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Dean Prof.RNDr. Jan Kratochvil
CSc Matematicko-fyzikalni fakulta
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Ke Karlova 3
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Dear Prof.RNDr. Kratochvil:

RE: Thesis "Extension properties of structures" by David Hartman

The thesis is exceptionally clear and represents a monograph which covers the work contained in a series of research articles covering original research of the candidate published with his colleagues. It is normally understood in such joint publications that the authors have each contributed significantly to the work. The original contributions to research have appeared, or are under review for publication in a sequence of eight publications. The list of co-authors includes J. Hlinka,J. Hubička, J. Kurths, N. Marwan, D. Mašulović, J. Nešetřil, D. Novotná, M Paluš, J. Runge, and M. Vejmelka, (in alphabetic order).

The thesis begins with a very good introduction providing standard definitions and notions in the area, as well as motivation linking a study of symmetry in large data sets to notions of symmetry that have been studied independently for more theoretical reasons, including introducing the random graph, and Fraïssé's methods for construction of more general ultrahomogenous relational structures. The author presents his adaptation of standard results on the extension property, amalgamation and the result that ultrahomogeneous structures are universal for the class of structures with âge contained in that of the given structure. In this section several standard families of structures are discussed, including digraphs and partial orders. The notion of homomorphism and core and basic properties are introduced. The introduction is also where the notion of multicolored graph is introduced—(both the vertices and the edges of a normal (undirected) graph may be coloured using a finite number of unary and binary relations).
A review of the literature of classification results for the families relevant to this thesis is also presented. In a final section the author introduces notions from first order logic which permit one to generalize the ideas of symmetry that are the theme of the thesis. The critical notion of ω-categorical structure and key results from the classical literature are discussed. This section ends introducing classes determined by forbidden substructure, and the hierarchies of homomorphism homogeneity introduced by Cameron and J. Nešetřil , including the modification of the notion of amalgamation and extension form the section on Fraïssé's method that is relevant to the context of the context of homomorphism homogeneity. While the results of this chapter are not new,

the presentation, setting context and providing key tools, is very well thought out.

The second chapter presents work from the four articles by the author and others dealing with "real world" situations that motivate interest in the more mathematical studies that follow. It also provides a very good base to understand how some seemingly very "pure" and abstract notions can be applied to other domains. To do this the thesis introduces the notion of complex network and reviews some of the notions and results in the area. Examples such as air transportation, street networks, computer networks, and the World Wide Web are sketched. Some standard "metrics" such as "betweeness centrality" are discussed. One important complex system studied in more detail is that of brain networks. The discussion of the distinction between structural and functional networks and how they relate to abstracting the problems was very informative. The studies engage in comparison of data sets with various linear surrogate sets and the effect on the standard notions of interest in the field. The other type of network studied in some detail by the authors are those associated with climate. Here actual data are used, and investigation of connectivity is a focus. All the results in this section have appeared in respected journals in the field.

The third section of the work is focused on bicolored graphs, which can be viewed as a pair of finite simple graphs on the same ground set. While these are a special case of the more general L-colored graphs studied in the next section, the research provides a more complete characterization than has been possible in the more general situation described in subsequent sections. It also features a proof technique, the "pumping argument" which is further developed later. The motivation was the search for examples of structures which distinguish the classes **HH** and **MH**, (every homomorphism of substructures extends to an endomorphism, and every monomoprhism extends to an endormorphism, respectively). The work provides a complete characterization of the (finite) homomorphism homogeneous bigraphs. As the author points out, this does not resolve whether the two classes coincide because this rests on the possibility of there being an induced path of length 2 (regardless of the color of edges). A joint article with D. Mašulović appeared in 2011. The work in this chapter is new and of interest.

The fourth part of the thesis provides finite examples distinguishing **HH** and **MH**, introduces several notions, and poses several interesting and significant questions, one being how much additional structure is needed in order to permit the distinction of the two classes. The notion of L-coloured graphs is introduced. These are multi-coloured graphs with colours in a partial order **L** with bottom, 0, and top 1. A homomorphism must preserve not only the graph structure but also the ordering on colours of (loopless) vertices and (undirected) edges. The pumping argument presented in part three is extended to this more general, and therefore complex setting. The main result is that if the partial order is linear then the two classes coincide. This is proven by the technique of proving a common structure for the two classes. When the partial order is allowed to be in the family of diamonds (partial orders with one middle level between 0 and 1) the authors construct a family of finite examples which are **MH** but not **HH** homogeneous. They study first the situation in which only one colour is allowed on vertices, and use this to obtain a partial classification giving the structure of **MH** and **HH** homogeneous graphs. They show that if the diamond is the four element diamond M_2, and all vertices have the same colour then the classes coincide (via a structural result). They conclude giving a simple example using more than one colour on vertices and on edges which provides the distinction. They also give an easy example of a digraph with loops which makes the distinction. Open problems include the classification of **MH** and **HH** homogeneous **L**-coloured graphs in general, and whether there are examples which are monochromatic on vertices. What happens in the infinite case is also open. The results appeared in a joint publication with J. Hubička and D. Mašulović in 2014. This section provides new results and new interesting problems.

The fifth part of the thesis introduces the hierarchy of **XY**-homogeneous graphs, and the differences between the finite and infinite. The work of Cameron and Lockett that introduced the program is surveyed as well as results of Rusinov and Schweitzer. The new contributions are centred on using the results from the previous sections of this thesis to analyse how the classes are modified for more general relational structures. New results of this section are introduced that are due to the candidate.

Part six of the thesis represents a departure from the exclusively finite and graph oriented world, by looking at general relational structures and setting out to throw some light on the question of how much (and how complicated) relational structure must be added to expand a given relational structure **A** to one **A'** which is ultrahomogeneous. It is obvious that by adding a unary predicate "naming" each element of the structure, one obtains an ultrahomogeneous structure, but the result bears little relation to the original—it is rigid, while the original may have been rich in automorphisms (symmetry). So, a goal is to preserve the automoprhism structure while adding the relations. This naturally gives rise to two different notions. One is to look for the least r (if it exists) such that by adding finitely many relations of arity at most r, the resulting expansion is ultrahomogeneous. The second is to ask for the least r, if it exists, so that one can add relations of arity at most r so that the resulting expansion is both ultrahomogeous and has the same automorphism group as the original. It is well known that by adding relations that pick out the r-types of elements for each r, one can do the latter, but this can lead to adding relations of arbitrary arity. It can also lead to adding infinitely many relations of bounded arity, as the example of a ray (ω, S), where S is the set of edges {n, n+1}. (0 is distinguished by being of degree one, and you must distinguish distances.) Because the author deals with metric graphs, and studies the ω-categorical case in some detail, I am convinced that the small lapse that occurs is because of an initial focus on the finite and a reordering of material that loses a necessary condition. Proposition 6.3.2 requires that the original structure be ω-categorical. Since one uses this proposition this oversight flows into other propositions in this part of the thesis. That quibble aside, the section contains a number of interesting and new results giving the complexity of some families of structures, such as metrically homogeneous finite graphs, and finite graph trees. Constructions are also adapted that provide interesting examples of large complexity. There are a number of very interesting examples and results specific to the countable case, and an introduction to general tools and notions, such as existential completeness that add to our knowledge. An example is Theorem 6.5.1. Section 6.5.1 of the work looks at producing bounds on complexity from knowledge about the amalgamation (or failure of amalgamation) for the âge. The results, examples, and questions posed in this section will be a source of study for colleagues for some time. The results appear in two articles with J. Hubička and J. Nešetřil.

In summary this thesis reports on significant contributions to the field, and represents clear promise of a successful research career for the candidate, David Hartman. I have no hesitation in recommending him for the doctorate.

Thank you for the opportunity to become better acquainted with his work.

Sincerely,

Dr. Robert Woodrow, Acting Head
Professor of Mathematics