_j - r_j|\, j < i\}$. We also choose r_i so that $\sum_{j=i+1}^{\infty} r_j < r_i/2$. The last condition on r_i is that for every $j < i$ the new interval $(q_i - r_i, q_i + r_i)$ is either a subset of $(q_j - r_j, q_j + r_j)$ or it is disjoint from $(q_j - r_j, q_j + r_j)$.

Choosing the quantity r_i according to all of the conditions above we can redefine v switching its value on $(q_i - r_i, q_i + r_i)$ from c to d, or from d to c. In other words, we set v on $(q_i - r_i, q_i + r_i)$ as follows:

$$v|_{(q_i - r_i, q_i + r_i)} := c \qquad \text{if the value of } v \text{ were } d \text{ on } (q_i - r_i, q_i + r_i),$$
$$v|_{(q_i - r_i, q_i + r_i)} := d \qquad \text{if the value of } v \text{ were } c \text{ on } (q_i - r_i, q_i + r_i).$$

We now prove that, at the end of the construction, h' has the desired properties. First, we observe that v is well define a.e. in $[0, 1]$. The function v is clearly well defined in those points that lie in only finite number of the intervals $(q_j - r_j, q_j + r_j)$. Since we have chosen r_i so as $\sum_{j=i+1}^{\infty} r_j < r_i/2$, we have $r_i < 1/2^i$ for every $i \in \mathbb{N}$. Thus, after the i-th step, v can be modified only on the set of measure less than $1/2^i$. Consequently, the set of points that lie in an infinite number of the intervals $(q_j - r_j, q_j + r_j)$ has measure zero.

Obviously, the sets C and D defined above are both Borel at the end of the construction, and thus measurable, because each of them is formed by a countable sequence of unions and differences of intervals. This implies that v is Lebesgue integrable.

Theorem 2.8 then says that h' exists a.e. in $[0, 1]$ and that $h' = v$. Consequently, the derivative h' is not continuous in any point of differentiability of h, since the sets C and D are both dense in $[0, 1]$.

For every $\gamma > 0$ and every $x \in (0, 1)$ such that $h'(x)$ exists there is $j \in \mathbb{N}$ with the property that $(q_j - r_j, q_j + r_j) \subset (x - \gamma, x + \gamma)$ and that the value of h' on the majority of $(q_j - r_j, q_j + r_j)$ is the opposite value to $h'(x)$. Thus, h' is not even essentially continuous at x.

Finally, h is bilipschitz, because for every $x, y \in [0, 1]$ we have $c\,|x - y| \leq |h(x) - h(y)| \leq d\,|x - y|$. **Q.E.D.**

Using this example it is now easy to construct a bilipschitz map in $\mathbb{R}^2$ such that its Jacobian is nowhere essentially continuous.

Corollary 2.10: *There is a bilipschitz map $g\colon [0, 1]^2 \to \mathbb{R}^2$ such that its Jacobian is nowhere essentially continuous.*

Proof. We take h from the theorem above and define $g((x, y)) := (h(x), y)$. Obviously, since h is bilipschitz, so is g. Moreover, we have $\mathrm{Jac}(g)(x, y) = h'(x)$. Thus, $\mathrm{Jac}(g)$ is nowhere essentially continuous in $[0, 1]^2$. **Q.E.D.**

Conclusion

The purpose of this thesis has been to introduce the open problem of Feige and the problem of bilipschitz nonrealizable densities, which was resolved by Burago, Kleiner, and McMullen, to relate these problems, and finally, to try to adapt the technique of Burago and Kleiner to a form applicable to Feige's problem.

We have analyzed the relation between Feige's problem and the problem of nonrealizable densities and we have proposed a way along which one can try to prove that the answer to Feige's question is negative. We have provided a continuous version of Feige's problem in a special case in Question 1 and have made several observations concerning this problem.

Moreover, we have presented the Burago–Kleiner anomalous density and have made additional observations concerning its properties beyond the work of Burago and Kleiner. Also, we have constructed a modified version of the Burago–Kleiner density that is bilipschitz nonrealizable even if restricted to an arbitrary subset of the domain with a nonempty interior.

Finally, I have conjectured that this density together with a uniform density form the answer to Question 1. However, my attempt to prove it has failed.

My impression of the densities provided by Burago and Kleiner, and by McMullen was that the property that makes them bilipschitz nonrealizable is that they oscillate by a quantity bounded below on smaller and smaller scales in the neighborhood of some points. Nevertheless, this idea has turned out to be wrong by the example devised by doc. Hencl [6], which has been presented at the end of Chapter 2.

This example suggests that the property of being bilipschitz nonrealizable is probably of more geometric nature. Therefore, in order to answer Question 1 one can try to make geometric observations similar to those presented in the construction of the Burago–Kleiner anomalous density.

I consider Question 1 to be a crucial problem on the way to the resolution of Feige's problem. The second important question is related to Jones' result [8] presented in the Introduction. How can Lipschitz maps relevant in Feige's problem look? Or rather, how can their continuous analogs on $[0,1]^2$ look in comparison with bilipschitz maps? Answering these questions seems to be critical in order to find an answer to Feige's question, at least as long as one tries to employ the technique of Burago and Kleiner.

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