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Structure of equality sets

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Název práce: Struktura ekvivalenčních množin

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Abstrakt:

Binární ekvivaleční množina dvou homomorfismů $g, h : \Sigma^* \to A^*$ je množina všech slov w nad dvouprkovou abecedou Σ splňujících rovnost $g(w) = h(w)$. Prvky této množiny se nazývají binární ekvivalenční slova.

Jedním z důležitých výsledků v oblasti ekvivalenčních množin je důkaz toho, že množina generátorů libovolné binární ekvivalenční množiny je maximálně dvouprvková za podmínky, že jsou oba homomorfismy g, h neperiodické. Pokud je navíc tato množina generována přesně dvěma prvky, je struktura těchto generátorů, a tím i celé množiny, jednoznačně dána.

Předložená práce se zabývá výzkumem struktury binárních ekvivalenčních množin s jedním generátorem. Důležitou vlastností těchto generátorů je možnost jejich rozkladu na jednodušší struktury. Generátory, které již nelze dále rozložit, se nazývají jednoduchá ekvivalenční slova. První část práce se věnuje struktuře jednoduchých ekvivalenčních slov a jejich podrobné klasifikaci. Hlavním výsledkem této části je nalezení přesné struktury jednoduchých ekvivalenčních slov, která jsou dostatečně dlouhá. Ve své druhé části práce popisuje iterační proces, který obecný generátor binární ekvivalenční množiny transformuje na jednoduché ekvivalenční slovo. Hlavním výsledkem druhé části práce je nalezení hranice pro maximální počet těchto iterací. Ve své poslední části práce syntetizuje veškeré dosud dosažené poznatky o binárních ekvivalenčních slovech a přináší jejich tabulkový přehled.

Klíčová slova: kombinatorika na slovech, Postův korespondenční problém, binární ekvivalenční množiny, slova vynucující periodicitu

Title: Structure of equality sets

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Abstract:

Binary equality set of two morphisms $g, h : \Sigma^* \to A^*$ is a set of all words w over two-letter alphabet Σ satisfying $g(w) = h(w)$. Elements of this set are called binary equality words.

One of the important results of research on binary equality sets is the proof of the fact that each binary equality set is generated by at most two words provided that both morphisms g and h are non-periodic. Moreover, if a binary equality set is generated by exactly two words, then the structure of both generators, and therefore of the whole set, is uniquely given.

This work presents the results of our research on the structure of binary equality sets with a single generator. Importantly, these generators can be decomposed into simpler structures. Generators which can not be further decomposed are called simple equality words. First part of the presented work describes the structure of simple equality words and introduces their detailed classification. The main result of the first part is a precise characterisation of sufficiently large simple equality words. In the second part, the work describes the iterative process which transforms a general generator of a binary equality set into a simple equality word. As the main result of the second part it has been proved that the number of iterations of the process is bounded by a constant. In the last part of the work, the results of our research on binary equality words are summed up and presented in a form of tabular listing.

Keywords: Combinatorics on words, Post Correspondence Problem, Binary equality sets, Periodicity forcing words

Contents

Introduction

Homomorphisms together with algebraic structures constitute the very heart of algebraic research. The essential property of homomorphisms is their ability to capture similarities between algebraic structures. In this sense, the description of any structure as a homomorphic image of another known structure is a valuable piece of information.

With the rise of computers half a century ago, basic algebraic structures, such as free monoids, have gained attention of computer scientists. Free monoids now constitute the basic computer language and homomorphisms can be seen as descriptions for computational processes. Based on this newly defined role for traditional structures, completely new tasks and questions arose.

One of the first questions proposed by E. Post was whether it is decidable that two different computational processes on the same input give us the same output ([19]). The famous problem, nowadays known as the *Post Correspondence Problem* (PCP), was immediately proved to be undecidable by Post himself. However, this simple yet intriguing problem was a jumping-off point for a whole variety of modifications of PCP, some of which turned out to be decidable. Most importantly for this work, the modified version of PCP with additional requirement for the input to be a word over two-letter alphabet (known as PCP(2)) was proved to be decidable in a series of works by A. Ehrenfeucht, J. Karhumäki and G. Rozenberg ([9], a complete proof given lately in [11]).

In algebraic terms, when exploring PCP, we are talking about equality sets, that is, about the sets of words on which two homomorphisms agree. In this sense, PCP becomes the task to decide whether an equality set contains a non-empty word. Similarly, the decidability of PCP(2) only says that this task is decidable on condition that the homomorphisms are binary.

Leaving apart decidability problems, equality sets constitute very interesting structures for a variety of reasons. For example, they provide characterisation for recursively enumerable languages. K. Culik proved that each recursively enumerable set is a homomorphic image of a set of generators of an equality set ([7]). Such purely homomorphic characterisation of recursively enumerable sets leaves us with natural interest in the structure of equality sets.

The task to find structure of the simplest nontrivial equality set, i.e. binary equality set, turned out to be a laborious work. The work by A. Ehrenfeucht, J. Karhumäki and G. Rozenberg ([10]) left us with the possibility of one, two or infinite number of generators of the binary equality set of non-periodic homomomorphisms. Karhümaki himself conjectured ([8]) that the set of generators should be finite. The conjecture was proved to be true by Š. Holub ([14]). Also, it was shown that the set $\{a^ib, ba^i\}^*$ is, up to the exchange of letters a and b, the only possible binary equality set with exactly two generators ([13]). Therefore, only two tasks remained: to find a generator of single generated binary equality set and to provide characterisation for binary equality languages of periodic homomorphisms. The generators of binary equality sets in periodic case were established in [15]. Describing the structure of single generators of binary equality sets is the main aim of this work.

The backbone of the presented work is shaped by our two articles, *Large simple binary equality words* [3] and *The block structure of successor morphisms* [1], and by results from our newest research. Together, they constitute a single entity which should provide the reader with the profound insight into the problematic of binary equality words.

The first chapter provides the reader with necessary terminology and general concepts of combinatorics on words. Most of the definitions which are not directly related to the problematic of equality sets are left to be found in secondary literature ([20, Chapter 6]).

In the second chapter we have a closer look at simple binary equality words, that is, the words that cannot be decomposed into simpler structures called letter blocks. These equality words constitute natural starting point of investigation of binary equality sets ([9]). We will show that, up to the special cases, the length of simple binary equality words is bounded by a constant. In the first part, we will prove that the only simple binary equality words that are arbitrary large in both letters a and b are, up to the exchange of letter a and b, the words a^ib^j and $(ab)^ia$. The proving methods as well as the background of this result is meticulously described in our article *Large simple binary equality words* ([3], extended versions [4] and [2]) and therefore, we focus mainly on explaining the result and on giving some interesting examples. The second part of the second chapter presents the results of our newest research. It concerns simple equality words with bounded number of at least one of the letters a or b. Since it has not yet been published, we have included there all details necessary for the proof. At the end of the second chapter we include a tabular listing and a graphical representation of simple binary equality words.

The third chapter is based on our article *The block structure of successor morphisms* [1]. We will present our results about non-simple binary equality words, that is, the words which allow decomposition into letter blocks. We will use the fact that every pair of binary morphisms with non-simple minimal equality word admits construction of a pair of successor morphisms which is simpler than the original pair of morphisms. This process can be repeated, leading eventually to a pair of morphisms that do not have any successors. We will show that, up to some special morphisms, the length of the sequence of successor morphisms is bounded by a constant.

The last chapter contains a tabular listing of all possible binary equality sets according to our current knowledge.

1. Basic concepts and terminology

We will use the terminology from [20, Chapter 6]. Particularly, the reader should recall that a set $X \subseteq A^*$ is a *code* if it is the minimal generating set of X^*. Importantly, two element set $\{x, y\}$ is a code iff $xy \neq yx$, i.e. x and y do not commute. Code $X \subseteq A^*$ is called *marked* if each code word starts with a different letter of the underlying alphabet A. Notice that every marked code is a prefix code, but the reverse implication does not hold. The set of all suffixes of a word u will be denoted by $\mathrm{suf}(u)$, the set of all prefixes of u by $\mathrm{pref}(u)$. The *maximal common prefix* of two words u and v will be denoted by $u \wedge v$. Similarly, $u \wedge_s v$ represents the *maximal common suffix* of two words u and v. A (one-way) *infinite word* composed of infinite number of copies of a word u will be denoted by u^ω. By the *length of a word* u we mean the number of its letters and we denote it by $|u|$. Similarly, $|u|_a$ stands for the number of occurrences of the letter a in u. It should also be mentioned that the *primitive root* of a word u, denoted by $\rho(u)$, is the shortest word p such that $u = p^k$ for some positive k. Words $u, v \in A^*$ are *conjugate* if $zu = vz$ for some word $z \in A^*$. Empty word will be denoted by 1.

The following lemma is an elementary tool for determining the commutativity (or conjugacy) of two words. We present here a new proof whose main idea, however, should be credited to T. Harju and D. Nowotka [12].

Lemma 1 (Periodicity lemma)**.** *Let p and q be primitive words. If p^ω and q^ω have a common factor of length at least $|p| + |q| - 1$, then p and q are conjugate. If, moreover, p and q are prefix (or suffix) comparable, then $p = q$.*

Proof. We will use the fact that every primitive word has an unbordered conjugate. We can suppose that the common factor of length at least $|p| + |q| - 1$ is a prefix of both words p^ω and q^ω. We will show that $|p| = |q|$. By symmetry, we suppose that $|p| > |q|$. Let p_1, p_2 be words such that $p = p_1p_2$ and p_2p_1 is an unbordered conjugate of p. Notice that both words p_1 and p_2 are non-empty since p is bordered. Clearly, $|p_1| \geq |q|$, otherwise p_2p_1 is a factor of q^ω, and thus bordered. But then $q' \leq_{\mathrm{s}} p_1$ and $q' \leq_{\mathrm{p}} p_2p_1$, for some conjugate q' of q, a contradiction with p_2p_1 being unbordered. $\square$

We get the following corollary:

Corollary 2. *Let $n, k \in \mathbb{N}$ be periods of a word w. If $|w| \geq n + k - 1$, then w has also a period* $\gcd(n, k)$.

Main objects of our interest are morphisms and their equality sets. We, therefore, start with some basic concepts and terminology related to them.

Marked morphisms. Morphism $g : \{a, b\}^* \to A^*$ is *non-periodic* if $\{g(a), g(b)\}$ is a code. Non-periodic morphism g is called *marked* if $\{g(a), g(b)\}$ is a marked code. Let us recall that for each non-periodic binary morphism $g : \{a, b\}^* \to A^*$ there is a uniquely given marked (non-periodic) binary morphism $g_{\mathrm{m}} : \{a, b\}^* \to A^*$ and a word z_g such that for all words $w \in \{a, b\}^*$ we have $g_{\mathrm{m}}(w) = z_g^{-1} g(w) z_g$.

Word z_g is in fact equal to $g(ab) \wedge g(ba)$. The definition of morphism g_{m} is correct since z_g is prefix comparable with all words from $\{g(a), g(b)\}^*$, as we can see from the following lemma and its corollary. Proof can be found in [20, Chapter 6, Lemma 3.1].

Lemma 3. *Let $X = \{x, y\}$ be a code and $u \in xX^*$, $v \in yX^*$ words such that $|u| \geq |xy \wedge yx|$ and $|v| \geq |xy \wedge yx|$. Then $u \wedge v = xy \wedge yx$.*

In accordance with the previous notation, if speaking about codes, word $xy \wedge yx$ will be denoted by z_X. Easy consequence of the previous lemma is the fact that z_X is prefix comparable with all words from X^*:

Corollary 4. *Let $X = \{x, y\}$ be a code. Then z_X is prefix comparable with all words from X^*. In particular, for any two words $u \in xX^*$, $v \in yX^*$ we have $u \wedge v \leq_{\mathrm{p}} x^+$, $u \wedge v \leq_{\mathrm{p}} y^+$ and $u \wedge v \leq_{\mathrm{p}} z_X$.*

The previous results can be formulated dually for $\wedge_s$. The word $xy \wedge_s yx$ will be denoted by $\underline{z}_X$ and it is suffix comparable with all words from X^*.

We would like to put emphasis on the fact that the maximal common prefix of two words from X^* which start with different code words is bounded, for this is the main reason of the decidability of PCP in binary case. Notice that this exceptional property of binary codes can not be found in codes with arity bigger than two, e.g. consider code $X = \{a, ab, bb\}$.

Next, we define an equality word of two binary morphisms.

Equality words. Let $g, h : \{a, b\}^* \to A^*$ be two different binary morphisms. A word w is an *equality word* of g, h if $g(w) = h(w)$. We say that an equality word w is *minimal* if no other equality word is a prefix of w. An equality word w is called *simple* if all overflows are unique. That is, if w is primitive and if w_1, $w_1 u$, w_2 and $w_2 u'$ prefixes of w^ω such that

$$g(w_1)z = h(w_2) \qquad \text{and} \qquad g(w_1 u)z = h(w_2 u')$$

for some word $z \in A^* \cup (A^{-1})^*$, then $|u| = |u'| = k|w|$ for some $k \in \mathbb{N}_+$. Notice that every simple equality word is minimal. However, the reverse implication does not hold.[1]

Let $g, h : \{a, b\}^* \to A^*$ be two binary morphisms and suppose that we have a pair of words (e, f) such that $g(e)$ is a conjugate of $h(f)$. Since in this situation it is also perfectly plausible to speak about overflows, we would like to extend our definition of an overflow and define simple pair of conjugate words. First, we will need the following terminology:

X-conjugate words. Let $X \subseteq A^*$ be a code. Word $x, x' \in X^*$ are *X-conjugate* if there is a word $z \in X^*$ such that $zx = x'z$. Notice that X-conjugate words are always conjugate but the reverse does not hold. Indeed, consider for example a code $X = \{ab, ba\}$. Then the words ab, ba are conjugate but not X-conjugate.

[1]Further in the text we will sometimes say that "a word w is a simple equality word" without specifying morphisms g and h. Since each word can be an equality word of various non-equivalent morphisms (and therefore w can be simple for one pair of morphisms and not simple for another), we should clarify what this expression means. By this expression we mean that there are morphisms g, h such that w is their simple equality word. Similarly, an expression "a word w is a non-simple equality word" means that there are morphisms g, h such that w is their equality word which is not simple.

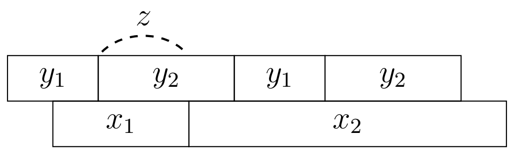

Figure 1: X-overflow z of $(x_1x_2, (y_1y_2)^2)$, $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$.

Now we can define overflows in conjugate words:

Overflows and simple conjugate pair. Let $X, Y \subseteq A^*$ be codes and $x \in X^*$, $y \in Y^*$ be primitive words such that x^i and y^j are conjugate for some $i, j \in \mathbb{N}$. We say that $z \in A^*$ is an X-*overflow* of (x^i, y^j) iff $|z| < |x^i|$ and there exists a word x', X-conjugate of x^i, and a word y', Y-conjugate of y^j, such that $zx' = y'z$ (see Fig. 1). We say that a pair of words (x^i, y^j) is *simple* (for X and Y) if $\gcd(i, j) = 1$, and $zx' = y'z$ and $zx'' = y''z$ imply $x' = x''$ and $y' = y''$, where $z \in (A \cup A^{-1})^*$, and x', x'' are X-conjugate of x^i, and y', y'' are Y-conjugate of y^j.

Similarly, we define Y-overflow of (x^i, y^j) as a word $z' \in A^*$ such that $|z'| < |x^i|$ and there exists a word x', X-conjugate of x^i, and a word y', Y-conjugate of y^j, such that $x'z' = z'y'$.

Importantly, if $G = \{g(a), g(b)\}$ and $H = \{h(a), h(b)\}$ are codes and $g(w)$ is primitive, our definition of overflow in conjugate pair o words corresponds to the definition of overflow in an equality word. Indeed, any overflow in the equality word w is an overflow of $(g(w), h(w))$. On the other hand, if we suppose that $g(w)$ is primitive, any two words w', w'' conjugate with w uniquely define an overflow in the equality word w. More precisely, if $z'g(w') = h(w'')z'$ and $zg(w') = h(w'')z$ for some z' such that $|z'| \geq |z|$, then $z^{-1}z' \in \rho(g(w'))^*$. Since we have supposed that $g(w)$ is primitive, $z^{-1}z' \in g(w')^*$ and from $|z| < |g(w)|$ we get $z' = z$.

Notice that in our definition of overflow we can dispense with equation $g(w) = h(w)$, since the property "being an overflow" is solely dependent on words $g(w)$ and $h(w)$. Therefore, we can speak for example about overflows arising from the equation $g(e)u = uh(f)$, where $u \in A^*$. This property will be found convenient when working with marked morphisms.

On the other hand, this more general definition is a trade-off with additional requirements imposed on G, H and w. First, we have to suppose that G and H are codes, that is, that g and h are non-periodical morphisms. Second, we have to suppose that $g(\rho(w))$ is primitive. Suppose now that $g(\rho(w))$ is not primitive. If w is a minimal equality word of g and h, then $\rho(w) = w$ and, according to [6, Theorem 4.2], we get that w is a conjugate of one of the words from the set $a^*b \cup ab^*$. Therefore, having in mind these additional equality words, in what follows, we can always suppose that $g(w)$ is primitive and work with the definition of overflow for conjugate pairs.

Equality sets. Equality set of morphisms $g, h : \{a, b\}^* \to A^*$ is a set of all their equality words, that is,

$$\mathrm{Eq}(g, h) = \{w, g(w) = h(w)\}.$$

Similarly, we define a set of cyclic equality words of two morphisms as a set of all words w, such that $g(w)$ and $h(w)$ are conjugate. We use notation

$$\mathrm{CEq}(g,h) = \{w, \exists z \in A^* : zg(w) = h(w)z\}.$$

If we consider marked versions $g_{\mathrm{m}}, h_{\mathrm{m}}$ of morphismsm g, h, we can see from the definition of marked version of a morphism that if $w \in \mathrm{Eq}(g,h)$, then $w \in \mathrm{CEq}(g_{\mathrm{m}}, h_{\mathrm{m}})$.

Example 1. Let $g, h : \{a,b\}^* \to A^*$ be two binary morphisms defined as follows:

$$\begin{aligned} g(a) &= a^3, & g(b) &= b, \\ h(a) &= a^2, & h(b) &= a^3 b. \end{aligned}$$

Then a^3b is unique minimal equality word of g, h. However, from $g(a^2) = h(a^3)$ and $g(a^3b) = h(a^3b)$ we can see that a^3b is not a simple equality word of g, h. Equality set of g, h is $(a^3b)^*$.

Letter blocks and block decomposition. According to [9] every minimal non-simple equality word decomposes into simple structures called *letter blocks*. A letter block of binary morphisms g, h is a (prefix) minimal pair of words (e, f) such that

$$g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f).$$

For each pair of binary morphisms there are at most two different letter blocks (e, f), (e', f') such that ef and $e'f'$ are non-empty. Moreover, $\mathrm{pref}_1 e \neq \mathrm{pref}_1 e'$ and $\mathrm{pref}_1 f \neq \mathrm{pref}_1 f'$.

For every minimal non-simple equality word w of binary morphisms g and h there is a sequence

$$(u_0, v_0), \ldots, (u_n, v_n)$$

such that $w = u_0 \ldots u_n = v_0 \ldots v_n$ and $(u_n u_0, v_n v_0)$ and (u_i, v_i), $0 < i < n$, are letter blocks of g and h. This sequence is called a *block decomposition* of the equality word w.

2. Simple binary equality words

Simple binary equality words constitute a natural starting point of investigation of equality words since they can not be decomposed into letter blocks. We will show that, up to the special cases, the length of simple binary equality words is bounded by a constant. Also, we will prove that the only simple binary equality words which are arbitrary large in both letters a and b are, up to exchange of letter a and b, the words $a^i b^j$ and $(ab)^i a$. Finally, we will investigate simple equality words with arbitrary large number of just one letter. We will close this chapter with the list of all simple equality words to the best of our knowledge.

Let $g, h : \{a, b\}^* \to A^*$ be non-periodic binary morphisms and w their simple equality word. To differentiate letters a and b, we will suppose that $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$. Quite naturally, we can expect that with rising length of the equality word the interaction between a g-image of the equality word and an h-image of the equality word becomes more sophisticated, giving way eventually to only few patterns which such long equality words have to respect.

The first part, called b-bound, presents the results from our article *Large simple binary equality words* [3] and shows that the only simple equality words with arbitrary large number of b's are words from the sets $(ab)^*a$, $(ba)^*b$ and b^*ab^*, or are equal to $b^i a^j$ or $a^j b^i$, where $j > i \geq 1$ and $\gcd(i, j) = 1$.

In the second part, called a-bound, we will restrict, again up to the aforementioned "large" equality words, the number of a's in w. Since this part was not published yet, we provide the reader with full proving methodology.

The last part of this chapter summarise the results concerning simple binary equality words. It also includes their tabular listing and a graphical representation.

2.1 b-bound

Let $g, h : \{a, b\}^* \to A^*$ be non-periodic binary morphisms and w their simple equality word. Suppose moreover that the length of $h(b)$ is maximal among the lengths of all four image words $g(a)$, $g(b)$, $h(a)$ and $h(b)$.

We will show that, up to the special cases, the number of b's in w is restricted. The detailed explanation and the proof can be found in our article *Large simple binary equality words* [3], or in its extended versions [4] and [2]. Here, we will limit ourselves to the presentations of the results together with some interesting examples.

First, we present the result we have proved for marked morphisms and simple pair of conjugate words. The proof and the details are given in [4, Lemma 14].

Theorem 5. *Let $g, h : \{a, b\}^* \to A^*$ be non-periodic marked binary morphisms and let w be a word such that $(g(w), h(w))$ is a simple pair of conjugate words. Suppose that $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$. If $|w|_b \geq 9$, then there are words e, f conjugate with w such that $g(e) = h(f)$ and*

$$e = f = (ab)^i a \quad \text{or} \quad e = f = (ba)^i b \quad \text{or} \quad e = f = ab^i \quad \text{or} \quad (e, f) = (b^k a^j, a^j b^k)$$

with $\gcd(j, k) = 1$ and $j > k$.

Notice that if w is an equality word of g, h, then from the definition of marked morphisms we have:

$$z_h^{-1} z_g g_{\mathrm{m}}(w) = h_{\mathrm{m}}(w) z_h^{-1} z_g,$$

and consequently $g_{\mathrm{m}}(w)$ and $h_{\mathrm{m}}(w)$ are conjugate words.

Suppose first that $g_{\mathrm{m}}(w)$ is not primitive. Since w is a minimal equality word, $\rho(w) = w$ and, according to [6, Theorem 4.2], we get that w is a conjugate of one of the words from the set $a^*b \cup ab^*$.

Consequently, up to the aforementioned words, we can suppose that $g_{\mathrm{m}}(w)$ (and $h_{\mathrm{m}}(w)$) is primitive. As a result, in case that w is simple, we obtain that $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$ is a simple pair of conjugate words.

Therefore, using the previous theorem, we can see that simple equality words with at least nine b's have to be conjugate of e (and f). Then, it is not so difficult to get the following result. The proof and the details are to be found in [4, Section 3.3].

Theorem 6. *Let $g, h : \{a, b\}^* \to A^*$ be non-periodic binary morphisms and let w be their simple equality word. Suppose that $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$. If $|w|_b \geq 9$, then*

$$w = (ab)^i a \quad \text{or} \quad w = (ba)^i b \quad \text{or} \quad w = a^j b^k \quad \text{or} \quad w = b^k a^j \quad \text{or} \quad w = b^m a b^n$$

with $\gcd(j, k) = 1$ and $j > k$.

Example 2. Each word mentioned in Theorem 6 is indeed a simple equality word. The word $w = (ab)^i a$ is a simple equality word for example of morphisms:

$$g(a) = (ab)^{i+1}a, \qquad g(b) = b,$$
$$h(a) = aba, \qquad h(b) = (ba)^{i+1}b.$$

The word $a^j b^k$ is a simple equality word for example of morphisms:

$$g(a) = p^\ell, \qquad g(b) = a,$$
$$h(a) = a^k b a^k, \qquad h(b) = s^m,$$

where

$$p = a^k (ba^{2k})^{j-1} b, \qquad s = (ba^{2k})^{j-1} b a^k,$$

and $\ell j - mk = 1$, $\ell > 1$. Notice that exchange of letters a and b in the previous examples gives us a pair of morphisms with a simple equality word $w = (ba)^i b$ (resp. $b^j a^k$). Finally, the word $w = b^m a b^n$ is a simple equality word for example of morphisms

$$g(a) = b^m a b^n, \qquad g(b) = b^{m+n},$$
$$h(a) = a, \qquad h(b) = b^{m+n+1}.$$

One should notice that the word $w = a^m b a^n$ is a simple equality word as well, but, since we were interested only in equality words w such that $|w|_b \geq 9$, it does not meet our criteria.

Notice also that each word a^jb^k with $\gcd(j,k) \neq 1$ is an equality word for some morphisms g, h. Indeed, in [8, Example 5.1], we can find the following pair of morphisms with the equality word a^jb^k, where j and k are arbitrary:

$$\begin{aligned} g(a) &= (a^{jk}b)^k, & g(b) &= a^j, \\ h(a) &= a^k, & h(b) &= (ba^{jk})^j. \end{aligned}$$

Observe, that this example does not contradict Theorem 6, since the word a^jb^k is not a *simple equality word* of the aforementioned morphisms.[1].

We finish this part with the weaker version of Theorem 6. The following theorem characterises simple binary equality words which are arbitrary large in both letters a and b. Notice that, in contrast with Theorem 6, we do not require that the word $h(b)$ is the longest word among the words $g(a), g(b), h(a)$ and $h(b)$.

Theorem 7. *Let $g, h : \{a, b\}^* \to A^*$ be non-periodic binary morphisms and let w be their simple equality word. If $|w|_b \geq 9$ and $|w|_a \geq 9$, then up to the exchange of letters a and b, either*

$$w = (ab)^i a,$$

or

$$w = a^j b^k$$

with $i, j, k \geq 9$ and $\gcd(j,k) = 1$.

2.2 a-bound

We have seen in the previous part that the number of b's in a simple equality word w is at most eight unless w is equal to one of the words specified in Theorem 6. We will now focus on number of a's in w.

Let again $g, h : \{a, b\}^* \to A^*$ be non-periodic binary morphisms and w their simple equality word. Suppose, moreover, that the length of $h(b)$ is maximal among the lengths of all four image words $g(a)$, $g(b)$, $h(a)$ and $h(b)$. We will show that if $|w|_b \geq 4$, then the number of a's in w is, up to the special cases, restricted. This result is the fruit of our newest work and has not been published yet. Therefore, it is presented here in full details.

We formulate the result in the following theorem:

Theorem 8. *Let $g, h : \{a, b\}^* \to A^*$ be non-periodic binary morphisms and let w be their simple equality word. Suppose that $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$. If $|w|_b \geq 4$ and $|w|_a \geq 30$, then*

$$w = (ab)^i a \quad \text{or} \quad w = (ba)^i b \quad \text{or} \quad w = a^j b^k \quad \text{or} \quad w = b^k a^j$$

with $\gcd(j,k) = 1$ and $j > k$.

[1]The only known simple equality word of the type a^jb^k with $\gcd(j,k) \neq 1$ is, up to the exchange of letters, the word $aabb$. Indeed, for example

$$\begin{aligned} g(a) &= aab, & g(b) &= ababa, \\ h(a) &= a, & h(b) &= baababa. \end{aligned}$$

is a pair of morphisms such that $aabb$ is their simple equality word.

Previous theorem restricts the number of a's in a simple equality word w only in case that $|w|_b \geq 4$. Notice that all words w such that $|w|_b = 1$ are simple equality words. Therefore, the only cases left for investigation are $|w|_b = 2$ or $|w|_b = 3$. Although we do not have definite results regarding these equality words, we will present some partial results at the end of the chapter.

The important role in the proof of Theorem 8 is played by *disjoint interpretations* of $h(b)$. Let us, therefore, first introduce some terminology:

X-interpretations. Let $X \subseteq A^*$ be a code. We define *X-interpretation* of a word $v \in A^*$ as a triple of words (q, d, p), such that q is a proper suffix of a word from X, p is a proper prefix of a word from X, $d \in X^*$ and $v = qdp$. Two X-interpretations (q, d, p) and (q', d', p') of v are *adjacent* if there exist words $d_1, d_2, d_1' \in X^*$ and $d_2' \in X^*$ such that $d = d_1 d_2$, $d' = d_1' d_2'$ and $q d_1 = q' d_1'$. X-interpretations which are not adjacent are called *disjoint*. We say that two X-interpretations (q, d, p) and (q', d', p') of v are *shifted by ρ* if there are words $d_1, d_2, d_1', d_2' \in X^*$ such that $d = d_1 d_2$, $d' = d_1' d_2'$ and $d_2 p (d_2' p')^{-1} \in \rho^*$ or $d_2' p' (d_2 p)^{-1} \in \rho^*$.

Before proceeding with the proof of Theorem 8, we have to strengthen our knowledge about X-interpretations and overflows. This will be the main aim of the first, resp. second, part of this section.

The first part is dedicated to interactions between disjoint $\{g(a), g(b)\}$ - interpretations of $h(b)$. Since these combinatorial properties do not depend on morphisms g and h, we will speak more generally about codes. In the second part, we will have a look at some special overflows in the simple pair of conjugate words and see what is their impact on the structure of an equality word. Finally, in the third part, we will proceed with the proof of Theorem 8.

2.2.1 Interpretations

This part is vastly inspired by the properties of interpretations discussed in [6]. The majority of lemmas in this part are not difficult to prove and they have merely technical character. Having said that, however, we would like to put emphasis on Lemma 21 which is the key lemma of this part and, at the same time, boasts a more complicated proof.

Let $X = \{x^{j_x}, y^{j_y}\} \subseteq A^*$ be a code. We are asking upon what conditions disjoint X-interpretations of a word $u \in A^*$ become shifted by the primitive root of a word from X^*. In [6] we can find that many words $u \in X^+$ with a sufficient length do not admit any interpretation disjoint from $(1, u, 1)$; e.g. if x and y are primitive, not conjugate words and $|x| \geq |y|$ then x^3 does not admit any X-interpretation disjoint from $(1, x^3, 1)$ ([6, Corollary 2.2]). We will present similar results: first we will be looking at interpretations of xy and then we will discuss interpretations of a word $uy^3v \in A^*$ such that $|uy^3v| \geq |x|$.

We first remind the reader one of the key results of E. Barbin-Le Rest and M. Le Rest which can be found, together with its proof, in [6, Theorem 2.1].

Lemma 9. *Let $Y = \{x, y\} \subseteq A^*$ be a code with x and y primitive, not conjugate words such that $|x| \geq |y|$. Suppose that (q, d, p) is an Y-interpretation of x^2 disjoint from $(1, x^2, 1)$. Then $q \not\leq_{\mathrm{s}} y$, $d = y$ and $p \not\leq_{\mathrm{p}} y$.*

The previous lemma yields the following two corollaries. First of them is again due to E. Barbin-Le Rest and M. Le Rest [6, Corollary 2.2].

Corollary 10. *Let $Y = \{x, y\} \subseteq A^*$ be a code with x and y primitive, not conjugate words such that $|x| \geq |y|$. Then all Y-interpretations of x^3 are adjacent to $(1, x^3, 1)$.*

Corollary 11. *Let $Y = \{x, y\} \subseteq A^*$ be a code with x and y primitive words such that $|x| \geq |y|$. Then x^2 admits at most one Y-interpretation disjoint from $(1, x^2, 1)$.*

Proof. If x and y are conjugate, then there are uniquely given words s_1, s_2 such that $x = s_1 s_2$ and $y = s_2 s_1$. From the primitivity of x, the only Y-interpretation disjoint from $(1, x^2, 1)$ of x^2 is (s_1, y, s_2).

Suppose now that x and y are not conjugate. If $(q, d, p) \neq (q', d', p')$ are two different Y-interpretations of x^2 disjoint from $(1, x^2, 1)$, then, by Lemma 9, we get $d = d' = y$, $q, q' \in \mathrm{suf}(x)$ and $p, p' \in \mathrm{pref}(x)$. Therefore, x^2 has a period $||q| - |q'|| < |x|$ and, by Corollary 2, x^2 has also a period $k = \gcd(|x|, ||q| - |q'||)$. Since $k < |x|$ and k divides $|x|$, we have a contradiction with x being primitive. $\square$

Since we would like to deal not only with codes with both code words primitive, we will start with few claims which are coping with imprimitive code words. Let $X = \{x^{j_x}, y^{j_y}\} \subseteq A^*$ and $Y = \{x, y\}$ be codes with x and y primitive words. Notice that each X-interpretation of a word w naturally induces an Y-interpretation of w. Formally, an Y-interpretation (q_Y, d_Y, p_Y) of w *is induced by* an X-interpretation (q_X, d_X, p_X) of w if $q_Y^{-1} q_X \in z^*$ for some $z \in Y$ such that q_X is a proper suffix of z^{j_z}, and $p_X p_Y^{-1} \in z'^*$ for some $z' \in Y$ such that p_X is a proper prefix of $z'^{j_{z'}}$. It follows from the definition of interpretations that $d_Y = q_Y^{-1} q_X d_X p_X p_Y^{-1}$. Notice at this point that one X-interpretations can induce various Y-interpretations. This is due to the ambiguity of q_X (p_X resp.) which can be a suffix (a prefix resp.) of both x^{j_x} and y^{j_y}. Words q_X, p_X in this case represent "an ambiguous part" of the factorisation of w into Y. Let us illustrate it by an example:

Example 3. Let $Y = \{a, baa\}$, $X = \{a^2, (baa)^2\}$ and $w = abaa$. Then $(a, 1, baa)$ is an X-interpretation of w. This interpretation induces two different Y - interpretations, namely $(a, baa, 1)$ and $(1, abaa, 1)$.

Also, we would like to point out that induced Y-interpretation only reflects the ambiguity already present in the original X-interpretation. If no such ambiguity is present (i.e. if $q_X \not\leq_{\mathrm{s}} x^{j_x} \wedge_s y^{j_y}$, and similarly for p_X), then induced Y-interpretation is unique. We illustrate it by the following example:

Example 4. Let $Y = \{a, baa\}$, $X = \{a^2, (baa)^2\}$ and $w = abaa$. Then $(abaa, 1, 1)$ is an X-interpretation of w. Since $abaa \not\leq_{\mathrm{s}} aa \wedge_s baabaa$, it induces unique Y-interpretation, namely $(a, baa, 1)$. Especially, notice that $(1, abaa, 1)$ is not its induced Y-interpretation since $abaa$ is unambiguously a suffix of $baabaa$.

In this sense, the set of all induced Y-interpretations corresponding to the same X-interpretation only reflects all possibilities of factorisation of the original interpretation in Y. Therefore, it should not be surprising that if there is an x in the induced Y-interpretation, then there was x^{j_x} "around" in the original X-interpretation. Formally, we formulate it as the next lemma:

Lemma 12. *Let $X = \{x^{j_x}, y^{j_y}\} \subseteq A^*$ and $Y = \{x, y\} \subseteq A^*$ be codes with x and y primitive words. Let (r, s, t) be an Y-interpretation of w induced by an*

X-interpretation (q,d,p) and let $s_1, s_2 \in Y^*$ be words such that $s = s_1 s_2$. Then there are words $d_1, d_2 \in X^*$ such that $d = d_1 d_2$ and

$$qd_1 = rs_1c^k, \qquad\qquad d_2p = c^{-k}s_2t,$$

for some $c \in Y$, $k \in \mathbb{Z}$ and $|k| < j_c$. Moreover, if $k \neq 0$, then $rs_1 \in \operatorname{suf}(Y^*c)$ and $s_2t \in \operatorname{pref}(cY^*)$.

Proof. From the definition of interpretation we get

$$s_1s_2 = r^{-1}qdpt^{-1}.$$

The definition of induced Y-interpretation yields

$$r^{-1}q = z_0^{i_0}, \qquad d = z_1^{i_1}z_2^{i_2}\cdots z_{m-1}^{i_{m-1}}, \qquad pt^{-1} = z_m^{i_m},$$

with $z_0, z_m \in Y \cup \{1\}$, $z_n \in Y$ for $n = 1, \ldots, m-1$, $i_n = j_{z_n}$ for $n = 1, 2, \ldots, m-1$, and $0 \le i_0 < j_{z_0}$ if $z_0 \in Y$, $0 \le i_m < j_{z_m}$ if $z_m \in Y$. Then

$$z_0^{i_0}z_1^{i_1}z_2^{i_2}\cdots z_m^{i_m},$$

is the unique decomposition of the word s_1s_2 into words from the code Y. Since $s_1, s_2 \in Y^*$, we have that

$$s_1 = z_0^{i_0}z_1^{i_1}\cdots z_{m'}^{\ell},$$

for some $0 \le m' \le m$ where $0 \le \ell \le i_{m'}$.

If $m' = 0$, then $rs_1z_0^k = q$ and $z_0^{-k}s_2t = dp$ where $k = i_0 - \ell < j_{z_0}$ in case that $z_0 \in Y$ and $k = 0$ otherwise. If $k \neq 0$, then $s_2t = z_0^k dp \in \operatorname{pref}(z_0Y^*)$ and $q \le_{\mathrm{s}} z_0^{j_0}$ yields $rs_1 \in \operatorname{suf}(Y^*z_0)$.

If $m' = m$, then $z_m^k s_2 t = p$ and $rs_1z_m^{-k} = qd$ where $k = \ell \le i_m < j_{z_m}$ in case that $z_0 \in Y$ and $k = 0$ otherwise. If $k \neq 0$, then $rs_1 = qdz_m^k \in \operatorname{suf}(Y^*z_m)$ and $p \le_{\mathrm{p}} z_m^{j_m}$ yields $s_2t \in \operatorname{pref}(z_mY^*)$.

Suppose that $m > m' > 0$. If $\ell = 0$ or $\ell = i_{m'}$, then $rs_1 \in qX^*$, $s_2t \in X^*p$ and our lemma holds with $k = 0$.

If $0 < \ell < i_{m'}$, then

$$rs_1z_{m'}^k = qz_1^{i_1}\cdots z_{m'}^{i_{m'}}, \qquad\qquad z_{m'}^{-k}s_2t = z_{i_{m'+1}}^{m'+1}\cdots z_{m-1}^{i_{m-1}}p,$$

where $k = i_{m'} - \ell < j_{z_{m'}}$. Then $rs_1 \in \operatorname{suf}(Y^*z_{m'}^{\ell})$ and $s_2t \in \operatorname{pref}(z_{m'}^kY^*)$. Since $k \neq 0$ and $\ell \neq 0$ we have finished the proof.

□

In the next lemma, we will have a closer look at interpretations of w which are shifted by x. We will see that if w allows two disjoint interpretations shifted by x, then w is a factor of x^+.

Lemma 13. *Let $X = \{x^{j_x}, y^{j_y}\} \subseteq A^*$ be a code with x and y primitive words. Suppose that a word $w \in A^*$ allows two disjoint X-interpretations which are shifted by x. Then w is a factor of x^+.*

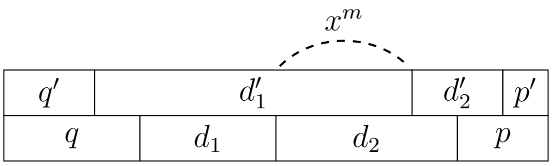

Figure 2: Two disjoint X-interpretation of w shifted by x.

Proof. Let (q, d, p), (q', d', p') be disjoint X-interpretation of w shifted by x. Then there are words $d_1, d_2, d'_1, d'_2 \in X^*$ such that $d = d_1 d_2$, $d' = d'_1 d'_2$ and $d_2 p (d'_2 p')^{-1} = x^m$, for some $m \geq 1$ (see Fig. 2). Assume that $d_2 p$ is not a prefix of x^+. Let u, u' be words such that $pu, p'u' \in X$. Without loss of generality, we can suppose that $d'_2 p' u' \in y^{j_y} X^*$. Notice that $d_2 pu \notin x^m y^{j_y} X^*$, otherwise j_x divides m and (p, d, q), (p', d', q') are adjacent. Therefore, by Corollary 4, $x^m d'_2 p'$ is a prefix of x^+. Similarly, we can show that $q d_1$ is a suffix of x^+.

□

One should notice that reverse implication from Lemma 13 does not hold. We can indeed find two disjoint X-interpretations of a factor of x^+ which are not shifted by x; just consider for example the case when x and y are conjugate.

In what follows, we will use the following abbreviation: We will say that two X-interpretations I, I' of w are Y-adjacent iff there are two adjacent Y-interpretations J, J' of w such that J is induced by I and J' is induced by I'. Otherwise, we will say that they are Y-disjoint. Notice that each pair of X-adjacent interpretations is also Y-adjacent. Now, we will be interested in X-interpretations which are X-disjoint and Y-adjacent at the same time.

In the next lemma we will consider a word w which is not a factor of y^+. We will show that if w allows two disjoint X-interpretations which are at the same time Y-adjacent, then these X-interpretations has to be shifted by x:

Lemma 14. *Let $X = \{x^{j_x}, y^{j_y}\} \subseteq A^*$ and $Y = \{x, y\} \subseteq A^*$ be codes with x and y primitive words. Suppose that a word $w \in A^*$ allows two disjoint X-interpretations I_1, I_2 which are Y-adjacent. If w is not a factor of y^+, then I_1, I_2 are shifted by x.*

Proof. Let $I_1 = (q, d, p)$, $I_2 = (q', d', p')$ and let (r, s, t), resp. (r', s', t'), be adjacent Y-interpretations induced by I_1, resp. I_2. By the definition of adjacent interpretations, there exist words $s_1, s_2, s'_1, s'_2 \in Y^*$ such that $s = s_1 s_2$, $s' = s'_1 s'_2$ and $r s_1 = r' s'_1$. By Lemma 12, there are $c, c' \in Y$ and exponents $e, e' \in \mathbb{Z}$ such that $|e| < j_c$, $|e'| < j_{c'}$,

$$q d_1 = r s_1 c^e, \qquad d_2 p = c^{-e} s_2 t,$$
$$q' d'_1 = r' s'_1 c'^{e'}, \qquad d'_2 p' = c'^{-e'} s'_2 t',$$

where $d_1, d'_1, d_2, d'_2 \in X^*$, $d = d_1 d_2$ and $d' = d'_1 d'_2$. Notice that $e \neq 0$ or $e' \neq 0$, otherwise (q, d, p), (q', d', p') would be X-adjacent. If now $c = c' = x$, or $e' = 0$ and $c = x$, or $e = 0$ and $c' = x$, then interpretations (q, d, p), (q', d', p') are shifted by x. If $c = c' = y$, or $e' = 0$ and $c = y$, or $e = 0$ and $c' = y$, then (q, d, p),

(q', d', p') are shifted by y and word w is a factor of y^+ by Lemma 13. Suppose therefore that $\{c, c'\} = Y$, $e \neq 0$ and $e' \neq 0$. By Lemma 12, we have

$$\begin{aligned} rs_1 &\in \mathrm{suf}(Y^*c), & s_2t &\in \mathrm{pref}(cY^*), \\ r's_1' &\in \mathrm{suf}(Y^*c'), & s_2't' &\in \mathrm{pref}(c'Y^*). \end{aligned}$$

Since $rs_1 = r's_1'$ and $s_2t = s_2't'$, we can use Corollary 4 and obtain that rs_1 is a suffix of y^+ and s_2t is a prefix of y^+. Then w is a factor of y^+. □

Notice that, according to the previous lemma, two X-interpretations of w which are Y-adjacent are not shifted by x or by y only if w is a factor of both x^+ and y^+. This fact implies the following corollary:

Corollary 15. *Let $X = \{x^{j_x}, y^{j_y}\} \subseteq A^*$ and $Y = \{x, y\} \subseteq A^*$ be a codes with x and y primitive words which are not conjugate. Let $w \in A^*$ be a word such that $|w| \geq \max\{j_x|x|, j_y|y|\}$ and let I_1, I_2 be two disjoint X-interpretation of w which are Y-adjacent. Then I_1, I_2 are shifted by x or by y.*

Proof. Let $I_1 = (q, d, p)$, $I_2 = (q', d', p')$ and let (r, s, t), resp. (r', s', t'), be adjacent Y-interpretations induced by I_1, resp. I_2. By the definition of adjacent interpretations, there exist words $s_1, s_2, s_1', s_2' \in Y^*$ such that $s = s_1s_2$, $s' = s_1's_2'$ and $rs_1 = r's_1'$. By Lemma 12, there are $c, c' \in Y$ and exponents $e, e' \in \mathbb{Z}$ such that $|e| < j_c$, $|e'| < j_{c'}$,

$$\begin{aligned} qd_1 &= rs_1c^e, & d_2p &= c^{-e}s_2t, \\ q'd_1' &= r's_1'c'^{e'}, & d_2'p' &= c'^{-e'}s_2't', \end{aligned}$$

where $d_1, d_1', d_2, d_2' \in X^*$, $d = d_1d_2$ and $d' = d_1'd_2'$. Notice that $e \neq 0$ or $e' \neq 0$, otherwise (q, d, p), (q', d', p') would be X-adjacent. Suppose that interpretations (q, d, p), (q', d', p') are not shifted by x nor by y. Then $Y = \{c, c'\}$ and $e \neq 0 \neq e'$. Since $|e| < j_c$, $|e'| < j_{c'}$ we have $j_x \geq 2$ and $j_y \geq 2$. By Lemma 14, word w is a factor of both y^+ and x^+. Then, from $|w| \geq \max\{j_y|y|, j_x|x|\} \geq |x| + |y|$, it follows that words x and y are conjugate, a contradiction.

□

Let us now focus on interpretations of the word $x^{j_x}y^{j_y}$:

Lemma 16. *Let $X = \{x^{j_x}, y^{j_y}\} \subseteq A^*$ be a code with x and y primitive words and suppose that $j_x|x| \geq j_y|y|$. Let*

$$\mathcal{I} = \{(I_0 = (1, x^{j_x}y^{j_y}, 1)\} \cup \{(I_i = (q_i, d_i, p_i), i \in \{1, \ldots, 3\}\}$$

be pairwise disjoint X-interpretations of $x^{j_x}y^{j_y}$. Then

- *there is $k \geq 1$ such that $x^{j_x}y^{kj_y}$ is not primitive and every pair of X-interpretations from $\mathcal{I}$ is shifted by the primitive root of $x^{j_x}y^{kj_y}$, or*
- *there are two X-interpretations $I_i, I_j \in \mathcal{I}$ shifted by x or by y.*

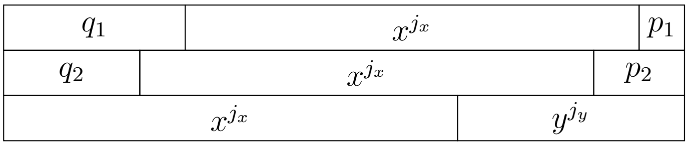

Figure 3: Case $I_1 = (q_1, x^{j_x}, p_1)$, $I_2 = (q_2, x^{j_x}, p_2)$ and $q_1, q_2 \in \mathrm{suf}(y^{j_y})$.

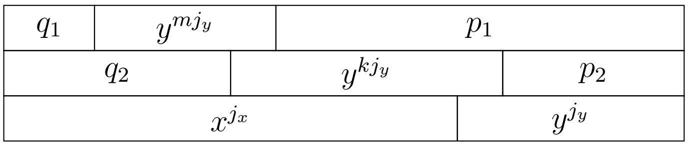

Figure 4: Case $I_1 = (q_1, y^{kj_y}, p_1)$ and $I_2 = (q_2, y^{mj_y}, p_2)$ and $p_1, p_2 \in \mathrm{pref}(x^{j_x})$.

Proof. Let $Y = \{x, y\}$.
1. Suppose that there are two X-interpretations from $\mathcal{I}$ which are at the same time Y-adjacent. Suppose moreover that they are not shifted neither by x nor by y. Then, by Lemma 14, the word $x^{j_x}y^{j_y}$ is a factor of both y^+ and x^+. Moreover, the primitivity of both words x and y yields that $x^{j_x}y^{j_y}$ is a prefix of x^+ and a suffix of y^+. Since $|x^{j_x}y^{j_y}| \geq |x| + |y|$, it follows from the Periodicity lemma that $x = y$, a contradiction with X being a code.
2. Suppose that all X-interpretations from $\mathcal{I}$ are Y-disjoint. Then for any X-interpretations $(q, d, p) \in \mathcal{I}\backslash\{I_0\}$, either $d = x^{j_x}$ and $q \not\leq_{\mathrm{s}} x^{j_x}$, or $d \in y^+$ and $p \not\leq_{\mathrm{p}} y^{j_y}$, otherwise (q, d, p) and I_0 would be Y-adjacent. Indeed, this is an easy consequence of the fact that if $z = z_1 z_2$ for some nonempty words z_1, z_2 such that $z_1 \leq_{\mathrm{s}} z$ and $z_2 \leq_{\mathrm{p}} z$, then z, z_1 and z_2 commute. In what follows, this fact will be used without explicit reference. Since $|\mathcal{I}\backslash\{I_0\}| = 3$, we can suppose, without loss of generality, that one of the following possibilities takes place:
2.1 Suppose that $I_1 = (q_1, x^{j_x}, p_1)$, $I_2 = (q_2, x^{j_x}, p_2)$ and $q_1 <_{\mathrm{s}} y^{j_y}$, $q_2 <_{\mathrm{s}} y^{j_y}$ (Fig. 3). Suppose that $|q_1| > |q_2|$. Then $p_2 <_{\mathrm{p}} y^{j_y}$ and, by Lemma 1, both $x^{j_x}p_2$ and $x^{j_x}y^{j_y}$ commute with q_2. Since q_1 has both periods $|q_2|$ and $|q_2^{-1}q_1|$, we obtain from Lemma 1 that q_1 and q_2 also commute. If now $I_3 = (q, x^{j_x}, p)$, with $q <_{\mathrm{s}} y^{j_y}$, is an X-interpretation of $x^{j_x}y^{j_y}$, then obviously q and q_2 commute. If $I_3 = (q, y^{mj_y}, p)$, with $p <_{\mathrm{p}} x^{j_x}$, is an X-interpretation of $x^{j_x}y^{j_y}$, then qy^{mj_y} has both periods $|q_2|$ and $|q_2^{-1}qy^{mj_y}|$. Therefore, by Lemma 1, words qy^{mj_y} and q_2 commute. We have shown that in this case every pair of X-interpretations from $\mathcal{I}$ is shifted by the primitive root of $x^{j_x}y^{j_y}$.
2.2 Suppose now that $I_1 = (q_1, y^{kj_y}, p_1)$ and $I_2 = (q_2, y^{mj_y}, p_2)$ and $p_1 <_{\mathrm{p}} x^{j_x}$, $p_2 <_{\mathrm{p}} x^{j_x}$ (Fig. 4). Suppose that $|p_1| > |p_2|$. Then $|q_1| < |q_2|$ and $q_2 <_{\mathrm{s}} x^{j_x}$. Since $p_2 \leq_{\mathrm{p}} p_1$, we get by [2, Lemma 8] that $x^{j_x}y^{j_y}$ has a period $\gcd(j_x|x| - |q_2|, |p_1| - |p_2|)$, and consequently $y^{kj_y} \in (s_2 s_1)^* s_2$, where $s_1 s_2$ is the primitive root of $p_1 p_2^{-1}$ and $s_2 s_1$ is the primitive root of $q_2^{-1}x^{j_x}$. Since x^{j_x} is a factor of $(s_1 s_2)^+$ and $s_2 s_1 \leq_{\mathrm{s}} x^{j_x}$, $s_1 s_2 \leq_{\mathrm{p}} p_1 <_{\mathrm{p}} x^{j_x}$, we get that $x^{j_x} \in (s_1 s_2)^+ s_1$. Notice that both words $q_1 y^{mj_y}$ and $q_2 y^{kj_y}$ commute with $s_1 s_2$. If now $I_3 = (q, y^{\ell j_y}, p)$, with $p <_{\mathrm{p}} x^{j_x}$, is an X-interpretation of $x^{j_x}y^{j_y}$, then obviously $qy^{\ell j_y}$ and $s_1 s_2$ commute. If $I_3 = (q, x^{j_x}, p)$, with $q <_{\mathrm{s}} y^{j_y}$, is an X-interpretation of $x^{j_x}y^{j_y}$, then from $s_1 s_2 \leq_{\mathrm{p}} x^{j_x}$ we obtain a commutativity of q and $(s_1 s_2)$. We have shown that in this case every pair of X-interpretations from $\mathcal{I}$ is shifted by the primitive root of $x^{j_x}y^{kj_y}$. $\square$

Slightly modified version of the previous lemma holds also for X-interpretations of $y^{j_y}x^{j_x}$. The proof matches the proof of Lemma 16 and therefore will be omitted.

Lemma 17. *Let $X = \{x^{j_x}, y^{j_y}\} \subseteq A^*$ be a code with x and y primitive words and suppose that $j_x|x| \geq j_y|y|$. Let*

$$\mathcal{I} = \{(I_0 = (1, y^{j_y}x^{j_x}, 1)\} \cup \{(I_i = (q_i, d_i, p_i), i \in \{1, \dots, 3\}\}$$

be pairwise disjoint X-interpretations of $y^{j_y}x^{j_x}$. Then

- *there is $k \geq 1$ such that $y^{kj_y}x^{j_x}$ is not primitive and every pair of X-interpretations from $\mathcal{I}$ is shifted by the primitive root of $y^{kj_y}x^{j_x}$, or*
- *there are two disjoint X-interpretations $I_i, I_j \in \mathcal{I}$ shifted by x or by y.*

Importantly, notice also that the conclusions of Lemma 16 and Lemma 17 hold even if $\mathcal{I}$ consists of more than four disjoint X-interpretations. Indeed, by [17, Section 3.5], the set $x^{j_x}(y^{j_y})^+ \cup (x^{j_x})^+y^{j_y}$ contains at most one imprimitive word.

A similar result as in Lemma 16 can be formulated for interpretations of a word $uy^{3j_y}v \in A^*$ longer than $j_x|x|$. We first formulate three auxiliary lemmas:

Lemma 18. *Let $u, w \in A^*$ be words such that u^2 is a factor of w. Suppose that*

$$w = q_1p_1 = q_2p_2,$$

where $q_1 <_{\mathrm{s}} q_2$ and $p_1 <_{\mathrm{p}} u$. Then $q_1^{-1}q_2$ and u commute.

Proof. Since $q_1 <_{\mathrm{s}} q_2$, q_2 has a period $|q_2| - |q_1| \leq |p_1| < |u|$. From u^2 being a factor of w and Lemma 1, it follows that primitive roots of $q_1^{-1}q_2$ and u are conjugate. The rest is the consequence of $q_1^{-1}q_2$ being a prefix of u.

□

Lemma 19. *Let u, v, w be words such that u^3 is a factor of w. Suppose $|w| \geq |v|$. Let*

$$w = q_1p_1 = q_2p_2$$

with $q_1 \leq_{\mathrm{s}} v$, $|p_1| \leq |u|$, $|q_2| \leq |u|$ and $p_2 \leq_{\mathrm{p}} v$. Then both words v and w are factors of u^+.

Proof. Both w and v have a period $q = |p_1| + |q_2| + |v| - |w| \leq 2|u|$. Since u^3 is a factor of w, we infer from Lemma 1, that the primitive root of u has length q. Since u is a factor of both w and v, they are both factors of u^+.

□

Lemma 20. *Let $X = \{x^{j_x}, y^{j_y}\} \subseteq A^*$ and $Y = \{x, y\} \subseteq A^*$ be a codes with x and y primitive words which are not conjugate. Let I_1, I_2 be two X-interpretations of y^{3j_y}. If $|y| \geq |x|$ or $j_y|y| \geq j_x|x|$, then I_1 and I_2 are Y-adjacent.*

Proof. **1.** Suppose first that $|y| \geq |x|$. Let $(r,s,t), (r',s',t')$ be two Y-interpretations of y^{3j_y} induced by interpretations I_1, I_2. By Corollary 10, interpretations (r,s,t), (r',s',t') are Y-adjacent to $(1, y^{3j_y}, 1)$. Therefore, there exist exponents $e, e' \in \mathbb{N}$ such that $e \leq 3j_y$, $e' \leq 3j_y$ and

$$\begin{aligned} rs_1 &= y^e, & s_2t &= y^{3j_y - e}, \\ r's'_1 &= y^{e'}, & s'_2t' &= y^{3j_y - e'}, \end{aligned} \tag{2.1}$$

for some words $s_1, s_2, s'_1, s'_2 \in Y^*$ such that $s = s_1s_2$, $s' = s'_1s'_2$. Let $e, e' \in \mathbb{N}$ be the smallest exponents satisfying Eq. (2.1). Then $e \leq 1$ and $e' \leq 1$. Indeed, if for example $e \geq 2$, then from $|r| < |y|$ and the minimality of e, it follows that $s_1 \in Y^*x$. By Corollary 4, we get that y^e is a suffix of x^+ and consequently, by Lemma 1, $x = y$, a contradiction with X being a code. If $e = e'$, then I_1, I_2 are Y-adjacent and we are through. Suppose therefore for example that $e < e'$. Then $e = 0$ and $s_2t = y^{3j_y}$. From $|t| < |y|$ we infer that $s_2 \neq 1$. If $s_2 \in yY^*$, interpretations I_1 and I_2 are Y-adjacent. On the other hand, if $s_2 \in xY^*$, then by Corollary 4, word y^{3j_y} is a prefix of x^+ and consequently, by Lemma 1, $x = y$, a contradiction with X being a code.

2. Suppose now that $j_y|y| \geq j_x|x|$ and $|x| > |y|$. Let $I_1 = (q,d,p)$ and $I_2 = (q',d',p')$. Since $j_y|y| \geq j_x|x|$ we have $|q| < j_y|y|$, $|q'| < j_y|y|$ and $|p| < j_y|y|$, $|p'| < j_y|y|$, and consequently both words d and d' are non-empty.

2.1 Suppose first that $d \in x^+$. If $q <_{\mathrm{s}} x^{j_x}$ or $|q^{-1}y^{j_y}| \geq |y|$, then x^+ and y^+ has a common factor of the length at least $(j_y+1)|y|$. Hence, Lemma 1 yields that words x and y are conjugate, a contradiction. Suppose that $q <_{\mathrm{s}} y^{j_y}$ and $|q^{-1}y^{j_y}| < |y|$. Inequalities $j_y|y| \geq j_x|x|$ and $|x| > |y|$ yield that $j_y \geq 2$. Then $y \leq_{\mathrm{s}} q$ and $q^{-1}y^{j_y} \in y^+$, a contradiction with $|q^{-1}y^{j_y}| < |y|$.

2.2 Finally, let us consider a case when $d \in d_1y^+d_2$ and $d' = d'_1y^+d'_2$ for some $d_1, d'_1, d_2, d'_2 \in X^*$. Then

$$\begin{aligned} qd_1 &= y^e, & d_1^{-1}dp &= y^{3j_y - e}, \\ q'd'_1 &= y^{e'}, & {d'_1}^{-1}d'p' &= y^{3j_y - e'}. \end{aligned} \tag{2.2}$$

Let $e, e' \in \mathbb{N}$ be the smallest exponents satisfying Eq. (2.2). Then $e \leq j_y$ and $e' \leq j_y$. Indeed, if for example $e > j_y$, then from $|q| < j_y|y|$ and the minimality of e, it follows that $d_1 \in X^*x^{j_x}$. Consequently, from Corollary 4, it follows that qd_1 is a suffix of x^+. Since $|qd_1| \geq (j_y+1)|y|$ and $j_y|y| \geq j_x|x|$, Lemma 1 yields commutativity of x and y, a contradiction. If $e = e'$, then I_1, I_2 are Y-adjacent. Suppose therefore that $e < e'$. Notice that ${d'_1}^{-1}d' \in y^{j_y}X^*$. Then from $e' - e \leq j_y$, it follows that I_1 and I_2 are Y-adjacent.

□

Let us now formulate the announced key result of this part which concerns interpretations of a word $uy^{3j_y}v \in A^*$ longer than $j_x|x|$:

Lemma 21. *Let $X = \{x^{j_x}, y^{j_y}\} \subseteq A^*$ be a code with x and y primitive words which are not conjugate. Let y^{3j_y} be a factor of a word w such that $|w| \geq j_x|x|$ and let*

$$\mathcal{I} = \{(I_i = (q_i, d_i, p_i), i \in \{0, \ldots, 3\}\}$$

be pairwise disjoint X-interpretations of w. Then

- *there is $k \geq 1$ such that $x^{j_x}y^{kj_y}$ is not primitive and for each X-interpretation $I \in \mathcal{I}$ there exists X-interpretation $J \in \mathcal{I}$, disjoint from I, such that I and J are shifted by the primitive root of $x^{j_x}y^{kj_y}$ or by the primitive root of $y^{kj_y}x^{j_x}$, or*

- *there are two X-interpretations $I_i, I_j \in \mathcal{I}$ shifted by x or by y.*

Proof. Let $Y = \{x, y\}$. Suppose that there are X-interpretations $I_i, I_j \in \mathcal{I}$ which are Y-adjacent. Then Corollary 15 yields that I_i, I_j are shifted by x or by y.

Suppose now that all X-interpretations from $\mathcal{I}$ are pairwise Y-disjoint. Then, from Lemma 20, it follows that $|x| > |y|$ and $j_x|x| > j_y|y|$.

Denote $D = \{d_0, d_1, d_2, d_3\}$. We continue by case analysis.

1. Let $(q, d, p) \in \mathcal{I}$ be an X-interpretation such that $d \in X^*x^{2j_x}X^*$. By Corollary 11, there is at most one Y-interpretation of x^2 Y-disjoint from $(1, x^2, 1)$. Since each X-interpretation from $\mathcal{I}$ induces Y-interpretation of w and $|\mathcal{I}| = 4$, we get a contradiction with our assumption that all X-interpretations from $\mathcal{I}$ are pairwise Y-disjoint.

2. Let $(q, d, p) \in \mathcal{I}$ be an X-interpretation such that $d \in X^*x^{j_x}y^{j_y}X^*$ for some $d \in D$. Then $\mathcal{I}$ yields four pairwise disjoint X-interpretations of $x^{j_x}y^{j_y}$, and the lemma holds by Lemma 16. Similarly, the claim holds by Lemma 17, if $d \in X^*y^{j_y}x^{j_x}X^*$.

We can therefore suppose that for each $d \in D$ we have $d = 1$, $d = x^{j_x}$ or $d \in y^+$. We now continue with our analysis:

3. $(\mathbf{q_i}, \mathbf{x^{j_x}}, \mathbf{p_i}), (\mathbf{q_j}, \mathbf{x^{j_x}}, \mathbf{p_j})$ Suppose that $d_i = d_j = x^{j_x}$ for $i \neq j$, $i, j \in \{0, 1, 2, 3\}$. Since interpretations are disjoint, we can suppose that $|q_i| < |q_j|$ and $|p_i| > |p_j|$. If now $q_j \leq_{\mathrm{s}} x^{j_x}$ (resp. $p_i \leq_{\mathrm{p}} x^{j_x}$), then $q_i^{-1}q_j$ (resp. $p_i{p_j}^{-1}$) and x commute, and (q_i, x^{j_x}, p_i), (q_j, x^{j_x}, p_j) are Y-adjacent, a contradiction. Suppose now that $q_j \leq_{\mathrm{p}} y^{j_y}$ and $p_i \leq_{\mathrm{p}} y^{j_y}$. Since $|x^{j_x}| \geq |y^{j_y}|$ and y^{3j_y} is a factor of w, words y^{3j_y} and $q_i^{-1}w{p_j}^{-1}$ have a common factor of the length at least $|q_j| - |q_i| + j_y|y|$. Since $|q_j| - |q_i|$ is a period of $q_i^{-1}w{p_j}^{-1}$, primitive roots of $q_i^{-1}q_j$ and y are conjugate. From $q_i^{-1}q_j$ being a suffix of y^{j_y}, we obtain the commutativity of y and $q_i^{-1}q_j$. Therefore, (q_i, x^{j_x}, p_i) and (q_j, x^{j_x}, p_j) are Y-adjacent, a contradiction.

4. $(\mathbf{q_i}, \mathbf{x^{j_x}}, \mathbf{p_i}), (\mathbf{suf}(\mathbf{x^{j_x}}), \mathbf{1}, \mathbf{pref}(\mathbf{y^{j_y}}))$ Suppose that for some $i, j \in \{0, 1, 2, 3\}$, we have $d_j = 1$, $q_j <_s x^{j_x}$, $p_j <_p y^{j_y}$ and $d_i = x^{j_x}$. Then $q_j^{-1}q_ix^{j_x}$ and y commute by Lemma 18 (with $\underline{q_1} = q_j$ and $\underline{q_2} = q_ix^{j_x}$). Therefore, (q_i, x^{j_x}, p_i) and $(q_j, 1, p_j)$ are Y-adjacent, a contradiction.

5. $(\mathbf{suf}(\mathbf{x^{j_x}}), \mathbf{1}, \mathbf{pref}(\mathbf{y^{j_y}})), (\mathbf{suf}(\mathbf{x^{j_x}}), \mathbf{1}, \mathbf{pref}(\mathbf{y^{j_y}}))$ If for $i \neq j$, $i, j \in \{0, 1, 2, 3\}$, we have $d_i = d_j = 1$, $q_i, q_j <_s x^{j_x}$ and $p_i, p_j <_p y^{j_y}$, then $q_j^{-1}q_i$ (or $q_i^{-1}q_j$) and y commute by Lemma 18. Therefore, $(q_i, 1, p_i)$ and $(q_j, 1, p_j)$ are Y-adjacent, a contradiction.

6. $(\mathbf{suf}(\mathbf{x^{j_x}}), \mathbf{1}, \mathbf{pref}(\mathbf{y^{j_y}})), (\mathbf{suf}(\mathbf{y^{j_y}}), \mathbf{1}, \mathbf{pref}(\mathbf{x^{j_x}}))$ Suppose that, for $i \neq j$, $i, j \in \{0, 1, 2, 3\}$, we have $d_i = d_j = 1$, $q_i <_s x^{j_x}$, $p_i <_p y^{j_y}$, $q_j <_s y^{j_y}$, $p_j <_p x^{j_x}$. By Lemma 19, both x^{j_x} and w are factors of y^+. Let $k \in \{0, 1, 2, 3\}$ and $k \neq i$, $k \neq j$.

6.1. If $d_k = x^{j_x}$, then we have **4**.

6.2. If $d_k = 1$, then we have **5** (up to symmetry).

6.3. It remains that $d_k, d_\ell \in y^+$ with $\{i, j, k, \ell\} = \{0, 1, 2, 3\}$. Let $w = y'y^\alpha y''$ where $y' \leq_{\mathrm{s}} y$, $y'' \leq_{\mathrm{p}} y$ and $\alpha \geq 3$. Then $q_k, q_\ell \in y'y^*$ and I_k, I_ℓ are shifted by y.

7. $((\mathbf{q_i}, \mathbf{x^{j_x}}, \mathbf{p_i}), (\mathbf{suf}(\mathbf{x^{j_x}}), \mathbf{1}, \mathbf{pref}(\mathbf{x^{j_x}}))$ If for $i, j \in \{0, 1, 2, 3\}$, we have $d_j = 1$, $q_j <_s x^{j_x}$, $p_j <_p x^{j_x}$ and $d_i = x^{j_x}$, then $q_i^{-1} q_j$ commute with x. Therefore, (q_i, x^{j_x}, p_i) and $(q_j, 1, p_j)$ are Y-adjacent, a contradiction.
8. $((\mathbf{suf}(\mathbf{x^{j_x}}), \mathbf{1}, \mathbf{pref}(\mathbf{x^{j_x}})), (\mathbf{suf}(\mathbf{x^{j_x}}), \mathbf{1}, \mathbf{pref}(\mathbf{x^{j_x}}))$ Similarly, if for $i \neq j$, $i, j \in \{0, 1, 2, 3\}$, we have $d_i = d_j = 1$, $q_i, q_j <_s x^{j_x}$ and $p_i, p_j <_p x^{j_x}$, then, since $|w| \geq |x^{j_x}|$, words $q_i^{-1} q_j$ (or $q_j^{-1} q_i$) commute with x. Therefore, $(q_i, 1, p_i)$ and $(q_j, 1, p_j)$ are Y-adjacent, a contradiction.
9. $(\mathbf{suf}(\mathbf{y^{j_y}}), \mathbf{1}, \mathbf{pref}(\mathbf{y^{j_y}}))$ Note that the case $d_i = 1$, $q_i <_s y^{j_y}$ and $p_i <_p y^{j_y}$ is impossible for length reasons.

The previous analysis shows that $\mathcal{I}$ contains at least two interpretations with $d_i \in y^+$ (see **3.**, **5.**, **6.**, **8.** and **9.**). Let therefore $d_0, d_1 \in y^+$, and suppose, without loss of generality, that $q_0 <_{\mathrm{p}} q_1$. This implies $q_1 <_s x^{j_x}$, otherwise $q_0^{-1} q_1$ and y commute and I_0, I_1 are Y-adjacent, a contradiction. Notice also that $q_0 d_0 <_{\mathrm{p}} q_1 y$, otherwise $q_0^{-1} q_1$ and y commute and I_0, I_1 are Y-adjacent, a contradiction. Therefore, $p_1 \leq_{\mathrm{s}} p_0$ and $p_0 <_{\mathrm{p}} x^{j_x}$, otherwise $p_0 p_1^{-1}$ commute with y and I_0, I_1 are Y-adjacent, a contradiction.

10. $(\mathbf{d_2} \in \mathbf{y^+})$ Suppose that $d_i \in y^+$ for $i \in \{2, 3\}$; say, $d_2 \in y^+$. We can suppose $q_0 <_p q_1 <_p q_2$, which implies, since the interpretations are Y-disjoint, that $p_2 <_s p_1 <_s p_0$ and $p_0, p_1 <_p x^{j_x}$ and $q_1, q_2 <_s x^{j_x}$. Let $d_i = y^{k_i j_y}$, $i = 0, 1, 2$. By Lemma [2, Lemma 8], w has a period $\gcd(|p_0| - |p_1|, |q_2| - |q_1|)$, and consequently $y^{k_1 j_y} \in (s_2 s_1)^* s_2$ where $s_1 s_2$ is the primitive root of $p_0 p_1^{-1}$ and $s_2 s_1$ is the primitive root of $q_1^{-1} q_2$. Since w has a period $|p_0| - |p_1|$, w is a factor of $(s_1 s_2)^+$.

Let $|s_1 s_2| \leq j_y |y|$. Since w contains $y^{3 j_y}$, Lemma 1 implies $|s_1 s_2| = |y|$, and $s_1 s_2 = s_2 s_1 = y$ follows from the presence of factors $y s_1 s_2$ and $s_2 s_1 y$ in w. Since the interpretations are Y-disjoint, words q_2 and p_0 have an overlap of length at least $|y|$, which implies, due to $|x^{j_x}| \leq |w|$, that x and y commute, a contradiction with X being a code.

Let therefore $|s_1 s_2| > j_y |y|$.
10.1. Suppose $d_3 = x^{j_x}$. This implies that x^{j_x} is a factor of $(s_1 s_2)^+$, and $x^{j_x} \in (s_1 s_2)^+ s_1$ holds since $s_1 s_2 <_p x^{j_x}$ and $s_2 s_1 <_s x^{j_x}$. Then $s_2 s_1$ is the primitive root of $y^{k_1 j_y} x^{j_x}$ and interpretations I_1, I_2, I_3 are pairwise shifted by $s_2 s_1$. Moreover, I_0 and I_1 are shifted by $s_1 s_2$.
10.2. Let $q_2 <_p q_3$ or $p_0 <_s p_3$, and suppose $d_3 \neq x^{j_x}$. If $q_2 <_{\mathrm{p}} q_3$, then $|q_3| > |s_1 s_2| > j_y |y|$. This implies that $q_3 <_s x^{j_x}$ and from $s_2 s_1 \leq_{\mathrm{s}} q_2 x^{j_x}$ follows that $q_3 \leq_{\mathrm{s}} (s_2 s_1)^+$. Then $s_2 s_1$ is the primitive root of $q_2^{-1} q_3$, and

$$|q_3| + |p_0| \geq |q_1| + 2|s_1 s_2| + |p_0| > |w| + |s_1 s_2|\,, \tag{2.3}$$

where the last inequality follows from $|s_1 s_2| > |y|$ and from $|q_1| + |p_0| + |y| > |w|$, that is, from the fact that $q_0 y^{j_y k_0}$ is a proper prefix of $q_1 y$. Since $q_3 <_{\mathrm{s}} x^{j_x}$, $p_0 <_{\mathrm{p}} x^{j_x}$ and $|w| \geq |x^{j_x}|$ we get from Eq. (2.3) that $x^{j_x} \in (s_1 s_2)^+ s_1$. Again, $s_2 s_1$ is the primitive root of $y^{k_1 j_y} x^{j_x}$ and interpretations I_1, I_2, I_3 are pairwise shifted by $s_2 s_1$ and interpretations I_0 and I_1 are shifted by $s_1 s_2$

The possibility $p_0 <_s p_3$ is symmetric. In particular, p_3 has periods $|s_1 s_2|$ and $p_3 p_0^{-1}$, $s_1 s_2$ is the primitive root of $p_3 p_0^{-1}$, and

$$|p_3| + |q_2| \geq |p_1| + 2|s_1 s_2| + |q_2| > |w| + |s_1 s_2|\,.$$

We obtain that $s_1 s_2$ is the primitive root of $x^{j_x} y^{k_1 j_y}$, interpretations I_0, I_1, I_3 are shifted by $s_1 s_2$ and interpretations I_2 and I_1 are shifted by $s_2 s_1$.

This case applies if $d_3 \in y^+$ since then we can suppose $q_2 <_p q_3$ without loss of generality. It also applies if $d_3 = 1$ and $p_3 <_p y^{j_y}$ or $q_3 <_s y^{j_y}$.

10.3. We can now suppose that $d_3 = 1$, $q_3 <_p q_2$, $q_3 <_s x^{j_x}$ and $p_3 <_s p_0$, $p_3 <_p x^{j_x}$. Then q_2 and p_0 overlap in w, and their overlap can be written as $p_0 p_3^{-1} q_3^{-1} q_2$. Since $p_0 p_3^{-1}$ is a period of p_0 and $q_3^{-1} q_2$ is a period of q_2, we deduce from Lemma 1 that $\gcd(|p_0| - |p_3|, |q_2| - |q_3|)$ is a period of w. From $q_2 <_{\mathrm{s}} x^{j_x}$, $p_0 <_{\mathrm{p}} x^{j_x}$ and $|w| \geq |x^{j_x}|$ we deduce that $q_3^{-1} q_2$ and x commute, and interpretations I_2 and I_3 are Y-adjacent, a contradiction.

We shall now study the last possibility. That is, when $d_2 \notin y^+$ and $d_3 \notin y^+$. Then $d_2 = d_3 = 1$ (see **3.**, **4.**, **7.** and **9.**). In such a case, we can suppose, without loss of generality, that $q_2 \leq_{\mathrm{s}} y^{j_y}$, $p_2 \leq_{\mathrm{p}} x^{j_x}$, and $q_3 \leq_{\mathrm{s}} x^{j_x}$, $p_3 \leq_{\mathrm{p}} x^{j_x}$ (see **5.**, **6.**, **8.** and **9.**).

11. $(\mathbf{d_0}, \mathbf{d_1} \in \mathbf{y}^+, (\mathbf{suf}(y^{j_y}), 1, \mathbf{pref}(x^{j_x})), ((\mathbf{suf}(x^{j_x}), 1, \mathbf{pref}(x^{j_x})))$ Let $d_2 = d_3 = 1$, $q_2 <_s y^{j_y}$, $p_2 <_p x^{j_x}$, $q_3 <_s x^{j_x}$ and $p_3 <_p x^{j_x}$. Suppose first that $|p_3| \leq j_y |y|$. Then, by Lemma 19, both w and x are factors of y^+. Then $q_1 \in q_0 y^+$ and $p_0 \in y^+ p_1$. Since $q_1 \leq_{\mathrm{s}} x^{j_x}$ and $p_0 \leq_{\mathrm{p}} x^{j_x}$, necessarily x and y commute, a contradiction with X being a code. Therefore, we suppose that $|p_3| > j_y |y|$.

11.1. If $|q_3| < |q_2|$, then p_3 has a period $|q_3^{-1} q_2|$, which is less than $j_y |y|$. Since $y^{3 j_y}$ is a factor of w, we get from Lemma 1 that $q_3^{-1} q_2$ and y commute. Therefore, I_2 and I_3 are Y-adjacent, a contradiction.

11.2. Let $|q_2| < |q_3| < |q_1|$. Then $q_2^{-1} q_1$ has a period $\gcd(|q_1| - |q_3|, |q_3| - |q_2|)$ which is also a period of p_2 and q_1. This implies that $q_3^{-1} q_1$ and x commute, and interpretations I_3 and I_1 are Y-adjacent, a contradiction.

11.3. If $|q_1| < |q_3| < |q_0 y^{k_0 j_y}|$, then q_3 has a period $|q_1^{-1} q_3|$ and p_3 has a period $|p_3 p_0^{-1}|$. Since $q_0 y^{j_y k_0} <_{\mathrm{p}} q_1 y$ we have

$$|q_3| + |p_3| = |w| < |q_1| + |p_0| + |y|,$$

and necessarily $|q_1^{-1} q_3| + |p_3 p_0^{-1}| < |y|$. Since $y^{3 j_y}$ is a factor of w, either $q_1^{-1} q_3 \in y^+$ or $p_3 p_0^{-1} \in y^+$, a contradiction.

11.4. Suppose that $|q_2| < |q_3|$, $|q_1| < |q_3|$ and that $|q_0 y^{k_0 j_y}| < |q_3|$. We point out two important factors of w. Denote by s the primitive root of $p_2 p_0^{-1}$, and by r the primitive root of $q_1^{-1} q_3$.

Note that w is a factor of s^+. From $|p_2| + |q_3| > |w| + |s|$ and $|x^{j_x}| \leq |w|$ we deduce that there are words s_1, s_2 such that $s = s_1 s_2$ and $x^{j_x} \in (s_1 s_2)^+ s_1$.

If $|s| \leq 2|y|$, then Lemma 1 implies, due to the presence of $y^{3 j_y}$ in w, that $|s| = |y|$. Suffix comparability of s and $y^{k_0 j_y}$ yields $s = y$. Since w is a factor of s^+, we have $q_1 \in q_0 y^+$. Since $q_1 <_{\mathrm{s}} x^{j_x}$ and x^{j_x} is a prefix of s^+, words x and y commute, a contradiction with x being a code. Therefore $|s| > 2|y|$.

11.4.1 Let $k_0 \geq k_1$. Denote by u the primitive root of $y^{k_1 j_y} p_0 (y^{k_1 j_y} p_1)^{-1}$. The word $y^{k_1 j_y} p_0 p_3^{-1}$ is a suffix of q_3 and therefore has a period $|r|$. Notice that the word $y^{k_1 j_y} p_0 p_3^{-1}$ has also a period $|u|$. Indeed, this is clear if $p_1 \leq_{\mathrm{p}} x^{j_x}$. On the other hand, if $p_1 <_{\mathrm{p}} y^{j_y}$, then from $|p_3| > j_y |y|$ we get also $|p_3| > |p_1|$ and $y^{k_1 j_y} p_0 p_3^{-1}$ is a prefix of $y^{k_1 j_y} p_0 p_1^{-1}$ and thus has a period $|u|$. Therefore $u = r$ and $w p_1^{-1}$ is a factor of r^+. Since $|w| - |p_1| \geq |p_2 p_0^{-1}| + |p_0 p_1^{-1}| \geq |s| + |r|$ and $r \leq_{\mathrm{s}} x^{j_x}$, Lemma 1 implies that $r = s_2 s_1$. Then interpretations I_0, I_1 and I_3 are shifted by $s_2 s_1$. Since $|p_0| \geq |u| = |r|$ and $p_0 <_{\mathrm{p}} x^{j_x}$, we get that $s_1 s_2 \leq_{\mathrm{p}} p_0$. Since $y^{k_1 j_y} p_0 \leq_{\mathrm{p}} r^+$, we have $y^{k_1 j_y} \in (s_2 s_1)^* s_2$ and interpretations I_0 and I_2 are shifted by $s_1 s_2$.

11.4.2. Let finally $k_0 < k_1$. Take $k = k_0$, in case that $q_0 \leq_s x^{j_x}$, and $k = k_0 + 1$, in case that $q_0 \leq_s y^{j_y}$. Denote by u the primitive root of $(q_0 y^{k_0 j_y})^{-1} q_1 y^{k j_y}$. From our choice of k, it follows that $q_0 y^{k_0 j_y}$ is a suffix of $q_1 y^{k j_y}$. Therefore, the word $q_1 y^{k j_y}$ has both periods $|s|$ and $|u|$. Since $q_1 y^{k j_y} \in q_2 s^+ u^+$, we obtain from the Periodicity lemma that $u = s$. Therefore, interpretations I_0, I_1, I_2 are shifted by s. It remains to show that I_3 and I_1 are shifted by $s_2 s_1$ and that $y^{k j_y} \in (s_2 s_1)^* s_2$.
11.4.2.1. If $|q_1| \geq |s|$, then both q_1 and q_3 have a suffix $s_2 s_1$ and I_3, I_1 are shifted by $s_2 s_1$, since w is a factor of s^+. Similarly, we deduce $y^{k j_y} \in (s_2 s_1)^* s_2$ from the fact that $s_2 s_1 y^{k j_y}$ is a factor of s^+, and the word $u = s$ is a suffix of $s_2 s_1 y^{k j_y} \leq_s q_1 y^{k j_y}$.
11.4.2.2. Suppose now that $|q_1| < |s|$. Since $|s| = |p_2| - |p_0|$, the word q_1 is a prefix of $q_0 y^{k_0 j_y}$. It follows from the definition of u and the fact that $s = u$, that $|s| < |y^{k j_y}|$. If $k < k_1$ then $|y^{k_1 j_y}| \geq |y| + |s|$ and we obtain from the Periodicity lemma that $|y| = |s|$, a contradiction with $|s| > 2|y|$. Therefore, $k = k_0 + 1 = k_1$, which implies $q_0 \leq_s y^{j_y}$ by the definition of k.

Let v denote the word $q_1^{-1} q_0 y^{k j_y}$. Note that v is an overlap of $y^{k_0 j_y}$ and $y^{k_1 j_y}$, which implies $|v| < |y|$. If $2|s| \leq |y^{k j_y}|$ then since $|s| > 2|y|$ the Periodicity lemma implies that $|y| = |s|$, a contradiction. Therefore, $y^{k j_y} = vs$.

Since $q_1 v$ is a suffix of $y^{k j_y}$, and of both $s_2 s_1 v$ and $v s_1 s_2$, it has periods $|y|$ and $|s_1 v|$.
11.4.2.2.1 If $|y| + |s_1 v| \leq |q_1 v|$, then we obtain from the Periodicity lemma that $s_1 v$ and y commute. Since $v s_1$ is a prefix of $y^{k j_y}$, we deduce that also v and y commute, a contradiction.
11.4.2.2.2 Therefore $|y| + |s_1 v| > |q_1 v| > |s| > 2|y|$, which implies $|s_1 v| > |y|$ and $|s_2| < |vy|$. Since $v s_1 \leq_p y^{k j_y}$ and $s_1 v \leq_s y^{k j_y}$, we deduce that $v s_1 v \in y^+$. From $v s_1 s_2 = y^{k j_y}$ and $|s_2| < |vy|$ we now have $s_2 = v$.

Therefore $y^{k j_y} = y^{k_1 j_y} = s_2 s_1 s_2$. Since w is a factor of s^+ and $s_2 s_1$ is a suffix of q_3, we also see that I_1 and I_3 are shifted by $s_2 s_1$.

This completes the whole proof.

□

Let us point out the differences between Lemma 16 and Lemma 21. In both, the first possibility yields that $x^{j_x} y^{j_y}$, resp. $u y^{3 j_y} v$, is a factor of $(s_1 s_2)^+$, where $s_1 s_2$ is the primitive root of $x^{j_x} y^{k j_y}$. Also, in both lemmas, we know that all interpretations from $\mathcal{I}$ have to follow the pattern implied by $(s_1 s_2)^+$. However, in the contrast with Lemma 16, where shifts can only be from $(s_1 s_2)^+$, the shifts in Lemma 21 can also be from $(s_2 s_1)^+$, or $s_2 (s_1 s_2)^*$, or $s_1 (s_2 s_1)^*$.

Notice that, same as in Lemma 16 and Lemma 17, the conclusions of Lemma 21 hold even if $\mathcal{I}$ consists of more than four disjoint X-interpretations. This is due to the fact that, by [17, Section 3.5], the set $x^{j_x} (y^{j_y})^+ \cup (x^{j_x})^+ y^{j_y}$ contains at most one imprimitive word.

If the words x and y are conjugate, then the situation is much easier. We will see that in case that w contains a word which is a factor of x^+ longer than $2|x|$, then w itself is a factor of x^+ and all X-interpretations of a word w follow the pattern imposed by x^+. First, we present the following easy lemma:

Lemma 22. *Let $Y = \{x_1 x_2, x_2 x_1\} \subseteq A^*$ be a code and suppose that $x_1 x_2$ is a primitive word. Let v be a factor of $(x_1 x_2)^+$ such that $|v| \geq 2|x_1 x_2|$ and let*

(q,d,p), (q',d',p') *be two disjoint Y-interpretations of v such that* $|q| > |q'|$. *Then*

$$q = x_i q',$$

for some $i \in \{1,2\}$.

Proof. It is an easy consequence of the fact that the primitive word cannot be a proper factor of its square. $\square$

Lemma 23. *Let* $X = \{x^{j_x}, y^{j_y}\} \subseteq A^*$ *be a code with x and y primitive words which are conjugate. Let* $u_1, u_2, v \in A^*$ *be words such that v is a factor of* x^+ *of the length at least* $2|x|$*, and suppose that* u_1vu_2 *admits two disjoint X-interpretations. Then* u_1vu_2 *is a factor of* x^+. *Let K be an arbitrary X-interpretation of* u_1vu_2. *If now* $I = (q, xd, p)$ *and* $J = (q', yd', p')$ *are X-interpretations of* u_1vu_2*, then either K and I are shifted by x, or K and J are shifted by y.*

Proof. Let x_1, x_2 be words such that $x = x_1x_2$ and $y = x_2x_1$. Let us denote $Y = \{x, y\}$ and $Z = \{x_1, x_2\}$. Let $(r, s, t), (r', s', t')$ be two Y-interpretations of u_1vu_2 induced by two disjoint X-interpretations of u_1vu_2. If they are Y-adjacent, then u_1vu_2 is a factor of x^+ by Lemma 13 and Lemma 14. Suppose therefore, that they are Y-disjoint. By Lemma 22, there are words $s_1, s_2, s_1', s_2' \in Y^*$ such that $s_1s_2 = s$, $s_1's_2' = s'$, and $s_2t = x_is_2't'$ or $s_2't' = x_is_2t$, for some $i \in \{1,2\}$.

Suppose, for example, that $s_2t = x_is_2't'$. We will show that $s_2t \leq_{\mathrm{p}} (x_ix_j)^+$, where $j \in \{1,2\}$, $j \neq i$. Let s_3 be a maximal prefix of s_2 from $(x_ix_j)^*$, and let s_3' be a maximal prefix of s_2' from $(x_jx_i)^*$. If $s_2t \not\leq_{\mathrm{p}} (x_ix_j)^+$, we obtain that

$$s_3^{-1}s_2t \leq_{\mathrm{p}} x_jx_iY^*, \qquad\qquad s_3'^{-1}s_2't' \leq_{\mathrm{p}} x_ix_jY^*.$$

If now $|x_is_3'| < |s_3|$, we have $(x_is_3')^{-1}s_3 \in (x_jx_i)^*x_j$ and

$$s_3'^{-1}s_2't' = s_3'^{-1}x_i^{-1}s_2t \in \mathrm{pref}(x_ix_jY^*) \cap \mathrm{pref}(x_jY^*) \in \mathrm{pref}(z_Z).$$

On the other hand, if $|s_3| < |x_is_3'|$, then $s_3^{-1}(x_is_3') \in (x_ix_j)^*x_i$ and

$$s_3^{-1}s_2t = s_3^{-1}x_is_2't' \in \mathrm{pref}(x_jx_iY^*) \cap \mathrm{pref}(x_iY^*) \in \mathrm{pref}(z_Z).$$

Since z_Z is prefix comparable with all words from Z^*, see Corollary 4, we get in both cases that $s_2t \leq_{\mathrm{p}} (x_ix_j)^+$. Similarly, we can show that $rs_1 \leq_{\mathrm{s}} (x_ix_j)^+$.

We have proved that u_1vu_2 is a factor of x^+. The remaining part can be easily deduced from the fact that u_1vu_2 is a factor of x^+ longer than $2|x|$.

$\square$

Let $X = \{x^{j_x}, y^{j_y}\}$ with x and y primitive words which are not conjugate. We have seen in Lemma 13 that if two disjoint X-interpretations of w are shifted by x, then w is a factor of x^+. Now we are interested in the reverse situation, that is, w being a factor of x^+ we are asking if there are any (and if so, how many) pairs of disjoint X-interpretations which are not shifted by x. Notice that not all interpretations of a factor of x^+ have to be pairwise shifted by x. This is obvious in case that w is shorter than x (then even no two X-interpretations of w are shifted by x). But also in case that $|w| > |x|$ we can construct examples with arbitrary many X-interpretations which are not shifted by x:

Example 5. Let $X = \{(aab)^i a, b(aab)^i\}$. Then $((aab)^k, 1, (aab)^{i-k}aa)$, where $1 \leq k \leq i$, are X-interpretations of $(aab)^i aa$, which are not pairwise shifted by $(aab)^i a$.

However, this construction becomes impossible when we add an additional requirement on the minimal length of w.

We will now generalise results of E. Barbin-Le Rest and M. Le Rest ([6, Theorem 2.1] and [6, Corollary 2.2]) and show that their observations hold also for arbitrary factors of x^+ longer than $2|x|$ (resp. $3|x|$):

Lemma 24. *Let $Y = \{x, y\} \subset A^*$ be a code with x and y primitive words which are not conjugate and $|x| \geq |y|$. Suppose that w is a word qxp, where $q <_{\mathrm{s}} x$ and $p <_{\mathrm{p}} x$ and $|w| \geq 2|x|$. Let $\mathcal{I}$ be a set of pairwise disjoint Y-interpretations of w. Then there is $i \geq 1$ and words q', p' such that $q' \not\leq_{\mathrm{s}} y$ or $p' \not\leq_{\mathrm{p}} y$ and*

$$\mathcal{I} \subseteq \{(q, x, p), (q', y^i, p')\}$$

Proof. Let $I' = (q', d', p') \in \mathcal{I}$ be a Y-interpretation of qxp. If $d' \in Y^* x Y^*$, then from the primitivity of the word x we get that $q = q'd_1$, $p \in d_2 p'$ and $d' = d_1 x d_2$, where $d_1, d_2 \in y^*$. If $q \in q'y^+$ or $p \in y^+ p'$, then I' and (q, x, p) are shifted by y. Then qxp is a factor of y^+, by Lemma 13, and words x, y are conjugate, a contradiction. Therefore, in case that $d' \in Y^* x Y^*$, we have $I' = (q, x, p)$. Suppose now that $d' \in y^*$. Since $|w| \geq 2|x|$ and $|x| \geq |y|$, necessarily $d' \in y^+$. Notice that $q' \not\leq_{\mathrm{s}} y$ or $p' \not\leq_{\mathrm{p}} y$, otherwise x and y are conjugate, a contradiction. In order to finish the proof we will show that all Y-interpretations of qxp different from (q, x, p) are Y-adjacent.

First, suppose that there is $i \geq 1$ such that an interpretation (q', y^i, p') is adjacent to (q, x, p). Then $xp \in y^+ p'$ or $qx \in q'y^+$ and interpretations (q', y^i, p') and (q, x, p) are shifted by y. By Lemma 13, qxp is a factor of y^+ and words x, y are conjugate, a contradiction.

Suppose therefore that $(q', y^i, p') \in \mathcal{I}$ and $(q'', y^j, p'') \in \mathcal{I}$ are two disjoint Y-interpretation of qxp which are both also disjoint from (q, x, p). We will show that $||q''| - |q'||$ or $||p'| - |p''||$ divides $|x|$. Since both $||q''| - |q'||$ and $||p'| - |p''||$ are strictly smaller than $|x|$, we get a contradiction with x being a primitive word. First notice that, since $|qxp| \geq 2|x|$ and $|x| \geq |y|$, words $q'y^i, q''y^j, y^i p', y^j p''$ are all longer than $|x|$. We have the following cases:

1 . $(\mathbf{suf}(\mathbf{x}), \mathbf{y^i}, \mathbf{p'}), (\mathbf{suf}(\mathbf{x}), \mathbf{y^j}, \mathbf{p''})$ Suppose first that both q' and q'' are suffixes of x and $|q'| < |q''|$. Since both interpretations are disjoint, we have $|q''y| > |q'y^i|$ and $p' \leq_{\mathrm{p}} x$. If now $i \leq j$, then $q''y^i$ has period $|q''| - |q'|$. Since w is a factor of x^+ and

$$|q''y^i| = |q'y^i| + |q''| - |q'| \geq |x| + |q''| - |q'|,$$

words x and ${q'}^{-1}q''$ commute. Since $|q''| - |q'| < |x|$, we can see that $|q''| - |q'|$ divides $|x|$. Suppose now that $i > j$. If $p'' \leq_{\mathrm{p}} y$, then $y^i p'$ has a period $|q''| - |q'|$ and $|q''| - |q'|$ divides $|x|$. If $p'' \leq_{\mathrm{p}} x$, then $y^j p'$ has a period $|p'| - |p''|$ and $|p'| - |p''|$ divides $|x|$. We can proceed similarly in case that p' and p'' are prefixes of x.

2 . $(\mathbf{suf}(\mathbf{x}), \mathbf{y^i}, \mathbf{pref}(\mathbf{y})), (\mathbf{suf}(\mathbf{y}), \mathbf{y^j}, \mathbf{pref}(\mathbf{x}))$ Suppose that $q' \leq_{\mathrm{s}} x$, $p' \leq_{\mathrm{p}} y$ and $q'' \leq_{\mathrm{s}} y$, $p'' \leq_{\mathrm{p}} x$. Since interpretations are disjoint, $|q'| > |q''|$ and $|q'y| > |q''y^j|$. First we will show that $i = j$. Suppose that $i > j$. Then $q'y^i$ has a period $|p''| - |p'|$. Since

$$|q'y^i| = |q''y^j| + |p''| - |p'| \geq |x| + |p''| - |p'|,$$

$|p''| - |p'|$ divides $|x|$. Similarly, if $j > i$, we will get that $|q'| - |q''|$ divides $|x|$. Let $i = j$. Notice that $y^j p'' p'^{-1}$ has a period $k = |p''| - |p'|$ and that xp has a period $h = |x| + |p| - |p''|$. If now $|q| \geq |q''|$, then xpp'^{-1} has both periods k and h and since $|xpp'^{-1}| = k + h$, the word $y^j p''$ has a period k. Since

$$|y^j p''| = k + |y^j p'| \geq k + |x|,$$

k divides $|x|$, a contradiction with x being primitive. Therefore, $|q| < |q''|$. Similarly, we can show that $|p| < |p'|$.

Suppose that $|p| < |p'| < |y|$ and $|q| < |q''| < |y|$. We have $j = i = 1$ otherwise $|x| \geq 2|y|$ and we get a contradiction with $|q| + |p| \geq |x|$. Since $|w| \geq 2|x| > |q'| + |p''|$, there are words r, s such that rs is a primitive word, $q''^{-1} q' = (rs)^m$, $p'' p'^{-1} = (sr)^m$ and $y = (rs)^{m+k} r$, for some $m \geq 1$ and $k \geq 0$. Notice that r and s are non-empty words. Let $u = q^{-1} q''$ and $v = p' p^{-1}$. Then $x = u(rs)^{2m+k} rv$ and either $|q|$ or $|p|$ is greater than $|rs|$. Suppose that $|q| \geq |rs|$. Since $q \leq_{\mathrm{s}} x$ and $(rs)^m \leq_{\mathrm{s}} q' \leq_{\mathrm{s}} x$, we get $rs \leq_{\mathrm{s}} q$. Consequently, $rsu \leq_{\mathrm{s}} q'' \leq_{\mathrm{s}} y = (rs)^{m+k} r$ and $u \in (rs)^* r$. From

$$(sr)^m \leq_{\mathrm{p}} p'' \leq_{\mathrm{p}} x = u(rs)^{2m+k} rv$$

and $u \in (rs)^* r$, it follows that r and s commute, a contradiction with rs being a primitive word.

$\square$

The previous lemma yields the following two corollaries:

Corollary 25. *Let $X = \{x^{j_x}, y^{j_y}\} \subseteq A^*$ be a code with x and y primitive words which are not conjugate and $|x| \geq |y|$. Suppose that $w \in A^*$ is a factor of x^+ such that $|w| \geq 3|x|$. Let $\mathcal{I}$ be a set of pairwise disjoint X-interpretations of w. Then all interpretations from $\mathcal{I}$ are pairwise shifted by x.*

Proof. Let $Y = \{x, y\}$. Since $|w| \geq 3|x|$, we can find words $q \leq_{\mathrm{s}} x$ and $p \leq_{\mathrm{p}} x$ such that $qx^2p \leq_{\mathrm{p}} w$ and $pq = x$. We will first show that all Y-interpretations of qx^2p are Y-adjacent.

1 . If $q = x$ or $p = x$, then $qx^2p = x^3$ and all Y-interpretations of qx^2p are Y-adjacent by Corollary 10.

2 . Assume that $q <_{\mathrm{s}} x$ and $p <_{\mathrm{p}} x$. Suppose, for a contradiction, that qx^2p admits Y-interpretation disjoint from (q, x^2, p). Such interpretation in particular defines two Y-interpretations qxp, disjoint from (q, x, p). By Lemma 24, the only interpretation of (q, x, p) disjoint form (q, x, p) is an interpretation (q', y^i, p') for some $i \geq 1$ and where $q' \not\leq_{\mathrm{s}} x$ or $p' \not\leq_{\mathrm{p}} x$, for some $i \geq 1$, $q' = q_2$, $p' = p_1$, and $p' \not\leq_{\mathrm{p}} y$ or $q' \not\leq_{\mathrm{s}} y$. Therefore, the only possible Y-interpretation of qx^2p disjoint from (q, x^2, p) is an interpretation $(q', y^i x y^i, p')$. By the primitivity of x we get either $q = q' y^i$ or $p = y^i p'$. Both cases yields a contradiction with our assumption that $(q', y^i x y^i, p')$ is disjoint from (q, x^2, p).

Let $I, I' \in \mathcal{I}$ be two disjoint X-interpretations of w. We have just seen that I, I' are Y-adjacent. By Lemma 14, they are shifted by x otherwise word w is a factor of y^+ and x, y are conjugate, a contradiction.

$\square$

Corollary 26. *Let $X = \{x^{j_x}, y^{j_y}\}$ be a code with x and y primitive words which are not conjugate and $|x| \geq |y|$. Suppose that $w \in A^*$ is a factor of x^+ such that $|w| \geq 2|x|$. Let $\mathcal{I}$ be a set of pairwise disjoint X-interpretations of w. Then at least $|\mathcal{I}| - 1$ of these interpretations are pairwise shifted by x.*

Proof. Let $Y = \{x, y\}$. If two disjoint X-interpretations of w are Y-adjacent, then they are shifted by x, otherwise, by Lemma 14, word w is a factor of y^+ and x, y are conjugate. Importantly, notice that "being shifted by x" is a reflexive, symmetric and transitive relation, that is an equivalence on $\mathcal{I}$. Since, by Lemma 24 or by Corollary 11, there are at most two X-interpretations of w which are Y-disjoint, $\mathcal{I}$ comprises at most two equivalence classes $\mathcal{I}_1, \mathcal{I}_2$. If $|\mathcal{I}_1| \geq 2$ and $|\mathcal{I}_2| \geq 2$, then, since w is a factor of x^+, each pair of X-interpretations $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$ is shifted by x. Consequently, $\mathcal{I}_1 = \mathcal{I}_2$ and our lemma holds. On the other hand, if $|\mathcal{I}_1| \leq 1$ or $|\mathcal{I}_2| \leq 1$, then at least $\max\{|\mathcal{I}_1|, |\mathcal{I}_2|\} \geq |\mathcal{I}| - 1$ interpretations from $\mathcal{I}$ are pairwise shifted by x.

□

In the next lemma, we will see that the lengths of words w and x^{j_x} depend on the number of disjoint X-interpretations of w which are pairwise shifted by x.

Lemma 27. *Let $X = \{x^{j_x}, y^{j_y}\} \subseteq A^*$ be a code with x and y primitive words. Let $w \in A^*$ and let $\mathcal{I}$ be a set of pairwise disjoint X-interpretations of w pairwise shifted by x. Then $|w| \geq (|\mathcal{I}| - 1)|x|$. Moreover, if X is marked, then $j_x \geq |\mathcal{I}|$.*

Proof. By the definition of disjoint X-interpretations shifted by x, we can find words $q_k, p_k \in A^*$ and $d_k, d'_k \in \{x, y\}^*$, $k \in \{0, \ldots, |\mathcal{I}| - 1\}$, such that $I_k = (q_k, d_k d'_k, p_k) \in \mathcal{I}$ and

$$q_k d_k = q_0 d_0 x^{r_k},$$

where $0 = r_0 < r_1 < \cdots < r_{|\mathcal{I}|-1}$. Then $|w| \geq r_{|\mathcal{I}|-1}|x| \geq (|\mathcal{I}| - 1)|x|$.

Observe that $q_\ell d_\ell = q_k d_k x^{r_\ell - r_k}$, for all $0 \leq k \leq \ell \leq |\mathcal{I}| - 1$. Suppose that the code X is marked. Then, for all $0 \leq k \leq \ell \leq |\mathcal{I}| - 1$, $d'_k \in x^{r_\ell - r_k}\{x, y\}^*$. We define an equivalence $\equiv$ on $\mathcal{I}$ as $I_k \equiv I_\ell$ iff $|r_\ell - r_k| \bmod j_x = 0$. Since interpretations from $\mathcal{I}$ are pairwise disjoint, $[I_k] = \{I_k\}$, and consequently $j_x \geq |\mathcal{I}/\equiv| = |\mathcal{I}|$.

□

As a consequence of Lemma 27 and Corollary 26, we observe that if the set $\mathcal{I}$ from Corollary 26 contains at least four X-interpretations, then Corollary 26 yields that at least three of them are pairwise shifted by x. Therefore, by Lemma 27 we have $|w| \geq 3|x|$, and by Corollary 25, all X-interpretations of w are shifted by x.

In the previous part we have explored possible Y-interpretations of sufficiently long factors x^+, with an assumption that $|x| \geq |y|$. Now, we would like to have similar results for Y-interpretations of words which are factors of y^+. The following lemma describes possible Y-interpretations of a word which is a factor of y^+ longer than $|x| + |y|$:

Lemma 28. *Let $Y = \{x, y\} \subset A^*$ be a code with x and y primitive words which are not conjugate and $|x| \geq |y|$. Suppose that w is a word qy^ip, where $i \in \mathbb{N}$,*

$q \leq_{\mathrm{s}} y$, $p \leq_{\mathrm{p}} y$ and $|w| \geq |x| + |y|$. Let $\mathcal{I}$ be a set of pairwise disjoint Y-interpretations of w. Then there is $d \in y^+ \cup xy^+ \cup y^+x$ and words q', p', q'', p'' such that (q'', d, p'') is Y-adjacent to (q, y^i, p), $q' \not\leq_{\mathrm{s}} x$ or $p' \not\leq_{\mathrm{p}} x$ and

$$\mathcal{I} \subseteq \{(q'', d, p''), (q', x, p')\}$$

Moreover, if $|w| \geq |x| + 2|y|$, then $\mathcal{I} = \{(q'', d, p'')\}$.

Proof. Let $I' = (q', d', p') \in \mathcal{I}$ be an Y-interpretation of qy^ip.

1. Suppose first that $d' \in Y^*yY^*$. Notice, that the case when $d' \in Y^*yx^+yY^* \cup Y^*xy^+xY^*$ yields a commutativity of x and y, a contradiction with Y being a code. Also notice, that x^2 cannot be a factor of w, otherwise x and y are conjugate. Therefore, $d' \in y^+ \cup xy^+ \cup y^+x$.

We will show now that if $I' = (q', d', p')$ and $I'' = (q'', d'', p'')$ are two Y-interpretations of w such that $d', d'' \in y^+ \cup xy^+ \cup y^+x$, then they have to be Y-adjacent:

$\mathbf{d}', \mathbf{d}'' \in \mathbf{y}^+$. Suppose first that $d', d'' \in y^+$. Suppose, for a contradiction, that I', I'' are not Y-adjacent and let $q' \leq_{\mathrm{p}} q''$. Then, since w is a factor of y^+, we obtain that $q'd' <_{\mathrm{p}} q''$ and the length reasons yield $q'' \leq_{\mathrm{s}} x$, $p' \leq_{\mathrm{p}} x$. Since w is a factor of y^+, both words $d''p''$ and p' are prefixes of y^+. Therefore, $d''p'' \leq_{\mathrm{p}} p' \leq_{\mathrm{p}} x$ and $w = q''d''p''$ is a factor of x^+. From $|w| \geq |x| + |y|$ we obtain that x and y are conjugate, a contradiction.

$\mathbf{d}', \mathbf{d}'' \in \mathbf{xy}^+$. Suppose that $d', d'' \in xy^+$ and let $q' \leq_{\mathrm{p}} q''$. Then $x \leq_{\mathrm{s}} y^+$ and $p' \leq_{\mathrm{p}} y^+$. Therefore, $|p'| < |y|$, otherwise $y \leq_{\mathrm{p}} p' \leq_{\mathrm{p}} x$ yields that x and y commute, a contradiction with Y being a code. But then $d' = xd_1d_2$ where $d_1, d_2 \in y^*$ and $d_2p' = d''p''$. We have proved that the interpretations I and I' are adjacent.

$\mathbf{d}' \in \mathbf{y}^+$ and $\mathbf{d}'' \in \mathbf{xy}^+$. Suppose that $d' \in y^+$, $d'' \in xy^+$. Suppose, for a contradiction, that I', I'' are not Y-adjacent. Since $|q'| < |x|$, we obtain that $y \leq_{\mathrm{p}} p' \leq_{\mathrm{p}} x$. From $d'' \in xy^+$ we get that $x \leq_{\mathrm{s}} y^+$, and x and y commute, a contradiction with Y being a code. We proceed similarly in case that $d' \in y^+$, $d'' \in y^+x$.

$\mathbf{d}' \in \mathbf{xy}^+$ and $\mathbf{d}'' \in \mathbf{y}^+\mathbf{x}$. Suppose that $d' \in xy^+$ and $d'' \in y^+x$. Then $x \leq_{\mathrm{p}} y^+$ and $x \leq_{\mathrm{s}} y^+$ yield that x and y commute, a contradiction with X being a code.

We have proved that any two Y-interpretations $I' = (q', d', p')$ and $I'' = (q'', d'', p'')$ of w such that $d', d'' \in Y^*yY^*$ are Y-adjacent. Now, we will have a look at the situation when $I' = (q', d', p') \in \mathcal{I}$ is an Y-interpretation of qy^ip such that $d' \in x^*$.

2. Suppose now that $d' \in x^*$. Notice that in fact $d' = x$. Indeed, if $|d''| \geq 2|x|$, then x^+ and y^+ have a common factor of the length at least $|x|+|y|$ and x and y are conjugate, a contradiction. On the other hand if $d'' = 1$, then $|w| \geq |x|+|y| \geq 2|y|$ yields that $q' \leq_{\mathrm{s}} x$ and $p' \leq_{\mathrm{p}} x$. Then again we get a contradiction with our assumption that x and y are not conjugate. Also, the same reasoning yields that $q' \not\leq_{\mathrm{s}} x$ or $p' \not\leq_{\mathrm{p}} x$.

Suppose now that $I' = (q', x, p')$ and $I'' = (q'', x, p'')$ are two Y-interpretation of w such that $q' \leq_{\mathrm{s}} y$ or $p' \leq_{\mathrm{p}} y$, and $q'' \leq_{\mathrm{s}} y$ or $p'' \leq_{\mathrm{p}} y$. We will show that they are adjacent.

$(\mathbf{suf}(\mathbf{y}), \mathbf{x}, \mathbf{p}'), (\mathbf{suf}(\mathbf{y}), \mathbf{x}, \mathbf{p}'')$ Suppose first that both q' and q'' are suffixes of y. Suppose for example that $|q'| < |q''|$. Then $q''x$ has a period $|q''| - |q'| < |y|$.

Since $q''x$ has also a period $|y|$ and

$$|q''x| \geq |q''| - |q'| + |x| \geq |q''| - |q'| + |y|,$$

$|q''| - |q'|$ divides $|y|$, a contradiction with primitivity of y. Therefore, $|q'| = |q''|$ and I', I'' are adjacent. We can proceed similarly in case that p' and p'' are prefixes of y.

$(\mathbf{suf}(\mathbf{y}), \mathbf{x}, \mathbf{pref}(\mathbf{x})), (\mathbf{suf}(\mathbf{x}), \mathbf{x}, \mathbf{pref}(\mathbf{y}))$ Suppose now that $q' \leq_{\mathrm{s}} y$, $p' \leq_{\mathrm{p}} x$ and $q'' \leq_{\mathrm{s}} x$, $p'' \leq_{\mathrm{p}} y$. If $|q''| > |q'|$, then $|q''| - |q'|$ divides $|x|$ and we have a contradiction with the primitivity of x. Suppose therefore that $|q'| > |q''|$. Then $q''^{-1}q'x$ has a period $|q'| - |q''| < |y|$. Since $q''^{-1}q'x$ has also a period $|y|$ and $|x| \geq |y|$, $|q'| - |q''|$ divides $|y|$, a contradiction with primitivity of y. This yields $|q'| = |q''|$ and I', I'' are adjacent.

We have shown that $\mathcal{I} \subseteq \{(q'', d, p''), (q', x, p')\}$, where $d \in y^+ \cup xy^+ \cup y^+x$ and $q' \not\leq_{\mathrm{s}} x$ or $p' \not\leq_{\mathrm{s}} x$.

If now $|w| \geq |x| + 2|y|$, then for any Y-interpretation (q, x, p) of w holds that $|q| \geq |y|$ or $|p| \geq |y|$. Therefore, q is a suffix of x or p is a prefix of x. Since w is a factor of y^+, the Periodicity lemma yields that x and y are conjugate. Therefore, $\mathcal{I} = \{(q'', d, p'')\}$.

□

We finish this part with two corollaries which are similar to Corollary 26.

Corollary 29. *Let $X = \{x^{j_x}, y^{j_y}\} \subset A^*$ be a code with x and y primitive words which are not conjugate and $|x| \geq |y|$. Suppose that $w \in A^*$ is a factor of y^+ such that $|w| \geq |x| + |y|$. Let $\mathcal{I}$ be a set of pairwise disjoint X-interpretations of w. Then at least $|\mathcal{I}| - 1$ of these interpretations are pairwise shifted by y.*

Corollary 30. *Let $X = \{x^{j_x}, y^{j_y}\} \subset A^*$ be a code with x and y primitive words which are not conjugate and $|x| \geq |y|$. Suppose that $w \in A^*$ is a factor of y^+ such that $|w| \geq |x| + 2|y|$. Let $\mathcal{I}$ be a set of pairwise disjoint X-interpretations of w. Then all interpretations from $\mathcal{I}$ are pairwise shifted by y.*

2.2.2 Overflows

We continue in this part with the preliminaries necessary for the proof of Theorem 8. In this part, as we have announced, we will focus on conjugate pairs of words and its overflows. Overflows in a conjugate pair are tightly connected with interpretations as we will see at the end of this part. The results presented here mainly explore the structure of so-called overflow languages. Their structure has a considerable impact on the final structure of the original conjugate pair. The angular result of this part is Lemma 34.

Let us first remind the definition of an overflow:

Overflows and simple conjugate pair. Let $X, Y \subseteq A^*$ be codes and $x \in X^*$, $y \in Y^*$ be primitive words such that x^i and y^j are conjugate for some $i, j \in \mathbb{N}$. We say that $z \in A^*$ is an X-*overflow* of (x^i, y^j) iff $|z| < |x^i|$ and there exists a word x', X-conjugate of x^i, and a word y', Y-conjugate of y^j, such that $zx' = y'z$. We say that a pair of words (x^i, y^j) is *simple* if $\gcd(i, j) = 1$, and $zx' = y'z$ and $zx'' = y''z$ imply $x' = x''$ and $y' = y''$, where $z \in A^* \cup (A^{-1})^*$, and x', x'' are X-conjugate of x^i, and y', y'' are Y-conjugate of y^j.

The following lemma (see also [4, Lemma 9]) uses the fact that different code words of marked code do not have any common prefix. The proof is not difficult and will be omitted.

Lemma 31. *Let $X \subseteq A^*$ be a marked code and let $z \in A^*$. If $w_1, w_2 \in zX^\omega \cup zX^*$ and $w_1 \neq w_2$, then $w_1 \wedge w_2 \in zX^*$.*

Notice that the previous lemma does not hold for prefix codes, e.g. $X = \{ab, aa\}$ is a prefix code, but $a = ab \wedge aa \notin X^*$.

Let $g, h : \{a, b\}^* \to A^*$ be non-periodic marked binary morphisms and let $(g(e), h(f))$ be a simple pair of conjugate words. We will set $G = \{g(a), g(b)\}$ and $H = \{h(a), h(b)\}$. Suppose that z is an H-overflow of $(g(e), h(f))$. The following lemma describes the structure of the language $G^\omega \cap zH^\omega$.

Lemma 32. *Let $g, h : \{a, b\}^* \to A^*$ be non-periodic marked binary morphisms and let $(g(e), h(f))$ be a pair of conjugate words such that $h(\rho(f)) = \rho(h(f))$. Let $G = \{g(a), g(b)\}$ and $H = \{h(a), h(b)\}$. Suppose that z is an H-overflow of $(g(e), h(f))$.*

- *If 1 is an overflow of $(g(e), h(f))$, then $G^\omega \cap zH^\omega$ is equal to $\alpha(G^\omega \cap H^\omega)$, where α is the shortest word from $G^* \cap zH^*$.*
- *If 1 is not an overflow of $(g(e), h(f))$, then $G^\omega \cap zH^\omega$ is equal to $\{g(e')^\omega\}$, where e' is a conjugate of e such that $g(e')z = zh(f')$ for some word f' conjugate of f.*

Proof. By the definition of H-overflow, we can suppose without loss of generality that $g(e)z = zh(f)$.

1. Suppose first that 1 is an overflow of $(g(e), h(f))$. We will show that there is a word $u \leq_{\mathrm{p}} e^+$ such that $g(u) \in zH^*$. Since 1 is an overflow of $(g(e), h(f))$, there are words $e_1 \leq_{\mathrm{p}} e$, $f_1 \leq_{\mathrm{p}} f$ such that

$$g(e_1^{-1} e e_1) = h(f_1^{-1} f f_1).$$

Let $u = ee_1$ and $r = z^{-1}g(u)$. Then

$$r^{-1}h(f)r = g(u)^{-1}zh(f)z^{-1}g(u) = g(e_1^{-1}ee_1) = h(f_1)^{-1}h(f)h(f_1),$$

and either $h(f_1) \in \rho(h(f))^+ r$ or $r \in \rho(h(f))^+ h(f_1)$. From our assumption that $h(\rho(f)) = \rho(h(f))$ we obtain in both cases that $r \in H^*$, and consequently that $g(u) \in zH^*$. We can therefore take the shortest word $\beta \leq_{\mathrm{p}} e^+$ such that $g(\beta) \in zH^*$ and set $\alpha = g(\beta)$. Notice that α is the shortest word from the set $G^* \cap zH^*$. Indeed, if there is $\alpha' \in G^* \cap zH^*$ such that $\alpha \neq \alpha'$, then by Lemma 31 also $\alpha \wedge \alpha' \in G^* \cap zH^*$. Since G is marked, we obtain that $\alpha \wedge \alpha' = g(\beta')$ for some $\beta' \leq_{\mathrm{p}} \beta$. Minimality of β yields $\beta' = \beta$, and consequently $\alpha \wedge \alpha' = \alpha$.

Let w be a word from $G^\omega \cap zH^\omega$. Then

$$w = g(\tilde{e}) = zh(\tilde{f})$$

for some $\tilde{e}, \tilde{f} \in \{a, b\}^\omega$. Let $e_1 = \tilde{e} \wedge e^\omega$ and $f_1 = \tilde{f} \wedge f^\omega$. If $\tilde{e} \neq e^\omega$ we get from Lemma 31 that

$$g(e_1) = g(\tilde{e}) \wedge g(e^\omega) = zh(\tilde{f}) \wedge zh(f^\omega) = zh(f_1).$$

Consequently, from the minimality of β, it follows that $\beta \leq_{\mathrm{p}} e_1 \leq_{\mathrm{p}} \tilde{e}$. We obtain

$$g(\tilde{e}) = g(\beta)g(\beta^{-1}\tilde{e}) \in \alpha G^{\omega}.$$

Since $\beta \in zH^*$, we can find $\beta' \in \{a,b\}^*$ such that $g(\beta) = zh(\beta')$. From $zh(\beta') = g(\beta) \leq_{\mathrm{p}} g(e_1) = zh(f_1)$ we infer that $\beta' \leq_{\mathrm{p}} f_1 \leq_{\mathrm{p}} \tilde{f}$. Then

$$zh(\tilde{f}) = \alpha g(\beta)^{-1} zh(\tilde{f}) = \alpha h(\beta'^{-1}\tilde{f}) \in \alpha H^{\omega}.$$

We have show that $w \in \alpha(G^{\omega} \cap H^{\omega})$ and therefore, $G^{\omega} \cap zH^{\omega} \subseteq \alpha(G^{\omega} \cap H^{\omega})$.

Let now w be a word from $\alpha(G^{\omega} \cap H^{\omega})$. By the definition of α as $g(\beta)$, we have $w \in G^{\omega} \cap zH^{\omega}$. Therefore, $\alpha(G^{\omega} \cap H^{\omega}) \subseteq G^{\omega} \cap zH^{\omega}$.

2. Suppose now that 1 is not an overflow of $(g(e), h(f))$. Let $\tilde{e} \in \{a,b\}^{\omega}$ be a word such that $g(\tilde{e}) \in zH^{\omega}$. We can find a word $\tilde{f} \in \{a,b\}^{\omega}$ such that $g(\tilde{e}) = zh(\tilde{f})$. Let $e_1 = \tilde{e} \wedge e^{\omega}$ and $f_1 = \tilde{f} \wedge f^{\omega}$. If $\tilde{e} \neq e^{\omega}$, then by Lemma 31 we get

$$g(e_1) = g(\tilde{e}) \wedge g(e^{\omega}) = zh(\tilde{f}) \wedge zh(f^{\omega}) = zh(f_1).$$

Therefore,

$$g(e_1^{-1}ee_1) = h(f_1^{-1}ff_1),$$

and 1 is an overflow of $(g(e), h(f))$. We have proved that $\tilde{e} = e^{\omega}$ and $G^{\omega} \cap zH^{\omega} = \{g(e)^{\omega}\}$. $\square$

As an easy consequence the reader should notice that 1 is an overflow of $(g(e), h(f))$ if and only if $G^* \cap zH^*$ is not empty. Moreover, if $G^* \cap zH^*$ is not empty, then $\alpha \leq_{\mathrm{p}} w$, for each $w \in G^* \cap zH^*$. We formulate it as follows:

Corollary 33. *Let $g, h : \{a,b\}^* \to A^*$ be non-periodic marked binary morphisms and let $(g(e), h(f))$ be a pair of conjugate words such that $h(\rho(f)) = \rho(h(f))$. Let $G = \{g(a), g(b)\}$ and $H = \{h(a), h(b)\}$ and suppose that z is an H-overflow of $(g(e), h(f))$. Then 1 is an overflow of $(g(e), h(f))$ if and only if $G^* \cap zH^*$ is not empty. Moreover, if $G^* \cap zH^*$ is not empty, then there is α, such that $G^{\omega} \cap zH^{\omega} = \alpha(G^{\omega} \cap H^{\omega})$ and $\alpha \leq_{\mathrm{p}} w$ for each $w \in G^* \cap zH^*$.*

In the following lemma we will consider a simple conjugate pair with two H-overflows which are shifted in a special way. We will see that such overflows have considerable impact on the final structure of the conjugate pair:

Lemma 34. *Let $g, h : \{a,b\}^* \to A^*$ be non-periodic marked binary morphisms and let $(g(e), h(f))$ be a simple pair of conjugate words such that $h(\rho(f)) = \rho(h(f))$. Let $G = \{g(a), g(b)\}$ and $H = \{h(a), h(b)\}$. Let $i \geq 0$ and let $z, (pq)^i pz$ be H-overflows of $(g(e), h(f))$. Suppose that $(pq)^{\omega} \in G^{\omega}$, $(qp)^{\omega} \in G^{\omega}$ and $(pq)^i p \notin G^*$. Then*

(a) either 1 is not an overflow of $(g(e), h(f))$ and $g(e') \in \rho(qp)^+$, for some conjugate e' of e, or

(b) 1 is an overflow of $(g(e), h(f))$ and there are $\alpha, \alpha' \in G^*$ such that

$$G^{\omega} \cap zH^{\omega} = \alpha(G^{\omega} \cap H^{\omega}) \quad \text{and} \quad G^{\omega} \cap (pq)^i pzH^{\omega} = \alpha'(G^{\omega} \cap H^{\omega}),$$

with either $\alpha \leq_{\mathrm{p}} (qp)^{\omega}$ or $\alpha' \leq_{\mathrm{p}} (pq)^{\omega}$. If, moreover, $pq = qp = \rho(g(c))$, for some $c \in \{a,b\}$, then $|G^{\omega} \cap H^{\omega}| = 1$ or $g(c)^{\omega} \in G^{\omega} \cap H^{\omega}$.

Proof. Let $\mathcal{L} = G^\omega \cap zH^\omega$ and $\mathcal{L}' = G^\omega \cap (pq)^i pzH^\omega$.
1. Suppose first that $(qp)^\omega \in zH^\omega$.
(a) If 1 is not an overflow of $(g(e), h(f))$, then from Lemma 32 we get that

$$(qp)^\omega \in G^\omega \cap zH^\omega = \{g(e')^\omega\},$$

where e' is a conjugate of e. Therefore, $g(e') \in \rho(qp)^+$.
(b) Suppose that 1 is an overflow of $(g(e), h(f))$. Then, by Lemma 32, $G^\omega \cap zH^\omega$ is equal to $\alpha(G^\omega \cap H^\omega)$, for some $\alpha \in G^* \cap zH^*$. Since $(qp)^\omega \in G^\omega \cap zH^\omega$, we have $\alpha \leq_{\mathrm{p}} (qp)^\omega$. If now $pq = qp = \rho(g(c))$, for some $c \in \{a, b\}$, then from $\alpha \in G^*$ and from g being marked, we get $\alpha \in g(c)^*$ and $g(c)^\omega = \alpha^{-1} g(c)^\omega \in G^\omega \cap H^\omega$.

2. Suppose now that $(qp)^\omega \notin zH^\omega$ and let $u \in zH^\omega$ be a word such that the word $\gamma = u \wedge (qp)^\omega$ is the longest prefix of $(qp)^\omega$. Consider words $w \in \mathcal{L}$ and $w' \in \mathcal{L}'$. By Lemma 31, we have

$$w \wedge (qp)^\omega \in G^*, \qquad\qquad w' \wedge (pq)^\omega \in G^*.$$

2.1 If $w \wedge (qp)^\omega <_p \gamma$, then $w \neq u$ and Lemma 31 implies $w \wedge u \in zH^*$. Since $w \wedge u = w \wedge (qp)^\omega$ we get $w \wedge (qp)^\omega \in G^* \cap zH^* \cap \mathrm{pref}(qp)^+$. Therefore, by Corollary 33, 1 is an overflow of $(g(e), h(f))$ and

$$\begin{aligned} G^\omega \cap zH^\omega &= \alpha(G^\omega \cap H^\omega), \\ G^\omega \cap (pq)^i pzH^\omega &= \alpha'(G^\omega \cap H^\omega), \end{aligned}$$

where $\alpha \leq_{\mathrm{p}} w \wedge (qp)^\omega$ and $\alpha, \alpha' \in G^*$.
2.2 Similarly, if $w' \wedge (pq)^\omega <_p (pq)^i p\gamma$, then $w' \wedge (pq)^\omega \in G^* \cap (pq)^i pzH^* \cap \mathrm{pref}(pq)^+$. Therefore, again by Corollary 33, 1 is an overflow of $(g(e), h(f))$ and

$$\begin{aligned} G^\omega \cap zH^\omega &= \alpha(G^\omega \cap H^\omega), \\ G^\omega \cap (pq)^i pzH^\omega &= \alpha'(G^\omega \cap H^\omega), \end{aligned}$$

where $\alpha' \leq_{\mathrm{p}} w' \wedge (pq)^\omega$ and $\alpha, \alpha' \in G^*$.
2.3 Suppose therefore that for all words in $w \in \mathcal{L}$ and all words $w' \in \mathcal{L}'$ we have

$$\begin{aligned} w \wedge (qp)^\omega &= \gamma, \\ w' \wedge (pq)^\omega &= (pq)^i p\gamma. \end{aligned}$$

Lemma 31 implies that $\gamma \in G^*$ and $(pq)^i p\gamma \in G^*$, and consequently there are words $u, u' \in \{a, b\}^*$ such that $\gamma = g(u)$ and $(pq)^i p\gamma = g(u')$. Since z, resp. $(pq)^i pz$, are overflows of $(g(e), h(f))$, we can find words e', e'', conjugates of e, and words f', f'', conjugates of f, such that

$$g(e')z = zh(f'), \qquad\qquad g(e'')(pq)^i pz = (pq)^i pzh(f'').$$

From $g(e')^\omega \in \mathcal{L}$, resp. $g(e'')^\omega \in \mathcal{L}'$, we obtain that $\gamma \leq_{\mathrm{p}} g(e')^\omega$, resp. $(pq)^i p\gamma \leq_{\mathrm{p}} g(e'')^\omega$, and consequently, since g is a marked morphism, we have $u \leq_{\mathrm{p}} e'^\omega$ and $u' \leq_{\mathrm{p}} e''^\omega$. Therefore,

$$g(u^{-1}e'u)\gamma^{-1}z = (\gamma^{-1}zh(f')z^{-1}\gamma)\gamma^{-1}z = \gamma^{-1}zh(f'), \tag{2.4}$$

and

$$g(u'^{-1}e''u')\gamma^{-1}z = \gamma^{-1}zh(f''). \tag{2.5}$$

Since we have assumed that $(g(e), h(f))$ is simple, we get from Eq. (2.4) and Eq. (2.5) that $u^{-1}e'u = u'^{-1}e''u'$, and consequently also for all $k \geq 1$

$$u^{-1}e'^k u = (u^{-1}e'u)^k = (u'^{-1}e''u')^k = u'^{-1}e''^k u'.$$

Therefore, for a sufficiently large $k \in \mathbb{N}$, we have $u \leq_{\mathrm{p}} e^k$ and $u' \leq_{\mathrm{p}} e''$ and thus, we can see that u and u' are suffix comparable. Then $(pq)^i pg(u) = g(u')$ yields that $(pq)^i p = g(uu'^{-1})$, a contradiction with our assumption that $(pq)^i p \notin G^*$.

Suppose finally, that $pq = qp = \rho(g(c))$. Suppose moreover that $|G^\omega \cap H^\omega| \neq 1$. Then morphisms g, h have two different letter blocks (s, t) and (s', t'). Recall that

$$G^\omega \cap H^\omega = \{g(s), g(s')\}^\omega = \{h(t), h(t')\}^\omega.$$

We will show that there are words $w \in \mathcal{L}$ and $w' \in \mathcal{L}'$ such that $w \wedge (qp)^\omega = \gamma$ and $w' \wedge (pq)^\omega = (pq)^i p\gamma$. Suppose that for all words $w \in \mathcal{L}$ we have $w \wedge (qp)^\omega \neq \gamma$. Let $w_m \in \mathcal{L}$ be a word such that $w_m \wedge (qp)^\omega$ is maximal among all words $w \wedge (qp)^\omega$, $w \in \mathcal{L}$. Since $w_m \wedge (qp)^\omega <_{\mathrm{p}} \gamma$, similarly as in the previous part, we infer that $w_m \wedge (qp)^\omega \in G^* \cap zH^*$, and consequently,

$$(w_m \wedge (qp)^\omega)(G^\omega \cap H^\omega) \subset \mathcal{L}.$$

Let $d \in \{a, b\}$ be a letter such that $(w_m \wedge (qp)^\omega)d \leq_{\mathrm{p}} (qp)^\omega$, and let us define a word $w_t = (w_m \wedge (qp)^\omega)u_d^\omega$, where $u_d = g(s)$ if $d \leq_{\mathrm{p}} g(s)$ and $u_d = g(s')$ otherwise. Then $w_t \in \mathcal{L}$ and $w_m \wedge (qp)^\omega <_{\mathrm{p}} w_t \wedge (qp)^\omega$, a contradiction with our choice of w_m. We have shown that $w \wedge (qp)^\omega = \gamma$ for some word $w \in \mathcal{L}$. Similarly, we can show that $w' \wedge (pq)^\omega = (pq)^i p\gamma$ or some word $w \in \mathcal{L}'$.

But then Lemma 31 implies that $\gamma \in g(c)^+$ and $(pq)^i p\gamma \in g(c)^+$. Then $(pq)^i p \in g(c)^+$, a contradiction with $(pq)^i p \notin G^*$. We have proved that $|G^\omega \cap H^\omega| = 1$.

□

Finally, let us have a look at the relation between disjoint shifted interpretations and overflows. We consider the following special case:

Let $g, h : \{a, b\}^* \to A^*$ be non-periodic marked binary morphisms and let $(g(e), h(f))$ be a simple pair of words such that $g(e) = h(f)$ and $h(\rho(f)) = \rho(h(f))$. Let $G = \{g(a), g(b)\}$ and $H = \{h(a), h(b)\}$. Let $s_1, s_2 \in A^*$ and $u_1, u_2 \in A^*$ be words such that $s_1 s_2$ the primitive root of $g(u_1)$ and $s_2 s_1$ is the primitive root of $g(u_2)$. Suppose there are words r, s, s' such that $h(v) = s'rs$, $|r| \geq |s_1 s_2|$ and r is a factor of $(s_1 s_2)^+$.

We will consider the situation when we have three different occurrences of $h(v)$ in $h(f)$ (see Fig. 5). We will be looking at they respective disjoint G-interpretations induced by $g(e) = h(f)$ such that their shifts are from $(s_1 s_2)^+$, or $(s_2 s_1)^+$, or from $s_2(s_1 s_2)^*$, or $s_1(s_2 s_1)^*$. More precisely, we have the following situation:

Let $vf_1 \leq_{\mathrm{s}} f_2$, $vf_2 \leq_{\mathrm{s}} f_3$ and vf_3 be different suffixes of f, and $I_k = (q_k, g(d'_k d_k), p_k)$ be disjoint G-interpretations of $h(v)$ such that for each $k \in \{1,2,3\}$, $d'_k d_k e_k \leq_{\mathrm{s}} e$ for a word $e_k \in \{a,b\}^*$ and $p_k = g(e_k)h(f_k)^{-1}$. We suppose moreover that for each $k \in \{1,2,3\}$ there is $i \in \{1,2\}$ and words r_k, r'_k such that $s'r'_k = q_k g(d'_k)$, $r_k s = g(d_k)p_k$ and $r'_k \leq_{\mathrm{s}} g(u_i)^+$, $r_k \leq_{\mathrm{p}} g(u_i)^+$. Notice that

$$s'r'_k r_k s = q_k g(d'_k d_k) p_k = h(v) = s'rs,$$

yields that $r'_k r_k = r$ for all $k \in \{1,2,3\}$.

Claim 1. $f_3 \leq_{\mathrm{p}} (f_2 f_1^{-1})^+$, and $d_3 e_3 \leq_{\mathrm{p}} u_i^+$ for some $i \in \{1,2\}$.

Before we proceed with the proof of Claim 1, we will formulate the following auxiliary lemma. The proof is straightforward and will be omitted.

Lemma 35. *Let $\tilde{r}, \tilde{t}$ be words such that $|\tilde{r}| \geq |\tilde{t}|$ and $\tilde{r} = \tilde{r}_1 \tilde{r}_2$ for some words $\tilde{r}_1 \leq_{\mathrm{s}} \tilde{t}^+$ and $\tilde{r}_2 \leq_{\mathrm{p}} \tilde{t}^+$. Then there are words t_1, t_2 such that $t_2 t_1 \leq_{\mathrm{s}} \tilde{r}$, $t_1 t_2 = \tilde{t}$ and $\tilde{r}_2 \in (t_1 t_2)^* t_1$.*

Now, we can proceed with the proof of Claim 1:

Proof. We will first show that

$$G^\omega \cap p_k H^\omega = g(e_k)(G^\omega \cap H^\omega). \tag{2.6}$$

Since 1 is an overflow of $(g(e), h(f))$, Lemma 32 yields that

$$G^\omega \cap p_k H^\omega = \alpha_k (G^\omega \cap H^\omega),$$

where α_k is the shortest word from $G^* \cap p_k H^*$. We will show that $\alpha_k = g(e_k)$. Suppose that $\alpha_k \neq g(e_k)$. Then, from $g(e_k) = p_k h(f_k)$ and Lemma 31, we obtain that

$$\alpha_k \wedge g(e_k) \in G^* \cap p_k H^*,$$

The minimality of α_k yields that $\alpha_k = \alpha_k \wedge g(e_k)$. Since g and h are marked, there are words $\beta_k \leq_{\mathrm{p}} e_k$ and $\gamma_k \leq_{\mathrm{p}} f_k$ such that

$$\alpha_k = g(\beta_k) = p_k h(\gamma_k).$$

Then,

$$g(\beta_k^{-1} e_k e e_k^{-1} \beta_k) = h(\gamma_k^{-1} f_k f f_k^{-1} \gamma_k),$$

and since $(g(e), h(f))$ is a simple pair of words, we get

$$(\beta_k^{-1} e_k) e (\beta_k^{-1} e_k)^{-1} = e,$$
$$(\gamma_k^{-1} f_k) f (\gamma_k^{-1} f_k)^{-1} = f.$$

Consequently, $e_k = \beta_k \rho(e)^{j_1}$ and $f_k \in \gamma_k \rho(f)^{j_2}$, for some $j_1, j_2 \in \mathbb{N}$. Then,

$$g(\rho(e)^{j_1}) = g(\beta_k^{-1} e_k) = (p_k h(\gamma_k))^{-1} p_k h(f_k) = h(\gamma_k^{-1} f_k) = h(\rho(f)^{j_2}).$$

Since $(g(e), h(f))$ is simple pair, the words e and f are minimal words satisfying the equality $g(e) = h(f)$. But then $|\rho(f)^{j_2}| \leq |f_k| < |vf_k| \leq |f|$ yields that

$j_2 = j_1 = 0$. Therefore, $e_k = \beta_k$ and $\alpha_k = g(e_k)$. We have proved that for each $k \in \{1,2,3\}$

$$G^\omega \cap p_k H^\omega = g(e_k)(G^\omega \cap H^\omega).$$

Let t' be a suffix of r of the length $|s_1 s_2|$. Then t' is conjugate of $s_1 s_2$.

1. Suppose first that $sh(f_2) \leq_{\mathrm{p}} t'^+$. Let $t = s^{-1}t's$. Then $h(f_2) \leq_{\mathrm{p}} t^+$ and from $h(f_2) = h(f'vf_1) \leq_{\mathrm{s}} g(e_2)$, for some $f' \leq_{\mathrm{p}} f_2$, we get $h(f'v) \in t^+$. Let γ be a maximal common prefix of $h(f_3)$ and t^ω. Then, by Lemma 31,

$$\gamma = h(f_3) \wedge h(f'v)^\omega \in H^*.$$

Since $|r| \geq |s_1 s_2|$ and $r = r_3' r_3$, where $r_3' \leq_{\mathrm{s}} g(u_i)^+$, $r_3 \leq_{\mathrm{p}} g(u_i)^+$ for some $i \in \{1,2\}$, we can use Lemma 35 and obtain that there are words t_1, t_2 such that $t' = t_2 t_1$, $\rho(g(u_i)) = t_1 t_2$ and $r_3 \in (t_1 t_2)^* t_1$. Then

$$r_3 s\gamma = g(d_3)p_3 h(f_3) \wedge r_3 sh(f'v)^\omega = g(d_3 e_3) \wedge (t_1 t_2)^\omega = g(d_3 e_3) \wedge g(u_i)^\omega,$$

and consequently, Lemma 31 yields that $r_3 s\gamma \in G^*$. From Eq. (2.6), we obtain

$$r_3 sH^\omega \cap G^\omega = g(d_3)(p_3 H^\omega \cap G^\omega) = g(d_3 e_3)(H^\omega \cap G^\omega).$$

Since $r_3 s\gamma \in r_3 sH^* \cap G^*$, Corollary 33 yields that $g(d_3 e_3) \leq_{\mathrm{p}} r_3 s\gamma$. Consequently, from $g(d_3 e_3) = r_3 sh(f_3)$, we obtain that $h(f_3) \leq_{\mathrm{p}} \gamma$. Then, $h(f_3) \leq_{\mathrm{p}} (f'v)^\omega$, and since morphisms h is marked, we have $f_3 \leq_{\mathrm{p}} (f'v)^+ = (f_2 f_1^{-1})^+$. Similarly, $g(d_3 e_3)$ is a prefix of $r_3 s\gamma \leq_{\mathrm{p}} g(u_i)^\omega$, and since g is marked, we obtain that $d_3 e_3 \leq_{\mathrm{p}} u_i^+$.

2. In order to finish the proof it remains to show that $sh(f_2) \leq_{\mathrm{p}} t'^+$. Suppose, for example, that $|r_3| > |r_2|$. From Lemma 35, we can infer that there is $k \geq 0$ such that $r_3 = (pq)^k p r_2$, where either $p = 1$ and $q \in \{s_1 s_2, s_2 s_1\}$, or $\{p,q\} = \{s_1, s_2\}$. Then,

$$g(d_3)p_3 = (pq)^k p g(d_2) p_2.$$

Notice that $(pq)^k p \notin G^*$, otherwise from G being marked would follow that interpretations I_2 and I_3 are adjacent. Since $g(d_3)p_3$ and $g(d_2)p_2$ are overflows of $(g(e), h(f))$ and $(pq)^\omega \in G^\omega$, $(qp)^\omega \in G^\omega$, we can use Lemma 34 (where we take $z = g(d_2)p_2$). Since, by Eq. (2.6), we have

$$G^\omega \cap g(d_k)p_k H^\omega = g(d_k e_k)(G^\omega \cap H^\omega),$$

we obtain from Lemma 34 that $g(d_3 e_3) \leq_{\mathrm{p}} g(u_i)^\omega$ or $g(d_2 e_2) \leq_{\mathrm{p}} g(u_{i'})^\omega$, where $i, i' \in \{1,2\}$. Both cases yield that $sh(f_2) \leq_{\mathrm{p}} t'^+$.

□

To illustrate the practical nature of the previous claim and its impact to the structure of the equality word, we would like to point out here one of its most common use we will employ in the proof of Theorem 8:

Let $v = b$, $u_1 = u_2 = c$, where $c \in \{a,b\}$, and suppose that assumptions of Claim 1 hold. Suppose, moreover, that $|h(b)| \geq |g(c)|$. Then, by Claim 1, $e_3 \in c^+$, and consequently, each occurrence of $h(b)$ in $h(f_3)$ is a factor of $g(c)^+$. Each $\{g(a), g(b)\}$-interpretation of $h(b)$ induced by $p_3 h(f_3) = g(e_3)$ therefore satisfies the assumptions of Claim 1. Therefore, instead of f_1, f_2, we can use Claim 1 on f_1', f_2' corresponding to the last two occurrences of $h(b)$ in $h(f_3)$ and conclude that there is $i \in \mathbb{N}$ such that $f_3 \in \mathrm{pref}((a^i b)^+)$, that is $f_3 \in (a^i b)^+ a^j$ for some $j \leq i$.

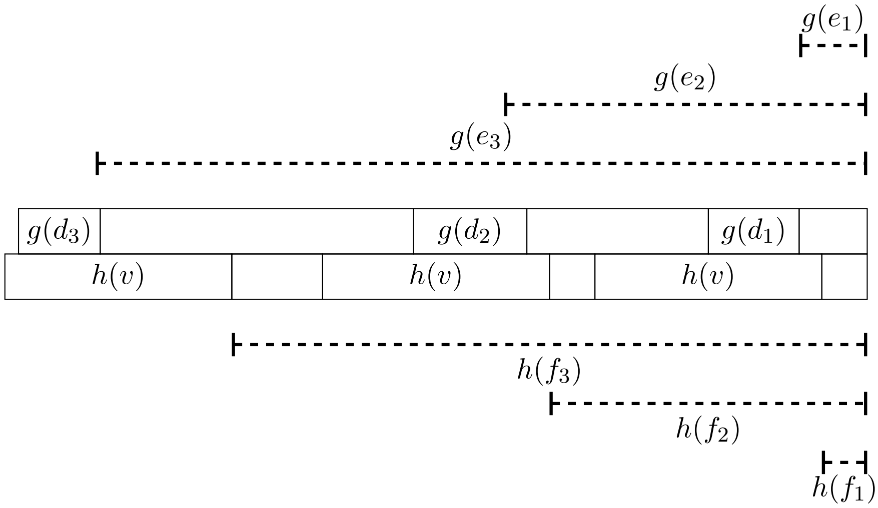

Figure 5: Conditions required by Claim 1.

2.2.3 Proof of Theorem 8

Finally, we will use the previous results and show that Theorem 8 holds. First, however, we will need the following technical lemma.

Lemma 36. *Let $g, h : \{a,b\}^* \to A^*$ be non-periodic binary morphisms and let w be their simple equality word. Suppose that $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$, and $1 \leq k|w|_b \leq |w|_a$, for some $k \geq 1$. If $|g(a)| \geq (k+1)|\rho(g(a))|$ and $|\rho(g(b))| \geq k|\rho(g(a))|$ then $|h(b)| \geq k|\rho(g(a))| + |\rho(g(b))|$.*

Proof. Let g_a, g_b, h_a, h_b be words $g(a), g(b), h(a), h(b)$, respectively. Since w is a simple equality word of g, h and $|w| \geq 2$, we have $|h_b| > |g_b|$ and $|h_a| < |g_a|$.

Suppose, for a contradiction, that $|h_b| < k|\rho(g_a)| + |\rho(g_b)|$ and $|g_a| \geq (k+1)|\rho(g_a)|$, $|\rho(g_b)| \geq k|\rho(g_a)|$. Then $|h_b| - |g_b| < k|\rho(g_a)|$. Since

$$|w|_b(|h_b| - |g_b|) = |w|_a(|g_a| - |h_a|),$$

and $k|w|_b \leq |w|_a$, we observe first that $|g_a| - |h_a| < |\rho(g_a)|$ and second that

$$j|h_a| + |h_b| \geq j|g_a| + |g_b|,$$

holds for all $j \leq k$. We will proceed by case analysis.

1. Suppose first that $a \leq_{\mathrm{p}} w$ and $a \leq_{\mathrm{s}} w$. Since

$$2|h_a| \geq |h_a| + (|g_a| - |\rho(g_a)|) \geq |h_a| + k|\rho(g_a)|,$$

and $h_a \leq_{\mathrm{p}} g_a$, the words g_a and h_a commute. We have a contradiction with $|g_a| - |h_a| < |\rho(g_a)|$. The same reasoning applies in case that $a^2 \leq_{\mathrm{p}} w$ or $a^2 \leq_{\mathrm{s}} w$.

2. Suppose that $ab \leq_{\mathrm{p}} w$ and $b \leq_{\mathrm{s}} w$. Let $t = (g_a g_b)^{-1}(h_a h_b)$. Notice that $|t| < |h_b| - |g_b| < k|\rho(g_a)|$.

2.1 If $b^2 \leq_{\mathrm{s}} w$, then from $|h_b| - |g_b| < k|\rho(g_a)|$ and from $|\rho(g(b))| \geq k|\rho(g_a)|$, it follows that h_b is a suffix of g_b^2. Then from $|g_a| > |h_a|$ we obtain $g_b t \leq_{\mathrm{s}} h_b \leq_{\mathrm{s}} \rho(g_b)^+$. Therefore, $t \in \rho(g_b)^+$, a contradiction with $|t| < k|\rho(g_a)|$.

2.2 If $ab^2 \leq_{\mathrm{p}} w$, then $t \leq_{\mathrm{p}} g_b$ and $t^{-1} g_b t$ is a suffix of h_b. Therefore, $t^{-1} g_b t = g_b$, and we have $t \in \rho(g_b)^+$, a contradiction with $|t| < k|\rho(g_a)|$.

2.3 Suppose $ab \leq_{\mathrm{s}} w$ and $aba \leq_{\mathrm{p}} w$. Then $h_a h_b \leq_{\mathrm{p}} g_a g_b g_a$ and $g_a g_b \leq_{\mathrm{s}} h_a h_b$. Therefore, by the Periodicity lemma, words $h_a h_b$, $g_a g_b$ and t commute. Since $|g_a| \geq (k+1)|\rho(g_a)| > |\rho(g_a)| + |t|$, also t and g_a commute. Therefore, g_a and g_b commute, a contradiction with g being a non-periodic morphism.

The case $ba \leq_{\mathrm{s}} w$ and $b \leq_{\mathrm{p}} w$ is symmetrical.

3. Suppose that $b \leq_{\mathrm{p}} w$ and $b \leq_{\mathrm{s}} w$. If $b^2 \leq_{\mathrm{p}} w$ or $b^2 \leq_{\mathrm{s}} w$, then g_b and h_b commute. We get a contradiction with $|h_b| - |g_b| < k|\rho(g_a)| \leq |\rho(g_b)|$.

3.1 Suppose $ba \leq_{\mathrm{p}} w$ and $ab \leq_{\mathrm{s}} w$ and let $s = h_a h_b (g_a g_b)^{-1}$, $t = (g_b g_a)^{-1} h_b h_a$.

3.1.1 Suppose that $baa \leq_{\mathrm{p}} w$. Then word $h_b^{-1} g_b g_a^2$ is a suffix of g_a^2 longer than

$$2|g_a| - (|h_b| - |g_b|) \geq 2|g_a| - k|\rho(g_a)| \geq |h_a| + |\rho(g_a)|.$$

Since $2|h_a| \geq |h_a| + k|\rho(g_a)|$ and $h_b^{-1} g_b g_a^2$ is a prefix comparable with h_a^2, the words g_a^2 and h_a^2 have a common factor of the length at least $|h_a| + |\rho(g_a)|$. Therefore, primitive roots of g_a and h_a are conjugate, a contradiction with $|g_a| - |h_a| < |\rho(g_a)|$. The case $aab \leq_{\mathrm{s}} w$ is symmetrical.

3.1.2 If $babb \leq_{\mathrm{p}} w$, then $t^{-1} g_b t = g_b$. Then t and g_b commute, a contradiction with $|t| < k|\rho(g_a)|$. The case $bbab \leq_{\mathrm{s}} w$ is symmetrical.

3.1.3 Suppose that $baba \leq_{\mathrm{p}} w$ and $abab \leq_{\mathrm{s}} w$. Let $p = h_b h_a h_b (g_b g_a g_b)^{-1}$. Notice that p is a prefix of $g_b g_a$ shorter than $2k|\rho(g_a)|$. From

$$|g_b g_a p| = |h_b h_a h_b| - |g_b|,$$

we obtain that

$$g_b g_a p = h_b h_a h_b g_b^{-1} = p g_b g_a.$$

Comparison of lengths $|p| < 2k|\rho(g_a)|$ and $|g_b g_a| \geq (2k+1)|\rho(g_a)|$ yields that $g_b g_a$ is not primitive and

$$|g_b g_a| - |t| \geq |g_b g_a| - k|\rho(g_a)| \geq \frac{1}{2}|g_b g_a| \geq |p|.$$

Since h_b is prefix comparable with $t^{-1} g_b g_a$, the words $(h_b h_a)^+$ and $(g_a g_b)^+$ have a common prefix of the length at least $|h_b h_a| + |p|$. Therefore, the words $h_b h_a$, $g_b g_a$ and t commute. From $|t| < k|\rho(g_a)|$ and $|g_a| \geq (k+1)|\rho(g_a)|$, it follows that g_a and t commute. Therefore, also g_a and g_b commute, a contradiction with our assumption that g is non-periodic. □

We now proceed with the proof Theorem 8. Let us first remind its formulation:

Theorem 8. *Let $g, h : \{a, b\}^* \to A^*$ be non-periodic binary morphisms and let w be their simple equality word. Suppose that $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$. If $|w|_b \geq 4$ and $|w|_a \geq 30$, then*

$$w = (ab)^i a \quad \text{or} \quad w = (ba)^i b \quad \text{or} \quad w = a^j b^k \quad \text{or} \quad w = b^k a^j$$

with $\gcd(j, k) = 1$ and $j > k$.

Proof. The main part of the proof of the theorem is occupied by (not so short) case analysis. In order that the reader does not get lost on the path, we provide him with the following "cookbook":

1. Work with marked versions of morphisms g, h. Instead of morphisms g, h, we will consider their marked versions $g_{\mathrm{m}}, h_{\mathrm{m}}$ (to remind you the definition see p. 4). Since $g_{\mathrm{m}}(w)z_g^{-1}z_h = z_g^{-1}z_h h_{\mathrm{m}}(w)$, we will have to work with a bit more complicated structure of a simple conjugate pair $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$. On the other hand, this rather technical inconvenience will be compensated for by the fact that both morphisms are marked.

2. Look at $\{g_{\mathrm{m}}(a), g_{\mathrm{m}}(b)\}$-interpretation of $h_{\mathrm{m}}(b)$. In the first place, we will be looking at $\{g_{\mathrm{m}}(a), g_{\mathrm{m}}(b)\}$-interpretation of occurrences of $h_{\mathrm{m}}(b)$ in $h_{\mathrm{m}}(w)$. Importantly, notice that any pair of such interpretations has to be disjoint since otherwise the pair $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$ would not be simple. We will show that the interpretations are shifted in a convenient way.

3. Narrow the "candidates" for the equality word. Having disjoint, but shifted, interpretations of $h_{\mathrm{m}}(b)$, will allow us to use Claim 1 and narrow our choice of the possible structure of equality words. We then proceed by a combinatorial analysis.

First notice that if $|w|_b \geq 9$, then our theorem holds by Theorem 6. Therefore, we suppose that $8 \geq |w|_b \geq 4$ and $|w|_a \geq 30$.

Let $g_{\mathrm{m}}, h_{\mathrm{m}} : \{a, b\}^* \to A^*$ be marked versions of morphisms g, h. Let g_a, g_b, h_a, h_b be words $g_{\mathrm{m}}(a), g_{\mathrm{m}}(b), h_{\mathrm{m}}(a), h_{\mathrm{m}}(b)$, respectively, and let $G = \{g_a, g_b\}$ and $H = \{h_a, h_b\}$. Since w is an equality word of g, h, we have $|h_b| > |g_b|$ and $|g_a| < |h_a|$. Also, $g_{\mathrm{m}}(w)z = zh_{\mathrm{m}}(w)$, where $z = z_g^{-1}z_h$. Therefore,

$$g_{\mathrm{m}}(w)^{\omega} = zh_{\mathrm{m}}(w)^{\omega}. \tag{2.7}$$

We will show that if $8 \geq |w|_b \geq 4$ and $|w|_a \geq 30$, then there are words e and f, conjugates of w, such that $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$ and e, f have to adopt one of the following forms:

Form (A)

$e \in e'a^+$, $f \in a^{\ell}b(a^ib)^+a^j$, where $j \leq i$ and $g_{\mathrm{m}}(e') \leq_{\mathrm{p}} h_{\mathrm{m}}(a^{\ell}b)$.

This case yields $(e, f) = (b^m a^{\ell}, a^{\ell} b^m)$, for some $\ell, m \geq 1$ (for details see Claim 2).

Form (B)

$e \in e'b^+$, $f \in a^{\ell}b(a^ib)^+a^j$, where $j \leq i$ and $g_{\mathrm{m}}(e') \leq_{\mathrm{p}} h_{\mathrm{m}}(a^{\ell}b)$.

This case yields $e = f = ab^m$, for some $m \geq 1$ (for details see Claim 3).

Form (C)

$e \in e'(ba^n)^+ba^m \cup e'(a^nb)^+a^m$, $f \in a^{\ell}b(a^ib)^+a^j$, where $j \leq i$, $m \leq n$, $n \neq 0$, $|h_b| > |\rho(g_{\mathrm{m}}(a^nb)|$ and $g_{\mathrm{m}}(e') \leq_{\mathrm{p}} h_{\mathrm{m}}(a^{\ell}b)$.

This case yields $e = f \in (ab)^+a$ or $e = f \in (ba)^+b$ (for details see Claim 4).

Form D

$e \in e'(ab^n)^+ab^m \cup e'(b^na)^+b^m$, $f \in a^\ell b(a^ib)^+a^j$, where $j \le i$, $m \le n$, $n \ne 0$, $|h_b| > |\rho(g_{\mathrm{m}}(b^na)|$ and $g_{\mathrm{m}}(e') \le_{\mathrm{p}} h_{\mathrm{m}}(a^\ell b)$.

This case yields $e = f \in (ab)^+a$ or $e = f \in (ba)^+b$ (for details see Claim 5).

Concerning now our equality word w, according to our article *Equation $x^iy^jx^k = u^iv^ju^k$ in words* [5, Theorem 1], Form A yields equality words $w = a^\ell b^m$ or $w = b^ma^\ell$. Form B does not satisfy our assumption that $|w|_a = |e|_a \ge 30$, and Form C and Form D yield, according to [4, Lemma 20], that $w = (ab)^ia$ or $w = (ba)^ib$. To finish the proof it remains to demonstrate that there are words e and f, conjugates of w, such that $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$ and e, f adopt one of the forms A - D.

Suppose first that $g_{\mathrm{m}}(w)$ is not primitive. Since w is a minimal equality word, $\rho(w) = w$ and, according to [6, Theorem 4.2], we obtain that w is a conjugate of one of the words from the set $a^*b \cup ab^*$, a contradiction with our assumption that $|w|_b \ge 4$ and $|w|_a \ge 30$. Therefore, we suppose that both $g_{\mathrm{m}}(w)$ and $h_{\mathrm{m}}(w)$ are primitive and $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$ is a simple pair of conjugate words.

We will now have a look at G-interpretations of h_b induced by Eq. (2.7). Let, therefore, $\mathcal{I}$ be a set of all such G-interpretations $(q, g_{\mathrm{m}}(d), p)$ of h_b for which there exist words e', f' and letters $c_q, c_p \in \{a, b\}$ satisfying $e'c_qdc_p \le_{\mathrm{p}} w^\omega$, $f'b \le_{\mathrm{p}} w^\omega$, $q <_{\mathrm{s}} g_{\mathrm{m}}(c_q)$, $p <_{\mathrm{p}} g_{\mathrm{m}}(c_p)$ and $g_{\mathrm{m}}(e'c_q) = zh_{\mathrm{m}}(f')q$. Since $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$ is a simple pair of conjugate words, all interpretations of h_b from $\mathcal{I}$ are pairwise disjoint.

We will now list four special situations. Further on we will show that those are the only situations which can in fact take place.

(A) Suppose that h_b admits two disjoint G-interpretations from $\mathcal{I}$ shifted by $\rho(g_a)$. Notice that since G is marked, we have $|g_a| \ge 2|\rho(g_a)|$ by Lemma 27. Moreover, Lemma 13 yields that the word h_b is a factor of g_a^+.

First, we will show that 1 is an overflow of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$. We employ Lemma 34 in the following way. Let $I_1, I_2 \in \mathcal{I}$ be two G-interpretations from $\mathcal{I}$ shifted by $\rho(g_a)$. Then $I_1 = (q, g(d_1d_2), p)$, $I_2 = (q', g(d_1'd_2'), p')$ where $d_1, d_2, d_1', d_2' \in \{a, b\}^*$ and $g(d_2)p(g(d_2')p')^{-1} = \rho(g_a)^i$ for some $i \ge 1$. Let word $\underline{q}, \underline{p}, \underline{z}$ from the formulation of Lemma 34 be the following: $\underline{q} = \rho(g_a)$, $\underline{p} = 1$ and $\underline{z} = g(d_2')p'$. Since $I_1, I_2 \in \mathcal{I}$, words $g(d_2')p'$ and $g(d_2)p = \rho(g_a)^ig(d_2')p'$ are H-overflows of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$. We have also $(\underline{pq})^\omega = (\underline{qp})^\omega = g_a^\omega \in G^\omega$. If $\rho(g_a)^i = g(u)$ for some $u \in \{a, b\}^*$, then since G is marked, we obtain that $u \le_{\mathrm{p}} d_2$, a contradiction with I_1 and I_2 being disjoint. Moreover, we have $h_{\mathrm{m}}(\rho(w)) = h_{\mathrm{m}}(w) = \rho(h_{\mathrm{m}}(w))$ since w is a simple equality word and $h_{\mathrm{m}}(w)$ is primitive.

Hence, Lemma 34 yields that either $g_{\mathrm{m}}(w') \in \rho(g_a)^+$ for some conjugate w' of w, or 1 is an overflow of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$. The former possibility yields that $g_{\mathrm{m}}(w') = \rho(g_{\mathrm{m}}(w')) = \rho(g_a)$, a contradiction with $|w'|_a = |w|_a \ge 30$. Therefore, 1 is an overflow of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$, and consequently there are words e, f, conjugates of w, such that $(g_{\mathrm{m}}(e), h_{\mathrm{m}}(f))$ is a simple pair of conjugate words such that $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$.

We will encounter the following situations:

(A1) $\rho(g_a)$ **and** $\rho(g_b)$ **are conjugate.** Suppose that primitive roots of g_a and g_b are conjugate. Then, since h_b is a factor of g_a^+ and $|h_b| \geq |g_a| \geq 2|\rho(g_a)|$, we can employ Lemma 23, and obtain that all interpretations from $\mathcal{I}$ satisfy the assumptions of Claim 1. Thus, we get Form A or Form B.

Suppose therefore that the **primitive roots of** g_a **and** g_b **are not conjugate**. If $|\rho(g_a)| \geq |\rho(g_b)|$, then we use our assumption that $|h_b| \geq |g_a|$, and consequently, Corollary 26 yields that at least $|\mathcal{I}| - 1$ G-interpretations from $\mathcal{I}$ are shifted by $\rho(g_a)$. In case that $|\rho(g_a)| < |\rho(g_b)|$, we use Lemma 36 for $k = 1$ and obtain that $|h_b| \geq |\rho(g_a)| + |\rho(g_b)|$. Then Corollary 29 yields that again at least $|\mathcal{I}| - 1$ G-interpretations from $\mathcal{I}$ are shifted by $\rho(g_a)$.

Thus, in both cases, Lemma 27 yields that $|g_a| \geq 3|\rho(g_a)|$. Notice that we have used that $|\mathcal{I}| - 1 = |w|_b - 1 \geq 3$.

(A2) $|\rho(g_a)| \geq |\rho(g_b)|$**.**
If $|\rho(g_a)| \geq |\rho(g_b)|$, then from $|h_b| \geq |g_a| \geq 3|\rho(g_a)|$ and Corollary 25, it follows that all G-interpretations from $\mathcal{I}$ are shifted by $\rho(g_a)$. Thus, by Claim 1, we get Form A.

Suppose now that $|\rho(g_a)| < |\rho(g_b)|$. We will list three sub-cases:

(A3) $|h_b| \geq 2|\rho(g_a)| + |\rho(g_b)|$**.**
Suppose that $|h_b| \geq 2|\rho(g_a)| + |\rho(g_b)|$. Then, according to Corollary 30, all G-interpretations from $\mathcal{I}$ are shifted by $\rho(g_a)$. Therefore, we can again use Claim 1 and obtain Form A.

(A4) $|h_b| \geq 4|\rho(g_a)|$**.**
Suppose that $|h_b| \geq 4|\rho(g_a)|$. If $|\rho(g_b)| \leq 2|\rho(g_a)|$, then $|h_b| \geq 2|\rho(g_a)| + |\rho(g_b)|$. In case that $|\rho(g_b)| > 2|\rho(g_a)|$, Lemma 36 yields that again $|h_b| \geq 2|\rho(g_a)| + |\rho(g_b)|$. We proceed by (A3).

(A5) $|h_b| \geq |g(a)| + |\rho(g_a)|$**.**
Suppose that $|h_b| \geq |g(a)| + |\rho(g_a)|$. Since $|g(a)| \geq 3|\rho(g_a)|$, we obtain that $|h_b| \geq 4|\rho(g_a)|$ and we proceed by (A4).

(B) Suppose that h_b admits two disjoint G-interpretations from $\mathcal{I}$ shifted by $\rho(g_b)$. Notice that since G is marked, we have $|g_b| \geq 2|\rho(g_b)|$ by Lemma 27. Moreover, Lemma 13 yields that the word h_b is a factor of g_b^+.

Similarly as in (A), we can prove that 1 is an overflow of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$. Thus, there are words e, f, conjugates of w, such that $(g_{\mathrm{m}}(e), h_{\mathrm{m}}(f))$ is a simple pair such that $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$.

We will encounter the following situations:

(B1) $\rho(g_a)$ and $\rho(g_b)$ are conjugate. Suppose that primitive roots of g_a and g_b are conjugate. As in (A1), we use Lemma 23 with Claim 1 and get Form (A) or Form (B).

Suppose, therefore, that the **primitive roots of g_a and g_b are not conjugate**. Notice that in this case, in contrast with the case (A), we cannot use Lemma 36.

(B2) $|\rho(g_b)| \geq |\rho(g_a)|$.
Suppose that $|\rho(g_b)| \geq |\rho(g_a)|$. From $|h_b| \geq |g_b| \geq 2|\rho(g_b)|$ and Corollary 26, it follows that at least $|\mathcal{I}| - 1 = |w|_b - 1$ G-interpretations from $\mathcal{I}$ are shifted by $\rho(g_b)$. Then Lemma 27 yields $|g_b| \geq 3|\rho(g_b)|$. Consequently, according to Corollary 25, all interpretations from $\mathcal{I}$ are shifted by $\rho(g_b)$. Thus, by Claim 1, we get Form (B).

Suppose that $|\rho(g_b)| < |\rho(g_a)|$.

(B3) $|h_b| \geq |\rho(g_a)| + 2|\rho(g_b)|$.
Suppose that $|h_b| \geq |\rho(g_a)| + 2|\rho(g_b)|$. Then, according to Corollary 30, all G-interpretations from $\mathcal{I}$ are shifted by $\rho(g_b)$. Therefore, we can use Claim 1 and obtain Form (B).

(C) Let $g_{\mathrm{m}}(cd^n) \in (s_1s_2)^+$, $g_{\mathrm{m}}(d^nc) \in (s_2s_1)^+$, where $\{c,d\} = \{a,b\}$, s_1s_2 is a primitive word and $n \geq 1$. Suppose that there are words s, r, s' such that $h_{\mathrm{m}}(b) = s'rs$ and r is a factor of $(s_1s_2)^+$ longer than $|s_1s_2|$. Suppose also that for each interpretation $I \in \mathcal{I}$ there are words d_1, d_2 and r_1, r_2 such that $I = (q, g_{\mathrm{m}}(d_1d_2), p)$, $r = r_1r_2$ and $qg_{\mathrm{m}}(d_1) = s'r_1$, $g_{\mathrm{m}}(d_2)p = r_2s$. Finally, we suppose that for each interpretation $I_1 = (q, g_{\mathrm{m}}(d_1d_2), p) \in \mathcal{I}$ there exists an interpretation $I_2 = (q', g_{\mathrm{m}}(d_1'd_2'), p') \in \mathcal{I}$ such that I_1, I_2 are shifted by s_1s_2 or by s_2s_1 in a way that $g_{\mathrm{m}}(d_2)p = (s_is_j)^k g_{\mathrm{m}}(d_2')p'$, $\{i,j\} = \{1,2\}$, $k \geq 1$, whenever $|g_{\mathrm{m}}(d_2)p| \geq |g_{\mathrm{m}}(d_2')p'|$ and $g_{\mathrm{m}}(d_2')p' = (s_is_j)^k g_{\mathrm{m}}(d_2)p$, $\{i,j\} = \{1,2\}$, $k \geq 1$, otherwise.

First, we will show that 1 is an overflow of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$. Since $|\mathcal{I}| = |w|_b \geq 4$, we have the following two cases:

(C1) Let $I_1, I_2 \in \mathcal{I}$ be two G-interpretation of h_b shifted by s_1s_2. Then $I_1 = (q, g_{\mathrm{m}}(d_1d_2), p)$, $I_2 = (q', g_{\mathrm{m}}(d_1'd_2'), p')$, where $d_1, d_2, d_1', d_2' \in \{a,b\}^*$, and we can suppose that

$$g_{\mathrm{m}}(d_2)p = (s_1s_2)^i g_{\mathrm{m}}(d_2')p',$$

for some $i \geq 1$.

We use Lemma 34 in the following way. Let word $\underline{q}, \underline{p}, \underline{z}$ from the formulation of the lemma be the following: $\underline{q} = s_1s_2$ and $\underline{p} = 1$ and $\underline{z} = g_{\mathrm{m}}(d_2')p'$. Since $I_1, I_2 \in \mathcal{I}$, we observe that $g_{\mathrm{m}}(d_2)p$ and $g_{\mathrm{m}}(d_2')p'$ are H-overflows of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$. Also, we have $(\underline{pq})^\omega = (\underline{qp})^\omega = g_{\mathrm{m}}(cd^n)^\omega \in G^\omega$, and $(\underline{pq})^i\underline{p} = (s_1s_2)^i \notin G^*$ since I_1 and I_2 are disjoint. Moreover, we have $h_{\mathrm{m}}(\rho(w)) = h_{\mathrm{m}}(w) = \rho(h_{\mathrm{m}}(w))$ since w is a simple equality word and $h_{\mathrm{m}}(w)$ is primitive.

Hence, Lemma 34 yields that either $g_{\mathrm{m}}(w') \in (s_1s_2)^+ = \rho(g_{\mathrm{m}}(cd^n))^+$ for some conjugate w' of w, or 1 is an overflow of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$. The former possibility yields that $g_{\mathrm{m}}(w') = \rho(g_{\mathrm{m}}(w')) = \rho(g_{\mathrm{m}}(cd^n))$. Consequently, since g_{m} is marked, we obtain that $w' = cd^n$, a contradiction with $|w'|_c \geq 4$. Therefore, 1 is an overflow of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$, and there are words e, f, conjugates of w, such that $(g_{\mathrm{m}}(e), h_{\mathrm{m}}(f))$ is a simple pair of conjugate words with $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$.

(C2) In case that $I_1, I_2 \in \mathcal{I}$ are two G-interpretations of h_b shifted by $s_2 s_1$ we proceed as in (C1). Again, we conclude that, 1 is an overflow of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$, and that there are words e, f, conjugates of w, such that $(g_{\mathrm{m}}(e), h_{\mathrm{m}}(f))$ is a simple pair of conjugate words with $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$.

We have shown that 1 is an overflow of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$ and that there are words e, f, conjugates of w, such that $(g_{\mathrm{m}}(e), h_{\mathrm{m}}(f))$ is a simple pair of conjugate words with $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$. Since we have assumed that $h_{\mathrm{m}}(b) = s'rs$, where r is a factor of $(s_1 s_2)^+$ longer than $|s_1 s_2|$ and all G-interpretations of h_b from $\mathcal{I}$ follow on r the pattern $(s_1 s_2)^+$, we can use Claim 1 and get Form C or Form D.

(D) Suppose that $\rho(g_a)^2$ is a factor of h_b and that primitive roots of g_a and g_b are conjugate. Then, according to Lemma 23, $h(b)$ is a factor of g_a^+ and all G-interpretations of h_b from $\mathcal{I}$ follow the pattern imposed by g_a^+ (and by g_b^+). We can then use again Lemma 34 and get that 1 is an overflow of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$. Therefore, there are words e, f, conjugates of w, such that $(g_{\mathrm{m}}(e), h_{\mathrm{m}}(f))$ is a simple pair such that $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$. Then Claim 1 yields Form A or Form B.

Now we will prove that (A) or (B) or (C) or (D) take place. Let us remind the reader that we still assume that the length of h_b is maximal among the lengths of the words g_a, g_b, h_a, h_b, and that we suppose that $|w|_a \geq 30$ and $4 \leq |w_b| \leq 8$.

(1) Suppose that primitive roots of g_a and g_b are conjugate. Let $(q, d, p) \in \mathcal{I}$ be a G-interpretation of h_b.

(1.1) Suppose that $d \in G^* g_a^2 G^*$. Then we have (D).

(1.2) Suppose that $d \in G^* g_b g_a G^*$ or $d \in G^* g_a g_b G^*$. Then, by [6, Proposition D, p.75], all G-interpretations from $\mathcal{I}$ are pairwise $\{\rho(g_a), \rho(g_b)\}$-adjacent. By Corollary 15, each pair of interpretations from $\mathcal{I}$ is shifted by $\rho(g_a)$ or $\rho(g_b)$, and consequently, by Lemma 13, h_b is a factor of $\rho(g_a)^+$ (or $\rho(g_b)^+$, but since $\rho(g(a))$ and $\rho(g(b))$ are conjugate these formulations are equal). Then our assumption that $d \in G^* g_b g_a G^*$ or $d \in G^* g_a g_b G^*$ yields that $\rho(g_a) = \rho(g_b)$, a contradiction with G being non-periodic.

Therefore, we are left with the cases when $d \in g_b^*$ and $d = g_a$.

(1.3) Suppose that there are two G-interpretations of h_b from $\mathcal{I}$ which are shifted by $\rho(g_a)$. Then we have the case (A1). Notice that this case applies if $d = g_a$ and $q \leq_{\mathrm{s}} g_a$, $p \leq_{\mathrm{p}} g_a$ and $d' = g_a$ for some G-interpretation $(q', d', p') \in \mathcal{I}$ of h_b disjoint from (q, d, p).

(1.4) Suppose that **no** two G-interpretations from $\mathcal{I}$ of h_b are shifted by $\rho(g_a)$. We will count the minimal number of occurrences of g_a inside $g_{\mathrm{m}}(w)$ which do not have any common factor with some occurrence of h_b in $h_{\mathrm{m}}(w)$. Since $d \in g_b^*$ or $d = g_a$, each occurrence of h_b in $h_{\mathrm{m}}(w)$ can overlap at most with three occurrences of g_a.

Suppose first that there is an occurrence of h_b which overlap with exactly three occurrences of g_a, that is $d = g_a$, $q <_{\mathrm{s}} g_a$ and $p <_{\mathrm{p}} g_a$. Then we have seen in **(1.3)** that each of the remaining occurrences of h_b overlap at most with two occurrences of g_a. Since $|w|_a \geq 30$ and $8 \geq |w|_b$, there are at least $30-3-7\cdot 2 = 13$ occurrences of g_a inside $g_{\mathrm{m}}(w)$ which are factors of h_a^+. Similarly, if no occurrence of h_b in $h_{\mathrm{m}}(w)$ overlap with three occurrences of g_a, there are at least $30 - 8 \cdot 2 = 14$ occurrences of g_a inside $g_{\mathrm{m}}(w)$ which are factors of h_a^+.

In all cases, there are at least 13 occurrences of g_a inside $g_{\mathrm{m}}(w)$ which are factors of h_a^+. Since $|g_a| > |h_a|$, they are all shifted by $\rho(h_a)$. We can then use Lemma 34 (in this case we work with a pair of G-overflows and the words $\underline{q}, \underline{p}$ from the lemma are $\underline{q} = \rho(h_a)$, $\underline{p} = 1$) and get that 1 is an overflow of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$. Therefore, there are words e, f, conjugates of w, such that $(g_{\mathrm{m}}(e), h_{\mathrm{m}}(f))$ is a simple pair of conjugate words satisfying $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$. Consequently, by Claim 1, there are words e', f' such that $e = e'(ab^i)^m ab^j$, $f \in f'a^+$, where $m \geq 12$, $j \leq i$ and $h_{\mathrm{m}}(f') \leq_{\mathrm{p}} g_{\mathrm{m}}(e'a)$. Since $|e|_b = |f|_b \leq 8$, we have $i = j = 0$, and the words g_a, h_a commute. By Claim 6, h_b either admits two disjoint G-interpretations from $\mathcal{I}$ shifted by $\rho(g_a)$, a contradiction with the assumption of the part **(1.4)**, or g_a^3 is a factor of h_b and we are in (D).

(2) Suppose that primitive roots of g_a and g_b are not conjugate.

(2.1) Suppose that g_a^3 is not a factor of h_b. Let $(q, d, p) \in \mathcal{I}$ be a G-interpretation of h_b.

(2.1.1) Suppose that $d \in G^* g_b g_a G^*$ or $d \in G^* g_a g_b G^*$. Then, by Lemma 16 or by Lemma 17, we are either in (C), or h_b admits two disjoint G-interpretations from $\mathcal{I}$ shifted by $\rho(g_a)$ or $\rho(g_b)$. If they are shifted by $\rho(g_a)$, then $|g(a)| \geq 2|\rho(g_a)|$ by Lemma 27, and we proceed by (A2) or by (A3). If they are shifted by $\rho(g_b)$, then $|g(b)| \geq 2|\rho(g_b)|$ by Lemma 27, and we proceed by (B2) or by (B3).

Therefore, we are left with the possibilities $d \in g_b^*$ or $d = g_a$ or $d = g_a^2$.

(2.1.2) Suppose that there are two G-interpretations of h_b from $\mathcal{I}$ which are shifted by $\rho(g_a)$. Suppose, moreover, that

$$|h_b| \geq \min\{4|\rho(g_a)|, |g(a)| + |\rho(g_a)|\}.$$

Then we obtain one of the cases (A2) or (A4) or (A5). Importantly, this case applies if $d = g_a^2$ and $q \leq_{\mathrm{s}} g_a$, $p \leq_{\mathrm{p}} g_a$ and $d' = g_a$ for some G-interpretation $(q', d', p') \in \mathcal{I}$ of h_b disjoint from (q, d, p). Indeed, in this case (q, d, p) and (q', d', p') are shifted by $\rho(g_a)$ and $|h(b)| \geq 2|g_a| \geq |g_a| + |\rho(g_a)|$. Similarly, if $d, d' \in g_a^+$ and $q, q' \leq_{\mathrm{s}} g_a$, then (q, d, p) and (q', d', p') are shifted by $\rho(g_a)$ and $|h_b| \geq |g_a| + ||q| - |q'|| \geq |g_a| + |\rho(g_a)|$. Same reasoning applies in case that $d, d' \in g_a^+$ and $p, p' \leq_{\mathrm{p}} g_a$.

(2.1.3) Suppose that **no** two G-interpretations from $\mathcal{I}$ of h_b are shifted by $\rho(g_a)$ or

$$|h_b| < \min\{4|\rho(g_a)|, |g(a)| + |\rho(g_a)|\}.$$

We will count the minimal number of occurrences of g_a inside $g_{\mathrm{m}}(w)$ which do not have any common factor with h_b. Since $d \in g_b^*$ or $d = g_a$ or $d = g_a^2$, each occurrence of h_b in $h_{\mathrm{m}}(w)$ can overlap at most with four occurrences of g_a. Suppose first that there is an occurrence of h_b in $h_{\mathrm{m}}(w)$ which overlap with exactly four occurrences of g_a, that is $d = g_a^2$, $q <_{\mathrm{s}} g_a$ and $p <_{\mathrm{p}} g_a$. Then we have seen in **(2.1.2)**, that each of the remaining occurrences of h_b overlaps at most with two occurrences of g_a. Since $|w|_a \geq 30$ and $8 \geq |w|_b$, there are at least $30 - 4 - 7 \cdot 2 = 12$ occurrences of g_a inside $g_{\mathrm{m}}(w)$ which are factors of h_a^+. Similarly, again by **(2.1.2)**, there are at most two occurrences of h_b which overlap with three occurrences of g_a, namely those are occurrences of h_b with interpretations $I, I' \in \mathcal{I}$ such that $d = d' = g_a^2$, and $q <_{\mathrm{s}} g_a$, $p' <_{\mathrm{p}} g_a$ or vice versa. Therefore again, there are at least $30 - 2 \cdot 3 - 6 \cdot 2 = 12$ occurrences of g_a inside $g_{\mathrm{m}}(w)$ which are factors of h_a^+.

In all cases, we have seen that there are at least 12 occurrences of g_a inside $g_{\mathrm{m}}(w)$ which are factors of h_a^+. Since $|g_a| > |h_a|$, they are all shifted by $\rho(h_a)$. We use again Lemma 34 (in this case we work with a pair of G-overflows) and get that 1 is an overflow of $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$. Therefore, there are words e, f, conjugates of w, such that $(g_{\mathrm{m}}(e), h_{\mathrm{m}}(f))$ is a simple pair of conjugate words such that $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$. From Claim 1, we infer that there are words e', f' such that $e = e'(ab^i)^m ab^j$, $f \in f'a^+$, where $m \geq 11$, $j \leq i$ and $h_{\mathrm{m}}(f') \leq_{\mathrm{p}} g_{\mathrm{m}}(e'a)$. Since $|e|_b = |f|_b \leq 8$, we have $i = j = 0$, and word g_a, h_a commute.

Since we have assumed in **(2.1)** that g_a^3 is not a factor of h_b, we infer from Claim 6 that h_b admits two disjoint G-interpretations from $\mathcal{I}$ shifted by $\rho(g_a)$. Therefore, an initial assumption of **(2.1.2)** yields that

$$|h_b| < \min\{4|\rho(g_a)|, |g(a)| + |\rho(g_a)|\}.$$

But this inequality does not hold, since, by Lemma 27, we have

$$|h_b| \geq |g_a| \geq 11|\rho(h_a)| = 11|\rho(g_a)|.$$

(2.2) Suppose that g_a^3 is a factor of h_b. If h_b allows two disjoint G-interpretations from $\mathcal{I}$ shifted by $\rho(g_a)$, we are in (A2) or in (A5). Suppose that h_b allows two disjoint G-interpretations from $\mathcal{I}$ shifted by $\rho(g_b)$. Notice that in this case $|g_b| \geq 2|\rho(g_b)|$ by Lemma 27, and consequently, h_b is a factor of g_b^+ greater than $2|\rho(g_b)|$. Since g_a^3 is a factor of h_b, and g_a, g_b are not conjugate, necessarily $|\rho(g_b)| \geq |\rho(g_a)|$. We then proceed by (B2). If h_b does not allow two disjoint G-interpretations shifted by $\rho(g_a)$ nor two disjoint G-interpretations shifted by $\rho(g_b)$, we can use Lemma 21 and proceed by (C).

We have proved that if $8 \geq |w|_b \geq 4$ and $|w|_a \geq 30$, then there are words e and f, conjugates of w, such that $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$ and e, f have adopt one of the forms A - D. This concludes the proof.

□

Forms (A) - (D)

We finish this part with the combinatorial analysis providing the exact structure of equality words for forms (A) - (D) from the previous lemma. We will again suppose that $g_{\mathrm{m}}, h_{\mathrm{m}} : \{a,b\}^* \to A^*$ are marked versions of morphisms g, h. We will denote by g_a, g_b, h_a, h_b words $g_{\mathrm{m}}(a), g_{\mathrm{m}}(b), h_{\mathrm{m}}(a), h_{\mathrm{m}}(b)$, respectively. Moreover, we suppose that h_b is of the maximal length among all four image words g_a, g_b, h_a, h_b and $|h_b| > |g_b|$ and $|h_a| < |g_a|$. Let $G = \{g_a, g_b\}$ and $H = \{h_a, h_b\}$ and let e, f be conjugate words such that (e, f) is a letter block of $g_{\mathrm{m}}, h_{\mathrm{m}}$.

Form (A)

Claim 2. Suppose that $e \in e'a^+$, $f \in a^\ell b(a^i b)^+ a^j$, where $j \leq i$ and $g_{\mathrm{m}}(e') \leq_{\mathrm{p}} h_{\mathrm{m}}(a^\ell b)$. If $|f|_b \geq 4$, then $e = b^m a^\ell$ and $f = a^\ell b^m$ for some $\ell, m \geq 1$.

Proof. Since $|f|_b \geq 4$ and $|g_a| \leq |h_b|$, we have

$$|h_{\mathrm{m}}(f)| - |h_{\mathrm{m}}(a^\ell b)| \geq 3|h_{\mathrm{m}}(a^{i-j}ba^j)| \geq |h_{\mathrm{m}}(a^{i-j}ba^j)| + |g_a|.$$

From our assumption that $g_{\mathrm{m}}(e') \leq_{\mathrm{p}} h_{\mathrm{m}}(a^\ell b)$, it follows that $h_{\mathrm{m}}((a^\ell b)^{-1} f)$ is a suffix of g_a^+. Thus, by the Periodicity lemma, words g_a and $h_{\mathrm{m}}(a^{i-j}ba^j)$ commute. Then $(a^+, \mathrm{pref}((a^{i-j}ba^j)^+))$ is one of the letter blocks of g_{m} and h_{m}. Since e and f are conjugate and $|f|_b \geq 4$, "our" letter block $(e,f) \neq (a^+, \mathrm{pref}((a^{i-j}ba^j)^+))$, and consequently, $b \leq_{\mathrm{p}} e'$. We can, without loss of generality, suppose that $b \leq_{\mathrm{s}} e'$.

Suppose first that $f \leq_{\mathrm{s}} (a^{i-j}ba^j)^+$. Then, since words g_a and $h_{\mathrm{m}}(a^{i-j}ba^j)$ commute, we have $g_{\mathrm{m}}(e') \leq_{\mathrm{s}} g_a^+$. Consequently, from $b \leq_{\mathrm{s}} e'$ and Corollary 4, it follows that

$$g_{\mathrm{m}}(e') = g_{\mathrm{m}}(e') \wedge_s g_a^+ \leq_{\mathrm{s}} \underline{z}_{g_m}.$$

From $|f|_b \geq 4$ and conjugacy of e and f, we obtain that ba^ib is a factor of e'. Since $|\underline{z}_{g_m}| < |g_a| + |g_b|$, we infer that $i = 0$ and $f \in b^+$, a contradiction with e and f being conjugate.

Therefore, $f \in a^{k+i}b(a^ib)^+a^j$ and $k \geq 1$. Since e and f are conjugate and each b in f is both followed and preceded by a^i, we have $a^ib \leq_{\mathrm{s}} e'$ and $ba^i \leq_{\mathrm{p}} e'$.

Suppose that $i \neq 0$. We will show that primitive roots of $g_{\mathrm{m}}(ba^i)$ and h_a are conjugate. Notice that since e and f are conjugate and $|f|_b \geq 4$, the word $(ba^i)^3$ is a factor of e'.

1. Suppose first that $g_{\mathrm{m}}(e') \leq_{\mathrm{p}} h_a^k$. Then $g_{\mathrm{m}}(ba^i)^3$ is a factor of h_a^+. Thus, from $|g_a| \geq |h_a|$ and the Periodicity lemma, it follows that primitive roots of $g_{\mathrm{m}}(ba^i)$ and h_a are conjugate.

2. Suppose now that $h_a^k \leq_{\mathrm{p}} g_{\mathrm{m}}(e')$. Then, since words g_a and $h_{\mathrm{m}}(a^{i-j}ba^j)$ commute, we have $h_a^{-k} g_{\mathrm{m}}(e') \leq_{\mathrm{s}} g_a^+$. Consequently, from $b \leq_{\mathrm{s}} e'$ and Corollary 4, it follows that $|h_a^{-k} g_{\mathrm{m}}(e')| \leq |\underline{z}_{g_m}|$. Since $|\underline{z}_{g_m}| < |g_a| + |g_b|$, the word $g_{\mathrm{m}}(e'(ab)^{-1})$ is a prefix of h_a^k. Then $g_{\mathrm{m}}(ba^i)^2$ is a factor of h_a^+. Therefore, by the Periodicity lemma, primitive roots of $g_{\mathrm{m}}(ba^i)$ and h_a are conjugate.

In both cases, we have obtained that primitive roots of $g_{\mathrm{m}}(ba^i)$ and h_a are conjugate. But since $h_a \leq_{\mathrm{p}} g_{\mathrm{m}}(ba^i)$, their primitive roots commute and $(\mathrm{pref}((ba^i)^+), a^+)$ is a letter block of g_{m} and h_{m}. Consequently, $(e,f) = (\mathrm{pref}((ba^i)^+), a^+)$, a contradiction with $|f|_b \geq 4$.

We have proved that $i = 0$, and consequently, $f = a^\ell b^m$ and $e = b^m a^\ell$, for some $m, \ell \geq 1$. □

Form (B)

Claim 3. Suppose that $e \in e'b^+$, $f \in a^\ell b(a^i b)^+ a^j$, where $j \le i$ and $g_{\mathrm{m}}(e') \le_{\mathrm{p}} h_{\mathrm{m}}(a^\ell b)$. If $|f|_b \ge 4$ and $|f|_a \ge 1$, then $e = f = ab^m$.

Proof. From $|f|_b \ge 4$ and $|g_b| \le |h_b|$, it follows that

$$|h_{\mathrm{m}}(f)| - |h_{\mathrm{m}}(a^\ell b)| \ge 3|h_{\mathrm{m}}(a^{i-j}ba^j)| \ge |h_{\mathrm{m}}(a^{i-j}ba^j)| + |g_b|.$$

From our assumption that $g_{\mathrm{m}}(e') \le_{\mathrm{p}} h_{\mathrm{m}}(a^\ell b)$, it follows that $h_{\mathrm{m}}((a^\ell b)^{-1}f)$ is a suffix of g_b^+. Thus, by the Periodicity lemma, we get the commutativity of words $g_{\mathrm{m}}(b)$ and $h_{\mathrm{m}}(a^{i-j}ba^j)$, and consequently, $(b^+, \mathrm{pref}((a^{i-j}ba^j)^+))$ is one of the letter blocks. Since e and f are conjugate and $|f|_a \ge 1$, "our" letter block $(e, f) \ne (b^+, \mathrm{pref}((a^i ba^j)^+))$. Hence, $a \le_{\mathrm{p}} e'$ and we can, without loss of generality, suppose that $a \le_{\mathrm{s}} e'$.

Let $k, m \in \mathbb{N}$ be an indices such that $e = e'b^k$ and $f = a^\ell b(a^i b)^m a^j$. Our assumption that $g_{\mathrm{m}}(e') \le_{\mathrm{p}} h_{\mathrm{m}}(a^\ell b)$ yields that

$$k|g_b| \ge m|h_{\mathrm{m}}(a^i b)|.$$

From $|f|_b \ge 4$, it follows that $m \ge 3$ and since $|g_b| \le |h_b|$ we get also that $k \ge m \ge 3$. Therefore, we conclude that $i = 0$, otherwise e and f are not conjugate.

Then $f = a^\ell b^{m+1}$, and consequently, conjugacy of e and f and our assumption that $a \le_{\mathrm{p}} e'$ yield that $e = f = a^\ell b^{m+1}$. Then

$$g_a^\ell = h_a^\ell \rho(g_b)^{(m+1)j},$$

where $m \ge 3$ and $j \ge 1$. Hence, by [18], $\ell = 1$, and $e = f = ab^{m+1}$.

□

Form (C)

Claim 4. Suppose that $e \in e'(ba^n)^+ba^m \cup e'(a^n b)^+ a^m$, $f \in a^\ell b(a^i b)^+ a^j$, where $j \le i$, $m \le n$, $n \ne 0$ and $g_{\mathrm{m}}(e') \le_{\mathrm{p}} h_{\mathrm{m}}(a^\ell b)$. Suppose moreover that $|h_{\mathrm{m}}(b)| > |\rho(g_{\mathrm{m}}(a^n b))|$. If $|f|_b \ge 4$, then $e = f \in (ab)^+ a$ or $e = f \in (ba)^+ b$.

Proof. Let $u = a^{n-m}ba^m$, $v = a^{i-j}ba^j$. Notice first that $g_{\mathrm{m}}(u)$ commutes with $h_{\mathrm{m}}(v)$. Indeed, since $|h_{\mathrm{m}}(v)| \ge |h_b| > |\rho(g_{\mathrm{m}}(a^n b)|$, $|f|_b \ge 4$ and $g_{\mathrm{m}}(e') \le_{\mathrm{p}} h_{\mathrm{m}}(a^\ell b)$, we have

$$|g_{\mathrm{m}}(e'^{-1}e)| \ge 3|h_{\mathrm{m}}(v)| \ge |h_{\mathrm{m}}(v)| + |\rho(g_{\mathrm{m}}(u))|.$$

The commutativity of $g_{\mathrm{m}}(u)$ and $h_{\mathrm{m}}(v)$ follows from the Periodicity lemma and the fact that $g_{\mathrm{m}}(e'^{-1}e)$ is a suffix of both $g_{\mathrm{m}}(u)^+$ and $h_{\mathrm{m}}(v)^+$. Let x be the common primitive root of $g_{\mathrm{m}}(u)$ and $h_{\mathrm{m}}(v)$. Notice that since we assume that $|h_b| > |\rho(g_{\mathrm{m}}(a^n b))|$, we have $|h_{\mathrm{m}}(v)| \ge 2|x|$.

We will first show that $|e'^{-1}e|_b \ge 2$. Suppose, for a contradiction, that $|e'^{-1}e|_b = 1$. Then $e = e'a^n ba^m$ and consequently,

$$2|g_a^n| \ge |g_a^n| + |g_a^m| = |g_{\mathrm{m}}(e'^{-1}e)| - |g_b| \ge 3|h_{\mathrm{m}}(v)| - |g_b| > 2|h_{\mathrm{m}}(v)|.$$

Therefore, $|g_a^n| > |h_{\mathrm{m}}(v)|$. From $|h_{\mathrm{m}}(v)| \geq 2|x|$, it follows that $n = 1$, otherwise, by the Periodicity lemma, words g_a and x are conjugate, which yields a contradiction with g_{m} being non-periodic. Therefore, we are left with $|g_a| > |h_m(v)| \geq |h_b|$, a contradiction with our assumption that $|h_b|$ is maximal among the lengths of g_a, g_b, h_a and h_b. We have shown that $|e'^{-1}e|_b \geq 2$.

We proceed by showing that u and v are conjugate. Suppose, for a contradiction, that $n \neq i$. Since $|e'^{-1}e|_b \geq 2$, the conjugacy of e and f yields that $n = j + \ell$, $m \leq \min\{i, n\}$ and $|e'^{-1}e|_b = 2$ (we have been counting the number of a's between two b's).

1. Suppose first that $n > i$. Then, since e and f are conjugate, we obtain that $e = e'ba^nba^m$. Notice that $e = e'a^nba^nba^m$ can be discarded again by counting the number of a's between two b's. Therefore,

$$|g_a^n| = |g_{\mathrm{m}}(e'^{-1}e)| - |g_b| - |g_{\mathrm{m}}(ba^m)|.$$

From $j + \ell = n > i$, $|g_{\mathrm{m}}(e)| = |h_{\mathrm{m}}(f)|$ and conjugacy of e and f, it follows that

$$(j + \ell - i)|h_a| + |f|_b|h_{\mathrm{m}}(v)| = |h_{\mathrm{m}}(f)| = |g_{\mathrm{m}}(e)| = (j + \ell - i)|g_a| + |f|_b|g_{\mathrm{m}}(v)|.$$

From $|g_a| > |h_a|$, we infer that $|g_{\mathrm{m}}(v)| < |h_{\mathrm{m}}(v)|$ and consequently, we obtain also $|g_{\mathrm{m}}(ba^m)| < |h_{\mathrm{m}}(v)|$. Hence, from $|h_{\mathrm{m}}(v)| \geq 2|x|$ and from $|g_{\mathrm{m}}(e'^{-1}e)| \geq 3|h_{\mathrm{m}}(v)|$, we infer that

$$|g_a^n| \geq 3|h_{\mathrm{m}}(v)| - |g_b| - |g_{\mathrm{m}}(ba^m)| \geq 2|h_{\mathrm{m}}(v)| - |g_b| > |h_{\mathrm{m}}(v)|.$$

Since $|g_a| \leq |h_b| \leq |h_{\mathrm{m}}(v)|$, necessarily $n \geq 2$. Also, since $|h_{\mathrm{m}}(v)| \geq 2|x|$, we have $|g_a^n| > 2|x|$. Therefore, g_a^n and x^+ have a common factor of the length at least $|g_a| + |x|$, and consequently, by the Periodicity lemma, words x and $\rho(g_a)$ are conjugate. Since x is the primitive root of $g(u)$, we have a contradiction with g_{m} being non-periodic morphism.

2. Suppose now that $n < i$. From $j + \ell = n < i$, $|g(e)| = |h(f)|$ and conjugacy of e and f, it follows that

$$(i-n)(|f|_b-1)|h_a|+|f|_b|h_{\mathrm{m}}(u)| = |h(f)| = |g(e)| = (i-n)(|f|_b-1)|g_a|+|f|_b|g_{\mathrm{m}}(u)|.$$

Then, from $|g_a| > |h_a|$, we infer that $|g_{\mathrm{m}}(u)| < |h_{\mathrm{m}}(u)|$, and consequently, also $|g_{\mathrm{m}}(u)| < |h_{\mathrm{m}}(v)|$. Since

$$|g_{\mathrm{m}}(e'^{-1}e)| \geq 3|h_{\mathrm{m}}(v)| \geq 3|g_{\mathrm{m}}(u)|,$$

we get $|e'^{-1}e|_b \geq 3$, a contradiction.

We have proved that u and v are conjugate. We proceed by showing that e is a suffix of u^+ and that f is a suffix of v^+.

We will show first that $|\underline{z}_{g_m}| < |x|$. By Corollary 4, $\underline{z}_{g_m}$ is suffix comparable with all words from G^*. If $|\underline{z}_{g_m}| \geq |x|$, then $x \leq_{\mathrm{s}} \underline{z}_{g_m}$, since $\underline{z}_{g_m}$ is suffix comparable with $g_{\mathrm{m}}(u)$. Hence, x is a suffix of both $g_{\mathrm{m}}(ba^n)$ and $g_{\mathrm{m}}(a^nb)$. Therefore, both $g_{\mathrm{m}}(ba^n) = g_{\mathrm{m}}(a^nb) \in x^+$, and consequently g_a and g_b commute, a contradiction with g_{m} being non-periodic.

Notice that $f \notin v^+$, otherwise $e \in u^+$ and from the minimality of (e, f), it would follow that $f = v$, a contradiction with $|f|_b \geq 4$. As a consequence, we have $\mathrm{pref}_1 e \neq \mathrm{pref}_1 u$ and $\mathrm{pref}_1 f \neq \mathrm{pref}_1 v$. Indeed, since $g_{\mathrm{m}}(u)$ and $h_{\mathrm{m}}(v)$ commute, one of the letter block has to be in $(\mathrm{pref}(u^+), \mathrm{pref}(v^+))$. But since $v \leq_{\mathrm{s}} f$ and $f \notin v^+$ necessarily $(e, f) \notin (\mathrm{pref}(u^+), \mathrm{pref}(v^+))$.

Let $e = u_1 u_2 u^r$ and $f = v_1 v_2 v^s$, where $u_2 = u_1 u_2 \wedge_s u <_{\mathrm{s}} u$ and $v_2 = v_1 v_2 \wedge_s v <_{\mathrm{s}} v$. From the structure of f we infer that $v_1 \in a^*$. Notice also that $|v_2|_b \leq |v|_b \leq 1$.

1. Suppose first that $|v_2|_b = 0$, i.e. that $v_2 \in a^*$. Since $|g_a| > |h_a|$, $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$ and e and f are conjugate, necessarily

$$s|g_{\mathrm{m}}(v)| = |g_{\mathrm{m}}(e)| - |g_{\mathrm{m}}(v_1 v_2)| < |h_{\mathrm{m}}(f)| - |h_{\mathrm{m}}(v_1 v_2)| = s|h_{\mathrm{m}}(v)|.$$

Hence, the conjugacy of u and v yields $|g_{\mathrm{m}}(u)| < |h_{\mathrm{m}}(v)|$. Consequently, $r \leq s$, otherwise $|f|_b = s < r \leq |e|_b$ yields a contradiction with e and f being conjugate.

1.1 Suppose that $r = s$. Then $u_1 u_2 = v_1 v_2 = a^k$, for some $k \geq 0$ and

$$g_{\mathrm{m}}(a^k) = h(a^k) x^{sk'},$$

for some $k' \geq 1$. From [18], it follows that $k = 1$, otherwise $s = |f_b| \geq 4$ yields that g_a commute with x, a contradiction with g_{m} being non-periodic. Since $\mathrm{pref}_1 e \neq \mathrm{pref}_1 u$ and $\mathrm{pref}_1 f \neq \mathrm{pref}_1 v$, necessarily $u = v = ba^i$. From $i = n \neq 0$, we get $u_1 = v_1 = 1$ and $e \leq_{\mathrm{s}} u^+$, $f \leq_{\mathrm{s}} v^+$.

1.2 Suppose that $r < s$. Then

$$|\underline{z}_{g_m}| \geq |h_{\mathrm{m}}(v_2 v^s)| - |g_{\mathrm{m}}(u_2 u^r)| \geq |x|,$$

a contradiction with $|\underline{z}_{g_m}| < |x|$.

2. Suppose that $|v_2|_b = 1$. Since $v_1 \in a^*$, $v_2 <_{\mathrm{s}} v$ and $v_2 = v_1 v_2 \wedge_s a^{i-j} b a^j$, necessarily $v_1 = 1$. Hence $f \leq_{\mathrm{s}} v^+$ and $g_{\mathrm{m}}(u_1) \leq_{\mathrm{s}} \underline{z}_{g_m}$.

It remain to show that $f \leq_{\mathrm{s}} u^+$. Since $|g_{\mathrm{m}}(u_1)| \leq |\underline{z}_{g_m}| < |x|$ and $|\underline{z}_{g_m}| < |g_a| + |g_b|$, we have $u_1 \in a^* \cup b^*$.

If now $|g_{\mathrm{m}}(u)| \leq |h_{\mathrm{m}}(v)|$, then

$$|v_2|_a (|g_a| - |h_a|) < n(|g_a| - |h_a|) \leq |h_b| - |g_b|,$$

and consequently, $|g_{\mathrm{m}}(v_2)| < |h_{\mathrm{m}}(v_2)|$. We obtain $|g_{\mathrm{m}}(v_2 v^s)| < |h_{\mathrm{m}}(v_2 v^s)|$, a contradiction with $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$.

Therefore, $|g_{\mathrm{m}}(u)| > |h_{\mathrm{m}}(v)|$. Since $|g_b| < |h_b| \leq |h_{\mathrm{m}}(v)|$, necessarily

$$|g_a^n| = |g_{\mathrm{m}}(u)| - |g_b| \geq |g_{\mathrm{m}}(u)| - |h_{\mathrm{m}}(v)|.$$

2.1 Suppose first that $n \geq 2$. Then $|g_a^n| < 2|x|$, otherwise from $g(u) \in x^+$, we get the commutativity of g_a and g_b, a contradiction with g_{m} being non-periodic. Consequently,

$$2|x| > |g_a^n| \geq |g_{\mathrm{m}}(u)| - |h_{\mathrm{m}}(v)|,$$

and $|g_{\mathrm{m}}(u)| > |h_{\mathrm{m}}(v)|$ and $|h_{\mathrm{m}}(v)| \geq 2|x|$ yields that

$$|g_{\mathrm{m}}(u)| = |h_{\mathrm{m}}(v)| + |x| \geq 3|x|.$$

Therefore, $|g_b| \geq |x|$, and consequently, from $|g_{\mathrm{m}}(u_1)| < |x|$, we obtain $u_1 \in a^*$. If e is not a suffix of u^+, then $u_1 = a^{m'}$, $m' \neq 0$, and $u_2 = a^m$. Since e and f are conjugate we obtain that $n + m' \leq i$, a contradiction with $i = n$. We have shown that $e \leq_{\mathrm{s}} u^+$.

2.2 Suppose that $n = 1$. Then $f = b(ab)^s$ and $u_2 \in 1 \cup a \cup b$. Suppose that $u_2 = 1$. From $u_1 \in a^* \cup b^*$ and the conjugacy of e and f, it follows that $u_1 = b$. Therefore, $e = b(ba)^s$, a contradiction with $\mathrm{pref}_1 e \neq \mathrm{pref}_1 u$. If $u_2 \neq 1$, then $u_2 = b$, $u = ab$ and $u_1 = 1$ (if $u_2 = a$, then $u = ba$ and $u_1 \in a^*$, and e and f are not conjugate). Therefore, $e \leq_{\mathrm{s}} u^+$.

We have proved that e is a suffix of u^+ and f is a suffix of v^+. Notice that in this case $r = s$, u_2 and v_2 are conjugate and $e = u_2 u^r$, $f = v_2 v^s$.

We will show that $i|g_a| \geq 2|x|$. Suppose, for a contradiction, that $i|g_a| < 2|x|$. We will consider cases according to the length of $h_{\mathrm{m}}(v)$.

Suppose first that $|h_{\mathrm{m}}(v)| < |g_{\mathrm{m}}(u)|$. Then $|g_{\mathrm{m}}(u_2)| < |h_m(v_2)|$ and from $|g_a| > |h_a|$ we have $|u_2|_b = |v_2|_b = 1$. Inequality $|h_{\mathrm{m}}(v)| < |g_{\mathrm{m}}(u)|$ yields

$$|h_b| - |g_b| < i(|g_a| - |h_a|) < 2|x|.$$

But then from $|u_2|_b = |v_2|_b = 1$ and $|h_a| < |g_a|$, we get

$$|h_{\mathrm{m}}(v_2)| - |g_m(u_2)| \leq |h_b| - |g_b| < 2|x|,$$

a contradiction with $r(|g_{\mathrm{m}}(u)| - |h_{\mathrm{m}}(v)|) \geq 3|x|$.

If $|h_{\mathrm{m}}(v)| > |g_{\mathrm{m}}(u)|$, then $v_2 = u_2 \in a^+$. But since

$$|g_{\mathrm{m}}(u_2)| - |h_{\mathrm{m}}(v_2)| < i|g_a| < 2|x|,$$

we have a contradiction with $r(|h_{\mathrm{m}}(v)| - |g_{\mathrm{m}}(u)|) \geq 4|x|$.

We have shown that $i|g_a| \geq 2|x|$. Then $|g_a| \geq |x|$, otherwise, by the Periodicity lemma, g_a and x are conjugate and we have a contradiction with g_{m} being non-periodic. Consequently, we obtain that $i = 1$. Therefore, $u, v \in ab \cup ba$. Since e and f are conjugate $u_2 = v_2 \in a \cup b$ and $e = f \in a(ba)^+$ or $e = f \in b(ab)^+$.

□

Form D

Claim 5. Suppose that $e \in e'(ab^n)^+ab^m \cup e'(b^na)^+b^m$, $f \in a^\ell b(a^ib)^+a^j$, where $j \leq i$, $m \leq n$, $n \neq 0$ and $g_{\mathrm{m}}(e') \leq_{\mathrm{p}} h_{\mathrm{m}}(a^\ell b)$. Suppose moreover that $|h_{\mathrm{m}}(b)| > |\rho(g_{\mathrm{m}}(b^na)|$. If $|f|_b \geq 4$, then $e = f \in (ab)^+a$ or $e = f \in (ba)^+b$.

Proof. We will show that $n = 1$. Then we have the situation from Claim 4. Let $u = b^{n-m}ab^m$ and $v = a^{i-j}ba^j$. Notice that since $|h_b| > |\rho(g_{\mathrm{m}}(b^na)|$, we have

$$|g_{\mathrm{m}}(e'^{-1}e)| \geq 3|h_{\mathrm{m}}(v)| \geq |h_{\mathrm{m}}(v)| + |\rho(g_{\mathrm{m}}(u)|.$$

Therefore, $g_{\mathrm{m}}(u)$ and $h_{\mathrm{m}}(v)$ commute. Let x be their common primitive root. Suppose, for a contradiction, that $n \geq 2$. Then $|g_b^n| < 2|x|$, otherwise we get that g_{m} is periodic.

Since e and f are conjugate, we obtain that $j + \ell = 0$ or $i = 0$. Then $f \in b(a^ib)^+$ or $f \in a^\ell b^+$. Let k be an index such that $e \in e'(ab^n)^kab^m \cup e'(b^na)^kb^m$. Then, since e and f are conjugate, we have $k = 1$.

1. Suppose first that $m = 0$. Then $e'^{-1}e \in ab^na \cup b^na$ and

$$|g_a| + |g_{\mathrm{m}}(u)| \geq |g_{\mathrm{m}}(e'^{-1}e)| \geq 3|h_{\mathrm{m}}(v)|.$$

Consequently, from $|g_a| \leq |h_b| \leq |h_{\mathrm{m}}(v)|$, it follows that $|g_{\mathrm{m}}(u)| \geq 2|h_{\mathrm{m}}(v)|$. If $|g_{\mathrm{m}}(u)| = 2|h_{\mathrm{m}}(v)|$, we have a contradiction with (e, f) being a letter block.

Therefore, $|g_{\mathrm{m}}(u)| > 2|h_{\mathrm{m}}(v)|$ and since $g_{\mathrm{m}}(u)$ and $h_{\mathrm{m}}(v)$ have the common primitive root x, we obtain $|g_{\mathrm{m}}(u)| \geq 2|h_{\mathrm{m}}(v)| + |x|$. Consequently,

$$|g_b^n| = |g_{\mathrm{m}}(u)| - |g_a| \geq 2|h_{\mathrm{m}}(v)| + |x| - |g_a| \geq |h_{\mathrm{m}}(v)| + |x| \geq 2|x|,$$

a contradiction.
2. Suppose that $m \geq 1$.
2.1 Suppose that $e = e'ab^nab^m$. Then $f \in b(a^ib)^+$. Conjugacy of e and f moreover yields that $f \in b(ab)^+$ and $e = e'ab^2ab$. Therefore, $e' \in (ab)^+$. Notice that from $|f|_b = |e|_b \geq 4$, it follows that $|e'|_b = |e'|_a \geq 2$. From $g_{\mathrm{m}}(e') \leq_{\mathrm{p}} h_{\mathrm{m}}(a^\ell b)$ we get $g_{\mathrm{m}}(e') \leq_{\mathrm{p}} h_b$, and consequently,

$$|h_b| \geq 2|g_a| + 2|g_b|.$$

Then,

$$|g_{\mathrm{m}}(e)| = |g_{\mathrm{m}}(e'ab^2ab)| \leq 2|h_b| + |g_b| < 3|h_b| < |h_{\mathrm{m}}(f)|,$$

a contradiction.
2.2 Suppose that $e = e'b^nab^m$. We have $f \in b(a^ib)^+ \cup a^\ell b^+$. Suppose first that $i = 0$ or $f \in a^\ell b^+$. If $f \in b^+$, we get a contradiction with e and f being conjugate. If $f \in a^\ell b^+$, then $e = b^nab^m$, and consequently, $f = ab^{m+n}$. Since

$$|g_b^n| < 2|x| \leq 2|h_{\mathrm{m}}(v)|,$$

and $|g_a| \leq |h_b| = |h_{\mathrm{m}}(v)|$, we have $|g_{\mathrm{m}}(u)| < 3|h_{\mathrm{m}}(v)|$. Then $g_{\mathrm{m}}(u) <_{\mathrm{s}} h_b^3$. If $h_b = x$, then (u, b^+) is a letter block and we get a contradiction with (e, f) being a letter block. Therefore, $|h_b| \geq 2|x|$ and $|g_b^m| \leq |g_b^n| < 2|x| \leq |h_{\mathrm{m}}(v)|$. Then

$$|g_{\mathrm{m}}(e)| = |g_{\mathrm{m}}(b^nab^m)| < 3|h_{\mathrm{m}}(v)| \leq |h_{\mathrm{m}}(f)|,$$

a contradiction with $|g_{\mathrm{m}}(e)| = |h_{\mathrm{m}}(f)|$.

Suppose now that $i \geq 1$ and $f \in b(a^ib)^+$. Then $f \in b(ab)^+$ and $e = e'b^2ab$, and consequently, $e' \in (ab)^+a$. From $g_{\mathrm{m}}(e') \leq_{\mathrm{p}} h_b$, we obtain that $|g_{\mathrm{m}}(ab)| \leq |h_b| < |h_{\mathrm{m}}(v)|$. Then

$$|g_b^2| = |g_{\mathrm{m}}(e'^{-1}e)| - |g_{\mathrm{m}}(ab)| \geq 3|h_{\mathrm{m}}(v)| - |g_{\mathrm{m}}(ab)| > 2|h_{\mathrm{m}}(v)| \geq 2|x|,$$

a contradiction with $|g_b^n| < 2|x|$.

We have proved that $n = 1$. Then

$$e \in e'a(ba)^+ \cup e'a(ba)^+b \cup e'(ba)^+ \cup e'(ba)^+b,$$

$f \in a^\ell b(a^ib)^+a^j$, where $j \leq i$, and we can proceed by Claim 4.

□

Extra case

Claim 6. Suppose that $e \in e'ba^i$, $f \in f'ba^j$, where $j \geq i \geq 4$ and g_a and h_a commute. If $8 \geq |f|_b \geq 4$ and $|e|_a \geq 30$, then g_a^3 is a factor of h_b or two G-interpretations of h_b induced by $g_{\mathrm{m}}(e) = h_{\mathrm{m}}(f)$ are shifted by $\rho(g_a)$.

Proof. Let x be the common primitive root of g_a and h_a, and let k, ℓ be numbers such that $g_a = x^k$ and $h_a = x^\ell$. Let

$$i_1 = \frac{\ell}{\gcd(k,\ell)}, \qquad j_1 = \frac{k}{\gcd(k,\ell)}.$$

Then (a^{i_1}, a^{j_1}) is a letter block of g_m and h_m. Since (e,f) is also a letter block of g_m which is not equal to (a^{i_1}, a^{j_1}), we obtain that $i_1 > i$ or $j_1 > j$. From $|g_a| > |h_a|$, we have $k > \ell$, and consequently, $k \geq j_1$ and $\ell \geq i_1$ yields that $k > i$, that is $|g_a| \geq 5|x|$.

Suppose, for a contradiction, that g_a^3 is not a factor of h_b and no two G-interpretations of h_b induced by $g_\mathrm{m}(e) = h_\mathrm{m}(f)$ are shifted by x.

First, we will show that a^4 is not a factor of e'. Suppose, for a contradiction, that it is, and let $(q, h_\mathrm{m}(d), p)$ be its corresponding H-interpretation induced by $g_\mathrm{m}(e) = h_\mathrm{m}(f)$. More precisely, we suppose that $(q, h_\mathrm{m}(d), p)$ is an H-interpretation of g_a^4 and that there are words e_1, f_1 such that $e_1 a^4 \leq_\mathrm{p} e'$, $f_1 d \leq_\mathrm{p} f'$ and $q = g_\mathrm{m}(e_1)^{-1} h_\mathrm{m}(f_1)$. Notice that such interpretation always exists since $|h_a| < |g_a|$ and g_a^3 is not a factor of h_b.

1. Suppose that $|d|_b \geq 2$. If $|d|_b \geq 2$, then there are two occurrences of h_b in $h_\mathrm{m}(f)$ which are factors of g_a^+. Then $|h_b| \geq |g_a|$ yields that there are two G-interpretations of h_b induced by $g_\mathrm{m}(e) = h_\mathrm{m}(f)$ that are shifted by x.

2. Suppose that $|d|_a \geq 1$. Suppose, moreover, without loss of generality, that $a \leq_\mathrm{p} d$. Since h_a and g_a commute, we obtain that

$$q(h_\mathrm{m}(f_1^{-1}f) \wedge h_a^\omega) = g_\mathrm{m}(e_1^{-1}e) \wedge g_a^\omega.$$

By Lemma 31, $h_\mathrm{m}(f_1^{-1}f) \wedge h_a^\omega \in h_a^+$ and $g_\mathrm{m}(e_1^{-1}e) \wedge g_a^\omega \in g_a^+$. Since (e,f) is a letter block, words $f_1^{-1}f$, $e_1^{-1}e$ are minimal words such that $q h_\mathrm{m}(f_1^{-1}f) = g_\mathrm{m}(e_1^{-1}e)$. Therefore, we obtain that

$$g_\mathrm{m}(e_1^{-1}e) = g_\mathrm{m}(e_1^{-1}e) \wedge g_a^\omega \in g_a^+,$$

and consequently $e_1^{-1}e \in a^+$, a contradiction with $e \in e'ba^i$.

We can use the same reasoning in case that $q \leq_\mathrm{s} h_a$ and $|q| \geq |x|$, or that $p \leq_\mathrm{p} h_a$ and $|p| \geq |x|$.

Suppose, therefore, that $q \leq_\mathrm{s} h_b$, if $|q| \geq |x|$, and that $p \leq_\mathrm{p} h_b$, if $|p| \geq |x|$.

3. Suppose that $d = b$. Since we have supposed that g_a^3 is not a factor of h_b, necessarily $|h_b| < 3|g_a| + |x|$. Therefore, from $|g_a| \geq 5|x|$, we obtain that $|p| > 2|x|$ or $|q| > 2|x|$. Suppose for example that $|q| > 2|x|$. Then $q \leq_\mathrm{s} h_b$, and consequently primitive roots of g_a and h_b are conjugate in a way that $q\rho(h_b)q^{-1} = x$. Therefore,

$$q(h_\mathrm{m}(f_1^{-1}f) \wedge h_b^\omega) = g_\mathrm{m}(e_1^{-1}e) \wedge g_a^\omega.$$

By Lemma 31, $h_\mathrm{m}(f_1^{-1}f) \wedge h_b^\omega \in h_b^+$ and $g_\mathrm{m}(e_1^{-1}e) \wedge g_a^\omega \in g_a^+$. Since (e,f) is a letter block, words $f_1^{-1}f$, $e_1^{-1}e$ are minimal words such that $q h_\mathrm{m}(f_1^{-1}f) = g_\mathrm{m}(e_1^{-1}e)$. Then, we obtain that

$$h_\mathrm{m}(f_1^{-1}f) = h_\mathrm{m}(f_1^{-1}f) \wedge h_b^\omega \in h_b^+,$$

and consequently $f_1^{-1}f \in b^+$, a contradiction with $f \in f'ba^j$, where $j \geq 4$.
4. Suppose that $d = 1$. If $q \leq_{\mathrm{s}} h_b$ and $p \leq_{\mathrm{p}} h_b$, then primitive roots of g_a and h_b are conjugate in a way that $q\rho(h_b)q^{-1} = x$. We can then proceed as in the previous part. Therefore, we suppose that $q \leq_{\mathrm{s}} h_a$ or $p \leq_{\mathrm{p}} h_a$, and consequently that $|q| < |x|$ or $|p| < |x|$. We are left with the possibilities that $p \leq_{\mathrm{p}} h_b$ such that $|p| > 3|g_a|$, or $q \leq_{\mathrm{s}} h_b$ such that $|q| > 3|g_a|$. Then g_a^3 is a factor of h_b, a contradiction.

We have proved that a^4 is not a factor of e'. Notice that b is also a prefix of $e'b$ and $f'b$ since $(e, f) \neq (a^{i_1}, a^{j_1})$. Since e and f are conjugate, $j \geq 4$ and a^4 is not a factor of e', we obtain that $i = j$ and $e'b = f'b$.

Let $s = |g_a| - |h_a|$ and $t = |h_b| - |g_b|$. We will show that h_b has a suffix from $\mathrm{suf}(x^+)$ longer than $2t$. Since $i = j$ and $e'b = f'b$, we have

$$g_{\mathrm{m}}(e'b)x^{j(k-\ell)} = h_{\mathrm{m}}(f'b).$$

Then, by Corollary 4,

$$x^{j(k-\ell)} \leq_{\mathrm{s}} h_{\mathrm{m}}(f'b) \wedge_s h_a^\omega \leq_{\mathrm{s}} \underline{z}_{h_m}.$$

By Corollary 4, words $\underline{z}_{h_m}$ and $h_{\mathrm{m}}(f')$ are suffix comparable. If $|h_b| \leq |x^{(j-1)(k-\ell)}|$, then $|h_{\mathrm{m}}(f')| > |x|$, and consequently, suffix comparability of $\underline{z}_{h_m}$ and $h_{\mathrm{m}}(f')$ yields that $x \leq_{\mathrm{s}} h_{\mathrm{m}}(f')$. Then, $h_b \in x^+$ and h_m is periodic, a contradiction. Therefore, we have $|h_b| > |x^{(j-1)(k-\ell)}|$.

Denote by r the longest common suffix of x^+ and h_b. Then, $|r| > |x^{(j-1)(k-\ell)}| = (j-1)|s|$. Since a^4 is not a factor of e' and $b \leq_{\mathrm{p}} e'$, we have $|e'|_a \leq 3|e'|_b = 3(|e|_b - 1)$. Consequently,

$$js = |e|_b t - |e'|_a s \geq |e|_b t - (3|e|_b - 2)s + s.$$

We will show that $(|e|_b - 2)t \geq (3|e|_b - 2)s$. Then $(j-1)s \geq 2t$ and r will be a suffix of x^+ longer than $2t$. Since $4 \leq |e|_b = |f|_b \leq 8$, we have

$$\frac{|e|_b(3|e|_b - 2)}{|e|_b - 2} < 30.$$

From $|e|_b t = |e|_a s \geq 30s$, we get

$$|e|_b(3|e|_b - 2)s < (|e|_b - 2)30s \leq (|e|_b - 2)|e|_b t.$$

Therefore,

$$|r| \geq (j-1)|s| \geq |e|_b t - 3|e|_b s + 2s = 2t + (|e|_b - 2)t - (3|e|_b - 2)s \geq 2t,$$

and h_b has a suffix from $\mathrm{suf}(x^+)$ longer than $2t$.

Notice that since $|e|_b t = |e|_a s$ and $|e|_a > 3|e|_b$, necessarily $t > 3s \geq 3|x|$. Suppose now that $ba \leq_{\mathrm{p}} e'ba^j = f'ba^j$ and let $u = g_b^{-1}h_b$. Since u is a suffix of h_b of the length t, we obtain from $r \in \mathrm{suf}(x^+)$ and $|r| \geq 2|t|$ that u is a suffix of x^+. Then $|u| = |t| \geq |x|$ yields that $x \leq_{\mathrm{p}} u$, and consequently $u \in x^+$. Therefore,

$$u(h_{\mathrm{m}}(b^{-1}f) \wedge h_a^\omega) = g_{\mathrm{m}}(b^{-1}e) \wedge g_a^\omega,$$

and by Lemma 31, $h_{\mathrm{m}}(b^{-1}f) \wedge h_a^\omega \in h_a^+$ and $g_{\mathrm{m}}(b^{-1}e) \wedge g_a^\omega \in g_a^+$. Since (e,f) is a letter block, words $b^{-1}f, b^{-1}e$ are minimal words such that $uh_{\mathrm{m}}(b^{-1}f) = g_{\mathrm{m}}(b^{-1}e)$. Then, we obtain that

$$h_{\mathrm{m}}(b^{-1}f) = h_{\mathrm{m}}(b^{-1}f) \wedge h_a^\omega \in h_a^+,$$

and consequently $b^{-1}f \in a^+$, a contradiction with $|f|_b \geq 4$.

Similarly, if $bba \leq_{\mathrm{p}} e'ba^j = f'ba^j$, then $b^{-2}f \in a^+$, a contradiction with $|f|_b \geq 4$. Finally, if $b^3 \leq_{\mathrm{p}} e'a^j = f'ba^j$, then h_b and g_b commute and $e = f \in b^+$, a contradiction with $|e|_a \geq 30$.

□

2.3 Classification

Let $g, h : \{a,b\}^* \to A^*$ be two non-periodic binary morphisms and suppose that $h(b)$ is of the maximal length among all four image words $g(a), g(b), h(a)$ and $h(b)$. Let w be a single generator of $\mathrm{Eq}(g,h)$ and suppose that w is a simple equality word of g, h. In the previous parts we have found the structure of equality set of g, h in case that $|w|_b \geq 9$, or $|w|_b \geq 4$ and $|w|_a \geq 30$. Namely, we then have:

$$\mathrm{Eq}(g,h) = \{(ab)^i a\}^*,$$

up to the exchange of letters, or

$$\mathrm{Eq}(g,h) = \{a^j b^k\}^*$$

or

$$\mathrm{Eq}(g,h) = \{b^k a^j\}^*$$

with $\gcd(j,k) = 1$ and $j > k$. Moreover, if $|w|_b \geq 9$, then, besides the aforementioned possibilities, equality set can be also the following:

$$\mathrm{Eq}(g,h) = \{b^i a b^j\}^*.$$

Therefore, our search for a single simple generator of $\mathrm{Eq}(g,h)$ is narrowed to simple equality words with at most 37 letters, or to simple equality words with at most three b's. To deal with the latter possibility, which allows arbitrary large equality words, we have to have a look at the equality words w such that $|w|_b = 2$ or $|w|_b = 3$, that is, we have to solve the following equations:

$$x^i y x^j y x^k y x^\ell = u^i v u^j v u^k v u^\ell,$$

and

$$x^i y x^j y x^k = u^i v u^j v u^k.$$

Notice at this point that the word $a^i b a^j$ is indeed a single simple generator of some $\mathrm{Eq}(g,h)$. The intricacies of the equation $x^i y^j x^k = u^i v^j u^k$, proved to have only periodic solutions ([5]) if $i + k \geq 3$ and $j \geq 3$, show us that solving previous equations is a laborious work, which we did not undertake. It should be mentioned that we should not expect the aforementioned equations to have only periodic solution. For example, equations

$$x^i y^2 x^{i+1} = u^i v^2 u^{i+1} \tag{2.8}$$

and

$$yx^{2i+1}y = vu^{2i+1}v, \tag{2.9}$$

for $i \geq 1$, have various non-periodic solutions, e.g. those hinted in [8, p.361]:

$$\begin{aligned} x &= (\alpha^i\beta)^2\alpha^{2i+1}, & u &= \alpha, \\ y &= \alpha^i\beta, & v &= (\beta\alpha^i\beta\alpha^{3i+1})^i(\beta\alpha^i)^2, \end{aligned}$$

is a solution of Eq. (2.8), and

$$\begin{aligned} x &= \alpha\beta^{2i+1}\alpha, & u &= \beta, \\ y &= \alpha, & v &= \alpha x^i\alpha. \end{aligned}$$

is a solution of Eq. (2.9). However, one should notice that neither of these solutions is simple since an overflow α^i (α resp.) occurs twice in the first (resp. second) solution.

In order to find simple solutions of both equations one should look for more complicated words, e.g.:

$$\begin{aligned} x &= (\alpha^i\beta)^2\alpha^{2i+1}, & u &= \alpha, \\ y &= \alpha^i\beta x^i\alpha^i\beta, & v &= (\beta\alpha^i\beta\alpha^{2i+1})x^{i-1}\alpha^i\beta x^i(\alpha^i\beta)^2\alpha^i. \end{aligned}$$

is a simple solution of Eq. (2.8), and

$$\begin{aligned} x &= (\alpha\beta^{2i+1}\alpha)^3, & u &= \beta, \\ y &= \alpha, & v &= \alpha x^i(\alpha\beta^{2i+1}\alpha)\alpha \end{aligned}$$

is a simple solutions of Eq. (2.9).

Therefore, words $a^ib^2a^j$ with $|i-j| = 1$ and $ba^{2i+1}b$ are also simple binary equality words. So far it seems to be only simple equality words with two or three b's and arbitrary many a's. Verification of this hypothesis could be laborious but not impossible.

One should also notice that all non-periodic solutions of Eq. (2.8) and Eq. (2.9) such that $|x| > |u|$ have to satisfy that $|v| \geq |x|$. This additional property of the aforementioned equations is not obvious, but the proof is straightforward. This asymmetry is reflected in Fig. 6.

We will finish this chapter with a tabular listing of simple binary equality words according to our current knowledge. The space of all simple binary equality words is depicted in Fig. 6. The listing as well as the graphical representation uses the assumption that $h(b)$ is the longest word among the words $g(a)$, $g(b)$, $h(a)$ and $h(b)$. Moreover, we emphasise that the characterisation of simple binary equality words inside the grey area on Fig. 6 is complete, that is, there is no word w on coordinates without a dot such that w is a simple equality word of some morhphisms g, h satisfying that $h(b)$ is the longest word among the words $g(a)$, $g(b)$, $h(a)$ and $h(b)$. Similarly, the dots in the grey area represent the only words which are simple equality words of some morhphisms g, h satisfying that $h(b)$ is the longest word among the words $g(a)$, $g(b)$, $h(a)$ and $h(b)$.

Types of morphisms	Simple equality words	Reason
A	a + b	J. Hadravová, Š. Holub [3],[4],[2]
B	b	New result, see Section 2.2
C	a + b + c + ⋇	
D	a + b + c + d	J. Karhümaki, K. Culik [8]
E	⊡	

Morphisms of the type A Non-periodic morphisms $g, h : \{a, b\}^* \to A^*$ with $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$ and a simple equality word w such that $|w|_b \geq 9$.

Morphisms of the type B Non-periodic morphisms $g, h : \{a, b\}^* \to A^*$ with $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$ and a simple equality word w such that $|w|_b \geq 4$ and $|w|_a \geq 30$.

Morphisms of the type C Non-periodic morphisms $g, h : \{a, b\}^* \to A^*$ with $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$ and a simple equality word w such that $|w|_b \leq 3$.

Morphisms of the type D Non-periodic morphisms $g, h : \{a, b\}^* \to A^*$ with a simple equality word w such that $|w| \leq 5$.

Morphisms of the type E Non-periodic morphisms $g, h : \{a, b\}^* \to A^*$ with $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$ and a simple equality word w such that $|w|_b \leq 8$ and $|w|_a \leq 29$.

Equality words of the type a Words w:

$$w = b^i a b^j,$$

with $i, j \geq 0$.

Equality words of the type b Words w:

$$w = (ab)^i a,$$
$$w = (ba)^i b,$$
$$w = a^j b^k,$$
$$w = b^k a^j,$$

with $\gcd(j, k) = 1$ and $j > k$.

Equality words of the type c Words w:

$$
\begin{aligned}
w &= 1,\\
w &= a^i b a^j,\\
w &= a^{i+1} b^2 a^i,\\
w &= a^i b^2 a^{i+1},\\
w &= b a^{2i+1} b,
\end{aligned}
$$

with $i, j \geq 0$.

Equality words of the type **d** Words w:

$$
\begin{aligned}
w &= a^2 b^2,\\
w &= b^2 a^2.
\end{aligned}
$$

Interestingly, words a^2b^2, b^2a^2 are simple equality words, even though $\gcd(2,2) = 2$. For example, a^2b^2 is a simple equality word of the following morphisms g, h:

$$
\begin{aligned}
g(a) &= aab, & g(b) &= ababa,\\
h(a) &= a, & h(b) &= baababa.
\end{aligned}
$$

Equality words of the type $\circledast$ Simple equality words w with $|w|_b \leq 3$ and arbitrary many a's. This is the only type that is not known and could contain equality words with more than 37 letters. However, we conjecture that only possible simple equality words of this type are words from a, b and c.

Equality words of the type $\boxdot$ Simple equality words w with $|w|_b \leq 8$ and $|w|_a \leq 29$.

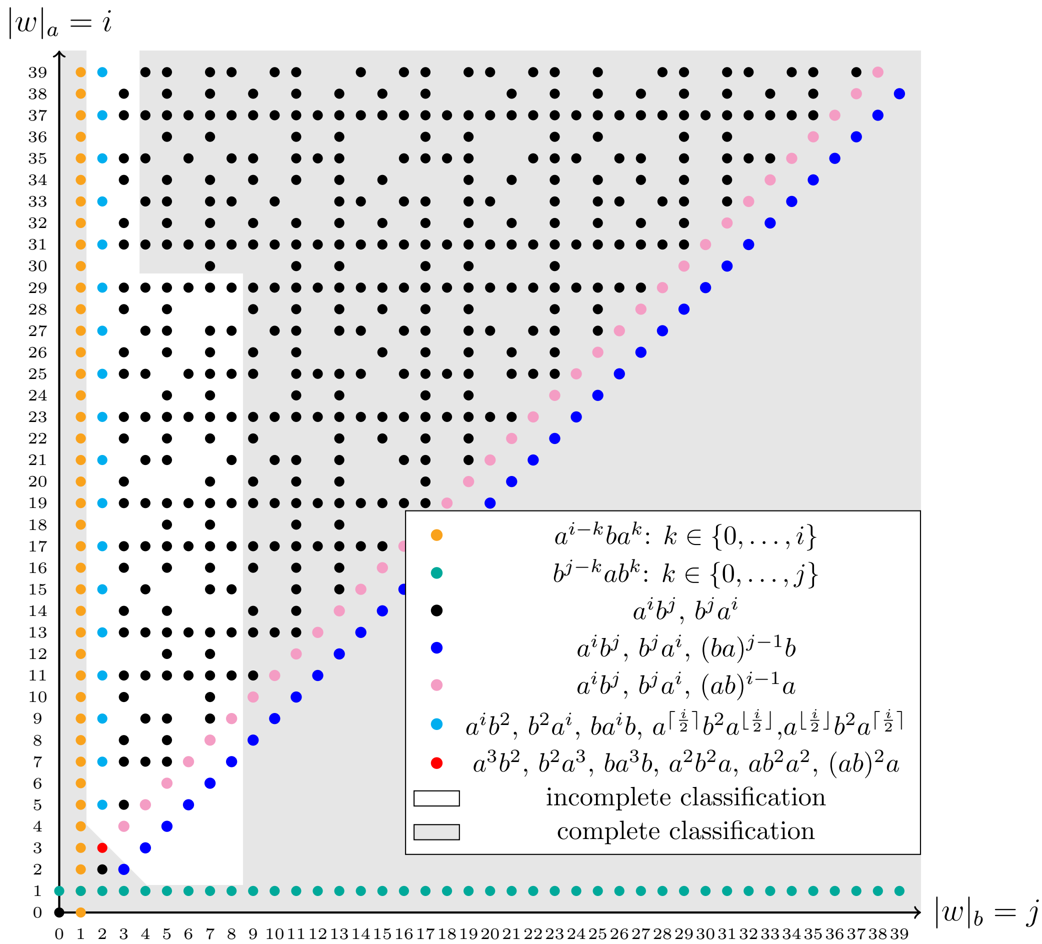

Figure 6: Simple equality words w of $g, h : \{a, b\}^* \to A^*$ with $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$.

3. Non-simple binary equality words

This chapter will discuss non-simple binary equality words, that is, binary equality words with repeated overflows. The main role in the analysis of non-simple binary equality words is played by successor morphisms, more precisely, by the reduction sequence of successor morphisms. The concept of a reduction sequence of successor morphisms was first introduced in the proof of the decidability of PCP(2) [10]. Starting with morphisms g, h, it allows us to simplify in each step the structure of the equality word. At the very end of the sequence we are left with an equality word with no letter blocks.

3.1 Reduction sequence of successor morphisms

The following section is based on our article *The block structure of successor morphisms* [1]. As the main result, we will show that, up to some special morphisms, the length of the sequence of successor morphisms is bounded by a constant. Moreover, we present here few additional results which justify the importance of the concept of reduction sequence of successor morphisms. The omitted proofs and more details can be found in [1].

Let us first briefly remind the definition of successor morphisms and letter blocks: Let $g, h : \{a, b\}^* \to A^*$ be non-periodic morphisms. The letter block of morphisms g, h is a (prefix) minimal pair of words (e, f) such that

$$g_{\rm m}(e) = h_{\rm m}(f) \ ,$$

where $g_{\rm m}, h_{\rm m}$ are marked versions of morphisms g, h resp. It is not difficult to see that there are at most two different letter blocks $(e, f), (e', f')$ and $e \wedge e' = 1$, $f \wedge f' = 1$. Indeed, by Lemma 31 we have

$$g_{\rm m}(e \wedge e') = g_{\rm m}(e) \wedge g_{\rm m}(e') = h_{\rm m}(f) \wedge h_{\rm m}(f') = h_{\rm m}(f \wedge f').$$

The rest follows from the fact that letter blocks are prefix minimal pairs.

Coincidence set. Let $g, h : \{a, b\}^* \to A^*$ be non-periodic morphisms. We define *coincidence set* of g, h as the set off all pairs of words on which g and h coincide, that is

$$\mathbb{C}(g, h) = \{(e, f), g(e) = h(f)\}.$$

Notice that $w \in {\rm Eq}(g, h)$ iff $(w, w) \in \mathbb{C}(g, h)$. Also, if g, h are marked, then $\mathbb{C}(g, h)$ is generated by letter blocks of g, h.

Successor morphisms. Morphisms $g_1, h_1 : \{a, b\}^* \to \{a, b\}^*$ are called *successor morphisms* if there exist non-periodic binary morphisms $g, h : \{a, b\}^* \to A^*$ such that $(g_1(a), h_1(a))$ and $(g_1(b), h_1(b))$ are their letter blocks. Morphisms g, h are called *predecessor morphisms* of g_1, h_1.

Notice that successor morphisms are marked and non-periodic. Each pair of successor morphisms can have various non-equivalent predecessors. Consider the following example:

Example 6. Let $g_1, h_1 : \{a,b\}^* \to \{a,b\}^*$ be morphisms defined as

$$g_1(a) = a, \qquad h_1(a) = b,$$
$$g_1(b) = bb, \qquad h_1(b) = ab.$$

Then both morphisms $g, h : \{a,b\}^* \to \{a,b\}^*$ and $g', h' : \{a,b\}^* \to \{a,b\}^*$ defined as follows:

$$g(a) = aba, \qquad h(a) = b,$$
$$g(b) = ba, \qquad h(b) = aba,$$

$$g'(a) = a, \qquad h'(a) = bab,$$
$$g'(b) = ba, \qquad h'(b) = a,$$

are predecessor morphisms of g_1, h_1. Notice, however, that there is no morphism $j : \{a,b\}^* \to \{a,b\}^+$ such that $j \circ g = g'$ and $j \circ h = h'$. Therefore, morphisms g, h and g', h' are not equivalent.

Reduction sequence of successor morphisms. Let $g, h : \{a,b\}^* \to A^*$ be non-periodic binary morphisms. Sequence of morphisms $\{(g_i, h_i)\}_{i \geq 0}$ is called *reduction sequence of successor morphisms* of g, h if $(g, h) = (g_0, h_0)$ and if it is the longest sequence such that for each $0 \leq i$, morphisms g_i, h_i are predecessors of g_{i+1}, h_{i+1}.

Reduction sequence of successor morphisms is uniquely given for each pair of non-periodic morphisms g, h. The reduction sequence of successor morphisms plays important role in the proof of decidability of PCP(2) [9]. The main idea of the proof is that either reduction sequence is finite or it is ultimately periodic. The ultimately periodic cases are classified in [16] and should be treated as special cases. Here, we will be interested in cases when reduction sequence of successor morphisms is finite.

There is a special connection between an equality words of successor morphisms and an equality words of the original pair of morphisms: If g_1 and h_1 are successor morphisms of g, h and there is a primitive word such that $(g(w), h(w))$ is not simple conjugate pair, then there is a word w_1 such that w, $g_1(w_1)$ and $h_1(w_1)$ are conjugate. We formulate it as the next lemma:

Lemma 37. *Let $g, h : \{a,b\}^* \to A^*$ be non-periodic binary morphisms and let $w \in \{a,b\}^*$ be a primitive word. Suppose that $(g(w), h(w))$ is a pair of conjugate words which is not simple. Then there are morphisms $g_1, h_1 : \{a,b\}^* \to \{a,b\}^*$ such that g_1, h_1 are successor morphisms of g, h and a primitive word $w_1 \in \{a,b\}^*$ such that w, $g_1(w_1)$ and $h_1(w_1)$ are conjugate.*

Proof. Notice first that if $(g(w), h(w))$ is a pair of conjugate words which is not simple, then $(g_{\mathrm{m}}(w), h_{\mathrm{m}}(w))$ is also a pair of conjugate words which is not simple. Since successor morphisms of g, h are exactly the same as successor morphisms of $g_{\mathrm{m}}, h_{\mathrm{m}}$ we can, without loss of generality, suppose that that morphisms g, h are marked.

Suppose that $(g(w), h(w))$ is not a simple pair of conjugate words. We will show that there are two different pairs of word $(e_1, f_1), (e_2, f_2) \in \mathbb{C}(g, h)$ such that e_1, e_2, f_1, f_2 and w are conjugate. Then there are two different generators $(e, f), (e', f')$ of $\mathbb{C}(g, h)$ and morphisms $g_1, h_1 : \{a, b\}^* \to \{a, b\}^*$ defined as

$$
\begin{aligned}
g_1(a) &= e, & h_1(a) &= f, \\
g_1(b) &= e', & h_2(a) &= f',
\end{aligned}
$$

are successor morphisms of g, h. Let $w_1 = g_1^{-1}(e_1)$. Then $g_1^{-1}(e_1) = h_1^{-1}(f_1)$ yields that $g_1(w_1)$, $h_1(w_1)$ and w are conjugate. Notice that since $g_1(w_1)$ is primitive, word w_1 is also primitive.

We will proceed by finding two different pairs of words $(e_1, f_1), (e_2, f_2) \in \mathbb{C}(g, h)$ such that e_1, e_2, f_1, f_2 and w are conjugate. Since $(g(w), h(w))$ is not simple and w is primitive, we can find a word $z \in A^*$ and words $u, u', v, v' \in \{a, b\}^*$ such that u, u', v, v' and w are conjugate, $u \neq u'$ or $v \neq v'$ and

$$
\begin{aligned}
zg(u) &= h(v)z, \\
zg(u') &= h(v')z.
\end{aligned}
$$

Notice that $u \neq u'$ iff $v \neq v'$. Indeed, if for example $v = v'$, then $g(u) = g(u')$, and consequently $u = u'$ otherwise we get a contradiction with g being a non-periodic morphism.

Denote $\bar{u} = u \wedge u'$ and $\bar{v} = v \wedge v'$. By Lemma 31, we have

$$
h(\bar{v}) = h(v) \wedge h(v') = zg(u) \wedge zg(u') = zg(\bar{u}).
$$

Then for words

$$
\begin{aligned}
e_1 &= \bar{u}^{-1} u \bar{u}, & e_2 &= \bar{u}^{-1} u' \bar{u}, \\
f_1 &= \bar{v}^{-1} v \bar{v}, & f_2 &= \bar{v}^{-1} v' \bar{v},
\end{aligned}
$$

we have

$$
\begin{aligned}
g(e_1) = g(\bar{u}^{-1} u \bar{u}^{-1}) &= h(\bar{v})^{-1} zg(u) z^{-1} h(\bar{v}) = h(\bar{v}^{-1} v \bar{v}) = h(f_1), \\
g(e_2) = g(\bar{u}^{-1} u' \bar{u}^{-1}) &= h(\bar{v})^{-1} zg(u') z^{-1} h(\bar{v}) = h(\bar{v}^{-1} v' \bar{v}) = h(f_2),
\end{aligned}
$$

Since $u \neq u'$ and $v \neq v'$, we have found two different pairs of words $(e_1, f_1), (e_2, f_2)$ from $\mathbb{C}(g, h)$ such that e_1, e_2, f_1, f_2 and w are conjugate.

□

Notice that the previous lemma gives us a powerful tool for simplifying the structure of the equality word. Starting with morphisms $g_0 = g$, $h_0 = h$ and a word $w_0 \in \mathrm{Eq}(g, h)$, along with the creation of the reduction sequence of successor morphisms g_i, h_i, $1 \leq i \leq k$, we get in each step a word w_i such that words $g_i(w_i)$, $h_i(w_i)$ and w_{i-1} are conjugate, or a pair of conjugate words $(g_{i-1}(w_{i-1}), h_{i-1}(w_{i-1}))$ is simple. Notice that, to comply with assumptions of Lemma 37, we have to suppose additionally that w_i is primitive for all $0 \leq i \leq k$. For that, it is enough to suppose that w_0 is primitive. We formulate it more precisely in the following corollary:

Corollary 38. *Let $(g, h) = (g_0, h_0)$, (g_1, h_1), $\ldots$, (g_k, h_k) be a finite reduction sequence of successor morphisms and let $w_0 \in \mathrm{Eq}(g, h)$ be a primitive word. Then there is $K \leq k$ such that for all $i \in \{1, \ldots, K\}$ there is a primitive word $w_i \in \{a, b\}^*$ such that words $g_i(w_i)$, $h_i(w_i)$ and w_{i-1} are conjugate, and $(g_K(w_K), h_K(w_K))$ is a simple pair of conjugate words.*

Proof. Let $0 \leq j < k$. We can suppose that w_j is primitive. If $(g_j(w_j), h_j(w_j))$ is a simple pair of conjugate words, then $K = j$ and we are done. Otherwise, by Lemma 37, there is a primitive word $w_{j+1} \in \{a,b\}^*$ such that $g_{j+1}(w_{j+1})$, $h_{j+1}(w_{j+1})$ and w_j are conjugate.

If $j = k$, then a pair of morphisms (g_j, h_j) does not have a successor. Therefore, by Lemma 37, $(g_j(w_j), h_j(w_j))$ is a simple pair of conjugate words and $K = k$.

□

Recall that our main goal is to determine the structure of binary equality word w in general case, that is, without supposing that w is simple as we did in Chapter 2. With help of Corollary 38, we are able to reduce this problem into the simpler one. We can gradually transform our original binary equality word w into simpler words w_i, ending up, finally, with the word w_K and morphisms g_K, h_K such that $(g_K(w_K), h_K(w_K))$ is a simple pair of conjugate words. This fact additionally clarifies our huge interest in simple binary equality words and simple structures in general, which we manifested in Chapter 2. More precisely, the relation between words w_K and w is the following:

Let $g, h : \{a,b\}^* \to A^*$ be non-periodic binary morphisms. Let $w \in \{a,b\}$ be a generator of the set $\mathrm{Eq}(g,h)$, that is $w \in \mathrm{Eq}(g,h) \backslash \{\mathrm{Eq}(g,h) \backslash 1\}^2$. Notice that, in particular, this implies that w is primitive.

If w is simple we can use classification of simple binary equality words from the end of Chapter 2.

Suppose that w is not simple. If $g(w)$ is not primitive, then by [6, Theorem 4.2], we get that $w \in a^*b \cup b^*a$. Suppose now that $g(w)$ is primitive. Let $(g,h) = (g_0,h_0)$, (g_1,h_1), $\ldots$, (g_k,h_k) be a finite reduction sequence of successor morphisms of g,h. By Corollary 38, we can find $K \leq k$ such that for all $i \in \{1,\ldots,K\}$ there is a word $w_i \in \{a,b\}^*$ such that words $g_i(w_i)$, $h_i(w_i)$ and w_{i-1} are conjugate, and $(g_K(w_K), h_K(w_K))$ is a simple pair of conjugate words. If we now iterate this process, we come up with the following interesting property: word w is a conjugate of all words

$$q_1 \circ \ldots q_K(w_K) \tag{3.1}$$

where $q_i \in \{g_i, h_i\}$, for all $i \leq K$, and $(g_K(w_K), h_K(w_K))$ is a simple pair of conjugate words.

According to Eq. (3.1) we can see that the length of the word w is then bounded as follows:

$$|w| \leq \min\{|g_1|, |h_1|\} \min\{|g_2|, |h_2|\} \ldots \min\{|g_K|, |h_K|\} |w_K|,$$

where $|g_i| = |g_i(a)| + |g_i(b)|$, $|h_i| = |h_i(a)| + |h_i(b)|$ for all $i \in \{1,\ldots,K\}$. This makes us to take an interest in the length of the sequence of successor morphisms and in the lengths of successor morphisms, i.e. $|g_i|$ and $|h_i|$.

In what follows we will prove that, up to some special cases, there is an uniform bound for $|g_i|$, resp. $|h_i|$ for all $i \geq 2$. Therefore, there is N such that $\min\{|g_i|, |h_i|\} \leq N$ for all $i \geq 2$. As a corollary, we will prove that the length of the sequence of successor morphisms is also bounded by a constant, denoted by M.

An important consequence of this result is that we are able, up to some special cases, establish the upper bound for number of letter blocks in which a binary equality word decomposes. This bound will depend only on the length of the final simple cyclic equality word w_K. From the proof of Lemma 37, we can see that number of letter blocks in which binary equality word decomposes is in fact given by the length of the word w_1. Therefore, from Eq. (3.1), we get the following bound:

$$|w_1| \leq \min\{|g_2|, |h_2|\} \dots \min\{|g_K|, |h_K|\}|w_K|.$$

Then,

$$|w_1| \leq N^M |w_K|,$$

and the number of blocks in which binary equality word w decomposes is bounded by $N^M|w_K|$.

First, we will need the following terminology.

Length of morphism. The length of a morphism $g : \{a, b\} \to A^*$ is defined as the sum of the lengths of its image words and denoted by $|g|$. That is,

$$|g| = |g(a)| + |g(b)|.$$

Suffix complexity. The suffix complexity $\sigma(g)$ of a morphism g is defined as the number of different non-empty suffixes of $g(a)$ or $g(b)$. That is, we have

$$\sigma(g) = |\{u, u \neq \varepsilon \text{ and } u \in \mathrm{suf}(g(a)) \cup \mathrm{suf}(g(b))\}|.$$

The concept was first introduced in the proof of the decidability of the binary PCP (see [9]). The suffix complexity of a pair of morphisms (g, h) is defined as the sum of their suffix complexities. It is known that suffix complexity of a pair of successor morphisms is less than or equal to the suffix complexity of its predecessors ([9, Lemma 6.2]). Moreover, the key argument in the proof of decidability of the PCP(2) [9] is that the suffix complexities of successors and predecessors are equal only in special cases which can be treated separately. In this sense, up to some special cases, each pair of successor morphisms is simpler than its predecessor.

The suffix complexity of a non-periodic morphism g is related to its length in the following way:

$$\frac{1}{2}(|g(a)| + |g(b)|) < \sigma(g) \leq |g(a)| + |g(b)|. \tag{3.2}$$

Moreover, the suffix complexity of successor morphisms is bounded by the number of certain types of overflows inside the letter blocks of their predecessor. Suppose that g_1, h_1 are successor morphisms of morphsims $g, h : \{a, b\}^* \to A^*$ and let $(e, f), (e', f')$ be letter blocks of g, h. Let $g_{\mathrm{m}}, h_{\mathrm{m}}$ be corresponding marked morphisms of g, h. We define a set O_g as a set of all non-empty suffixes of $g_{\mathrm{m}}(a)$ or $g_{\mathrm{m}}(b)$ which occur as overflows inside at least one of the letter blocks. More formally, $u \in O_g$ if u is a non-empty suffix of $g_{\mathrm{m}}(a)$ or $g_{\mathrm{m}}(b)$ and there are words e_1, e_2, f_1 and f_2 such that

$$g_{\mathrm{m}}(e_1) = h_{\mathrm{m}}(f_1)u \qquad \text{and} \qquad ug_{\mathrm{m}}(e_2) = h_{\mathrm{m}}(f_2).$$

Each $u \in O_g$ determines uniquely a non-empty word $f_2 \in \mathrm{suf}(f) \cup \mathrm{suf}(f')$. This defines a mapping

$$\pi_g : O_g \to \{\mathrm{suf}(f) \cup \mathrm{suf}(f')\} \backslash \{\varepsilon\}.$$

Since words f and f' are in fact words $h_1(a)$ and $h_1(b)$, we have obtained a mapping between O_g and $\sigma(h_1)$. Notice that this mapping is surjective, since for each non-empty $s \in \mathrm{suf}(f)$ we can find $u \in O_g$ as $g_{\mathrm{m}}(e_1)(h_{\mathrm{m}}(fs^{-1}))^{-1}$ where e_1 is a minimal prefix of e such that $|g_{\mathrm{m}}(e_1)| > |h_{\mathrm{m}}(fs^{-1})|$. Surjectivity of π_g immediately implies the following bound for the suffix complexity of h_1:

$$\sigma(h_1) \leq |O_g|. \tag{3.3}$$

Finally, notice also that since words $u \in O_g$ are non-empty suffixes of $g_{\mathrm{m}}(a)$ or $g_{\mathrm{m}}(b)$, we have

$$|O_g| \leq \sigma(g_{\mathrm{m}}). \tag{3.4}$$

Saturated morphisms. A pair of successor morphisms (g_1, h_1) is called *saturated* if at least one pair of their predecessor morphisms has a non-simple minimal equality word.

The following example shows that the condition that at least one pair of predecessor morphisms has an equality word does not imply that other pairs of predecessor morphisms of g_1 and h_1 have to possess an equality word as well.

Example 7. Let $g_1, h_1 : \{a,b\}^* \to \{a,b\}^*$ be binary morphisms defined in the following way:

$$g_1(a) = a, \qquad h_1(a) = ab,$$
$$g_1(b) = bb, \qquad h_1(b) = b.$$

It is easy to check that the following two pairs of morphisms (g, h) and (g', h') are predecessor morphisms of (g_1, h_1):

$$g(a) = abb, \qquad h(a) = a,$$
$$g(b) = b, \qquad h(b) = bb,$$

$$g'(a) = babab, \qquad h'(a) = b,$$
$$g'(b) = ab, \qquad h'(b) = baba.$$

The morphisms g, h have a minimal equality word $w = abb$. Moreover, this equality word is non-simple since it has two overflows which are the same. Namely, for $w_1 = a, w_2 = ab$ and $u = b^2, u' = b$ we have

$$g(w_1) = h(w_2) \qquad \text{and} \qquad g(w_1u) = h(w_2u'),$$

but $|u|$ is not a multiple of $|w|$. Therefore, the pair of morphisms (g_1, h_1) is saturated. However, it can be checked easily that the pair (g', h') does not have any equality word.

Combinatorially rich morphisms. Let $g, h : \{a, b\}^* \to \{a, b\}^*$ be binary morphisms and let (e, f), (e', f') be their letter blocks. A pair of morphisms (g, h) is called *combinatorially rich* if the primitive roots of words $\{g(a), g(b), h(a), h(b), e, f, e', f'\}$ have the length at least two.

Notice that since the length of the primitive roots is at least two, each image word as well as each letter block word of combinatorially rich morphisms have to contain both letters a and b at least once.

We can now formulate the main result of this chapter. Up to the morphisms which are not combinatorially rich, both O_g and O_h are bounded by a constant. An interested reader can find a proof and more information in our article *The block structure of successor morphisms* [1, Lemma 8].

Theorem 39. *Let $g, h : \{a, b\}^* \to \{a, b\}^*$ be a pair of combinatorially rich saturated successor morphisms. Then $|O_g| \leq 706$ and $|O_h| \leq 34$ or vice versa.*

This theorem has some interesting corollaries. First, if $\{g_i, h_i\}_{i \geq 0}$ is a reduction sequence of successor morphisms of g, h, we can find a uniform bound for $|g_i|$ and $|h_i|$. We formulate it as the next theorem:

Theorem 40. *Let $g, h : \{a, b\}^* \to A^*$ be non-periodic binary morphisms and let $\{g_i, h_i\}_{i \geq 0}$ be its reduction sequence of successor morphisms. If (g_1, h_1) is a pair of combinatorially rich saturated morphisms, then for all $i \geq 2$ we have $|g_i| < 68$ and $|h_i| < 1412$ or vice versa.*

Proof. Since successor morphisms are all marked, we can see from Eq. (3.4) and Eq. (3.3)

$$\begin{aligned} \sigma(g_{i+1}) &\leq |O_{h_i}| \leq \sigma(h_i), \\ \sigma(h_{i+1}) &\leq |O_{g_i}| \leq \sigma(g_i), \end{aligned}$$

for all $i \geq 1$. Therefore, according to Eq. (3.2), we have

$$\begin{aligned} |g_{2k}| &< 2\sigma(g_2), & |h_{2k}| &< 2\sigma(h_2), \\ |g_{2k+1}| &< 2\sigma(h_2), & |h_{2k+1}| &< 2\sigma(g_2), \end{aligned}$$

for all $k \geq 1$. Since g_2, h_2 are successor morphisms of g_1, h_1, we get from Eq. (3.3) and Theorem 39 that

$$\begin{aligned} \sigma(g_2) &\leq |O_{h_1}| \leq 34, \\ \sigma(h_2) &\leq |O_{g_1}| \leq 706, \end{aligned}$$

or vice versa. □

Another consequence of Theorem 39 is the existence of a bound for the length of the reduction sequence of successor morphisms:

Theorem 41. *Let $g, h : \{a, b\}^* \to A^*$ be non-periodic binary morphisms and let $\{g_i, h_i\}_{i \geq 0}$ be its reduction sequence of successor morphisms with decreasing suffix complexity, that is, we suppose that $\sigma(g_i) + \sigma(h_i) < \sigma(g_{i+1}) + \sigma(h_{i+1})$ for all $i \geq 0$. Suppose, moreover, that (g_1, h_1) is a pair of combinatorially rich saturated morphisms. Then $|\{g_i, h_i\}_{i \geq 0}| \leq 738$.*

Proof. The decreasing suffix complexity of $\{g_i, h_i\}_{i\geq 0}$ and Theorem 39 yield

$$\sigma(g_{j+1}) + \sigma(h_{j+1}) \leq \sigma(g_1) + \sigma(h_1) - j \leq 740 - j,$$

for all $j \geq 0$. Since a suffix complexity of a non-erasing non-periodic morphism is at least two, we have

$$4 \leq \sigma(g_{j+1}) + \sigma(h_{j+1}) \leq 740 - j,$$

for all $j \geq 0$. Then $j \leq 736$, and consequently, $|\{g_i, h_i\}_{i\geq 0}| \leq 738$. $\square$

Notice that in the previous lemma we have been working up to some special cases. First we have assumed that $\{g_i, h_i\}_{i\geq 0}$ is a sequence of successor morphisms with decreasing suffix complexity and second that (g_1, h_1) is a pair of combinatorially rich saturated morphisms. The cases when the reduction sequence of morphisms $\{g_i, h_i\}_{i\geq 0}$ does to have decreasing suffix complexity are classified in [16].

3.2 Examples of non-simple equality words

We have seen in the previous part that, up to some special cases, we can find a bound for number of blocks in which a non-simple equality words decomposes. This bound depends on the length of simple equality word w_K created at the end of the reduction sequence.

By Theorem 5, we know that if we suppose that

$$|h_K(b)| = \max\{|g_K(a)|, |g_K(b)|, |h_K(a)|, |h_K(b)|\}$$

and that $|w_K|_b \geq 9$, then there are words e, f conjugate with w_K such that $g_K(e) = h_K(f)$ and

$$e = f = (ab)^i a \quad \text{or} \quad e = f = (ba)^i b \quad \text{or} \quad e = f = ab^i \quad \text{or} \quad (e, f) = (b^i a^j, a^j b^i)$$

with $\gcd(i, j) = 1$ and $j > i$. In context of reduction sequence this leads us to the question: Can a pair of morphisms (g_K, h_K) with aforementioned equality word w_K have predecessors? We are not going to resolve this question here, but, for example, it is not so difficult to verify that non-periodic morphisms $g, h : \{a, b\}^* \to A^*$ such that $g((ab)^i a) = h((ab)^i a)$ and $i \geq 2$ cannot have predecessors. Therefore, there is no non-simple equality word whose reduction sequence could terminate with morphisms g_K, h_K such that $g_K((ab)^i a) = h_K((ab)^i a)$. The remaining cases form a good starting point for the further research.

In the previous paragraph, we have seen an example of morphisms which cannot have any predecessors. In fact, despite the very generous bound from Theorem 41 for the length of the sequence of successor morphisms, the only known examples of non-simple minimal equality words are part of our special cases, that is, their reduction sequence $\{g_i, h_i\}_{i\geq 0}$ of successor morphisms does not satisfy that g_1, h_1 are combinatorially rich:

Example 8. Words $a^{i+1}b^{j+1}$, $a^i b^2 a^{i+1}$, $a^{i+1} b^2 a^i$ and $ba^{2i+1}b$, where $i, j \geq 1$, are, up to the exchange of letters, examples of non-simple minimal binary equality words. Notice that words $a^{i+1}b^{j+1}$ with $\gcd(i+1, j+1) = 1$ as well as words $a^i b^2 a^{i+1}$,

$a^{i+1}b^2a^i$ and $ba^{2i+1}b$ are also examples of simple binary equality words. This is the consequence of the fact that different (non equivalent) pairs of morphisms can share the same equality word. In this sense only "pure" simple equality words are, up to the exchange of letters, $(ab)^ia$, with $i \geq 1$.

4. Classification of binary equality sets

Finally, we put our results into a perspective of the current state of knowledge of binary equality sets in general. Our work has significantly contributed to the understanding of equality sets generated by a unique word, in particular if the word is simple. Non-simple equality words remain the main challenge for the future.

We present a tabular listing of the current state of research of the structure of binary equality sets. In order to provide the reader with the complex view on the problem, we have included also results from works by other authors ([8], [13], [14] and [15]). The following list extends the classification of simple binary equality words made in Chapter 2. Particularly, we have included also binary equality sets of non-periodic morphisms and non-simple binary equality words.

Types of morphisms	Equality sets	Reason
1	i	A. Ehrenfeucht, J. Karhumäki and G. Rozenberg, 1983 [10]
2	ii	Š. Holub, 2004 [15]
3	iii	Š. Holub, 2003 [13]
3.1 3.2	A	Š. Holub, 2003 [14]
3.2.1	a + b	J. Hadravová, Š. Holub, 2008 [3], 2012 [4], 2011 [2]
3.2.2	b	New result, see Chapter 2
3.2.3	a + b + c + ⊛	
3.2.4	a + b + c + d	J. Karhümaki, K. Culik, 1980 [8]
3.2.5	$\boxdot$	
3.2.6	e	New result, see Chapter 3, J. Hadravová, 2011 [1]

Morphisms of the type 1 Morphisms $g, h : \{a, b\}^* \to A^*$ such that both g and h are periodic.

Morphisms of the type 2 Morphisms $g, h : \{a, b\}^* \to A^*$ such that g is periodic and h is non-periodic.

Morphisms of the type 3 Morphisms $g, h : \{a, b\}^* \to A^*$ such that both g and h are non-periodic.

Morphisms of the type 3.1 Morphisms $g, h : \{a, b\}^* \to A^*$ such that both g and h are non-periodic, $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$ and $\mathrm{Eq}(g, h) = \{\alpha, \beta\}^*$ for some non-empty words $\alpha, \beta \in \{a, b\}^+$ such that $\alpha \neq \beta$.

Morphisms of the type 3.2 Morphisms $g, h : \{a, b\}^* \to A^*$ such that both g and h are non-periodic and $\mathrm{Eq}(g, h) = \{\alpha\}^*$ for some non-empty word $\alpha \in \{a, b\}^+$.

Morphisms of the type 3.2.1 Non-periodic morphisms $g, h : \{a, b\}^* \to A^*$ with $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$ and a simple equality word α such that $|\alpha|_b \geq 9$.

Morphisms of the type 3.2.2 Non-periodic morphisms $g, h : \{a, b\}^* \to A^*$ with $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$ and a simple equality word α such that $|\alpha|_b \geq 4$ and $|\alpha|_a \geq 30$.

Morphisms of the type 3.2.3 Non-periodic morphisms $g, h : \{a, b\}^* \to A^*$ with $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$ and a simple equality word α such that $|\alpha|_b \leq 3$.

Morphisms of the type 3.2.4 Non-periodic morphisms $g, h : \{a, b\}^* \to A^*$ with a simple equality word α such that $|\alpha| \leq 5$.

Morphisms of the type 3.2.5 Non-periodic morphisms $g, h : \{a, b\}^* \to A^*$ with $|h(b)| = \max\{|g(a)|, |g(b)|, |h(a)|, |h(b)|\}$ and a simple equality word w such that $|\alpha|_b \leq 8$ and $|\alpha|_a \leq 29$.

Morphisms of the type 3.2.6 Non-periodic morphisms $g, h : \{a, b\}^* \to A^*$ with a non-simple minimal equality word α. We suppose moreover that their reduction sequence of morphisms $\{g_i, h_i\}_{i \geq 0}$ has a decreasing suffix complexity and g_1, h_1 are combinatorially rich.

Equality sets of the type i

$$\mathrm{Eq}(g, h) = \{1\},$$

$$\mathrm{Eq}(g, h) = \{1\} \cup \{w \in \{a, b\}^*, \frac{|w|_a}{|w|_b} = k\},$$

with $k \geq 0$, or $k = \infty$.

Equality sets of the type ii

$$\begin{aligned}\mathrm{Eq}(g,h) &= \{1\}\\ \mathrm{Eq}(g,h) &= (a^i b a^j)^*,\\ \mathrm{Eq}(g,h) &= (b^i a b^j)^*,\end{aligned}$$

with $i, j \geq 0$.

Equality sets of the type iii

$$\mathrm{Eq}(g,h) = \{\alpha, \beta\}^*,$$

where $\alpha, \beta \in \{a, b\}^*$.

Equality sets of the type A

$$\mathrm{Eq}(g,h) = \{a^i b, b a^i\}^*,$$

with $i \geq 1$.

Equality sets of the type a

$$\mathrm{Eq}(g,h) = (b^i a b^j)^*,$$

with $i, j \geq 0$.

Equality sets of the type b

$$\begin{aligned}\mathrm{Eq}(g,h) &= ((ab)^i a)^*,\\ \mathrm{Eq}(g,h) &= ((ba)^i b)^*,\\ \mathrm{Eq}(g,h) &= (a^j b^k)^*,\\ \mathrm{Eq}(g,h) &= (b^k a^j)^*,\end{aligned}$$

with $\gcd(j,k) = 1$ and $j > k$.

Equality sets of the type c

$$\begin{aligned}\mathrm{Eq}(g,h) &= (a^i b a^j)^*,\\ \mathrm{Eq}(g,h) &= (a^{i+1} b^2 a^i)^*,\\ \mathrm{Eq}(g,h) &= (a^i b^2 a^{i+1})^*,\\ \mathrm{Eq}(g,h) &= (b a^{2i+1} b)^*,\end{aligned}$$

with $i, j \geq 0$.

Equality sets of the type d

$$\begin{aligned}\mathrm{Eq}(g,h) &= (a^2 b^2)^*,\\ \mathrm{Eq}(g,h) &= (b^2 a^2)^*.\end{aligned}$$

Equality sets of the type (e)

$$\mathrm{Eq}(g,h) = \{\alpha\}^*,$$

with $|\alpha| \leq 68^{738}|\alpha_K|$ and α_K is a word such that $(g_K(\alpha_K), h_K(\alpha_K))$ is a simple conjugate pair of non-periodic binary morphisms g_K, h_K.

Equality sets of the type (⋇)

$$\mathrm{Eq}(g,h) = \{\alpha\}^*,$$

with $|\alpha|_b \leq 3$. We conjecture that all binary equality sets of this type are included already in (a), (b) and (c).

Equality sets of the type (⊡)

$$\mathrm{Eq}(g,h) = \{\alpha\}^*,$$

with $|\alpha|_b \leq 8$ and $|\alpha|_a \leq 29$.

Conclusion

The main aim of this work was to give the most up-to-date information about binary equality words and also, importantly, present the results of our original research in the field of single generators of binary equality languages. We have gathered in one place the results from our articles together with some new results, and, taking advantage of larger scale of this work, we could present the more complex view on the subject.

The main contribution of this work is the classification of simple binary equality words presented in Chapter 2. In Chapter 3 we have made some important conclusions about non-simple binary equality words. We hope, that these results will together make a solid base for further research of equality languages.

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List of Abbreviations

PCP	Post Correspondence Problem
X^*	Free monoid generated by X
$\rho(u)$	Primitive root of u
1	Empty word
$u \leq_{\mathrm{p}} v$	u is a prefix of v
$u \leq_{\mathrm{s}} v$	u is a suffix of v
$u <_{\mathrm{p}} v$	u is a proper prefix of v
$u <_{\mathrm{s}} v$	u is a proper suffix of v
$\mathrm{pref}(u)$	Set of all prefixes of u
$\mathrm{suf}(u)$	Set of all suffixes of u
$u \wedge v$	Maximal common prefix of u and v
$u \wedge_s v$	Maximal common suffix of u and v
u^ω	One-way infinite word composed of infinite numbers of copies of u
$\|u\|$	Length of u
$\|u\|_a$	Number of occurrences of the letter a in u