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COHENŮV FORCING A JEHO VLASTNOSTI
COHEN FORCING AND ITS PROPERTIES

Bakalářská práce

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2015

Tímto děkuji svému vedoucímu práce, Mgr. Radku Honzíkovi, Ph.D, za cenné postřehy a rady, bez nichž by tato práce nemohla vzniknout.

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V Praze 24. května 2015

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Abstract

Bakalářská práce se zabývá vlastnostmi Cohenova forcingu a jeho vztahu k nedokazatelnosti Hypotézy kontinua a Obecné hypotézy kontinua. Práce je rozdělena na čtyři části. V první části je zavedena technika forcingu pomocí částečných uspořádání. Druhá část zavádí pojem Cohenovského forcingu, ukazuje vlastnosti kardinální aritmetiky postačující k zachovávání kardinálů Cohenovským forcingem a zejména se pak soustředí na generické množiny přidané konkrétními variacemi Cohenovského forcingu. Nakonec tato část ukazuje některé vlastnosti Cohenovských reálných čísel. Třetí část rekonstruuje důkaz nedokazatelnosti Hypotézy kontinua a ukazuje užití Cohenovského forcingu při důkazu tvrzeních o Obecné hypotéze kontinua. Poslední část se krátce zmiňuje o neminimalitě generických filtrů na Cohenovském forcingu a zavádí Sacksův forcing, na kterém dokládá, že existují forcingy, jejichž generické filtry jsou minimální.

Klíčová slova

Cohenův forcing, CH, GCH, Cohenova reálná čísla.

Abstract

This bachelor thesis studies properties of Cohen Forcing and its relation to the unprovability of Continuum Hypothesis and Generalised Continuum Hypothesis. The thesis is divided into four parts. In the first part the technique of forcing based on partial orders is introduced. The second part introduces a notion of Cohen forcing, shows properties of cardinal arithmetic sufficient to preservation of cardinals by Cohen forcing and focuses mainly on generic sets added by concrete variations of Cohen Forcing. Finally some of the properties of Cohen reals are shown in this part. The third part reconstruct a proof of unprovability of Continuum Hypothesis and shows a use of Cohen Forcing in relation to the statements about the Generalised Continuum Hypothesis. The last part discusses briefly a non-minimality of generic filters on Cohen forcing and introduce a notion of Sacks forcing in order to show an existence of forcing notion whose generic filters are minimal.

Keywords

Cohen forcing, CH, GCH, Cohen reals.

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1 Introduction

In 1900 David Hilbert published a list of, from his point of view, most problematic questions of mathematics including Continuum Hypothesis (CH) on the first place. In 1938 Kurt Gödel proved that CH is consistent with ZFC by constructing a model where CH holds. Finally in 1963 Paul Cohen finished the prove of independence of CH on ZFC by constructing a model of ZFC where CH fails (see [Coh63]). In the same year Robert M. Solovay generalised in [Sol63] the result of Cohen so that for a regular kardinal κ assuming $2^{<\lambda} = \kappa$ and $\lambda^{\kappa} = \lambda$ in the ground model one obtains a model with $2^{\kappa} = \lambda$. However, the biggest contribution of Paul Cohen's prove lied in introducing a universal technique for constructing models of ZFC. Following the result of Cohen a lot of forcing notions were introduced to prove or disprove various statements about ZFC. Nevertheless as more complicated the statements became as more complicated the forcing notions needed to be. Although the forcing notion of Paul Cohen remains one of the simplest, there still remains a lot to show about characteristic of models it produces which will be the main aim of this thesis.

In section 2 and 3 of this thesis we introduce a general forcing technique according to the [Kun06] and [Jech02] with emphasis on a forcing equivalence in section 3 which will be widely used in section 4 and 5. In section 4 we show properties of cardinal arithmetics ensuring the preservation of cardinals for different variations of Cohen forcing. We focus further in this section on collapsing cardinals by various notions of Cohen forcing including product of forcing notions and show some properties of Cohen reals.

In the first part of section 5 we reconstruct the result of Paul Cohen and give its generalised version which allows us to violate GCH in arbitrary finitely many cases. In the next part we investigate possibilities of constructing a model of ZFC where 'GCH fails on $\aleph_\omega$ but holds below' via the Cohen forcing and show that it does not seem possible considering the properties of product of Cohen forcings. We show further using an argumentation of Easton forcing that there exists a model of ZFC where 'GCH holds on $\aleph_\omega$ but does not hold below'.

Finally in section 6 we briefly discus the non-minimality of generic filters on Cohen forcing and give an example of forcing notion (Sacks forcing) whose generic filters are minimal.

2 Preliminaries

In this section we establish our terminology and notation and we also give basic facts about forcing covering Chapter VII and VIII of Kunen's book [Kun06]. We consider ZFC as the primary theory based on Kunen's book [Kun06] and therefore we are not going to stress where AC (Axiom of Choice) is or is not needed. The set-theoretic notation used is mostly that of [Kun06] with some additional terminology from [Jech02] when the terminology is not discussed in [Kun06].

2.1 General Set Theory

Definition 2.1. *A binary relation $\leq$ is a partial ordering on a set P if*

(i) $(\forall p \in P)(p \leq p)$

(ii) $(\forall p, q, r \in P)((p \leq q \;\&\; q \leq r) \rightarrow p \leq r)$

(iii) $(\forall p, q \in P)((p \leq q \;\&\; q \leq p) \rightarrow p = q)$.

For any p,q in $\mathbb{P}$ we define further $p < q$ if and only if $p \leq q \;\&\; p \neq q$.

A special example of a partial order on a set P is $\leq$ called 'reverse inclusion' defined as $x \leq y$ if and only if $y \subseteq x$ for any $x, y \in P$, where $\subseteq$ is the set-theoretical inclusion.

Definition 2.2. *A triple $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ is a partial order if $\leq$ is a partial ordering on $\mathbb{P}$, $1_{\mathbb{P}} \in \mathbb{P}$ and $(\forall p \in \mathbb{P})(p \leq 1_{\mathbb{P}})$.*

Note that if $\mathbb{P}$ is a set partially ordered by $\leq$ but there is no element q in $\mathbb{P}$ such that $(\forall p \in \mathbb{P})(p \leq q)$ then we can always add a new element 1 to $\mathbb{P}$ such that $\langle \mathbb{P} \cup \{1\}, \leq, 1 \rangle$ is a partial order according to the previous definition.

We often abuse notation by referring to "the partial order $\mathbb{P}$" or "the partial order $\leq$" if $\leq$ or $\mathbb{P}$ are arbitrary or clear from context.

Definition 2.3. *Let $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ be a partial order then we say that $p, q \in \mathbb{P}$ are compatible and write $p \| q$ if $(\exists r \in \mathbb{P})(r \leq p \;\&\; r \leq q)$, otherwise we say that p, q are incompatible and write $p \perp q$.*

Definition 2.4. *We say that a partial order $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ is:*

(i) *atomless if* $(\forall p \in \mathbb{P})(\exists q, q' \in \mathbb{P})(q \perp q' \ \&\ q < p \ \&\ q' < p)$.

(ii) *separative if* $(\forall p, q \in \mathbb{P})(p \not\leq q \rightarrow (\exists r \in \mathbb{P})(r \leq p \ \&\ r \perp q))$.

Definition 2.5. *Let $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ be a partial order and $p \in \mathbb{P}$ then :*

(i) *a set $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ if* $(\forall p \in \mathbb{P})(\exists q \in D)(q \leq p)$.

(ii) *a set $D \subseteq \mathbb{P}$ is dense below p if* $(\forall q \leq p)(\exists r \leq q)(r \in D)$.

(iii) *a set $A \subseteq \mathbb{P}$ is an antichain in $\mathbb{P}$ if* $(\forall p, q \in A)(p \perp q)$.

(iv) *a set $A \subseteq \mathbb{P}$ is maximal antichain in $\mathbb{P}$ if every $A' \subseteq \mathbb{P}$ such that $A \subsetneq A'$ is not an antichain in $\mathbb{P}$.*

Definition 2.6. *Let $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ be a partial order then a set $G \subseteq \mathbb{P}$ is a filter if the following conditions hold:*

(i) *G is centred, i.e.* $(\forall p, q \in G)(\exists r \in G)(r \leq p \ \&\ r \leq q)$.

(ii) *G is upward closed, i.e* $(\forall q \in G)(\forall p \in \mathbb{P})(q \leq p \rightarrow p \in G)$.

2.2 Relativisation

Definition 2.7. *Let M be any class then for any formula ϕ of language of ZFC we define ϕ^M, the relativization of ϕ to M, by induction on ϕ by:*

(i) $(x = y)^M$ *is* $x = y$.

(ii) $(x \in y)^M$ *is* $x \in y$.

(iii) $(\phi \ \&\ \psi)^M$ *is* $\phi^M \ \&\ \psi^M$.

(iv) $(\neg\phi)^M$ *is* $\neg(\phi^M)$.

(v) $(\exists x \phi)^M$ *is* $(\exists x)(x \in M \ \&\ \psi^M)$.

Definition 2.8. *Let M be any class then:*

(i) *for a sentence ϕ, 'ϕ is true in M' or '$M \models \phi$' means ϕ^M.*

(ii) *for a set of sentences S, 'S is true in M' or 'M is a model for S' means that each sentence from S is true in M.*

We say that a class M is a model for ZFC if each axiom of ZFC is true in M.

Definition 2.9. *Let ϕ be a formula with all free variables shown between $x_1, x_2, ..., x_n$ and M,N arbitrary classes, then:*

(i) if $M \subseteq N$: ϕ is absolute for (or between) M,N if and only if

$$(\forall x_1, x_2, ..., x_n \in M)(\phi^M(x_1, ..., x_n) \leftrightarrow \phi^N(x_1, ..., x_n)).$$

(ii) ϕ is absolute for M if and only if ϕ is absolute for M,V i.e.:

$$(\forall x_1, x_2, ..., x_n \in M)(\phi^M(x_1, ..., x_n) \leftrightarrow \phi(x_1, ..., x_n)).$$

Note that if ϕ is absolute for M and ϕ is absolute for N and $M \subseteq N$, then ϕ is absolute between M,N.

Definition 2.10. *A set or a class M is transitive if $(\forall x)(x \in M \rightarrow x \subseteq M)$.*

Lemma 2.11. *The following formulas are absolute between any two transitive models for ZFC:*

(i) $x = \{u, v\}$, $x = \langle u, v\rangle$, *'x is empty'*, $x \subseteq y$.

(ii) $z = x \times y$, $z = x - y$, $z = x \cap y$, $z = \cup x$.

(iii) 'x is transitive', 'x is an ordinal', 'x is a limit ordinal', 'x is a natural number', '$x = \omega$'.

(iv) 'x is a relation', 'f is function', $y = f(x)$ (i.e. $\langle x, y\rangle \in f$), $g = f\restriction z = \{\langle x, y\rangle \in f \mid x \in z\}$.

Proof. See [Jech02] Lemma 12.10. □

2.3 Basic Forcing Definitions and Lemmas

Definition 2.12. *Let $\langle \mathbb{P}, \leq, 1_{\mathbb{P}}\rangle$ be a partial order in M, we say that a set G is $\mathbb{P}$-generic over M if and only if $G \subseteq \mathbb{P}$, G is a filter on $\mathbb{P}$ and for for all sets D dense in $\mathbb{P}$, $D \in M \rightarrow G \cap D \neq \emptyset$.*

Definition 2.13. *Let M be a transitive model of ZFC, and $\langle \mathbb{P}, \leq, 1_{\mathbb{P}}\rangle$ a partial order in M. We define a class $M^{\mathbb{P}}$ by induction on* ORD *as follows:*

$$M_0^{\mathbb{P}} = \emptyset$$

$$M^{\mathbb{P}}_{\alpha+1} = \{\langle x, p\rangle \mid x \in M^{\mathbb{P}}_{\alpha} \;\&\; p \in \mathbb{P}\}$$

$$M^{\mathbb{P}}_{\alpha} = \bigcup_{\beta<\alpha} M^{\mathbb{P}}_{\beta}, \textit{ for } \alpha \textit{ limit}$$

and $M^{\mathbb{P}} = \{\tau \mid (\exists\alpha \in \mathrm{ORD})(\tau \in M^{\mathbb{P}}_{\alpha})\}$. *We call the elements of* $M^{\mathbb{P}}$ $\mathbb{P}$*-names and we say that* $\mathcal{L}_{\mathbb{P}}$ *defined by*

$$\mathcal{L}_{\mathbb{P}} = \{\in\} \cup \{\tau \mid \tau \in M^{\mathbb{P}}\}$$

is a forcing language of $\mathbb{P}$.

Definition 2.14. *Let* M *be a transitive model of ZFC , we define a function* $\check{}: M \to M^{\mathbb{P}}$ *by* $\in$*-recursion as:*

$$\check{\emptyset} = \emptyset$$

$$\check{x} = \{\langle \check{y}, 1_{\mathbb{P}}\rangle \mid y \in x\}$$

and we say that $\check{x}$ *is a standard* $\mathbb{P}$*-name of a set* $x \in M$.

Definition 2.15. *Let* M *be a transitive model for ZFC,* $\langle \mathbb{P}, \leq, 1_{\mathbb{P}}\rangle$ *a partial order in* M *and* G *a filter on* $\mathbb{P}$. *We define a function* $\;_G : M^{\mathbb{P}} \to V$ *by*

$$\emptyset_G = \emptyset$$

$$\tau_G = \{\sigma_G \mid \exists p \in G(\langle \sigma, p\rangle \in \tau)\},$$

and we say that τ_G *is a valuation of (a name)* τ *by* G.

For more informative proofs of the following theorem and two lemmas see [Kun06] page 187 respectively pages 190-191.

Theorem 2.16. *(Rasiowa-Sikorski) Let* M *be a countable transitive model for ZFC,* $\langle \mathbb{P}, \leq, 1_{\mathbb{P}}\rangle$ *an atomless partial order in* M *and* $p \in \mathbb{P}$*, then there exists* G *a* $\mathbb{P}$*-generic filter over* M *with* $p \in G$.

Proof. Let $p \in \mathbb{P}$ be given. Suppose $D_0, D_1, D_2, \ldots$ is enumeration of all dense subsets of $\mathbb{P}$ which are in M. Since M is countable the enumeration is also countable and we can easily construct a sequence $\langle p_n \mid n \in \omega\rangle$ defined by $p_0 = p$ and $p_{n+1} \in D_n$ such that $p_{n+1} \leq p_n$. Then $G = \{q \in \mathbb{P} \mid (\exists n \in \omega)(p_n \leq q)\}$ is a filter which is

$\mathbb{P}$-generic generic over M. Indeed, otherwise if $G \in M$, $(\mathbb{P} - G) \in M$ is a dense subset of $\mathbb{P}$ which could not be enumerated above. $\square$

Definition 2.17. *Let M be a transitive model of ZFC, $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ in M and $G \subseteq \mathbb{P}$ then we define:*

$$M[G] = \{\tau_G \mid \tau \in M^{\mathbb{P}}\}$$

and say that 'M[G] is generic extension of M' and 'M is a ground model of generic extension'.

Note. *Let N be a model for ZFC. Recall the definition of* ORD^N *- a class of all ordinals in N and* CARD^N *- a class of all cardinals in N.*

Lemma 2.18. *If M is a transitive model of ZFC, $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ is a partial order in M and G a non-empty filter on $\mathbb{P}$, then :*

(a) $(\forall x \in M)(\check{x} \in M^{\mathbb{P}}\ \&\ \check{x}_G = x)$
(b) $M \subseteq M[G]$
(c) $\mathrm{ORD}^{M[G]} = \mathrm{ORD}^M$

Lemma 2.19. *Let M be a transitive model of ZFC, $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ a partial order in M and G a non-empty filter on $\mathbb{P}$. Then the Axiom of Extensionality, Foundation, Infinity, Pairing and Union are true in $M[G]$.*

Proof. We observe that by definition of G and $M[G]$, $M[G]$ is transitive and whence Extensionality holds in $M[G]$. Foundation is true relativized to any class and Infinitiy holds by $\omega = \check{\omega}_G$. To show Pairing let $a, b \in M[G]$, $a = \tau^a_G$ and $b = \tau^b_G$ and consider a name $\{\langle \tau^a, 1_{\mathbb{P}} \rangle, \langle \tau^b, 1_{\mathbb{P}} \rangle\}$. For Axiom of Union let $a \in M[G]$, $a = \tau_G$, define $\pi = \{\langle \rho, p \rangle \mid (\exists \langle \sigma, q \rangle)(\exists r)(\langle \rho, r \rangle \in \sigma\ \&\ p \leq r\ \&\ p \leq q)\}$ and argue that since $\pi \in M^{\mathbb{P}}$, $\pi_G = \bigcup \tau_G = \bigcup a$. $\square$

Definition 2.20. *Let $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ be a partial order and G a filter on $\mathbb{P}$. Then for every formula φ in a language $\mathcal{L}_{\mathbb{P}}$ with all free variables between $\tau^0, \tau^1, ..., \tau^n$ we define a formula φ_G to be a formula which arises from φ by substitution of τ^i for τ^i_G for every $i < n + 1$.*

Note that if φ is a sentence of $\mathcal{L}_{\mathbb{P}}$ then $\varphi \leftrightarrow \varphi_G$ and we can write φ instead of φ_G.

Definition 2.21. *Let M be a countable transitive model for ZFC, $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ a partial order in M and G a $\mathbb{P}$-generic over M. Let further φ be a formula in language $\mathcal{L}_{\mathbb{P}}$. We define a relation $\models$ by:*

$$M[G] \models \varphi_G \text{ if and only if } \varphi_G \text{ holds in } M[G].$$

Definition 2.22. *Let $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ be a partial order in M, φ a formula of $\mathcal{L}_{\mathbb{P}}$ and $p \in \mathbb{P}$ we define a forcing relation $\Vdash$ as follows:*

$$p \Vdash \varphi \text{ if and only if } (\forall G\ \mathbb{P}\text{-generic over } M)(p \in G \rightarrow M[G] \models \varphi_G)$$

and say that 'p forces φ' if and only if $p \Vdash \varphi$ and 'p decides φ' if and only if $p \Vdash \varphi$ or $p \Vdash \neg\varphi$.

The previous definition of forcing does take place outside of M. However, the core of the method of forcing is based on a definition of forcing relation which takes place inside of M and which is equivalent to the relation from the previous definition. The following definition and theorem is given in [Kun06] page 195-201.

Definition 2.23. *Let $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ be a partial order. The following clauses define the notion $p \Vdash^* \phi(\tau_1, \tau_2, ..., \tau_n)$ where $\phi(x_1, x_2, ..., x_n)$ is a formula with all free variables shown, $p \in \mathbb{P}$ and $\tau_1, \tau_2, ..., \tau_n \in V^{\mathbb{P}}$.*

(i) $p \Vdash^* \tau_1 = \tau_2$ *if and only if*

- *a.* $\forall \langle \pi_1, s_1 \rangle \in \tau_1$ *the set* $\{q \leq p \mid q \leq s_1 \rightarrow (\exists \langle \pi_2, s_2 \rangle \in \tau_2)(q \leq s_2\ \&\ q \Vdash^* \pi_1 = \pi_2)\}$ *is dense below p and*
- *b.* $\forall \langle \pi_2, s_2 \rangle \in \tau_2$ *the set* $\{q \leq p \mid q \leq s_2 \rightarrow (\exists \langle \pi_1, s_1 \rangle \in \tau_1)(q \leq s_1\ \&\ q \Vdash^* \pi_1 = \pi_2)\}$ *is dense below p.*

(ii) $p \Vdash^* \tau_1 \in \tau_2$ *if and inly if* $\{q \mid (\exists \langle \pi, s \rangle \in \tau_2)(q \leq s\ \&\ q \Vdash^* \pi = \tau_1)\}$ *is dense below p.*

(iii) $p \Vdash^* \varphi\ \&\ \psi$ *if and only if* $p \Vdash^* \varphi$ *and* $p \Vdash^* \psi$.

(iv) $p \Vdash^* \neg\varphi$ *if and only if there is no* $q \leq p$ *such that* $q \Vdash^* \varphi$.

(v) $p \Vdash^* \exists x \varphi(x)$ *if and only if a set* $\{r \mid (\exists \dot{x} \in M^{\mathbb{P}})(r \Vdash^* \varphi(\dot{x}))\}$ *is dense below p.*

Theorem 2.24. *(The Forcing Theorem) Let M be a countable transitive model of ZCF and $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ a partial order in M; let $\psi(x_1, ..., x_n)$ be a formula with all free variables shown; let $\tau_1, ..., \tau_n \in M^{\mathbb{P}}$ then:*

(1) for all $p \in \mathbb{P}$,

$$p \Vdash \psi(\tau_1, ..., \tau_n) \leftrightarrow (p \Vdash^* \psi(\tau_1, ..., \tau_n))^M.$$

(2) for all G which are $\mathbb{P}$-generic over M,

$$\psi(\tau_{1_G}, ..., \tau_{n_G})^{M[G]} \leftrightarrow \exists p \in G(p \Vdash \psi(\tau_1, ..., \tau_n)).$$

The following theorem showing more properties of forcing relation is contained in Theorem 14.7 of [Jech02].

Theorem 2.25. *Let M be a countable transitive model for ZFC, $\mathbb{P}$ a partial order in M and φ, ψ formulas of $\mathcal{L}_{\mathbb{P}}$, then for all $p \in \mathbb{P}$ the forcing relation satisfies:*

(i) If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$.

(ii) No p forces φ and $\neg\varphi$.

(iii) $\exists q \leq p$ for which $q \Vdash \varphi$ or $q \Vdash \varphi$.

(iv) $p \Vdash \varphi \vee \psi$ if and only if $(\forall q \leq)(\exists r \leq q)(r \Vdash \varphi$ or $r \Vdash \psi)$.

(v) $p \Vdash \forall x \varphi(x)$ if and only if $p \Vdash \varphi(\dot{x})$ for every $\dot{x} \in M^{\mathbb{P}}$.

(vi) (Maximal Principle) If $p \Vdash \exists x \varphi(x)$, then there exists $\dot{x} \in M^{\mathbb{P}}$ such that $p \Vdash \varphi(\dot{x})$.

For more informative proof of the next main theorem see [Kun06] pages 200-203.

Theorem 2.26. *Let M be a countable transitive model for ZFC, $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ a partial order in M and G a $\mathbb{P}$-generic over M, then $M[G]$ satisfies ZFC.*

Proof. According to the Lemma 2.19 given above, it remains to show that Axiom of Comprehension, Replacement, Power Set and Axiom of Choice holds in $M[G]$.

To show that Axiom of Comprehension holds in $M[G]$ let $\sigma, \tau^1, \ldots, \tau^n \in M^{\mathbb{P}}$ and a formula $\psi(x, \sigma, y_1, \ldots, y_n)$ be given and define $\rho = \{\langle \pi, p \rangle \in dom(\sigma) \times \mathbb{P} \mid p \Vdash \pi \in \sigma \ \& \ \psi(\pi, \sigma, \tau^1, \ldots, \tau^n)\}$. As $\rho \in M^{\mathbb{P}}$ and it can be shown that $\rho_G = \{a \in \sigma_G \mid \psi(a, \sigma_G, \tau^1_G, \ldots, \tau^n_G)^{M[G]}\}$, we are done.

To verify Axiom of Replacement let $\sigma_G, \tau_G^1, \ldots, \tau_G^n \in M[G]$ and a formula $\psi(x, v, r, z_1, \ldots, z_n)$ be given. Moreover assume $(\forall x \in \sigma_G)(\exists! y)\psi(x, y, \sigma_G, \tau_G^1, \ldots, \tau_G^n)$ holds in $M[G]$. To show there exists a name $\rho \in M^{\mathbb{P}}$ such that $[(\forall x \in \sigma_G)(\exists y \in \rho_G)\psi(x, y, \sigma_G, \tau_G^1, \ldots, \tau_G^n)]^{M[G]}$ let $S \in M$ be a set with $S \subseteq M^{\mathbb{P}}$ and (for brevity we suppress mention of τ's)

$$(\forall \pi \in dom(\sigma))(p \in \mathbb{P})(\exists \mu \in M^{\mathbb{P}} : p \Vdash \psi(\pi, \mu) \rightarrow \exists \mu \in S : p \Vdash \psi(\pi, \mu)).$$

As such S exists by reflection theorem and $p \Vdash \psi(\pi, \mu)$ being definable in M, we can define ρ by $\rho = S \times \{1_{\mathbb{P}}\}$ and we are done.

To argue for Power Set Axiom, let $\sigma_G \in M[G]$ be given. We define a name $\rho = S \times \{1_{\mathbb{P}}\}$ where $S = \{\tau \in M^{\mathbb{P}} \mid dom(\tau) \subseteq dom(\sigma)\}$ and derive that whenever $\mu_G \subseteq \sigma_G$ for $\mu \in M^{\mathbb{P}}$, $\mu_G \in \rho_G$. Indeed, let $\tau = \{\langle \pi, p \rangle \mid \pi \in dom(\sigma) \ \& \ p \Vdash \pi \in \mu\}$ then $\tau \in S$ whence $\tau_G \in \rho_G$ and as it can be shown that $\mu_G = \tau_G$ we are done.

Finally we verify a statement equivalent to Axiom of Choice (see [Kun06] page 201, Lemma 4.1). The statement is

$$(\forall x)(\exists \alpha \in ORD^{M[G]})(\exists f)(f \text{ is a function } \ \& \ dom(f) = \alpha \ \& \ x \subseteq rng(f)).$$

For a given $x = \sigma_G \in M[G]$ let $dom(\sigma) = \{\pi_\gamma \mid \gamma < \alpha\}$ for some $\alpha \in ORD^M$ which is possible by Axiom of Choice in M. Let $\tau = \{\mu \mid \mu \text{ is a standart name of } \langle \check{\gamma}, \pi_\gamma \rangle \ \& \ \gamma < \alpha\} \times \{1_{\mathbb{P}}\}$. Then τ_G is a function with domain α and $x \subseteq rng(\tau_g)$ and we done. □

Note that unlike 'α is an ordinal', 'κ is a cardinal' is not absolute between ground model and its generic extension. Whence it does not always hold that $\mathrm{CARD}^M = \mathrm{CARD}^{M[G]}$ where M is a countable transitive model of ZFC and $M[G]$ its generic extension. therefore there is a reason to state the following definition.

Definition 2.27. *Let M be a model for ZFC, $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ a partial order in M and θ cardinal in M then:*

(i) $\mathbb{P}$ preserves cardinals $\geq \theta$ if and only if whenever $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over M:

$$(\forall \beta \in \mathrm{ORD}^M, \beta \geq \theta)((\beta \textit{ is a cardinal})^M \leftrightarrow (\beta \textit{ is a cardinal})^{M[G]})$$

(ii) $\mathbb{P}$ *preserves cofinalities* $\geq \theta$ *if and only if whenever* $G \subseteq \mathbb{P}$ *is a* $\mathbb{P}$*-generic filter over M,* γ *limit ordinal in* M *and* $cf(\gamma) \geq \theta$*:*

$$cf(\gamma)^M = cf(\gamma)^{M[G]}$$

The same is defined for '$\leq \theta$*' replacing '*$\geq \theta$*'.*

Definition 2.28. *Let* $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ *be a partial order in* M*,* G *a* $\mathbb{P}$*-generic over* M *and* $\kappa \in \mathrm{CARD}^M$ *then we say that:*

'κ *is collapsed in* $M[G]$*' if and only if* $\kappa \notin \mathrm{CARD}^{M[G]}$

'κ *is collapsed to* $\lambda \in \mathrm{CARD}^{M[G]}$*' if and only if* κ *is collapsed in* $M[G]$ *and* $(\lambda = |\kappa|)^{M[G]}$

Before we start with proving basic properties of forcing we need to introduce one basic combinatorial tool (see [Kun06] page 49, Theorem 1.6) which will be used very often from now on.

Definiton 2.29. *A family* $\mathcal{A}$ *of sets is called a* Δ*-system if there is a fixed set* r*, called root of the* Δ*-system, such that* $a \cap b = r$ *for all* a, b *distinct members of* $\mathcal{A}$*.*

Lemma 2.30. *Let* κ *be any infinite cardinal and* $\mathcal{A}$ *a family of sets. Let* $\theta > \kappa$ *be regular and satisfy* $(\forall \alpha < \theta)(|\alpha|^{<\kappa} < \theta)$*. Assume* $|A| \geq \theta$ *and* $(\forall x \in A)(|x| < \kappa)$*, then there is a family of sets* $\mathcal{B} \subseteq \mathcal{A}$*, such that* $|\mathcal{B}| = \theta$ *and* $\mathcal{B}$ *forms a* Δ*-system.*

From now further by *'forcing'* we mean *'partially ordered set (used for producing generic extensions)'*, by *'conditions'* we mean *'elements of some forcing'* and whenever $\mathbb{P}$ is a forcing notion, we say that *'* $\mathbb{P}$ *adds a set* A*'* if A is an element of every generic extension by $\mathbb{P}$ but not an element of a ground model.

3 General Forcing Properties

In this section we show some facts about general properties of forcing.

3.1 Generic Filters

Lemma 3.1.1. *(Equivalent definition of genericity) Let $\mathbb{P}$ be partial order then the followings are equivalent for a set G:*

(i) *G is a generic filter.*

(ii) *G is a filter intersecting every maximal antichain in $\mathbb{P}$.*

(iii) *a.* $(\forall p, q \in G)(\exists r \in \mathbb{P})(r \leq p, q)$ *and*

b. $(\forall p \in G)(\forall q \in \mathbb{P})(q \geq p \rightarrow q \in G)$ *and*

c. $(\forall D \subseteq \mathbb{P}, D \in M)(D$ *is dense* $\rightarrow G \cap D \neq \emptyset)$.

Proof.

(i)$\rightarrow$(ii): Let A,a maximal antichain in $\mathbb{P}$, be given. Then we define a set $D = \{p \in \mathbb{P} \mid (\exists a \in A)(p \leq a)\}$ and show it is dense: Let $p \in \mathbb{P}$ be given. As A is a maximal antichain there exists some $q \in A$ such that $p \| q$ otherwise there is some $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$. By definition of D, r is in D and D is dense. By density of D there is some $p \in D \cap G$ and hence $a \in A$ such that $p \leq a$. From G being upward closed we get $a \in G$ and we are done.

(ii)$\rightarrow$(i): Let D be a set dense in $\mathbb{P}$. Let $A_D \subseteq D$ be an antichain in $\mathbb{P}$ which is maximal in D, i.e. whenever $A'_D \subseteq D$ and $A_D \subsetneq A'_D$ then A'_D is not an antichain in $\mathbb{P}$. Note that A_D exists by *Zorn lemma*. It is sufficient to show that in fact A_D is maximal in $\mathbb{P}$ and argue that $\emptyset \neq A_D \cap G$. Let $p \in \mathbb{P} - A_D$, by density of D there is a $q \in D$ such that $q \leq p$ and $q \in A_D$ or there is some $r \in A_D$ such that $q \| r$ since A_D is a maximal antichain in D. In the first case $q \leq p$ hence $A_D \cup \{p\}$ is not an antichain. In the second case there is some $r' \in \mathbb{P}$ such that $r' \leq q, r$ and again from $r \| p$, $A_D \cup \{p\}$ is not an antichain. Whence A_D is a maximal antichain in $\mathbb{P}$ and we are done

(i)$\leftrightarrow$(iii): Left right implication is obvious since condition a. results from G being centred system and the rest is the same as in Definition 2.6 and 2.12. For the right left implication we show at first that

$$D_{p,q} = \{r \in \mathbb{P} \mid r \perp p \ \vee \ r \perp q \vee r \leq p, q\}$$

is dense for all $p, q \in \mathbb{P}$. Let $r \in \mathbb{P} - D$ then from $r \| p$ and $r \| q$ we have $r_1 \leq r, p$ and $r_2 \leq r, q$. Now if $r_1 \perp q$ then $r_1 \in D_{p,q}$ and we are done otherwise there exists $r_3 \leq r_1, q$ i.e $r_3 \leq r, p, q$ hence $r_3 \in D_{p,q}$. Since $(\forall p, q \in G)(\exists r \in D_{p,q} \cap G)$ means $r \leq p, q$ otherwise (i.e $r \perp p$ or $r \perp q$) G would not be a filter, we are done. □

Claim 3.1.2. *Assume that M is a countable transitive model of ZFC, $\mathbb{P} \in M$ a partial order, $E \subseteq \mathbb{P}$ and $E \in M$. Let G be a $\mathbb{P}$-generic filter over M then either $G \cap E \neq \emptyset$ or $(\exists p \in G)(\forall r \in E)(r \perp p)$.*

Proof. We define

$$D = \{p \mid (\exists r \in E)(p \leq r)\} \cup \{p \mid (\forall r \in E)(r \perp p)\}$$

and show it is dense. Let $q \in \mathbb{P} - D$ then there exists $r \in E$ such that $r \| q$ hence there is $p \in \mathbb{P}$, $p \leq r, q$ and $p \in D$. Thus neither there exists $p \in G \cap D$ for which $p \in E$ or $(\forall r \in E)(r \perp p)$, in both cases we are done. □

Note that by choosing E from the previous claim to be $\{p\}$ for some $p \in \mathbb{P}$, we get $p \in G$ if and only if $(\forall q \in G)(p \| q)$.

Claim 3.1.3. *Let $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ be a partial order, G a $\mathbb{P}$-generic filter and $p \in G$. Then if D is a subset of $\mathbb{P}$ dense below p, $G \cap D \neq \emptyset$ and if A a maximal antichain below p, $G \cap A \neq \emptyset$.*

Proof. Let p, D be given and let B be a maximal antichain in $\mathbb{P}$ which contains p (this is possible by *Zorn Lemma*).We define a set

$$D_B = D \cup \{q \in \mathbb{P} \mid (\exists a \in B)(a \neq p \ \& \ q \leq a)\}.$$

It is immediate that D_B is dense in $\mathbb{P}$ and hence $G \cap D_B \neq \emptyset$.Since there is $q \in G \cap D_B$ and $G \cap B = \{p\}$ (G intersects B at exactly one point, otherwise G would not be a filter) it must be the case that $q \in D$.

To argue that G intersects every maximal antichain A below $p \in G$, let A' be a maximal antichain in $\mathbb{P}$ containing p (again it exists by *Zorn Lemma*). Then

$A'' = A \cup A' - \{p\}$ (replace p in A' by elements from A) is a maximal antichain in $\mathbb{P}$ and since $G \cap A' = \{p\}$ it must be the case that $G \cap A'' = \{a\}$ for some $a \in A$ and we are done. □

3.2 Forcing Equivalence

In this section we investigate a notion of forcing equivalence which helps us to understand more precisely the notion of Cohen forcing defined in section 4.

Definition 3.2.1. *Let M be a countable transitive model for ZFC, and $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$, $\langle \mathbb{Q}, \leq, 1_{\mathbb{Q}} \rangle$ be partial orders in M. We say that $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent if and only if for every G a $\mathbb{P}$-generic filter over M there exist H a $\mathbb{Q}$-generic filter over M such that $M[G] = M[H]$ and vice versa.*

The most common way how to show forcing equivalence of two partial orders is to show that first one is densely embeddable (see below) into the second one. This technique using the notion of dense embedding will be discussed in this section.

Definition 3.2.2. *Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, 1_{\mathbb{Q}} \rangle$ be any partial orders, we say that a function $i : \mathbb{P} \to \mathbb{Q}$ is an embedding if:*

(i) $(\forall p, p' \in P)(p \leq_{\mathbb{P}} p' \to i(p) \leq_{\mathbb{Q}} i(p'))$.

(ii) $(\forall p, p' \in P)(p \perp_{\mathbb{P}} p') \leftrightarrow i(p) \perp_{\mathbb{Q}} i(p')$.

Definition 3.2.3. *Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, 1_{\mathbb{Q}} \rangle$ be any partial orders and $i : \mathbb{P} \to \mathbb{Q}$ an embedding. We say that:*

(1) i is a dense embedding if $i''P$ is dense in $\mathbb{Q}$.

(2) i is a complete embedding if $(\forall q \in \mathbb{Q})(\exists p \in \mathbb{P})(\forall p' \in \mathbb{P})(p' \leq_{\mathbb{P}} p \to i(p') \| _{\mathbb{Q}} q)$

Claim 3.2.4. *Every dense embedding is complete.*

Proof. Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, 1_{\mathbb{Q}} \rangle$ be any partial orders and $i : \mathbb{P} \to \mathbb{Q}$ be a given dense embedding. According to the Definition 3.2.3 we need to show that (2) holds. Suppose $q \in \mathbb{Q}$ then from density of $i''\mathbb{P}$ there is $i(p) \leq_{\mathbb{Q}} q$ for some $p \in \mathbb{P}$. If $p' \in \mathbb{P}$ and $p' \leq_{\mathbb{P}} p$ then $i(p') \leq_{\mathbb{Q}} i(p)$ hence $i(p') \leq_{\mathbb{Q}} q$ and finally $i(p') \| _{\mathbb{Q}} q$. □

Claim 3.2.5. *A composition of complete embeddings is complete and a composition of dense embeddings is dense.*

Proof. Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$, $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, 1_{\mathbb{Q}} \rangle$, $\langle \mathbb{R}, \leq_{\mathbb{R}}, 1_{\mathbb{R}} \rangle$ be partial orders. Suppose $i : \mathbb{Q} \to \mathbb{P}$ and $j : \mathbb{P} \to \mathbb{R}$ are embeddings, then it can be easily seen that (i) and (ii) from Definition 3.2.2 holds. Suppose i, j are complete embeddings and let $r \in \mathbb{R}$ be given, then there is some $p \in \mathbb{P}$ with $(\forall p' \leq p)(j(p') \|_{\mathbb{R}} r)$ and $q \in \mathbb{Q}$ with the same property for p. Since $i \circ j$ is an embedding it follows that $(\forall q' \leq q)(j \circ i(q') \|_{\mathbb{R}} r)$. If we suppose i, j are dense embeddings and $r \in \mathbb{R}$ then there is some $p \in \mathbb{P}$ with $j(p) \leq_{\mathbb{R}} r$ and since the same holds for $j(p)$ and $\mathbb{Q}$ there is some $q \in \mathbb{Q}$ such that $j \circ i(q) \leq_{\mathbb{R}} r$. □

Claim 3.2.6. *Let $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq, 1_{\mathbb{Q}} \rangle$ be partial orders, $i : \mathbb{P} \to \mathbb{Q}$ dense embedding and D a set dense in $\mathbb{P}$ then $i''D$ is dense in $\mathbb{Q}$.*

Proof. Let $q \in \mathbb{Q}$ then from density of $i''\mathbb{P}$ we have $p \in \mathbb{P}$ such that $i(p) \leq q$. Since D is dense in $\mathbb{P}$ we have $p' \in D$ such that $p' \leq p$ hence $i(p') \leq i(p) \leq q$ and $i(p') \in i''D$. □

There is of course as always when we are talking about any kind of equivalence a special case of dense embedding – an isomorphism. It would be possible to show directly from definition of forcing (the recursive one - Theorem 2.24) that whenever $\mathbb{P} \cong \mathbb{Q}$ via $i \in M$ (M countable transitive model of ZFC) and $\varphi(\tau_1, \tau_2, \dots, \tau_n)$ is a formula of language $M^{\mathbb{P}}$ with all variables shown between $\tau_1, \tau_2, \dots, \tau_n$ then

$$(\forall p \in \mathbb{P})(p \Vdash_{\mathbb{P}} \varphi(\tau_1, \tau_2, \dots, \tau_n) \leftrightarrow i(p) \Vdash_{\mathbb{Q}} \varphi(\tau_1^*, \tau_2^*, \dots, \tau_n^*))$$

where τ_i^* is defined recursively as $\tau_i^* = \{\langle \sigma^*, i(p) \rangle \mid \langle \sigma, p \rangle \in \tau_i\}$. Hence moreover from

$$\tau_H = \{\sigma_H \mid (\exists p \in H)(\langle \sigma, p \rangle \in \tau)\} = \{\sigma_{i''H} \mid (\exists p \in i''H)(\langle \sigma^*, i(p) \rangle \in \tau^*_{i''H})\} = \tau^*_{i''H}$$

we can infer $M[H] = M[i''H]$ for H $\mathbb{Q}$-generic over M. However, as it will be shown an isomorphism is unnecessarily strong prerequisite. To show that non-isomorphic partial orders are forcing equivalent we introduce more general tools.

Lemma 3.2.7. *Let M be a countable transitive model for ZFC, $\langle \mathbb{P}, \leq, 1_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq, 1_{\mathbb{Q}} \rangle$ any partial orders in M. Let further $i : \mathbb{P} \to \mathbb{Q}$ a complete embedding in M and H a $\mathbb{Q}$-generic over M. Then $i^{-1\prime\prime}H$ is $\mathbb{P}$-generic over M.*

Proof. We use an equivalent definition of genericity given by Lemma 3.1.1 (iii). To check a. and b. it is sufficient to use clause (ii) and (i) respectively of Definition 3.2.2. To verify c. in Lemma 3.1.1 let D be a set dense in $\mathbb{P}$ and suppose for a contradiction $D \cap i^{-1\prime\prime}H = \emptyset$ and hence $i''D \cap H = \emptyset$. By Claim 3.1.2 there exists $q \in H$ such that $(\forall r \in i''D)(q \perp r)$. From i being a complete embedding it follows that $(\forall r' \in D)(i(r') \perp q)$ and again from i being complete there exists $p \in \mathbb{P}$ such that $(\forall p' \leq p)(i(p') \| q)$ i.e. $(\forall p' \leq p)(p' \notin D)$, but this is the contradiction for density of D. □

Note that previous Lemma give us possibility to in some sense order the forcing notions by size of generic extensions they produce. Indeed, whenever $\mathbb{Q}$ is completely embeddable into $\mathbb{P}$ via i, H is a $\mathbb{P}$ generic over M (countable transitive model for ZFC) then $i^{-1\prime\prime}H$ is $\mathbb{Q}$-generic over M definable by parameters from $M[H]$ and hence $M[i^{-1\prime\prime}H] \subseteq M[H]$. In other words, an extension of M by $\mathbb{Q}$ is included in every generic extension of M by $\mathbb{P}$.

Lemma 3.2.8. *Let M be a countable transitive model for ZFC, $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$, $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, 1_{\mathbb{Q}} \rangle$ any partial orders in M and $i : \mathbb{P} \to \mathbb{Q}$ a dense embedding. Then:*

(i) if G is $\mathbb{P}$-generic over M, then $F = \{q \in Q \mid (\exists p \in G)(i(p) \leq_{\mathbb{Q}} q)\}$ is $\mathbb{Q}$-generic over M.

(ii) if F is $\mathbb{Q}$-generic over M, then $i^{-1\prime\prime}F$ is $\mathbb{P}$-generic over M.

Proof. ad (i) Let F be as stated then it is by definition upwards closed. If there are $q_1, q_2 \in F$ then there exist $p_1, p_2 \in G$ such that $i(p_1) \leq_{\mathbb{Q}} q_1$, $i(p_2) \leq_{\mathbb{Q}} q_2$ and from G being centred there exists some r under p_1, p_2 hence $i(r) \in F$ and $i(r) \leq_{\mathbb{Q}} q_1, q_2$ i.e. F is centred. We show now that F intersects every dense subset of $\mathbb{Q}$ which are in M: Let $D_{\mathbb{Q}}$ be a set dense in $\mathbb{Q}$ which is in M. We define a set

$$D_{\mathbb{P}} = \{p \in \mathbb{P} \mid (\exists q \in D_{\mathbb{Q}})(i(p) \leq_{\mathbb{Q}} q)\}$$

and show it is dense in $\mathbb{P}$. Let $p \in \mathbb{P}$. From density of $D_{\mathbb{Q}}$ we have $q \in D_{\mathbb{Q}}$ such that $q \leq i(p)$ and from density of $i''\mathbb{P}$ we have $p' \in \mathbb{P}$ such that $i(p') \leq_{\mathbb{Q}} q \leq_{\mathbb{Q}} i(p)$.

It must holds that p and p' are compatible otherwise $i(p')$ and $i(p)$ would be incompatible since i is an embedding. Take r which witnesses compatibility of p and p'. It follows from $i(r) \leq_{\mathbb{Q}} i(p') \leq_{\mathbb{Q}} q$ and definition of $D_{\mathbb{P}}$ that $r \in D_{\mathbb{P}}$ and whence $D_{\mathbb{P}}$ is dense. Now we finish the proof by considering existence of some $t \in D_{\mathbb{P}} \cap G$ for which $i(t) \leq_{\mathbb{Q}} q$ for some $q \in D_{\mathbb{Q}}$ but from definition of F we have $q \in F$ what finishes the proof of (i).

ad (ii) By Claim 3.2.4 i is a complete embedding and hence by Lemma 3.2.7 $i^{-1}{}''F$ is $\mathbb{Q}$-generic over M.

□

Lemma 3.2.9. *Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$, $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, 1_{\mathbb{Q}} \rangle$ be any partial orders in M, countable transitive model for ZFC, and $i \in M$ $i : \mathbb{P} \to \mathbb{Q}$ dense embedding, then $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent.*

Proof. Let G be a $\mathbb{P}$-generic filter over M be given. We define a set $H = \{q \in \mathbb{Q} \mid (\exists p \in G)(i(p) \leq_{\mathbb{Q}} q\}$ which is in $M[G]$ since H is defined by G and parameters from M. Since $M[H]$ is the smallest countable transitive model for ZFC containing H such that $M \subseteq M[G]$, it must holds that $M[H] \subseteq M[G]$. It is clear by definition of H that $G \subseteq i^{-1}{}''H$. To show $i^{-1}{}''H \subseteq G$ let $p \in i^{-1}{}''H$ be given then $p \in G$, otherwise $(\exists q \in G)(q \perp p)$ and hence $i(q) \perp i(p)$ and $i(p), i(q) \in H$ which contradicts H being a filter by the previous lemma. Whence G is definable by H and parameters from M and according to the same argument as in the previous case, $M[G] \subseteq M[H]$. □

3.3 Cardinal Preservation

In this section we show some basic facts about preservation of cardinals following the proofs given in [Kun06], Chapter VII.

Lemma 3.3.1. *Let M be a countable transitive model for ZFC, $\mathbb{P} \in M$ a partial order and G a $\mathbb{P}$-generic over M. Suppose further that in M: κ, δ are cardinals, κ is limit and $\delta < \kappa$. Then $(\kappa$ is a cardinal $)^{M[G]}$ if $(\forall \gamma \in \mathrm{CARD}^M)((\delta \leq \gamma < \kappa)^M \to$ $(\gamma$ is a cardinal $)^{M[G]})$.*

Proof. Suppose for a contradiction κ, δ are as required but κ is collapsed to some λ in $M[G]$. As it holds that $(\lambda < \kappa)^M$ and $(\gamma$ is a cardinal $)^{M[G]}$ for all $\delta \leq \gamma < \kappa$ if we set $\gamma = max(\lambda, \delta)$, we derive from $(\lambda \leq \gamma < \gamma^+ \leq |\kappa| = \lambda)^{M[G]}$ that $(\gamma = \gamma^+)^{M[G]}$ – contradiction. □

Lemma 3.3.2. *Let M be a countable transitive model for ZFC, $\mathbb{P} \in M$ a partial order in M and θ an infinite cardinal in M.Let further G be a $\mathbb{P}$-generic over M. Then $\mathbb{P}$ preserves cardinals $\leq \theta$ whenever $\mathbb{P}$ preserves cofinalities $\leq \theta$ and $\mathbb{P}$ preserves cardinals $\geq \theta$ whenever $(\theta$ is regular$)^M$ and $\mathbb{P}$ preserves cofinalities $\geq \theta$.*

Proof. Let θ be an infinite cardinal, we show that if $\mathbb{P}$ preserves cofinalities $\geq \theta$ and $(\theta$ is regular$)^M$ then $\mathbb{P}$ preserves cardinalities $\geq \theta$. Proof of "$\leq \theta$" version is similar. Suppose $\kappa \geq \theta$ is a regular cardinal in M, we show that it is still a cardinal in $M[G]$. Since $ZFC \vdash (\forall \alpha \in \text{ORD})(cf(\alpha) \in \text{CARD})$; $cf(\kappa)^M = \kappa$ and $cf(\kappa)^M = cf(\kappa)^{M[G]}$ it follows that κ is a cardinal in $M[G]$ and θ is preserved too. If κ is limit cardinal and we know that $\mathbb{P}$ preserves successor cardinals $\geq \theta$ it follows from Lemma 3.3.1 that $\mathbb{P}$ preserves κ. □

Lemma 3.3.3. *Let M be a countable transitive model for ZFC, $\mathbb{P} \in M$ a partial order in M, G a $\mathbb{P}$-generic over M and $\theta \in \text{CARD}^M$. Assume moreover it holds $(\kappa$ is regular cardinal $)^{M[G]}$ whenever $(\kappa$ is regular cardinal and $\kappa \geq \theta)^M$. Then $\mathbb{P}$ preserves cofinalities $\geq \theta$. Likewise with $\leq \theta$ replacing $\geq \theta$.*

Proof. Let γ be limit ordinal such that $(cf(\gamma) \geq \theta)^M$. Since $cf(\gamma)$ is regular cardinal in M and $\mathbb{P}$ preserves regularity of cardinals $\geq \theta$ we have $cf(\gamma)^M = cf(cf(\gamma))^M = cf(cf(\gamma))^{M[G]} = cf(\gamma)^{M[G]}$. □

Note 3.3.4. *By Definition 2.27, in previous lemma there holds an inverse implication too. Hence every partial order $\mathbb{P}$ preserves cofinalities $\geq \theta$ if and only if it preserves regularity of cardinals $\geq \theta$ and every partial order $\mathbb{P}$ preserves cofinalities $\leq \theta$ if and only if it preserves regularity of cardinals $\leq \theta$.*

Now we investigate which conditions make a partial order to preserve cofinalities and hence cardinals.

Definition 3.3.5. *We say that a partial order $\mathbb{P}$ is a κ-cc or satisfies 'κ chain condition' if and only if every antichain in $\mathbb{P}$ has size strictly smaller than κ. If $\kappa = \omega_1$ we say that $\mathbb{P}$ is ccc instead of ω_1-cc.*

Lemma 3.3.6. *Let M be a countable transitive model for ZFC. Assume $\kappa \in \mathrm{CARD}^M$, $\mathbb{P}$ is a κ-cc partial order in M and $A, B \in M$; let G be a $\mathbb{P}$-generic over M and let $f : A \to B$ be a function in $M[G]$. Then there exists a map $F : A \to \mathcal{P}(B)$, $F \in M$ with $(\forall a \in A)(f(a) \in F(a))$ and $(\forall a \in A)((|F(a)| < |\kappa|)^M)$.*

Proof. Let f,G be a as stated and let $\dot{f}$ be some $\mathbb{P}$-name for f, then there is $p_0 \in G$ such that $p_0 \Vdash$ '$\dot{f}$ is a function from $\check{A}$ to $\check{B}$'. For every $a \in A$ we define a set

$$X_a = \{p \in \mathbb{P} \mid p \leq p_0 \;\&\; (\exists b_p \in \check{B})(p \Vdash \dot{f}(\check{a}) = \check{b}_p)\}.$$

Let $Y_a \subset X_a$ be a maximal antichain in X_a (i.e. there exists no antichain $Y'_a \subseteq X_a$ with $Y_a \subsetneq Y'_a$). At first we show that Y_a is in fact a maximal antichain below p_0: Suppose for a contradiction there exists $p \in \mathbb{P}, p \leq p_0$ such that $Y_a \cup \{p\}$ is an antichain. Then $(\forall p' \leq p)(\forall b \in B)(p' \nVdash \dot{f}(\check{a}) = \check{b})$ otherwise if there were $p' \leq p$ with $\exists b \in B)(p' \Vdash \dot{f}(\check{a}) = \check{b})$ then $p' \in X_a$ and $(\exists q \in Y_a)(q \| p')$ hence $p \| q$ and $Y_a \cup \{p\}$ would not be an antichain. Since $(\forall p' \leq p)(\forall b \in B)(p' \nVdash \dot{f}(\check{a}) = \check{b})$ i.e. (by Theorem 2.23) $(\forall p' \leq p)(p' \nVdash (\forall b \in \check{B})(\dot{f}(\check{a}) = b)$ and (again by Theorem 2.23) it holds that $p \Vdash (\forall b \in \check{B})(\dot{f}(\check{a}) \neq b)$ which contradicts $p_0 \Vdash$ '$\dot{f}$ is a function from $\check{A}$ to $\check{B}$' since $p \leq p_0$.

Now we define for every $a \in A$ a function F by $F(a) = \{b_p \mid p \in Y_a\} \subseteq B$. From $\mathbb{P}$ being κ-cc we have that $|F(a)| = |Y_a| < \kappa$ for all $a \in A$ and from genericity of G and $p_0 \in G$, $G \cap Y_a = \{p\}$ hence by Theorem 2.24 $M[G] \models f(a) = b_p$ and by definition of F, $M[G] \models f(a) \in F(a)$. □

Lemma 3.3.7. *Assume M is a countable transitive model of ZFC, $\mathbb{P} \in M$ is a partial order, $\theta \in \mathrm{CARD}^M$ and $(\mathbb{P}$ is θ-cc$)^M$. Then $\mathbb{P}$ preserves cofinalities $\geq \theta$.*

Proof. Recall Lemma 3.3.3 and suppose for a contradiction there exists a regular ordinal $\kappa \geq \theta$ in M which is singular in $M[G]$. Let $\lambda \in \mathrm{CARD}^{M[G]}$ be such that $(\lambda = cf(\kappa))^{M[G]}$, then there exists a cofinal map $c : \lambda \to \kappa$ in $M[G]$. By $\mathbb{P}$ being $\theta - cc$ and Lemma 3.3.6 we have a function $C : \lambda \to P(\kappa)$ such that $(\forall \alpha \in \lambda)((c(\alpha) \in C(\alpha))^{M[G]} \;\&\; (|C(\alpha)| < \kappa)^M)$. Since $rng(c) \subseteq rng(C)$ and $rng(c)$ is cofinal in κ, $rng(C)$ is cofinal in κ too. However, as it holds in M that $rng(C)$ is union of λ many sets of size strictly smaller then κ and κ is regular, we have $(|rng(C)| < max(\lambda, \kappa) = \kappa)^M$ which contradicts the regularity of κ in M. Since we know $\mathbb{P}$ preserves regularity of cardinals $\geq \theta$ we derive from Lemma 3.3.3 that $\mathbb{P}$ preserves cofinalities $\geq \theta$. $\square$

Corollary 3.3.8. *Assume M is a countable transitive model of ZFC, $\mathbb{P} \in M$ is a partial order, (θ is regular cardinal)^M and ($\mathbb{P}$ is θ-cc)^M. Then $\mathbb{P}$ preserves cardinals $\geq \theta$.*

Corollary 3.3.9. *Every partial order $\mathbb{P}$ preserves at least cardinals $\geq |\mathbb{P}|^+$.*

In the following part of this section show an "inverse" condition to κ-cc which makes partial orders preserve cardinals smaller then some bound.

Definition 3.3.10. *Let $\mathbb{P}$ be a partial order and κ a cardinal, we say that $\mathbb{P}$ is κ-closed if for every ordinal λ with $|\lambda| < \kappa$ and every decreasing sequence $\langle p_\xi \in \mathbb{P} \mid \xi < \lambda \rangle$ of conditions from $\mathbb{P}$ there exists a $p \in \mathbb{P}$ such that $(\forall \xi < \lambda)(p \leq p_\xi)$.*

Theorem 3.3.11. *Suppose M is a countable transitive model for ZFC, $\mathbb{P} \in M$ a partial order and $A, B \in M$. Assume moreover that in M holds: λ is a cardinal, $\mathbb{P}$ is λ-closed and $|A| < \lambda$. Let further G be a $\mathbb{P}$-generic over M and $f : A \to B$ a function in $M[G]$. Then $f \in M$.*

Proof. Let G, f be as stated and let $\dot{f}$ be some name $\mathbb{P}$-name for f. Then there is $q_0 \in G$ with $q_0 \Vdash$'$\dot{f}$ is a function from $\check{A}$ to $\check{B}$ with domain $\check{A}$'. It is sufficient to show that a set

$$D = \{q \in \mathbb{P} \mid q \leq q_0 \ \& \ (\exists g \in M)(q \Vdash \check{g} = \dot{f})\}$$

is dense below q_0. After this we argue as follows: If D is dense below q_0 and $q_0 \in G$ then $\exists p \in D \cap G$ i.e. $p \Vdash \check{g} = \dot{f}$ for some $g \in M$ and hence by Theorem 2.24 $M[G] \models \check{g}_G = \dot{f}_G$ i.e. $M[G] \models g = f$.

To show that D is dense let $q \in \mathbb{P}, q \leq q_0$ be arbitrary. We construct a decreasing sequence $\langle p_i \mid i < \gamma \rangle$ of elements of $\mathbb{P}$ such that $(\forall i < \gamma)(p_i \Vdash \dot{f}(\check{a}_i) = \check{b}_i)$ and show that $p = \bigcup_{i<\gamma} p_i$ is in D and $p \leq q$. Since for all $i < \gamma$, a set $D_i = \{q \in \mathbb{P} \mid q \leq q_0 \ \& \ (\exists b_i \in B)(p \Vdash \dot{f}(\check{a}_i) = \check{b}_i)\}$ is dense (using Theorem 2.25 (vi) and the fact, that 1 forces that $\dot{f}$ is a function with domain A) there exists $p_0 \in D_0 : p_0 \leq q$ and by the same reasons there exists $p_{i+1} \in D_{i+1}$ whenever $i < \gamma$ is a successor cardinal and p_i

is constructed. If $i < \gamma$ is a limit cardinal we take $p'_i = \bigcup_{j<i} p_j$ which is in $\mathbb{P}$ as $\mathbb{P}$ is λ-closed and again find some $p_i \in D_i : p_i \leq p'_i$. Now a function defined by

$$g = \{\langle a_i, b_i \rangle \mid i < \gamma \ \& \ p_i \Vdash \dot{f}(\check{a}_i) = \check{b}_i)\}$$

is in M. If we take $p = \bigcup_{i<\gamma} p_i$, then $(\forall a_i \in A)(\exists b \in B)(p \Vdash \dot{f}(\check{a}_i) = \check{b}_i)$ and we derive $p \Vdash (\forall i < \check{\gamma})(\check{\langle a_i, b_i \rangle} \in \dot{f})$ i.e. $p \Vdash \check{g} \subseteq \dot{f}$. As it is $dom(g) = A$ we have $p \Vdash \check{g} = \dot{f}$ and whence $p \in D$ which finishes the proof. □

Lemma 3.3.12. *Assume $\mathbb{P} \in M$ is a partial order, $\theta \in \mathrm{CARD}^M$ and ($\mathbb{P}$ is θ-closed)^M. Then $\mathbb{P}$ preserves cofinalities $\leq \theta$.*

Proof. Recall Lemma 3.3.3 and suppose for a contradiction there is a regular cardinal $\kappa \leq \theta$ in M which is singular in $M[G]$. Then there is a regular cardinal λ in $M[G]$ such that $(\lambda < \kappa)^M$, $(\lambda = cf(\kappa))^{M[G]}$ and hence a cofinal map $c : \lambda \to \kappa$, $c \in M[G]$. Since $\mathbb{P}$ is θ-closed and $(\lambda < \theta)^M$ we have by Lemma 3.3.11 that $c \in M$ and hence $cf(\kappa)^M \leq \lambda$ – contradiction. □

Corollary 3.3.13. *Let M be a countable transitive model for ZFC, $\mathbb{P} \in M$ a λ-closed partial order and G a $\mathbb{P}$-generic over M. Then for all $\gamma < \lambda$ in M, $({}^{\gamma}2)^M = ({}^{\gamma}2)^{M[G]}$.*

Proof. We have $({}^{\gamma}2)^M \subseteq ({}^{\gamma}2)^{M[G]}$ by definition of $M[G]$. If $f \in ({}^{\gamma}2)^{M[G]}$ then $f \in M$ by Theorem 3.3.11 and hence $f \in ({}^{\gamma}2)^M$. □

Corollary 3.3.14. *Suppose $\mathbb{P}$ is λ-closed and λ^+-cc then it preserves cardinals.*

3.4 Product of Forcing

In this section we focus on some basic properties of product of forcing notions which will be used in section 5.

Definition 3.4.1. *Let $\mathbb{P}_i, i \in I$ be partial orders and κ a cardinal. For $p \in \prod_{i \in I} \mathbb{P}_i$ we define a set $supt(p) = \{i \in I \mid p_i \neq 1_i\}$ and further we say that $\mathbb{P} = \{p \in \prod_{i \in I} \mathbb{P}_i \mid |supt(p)| < \kappa\}$ with ordering:*

$$(\forall p, q \in \mathbb{P})(p \leq q) \text{ if and only if } (\forall i \in I)(p(i) \leq_{\mathbb{P}_i} q(i)).$$

is a κ-support product forcing. We say that $\mathbb{P}$ is finite-support if $\kappa = \omega$, countable-support if $\kappa = \aleph_1$ and full-support if $\kappa = |I|^+$.

To make the reasons for introducing product of forcing notions more clear, we proof following statements, which shows that forcing with finite product can be understood as forcing with finitely many forcing notions one by one and vice versa.

The contain of the following lemmas was first published in [Sol70] and is due to Robert M.Solovay, however, the proofs here follow that of [Kun06], pages 252-254.

Claim 3.4.2. *Let $\langle \mathbb{P}_0, \leq_0, 1_0 \rangle$, $\langle \mathbb{P}_1, \leq_1, 1_1 \rangle$ be partial orders in M and $i_0 : \mathbb{P}_0 \to \mathbb{P}_0 \times \mathbb{P}_1$, $i_1 : \mathbb{P}_1 \to \mathbb{P}_0 \times \mathbb{P}_1$ functions defined by $i_0(p_0) = \langle p_0, 1_1 \rangle$ and $i_1(p_1) = \langle 1_0, p_1 \rangle$. Then i_0,i_1 are complete embeddings.*

Proof. Clauses (i) and (ii) of Definition 3.2.2 are immediate. To verify clause (2) of Definition 3.2.3 for i_0 argue as follows: Let $\langle p_0, p_1 \rangle \in \mathbb{P}_0 \times \mathbb{P}_1$ be given and take $p_0 \in \mathbb{P}_0$. It is immediate that $(\forall p' \leq_0 p_0)(i_0(p') \|_1 \langle p_0, p_1 \rangle)$ since $i_0(p') = \langle p', 1_1 \rangle$ and $\langle p', p_1 \rangle \leq \langle p', 1_1 \rangle, \langle p_0, p_1 \rangle$. For i_1 argumentation is the same. □

Lemma 3.4.3. *Let $\langle \mathbb{P}_0, \leq_0, 1_0 \rangle$, $\langle \mathbb{P}_1, \leq_1, 1_1 \rangle$ be partial orders in M and $i_0 : \mathbb{P}_0 \to \mathbb{P}_0 \times \mathbb{P}_1$, $i_1 : \mathbb{P}_1 \to \mathbb{P}_0 \times \mathbb{P}_1$ functions defined by $i_0(p_0) = \langle p_0, 1_1 \rangle$ and $i_1(p_1) = \langle 1_0, p_1 \rangle$. Assume G is a $\mathbb{P}_0 \times \mathbb{P}_1$-generic over M. Then $i_0^{-1\prime\prime}G$ is $\mathbb{P}_0$-generic over M, $i_1^{-1\prime\prime}G$ is $\mathbb{P}_1$-generic over M and $G = i_0^{-1\prime\prime}G \times i_1^{-1\prime\prime}G$.*

Proof. Since i_0, i_1 are complete embeddings by Claim 3.4.2, it follows from Lemma 3.2.7 that $i_0^{-1\prime\prime}G$ is a $\mathbb{P}_0$-generic filter over M and $i_1^{-1\prime\prime}G$ is a $\mathbb{P}_1$-generic filter over M.

Now if $\langle p_0, p_1 \rangle \in G$ then $\langle p_0, 1_1 \rangle, \langle 1_0, p_1 \rangle \in G$ hence $p_0 \in i_0^{-1\prime\prime}G$, $p_1 \in i_1^{-1\prime\prime}G$ and finaly$\langle p_0, p_1 \rangle \in i_0^{-1\prime\prime}G \times i_1^{-1\prime\prime}G$. If $\langle p_0, p_1 \rangle \in i_0^{-1\prime\prime}G \times i_1^{-1\prime\prime}G$ then $\langle p_0, 1_1 \rangle, \langle 1_0, p_1 \rangle \in G$ and hence there must be $\langle q_0, q_1 \rangle \in G$ such that $\langle q_0, q_1 \rangle \leq \langle p_0, 1_1 \rangle, \langle 1_0, p_1 \rangle$. Since G is upward closed, and $\langle p_0, p_1 \rangle$ above $\langle q_0, q_1 \rangle$, $\langle p_0, p_1 \rangle$ is in G and we are done. □

Lemma 3.4.4. *Suppose $\langle \mathbb{P}_0, \leq_0, 1_0 \rangle$, $\langle \mathbb{P}_1, \leq_1, 1_1 \rangle$ are partial orders in M and $G_0 \subseteq \mathbb{P}_0$, $G_1 \subseteq \mathbb{P}_1$. Then the followings are equivalent:*

(1) $G_0 \times G_1$ is $\mathbb{P}_0 \times \mathbb{P}_1$-generic over M.

(2) G_0 is $\mathbb{P}_0$-generic over M and G_1 is $\mathbb{P}_1$-generic over $M[G_0]$.

(3) G_1 is $\mathbb{P}_1$-generic over M and G_0 is $\mathbb{P}_0$-generic over $M[G_1]$.

Furthermore, if (1)-(3) hold, then $M[G_0 \times G_1] = M[G_0][G_1] = M[G_1][G_0]$.

Proof. We show (1)$\leftrightarrow$(2), the proof of (1) $\leftrightarrow$ (3) is the same.

(1)$\rightarrow$(2): Let i_0 be as in Lemma 3.4.3 then by the same lemma $G_0 = i''^{-1}(G_0 \times G_1)$ is $\mathbb{P}_0$-generic over M. To show that G_1 is $\mathbb{P}_1$-generic over $M[G_0]$ let $D \in M[G_0]$ be a dense subset of $\mathbb{P}_1$ and $\tau \in M^{\mathbb{P}_0}$ its name i.e. $\tau_{G_0} = D$. Since $\tau_{G_0} = D$ there must be a $p_0 \in G_0$ such that $p_0 \Vdash_{\mathbb{P}_0} (\tau \text{ is dense in } \check{\mathbb{P}}_1)$. Now we define

$$D' = \{\langle q_0, q_1\rangle \in \mathbb{P}_0 \times \mathbb{P}_1 \mid q_0 \leq_0 q_1 \ \& \ q_0 \Vdash_{\mathbb{P}_0} \check{q}_1 \in \tau\}$$

and show it is dense below $\langle p_0, 1_1\rangle$ in $\mathbb{P}_0 \times \mathbb{P}_1$. Let $\langle r_0, r_1\rangle \leq \langle p_0, 1_1\rangle$ i.e $r_0 \leq p_0$ i.e $r_0 \Vdash_{\mathbb{P}_0} (\exists x \in \check{\mathbb{P}}_1)(x \in \tau \ \& \ x \leq \check{r}_1)$ and whence there is a $q_1 \in \mathbb{P}_1$ and $q_0 \leq_0 r_0$ such that $q_0 \Vdash_{\mathbb{P}_0} (\check{q}_1 \in \tau \ \& \ \check{q}_1 \leq \check{r}_1)$ which leads to $\langle q_0, q_1\rangle \in D'$ and $\langle q_0, q_1\rangle \leq_{0\times 1} \langle r_0, r_1\rangle$.

Since D' is dense below $\langle p_0, 1_1\rangle$ and $\langle p_0, 1_1\rangle \in G_0 \times G_1$ there is by Claim 3.1.3 some $\langle q_0, q_1\rangle \in D' \cap G_0 \times G_1$. Then $q_0 \Vdash_{\mathbb{P}_0} (\check{q}_1 \in \tau)$ i.e $q_1 \in D$ and $q_1 \in G_1$ and we are done.

(2)$\rightarrow$(1): It can be easily seen that $G_0 \times G_1$ is a filter. We show it is generic: Let $D \in M$ be a dense subset of $\mathbb{P}_0 \times \mathbb{P}_1$ we define a set

$$D_1 = \{p_1 \in \mathbb{P}_1 \mid (\exists p_0 \in G_0)(\langle p_0, p_1\rangle \in D)\}$$

and show it is dense in $\mathbb{P}_1$. Let $p \in \mathbb{P}_1$ and define in M a set

$$D_0 = \{p_0 \in \mathbb{P}_0 \mid (\exists p_1 \in \mathbb{P}_1)(p_1 \leq p \ \& \ \langle p_0, p_1\rangle \in D)\}$$

which is in M. It is easy to prove that D_1 is dense after we show that D_0 is dense. To show D_0 is dense argue as follows: Let $p_0 \in \mathbb{P}_0$ be given, then from $\langle p_0, p\rangle \in \mathbb{P}_0 \times \mathbb{P}_1$ and density of D there exists $\langle q_0, q_1\rangle \in D$ such that $\langle q_0, q_1\rangle \leq \langle p_0, p\rangle$ and hence $q_0 \leq p_0$ and $q_0 \in D_0$.

Now we finish the argument about density of D_1. Since D_0 is dense in $\mathbb{P}_0$ and G_0 is a $\mathbb{P}_0$- generic over M there exists $q_0 \in D_0 \cap G_0$ and $q_1 \leq p_1$ such that $\langle q_0, q_1\rangle \in D$ i.e $q_1 \in D_1$ and hence D_1 is dense.

Finally to show that $D \cap G_0 \times G_1$ is non-empty we argue that since G_1 is $\mathbb{P}_1$-generic over $M[G_0]$ and D_1 is in $M[G_0]$, $D_1 \cap G_1$ is non-empty and by the definition of D_1, there is a $\langle p_0, p_1 \rangle \in D \cap G_0 \times G_1$.

Assume now that (1)-(3) holds. To show $M[G_0 \times G_1] = M[G_0][G_1] = M[G_1][G_0]$ we argue by minimality of generic extensions. Since $G_0 \times G_1 \in M[G_0][G_1]$ it holds that $M[G_0 \times G_1] \subseteq M[G_0][G_1]$. Conversely by $G_0 \in M[G_0 \times G_1]$ we have $M[G_0] \subseteq M[G_0 \times G_1]$ and by $G_1 \in M[G_0 \times G_1]$ and $\mathbb{P}_1$-genericity over $M[G_0]$ of G_1 and hence minimality of extension by G_1 we have $M[G_0][G_1] \in M[G_0 \times G_1]$. To show $M[G_0 \times G_1] = M[G_1][G_0]$ argumentation is the same. □

Finally we introduce useful lemma concerning Knaster (see below) property of product of forcings. While taking of two κ-cc forcing notions does not imply product of them being again κ-cc forcing, it can be shown that if the forcing notions satisfy κ-Knaster property, their product is again κ-cc.

Definition 3.4.1. *We say that a partial order $\mathbb{P}$ has κ-Knaster property (or is κ-Knaster) if and only if*

$$(\forall A \in [\mathbb{P}]^\kappa)(\exists B \in [A]^\kappa)(\forall p, q \in B)(p \| q)$$

Beside ω-Knaster we sometimes say simply Knaster.

It is straightforward that if $\mathbb{P}$ has κ-Knaster property then it is κ-cc.

Lemma 3.4.5. *Let κ be a regular cardinal and $\mathbb{P}$, $\mathbb{Q}$ any partial orders then if both of them are κ- Knaster, $\mathbb{P} \times \mathbb{Q}$ is also κ-Knaster.*

Proof. Let $W \subseteq \mathbb{P} \times \mathbb{Q}$ be such that $|W| = \kappa$, then one of the following two cases holds:

(i) $\exists p \in \mathbb{P}$ such that $X_p = \{q \in \mathbb{Q} \mid \langle p, q \rangle \in W\}$ has size κ. Then from $\mathbb{Q}$ being κ-Knaster there exists some $Y \subseteq X_p$ of pairwise compatible elements having size κ. Hence there is a set $\{\langle p, q \rangle \in W \mid q \in Y\} \subseteq W$ of size κ of pairwise compatible elements and we are done. If there is no such $p \in \mathbb{P}$ but there exist such $q \in \mathbb{Q}$ an argument is the same.

(ii) $\forall p \in \mathbb{P}$ the set X_p from (i) is smaller than κ and the same holds for every q in $\mathbb{Q}$. Then there exists a function $f : \mathbb{P} \to P(\mathbb{Q})$ such that $(\forall p \in \mathbb{P})(f(p) \in X_p)$ and whenever f is constant on $A \subseteq \mathbb{P}$ (i.e. $f''A = \{c\}$ for some $c \in \mathbb{Q}$) then $|A| < \kappa$. Indeed, if there were A of size κ and $f''A = \{c\}$ for some $c \in \mathbb{Q}$ then $\{p \in \mathbb{P} \mid \langle p, c\rangle \in W\}$ has size κ but this is case (i) for $\mathbb{Q}$ which does not hold in case (ii). From regularity of κ there is a set $C \subseteq dom(f)$ of size κ such that $f \restriction C$ is injection and from $\mathbb{P}$ being κ-Knaster there exists $C' \subseteq C$ of size κ of pairwise compatible elements. Since $f \restriction C'$ is injection $R = rng(f \restriction C')$ have size κ and from $\mathbb{Q}$ being κ-Knaster there exists $R' \subseteq R$ of size κ of pairwise compatible elements. To finish the proof we observe that $C' \times R'$ is a subset of W of size κ of pairwise compatible elements and we are done.

$\square$

4 Cohen Forcing

In this section we investigate properties of historically very first forcing called Cohen forcing after his inventor Paul Cohen.

Definition 4.0.1. *Let I,J be any sets and λ a cardinal such that $|I| \geq \lambda$, then we define:*

$$Fn(I, J) = \{f \mid (f \textit{ is a function } f : I \to J) \;\&\; (|dom(f)| < \omega)\}$$
$$Fn(I, J, \lambda) = \{f \mid (f \textit{ is a function } f : I \to J) \;\&\; (|dom(f)| < \lambda)\}.$$

to be sets ordered by reverse inclusion with a maximal element $\emptyset$.

Note that as a consequence of section 3.2 we do not need to make any distinction between $\mathbb{P} = Fn(I, J, \lambda)$ for some sets I, J and $\mathbb{U} = Fn(\kappa, \gamma, \lambda)$ whenever $|I| = \kappa$ via bijection b_I and $|J| = \gamma$ via bijection b_J. Indeed, there is always a function $i : \mathbb{U} \to \mathbb{P}$ in M defined by $f \mapsto \{\langle b_I(\alpha), b_J(\beta)\rangle \mid \langle \beta, \alpha\rangle \in f\}$ which is an isomorphism and whence dense embedding that makes $\mathbb{U}$ forcing equivalent with $\mathbb{P}$.

To understand the notion of Cohen forcing more precisely, we show at first that some of the notions are forcing equivalent.

Theorem 4.0.2. *Any two countable atomless partial orders are forcing equivalent.*

Proof. We show that $\mathbb{Q} = \{f \mid f : \omega \to \omega \;\&\; dom(f) \in \omega\}$ ordered by reverse inclusion is densely embeddable to every countable atomless partial order $\mathbb{P}$.

Let $\mathbb{P} = \{1_{\mathbb{P}}, p_1, p_2, \ldots\}$ be countable atomless partial order . We are going to construct inductively an increasing sequence $\langle e_i | i \in \omega\rangle$ of embeddings from $\mathbb{Q}$ to $\mathbb{P}$ such that for every $i \in \omega$:

(i) $e_i \subseteq e_{i+1}$
(ii) $rng(e_{i+1} - e_i)$ is maximal antichain in $\mathbb{P}$
(iii) e_i is an embedding
(iv) $(\forall j \leq i)(\exists r \in rng(e_i))(p_j \geq r)$

For 0 take $e_0 = \langle\{\emptyset\}, 1_{\mathbb{P}}\rangle$. Suppose e_j is constructed,we construct e_{j+1} as follows: Since $A = rng(e_j - e_{j-1}) = \{a_i | i \in \omega\}$ or $A = rng(e_j) = \{a_i | i \in \omega\}$ if j=0 is a maximal antichain there exists r in $\mathbb{P}$ which witnesses $p_{j+1} || a_k$ for some $a_k \in A$. Let for every $i \in \omega, i \neq n$ $A_i = \{a^i_n \in \mathbb{P} \mid n \in \omega\}$ be any maximal antichain in $\mathbb{P}$ below a_i which exist from $\mathbb{P}$ being atomless and which is countable from $\mathbb{P}$ being countable and atomless. For a_k take A_k such that $r \in A_k$, this is possible from *Zorn Lemma*. Now we define further a set

$$B_i = \{b^i_n \in \mathbb{Q} \mid n \in \omega\}$$

where b^i_n is a function form ω to ω with $dom(b^i_n) = j + 1$ such that $b^i_n(j) = i$ and $b^i_n(j + 1){=} n$. At least define $e^i_{j+1} = e_{j+1} \cup \{\langle b^i_n, a^i_n\rangle \mid n \in \omega\}$ and finaly $e_{j+1} = \bigcup_{i\in\omega} e^i_{j+1}$. It is a routine to check that conditions (i)-(iv) holds for e_{j+1} and hence $e = \bigcup_{j\in\omega} e_j$ is dense embedding. □

Since for arbitrary sets I, J and a cardinal λ such that $|I| \geq \lambda$, $Fn(I, J, \lambda)$ and $Fn(I, J)$ are clearly atomless. It follows from the above stated lemma that all partial orders of the form $\mathbb{P} = Fn(I, J)$ where $|I|, |J| \leq \omega$ are forcing equivalent.

Claim 4.0.3. *Let $\kappa, \theta, \gamma, \lambda$ be infinite cardinals such that $\lambda \leq \kappa \leq \theta$, then $Fn(\kappa, \gamma, \lambda)$ is completely embeddable into $Fn(\theta, \gamma, \lambda)$.*

Proof. Since $Fn(\kappa, \gamma, \lambda) \subseteq Fn(\theta, \gamma, \lambda)$ it is sufficient to show that identity function $i : Fn(\kappa, \gamma, \lambda) \to Fn(\theta, \gamma, \lambda)$ is the complete embedding. (i) and (ii) from Definition 3.2.2 follows immediately and hence it is an embedding. To argue that i is a complete embedding let $q \in Fn(\theta, \gamma, \lambda)$ be given and let $p = q\restriction\kappa$ then p is in $Fn(\kappa, \gamma, \lambda)$ and $(\forall p' \leq p)(i(p') \geq q)$.

□

The following statement shows that some of the Cohen forcings are very universal in the sense that the generic extensions they produce are included in generic extensions of lot of another forcings.

Lemma 4.0.4. *Let κ be a cardinal, $\mathbb{P}_i$ for $i \in \kappa$ an atomless forcing notion and $\mathbb{P} = \prod_{i\in\kappa} \mathbb{P}_i$ a κ-support product forcing notion. Then every generic extension by $\mathbb{P}$ contains a generic extension by $Fn(\kappa, 2, \kappa)$.*

Proof. Observe that once we show there exists for every $i \in \omega$ a partial order $\mathbb{R}_i$ such that $\mathbb{R}_i$ is forcing equivalent with $\mathbb{P}_i$ and $\mathbb{R}_i$ contains two distinct elements a^i_0, a^i_1 which forms a maximal antichain in $\mathbb{R}_i$, we are done. Indeed, then $\mathbb{P} = \prod_{i\in\omega} \mathbb{P}_i$

is forcing equivalent with $\mathbb{R} = \prod_{i\in\omega} \mathbb{R}_i$ and letting $\mathbb{F} = Fn(\kappa, 2, \kappa)$ we can define a complete embedding $e : \mathbb{F} \to \mathbb{R}$ by $e(p) = r$ if $dom(p) = supt(r)$ and $r(i) = a^i_{p(i)}$ whenever $i \in supt(r)$. It can be easy seen that (i) and (ii) from Definition 3.2.2 holds for e. To argue that (2) in Definition 3.2.3 holds let $r \in \mathbb{R}$ be given and assume $p \in \mathbb{F}$ is such that $dom(p) = supt(r)$ and for every $i \in dom(p)$, $p(i) = 0$ if $r(i)\|a^i_0$ and $p(i) = 1$ otherwise. Since for every $i \in \omega$ either $r(i)\|a^i_0$ or $r(i) \perp a^i_0$ and $r(i)\|a^i_1$, in both cases $e(p)(i)\|r(i)$ by definition of e and by the same argument it holds for every $p' \leq p$ and hence e is a complete embedding.

Now it would be sufficient to let, for every $i \in \omega$, $\mathbb{R}_i$ be a completion of $\mathbb{P}_i$ (see [Monk89], pages 55-56) however, we can show a straightforward construction where the notion of completion is not needed:

For every $p \in \mathbb{P}_i$ let $u_p = \{q \in \mathbb{P}_i \mid q \leq p\}$, then we define a set $\mathbb{R}_i = \{\bigcup_{p\in I} u_p \mid I \subseteq \mathbb{P}_i \ \&\ I \neq \emptyset\}$ order by inclusion (i.e $\mathbb{R}_i$ consists in fact exactly of the non-empty open sets of topology generated by $\{u_p \mid p \in \mathbb{P}_i\}$). It can be easily seen that $j : \mathbb{P}_i \to \mathbb{R}_i$ defined by $j(p) = u_p$ is an embedding. Moreover j is a dense embedding since every $r \in \mathbb{R}_i$ is a non-empty set of the form $r = \bigcup_{p\in I} u_p$ for suitable $I \subseteq \mathbb{P}_i$, therefore $j(p) = u_p \subseteq r$ for arbitrary $p \in I$. It remains to show there exists a maximal antichain of size 2 in $\mathbb{R}_i$. Fix A, a maximal antichain in $\mathbb{P}_i$ and let $x \in A$ be arbitrary. Then $|A| \geq 2$ since $\mathbb{P}_i$ is atomless and thus the set $\{a^i_0, a^i_1\}$ where $a^i_0 = u_x, a^i_1 = \bigcup_{b\in A, b\neq x} u_b$ is a maximal antichain in $\mathbb{R}_i$. Indeed, $\{a^i_0, a^i_1\}$ is clearly an antichain and if there were $r \in \mathbb{R}_i$ with $r \cap a^i_0 = \emptyset$ and $r \cap a^i_1 = \emptyset$ i.e. $r \cap u_x = \emptyset$ for all $x \in A$ then by density of $j''\mathbb{P}_i$ there is some p such that $j(p) \subseteq r$ and hence $A \cup \{p\}$ is an antichain–contradiction. Thus $\mathbb{R}_i$ is forcing equivalent with $\mathbb{P}_i$ and contains two distinct elements which form a maximal antichain in $\mathbb{R}_i$. □

Now we turn back to the purpose of defining Cohen forcing. Although the nowadays technical concept of forcing technique differs from the Cohen one‘s the intuition behind getting a model of ZFC where Continuum Hypothesis fails is still the same:

Assume we have an M, countable transitive model for ZFC, and define there a partial order consisting of approximations of objects which we want to add. In case of violating Continuum Hypothesis we should add at least $\aleph_2$ subsets of ω. For this reasons we use $\mathbb{P} = Fn(\aleph_2 \times \omega, 2, \omega)$ and take G a $\mathbb{P}$-generic filter over M (it exists by *Rasiowa-Sikorski Theorem*) whose union is a function defining $\aleph_2^M$ subset of ω

in $M[G]$ as it will be shown in section 4.2. At the end we argue that $\aleph_2$ in M is the same ordinal as $\aleph_2$ in $M[G]$ and whence Continuum Hypothesis fails in $M[G]$.

To be able to generalise this process for uncountable cardinals and violate General Continuum Hypothesis on uncountable cardinal, we will focus in the following section on properties of cardinal arithmetic sufficient which allow to Cohen forcing preserve cardinals.

4.1 Cardinal Preservation

Lemma 4.1.1. *Let I, J be arbitrary sets, λ a cardinal with $|J| \leq \lambda \leq |I|$ and $\mathbb{Q}$=Fn(I,J,λ) a partial order. Then $\mathbb{Q}$ is $(|J|^{<\lambda})^+$-Knaster.*

Proof. Let $\theta = (|J|^{<\lambda})^+$ and $A \subseteq \mathbb{P}$ such that $|A| = \theta$. We show that there exists $A' \subseteq A$ of size θ of pairwise compatible elements. Our argumentation depends on cofinality of λ:

(i) λ is regular:
If λ is regular then $(|J|^{<\lambda})^{<\lambda} = |J|^{<\lambda} < \theta$ thus a set $A^{dom} = \{dom(p)|\, p \in A\}$ has cardinality θ, otherwise if $|A^{dom}| < |A|$ then $|A| \leq |A^{dom}|^{<\lambda} < \theta$ which is a contradiction. Since $(\forall\alpha < \theta)(|\alpha^{<\lambda}| < \theta)$, $|A^{dom}| \geq \theta$ and $(\forall x \in A^{dom})(|x| < \theta)$ there exist by Δ-*system lemma* a subset $R \subseteq A^{dom}$ of size θ which forms a Δ-system with root r. Now we define:

(a) $F = \{f|\, f : r \to J\}$, hence $|F| = |J|^{|r|} \leq |J|^{<\lambda} < \theta$,
(b) $B = \{p \in A|\, dom(p) \in R\}$, hence $|B| \geq |R| = \theta$,
(c) $A_f = \{p \in B|\, p\restriction r = f\}$ for every $f \in F$.

Note that we can assume r being non-empty because otherwise B contains pairwise compatible elements and we are done. It follows that system of A_f's is a partion of $B \subseteq A$ i.e. $B = \bigcup_{f\in F} A_f$. Since θ is regular and B is a union of less than θ many sets and B having size θ, there must exists some $f \in F$ with $|A_f| = \theta = (|J|^{<\lambda})^+$ and hence for this f, A_f is a subset of A of pairwise compatible elements having size θ.

(ii) λ is singular:
Suppose for a contradiction A is an antichain. If λ is singular then from

regularity of θ we have $\theta > \lambda$ and from $\theta = \bigcup_{\gamma<\lambda}\{\xi \mid |p_\xi| \leq \gamma\}$ there exists $\gamma < \lambda$ such that $Y = \{\xi \mid |p_\xi| \leq \gamma\} = \{\xi \mid |p_\xi| < \gamma^+\}$ has size θ (otherwise θ would not be regular or bigger than λ). As λ is singular, it is not a successor cardinal and hence we have $\gamma < \gamma^+ < \lambda$. Now observe that $Fn(I, J, \gamma^+)$ is by (i) $(|J|^\gamma)^+$-Knaster but this is a contradiction since $\{p_\xi \mid \xi \in Y\}$ is a set of pairwise compatible elements having size θ and $\theta \geq (|J|^\gamma)^+$.

□

Lemma 4.1.2. *Let I, J be arbitrary sets with $|J| \geq 2$ and λ a with $|I| \leq \lambda$. Then $\mathbb{P} = Fn(I, J, \lambda)$ is $cf(\lambda)$-closed.*

Proof. Suppose λ is regular: Assume $\gamma < \lambda$, $\langle p_\xi \mid \xi < \gamma\rangle$ is a decreasing sequence of elements from $\mathbb{P}$ and take $p = \bigcup\{p_\xi \mid \xi < \gamma\}$. Since p is a function from I to J and $(\forall\xi < \gamma)(|dom(p_\xi)| < \lambda)$ we have by regularity of λ that $|dom(p)| < \lambda$. Whence $p \in \mathbb{P}$ and $p \leq p_\xi$ for all $\xi < \lambda$ by definition of p.

Suppose λ is singular: Let $cf(\lambda) = \theta$ and let $\langle p_\xi \mid \xi < \gamma\rangle$ for $\gamma < \theta$ be a decreasing sequence of elements from $\mathbb{P}$. Let again $p = \bigcup\{p_\xi \mid \xi < \gamma\}$ then by the same argument as in (i), p is in $\mathbb{P}$. Indeed, it cannot be the case that $|p| \geq \lambda$ since p is a union of $\gamma < cf(\lambda)$ sets of a size strictly smaller than λ. □

Corollary 4.1.3. *Let M be a countable transitive model of ZFC and in M let : I, J be arbitrary sets, λ a regular cardinal, $2^{<\lambda} = \lambda$ and $|J| \leq \lambda$. Then $Fn(I, J, \lambda)^M$ preserves cofinalities, regularity of cardinals and cardinals.*

Proof. By Lemma 4.1.1 we have that $Fn(I, J, \lambda)^M$ is $|J^{<\lambda}|^+$-cc and from $2^{<\lambda} = \lambda$ and $|J| \leq \lambda$ it is λ^+-cc. By regularity of λ and Lemma 4.1.2 is $Fn(I, J, \lambda)^M$ λ-closed. Whence it preserves cofinalities $\geq \lambda^+$ and $\leq \lambda$ which means it preserves cofinalities and hence by Lemma 3.3.3 regularity of cardinals and hence by Lemma 3.3.2 cardinals. □

4.2 Adding New Sets

Now we move to the second step of reconstruction of the Cohen's result and show among other that Cohen forcing adds arbitrary many new subsets of any ordinal in the sense of ground model.

Lemma 4.2.1. *Let M be a countable transitive model for ZFC, $\kappa, \gamma, \lambda \in \mathrm{CARD}^M$ such that $\kappa \geq \lambda$, $\mathbb{P} = Fn(\kappa \times \lambda, \gamma, \lambda)^M$ and G a $\mathbb{P}$-generic over M. Then $\bigcup G$ is a function in $M[G]$ from $\kappa \times \lambda$ to γ and whenever $g_\alpha = \{\langle \xi, \delta\rangle \mid \langle\langle \alpha, \xi\rangle, \delta\rangle \in \bigcup G\}$ for $\alpha \in \kappa$ then $g_\alpha \neq f$ for each functions f from λ to γ in M.*

Proof. Let G be given, then

(i) $\bigcup G$ is a function: Suppose for a contradiction there exist $\langle\langle \xi, \delta\rangle, \alpha\rangle, \langle\langle \xi, \delta\rangle, \beta\rangle \in \bigcup G$ for $\alpha \neq \beta$, than there exist $p_0, p_1 \in G$ such that $\langle\langle \xi, \delta\rangle, \alpha\rangle \in p_0$ and $\langle\langle \xi, \delta\rangle, \beta\rangle \in p_1$, but Since G is a filter, there exists some $p \in G$ such that $p \leq p_0$ and $p \leq p_1$ – contradiction.

(ii) $dom(\bigcup G) = \kappa \times \lambda$: For arbitrary $\xi \in \kappa, \delta \in \gamma$ we define (within M) a set $D^\xi_\delta = \{g \in \mathbb{P} \mid \langle \xi, \delta\rangle \in dom(g)\}$ and show it is dense in $\mathbb{P}$. Assume $p \in \mathbb{P}$ then either $\langle \xi, \delta\rangle \in dom(p)$ and hence $p \in D^\xi_\delta$ or $p' = p \cup \{\langle\langle \xi, \delta\rangle, 0\rangle\} \leq p$ and $p' \in D^\xi_\delta$. It follows from $\mathbb{P}$-genericity of G that for every $\xi \in \kappa, \delta \in \gamma$ we have $D^\xi_\delta \cap G \neq \emptyset$ and hence $\langle \xi, \delta\rangle \in dom(\bigcup G)$.

(iii) For $f \in M$ function from λ to γ and arbitrary $\alpha \in \kappa$ we have $f \neq g_\alpha$: We define (within M) a set $D^\alpha_f = \{g \in \mathbb{P} | (\exists \xi \in \gamma)(\langle \alpha, \xi\rangle \in dom(g)\ \&\ g(\langle \alpha, \xi\rangle) \neq f(\xi))\}$ and show again it is dense in $\mathbb{P}$. Let $p \in \mathbb{P}$ be given, since $dom(p) < \lambda$ there exists ξ such that $\langle \alpha, \xi\rangle \notin dom(p)$ and $p' = p \cup \{\langle\langle \alpha, \xi\rangle, \beta\rangle\}$ for some $\beta \neq f(\xi)$ is an element of D^α_f. It follows from $\mathbb{P}$-genericity of G that $D^\alpha_f \cap G \neq \emptyset$ which means there is some $p \in G$ such that $(\exists \xi \in \gamma)(p(\langle \alpha, \xi\rangle) \neq f(\xi))$ and hence $g_\alpha \neq f$.

□

Corollary 4.2.2. *$Fn(\kappa, \gamma, \lambda)$ especially $Fn(\kappa, 2, \lambda)$ add κ many new subsets of any θ such that $\lambda \leq \theta \leq \kappa$, in the sense of ground model.*

Using the previous lemma and Corollary 4.1.3 we are already able to state the followings:

Corollary 4.2.3. *Let M be a countable transitive model of ZFC, $\kappa, \lambda \in \mathrm{CARD}^M$, $\mathbb{P} = Fn(\kappa, 2, \lambda)^M$ and G a $\mathbb{P}$-generic over M. Then $M[G] \models 2^\lambda \geq \kappa$ whenever $(2^{<\lambda} = \lambda)^M$.*

Lemma 4.2.4. *Assume M is a countable transitive model of ZFC, $\kappa, \lambda \in \mathrm{CARD}^M$, $\mathbb{P} = Fn(\kappa, \lambda, \kappa)^M$ and G a $\mathbb{P}$-generic over M. Then in $M[G]$ exists an injection from λ to $cf(\kappa)^M$.*

Proof. Let in M, $\kappa_0, \kappa_1, \kappa_2, \ldots$ be a strictly increasing cofinal sequence of elements from κ. For every $\alpha \in \lambda$ we define a set

$$D_\alpha = \{p \in \mathbb{P} \mid (\exists n \in cf(\kappa))(dom(p) = \kappa_{n+1} \ \& \ |\{\xi \in (\kappa_{n+1} - \kappa_n) \mid p(\xi) \neq \alpha\}| \leq \kappa_n\}$$

and show it is dense in $\mathbb{P}$. Let $p \in \mathbb{P}$ be given, since $|dom(p)| < \kappa$ there is $n \in cf(\kappa)$ such that $|dom(p)| \leq \kappa_n$. Suppose further we define $q \in \mathbb{P}$ as follows: $dom(q) = dom(p) \cup \kappa_{n+1}$ and $q(x) = p(x)$ if $x \in dom(p)$, otherwise $q(x) = \alpha$. Since $|dom(p) \cap \kappa_{n+1}| \leq \kappa_n$, q is in D_α.

Now a function g defined by

$$g(\alpha) = min\{n \in cf(\kappa)^M \mid (|\{\xi \in \kappa_{n+1} - \kappa_n \mid \bigcup G(\xi) \neq \alpha\}| \leq \kappa_n)^M\}$$

is by genericity of G an injection from λ to $cf(\kappa)^M$. □

Corollary 4.2.5. *For an arbitrary cardinal λ, $Fn(\aleph_\omega, \lambda, \aleph_\omega)$ collapses λ to ω.*

Lemma 4.2.6. *Assume M is a countable transitive model for ZFC, $\kappa, \gamma, \lambda \in \mathrm{CARD}^M$ such that $\kappa \geq \lambda$ and $\mathbb{P} = Fn(\kappa, \gamma, \lambda)^M$. Then if G is $\mathbb{P}$-generic over M, $\bigcup G$ is a function in $M[G]$ which is from κ onto γ.*

Proof. By Lemma 4.2.1 $\bigcup G$ is a function from κ to γ. We show it is onto: For every $\alpha \in \gamma$ we define a set $D_\alpha = \{g \in \mathbb{P} \mid (\exists \xi \in dom(g))(g(\xi) = \alpha)\}$. It is easily seen that D_α is dense for every $\alpha \in \gamma$ since every function $g \in \mathbb{P}$ can be always extended to $g' = g \cup \{\langle \xi, \alpha \rangle\}$ where ξ is from $\kappa - dom(g)$ which is non-empty by $\kappa \geq \lambda$. □

Note. Assuming an inaccessible cardinal μ (i.e $\mu > \aleph_0$, μ is regular and $\forall \lambda < \mu : 2^\lambda < \mu$) and a regular cardinal κ the forcing notion

$$Coll(\kappa, < \mu) = \{p \in Fn(\kappa \times \mu, \mu, \kappa) \mid (\forall \langle \alpha, \beta \rangle \in dom(p))(p(\langle \alpha, \beta \rangle) < \beta)\}$$

ordered by reverse inclusion which is isomorphic to the κ-support product forcing notion $\mathbb{P} = \prod_{\lambda < \mu} Fn(\kappa, \lambda, \kappa)$ is called Lévy collapse due to Azriel Lévy who us it for the first time in [Levy70].

$Col(\kappa, < \mu)$ is clearly κ-closed and it can be shown that due to an inaccessibility of μ it is also μ-cc (see [Jech02] Lemma 15.4 and Theorem 15.17 (iii)). Whence

$Col(\kappa, \mu)$ makes μ into the successor of κ since $\mathbb{P}$ adds by the previous lemma a function from κ onto λ for every $\lambda < \mu$.

Corollary 4.2.7. *Let M be a countable transitive model for ZFC $\kappa, \gamma \in \mathrm{CARD}^M$ and $\mathbb{P} = Fn(\kappa, \gamma, \kappa)^M$. Then $\mathbb{P}$ collapses γ to κ whenever $\kappa < \gamma$ and κ is regular.*

Proof. By previous lemma $(\gamma \leq \kappa)^{M[G]}$ whenever G is $\mathbb{P}$-generic over M and by Lemma 4.1.2 $\mathbb{P}$ is κ-closed and hence $\kappa \in \mathrm{CARD}^{M[G]}$. □

Lemma 4.2.1. *Let λ be a successor cardinal and let $\mathbb{U} = Fn(\lambda, 2^{<\lambda}, \lambda)$ be a forcing notion. Then $\mathbb{U}$ is forcing equivalent to $\mathbb{P} = Fn(\lambda, J, \lambda)$ whenever J is a set with $2 \leq |J| \leq 2^{<\lambda}$.*

Proof. Let $\lambda = \theta^+$ for suitable cardinal θ. By $|J|^{<\lambda} = 2^{<\lambda}$ we have that $\mathbb{U}^* = \{p \in Fn(\lambda, 2^{<\lambda}, \lambda) \mid dom(p) \in \lambda\}$ ordered by reverse inclusion is a dense subset of $\mathbb{U}$ and therefore forcing equivalent with $\mathbb{U}$. Whence it is sufficient to show that $\mathbb{U}^*$ is isomorphic with $\mathbb{P}^* = \{p \in \mathbb{P} \mid (\exists \kappa \in \mathrm{CARD})(dom(p) = (\kappa + 1) \cdot \theta)\}$ which is clearly a dense subset of $\mathbb{P}$. We will proceed similarly as in the proof of Theorem 4.0.2 and construct an increasing sequence of embeddings $\langle e_i \mid i \in \lambda \rangle$ as follows: For every $u \in \mathbb{U}^*$ and $p \in \mathbb{P}^*$ we define sets

$$A_u = \{q \in \mathbb{U}^* \mid dom(q) = dom(u) + 1 \ \&\ q \restriction dom(u) = u\} \text{ and}$$

$$B_p = \{q \in \mathbb{P}^* \mid dom(q) = dom(p) + \theta \ \&\ q \restriction dom(p) = p\}.$$

Since it is clearly the case that for every $u \in \mathbb{U}^*$ and every $p \in \mathbb{P}^*$, $|A_u| = |B_p| = |J|^{<\lambda}$ we can write $A_u = \{a_u^i \mid i \in J^{<\lambda}\}$ and $B_p = \{b_p^i \mid i \in J^{<\lambda}\}$. Now we let:

$$e_0 = \{\langle 1_{\mathbb{U}^*}, 1_{\mathbb{P}^*} \rangle\},$$

$$e_{\beta+1} = e_\beta \cup \{\langle a_p^i, b_{e_n(p)}^i \rangle \mid i {\in} J^{<\lambda} \ \&\ p {\in} X_\beta\}, \text{ where} X_\beta = e_\beta - \bigcup_{\gamma < \beta} e_\gamma$$

and for limit α

$$e_\alpha = \bigcup_{\beta < \alpha} e_\beta \cup \{\langle a_p^i, b_{e_n(p)}^i \rangle \mid i {\in} J^{<\lambda} \ \&\ p {\in} X_\alpha\}$$

where

$$X_\alpha = \{\bigcup_{i<\alpha} p_i \mid \langle p_i \mid i < \alpha\rangle \text{ is a decreasing sequence of elements from } dom(\bigcup_{\beta<\alpha} e_\beta)\}.$$

Finally we let $e = \bigcup_{\gamma<\lambda} e_\gamma$. By $|A_u| = |B_p|$ for every $u \in \mathbb{U}^*$, $p \in \mathbb{P}^*$ and $X_\alpha \subseteq \mathbb{P}^*$ (by definiton of $\mathbb{P}^*$ and $\mathbb{P}$ being λ-closed), e is clearly an isomorphism between $\mathbb{U}^*$ and $\mathbb{P}^*$ and we are done. □

Corollary 4.2.8. *Let λ be a cardinal and I, J arbitrary sets with $2 \leq |I|, |J| \leq 2^\lambda$. Then $Fn(\lambda^+, I, \lambda^+)$ is forcing equivalent with $Fn(\lambda^+, J, \lambda^+)$.*

Corollary 4.2.9. *Let λ be a cardinal, then $Fn(\lambda^+, 2, \lambda^+)$ adds a surjection from λ^+ to 2^λ. Especially $Fn(\omega_1, 2, \omega_1)$ adds a surjection from ω_1 to 2^ω.*

Proof. By Lemma 4.2.6 $Fn(\lambda^+, 2^\lambda, \lambda^+)$ adds a surjection from λ^+ to 2^λ. By the previous lemma $Fn(\lambda^+, 2, \lambda^+)$ and $Fn(\lambda^+, 2^{<\lambda^+}, \lambda^+) = Fn(\lambda^+, 2^\lambda, \lambda^+)$ are forcing equivalent. □

Lemma 4.2.10. *Let M be a countable transitive model of ZFC, κ, γ cardinals, κ singular and $\mathbb{P} = Fn(\kappa, \gamma, \kappa)^M$. Then if G is a $\mathbb{P}$-generic over M, there exists in $M[G]$ an injective function from κ to $cf(\kappa)^M$.*

Proof. Let $\langle \kappa_i \mid i \in cf(\kappa)^M\rangle$ be an increasing cofinal sequence of κ and assume further that it contains only regular cardinals. Indeed, if there were singular κ_i then we can always replace it by $\kappa_i^+ < \kappa$ since κ is limit. Consider that once we show in M that for every $\alpha \in \kappa$ a set

$$D_\alpha = \{p \in \mathbb{P} \mid (\exists \beta \in cf(\kappa))(dom(q) \supseteq \kappa_{\beta+1} - \kappa_\beta)$$
$$(ot(\{\xi \in \kappa_{\beta+1} - \kappa_\beta \mid q(\xi) = 1\}) = \kappa_\beta + \alpha)\}$$

is dense, we can define $g : \kappa \to cf(\kappa)$ by

$$g(\alpha) = min\{\beta \in cf(\kappa)^M \mid ot(\{\xi \in \kappa_{\beta+1} - \kappa_\beta \mid \bigcup G(\xi) = 1\}) = \kappa_\beta + \alpha\}$$

which is injective since $g(\alpha) = g(\beta)$ implies $\kappa_{g(\alpha)} + \alpha = \kappa_{g(\beta)} + \beta$ and thus $\alpha = \beta$.

Suppose $\alpha \in \kappa$, we show D_α is dense:

Let $p \in \mathbb{P}$ be given then from $|dom(p)| < \kappa$ there is some $\beta \in cf(\kappa)$ such that $|dom(p)| < \kappa_\beta$ and $\alpha < \kappa_\beta$. Let $\gamma = ot(\{\xi \in \kappa_{\beta+1} - \kappa_\beta \mid \xi \in dom(p) \ \& \ p(\xi) = 1\})$ than it holds $\gamma < |\gamma|^+ \leq |dom(p)|^+ \leq \kappa_\beta$ i.e. $\gamma \leq \kappa_\beta + \alpha \leq \kappa_{\beta+1}$.

Now let $\delta \in$ ORD be such that $\gamma + \delta = \kappa_\beta + \alpha$, then from $\gamma + \delta < \kappa_\beta$ we have that $\delta < \kappa_{\beta+1}$. Suppose $\lambda = max(\kappa_\beta, sup\{\xi \in \kappa_{\beta+1} - \kappa_\beta \mid \xi \in dom(p)\})$ then $\lambda < \kappa_{\beta+1}$ by regularity of $\kappa_{\beta+1}$ and by $|dom(p)| < \kappa_\beta$. Let $\langle \delta_i \mid i \in \delta \rangle$ be an increasing sequence of ordinals from $\kappa_{\beta+1} - \lambda$ of order type δ (i.e. $ot(\{\delta_i \mid i \in \delta\}) = \delta$) which exists since $|\kappa_{\beta+1} - \lambda| = \kappa_{\beta+1} > \delta$.

Now let $q \in \mathbb{P}$ be such that $dom(q) = dom(p) \cup (\kappa_{\beta+1} - \kappa_\beta)$ (i.e. $|dom(p)| < \kappa$) and $q(x) = p(x)$ whenever $x \in dom(p)$, $q(x) = 1$ whenever $x = \delta_i$ for some $i \in \delta$ and $q(x) = 0$ otherwise. We have $q \leq p$ by definition of q and $q \in D_\alpha$ by $ot(\{\xi \in \kappa_{\beta+1} - \kappa_\beta \mid q(\xi) = 1\}) = \gamma + \delta = \kappa_\beta + \alpha$. □

To illustrate the importance of support in case of product of forcing notions we introduce the following example. Although $\mathbb{P} = \prod_{i\in\omega} \mathbb{P}_i$ for $\mathbb{P}_i = Fn(\omega, 2, \omega)$ with finite support preserves cardinals since it is isomorphic to $Fn(\omega \times \omega, 2, \omega)$, we can show that if we consider $\mathbb{P}$ with countable support it is not the case:

Lemma 4.2.11. *Let $\mathbb{P}_i = Fn(\omega, 2, \omega)$ for $i \in \omega$ and let $\mathbb{P} = \prod_{n\in\omega} \mathbb{P}_i$ be a full support product forcing. Then $\mathbb{P}$ collapses $\aleph_1$.*

Proof. Let M, a countable transitive model for ZFC, be a ground model and G a $\mathbb{P}$-generic filter over M. We show that $\mathbb{P}$ adds an injection from $({}^\omega 2)^M$ into ω. Let us define for every $i \in \omega$ a sequence $s_i = \bigcup G{\restriction}\{i\}$ and assume the following function $c : \omega \to {}^\omega 2$ defined (in $M[G]$) for every $k \in \omega$ by $c(k)(0) = k$ and

$$c(k)(n+1) = \max\{x \in \omega \mid (\forall j \in \omega) \\ ((c(k)(n) \leq j \leq c(k)(n) + x) \to (s_{n+1}(j) = s_{n+1}(c(k)(n)))\}.$$

Then we can define a coding function $C : \omega \to {}^\omega 2$ by $C(k)(0) = s_0(k)$ and $C(k)(n+1) = s_n(c(k)(n+1))$ for every $k \in \omega$. The meaning of the coding function is following: let $k \in \omega$ be given (i.e. $k = c(k)(0)$), assume a sequence s_0 and let $a_0 = s_0(k)(= C(k)(0))$. Let further $n_1(= c(k)(1))$ be the number of a_0's counted from the k-th position of s_0 upwards until the first non a_0 is reached. Formally let n_1 be the biggest number such that $(\forall n \in \omega)(k \leq n \leq k + n_1 \to s_0(n) = a_0)$.

Whence a_0 is the first number of a sequence coded by k. Since the n_{m+1} is known we can repeat this procedure replacing k by n_{m+1} and s_0 by s_{m+1} to get a_{m+1} and n_{m+2}. It is clear that using this procedure which is equivalent to the above stated codding function we get a sequence $\langle a_i \mid i \in \omega\rangle = \langle C(k)(i) \mid i \in \omega\rangle$ from $({}^{\omega}2)^M$ and it is also clear that every $k \in \omega$ codes exactly one sequence from $({}^{\omega}2)^M$.

Note that it is possible to say in M that $p \in P$ codes $s \in {}^{\omega}2$ according to the above stated coding function. Indeed, we can say that p codes s if there exists $k \in \omega$ such that $C'(k)$ (replace s_i by $p\restriction\{i\}$ in c and C to get c' and C') is correctly defined function i.e. for every $i \in \omega$

$$p\restriction\{i\} \supseteq (\{c'(k)(i), c'(k)(i)+1, \dots, c'(k)(i)+c'(k)(i+1)\} \times \{C'(k)(i)\}) \\ \cup \{\langle c'(k)(i)+c'(k)(i+1)+1, x\rangle\}$$

where $\{x, C'(k)(i)\} = \{0,1\}$. Whence it remains to show that for every $s \in ({}^{\omega}2)^M$ there is some $k \in \omega$ coding s. It is sufficient to argue that for every $s \in ({}^{\omega}2)^M$ a set $D_s = \{p \in \mathbb{P} \mid p \text{ codes } s\}$ is dense in $\mathbb{P}$, then C is surjective and we are done. Let $p \in \mathbb{P}$ be given and let $n_i = max\{k+1 \mid k \in dom(p\restriction\{i\})\}$ then we can define p' by: for every x whenever $s(x) = i$ and $\{i,j\} = \{0,1\}$, $p'\restriction\{x\} = p\restriction\{x\} \cup \{\langle n_x, i\rangle, \langle n_x+1, i\rangle, \dots, \langle n_x+n_{x+1}-1, i\rangle, \langle n_x+n_{x+1}, i\rangle\langle n_x+n_{x+1}+1, j\rangle\}$; clearly $p' \in D_s$ and we are done. □

Summary 4.2.12. *Let $\kappa, \gamma, \lambda, \theta$ be cardinals such that $\lambda \leq \theta \leq \kappa$ then:*

1. *$Fn(\kappa, \gamma, \lambda)$ adds κ many functions from θ to λ in the sense of ground model*
2. *$Fn(\kappa, \gamma, \lambda)$ adds a surjective function from κ to γ*
3. *$Fn(\kappa, \gamma, \kappa)$ adds an injective function from γ to $cf(\kappa)$ in the sense of ground model*
4. *$Fn(\kappa, \gamma, \kappa)$ adds an injective function from κ to $cf(\kappa)$ in the sense of ground model*

4.3 Nice Names

By the Corollary 4.2.3 we have a lower bound on number of subsets that are added by concrete Cohen forcing. To get the upper bound we use argumentation based on counting names and a definition of generic extension. Recall that according to the Definition 2.17 a generic extension is a class of all valuations of names from a ground model.

Definition 4.3.1. *Let $\mathbb{P}$ be a partial order and $V^{\mathbb{P}}$ class of all $\mathbb{P}$-names, then if $\sigma \in V^{\mathbb{P}}$ we say that $\tau \in V^{\mathbb{P}}$ is a nice name for a subset of σ if it is of the form $\bigcup\{\{\pi\} \times A_\pi \mid \pi \in dom(\sigma)\}$, where each A_π is an antichain in $\mathbb{P}$.*

Lemma 4.3.1. *Let M be a countable transitive model for ZFC, $\mathbb{P} \in M$ a partial order and $\sigma, \mu \in M^{\mathbb{P}}$, then there is a nice name $\tau \in M^{\mathbb{P}}$ for a subset of σ such that $1 \Vdash (\mu \subseteq \sigma \rightarrow \mu = \tau)$.*

Proof. For every $\pi \in dom(\sigma)$ let $A_\pi \subseteq \mathbb{P}$ such that:

(1) $(\forall p \in A_\pi)(p \Vdash \pi \in \mu)$
(2) A_π is an antichain in $\mathbb{P}$
(3) A_π is maximal in inclusion with respect to conditions (1) and (2).

By Zorn lemma there exists such A_π for every $\pi \in dom(\sigma)$ and by $\Vdash$ being definable in M, $\{A_\pi \mid \pi \in dom(\sigma)\} \in M$. Whence we can define in M for μ a nice name τ by

$$\tau = \bigcup\{\{\pi\} \times A_\pi \mid \pi \in dom(\sigma)\}$$

Now we show that for every G a $\mathbb{P}$-generic over M, $M[G] \models (\mu_G \subseteq \sigma_G \rightarrow \mu_G = \tau_G)$ which implies $1 \Vdash (\mu \subseteq \sigma \rightarrow \mu = \tau)$ by Theorem 2.24. Let G a $\mathbb{P}$-generic over M be given and $M[G] \models \mu_G \subseteq \sigma_G$, we show that $M[G] \models \mu_G = \tau_G$:

(i) $M[G] \models \tau_G \subseteq \mu_G$:
Fix $x \in \tau_G$ then there is $\langle \pi, p \rangle \in \tau$ such that $x = \pi_G$ and whence $p \in G \cap A_\pi$, but as it is $p \in A_\pi$ it holds by (1) that $p \Vdash \pi \in \mu$. Since $p \in G$, $M[G] \models \pi_G \in \mu_G$ hence $M[G] \models x \in \mu_G$.

(ii) $M[G] \models \mu_G \subseteq \tau_G$:
Fix $x \in \mu_G$ then $x \in \sigma_G$ and there is some $\pi \in dom(\sigma)$ such that $x = \pi_G$. If $\exists p \in A_\pi \cap G$ we have $p \Vdash \pi \in \mu$ and from $p \in G$ and definition of π we can derive that $M[G] \models x \in \tau_G$. However, if there is no p in $A_\pi \cap G$ we have by Lemma 3.1.2 some $p \in G$ such that $(\forall q \in A_\pi)(q \perp p)$. Let $p' \in G$ be such that $p' \Vdash \pi \in \mu$ and let $r \in G$ be a witness for p, p' being compatible in G. Now

$A_\pi \cup \{r\}$ satisfies conditions (1) and (2) which contradicts the maximality of A_π given by (3) and whence $A_\pi \cap G \neq \emptyset$.

$\square$

Lemma 4.3.2. *Assume M is a countable transitive model for ZFC, $\theta, \lambda, \kappa \in \mathrm{CARD}^M$, $\mathbb{P} \in M$ a κ-cc partial order such that $|\mathbb{P}| = \theta$. Then if G be a $\mathbb{P}$-generic over M, $M[G] \models |2^\lambda| \leq ||[\theta]^{<\kappa}|^\lambda|$.*

Proof. At first note that $\mathbb{P}$ preserves $||[\theta]^{<\kappa}|^\lambda|$ as $\mathbb{P}$ is κ-cc. Assume $M[G] \models \dot{x}_G \subseteq \lambda$, by previous lemma there is a nice name $\tau \in M^{\mathbb{P}}$ such that $M[G] \models \dot{x}_G = \tau_G$ and whence it is sufficient to show in M that number of nice names for subsets of λ is at most $||[\theta]^{<\kappa}|^\lambda|$. As $\mathbb{P}$ is κ-cc there exist at most $|[\theta]^{<\kappa}|$ antichains and since every nice name is of the form $\bigcup\{\{\check{\alpha}\} \times A_{\check{\alpha}} \mid \check{\alpha} \in dom(\check{\lambda})\}$, the number of nice names for subsets of λ is at most $||[\theta]^{<\kappa}|^\lambda| = \gamma$. Now let $\langle \tau_i \mid i < \gamma \rangle$ enumerate all such a nice names for subsets of λ and define

$$\pi = \{\langle \langle \check{i}, \tau_i \rangle, 1 \rangle \mid i < \gamma\}$$

then π_G is a function with $dom(\pi_G) = \gamma$, $P(\lambda)^{M[G]} \subseteq rng(\pi_G)$ and whence $M[G] \models |2^\lambda| \leq ||[\theta]^{<\kappa}|^\lambda|$. $\square$

4.4 Properties of Cohen Reals

By section 4.1 we already know that $Fn(\omega, 2, \omega)$ adds ω many new subsets of ω. Since we know further, that ${}^\omega 2$ (and thus ${}^\omega\omega$ too) is isomorphic to the set of real numbers, it might be tempting to know what the properties of new reals are which will be the contain of this section.

Definition 4.4.1. *Let κ be a cardinal, G a $Fn(\omega, 2, \omega)$-generic filter and G' a $Fn(\kappa \times \omega, 2, \omega)$-generic filter. We say that a function g is a Cohen real if $g = \bigcup G$ and that a function g_α is the α-th Cohen real if $g_\alpha = \bigcup G' \upharpoonright \{\alpha\} \times \omega$. The same is defined for ω replaceing '2'.*

For arbitrary functions f, g from ω to ω recall the notation of $<_{FIN}, \leq_{FIN}$ and $=_{FIN}$ defined (for $\triangle \in \{<, \leq, =\}$) by: $f \triangle_{FIN} g$ if $f(n) \triangle g(n)$ for all but finitely many n's.

Definition 4.4.2. *Let M, N be any countable transitive models for ZFC such that $M \subseteq N$. A function $f \in {}^{\omega}\omega \cap N$ is:*

(i) a bounded real if $f \leq_{FIN} g$ for some $g \in {}^{\omega}\omega \cap M$, otherwise it is unbounded.

(ii) a dominating real if $g <_{FIN} f$ for every $g \in {}^{\omega}\omega \cap M$.

(iii) an eventually different if the set $\{n \mid g(n) = f(n)\}$ is finite for each $g \in {}^{\omega}\omega \cap M$.

(iv) an independent real if both $rng(f) \cap X$ and $X - rng(f)$ are infinite for each $X \in [\omega]^{\omega} \cap M$.

Note that if g witnesses that f is bounded then $f <_{FIN} g'$ for g' defined by $g'(n) = g(n) + 1$ which is in ${}^{\omega}\omega \cap M$ if and only if g is and hence $\leq_{FIN}$ in definition of bounded real can by replaced by $<_{FIN}$. Observe further that every dominating real is eventually different and unbounded.

Lemma 4.4.3. *$Fn(\omega, 2, \omega)$ adds an independent real and an unbounded real but does not add an eventually different real and hence a dominating real.*

Proof. We show that $Fn(\omega, \omega, \omega)$ adds an unbounded real and argue that $Fn(\omega, 2, \omega)$ does too since $Fn(\omega, \omega, \omega)$ is forcing equivalent with $Fn(\omega, 2, \omega)$. Let G be a $Fn(\omega, \omega, \omega)$-generic filter, g a Cohen real, and f a fuction from ω to ω in ground model. Then for every $n_0 \in \omega$ a set

$$D_f^{n_0} = \{p \in Fn(\omega, \omega, \omega) \mid (\exists n > n_0)(n \in dom(p))(p(n) > f(n))\}$$

is clearly dense in $Fn(\omega, \omega, \omega)$ and whence by genericity of G, g is unbounded.

To show that $Fn(\omega, 2, \omega)$ adds an independent real we argue as follows: Let X, an infinite subset of ω in ground model, be given. It is sufficient to show that above every $n \in \omega$ there exist some $n_0, n_1 \in X$ such that $n_0 < n_1$, $g(n_0) = 0$ and $g(n_1) = 1$. Indeed, we define an independent real f inductively by $f_0 = \{\langle 0, 0\rangle\}$, $f_{n+1} = f_n \cup \{\langle n+1, m\rangle\}$ where $m \in X$ is any number strictly above $f_n(n)$ with $g(m) = 0$ and finally we let $f = \bigcup_{n\in\omega} f_n$. To show there exist such $n_0, n_1 \in X$ above every n it is sufficient to argue that a set

$$D_n^X = \{p \in Fn(\omega, 2, \omega) \mid (\exists n_0, n_1 \in X - n)$$
$$(n_0, n_1 \in dom(p) \ \&\ p(n_0) = 0 \ \&\ p(n_1) = 1)\}$$

is dense in $Fn(\omega, 2, \omega)$. However, this is obvious since the conditions in $Fn(\omega, 2, \omega)$ are finite and can be always extended into the element of D_n^X.

To finish the proof we show that $Fn(\omega, 2, \omega)$ does not add an eventually different real: Suppose for a contradiction $\dot{f}$ is a name for eventually different real i.e. there is some $p \in Fn(\omega, 2\omega)$ such that for all $g \in {}^{\omega}\omega$, $p \Vdash (\exists n_0)(\forall n \geq n_0)(\dot{f}(n) \neq \check{g}(n))$. Let $\{p_i \mid i \in \omega\} = Fn(\omega, 2, \omega)$, then below every p_i there is some q_i such that $q_i \Vdash \dot{f}(\check{i}) = \check{y}_i$ for some $y_i \in \omega$. It is sufficient to define a function g by $g(i) = y_i$ and argue that since there is infinitely many conditions below p, there is some $i > n_0$ for every $n_0 \in \omega$ such that $p_i \leq p$ and whence there is some $q_i \leq p$ such that $q_i \Vdash (\exists i \geq \check{n}_0)(\dot{f}(i) = \check{g}(i))$ by definition of g and choice of i which contradicts $p \Vdash (\exists n_0)(\forall n \geq n_0)(\dot{f}(n) \neq \check{g}(n))$. $\square$

5 Applications to Continuum function

After section 3 and 4 we know enough to produce consistency results about CH and GCH.

5.1 Cohen's Result

The first consistency result we are going to show is generalised version of the Cohen's consistency result about CH. This result was for the first time published in [Sol63] immediately after Cohen's result. It says that not only $|2^\omega|$ can be of arbitrary size but that also $|2^\lambda|$ for some regular λ's can be.

Theorem 5.1.1. *Assume that following hold in M, countable transitive model ZFC: $\lambda < \kappa$, $|2^\lambda| \leq \kappa$, λ is regular, $2^{<\lambda} = \lambda$ and $\kappa^\lambda = \kappa$. Let $\mathbb{P} = Fn(\kappa \times \lambda, 2, \lambda)^M$ and G be a $\mathbb{P}$ -generic over M, then $M[G] \models 2^\lambda = \kappa$.*

Proof. First observe that $\mathbb{P}$ preserves cardinals by Corollary 4.1.3 and by Corollary 4.2.3 it holds that $M[G] \models 2^\lambda \geq \kappa$. From the upper bound on number of nice names for subsets of λ given in Lemma 4.3.2 there is at most $||[\mathbb{P}]^{<\lambda}|^\lambda|$ subsets of λ in $M[G]$. Finally from calculating $|\mathbb{P}| \leq |[\kappa \times \lambda \times 2]^{<\lambda}| = |[\kappa]^{<\lambda}| = \kappa$ we have $|[\mathbb{P}]^{<\lambda}|^\lambda = |[\kappa]^{<\lambda}|^\lambda = \kappa^\lambda = \kappa$ and hence $M[G] \models 2^\lambda \leq \kappa$ which finishes the proof. □

The previous theorem gives us possibility to find a model of ZFC where is GCH violated by concrete values of one regular cardinal exponent. However, there is a possibility that if we start with suitable model of ZFC and work carefully, we are able to to violate GCH by concrete values on finitely many regular cardinal exponents. This fact is stated in the following theorem:

Theorem 5.1.2. *Let M be a countable transitive model for ZFC and assume in M: $\langle \lambda_i \mid i < n \rangle$ is a strictly decreasing sequence of regular cardinals such that $(\forall i < n)(2^{<\lambda_i} = \lambda_i)$ and $\langle \kappa_i \mid i < n \rangle$ is a decreasing sequence of cardinals such that $(\forall i < n)(|2^{\lambda_i}| \leq \kappa_i \;\&\; \lambda_i < \kappa_i \;\&\; \kappa_i^{\lambda_i} = \kappa_i)$ and Then there exists $N \supseteq M$ countable transitive model for ZFC such that $N \models \ (\forall i < n)(|2^{\lambda_i}| = \kappa_i)$ and 'being cardinal', 'being regular cardinal' is absolute between N and M.*

Proof. We show using Theorem 5.1.1 n-times that there is a sequence $\langle M_i \mid i < n \rangle$ of generic extensions of M and a sequence $\langle \mathbb{P}_i \mid i < n \rangle$ where $\mathbb{P}_i = Fn(\kappa_i \times \lambda_i, 2, \lambda_i)$ of partial orders in M such that $M_0 = M$, $M_{n-1} = N$ and $(\forall i < n-1)(M_{i+1} = M_i[G_i])$ where G_i is $\mathbb{P}_i$-generic over M_i.

For $i = 0$ we have by Theorem 5.1.1 that $M_1 = M_0[G_0] \models |2^{\lambda_0}| = \kappa_0$, $\mathrm{CARD}^{M_0} = \mathrm{CARD}^{M_1}$ and cofinalities are preserved hence λ_i is regular for all $i < n$. Since $\mathbb{P}_)$ is by Lemma 4.1.2 λ_0-closed it holds by Corrolary 3.3.13 that $(2^{\lambda_i})^{M_1} = (2^{\lambda_i})^{M_0}$ for all $0 < i < n$ and hence $|2_i^{\lambda}| \leq \kappa_i$ and $2^{<\lambda_i} = \lambda_i, \kappa_i^{\lambda_i} = \kappa_i, \lambda_i < \kappa_i$ for all $0 < i < n$. At the end from that all $(\mathbb{P}_i)^{M_0} = (\mathbb{P}_i)^{M_1}$ for all $0 < i < n$.

For construction of $i + 1$ argumentation is the same as in the case for $i = 0$. $\square$

5.2 Violating GCH

With the Theorem 5.1.2 we can finally start thinking about what we are able to force about GCH, but before we start, we would like to show an example of reckless use of forcing. To avoid fruitless ideas we should always bear in our minds that we are just producing models of ZFC. This means everything provable in ZFC must holds in these models too as illustrate the following example:

Question 5.2.1. *Is it possible to find a generic extension N of M,countable transitive model for ZFC, such that $N \models [2^{\aleph_0} = \aleph_2, 2^{\aleph_3} = \aleph_7, GCH$ on the rest of regular cardinals$]$?*

Answer. In this case the answer is simply no. Although we could from M construct according to the Theorem 5.1.2 model N where $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_3} = \aleph_7$ holds, since $ZFC \vdash \aleph_\alpha < \aleph_\beta \rightarrow 2^{\aleph_\alpha} \leq 2^{\aleph_\beta}$ the same must holds in every model of ZFC and hence also in model N. However, in N holds $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_3} = \aleph_7$ which implies $N \models \aleph_5 < \aleph_7 = 2^{\aleph_3} \leq 2^{\aleph_4}$ but since $\aleph_5$ is regular and $2^{\aleph_4} \neq \aleph_5$ in N, GCH does not hold on the rest of regular cardinals. $\square$

Note however, that if we replace 'GCH on the rest of regular cardinals' by 'GCH on the rest of regular cardinals when it is possible' in the previous question the answer would be by Theorem 5.1.2 'yes'. In other words, we are able to find a model of ZFC where GCH does not hold only in finitely many cases. Now we show an example of formula which can be forced but not with Cohen forcing while using arbitrary models of ZFC+GCH.

Question 5.2.2. *Supose M is a model for ZFC. Is it possible to use Cohen forcing to find a generic extension of M which satisfies $2^{\aleph_\omega} = \aleph_{\omega+2}$ and GCH below $\aleph_\omega$?*

We divide investigation of this question into two cases:

(i) We try to start with M, countable transitive model for ZFC+ 'GCH below $\aleph_\omega$', and find its generic extension N where $2^{\aleph_\omega} = \aleph_{\omega+2}$ and GCH below $\aleph_\omega$ holds.

(ii) We try to start with M, countable transitive model for ZFC+'¬GCH below $\aleph_\omega$', and find its generic extension N where $2^{\aleph_\omega} = \aleph_{\omega+2}$ and GCH below $\aleph_\omega$ holds.

Case (i): Let M, N be as stated. Recall that first from all, the forcing notion we are going to use must preserves every equality of the form $2^{\aleph_n} = \aleph_{n+1}$ for every $n \in \omega$ i.e. :

(a) the forcing notion should not add more than $\aleph_{n+1}$ many new subsets of $\aleph_n$ for every $n \in \omega$ and

(b) the forcing notion should preserves cardinals $< \aleph_\omega$.

Since $2^{\aleph_\omega} = \aleph_{\omega+2}$ must hold in N, we get two more conditions:

(c) the forcing notion should add exactly $\aleph_{\omega+2}$ many new subsets of $\aleph_\omega$ in sense of generic extension whenever $(2^{\aleph_\omega} = \aleph_{\omega+1})^M$ and

(d) the forcing notion should add surjection from $(\aleph_{\omega+2})^M$ to $(2^{\aleph_\omega})^N$ and further ensure $\aleph^M_{\omega+2} = \aleph^N_{\omega+2}$ whenever $(2^{\aleph_\omega} > \aleph_{\omega+2})^M$.

Now suppose $2^{\aleph_\omega} = \aleph_{\omega+1}$ holds in M. By condition (c), Corollary 4.2.2 and Lemma 4.3.2 we need to use $\mathbb{P} = Fn(\kappa, \gamma, \lambda)$ for $\kappa = \aleph_{\omega+2}$, $\lambda = \aleph_\omega$. Indeed, if $\lambda < \aleph_\omega$ then since $\mathbb{P}$ adds $\aleph_{\omega+2}$ many new subsets of λ^+, condition (a) is violated and if $\aleph_{\omega+2} \geq \lambda > \aleph_\omega$ then $\mathbb{P}$ adds no new subsets of $\aleph_\omega$.

According to these, it is sufficient to show that forcing with $\mathbb{P} = Fn(\aleph_{\omega+2}, \gamma, \aleph_\omega)$ for arbitrary cardinal γ violate GCH below $\aleph_\omega$ which shows that this part of the case (i) is not solvable by Cohen forcing: Let $\gamma \in \mathrm{CARD}^M$ be arbitrary. We know by Claim 4.0.3 that $Fn(\aleph_\omega, \gamma, \aleph_\omega)$ is completely embeddable into $Fn(\aleph_{\omega+2}, \gamma, \aleph_\omega)$ and by Lemma 4.2.10 that $Fn(\aleph_\omega, \gamma, \aleph_\omega)$ collapses $\aleph_\omega$ to ω. Whence $Fn(\aleph_{\omega+2}, \gamma, \aleph_\omega)$ collapses $\aleph_\omega$ to ω which means it does not preserves GCH below $\aleph_\omega$.

If we suppose $2^{\aleph_\omega} > \aleph_{\omega+2}$ holds in M, then using of $Fn(\aleph_{\omega+2}, 2^{\aleph_\omega}, \aleph_{\omega+2})$ ensure by Lemma 4.2.6 $\aleph_{\omega+2} = 2^{\aleph_\omega}$ and GCH below $\aleph_\omega$ in generic extension, whence this part of the case (i) can be solved using Cohen forcing.

Case (ii): Let M, N be as stated. Recall we have to force $2^{\aleph_\omega} = \aleph_{\omega+2}$ at first and after that GCH below $\aleph_\omega$ otherwise it would be the case (i). Suppose for now, that we are able to find a generic extension N_0 of M using Cohen forcing such that $2^{\aleph_\omega} = \aleph_{\omega+2}$ and '$\neg$GCH below $\aleph_\omega$' hold in N_0. To get the desired model N we have to:

(a) add a surjection from $\aleph_{n+1}^{N_0}$ to $(2^{\aleph_n})^N$ and ensure $\aleph_{n+1}^{N_0} = \aleph_{n+1}^N$, whenever $2^{\aleph_n} \geq \aleph_{n+2}$ in N_0.

Notice, that if there is only a finite number of cases in N_0, when the surjection from (a) need to be add and n is the biggest such that surjection from $\aleph_{n+1}^{N_0}$ to $(2^{\aleph_n})^N$ need to be add, then GCH above $\aleph_n$ cannot hold in N. Indeed, whenever we use forcing that adds finitely many surjections from (a) we collapse cardinals above $\aleph_n$ and whence violate GCH above $\aleph_n$. Whence it must be the case that the desired forcing notion adds countably many surjections from (a). Assume $I \subseteq \omega$ is an infinite set of all $i's$ such that a surjection from $\aleph_{i+1}^{N_0}$ to $(2^{\aleph_i})^N$ is added. Since only possible Cohen forcing adding a surjection from $\aleph_{n+1}^{N_0}$ to $(2^{\aleph_n})^N$ and ensuring $\aleph_{n+1}^{N_0} = \aleph_{n+1}^N$ is $Fn(\aleph_{i+1}, 2^{\aleph_i}, \aleph_{i+1})$ it is sufficient to consider the following two cases:

(1) we use $\mathbb{P} = \prod_{i \in I} Fn(\aleph_{i+1}, 2^{\aleph_i}, \aleph_{i+1})$ with finite support or

(2) we use $\mathbb{P} = \prod_{i \in I} Fn(\aleph_{i+1}, 2^{\aleph_i}, \aleph_{i+1})$ with full support.

Before we analyse both cases, we make a definition first.

Definition 5.2.3. *Let κ be a cardinal, $\alpha \in \kappa$ and $p \in Fn(\kappa, 2, \kappa)$. We say that 'p codes α' if $\alpha \subseteq dom(p)$ and $(\forall \beta \in \alpha)(p(\beta) = 1)$ & $p(\alpha) = 0$*

ad. (1): We show that $\mathbb{P}$ adds a surjection from $\aleph_\omega$ to ω:
Assume G is a $\mathbb{P}$-generic over N_0, once we show that

$$D_\alpha = \{p \in \mathbb{P} \mid (\exists n \in I)(p(n) \text{ codes } \alpha)\}$$

is dense in $\mathbb{P}$, we can define a function $g : \aleph_\omega \to I$ by $g(\alpha) = min\{n \in I \mid \bigcup G(n) \text{ codes } \alpha\}$ for every $\alpha \in \aleph_\omega$. Then $|rng(g)| = \omega$ since g is clearly injective and whence there exists a surjective map from $\aleph_\omega^{N_0}$ to ω in N. To show D_α is dense argue as follows: Suppose $\alpha \in \aleph_\omega$ and $p \in \mathbb{P}$ are given. Since $|supt(p)| < \omega$, there exists $i \in I$ such that $\alpha \in \aleph_{i+1}$ and $p(i) = 1_i$. It is immediate that there is some $p' \in \mathbb{P}$: $p'(n) = p(n)$ if $n \neq i$ and $p'(i) = q$ where $q \in Fn(\aleph_{i+1}, 2^{\aleph_i}, \aleph_{i+1})$ and $q = \{\langle \beta, 1\rangle \mid \beta \in \alpha\} \cup \{\langle \alpha, 0\rangle\}$ i.e. q codes α i.e. $p'(i)$ codes α and whence $p' \in D_\alpha$.

ad. (2): We prove the following theorem given in [Hon10] and conclude that $\mathbb{P}$ collapses $\aleph_{\omega+2}$ to $\aleph_{\omega+1}$:

Theorem 5.2.4. *Assume in M, countable transitive model for ZFC, κ is a strong limit cardinal of cofinality ω and $2^\kappa = \kappa^{++}$. Let further $\langle \lambda_i \mid i < \omega\rangle$ be a sequence of regular cardinals cofinal in κ, then a product forcing notion $\mathbb{P} = \prod_{n\in\omega} Fn(\lambda_i, 2, \lambda_i)$ with full support collapses 2^κ to κ^+.*

To proof this theorem we use a Theorem of Shelah, see for instance [Jech02] Theorem 24.8, which we leave without proof.

Definition 5.2.5. *Let α, β be ordinals and f, g be functions from α to β then we define an ordering $<_{FIN}$ by:*

$$f <_{FIN} g \text{ if and only if } \{\xi \in \alpha \mid f(\xi) \geq g(\xi)\} \text{ is finite.}$$

Theorem 5.2.6. *(Shelah) Let κ be a strong limit cardinal of cofinality ω. There exists a sequence $\langle \lambda_i \mid i < \omega\rangle$ of regular cardinals from κ which is cofinal in κ and a sequence $\langle f_\xi \mid \xi \in \kappa^+\rangle$ of elements from $\prod_{n\in\omega} \lambda_i$ which is cofinal in ordering $<_{FIN}$.*

Proof of Theorem 5.2.4. Let $\langle \lambda_i \mid i < \omega\rangle$ and $\langle f_\xi \mid \xi < \kappa^+\rangle$ be sequences given by Theorem 5.2.6. Let G be a $\mathbb{P}$-generic over M and for every $i \in \omega$, G_i a $Fn(\lambda_i, 2, \lambda_i)$-generic over M such that $G = \prod_{i\in\omega} G_i$ which is possible by Lemma 3.4.3. We define a function $h : \kappa^+ \to \prod_{n\in\omega} \lambda_i$ by

$$h(\xi) = \{\langle\alpha_i \mid i < \omega\rangle \in \prod_{n\in\omega} \lambda_i \mid (\exists n_0 \in \omega)(\forall n \geq n_0)$$
$$(\bigcup G_n \restriction [f_\xi(n), f_\xi(n) + \xi + 1) \text{ codes } \alpha_n)\}.$$

Assuming κ is strong limit, $|\prod_{n\in\omega} \lambda_i| = \kappa^\omega = 2^\kappa$ by $2^\kappa \leq 2^{\sum_{i\in\omega}\lambda_i} = \prod_{i\in\omega} 2^\kappa \leq \prod_{i\in\omega} \kappa = \kappa^\omega$ and $\kappa^\omega \leq \kappa^\kappa = 2^\kappa$. Thus it is sufficient to show that $(\forall\xi \in \kappa^+)(|h(\xi)| \leq \kappa)$ and $\bigcup_{\xi\in\kappa^+} h(\xi) = \prod_{n\in\omega} \lambda_i$. Indeed, then $\kappa^+ = \kappa^+ \times \kappa \geq |\bigcup_{\xi\in\kappa^+} h(\xi)| = |\prod_{n\in\omega} \lambda_i| = 2^\kappa$ and we are done.

To show $(\forall\xi \in \kappa^+)(|h(\xi)| \leq \kappa)$ let

$$[n_0] = \{\langle\alpha_i \mid i < \omega\rangle \in h(\xi) \mid n_0 \text{ is the least number such that}$$
$$(\forall n \geq n_0)(\bigcup G_n \restriction [f_\xi(n), f_\xi(n) + \xi + 1) \text{ codes } \alpha_n)\}.$$

Observe that $(\forall s \in [n])(s = s_n{}^\frown\langle\alpha_i \mid n \leq i < \omega\rangle)$ for some fix $\langle\alpha_i \mid n \leq i < \omega\rangle$ and suitable s_n. Since $|s_n| = n$ it is easily seen that $|[n]| = \kappa^n = \kappa$ for every $n \in \omega$ and hence $|h(\xi)| = |\bigcup_{n\in\omega}[n]| \leq \kappa$.

Now we show that $\bigcup_{\xi\in\kappa^+} h(\xi) = \prod_{n\in\omega} \lambda_i$:
For every $p \in \mathbb{P}$ we define a set

$$Code_\xi(p) = \{\langle\alpha_i \mid i < \omega\rangle \in \prod_{n\in\omega} \lambda_i \mid (\exists n_0 \in \omega)(\forall n \geq n_0)$$
$$(p(n) \restriction [f_\xi(n), f_\xi(n) + \xi + 1) \text{ codes } \alpha_n\}.$$

Let $s = \langle\beta_i | i \in \omega\rangle \in \prod_{n\in\omega} \lambda_i$ be given, we show that a set $D_s = \{p \in \mathbb{P} \mid (\exists\xi \in \kappa^+) (s \in Code_\xi(p))\}$ is dense which implies $s \in h(\xi)$ for some ξ. Let $p \in \mathbb{P}$ be given. Let $\gamma_i < \lambda_i$ be an upper bound of $rng(p(i))$, then a function $g : \omega \to \kappa$ defined by $g(i) = \gamma_i$ is in $\prod_{i\in\omega} \lambda_i$. By Theorem 5.2.6 (i.e. by cofinality of $\langle f_\xi \mid \xi \in \kappa^+\rangle$ with respect to $<_{FIN}$) there exists some $\xi \in \kappa^{++}$ such that $g <_{FIN} f_\xi$. Then a function p' defined by $\forall i \in \omega$: $p'(i) = p(i) \cup \{\langle\delta, 1\rangle \mid \delta \in [f_\xi(i) + \beta_i + 1)\} \cup \{\langle(f_\xi(i) + \beta_i + 1), 0\rangle\}$ is an element of D_s below p. □

Note. *Let M be a model for ZFC, recall a cardinal κ is A-hypermeasurable if there exists an elementary embedding $j : V \to M$ with critical point κ (i.e $j\restriction\kappa = id$ and $j(\kappa) > \kappa$), for which $A \in M$. Recall further $\mathcal{P}_3(\kappa) = \mathcal{P}(\mathcal{P}(\mathcal{P}(\kappa)))$. As was shown by James Cummings [Cum92], it is possible to find a generic extension of ZFC $+\mathcal{P}_3\kappa$ - hypermeasurable, where GCH fails on $\aleph_\omega$ but holds below.*

After we introduced two examples of formula which cannot be forced by trivial use of Cohen forcing, we show an interesting example of formula which is not as trivial but is forcable by Cohen forcing.

Since we are by Theorem 5.1.2 able to find model of ZFC where GCH fails on arbitrary finitely many $\aleph_n$ smaller than $\aleph_\omega$ there rise a question whether we are able to find a model where GCH fails on each (or infinitely many) $\aleph_n$ smaller than $\aleph_\omega$. Once we know that $Fn(\aleph_{n+2} \times \aleph_n, 2, \aleph_n)$ ensure $2^{\aleph_n} = \aleph_{n+2}$ it seems intuitively that full support product forcing $\mathbb{P} = \prod_{n\in\omega} Fn(\aleph_{n+2} \times \aleph_n, 2, \aleph_n)$ might force $(\forall n \in \omega)(2^{\aleph_n} = \aleph_{n+2})$ which is a content of the following theorem:

Theorem 5.2.7. *Let M be a countable transitive model for ZFC+GCH, $f : \omega \to \omega$ a strictly increasing map with $f(0) > 0$ and $\mathbb{P} = \prod_{n\in\omega} Fn(\aleph_{f(n)}, 2, \aleph_n)^M$ a full support product forcing. Then if G is a $\mathbb{P}$-generic over M, $M[G] \models (\forall n \in \omega)(2^{\aleph_n} = \aleph_{f(n)})$.*

Proof. At first to make a notation of the proof easier we introduce following short cuts:

$$\mathbb{P}_n = Fn(\aleph_{f(n)}, 2, \aleph_n), \quad \mathbb{P}_k^+ = \prod_{k<n\in\omega} Fn(\aleph_{f(n)}, 2, \aleph_n), \quad \mathbb{P}_k^- = \prod_{n\leq k, n\in\omega} Fn(\aleph_{f(n)}, 2, \aleph_n).$$

In the first part of this proof we show that $G_n = \{p(n) |\ p \in G\}$ is $\mathbb{P}_n$- generic over M whenever G is a $\mathbb{P}$-generic over M. In a next part we show that $\mathbb{P}$ preserves cardinals below $\aleph_\omega$ including $\aleph_\omega$ and in the last part we conclude that $M[G] \models (\forall n \in \omega)(2^{\aleph_n} = \aleph_{f(n)})$ whenever G is a $\mathbb{P}$-generic over M.

Let G be a $\mathbb{P}$-generic over M and $n \in \omega$ be given, then $G_n = \{p(n) |\ p \in G\}$ is $\mathbb{P}_n$- generic over M: We define a function $i_n : \mathbb{P}_n \to \mathbb{P}$ by $i_n(p) = q$ if $q(x) = p$ whenever $x = n$ and $q(x) = 1_x$ otherwise, which is by the same argumentation as in Lemma 3.4.2 a complete embedding. Whence by Lemma 3.2.7 $G_n = i_n''^{-1}G$ is a $\mathbb{P}_n$-generic over M.

Now we show that $\mathbb{P}$ preserves cardinals below $\aleph_\omega$. Let $n \in \omega$ be given, then:

(i) $\mathbb{P}_n^+$ is $\aleph_{n+1}$-closed: Let $\aleph_j < \aleph_{n+1}$ and $\langle p_\alpha \mid \alpha < \aleph_j \rangle$ be arbitrary decreasing sequence of elements from $\mathbb{P}_n^+$. Define

$$p : \omega - (n+1) \to \bigcup_{n<k\in\omega} \mathbb{P}_n \times \{n\} \text{ by } p(k) = \bigcup_{\alpha<\aleph_j} p_\alpha(k).$$

Since $|p(k)| = |\bigcup_{\alpha<\aleph_j} p_\alpha(k)| < max(\aleph_j, \aleph_k) = \aleph_k < \aleph_{n+1}$ we have $p(k) \in \mathbb{P}_k$ for all $k \in (\omega - (n+1))$ and hence $p \in \mathbb{P}_n^+$. It is straightforward according to the definition of p that $(\forall \alpha < \aleph_j)(p \le p_\alpha)$.

(ii) P_n^- is $\aleph_{n+1}$-Knaster and hence $\aleph_{n+1}$-cc:
Since for every $i \in \omega$ it holds that $\mathbb{P}_i$ is by Lemma 4.1.1 $\aleph_{i+1}$-Knaster whenever $2^{<\aleph_i} = \aleph_i$ we have by Lemma 3.4.5 that $\mathbb{P}_n^-$ is $\aleph_{n+1}$-Knaster and hence $\aleph_{n+1}$-cc as $2^{<\aleph_i} = \aleph_i$ for all $i \le n$ by GCH.

(iii) $\mathbb{P}$ preserves $\aleph_n$: Let K be $\mathbb{P}$-generic over M and define $i_- : \mathbb{P}_{n-1}^- \to \mathbb{P}_{n-1}^- \times \mathbb{P}_{n-1}^+$ and $i_+ : \mathbb{P}_{n-1}^+ \to \mathbb{P}_{n-1}^- \times \mathbb{P}_{n-1}^+$ by $i_-(p) = \langle p, 1_+ \rangle$ respectively $i_+(p) = \langle 1_-, p \rangle$ which are by Claim 3.4.2 complete embedings. Let further j be any isomorphism from $\mathbb{P}$ to $\mathbb{P}_{n-1}^- \times \mathbb{P}_{n-1}^+$ and $G^- = i_-''^{-1}(j''K)$ and $G^+ = i_+''^{-1}(j''K)$. As $j''K = G^- \times G^+$ is $\mathbb{P}_0 \times \mathbb{P}_+$ generic over M and $M[K] = M[j''K]$ by isomorphicity of j, we derive by Lemma 3.4.4 that $M[K] = M[G^+][G^-]$. Now from $\mathbb{P}_{n-1}^+$ being $\aleph_n$-closed by (i) we derive $\aleph_n^M = \aleph_n^{M[G^+]}$. Since $\mathbb{P}_{n-1}^+$ is $\aleph_n$-closed and GCH holds in M, $\mathbb{P}_{n-1}^+$ preserves for every $i \le n$ the equality of the form $2^{<\aleph_i} = \aleph_i$ and it follows by (ii) that ($\mathbb{P}_{n-1}^-$ is $\aleph_n$-cc)$^{M[G^+]}$ and whence $\aleph_n^{M[G^+]} = \aleph_n^{M[G^+][G^-]}$ i.e. $\aleph_n^M = \aleph_n^{M[K]}$.

(iv) $\mathbb{P}$ preserves $\aleph_\omega$: Since $\mathbb{P}$ preserves $\aleph_n$ for every $n \in \omega$ it preserves by Lemma 3.3.1 $\aleph_\omega$.

Let G be a $\mathbb{P}$-generic over M and let $n \in \omega$ be given. We show that $(2^{\aleph_n} = \aleph_{f(n)})^{M[G]}$ which finishes the proof. Since $G_n = \{p(n) | \ p \in G\}$ is $\mathbb{P}_n$ - generic over M, $\bigcup G(n) = \bigcup G_n$ is by Lemma 4.2.1 a function adding $\aleph_{f(n)}^M$ new subsets of $\aleph_n^M$. By (iii) $\aleph_{f(n)}^M = \aleph_{f(n)}^{M[K]}$ and $\aleph_n^M = \aleph_n^{M[K]}$ and whence $(2^{\aleph_n} \ge \aleph_{f(n)})^{M[G]}$. To show $(2^{\aleph_n} \le \aleph_{f(n)})^{M[G]}$ we follow the notation and argumentation of (iii) replacing n by $n+1$. As it is $\mathbb{P}_n^M = \mathbb{P}_n^{M[G^+]}$, $\mathbb{P}_n^{M[G^+]}$ is $\aleph_{n+1}^{M[G^+]}$-cc and whence by $|\mathbb{P}_n| = \aleph_{f(n)}^{M[G^+]}$ there exist at most $(\aleph_{f(n)}^{<\aleph_n})^{\aleph_n} = \aleph_{f(n)}$ nice names for subsets of $\aleph_n$ in the sense of $M[G^+]$. Finally since $\mathbb{P}$ preserves cardinals below $\aleph_\omega$, $M[G] = M[G^+][G^-] \models 2^{\aleph_n} \le \aleph_{f(n)}$ and we are done. □

6 Minimality of Forcing Extensions

Definiton 1. *A generic filter G is minimal over the ground model M if for every set of ordinals X in $M[G]$, either $X \in M$ or $G \in M[X]$.*

We make a short note to the minimality of forcing extensions in this section showing that extensions produced by Cohen forcing are not minimal and defining a forcing notion whose generic extensions are minimal.

6.1 Non-minimality of Cohen Extensions

Definition 1. *Let g be a set and M a countable transitive model of ZFC. Then we define $M[g]$ to be the least countable transitive model of ZFC such that $\mathrm{ORD} \cap M[g] = \mathrm{ORD} \cap M$, $M \subseteq M[g]$ and $g \in M[g]$.*

Lemma 1. *A Cohen real is not minimal over the ground model.*

Proof. Let M be a countable transitive model for ZFC, $\mathbb{P} = Fn(\omega, \omega, \omega)^M$, G a $\mathbb{P}$-generic filter over M and g a Cohen real. Due to an isomorphism of $Fn(\omega, \omega, \omega)$ and $Fn(\omega, \omega, \omega) \times Fn(\omega, \omega, \omega)$, Lemma 3.4.3 and Lemma 3.4.4 there exists G_0 a $Fn(\omega, \omega, \omega)$-generic over M and G_1 a $Fn(\omega, \omega, \omega)$-generic over $M[G_0]$ such that $M[G] = M[G_0][G_1]$. Since a generic filter G can be reconstructed from the Cohen real g by $G = \{p \in \mathbb{P} \mid p \subseteq g\}$ and the same holds for G_0, G_1, we get $M \subsetneq M[g_0] \subsetneq M[g_0][g_1] = M[g]$ and whence for g_0 (identified with a set of ordinals): $g_0 \notin M$ and $g \notin M[g_0]$. □

Since the Cohen extensions are not minimal, there rise a question, whether there exists any forcing notion whose extensions are minimal. The next part shows there exists such, however, it also shows that the forcing notion becomes more complicated while demanding minimality.

6.2 Sacks Forcing

Definition 6.2.1. *Let λ be a cardinal. We say that a set $p \in {}^{<\lambda}2$ is a λ-tree if it satisfies $(\forall t \in p)(\exists t' \in p)(t \subsetneq t')$ and whenever $t \in p$ and $s = t \restriction \alpha$ for some $\alpha \in \lambda$, then $s \in p$.*

We say further that a set $p \in^{<\lambda} 2$ is a perfect tree at λ if it is a λ-tree whenever $t \in p$ there exists $s \in p$ such that $t \subseteq s$ and $s^\frown 0, s^\frown 1 \in p$. Finally we call nodes the elements of a tree.

Definiton 6.2.2. *Let λ be a cardinal. We define a set*

$$S(\lambda) = \{p \in {}^{<\lambda}2 \mid \ p \textit{ is a perfect tree at } \lambda\}$$

and say that $S(\lambda)$ ordered by inclusion is a Sacks forcing at λ.

Note that $f = \bigcup\{s \mid (\forall p \in G)(s \in p)\}$ called a *Sacks real* is a function from ω to 2 whenever G is a $S(\omega)$-generic over a ground model. Indeed, since G is a filter it is the case that $\forall p \in G : \langle 0 \rangle \in p$ or $\forall p \in G : \langle 1 \rangle \in p$ and further by induction: if there exists a sequence s of length n which is a node of every $p \in G$ then $s^\frown 0$ or $s^\frown 1$ is a node of every tree in G otherwise G would not be a filter or would include nonperfect trees. Since $G = \{p \in S(\omega) \mid (\forall n \in \omega)(f{\restriction}n \in p)\}$ it holds that $M[f] = M[G]$.

Definition 6.2.3. *Let p be a tree. A node $s \in p$ is a splitting node if both $s^\frown 0, s^\frown 1$ are in p. If p is a ω-tree, we say that $s \in p$ is n-th splitting node if there are exactly n splitting nodes t in p such that $t \subseteq s$.*

Definition 6.2.4. *Let $p, q \in S(\omega)$. For every $n \in \omega$ define an ordering $\leq_n$ by: $p \leq_n q$ if and only if $p \leq q$ and if $s \in q$ is an n-th splitting node in q, then s is an n-th splitting node of p.*

Definition 6.2.5. *We say that a sequence $\langle p_i \mid i \in \omega\rangle$ of elements of $S(\omega)$ is a fusion sequence if $p_n \leq_n p_{n-1}$ for every $n \geq 1$.*

Lemma 6.2.6. *If $\{p_n\}_{n\in\omega}$ is a fusion sequence of elements from $S(\omega)$ then $p = \bigcap_{n\in\omega} p_n$ is a perfect tree.*

Proof. Observe that p is clearly a tree at ω. To show it is a perfect tree suppose $s \in p$ and $|s| = m$. Since the rank of n-th splitting node in p_0 can only decrease, it must be the case that at least from p_{m+1} there is some splitting node t above s which is preserved by the ordering of fusion sequence. □

Definition 6.2.7. *Let p be a perfect tree and $s \in p$ then we define a perfect tree $p \restriction s = \{t \in p \mid t \subseteq s$ or $t \supseteq s\}$. Let A be a set of pairwise incompatible nodes of p and for every $s \in A$, $q_s \subseteq p \restriction s$ is a perfect tree. Then amalaganation of $\{q_s \mid s \in A\}$ into p is a tree $\{t \in p \mid (\exists s \in t)(s \subseteq t \rightarrow t \in q_s)\}$ i.e. a tree derived from p by replacing every $p \restriction s$ by q_s.*

The following theorem is due to Gerald E. Sacks and was first published in [SACKS71]. However, the proof given here follows [Jech02] Theorem 15.34.

Theorem 6.2.8. *(Sacks) Let M be a countable transitive model for ZFC and G a $S(\omega)$-generic over M. Then G is minimal over M.*

Proof. Let X be a set of ordinals in $M[G]$ which is not in M and $\dot{X}$ any of its name. let further $p \in S(\omega)$ be such that for every $A \in M$ $p \Vdash \dot{X} \neq \check{A}$. We find a condition $q \leq p$ and a set of ordinals $\{\gamma_s \mid s$ is a splitting node of $q\}$ such that $q \restriction s^\frown 0$, $q \restriction s^\frown 1$ decides $\gamma_s \in \dot{X}$ but in opposite ways. Then we are able to reconstruct Sacks real f by $f = \bigcup \{s \mid (q \restriction s \Vdash \check{\gamma}_s \in \dot{X})$ & $(\gamma_s \in X)\}$ and argue that $f \in M[\dot{X}_G]$ i.e. $G \in M[\dot{X}_G]$.

To find such a condition q, we construct a fusion sequence $\{p_i\}_{i \in \omega}$. Let $p_0 = p$. To construct p_n for $n \geq 1$ proceed as follows: Let S_n be the set of all n-th splitting nodes in p_{n-1}. For every $s \in S_n$ there is an ordinal γ_s such that $p_{n-1} \restriction s$ does not decides $\check{\gamma}_s \in \dot{X}$. Indeed, otherwise $Y = \{\gamma_s \mid p_{n-1} \restriction s \Vdash \check{\gamma}_s \in \dot{X}\}$ would be a set definable in M, $Y = \dot{X}_G$ and whence $X \in M$. Let further $q_{s^\frown 0} \leq p_{n-1} \restriction s^\frown 0$ and $q_{s^\frown 1} \leq p \restriction s^\frown 1$ be conditions deciding $\check{\gamma}_s \in \dot{X}$ in the opposite way (again they exist otherwise $p_{n-1} \restriction s$ would decide $\check{\gamma}_s \in \dot{X}$). Let p_n be an amalganation of $\{q_{s^\frown i} \mid i \in \{0,1\}$ & $s \in S_n\}$ into p_{n-1}. Then clearly $p_n \leq p_{n-1}$ and whence $\{p_i\}_{i \in \omega}$ is a fusion sequence. Finally it is sufficient to let $q = \bigcap_{n \in \omega} p_n$. □

The last thing that remain before we conclude that Sacks real is the real we are looking for is to show that $S(\omega)$ preserves cardinals.

Lemma 6.2.9. *Let M be a countable transitive model for ZFC, G a $S(\omega)$-generic filter over M and X a set of ordinals in $M[G]$. Then there exists a set $A \in M$, countable in M, such that $X \subseteq A$.*

Proof. We will proceed similarly as in the proof of Theorem 6.2.8. Let X be a countable set of ordinals in $M[G]$ and $\dot{F}$ a name for an enumeration function of X. Let

$p \in S(\omega)$ be such that $p \Vdash$ '$\dot{F}$ is a function from ω to ORD'. We are going to find a condition $q \leq p$ and a set A in M such that $q \Vdash rng(\dot{F}) \subseteq \check{A}$. Assume a fusion sequence constructed as follows: Let $p_0 = p$ and for given $n \geq 1$ let S_n be a set of all n-th splitting nodes of p_{n-1}. For every $s \in S_n$ there exist (for $i = 0$ and for $i = 1$) a $q_{s^\frown i}$ and $a_{s^\frown i}$ such that $q_{s^\frown i} \leq p_{n-1} \restriction s$ and $q_{s^\frown i} \Vdash \dot{F}(n \check{-} 1) = \check{a}_{s^\frown i}$. Indeed, it cannot be the case that $(\forall a_{s^\frown i} \in \text{ORD})(p_{n-1} \restriction s \Vdash \dot{F}(n \check{-} 1) \neq \check{a}_{s^\frown i})$ since $p_{n-1} \restriction s \leq p$. Let p_n be an amalgamation of $\{q_{s^\frown i} \mid s \in S_n \ \& \ i \in \{0,1\}\}$ into p_{n-1}. Now if we let $q = \bigcap_{n \in \omega} p_n$ and
$A = \bigcup \{a_{s^\frown i} \mid s \in S_n \ \& \ i \in \{0,1\}\}$, then since every S_n is finite, A is countable in M and it follows that $X \subseteq A$ by $q \Vdash rng(\dot{F}) \subseteq \check{A}$. □

Corollary 6.2.10. *If CH holds in a ground model then $S(\omega)$ preserves cardinals.*

Proof. Assuming CH and $|S(\omega)| = 2^{\aleph_0}$ we can infer that $S(\omega)$ is $\aleph_2$-cc, whence it remains to show that $S(\omega)$ preserves $\aleph_1$. However, according to the previous lemma, an existence of a map in generic extension collapsing $\aleph_1$ of ground model to ω would lead to a contradiction since in ground model this map is and is not countable in the same time. □

7 Conclusion

In this bachelor thesis we introduced the technique of forcing with aim to investigate properties of Cohen forcing.

In section 4 we showed that many of variations of Cohen forcing collapse cardinals and thus are not suitable for use in case of proving of an independence of statements about ZFC. In section 5 we analysed whether it is possible to find a model of ZFC where 'GCH fails on $\aleph_\omega$ but holds below' and showed some pathologies while using different support of product of Cohen forcings. After that we find a model of ZFC where 'GCH holds on $\aleph_\omega$ but fails below' using simple ad hoc argumentation following the technique of Easton forcing. In the last section we gave a brief note on non-minimality of generic filters on Cohen forcing and showed that generic filters on Sacks forcing are minimal. We illustrate in the same time how simple the notion of Cohen forcing is in compare to another forcing notion.

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