Charles University in Prague
Faculty of Mathematics and Physics

MASTER THESIS

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Effective interactions of the Euler-Heisenberg type in models of quantum field theory

Institute of Theoretical Physics

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Efektivní interakce Euler-Heisenbergova typu v modelech kvantové teorie pole

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Katedra: Ústav teoretické fyziky

Vedoucí diplomové práce: Prof. RNDr. Jiří Hořejší, DrSc.

Abstrakt: V předložené práci studujeme rozptyl světla na světle, což je nelineární efekt objevující se v kvantové elektrodynamice (QED). Cílem práce je studovat nízkoenergetickou efektivní teorii (lagrangiány Euler-Heisenbergova typu).

První část práce je věnována odvození efektivního lagrangiánu ve spinorové, skalární a vektorové QED pomocí přímého fitování amplitud jednosmyčkových diagramů. Výpočet pro případ vektorové QED je proveden s použitím unitární kalibrace, což pravděpodobně zatím nebylo zpracováno žádným jiným autorem.

Ve druhé části je efektivní lagrangián pro spinorovou QED odvozen pomocí funkcionálních metod. Podstatný bod odvození je spočtení determinantu Diracova operátoru v přítomnosti konstantního pozaďového elektromagnetického pole.

Klíčová slova: efektivní teorie, rozptyl světla na světle, kvantová elektrodynamika, standardní model

Title: Effective interactions of the Euler-Heisenberg type in models of quantum field theory

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Abstract: In the present thesis we study light-light scattering, which is a nonlinear effect occurring in quantum electrodynamics. The goal of this thesis is to study the low-energy effective theory (Lagrangians of Euler-Heisenberg type).

The first part of the work is devoted to the derivation of the effective Lagrangian in spinor, scalar and vector QED using the amplitude matching of one-loop diagrams. The calculation for the case of vector QED is performed using the unitary gauge, which probably has not been done yet so far by other authors.

In the second part, the effective Lagrangian for spinor QED is derived using functional methods. The essential point of the derivation is to calculate the determinant of the Dirac operator in constant background electromagnetic field.

Keywords: effective theories, light by light scattering, quantum electrodynamics, standard model

Contents

Preface

It is a well known fact that the classical electrodynamics is a linear theory. There is no interaction between two electromagnetic waves in the classical vacuum, which means one wave simply passes through another without any mutual influence. In other words, the waves behave like they do not know about each other. However, the situation is quite different in quantum electrodynamics due to the polarizability of vacuum which is an purely quantum effect.

In quantum electrodynamics (QED), charged virtual pairs can emerge from "nothingness" and mediate interaction between photons. This means quantum electrodynamics gives some nonlinear corrections to the classical theory. These correction can cause a series of interesting and fascinating non-classical effects including light by light scattering, photon splitting in vacuum or vacuum birefringence.

This area of study proved to be a fertile soil for both the experimental and the theoretical physics. First thoughts on this subject appeared shortly after Dirac proposed his hole theory in 1928, and the discovery of positron in 1932. A brief historical overview of this research area is given in the first chapter.

In this thesis we are interested in Lagrangians of Euler-Heisenberg type, which function as the low-energy effective description of those nonlinear corrections. The first part of the thesis is devoted to a detailed derivation of one-loop effective Lagrangians of Euler-Heisenberg type in the lowest (four-photon) order in cases of various versions of QED (spinor QED, scalar QED and vector QED). This is accomplished by the direct calculation of one-loop diagrams and subsequent matching the calculated amplitudes to the amplitude given by the effective theory. The case of vector QED is calculated using the unitary gauge, which probably has not been done to this date.

In the second part, we demonstrate an alternative approach that is based on the path integral formulation. We derive the Euler-Heisenberg Lagrangian for spinor QED by the calculation of the determinant of the Dirac operator in constant background electromagnetic field.

1. Historical overview

Since there is a great number of published articles and other literature on this subject, we restrict ourselves only to a few. A part of this overview is based on [1].

The possibility of quantum-induced nonlinear corrections was first proposed by Halpern who published a brief note that light by light scattering could occur.

Debye and Heisenberg privately discussed this, and Heisenberg then assigned his student Hans Euler a task to study this problem using the density matrix formalism Heisenberg had developed. This later became Euler's PhD thesis that he defended in 1936 at Leipzig [2].

A year later, in 1935, Euler and another Heisenberg's student, Bernhard Kockel, published a paper, where they gave results for the light-light scattering amplitudes in the low frequency limit [3] (meaning the energy of the scattering photon is small compared to the mass of electron). In their paper, they computed the leading quantum correction to the Maxwell Lagrangian

$$\mathscr{L} = \frac{\mathbf{E}^2 - \mathbf{B}^2}{2} + \frac{1}{90\pi}\frac{\hbar c}{e^2}\frac{1}{E_0^2}\left[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 7(\mathbf{E}\cdot\mathbf{B})^2\right], \tag{1.1}$$

where $E_0 \equiv e/(e^2/mc^2)^2$. They also calculated the light-light scattering cross-section

$$\sigma \sim \left(\frac{e^2}{\hbar c}\right)^4 \left(\frac{\hbar^4}{mc}\right)^4 \frac{1}{\lambda^2} \tag{1.2}$$

In modern language, we can say that they studied QED vacuum polarization in the limit of constant background field.

Not long after that Akhieser, Landau and Pomeranchuk published similar results for the high frequency limit, which later became Akhieser's PhD thesis [4].

Euler and Heisenberg published a paper [5] in 1936 in which they significantly extended the Euler-Kockel results. They obtained a closed-form expression for the full nonlinear correction to the Maxwell Lagrangian

$$\begin{aligned}\mathscr{L} =& \frac{e^2}{\hbar c}\int_0^\infty \frac{\mathrm{d}\eta}{\eta^3}\mathrm{e}^{-\eta} \\ & \left\{ i\eta^2(\mathbf{E}\cdot\mathbf{B})\frac{\cos\left[\frac{\eta}{\mathcal{E}_c}\sqrt{\mathbf{E}^2 - \mathbf{B}^2 + 2i(\mathbf{E}\cdot\mathbf{B})}\right] + \text{c.c.}}{\cos\left[\frac{\eta}{\mathcal{E}_c}\sqrt{\mathbf{E}^2 - \mathbf{B}^2 + 2i(\mathbf{E}\cdot\mathbf{B})}\right] - \text{c.c.}} + \mathcal{E}_c^2 + \frac{\eta^2}{3}(\mathbf{B}^2 - \mathbf{E}^2)\right\},\end{aligned} \tag{1.3}$$

where $\mathcal{E}_c \equiv m^2c^3/e\hbar \approx 10^{16}$ V/cm is the critical field strength.

This correction is non-perturbative – it incorporates all orders in the constant background electromagnetic field. One can recover the original Euler-Kockel result (1.1) by expanding this formula in a weak-field field expansion to quartic order.

Remarkably, they were able to identify the physical significance of the subtraction terms in the formula. The first term corresponds to the subtraction of the infinite free-field effective action. The second term is related to the charge renormalization.

Finally, the formula has the form that we nowadays refer to as the "proper-time form"

Euler and Heisenberg used a brute force in their calculation, working with exact solutions of Dirac equation in constant background. Soon after their paper, Weisskopf presented [6] a significantly simplified computation of the effective action not just for spinor QED, but for scalar QED as well. He worked directly with the spectrum of the Dirac and the Klein-Gordon operators rather than with their eigenfunctions.

Later the problem was studied by Feynman [7] who viewed QED processes as evolution in proper-time (thanks to Fock) and tried to extend his path integral formulation of non-relativistic quantum mechanics to the relativistic theory of Dirac. His findings are now known as the worldline path integral formalism.

Soon after Feynman's work, Schwinger published an essential paper [8], where he reformulated the results of Euler and Heisenberg in the new language of renormalized QED. He also viewed QED processes as evolution in proper-time, but instead of the path integral method, he used operator solutions.

The paper also presents the exact result for two special cases, first, the uniform background treated by Euler and Heisenberg, and second, the plane-wave background, for which the Dirac equation had been solved by Volkov [9].

There is also a more recent paper by Dittrich [10], where the effective action is derived using Schwinger's proper-time method for the case of constant magnetic field and a laser field.

The actual problem of light-light scattering in diagramatically formulated spinor QED was attacked by Karplus and Neuman in 1950 [11]. They calculated the relevant one-loop box Feynman diagrams and gave expressions for various form factors in the corresponding amplitude as integrals over Feynman parameters. In the end, they performed low-energy expansion of the amplitude and obtained a result which perfectly matched the amplitude calculated using the effective Lagrangian (1.1).

Later was the same problem studied by Constantini, De Tollis and Pistoni [12], [13]. They exactly calculated the rank-4 polarization tensor in terms of rational, logarithm and dilogarithm function using dispersion relations and they also provided the exact amplitudes for light-light scattering, photon splitting and photon coalescence into photons on nuclei. Delbrück scattering (the deflection of high-energy photons in the Coulomb field of nuclei) was studied as well.

Recently, these long-established results for the low-energy photon-photon scattering have been questioned by Kanda and Fujita [14], [15] who claimed that the differential cross-section formula should read

$$\frac{\mathrm{d}\sigma_{\mathrm{FK}}}{\mathrm{d}\Omega} = \frac{\alpha^4}{(12\pi)^2\omega^2}(3 + 2\cos^2\theta + \cos^4\theta), \tag{1.4}$$

which is in contradiction with the well-known result (see for example the article itself, or any good book on QED)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{139\alpha^4}{(180\pi)^2}\frac{\omega^6}{m^8}(3 + \cos^2\theta)^2. \tag{1.5}$$

This has been rebutted by Liang and Czarnecki [16]. They have shown that this different result originated from erroneous manipulations with unregulated divergent integrals. The result (1.4) is also in contradiction with the current experimental upper limit [17].

So far, we have talked only about spinor QED (apart from Weisskopf). There is a paper by König [18], in which he calculated the polarization tensor for the process $Z \to \gamma\gamma\gamma$ via scalar loops in MSSM in the low-energy limit. This particular process is very similar to the scalar light-light scattering.

In a related paper by Jiang and Zhou [19], the polarization tensors for the processes $Z \to \gamma\gamma\gamma$ and $\gamma\gamma \to \gamma\gamma$ were calculated via vector boson loops in the standard model.

Jikia and Tkabladze [20] calculated the process $\gamma\gamma \to \gamma\gamma$ as well and presented explicit formulae for the helicity amplitudes.

And finally, there is a quite old article by Vanyashin and Terent'ev [21]. They calculated the nonlinear corrections to the Lagrangian of constant electromagnetic field, caused by the vacuum polarization of charged vector field. In other words, they computed the Lagrangian of Euler-Heisenberg type for vector QED.

2. Diagrammatic approach

In this chapter, we derive one-loop effective Lagrangians of Euler-Heisenberg type in the lowest (four-photon) order for spinor, scalar and vector QED. This is carried out through direct matching of one-loop four-photon amplitudes to the effective theory.

First the relevant one-loop amplitudes are calculated in the underlying theory in a low-energy approximation (meaning the energy of the participating photons is considered much lower than the mass of the mediating charged particle). Subsequently the results are fitted to the corresponding amplitude obtained via the effective theory.

We start with a preparatory analysis of the effective theory.

2.1 Effective theory of Euler-Heisenberg type

2.1.1 Lagrangian form

It is clear that the effective theory that describes light by light scattering and related processes should be characterized by a Lagrangian that is formed only of from the electromagnetic fields A_μ since there are no other fields effectively present.

Also the theory should be gauge invariant (the underlying theory certainly is), therefore, to automatically ensure this, the gauge invariant field strength tensor $F_{\mu\nu}$ should be used instead of the potential A_μ for the construction of the Lagrangian.

We consider only four-photon interactions, therefore, there should be in total four $F_{\mu\nu}$ tensors in the Lagrangian. Obviously all their Lorentz indices have to be contracted since the Lagrangian is a scalar.

We are not interested in terms containing derivatives of the field strength tensor since these are terms of a higher mass dimension. A general structure of suitable Lagrangian terms then should be

$$F^{\circ}{}_{\circ}F^{\circ}{}_{\circ}F^{\circ}{}_{\circ}F^{\circ}{}_{\circ}, \tag{2.1}$$

with various contractions of the indices (which are represented by $^\circ$ and $_\circ$). There are many possible contractions, however, many of them are zero (as a consequence of the anti-symmetry of $F_{\mu\nu}$). It turns out that there are only three independent terms in total. Their mass dimension is 8 and they are (up to some numerical constants)

$$\mathcal{F}^2, \qquad \mathcal{G}^2, \qquad \mathcal{F}\mathcal{G}, \tag{2.2}$$

where

$$\begin{aligned} \mathcal{F} &\equiv F_{\mu\nu}F^{\mu\nu} = -2(\mathbf{E}^2 - \mathbf{B}^2) \\ \mathcal{G} &\equiv \star F_{\mu\nu}F^{\mu\nu} = 4(\mathbf{E}\cdot\mathbf{B}) \end{aligned} \tag{2.3}$$

are the two fundamental invariants of electromagnetic field and

$$\star F_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\varrho\sigma}F^{\varrho\sigma} \tag{2.4}$$

is the Hodge dual of $F_{\mu\nu}$. The last term in (2.2) is a pseudoscalar, hence, it must be dropped if one wishes to maintain parity conservation (parity is conserved in the underlying theory).

This reasoning leads us to the final form of the effective Lagrangian[1]

$$\mathscr{L}_{\text{eff.}}(g_1, g_2) \equiv g_1(F_{\mu\nu}F^{\mu\nu})^2 + g_2(\star F_{\mu\nu}F^{\mu\nu})^2. \tag{2.5}$$

The coupling constants g_1, g_2 of dimension -4 have been added to keep the correct dimension of the Lagrangian (which must be 4). Our task is to find their values for the case of spinor, scalar and vector QED. The Lagrangian then can be visualized as a combination of two effective vertices, which can be seen in Fig. 2.1.

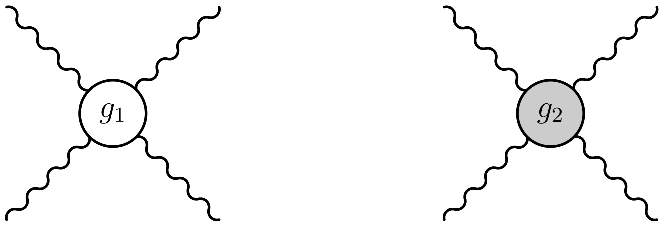

Figure 2.1: The effective vertices

2.1.2 Invariant amplitude

We now proceed with evaluation of the amplitude of a four-photon process. The initial state and the final state are taken as

$$\begin{aligned} |\mathrm{i}\rangle &= |0\rangle \\ |\mathrm{f}\rangle &= \left[\prod_{i=1}^{4} a^\dagger(p_i, \lambda_i)\right] |0\rangle, \end{aligned} \tag{2.6}$$

where p_i are the momenta and $\lambda_i = 1, 2$ are the polarizations of the photons. The photons are taken as outgoing, so

$$\sum_{i=1}^{4} p_i = 0, \tag{2.7}$$

and as on-shell, so

$$p_i^2 = 0, \qquad i = 1, 2, 3, 4. \tag{2.8}$$

Conservation (2.7) together with the on-shell condition (2.8) implies

$$\begin{aligned} p_3 \cdot p_4 &= p_1 \cdot p_2 \\ p_2 \cdot p_4 &= p_1 \cdot p_3 \\ p_1 \cdot p_4 &= p_2 \cdot p_3, \end{aligned} \tag{2.9}$$

and also

$$(p_1 \cdot p_2) + (p_1 \cdot p_3) + (p_2 \cdot p_3) = 0. \tag{2.10}$$

[1]Alternatively, we could have used $(F_{\mu\nu}F^{\mu\nu})^2$ and $F_{\mu\nu}F^{\nu\varrho}F_{\varrho\sigma}F^{\sigma\mu}$ as a different tensor basis instead of $(F_{\mu\nu}F^{\mu\nu})^2$ and $(\star F_{\mu\nu}F^{\mu\nu})^2$.

The potential A_μ is decomposed as follows [22]

$$A_\mu(x) = \sum_{\lambda=1}^{2} \int \widetilde{\mathrm{d}^3 p} \left[\varepsilon_\mu(p,\lambda) a(p,\lambda) \mathrm{e}^{-ip\cdot x} + \varepsilon^*_\mu(p,\lambda) a^\dagger(p,\lambda) \mathrm{e}^{ip\cdot x} \right], \tag{2.11}$$

where ε_μ is the polarization vector, $E \equiv p_0$ is the photon energy, and

$$\widetilde{\mathrm{d}^3 p} \equiv \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}(2E)^{1/2}}. \tag{2.12}$$

By differentiation we get

$$\partial_\varrho A_\mu(x) = i \sum_{\lambda=1}^{2} \int \widetilde{\mathrm{d}^3 p} \left[-p_\varrho \varepsilon_\mu(p,\lambda) a(p,\lambda) \mathrm{e}^{-ip\cdot x} + p_\varrho \varepsilon^*_\mu(p,\lambda) a^\dagger(p,\lambda) \mathrm{e}^{ip\cdot x} \right]. \tag{2.13}$$

We now begin analyzing the first term in (2.5). The second term is processed analogously later. Using

$$F_{\mu\nu}F^{\mu\nu} = 2(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu), \tag{2.14}$$

we have

$$\begin{aligned} \mathscr{L}_1 &\equiv g_1 (F_{\mu\nu}F^{\mu\nu})^2 \\ &= 4g_1 (\partial_\mu A_\nu \partial^\mu A^\nu \partial_\varrho A_\sigma \partial^\varrho A^\sigma \\ &\qquad - 2\partial_\mu A_\nu \partial^\mu A^\nu \partial_\varrho A_\sigma \partial^\sigma A^\varrho \\ &\qquad + \partial_\mu A_\nu \partial^\nu A^\mu \partial_\varrho A_\sigma \partial^\sigma A^\varrho). \end{aligned} \tag{2.15}$$

From this, using (2.13) and (2.15), we compute the element of the S-matrix in the first order of the perturbation theory (the superscript denotes the first order and the subscript denotes the first term of the Lagrangian)

$$\begin{aligned} \langle \mathrm{f}|S_1^{(1)}|\mathrm{i}\rangle &= i \int \mathrm{d}^4 x \; \langle 0| \left[\prod_{i=1}^{4} a(p_i,\lambda_i) \right] \mathscr{L}_1(x) |0\rangle \\ &= i\mathcal{M}_1 (2\pi)^4 \delta^{(4)}(p_1+p_2+p_3+p_4) \prod_{i=1}^{4} \frac{1}{(2\pi)^{3/2}(2E_i)^{1/2}}, \end{aligned} \tag{2.16}$$

where $\mathcal{M}_1$ is the invariant amplitude and E_i are the photon energies.

For the amplitude $\mathcal{M}_1$, we find

$$\begin{aligned} \mathcal{M}_1 = 4g_1 \sum_\pi [&(p_{\pi_1}\cdot p_{\pi_2})(\varepsilon_{\pi_1}\cdot\varepsilon_{\pi_2})(p_{\pi_3}\cdot p_{\pi_4})(\varepsilon_{\pi_3}\cdot\varepsilon_{\pi_4}) \\ &- 2(p_{\pi_1}\cdot p_{\pi_2})(\varepsilon_{\pi_1}\cdot\varepsilon_{\pi_2})(p_{\pi_3}\cdot\varepsilon_{\pi_4})(\varepsilon_{\pi_3}\cdot p_{\pi_4}) \\ &+ (p_{\pi_1}\cdot\varepsilon_{\pi_2})(\varepsilon_{\pi_1}\cdot p_{\pi_2})(p_{\pi_3}\cdot\varepsilon_{\pi_4})(\varepsilon_{\pi_3}\cdot p_{\pi_4})], \end{aligned} \tag{2.17}$$

where $\varepsilon_i \equiv \varepsilon(p_i,\lambda_i)$. The summation runs over all permutations of four elements and each permutation π_k is understood to be a function of its index, i.e.[2] $\pi_k \equiv \pi(k)$. The summation is a consequence of the fact that there are $4! = 24$

[2]For instance, we have $\pi_k = k$ for the identity permutation.

possible Wick contractions in (2.16) and each possible contraction corresponds to a permutation. One can easily see that the result reflects (in a way) the structure of (2.15). Aside from the summation, it is basically its Fourier transform.

The amplitude can be further rewritten as

$$\mathcal{M}_1 = \Gamma^1_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4)\varepsilon_1^\mu \varepsilon_2^\nu \varepsilon_3^\varrho \varepsilon_4^\sigma, \tag{2.18}$$

where

$$\begin{aligned}\Gamma^1_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) &\equiv \mathrm{sym}_4\, \Theta^1_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) \\ &\equiv \Theta^1_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) + \Theta^1_{\nu\mu\varrho\sigma}(p_2, p_1, p_3, p_4) \\ &\quad + \Theta^1_{\varrho\nu\mu\sigma}(p_3, p_2, p_1, p_4) + \ldots,\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}\Theta^1_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) \equiv 4g_1[&(p_1 \cdot p_2)(p_3 \cdot p_4) g_{\mu\nu} g_{\varrho\sigma} \\ &- 2(p_1 \cdot p_2)(p_3)_\sigma (p_4)_\varrho g_{\mu\nu} \\ &+ (p_1)_\nu (p_2)_\mu (p_3)_\sigma (p_4)_\varrho].\end{aligned} \tag{2.20}$$

The "sym_4" operator denotes the complete tensor symmetrization (without any numerical prefactors) that is simultaneously acting on both Lorentz indices and arguments (which are the momenta). The tensor $\Gamma^1_{\mu\nu\varrho\sigma}$ is called the polarization tensor and it is (as we can see) the invariant amplitude stripped off the polarization vectors.

Owing to the following symmetries of $\Theta^1_{\mu\nu\varrho\sigma}$

$$\begin{aligned}\Theta^1_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) &= \Theta^1_{\nu\mu\varrho\sigma}(p_2, p_1, p_3, p_4) \\ &= \Theta^1_{\mu\nu\sigma\varrho}(p_1, p_2, p_4, p_3) \\ &= \Theta^1_{\varrho\sigma\mu\nu}(p_3, p_4, p_1, p_2),\end{aligned} \tag{2.21}$$

we easily obtain that

$$\begin{aligned}\Gamma^1_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) = 32g_1[&(p_1 \cdot p_2)(p_3 \cdot p_4) g_{\mu\nu} g_{\varrho\sigma} \\ &+ (p_1 \cdot p_3)(p_2 \cdot p_4) g_{\mu\varrho} g_{\nu\sigma} \\ &+ (p_1 \cdot p_4)(p_2 \cdot p_3) g_{\mu\sigma} g_{\nu\varrho} \\ &- 2(p_1 \cdot p_2)(p_3)_\sigma (p_4)_\varrho g_{\mu\nu} \\ &- 2(p_1 \cdot p_3)(p_2)_\sigma (p_4)_\nu g_{\mu\varrho} \\ &- 2(p_1 \cdot p_4)(p_2)_\varrho (p_3)_\nu g_{\mu\sigma} \\ &+ (p_1)_\nu (p_2)_\mu (p_3)_\sigma (p_4)_\varrho \\ &+ (p_1)_\varrho (p_3)_\mu (p_2)_\sigma (p_4)_\nu \\ &+ (p_1)_\sigma (p_4)_\mu (p_2)_\varrho (p_3)_\nu].\end{aligned} \tag{2.22}$$

Now, for the second term of the effective Lagrangian (2.5), the situation is a bit more complex. Let us similarly define

$$\mathscr{L}_2 \equiv g_2(\star F_{\mu\nu} F^{\mu\nu})^2. \tag{2.23}$$

Using (2.4) we get

$$\mathscr{L}_2 = \frac{g_2}{4}\varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\varrho\sigma}F^{\alpha\beta\gamma\delta}F^{\mu\nu\varrho\sigma}, \tag{2.24}$$

where

$$F_{\mu\nu\varrho\sigma} \equiv F_{\mu\nu}F_{\varrho\sigma}. \tag{2.25}$$

Since it holds

$$\det(G) \equiv \det\begin{pmatrix} g_{\alpha\mu} & g_{\beta\mu} & g_{\gamma\mu} & g_{\delta\mu} \\ g_{\alpha\nu} & g_{\beta\nu} & g_{\gamma\nu} & g_{\delta\nu} \\ g_{\alpha\varrho} & g_{\beta\varrho} & g_{\gamma\varrho} & g_{\delta\varrho} \\ g_{\alpha\sigma} & g_{\beta\sigma} & g_{\gamma\sigma} & g_{\delta\sigma} \end{pmatrix} = -\varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\varrho\sigma}, \tag{2.26}$$

we can use the Leibniz determinant formula

$$\det(G) = \sum_{\pi}\operatorname{sgn}(\pi)\prod_{k=1}^{4} G_{k\pi_k} \tag{2.27}$$

to deduce that

$$\begin{aligned}\mathscr{L}_2 &= -\frac{g_2}{4}F^{\mu\nu\varrho\sigma}(F_{\mu\nu\varrho\sigma} - F_{\nu\mu\varrho\sigma} + F_{\nu\varrho\mu\sigma} + \ldots) \\ &= -6g_2F^{\mu\nu\varrho\sigma}F_{[\mu\nu\varrho\sigma]},\end{aligned} \tag{2.28}$$

where [ ] denotes total tensor anti-symmetrization (including the 1/4! prefactor). Again, because of the symmetries of $F_{\mu\nu\varrho\sigma}$

$$\begin{aligned}F_{\mu\nu\varrho\sigma} &= -F_{\nu\mu\varrho\sigma} \\ &= -F_{\mu\nu\sigma\varrho} \\ &= F_{\varrho\sigma\mu\nu},\end{aligned} \tag{2.29}$$

we get

$$\mathscr{L}_2 = -2g_2F^{\mu\nu\varrho\sigma}(F_{\mu\nu\varrho\sigma} + F_{\mu\sigma\nu\varrho} + F_{\mu\varrho\sigma\nu}). \tag{2.30}$$

Finally, inserting

$$F_{\mu\nu\varrho\sigma} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\varrho A_\sigma - \partial_\sigma A_\varrho) \tag{2.31}$$

into the expression yields

$$\begin{aligned}\mathscr{L}_2 = -8g_2\partial^\mu A^\nu\partial^\varrho A^\sigma(&\partial_\mu A_\nu\partial_\varrho A_\sigma - 2\partial_\mu A_\nu\partial_\sigma A_\varrho + \partial_\nu A_\mu\partial_\sigma A_\varrho \\ &+ \partial_\mu A_\sigma\partial_\nu A_\varrho + \partial_\sigma A_\mu\partial_\varrho A_\nu + \partial_\varrho A_\mu\partial_\nu A_\sigma \\ &+ \partial_\mu A_\varrho\partial_\sigma A_\nu - \partial_\sigma A_\mu\partial_\nu A_\varrho - \partial_\mu A_\sigma\partial_\varrho A_\nu \\ &- \partial_\mu A_\varrho\partial_\nu A_\sigma - \partial_\varrho A_\mu\partial_\sigma A_\nu).\end{aligned} \tag{2.32}$$

Analogously as in the previous case we have

$$\mathcal{M}_2 = \Gamma^2_{\mu\nu\varrho\sigma}(p_1,p_2,p_3,p_4)\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\varrho\varepsilon_4^\sigma, \tag{2.33}$$

where

$$\Gamma^2_{\mu\nu\varrho\sigma}(p_1,p_2,p_3,p_4) \equiv \operatorname{sym}_4\Theta^2_{\mu\nu\varrho\sigma}(p_1,p_2,p_3,p_4), \tag{2.34}$$

and from the Lagrangian (2.32) we extract

$$
\begin{aligned}
\Theta^2_{\mu\nu\varrho\sigma}(p_1,p_2,p_3,p_4) \equiv -8g_2[&(p_1\cdot p_2)(p_3\cdot p_4)g_{\mu\nu}g_{\varrho\sigma}\\
&-2(p_1\cdot p_2)(p_3)_\sigma(p_4)_\varrho g_{\mu\nu}\\
&+(p_1)_\nu(p_2)_\mu(p_3)_\sigma(p_4)_\varrho\\
&+(p_1\cdot p_2)(p_3)_\sigma(p_4)_\mu g_{\nu\varrho}\\
&+(p_1)_\nu(p_2)_\varrho(p_3\cdot p_4)g_{\mu\sigma}\\
&+(p_1)_\nu(p_2\cdot p_3)(p_4)_\mu g_{\varrho\sigma}\\
&+(p_1\cdot p_2)(p_3)_\nu(p_4)_\varrho g_{\mu\sigma}\\
&-(p_1)_\nu(p_2)_\varrho(p_3)_\sigma(p_4)_\mu\\
&-(p_1\cdot p_2)(p_3\cdot p_4)g_{\mu\sigma}g_{\nu\varrho}\\
&-(p_1\cdot p_2)(p_3)_\nu(p_4)_\mu g_{\varrho\sigma}\\
&-(p_1)_\nu(p_2\cdot p_3)(p_4)_\varrho g_{\mu\sigma}]. \qquad (2.35)
\end{aligned}
$$

Unfortunately, computing the polarization tensor (2.34) is not so easy as in the previous case because $\Theta^2_{\mu\nu\varrho\sigma}$ does not possess any apparent symmetries (unlike $\Theta^1_{\mu\nu\varrho\sigma}$).

After summing over all of the 24 permutations of $\Theta^2_{\mu\nu\varrho\sigma}$ (which was done on the computer in `FeynCalc` and `Wolfram Mathematica`), we found that the polarization tensor $\Gamma^2_{\mu\nu\varrho\sigma}$ consists of 60 terms (see Fig. 2.2).

$$
\begin{aligned}
-32\,\mathrm{g2}\,(&2\,\mathrm{p4}^\mu\,\mathrm{p3}^\nu\,\mathrm{p2}^\rho\,\mathrm{p1}^\sigma-\mathrm{p3}^\mu\,\mathrm{p4}^\nu\,\mathrm{p2}^\rho\,\mathrm{p1}^\sigma-\mathrm{p2}^\mu\,\mathrm{p3}^\nu\,\mathrm{p4}^\rho\,\mathrm{p1}^\sigma-\\
&2\,\mathrm{p4}^\mu\,g^{\nu\rho}\,\mathrm{p2}\cdot\mathrm{p3}\,\mathrm{p1}^\sigma+g^{\mu\rho}\,\mathrm{p4}^\nu\,\mathrm{p2}\cdot\mathrm{p3}\,\mathrm{p1}^\sigma+g^{\mu\nu}\,\mathrm{p4}^\rho\,\mathrm{p2}\cdot\mathrm{p3}\,\mathrm{p1}^\sigma+\\
&\mathrm{p3}^\mu\,g^{\nu\rho}\,\mathrm{p2}\cdot\mathrm{p4}\,\mathrm{p1}^\sigma-g^{\mu\rho}\,\mathrm{p3}^\nu\,\mathrm{p2}\cdot\mathrm{p4}\,\mathrm{p1}^\sigma+\mathrm{p2}^\mu\,g^{\nu\rho}\,\mathrm{p3}\cdot\mathrm{p4}\,\mathrm{p1}^\sigma-\\
&g^{\mu\nu}\,\mathrm{p2}^\rho\,\mathrm{p3}\cdot\mathrm{p4}\,\mathrm{p1}^\sigma-\mathrm{p4}^\mu\,\mathrm{p3}^\nu\,\mathrm{p1}^\rho\,\mathrm{p2}^\sigma+2\,\mathrm{p3}^\mu\,\mathrm{p4}^\nu\,\mathrm{p1}^\rho\,\mathrm{p2}^\sigma-\\
&\mathrm{p3}^\mu\,\mathrm{p1}^\nu\,\mathrm{p4}^\rho\,\mathrm{p2}^\sigma-\mathrm{p2}^\mu\,\mathrm{p4}^\nu\,\mathrm{p1}^\rho\,\mathrm{p3}^\sigma-\mathrm{p4}^\mu\,\mathrm{p1}^\nu\,\mathrm{p2}^\rho\,\mathrm{p3}^\sigma+\\
&2\,\mathrm{p2}^\mu\,\mathrm{p1}^\nu\,\mathrm{p4}^\rho\,\mathrm{p3}^\sigma-\mathrm{p4}^\mu\,\mathrm{p3}^\nu\,g^{\rho\sigma}\,\mathrm{p1}\cdot\mathrm{p2}-\mathrm{p3}^\mu\,\mathrm{p4}^\nu\,g^{\rho\sigma}\,\mathrm{p1}\cdot\mathrm{p2}+\\
&\mathrm{p3}^\mu\,g^{\nu\sigma}\,\mathrm{p4}^\rho\,\mathrm{p1}\cdot\mathrm{p2}+g^{\mu\sigma}\,\mathrm{p3}^\nu\,\mathrm{p4}^\rho\,\mathrm{p1}\cdot\mathrm{p2}+\mathrm{p4}^\mu\,g^{\nu\rho}\,\mathrm{p3}^\sigma\,\mathrm{p1}\cdot\mathrm{p2}+\\
&g^{\mu\rho}\,\mathrm{p4}^\nu\,\mathrm{p3}^\sigma\,\mathrm{p1}\cdot\mathrm{p2}-2\,g^{\mu\nu}\,\mathrm{p4}^\rho\,\mathrm{p3}^\sigma\,\mathrm{p1}\cdot\mathrm{p2}+\mathrm{p2}^\mu\,\mathrm{p4}^\nu\,g^{\rho\sigma}\,\mathrm{p1}\cdot\mathrm{p3}-\\
&\mathrm{p4}^\mu\,g^{\nu\sigma}\,\mathrm{p2}^\rho\,\mathrm{p1}\cdot\mathrm{p3}+g^{\mu\sigma}\,\mathrm{p4}^\nu\,\mathrm{p2}^\rho\,\mathrm{p1}\cdot\mathrm{p3}-\mathrm{p2}^\mu\,g^{\nu\sigma}\,\mathrm{p4}^\rho\,\mathrm{p1}\cdot\mathrm{p3}+\\
&\mathrm{p4}^\mu\,g^{\nu\rho}\,\mathrm{p2}^\sigma\,\mathrm{p1}\cdot\mathrm{p3}-2\,g^{\mu\rho}\,\mathrm{p4}^\nu\,\mathrm{p2}^\sigma\,\mathrm{p1}\cdot\mathrm{p3}+g^{\mu\nu}\,\mathrm{p4}^\rho\,\mathrm{p2}^\sigma\,\mathrm{p1}\cdot\mathrm{p3}+\\
&\mathrm{p2}^\mu\,\mathrm{p3}^\nu\,g^{\rho\sigma}\,\mathrm{p1}\cdot\mathrm{p4}+\mathrm{p3}^\mu\,g^{\nu\sigma}\,\mathrm{p2}^\rho\,\mathrm{p1}\cdot\mathrm{p4}-2\,g^{\mu\sigma}\,\mathrm{p3}^\nu\,\mathrm{p2}^\rho\,\mathrm{p1}\cdot\mathrm{p4}-\\
&\mathrm{p3}^\mu\,g^{\nu\rho}\,\mathrm{p2}^\sigma\,\mathrm{p1}\cdot\mathrm{p4}+g^{\mu\rho}\,\mathrm{p3}^\nu\,\mathrm{p2}^\sigma\,\mathrm{p1}\cdot\mathrm{p4}-\mathrm{p2}^\mu\,g^{\nu\rho}\,\mathrm{p3}^\sigma\,\mathrm{p1}\cdot\mathrm{p4}+\\
&g^{\mu\nu}\,\mathrm{p2}^\rho\,\mathrm{p3}^\sigma\,\mathrm{p1}\cdot\mathrm{p4}+\mathrm{p4}^\mu\,\mathrm{p1}^\nu\,g^{\rho\sigma}\,\mathrm{p2}\cdot\mathrm{p3}+\mathrm{p4}^\mu\,g^{\nu\sigma}\,\mathrm{p1}^\rho\,\mathrm{p2}\cdot\mathrm{p3}-\\
&g^{\mu\sigma}\,\mathrm{p4}^\nu\,\mathrm{p1}^\rho\,\mathrm{p2}\cdot\mathrm{p3}-g^{\mu\sigma}\,\mathrm{p1}^\nu\,\mathrm{p4}^\rho\,\mathrm{p2}\cdot\mathrm{p3}+2\,g^{\mu\sigma}\,g^{\nu\rho}\,\mathrm{p1}\cdot\mathrm{p4}\,\mathrm{p2}\cdot\mathrm{p3}-\\
&g^{\mu\rho}\,g^{\nu\sigma}\,\mathrm{p1}\cdot\mathrm{p4}\,\mathrm{p2}\cdot\mathrm{p3}-g^{\mu\nu}\,g^{\rho\sigma}\,\mathrm{p1}\cdot\mathrm{p4}\,\mathrm{p2}\cdot\mathrm{p3}+\mathrm{p3}^\mu\,\mathrm{p1}^\nu\,g^{\rho\sigma}\,\mathrm{p2}\cdot\mathrm{p4}-\\
&2\,\mathrm{p3}^\mu\,g^{\nu\sigma}\,\mathrm{p1}^\rho\,\mathrm{p2}\cdot\mathrm{p4}+g^{\mu\sigma}\,\mathrm{p3}^\nu\,\mathrm{p1}^\rho\,\mathrm{p2}\cdot\mathrm{p4}-g^{\mu\rho}\,\mathrm{p1}^\nu\,\mathrm{p3}^\sigma\,\mathrm{p2}\cdot\mathrm{p4}+\\
&g^{\mu\nu}\,\mathrm{p1}^\rho\,\mathrm{p3}^\sigma\,\mathrm{p2}\cdot\mathrm{p4}-g^{\mu\sigma}\,g^{\nu\rho}\,\mathrm{p1}\cdot\mathrm{p3}\,\mathrm{p2}\cdot\mathrm{p4}+2\,g^{\mu\rho}\,g^{\nu\sigma}\,\mathrm{p1}\cdot\mathrm{p3}\,\mathrm{p2}\cdot\mathrm{p4}-\\
&g^{\mu\nu}\,g^{\rho\sigma}\,\mathrm{p1}\cdot\mathrm{p3}\,\mathrm{p2}\cdot\mathrm{p4}-2\,\mathrm{p2}^\mu\,\mathrm{p1}^\nu\,g^{\rho\sigma}\,\mathrm{p3}\cdot\mathrm{p4}+\mathrm{p2}^\mu\,g^{\nu\sigma}\,\mathrm{p1}^\rho\,\mathrm{p3}\cdot\mathrm{p4}+\\
&g^{\mu\sigma}\,\mathrm{p1}^\nu\,\mathrm{p2}^\rho\,\mathrm{p3}\cdot\mathrm{p4}+g^{\mu\rho}\,\mathrm{p1}^\nu\,\mathrm{p2}^\sigma\,\mathrm{p3}\cdot\mathrm{p4}-g^{\mu\nu}\,\mathrm{p1}^\rho\,\mathrm{p2}^\sigma\,\mathrm{p3}\cdot\mathrm{p4}-\\
&g^{\mu\sigma}\,g^{\nu\rho}\,\mathrm{p1}\cdot\mathrm{p2}\,\mathrm{p3}\cdot\mathrm{p4}-g^{\mu\rho}\,g^{\nu\sigma}\,\mathrm{p1}\cdot\mathrm{p2}\,\mathrm{p3}\cdot\mathrm{p4}+2\,g^{\mu\nu}\,g^{\rho\sigma}\,\mathrm{p1}\cdot\mathrm{p2}\,\mathrm{p3}\cdot\mathrm{p4})
\end{aligned}
$$

Figure 2.2: The polarization tensor $\Gamma^2_{\mu\nu\varrho\sigma}$

In conclusion, for the total invariant amplitude we then have

$$\mathcal{M} \equiv \mathcal{M}_1 + \mathcal{M}_2 = \Gamma_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4)\varepsilon_1^{\mu}\varepsilon_2^{\nu}\varepsilon_3^{\varrho}\varepsilon_4^{\sigma}, \tag{2.36}$$

where

$$\Gamma_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) \equiv \Gamma^1_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) + \Gamma^2_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) \tag{2.37}$$

is the total polarization tensor, which is shown in Fig. 2.3.

$$\begin{aligned}
32\,(&g1\, p4^{\mu}\, p3^{\nu}\, p2^{\rho}\, p1^{\sigma} - 2\, g2\, p4^{\mu}\, p3^{\nu}\, p2^{\rho}\, p1^{\sigma} + g2\, p3^{\mu}\, p4^{\nu}\, p2^{\rho}\, p1^{\sigma} + \\
&g2\, p2^{\mu}\, p3^{\nu}\, p4^{\rho}\, p1^{\sigma} - g1\, p4^{\mu}\, g^{\nu\rho}\, p2\cdot p3\, p1^{\sigma} + 2\, g2\, p4^{\mu}\, g^{\nu\rho}\, p2\cdot p3\, p1^{\sigma} - \\
&g2\, g^{\mu\rho}\, p4^{\nu}\, p2\cdot p3\, p1^{\sigma} - g2\, g^{\mu\nu}\, p4^{\rho}\, p2\cdot p3\, p1^{\sigma} - g2\, p3^{\mu}\, g^{\nu\rho}\, p2\cdot p4\, p1^{\sigma} + \\
&g2\, g^{\mu\rho}\, p3^{\nu}\, p2\cdot p4\, p1^{\sigma} - g2\, p2^{\mu}\, g^{\nu\rho}\, p3\cdot p4\, p1^{\sigma} + g2\, g^{\mu\nu}\, p2^{\rho}\, p3\cdot p4\, p1^{\sigma} + \\
&g2\, p4^{\mu}\, p3^{\nu}\, p1^{\rho}\, p2^{\sigma} + g1\, p3^{\mu}\, p4^{\nu}\, p1^{\rho}\, p2^{\sigma} - 2\, g2\, p3^{\mu}\, p4^{\nu}\, p1^{\rho}\, p2^{\sigma} + \\
&g2\, p3^{\mu}\, p1^{\nu}\, p4^{\rho}\, p2^{\sigma} + g2\, p2^{\mu}\, p4^{\nu}\, p1^{\rho}\, p3^{\sigma} + g2\, p4^{\mu}\, p1^{\nu}\, p2^{\rho}\, p3^{\sigma} + \\
&g1\, p2^{\mu}\, p1^{\nu}\, p4^{\rho}\, p3^{\sigma} - 2\, g2\, p2^{\mu}\, p1^{\nu}\, p4^{\rho}\, p3^{\sigma} + g2\, p4^{\mu}\, p3^{\nu}\, g^{\rho\sigma}\, p1\cdot p2 + \\
&g2\, p3^{\mu}\, p4^{\nu}\, g^{\rho\sigma}\, p1\cdot p2 - g2\, p3^{\mu}\, g^{\nu\sigma}\, p4^{\rho}\, p1\cdot p2 - g2\, g^{\mu\sigma}\, p3^{\nu}\, p4^{\rho}\, p1\cdot p2 - \\
&g2\, p4^{\mu}\, g^{\nu\rho}\, p3^{\sigma}\, p1\cdot p2 - g2\, g^{\mu\rho}\, p4^{\nu}\, p3^{\sigma}\, p1\cdot p2 - g1\, g^{\mu\nu}\, p4^{\rho}\, p3^{\sigma}\, p1\cdot p2 + \\
&2\, g2\, g^{\mu\nu}\, p4^{\rho}\, p3^{\sigma}\, p1\cdot p2 - g2\, p2^{\mu}\, p4^{\nu}\, g^{\rho\sigma}\, p1\cdot p3 + g2\, p4^{\mu}\, g^{\nu\sigma}\, p2^{\rho}\, p1\cdot p3 - \\
&g2\, g^{\mu\sigma}\, p4^{\nu}\, p2^{\rho}\, p1\cdot p3 + g2\, p2^{\mu}\, g^{\nu\sigma}\, p4^{\rho}\, p1\cdot p3 - g2\, p4^{\mu}\, g^{\nu\rho}\, p2^{\sigma}\, p1\cdot p3 - \\
&g1\, g^{\mu\rho}\, p4^{\nu}\, p2^{\sigma}\, p1\cdot p3 + 2\, g2\, g^{\mu\rho}\, p4^{\nu}\, p2^{\sigma}\, p1\cdot p3 - g2\, g^{\mu\nu}\, p4^{\rho}\, p2^{\sigma}\, p1\cdot p3 - \\
&g2\, p2^{\mu}\, p3^{\nu}\, g^{\rho\sigma}\, p1\cdot p4 - g2\, p3^{\mu}\, g^{\nu\sigma}\, p2^{\rho}\, p1\cdot p4 - g1\, g^{\mu\sigma}\, p3^{\nu}\, p2^{\rho}\, p1\cdot p4 + \\
&2\, g2\, g^{\mu\sigma}\, p3^{\nu}\, p2^{\rho}\, p1\cdot p4 + g2\, p3^{\mu}\, g^{\nu\rho}\, p2^{\sigma}\, p1\cdot p4 - g2\, g^{\mu\rho}\, p3^{\nu}\, p2^{\sigma}\, p1\cdot p4 + \\
&g2\, p2^{\mu}\, g^{\nu\rho}\, p3^{\sigma}\, p1\cdot p4 - g2\, g^{\mu\nu}\, p2^{\rho}\, p3^{\sigma}\, p1\cdot p4 - g2\, p4^{\mu}\, p1^{\nu}\, g^{\rho\sigma}\, p2\cdot p3 - \\
&g2\, p4^{\mu}\, g^{\nu\sigma}\, p1^{\rho}\, p2\cdot p3 + g2\, g^{\mu\sigma}\, p4^{\nu}\, p1^{\rho}\, p2\cdot p3 + g2\, g^{\mu\sigma}\, p1^{\nu}\, p4^{\rho}\, p2\cdot p3 + \\
&g1\, g^{\mu\sigma}\, g^{\nu\rho}\, p1\cdot p4\, p2\cdot p3 - 2\, g2\, g^{\mu\sigma}\, g^{\nu\rho}\, p1\cdot p4\, p2\cdot p3 + \\
&g2\, g^{\mu\rho}\, g^{\nu\sigma}\, p1\cdot p4\, p2\cdot p3 + g2\, g^{\mu\nu}\, g^{\rho\sigma}\, p1\cdot p4\, p2\cdot p3 - \\
&g2\, p3^{\mu}\, p1^{\nu}\, g^{\rho\sigma}\, p2\cdot p4 - g1\, p3^{\mu}\, g^{\nu\sigma}\, p1^{\rho}\, p2\cdot p4 + 2\, g2\, p3^{\mu}\, g^{\nu\sigma}\, p1^{\rho}\, p2\cdot p4 - \\
&g2\, g^{\mu\sigma}\, p3^{\nu}\, p1^{\rho}\, p2\cdot p4 + g2\, g^{\mu\rho}\, p1^{\nu}\, p3^{\sigma}\, p2\cdot p4 - g2\, g^{\mu\nu}\, p1^{\rho}\, p3^{\sigma}\, p2\cdot p4 + \\
&g2\, g^{\mu\sigma}\, g^{\nu\rho}\, p1\cdot p3\, p2\cdot p4 + g1\, g^{\mu\rho}\, g^{\nu\sigma}\, p1\cdot p3\, p2\cdot p4 - \\
&2\, g2\, g^{\mu\rho}\, g^{\nu\sigma}\, p1\cdot p3\, p2\cdot p4 + g2\, g^{\mu\nu}\, g^{\rho\sigma}\, p1\cdot p3\, p2\cdot p4 - \\
&g1\, p2^{\mu}\, p1^{\nu}\, g^{\rho\sigma}\, p3\cdot p4 + 2\, g2\, p2^{\mu}\, p1^{\nu}\, g^{\rho\sigma}\, p3\cdot p4 - g2\, p2^{\mu}\, g^{\nu\sigma}\, p1^{\rho}\, p3\cdot p4 - \\
&g2\, g^{\mu\sigma}\, p1^{\nu}\, p2^{\rho}\, p3\cdot p4 - g2\, g^{\mu\rho}\, p1^{\nu}\, p2^{\sigma}\, p3\cdot p4 + g2\, g^{\mu\nu}\, p1^{\rho}\, p2^{\sigma}\, p3\cdot p4 + \\
&g2\, g^{\mu\sigma}\, g^{\nu\rho}\, p1\cdot p2\, p3\cdot p4 + g2\, g^{\mu\rho}\, g^{\nu\sigma}\, p1\cdot p2\, p3\cdot p4 + \\
&g1\, g^{\mu\nu}\, g^{\rho\sigma}\, p1\cdot p2\, p3\cdot p4 - 2\, g2\, g^{\mu\nu}\, g^{\rho\sigma}\, p1\cdot p2\, p3\cdot p4)
\end{aligned}$$

Figure 2.3: The total polarization tensor $\Gamma_{\mu\nu\varrho\sigma}$

The total polarization tensor can be also further rewritten by eliminating the momentum p_4 using (2.7) and (2.9), see Fig. 2.4.

$$
\begin{aligned}
32\,\big(& g1\, g^{\mu\nu} g^{\rho\sigma}\, p1\cdot p2^2 - g1\, p2^{\mu}\, p1^{\nu} g^{\rho\sigma}\, p1\cdot p2 + g1\, g^{\mu\nu} (p1^{\rho}+p2^{\rho}+p3^{\rho})\, p3^{\sigma}\, p1\cdot p2 + \\
& g1\, g^{\mu\rho} g^{\nu\sigma}\, p1\cdot p3^2 + g1\, g^{\mu\sigma} g^{\nu\rho}\, p2\cdot p3^2 - g1\,(p1^{\mu}+p2^{\mu}+p3^{\mu})\, p3^{\nu}\, p2^{\rho}\, p1^{\sigma} - \\
& g1\, p3^{\mu}\,(p1^{\nu}+p2^{\nu}+p3^{\nu})\, p1^{\rho}\, p2^{\sigma} - g1\, p2^{\mu}\, p1^{\nu}\,(p1^{\rho}+p2^{\rho}+p3^{\rho})\, p3^{\sigma} - \\
& g1\, p3^{\mu} g^{\nu\sigma}\, p1^{\rho}\, p1\cdot p3 + g1\, g^{\mu\rho}\,(p1^{\nu}+p2^{\nu}+p3^{\nu})\, p2^{\sigma}\, p1\cdot p3 - \\
& g1\, g^{\mu\sigma}\, p3^{\nu}\, p2^{\rho}\, p2\cdot p3 + g1\,(p1^{\mu}+p2^{\mu}+p3^{\mu})\, g^{\nu\rho}\, p1^{\sigma}\, p2\cdot p3 - \\
& g2\,\big(-g^{\mu\sigma} g^{\nu\rho}\, p1\cdot p2^2 - g^{\mu\rho} g^{\nu\sigma}\, p1\cdot p2^2 + 2\, g^{\mu\nu} g^{\rho\sigma}\, p1\cdot p2^2 - \\
&\quad 2\, p2^{\mu}\, p1^{\nu} g^{\rho\sigma}\, p1\cdot p2 + (p1^{\mu}+p2^{\mu}+p3^{\mu})\, p3^{\nu} g^{\rho\sigma}\, p1\cdot p2 + \\
&\quad p3^{\mu}\,(p1^{\nu}+p2^{\nu}+p3^{\nu})\, g^{\rho\sigma}\, p1\cdot p2 + p2^{\mu} g^{\nu\sigma}\, p1^{\rho}\, p1\cdot p2 + \\
&\quad g^{\mu\sigma}\, p1^{\nu}\, p2^{\rho}\, p1\cdot p2 - p3^{\mu} g^{\nu\sigma}\,(p1^{\rho}+p2^{\rho}+p3^{\rho})\, p1\cdot p2 - \\
&\quad g^{\mu\sigma}\, p3^{\nu}\,(p1^{\rho}+p2^{\rho}+p3^{\rho})\, p1\cdot p2 + p2^{\mu} g^{\nu\rho}\, p1^{\sigma}\, p1\cdot p2 - \\
&\quad g^{\mu\nu}\, p2^{\rho}\, p1^{\sigma}\, p1\cdot p2 + g^{\mu\rho}\, p1^{\nu}\, p2^{\sigma}\, p1\cdot p2 - g^{\mu\nu}\, p1^{\rho}\, p2^{\sigma}\, p1\cdot p2 - \\
&\quad (p1^{\mu}+p2^{\mu}+p3^{\mu})\, g^{\nu\rho}\, p3^{\sigma}\, p1\cdot p2 - g^{\mu\rho}\,(p1^{\nu}+p2^{\nu}+p3^{\nu})\, p3^{\sigma}\, p1\cdot p2 + \\
&\quad 2\, g^{\mu\nu}\,(p1^{\rho}+p2^{\rho}+p3^{\rho})\, p3^{\sigma}\, p1\cdot p2 - g^{\mu\sigma} g^{\nu\rho}\, p1\cdot p3^2 + \\
&\quad 2\, g^{\mu\rho} g^{\nu\sigma}\, p1\cdot p3^2 - g^{\mu\nu} g^{\rho\sigma}\, p1\cdot p3^2 + 2\, g^{\mu\sigma} g^{\nu\rho}\, p2\cdot p3^2 - \\
&\quad g^{\mu\rho} g^{\nu\sigma}\, p2\cdot p3^2 - g^{\mu\nu} g^{\rho\sigma}\, p2\cdot p3^2 - 2\,(p1^{\mu}+p2^{\mu}+p3^{\mu})\, p3^{\nu}\, p2^{\rho}\, p1^{\sigma} + \\
&\quad p3^{\mu}\,(p1^{\nu}+p2^{\nu}+p3^{\nu})\, p2^{\rho}\, p1^{\sigma} + p2^{\mu}\, p3^{\nu}\,(p1^{\rho}+p2^{\rho}+p3^{\rho})\, p1^{\sigma} + \\
&\quad (p1^{\mu}+p2^{\mu}+p3^{\mu})\, p3^{\nu}\, p1^{\rho}\, p2^{\sigma} - 2\, p3^{\mu}\,(p1^{\nu}+p2^{\nu}+p3^{\nu})\, p1^{\rho}\, p2^{\sigma} + \\
&\quad p3^{\mu}\, p1^{\nu}\,(p1^{\rho}+p2^{\rho}+p3^{\rho})\, p2^{\sigma} + p2^{\mu}\,(p1^{\nu}+p2^{\nu}+p3^{\nu})\, p1^{\rho}\, p3^{\sigma} + \\
&\quad (p1^{\mu}+p2^{\mu}+p3^{\mu})\, p1^{\nu}\, p2^{\rho}\, p3^{\sigma} - 2\, p2^{\mu}\, p1^{\nu}\,(p1^{\rho}+p2^{\rho}+p3^{\rho})\, p3^{\sigma} + \\
&\quad p3^{\mu}\, p1^{\nu} g^{\rho\sigma}\, p1\cdot p3 - p2^{\mu}\,(p1^{\nu}+p2^{\nu}+p3^{\nu})\, g^{\rho\sigma}\, p1\cdot p3 - \\
&\quad 2\, p3^{\mu} g^{\nu\sigma}\, p1^{\rho}\, p1\cdot p3 + g^{\mu\sigma}\, p3^{\nu}\, p1^{\rho}\, p1\cdot p3 + \\
&\quad (p1^{\mu}+p2^{\mu}+p3^{\mu})\, g^{\nu\sigma}\, p2^{\rho}\, p1\cdot p3 - g^{\mu\sigma}\,(p1^{\nu}+p2^{\nu}+p3^{\nu})\, p2^{\rho}\, p1\cdot p3 + \\
&\quad p2^{\mu} g^{\nu\sigma}\,(p1^{\rho}+p2^{\rho}+p3^{\rho})\, p1\cdot p3 + p3^{\mu} g^{\nu\rho}\, p1^{\sigma}\, p1\cdot p3 - \\
&\quad g^{\mu\rho}\, p3^{\nu}\, p1^{\sigma}\, p1\cdot p3 - (p1^{\mu}+p2^{\mu}+p3^{\mu})\, g^{\nu\rho}\, p2^{\sigma}\, p1\cdot p3 + \\
&\quad 2\, g^{\mu\rho}\,(p1^{\nu}+p2^{\nu}+p3^{\nu})\, p2^{\sigma}\, p1\cdot p3 - g^{\mu\nu}\,(p1^{\rho}+p2^{\rho}+p3^{\rho})\, p2^{\sigma}\, p1\cdot p3 - \\
&\quad g^{\mu\rho}\, p1^{\nu}\, p3^{\sigma}\, p1\cdot p3 + g^{\mu\nu}\, p1^{\rho}\, p3^{\sigma}\, p1\cdot p3 - (p1^{\mu}+p2^{\mu}+p3^{\mu})\, p1^{\nu} g^{\rho\sigma} \\
&\quad\; p2\cdot p3 + p2^{\mu}\, p3^{\nu} g^{\rho\sigma}\, p2\cdot p3 - (p1^{\mu}+p2^{\mu}+p3^{\mu})\, g^{\nu\sigma}\, p1^{\rho}\, p2\cdot p3 + \\
&\quad g^{\mu\sigma}\,(p1^{\nu}+p2^{\nu}+p3^{\nu})\, p1^{\rho}\, p2\cdot p3 + p3^{\mu} g^{\nu\sigma}\, p2^{\rho}\, p2\cdot p3 - \\
&\quad 2\, g^{\mu\sigma}\, p3^{\nu}\, p2^{\rho}\, p2\cdot p3 + g^{\mu\sigma}\, p1^{\nu}\,(p1^{\rho}+p2^{\rho}+p3^{\rho})\, p2\cdot p3 + \\
&\quad 2\,(p1^{\mu}+p2^{\mu}+p3^{\mu})\, g^{\nu\rho}\, p1^{\sigma}\, p2\cdot p3 - g^{\mu\rho}\,(p1^{\nu}+p2^{\nu}+p3^{\nu})\, p1^{\sigma}\, p2\cdot p3 - \\
&\quad g^{\mu\nu}\,(p1^{\rho}+p2^{\rho}+p3^{\rho})\, p1^{\sigma}\, p2\cdot p3 - p3^{\mu} g^{\nu\rho}\, p2^{\sigma}\, p2\cdot p3 + \\
&\quad g^{\mu\rho}\, p3^{\nu}\, p2^{\sigma}\, p2\cdot p3 - p2^{\mu} g^{\nu\rho}\, p3^{\sigma}\, p2\cdot p3 + g^{\mu\nu}\, p2^{\rho}\, p3^{\sigma}\, p2\cdot p3\big)\big)
\end{aligned}
$$

Figure 2.4: The total polarization tensor – the momentum p_4 eliminated

2.1.3 Transversality

Using (2.7) and (2.8) it can be checked (by hand or via `FeynCalc`) that both (2.22) and (2.34) are transverse, i.e.

$$
\begin{aligned}
\Gamma^1_{\mu\nu\varrho\sigma} p_1^{\mu} &= 0, & \Gamma^2_{\mu\nu\varrho\sigma} p_1^{\mu} &= 0 \\
\Gamma^1_{\mu\nu\varrho\sigma} p_2^{\nu} &= 0, & \Gamma^2_{\mu\nu\varrho\sigma} p_2^{\nu} &= 0 \\
\Gamma^1_{\mu\nu\varrho\sigma} p_3^{\varrho} &= 0, & \Gamma^2_{\mu\nu\varrho\sigma} p_3^{\varrho} &= 0 \\
\Gamma^1_{\mu\nu\varrho\sigma} p_4^{\sigma} &= 0, & \Gamma^2_{\mu\nu\varrho\sigma} p_4^{\sigma} &= 0
\end{aligned}
\tag{2.38}
$$

Obviously (2.37) is then transverse as well. This is quite natural since we are working with a gauge invariant theory.

2.1.4 Cross sections

Furthermore, we use the previous results to derive some formulae for differential and total cross sections respectively.

The general formula for elastic binary process differential cross section (in center-of-mass system) is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 s}, \tag{2.39}$$

where $|\mathcal{M}|^2$ is the square of the invariant amplitude and s is the square of the center-of-mass system energy [22].

We now need to compute $|\mathcal{M}|^2$. For the sake of simplicity, suppose that the photons are unpolarized (i.e. we are averaging over the initial polarizations and summing over the final ones). The unpolarized amplitude then reads

$$\overline{|\mathcal{M}|^2} = \frac{1}{4} \sum_{\substack{\lambda_i=1 \\ i=1,2,3,4}}^{2} \Gamma_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4)\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)\varepsilon_1^{\mu}\varepsilon_2^{\nu}\varepsilon_3^{\varrho}\varepsilon_4^{\sigma}\varepsilon_1^{*\alpha}\varepsilon_2^{*\beta}\varepsilon_3^{*\gamma}\varepsilon_4^{*\delta}. \tag{2.40}$$

The photon polarization sum has the following form [23]

$$P^{\mu\nu} \equiv \sum_{\lambda=1}^{2} \varepsilon^{\mu}(p, \lambda)\varepsilon^{*\nu}(p, \lambda) = -g^{\mu\nu} + \Lambda^{\mu\nu}(p), \tag{2.41}$$

where $\Lambda^{\mu\nu}$ is some longitudinal part (which is gauge-dependent). If $\Lambda^{\mu\nu}$ is contracted with any transverse tensor, the result is zero. Using this simple fact, together with the identities (2.38), we get the final answer

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\Gamma_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4)\Gamma^{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4). \tag{2.42}$$

Using `FeynCalc`, we find

$$\begin{aligned}
\overline{|\mathcal{M}|^2} = 512\{(5g_1^2 - 6g_1g_2 + 5g_2^2)[&(p_1 \cdot p_2)^2(p_3 \cdot p_4)^2 \\
&+ (p_1 \cdot p_3)^2(p_2 \cdot p_4)^2 \\
&+ (p_1 \cdot p_4)^2(p_2 \cdot p_3)^2] \\
- 4(g_1 - g_2)^2[&(p_1 \cdot p_2)(p_1 \cdot p_3)(p_2 \cdot p_4)(p_3 \cdot p_4) \\
&+ (p_1 \cdot p_3)(p_1 \cdot p_4)(p_2 \cdot p_3)(p_2 \cdot p_4) \\
&+ (p_1 \cdot p_2)(p_1 \cdot p_4)(p_2 \cdot p_3)(p_3 \cdot p_4)]\},
\end{aligned} \tag{2.43}$$

where we used (2.8). After elimination of all the scalar products containing p_4 via (2.9) and by subsequent application of (2.10), one gets

$$\overline{|\mathcal{M}|^2} = 1024(3g_1^2 - 2g_1g_2 + 3g_2^2)[(p_1 \cdot p_2)^2 + (p_1 \cdot p_3)^2 + (p_1 \cdot p_2)(p_1 \cdot p_3)]^2. \tag{2.44}$$

Now we can finally substitute (the signs might seem a bit weird since momenta p_1, p_2 are outgoing instead of incoming)

$$\begin{aligned} p_1 \cdot p_2 &= \frac{s}{2} \\ p_1 \cdot p_3 &= \frac{s(\cos\vartheta - 1)}{4}, \end{aligned} \tag{2.45}$$

which yields

$$\overline{|\mathcal{M}|^2} = s^4(3g_1^2 - 2g_1g_2 + 3g_2^2)(7 + \cos 2\vartheta)^2, \tag{2.46}$$

where ϑ is the angle between the spatial parts of p_1 and p_3. Then (2.39) implies

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{s^3(3g_1^2 - 2g_1g_2 + 3g_2^2)(7 + \cos 2\vartheta)^2}{64\pi^2} \\ &= \frac{s^3(3g_1^2 - 2g_1g_2 + 3g_2^2)(3 + \cos^2\vartheta)^2}{16\pi^2}. \end{aligned} \tag{2.47}$$

After angular integration, we end up with (the 1/2! prefactor comes from the fact that there are two identical particles in the final state)

$$\sigma = \frac{1}{2!}\int_{\mathcal{S}^2} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = \frac{7s^3(3g_1^2 - 2g_1g_2 + 3g_2^2)}{5\pi}. \tag{2.48}$$

In the next sections, we start studying our four-photon process in the context of various versions of QED and fitting the obtained amplitudes to the amplitude that we found in this section. This will enable us to find the values of the coupling constants in (2.5).

2.2 Case of spinor QED

The interaction Lagrangian of spinor QED reads[3]

$$\mathscr{L}_{\text{int.}} = -e\bar{\psi}\gamma^\mu A_\mu \psi = e\bar{\psi}\not{A}\psi, \tag{2.49}$$

where ψ is some fermionic field (electron, muon, quark, etc.), A_μ is the electromagnetic four-potential, and e is the coupling constant (electric charge).

It is self-evident from the Lagrangian, the lowest order contributions to our four-photon process are given by fourth-order box diagrams with one closed fermion loop (see Fig. 2.5).

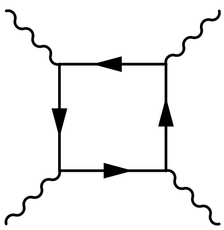

Figure 2.5: A box diagram

[3]The normal ordering symbol is omitted.

2.2.1 Initial analysis

Let us write down the expression for the relevant element of the S-matrix

$$\langle \mathrm{f}|S^{(4)}|\mathrm{i}\rangle = \frac{1}{4!}\int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3\mathrm{d}^4x_4\ \langle 0|\left[\prod_{i=1}^{4} a(p_i,\lambda_i)\right] \\ \mathcal{T}[\mathscr{L}_{\mathrm{int.}}(x_1)\mathscr{L}_{\mathrm{int.}}(x_2)\mathscr{L}_{\mathrm{int.}}(x_3)\mathscr{L}_{\mathrm{int.}}(x_4)]|0\rangle. \tag{2.50}$$

The integrand has the following structure

$$a_1a_2a_3a_4\ \mathcal{T}[\bar{\psi}_1\not{A}_1\psi_1\ \bar{\psi}_2\not{A}_2\psi_2\ \bar{\psi}_3\not{A}_3\psi_3\ \bar{\psi}_4\not{A}_4\psi_4], \tag{2.51}$$

where

$$\begin{aligned} a_i &\equiv a(p_i,\lambda_i)\\ \psi_j &\equiv \psi(x_j)\\ \not{A}_j &\equiv \not{A}(x_j). \end{aligned} \tag{2.52}$$

As we can see, there are $4! = 24$ possible Wick contractions of the photon annihilation operators with the A operators [see the decomposition (2.11)]. There are also 4! possible contractions of the fermionic operators $\bar{\psi},\psi$, however, only 6 of them form a full cycle. This gives us $24\times 6 = 144$ possible ways how to contract the operators.

They are not distinct though. In fact, there are only 6 topologically distinct diagrams and each one of them is contained 24 times in those 144. This factor $24 = 4!$ is exactly canceled by the Dyson expansion prefactor in (2.50).

If we denote the contribution of one of these diagrams by $\Theta_{\mu\nu\varrho\sigma}(p_1,p_2,p_3)$ [momentum p_4 is understood to be dependent on the other momenta through the conservation law (2.7)], then for the polarization tensor one has

$$\begin{aligned} \Gamma_{\mu\nu\varrho\sigma}(p_1,p_2,p_3) &= \Theta_{\mu\nu\varrho\sigma}(p_1,p_2,p_3) + \Theta_{\mu\varrho\nu\sigma}(p_1,p_3,p_2)\\ &\quad + \Theta_{\nu\mu\varrho\sigma}(p_2,p_1,p_3) + \Theta_{\nu\varrho\mu\sigma}(p_2,p_3,p_1)\\ &\quad + \Theta_{\varrho\mu\nu\sigma}(p_3,p_1,p_2) + \Theta_{\varrho\nu\mu\sigma}(p_3,p_2,p_1)\\ &= \mathrm{sym}_3\,\Theta_{\mu\nu\varrho|\sigma}(p_1,p_2,p_3), \end{aligned} \tag{2.53}$$

and the invariant amplitude of the process is given by contraction of the polarization tensor with the polarization vectors (in the same way as in the previous section).

This identity can be proved by writing down all the contractions and then performing various renaming of the variables x_i or swapping the momenta (and the polarizations). For instance, a contraction

$$a_1a_2a_3a_4\ \mathcal{T}[\bar{\psi}_1\not{A}_1\psi_1\ \bar{\psi}_2\not{A}_2\psi_2\ \bar{\psi}_3\not{A}_3\psi_3\ \bar{\psi}_4\not{A}_4\psi_4] \tag{2.54}$$

is the same as

$$a_1a_2a_3a_4\ \mathcal{T}[\bar{\psi}_1\not{A}_1\psi_1\ \bar{\psi}_2\not{A}_2\psi_2\ \bar{\psi}_3\not{A}_3\psi_3\ \bar{\psi}_4\not{A}_4\psi_4] \tag{2.55}$$

if we rename $x_1 \leftrightarrow x_2$. And it is also the same as

$$a_1 a_2 a_3 a_4 \ \mathcal{T}[\bar{\psi}_1 \not{A}_1 \psi_1 \ \bar{\psi}_2 \not{A}_2 \psi_2 \ \bar{\psi}_3 \not{A}_3 \psi_3 \ \bar{\psi}_4 \not{A}_4 \psi_4] \tag{2.56}$$

if we swap $\{p_1, \lambda_1\} \leftrightarrow \{p_2, \lambda_2\}$.

There is also another, sort of intuitive, approach. There are 144 possible ways how to contract the fermionic and photon operators. Each possible way gives us a contribution to the polarization tensor. As before, we denote one of those contributions by $\Theta_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4)$ (now we have also included the fourth momentum[4]).

Since the external lines are bosonic, we can guess that the polarization tensor must be some kind of symmetrization (meaning there are no sign changes). The easiest is to make total symmetrization of $\Theta_{\mu\nu\varrho\sigma}$, which consists of $4! = 24$ terms. It is also clear that all of the 144 contributions should be treated equally. Based on this, we can theorize that the correct formula should be (the $1/4!$ prefactor comes from the Dyson expansion)

$$\begin{aligned} \Gamma_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) &= \frac{6}{4!} \operatorname{sym}_4 \Theta_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) \\ &= \frac{1}{4} \operatorname{sym}_4 \Theta_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4). \end{aligned} \tag{2.57}$$

In other words, each permutation should contribute 6 times (because $144/24 = 6$). We will see later that this sort of handwaving argument actually works pretty well and gives correct results.

2.2.2 Ward identities

This section is more or less independent of the rest – we demonstrate that the polarization tensor is transverse, i.e. it satisfies the Ward identities. We directly prove the following identity (the remaining three identities can be proved similarly)

$$\Gamma_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) p_4^\sigma = 0. \tag{2.58}$$

First we show that the contribution of a box diagram is independent on the sense of the loop momentum circulation, which comes in handy later. The contribution of the diagram on the left (Fig. 2.6) is proportional to

$$T_{\mu\nu\varrho\sigma}(p_1, p_2, p_3) \equiv \int \mathrm{d}^4\ell \frac{\mathrm{Tr}[(\not{k}_1 + m)\gamma_\mu(\not{k}_2 + m)\gamma_\nu(\not{k}_3 + m)\gamma_\varrho(\not{k}_4 + m)\gamma_\sigma]}{\prod_{i=1}^4 (k_i^2 - m^2)}, \tag{2.59}$$

whereas the contribution of the diagram on the right is proportional to (R stands for reverse)

$$T^{\mathrm{R}}_{\mu\nu\varrho\sigma}(p_1, p_2, p_3) \equiv \int \mathrm{d}^4\ell \frac{\mathrm{Tr}[\gamma_\sigma(\not{k}'_4 + m)\gamma_\varrho(\not{k}'_3 + m)\gamma_\nu(\not{k}'_2 + m)\gamma_\mu(\not{k}'_1 + m)]}{\prod_{i=1}^4 (k_i'^2 - m^2)}, \tag{2.60}$$

[4]Even though the fourth momentum might not be present in the particular formula, we can still consider it a function of the momentum.

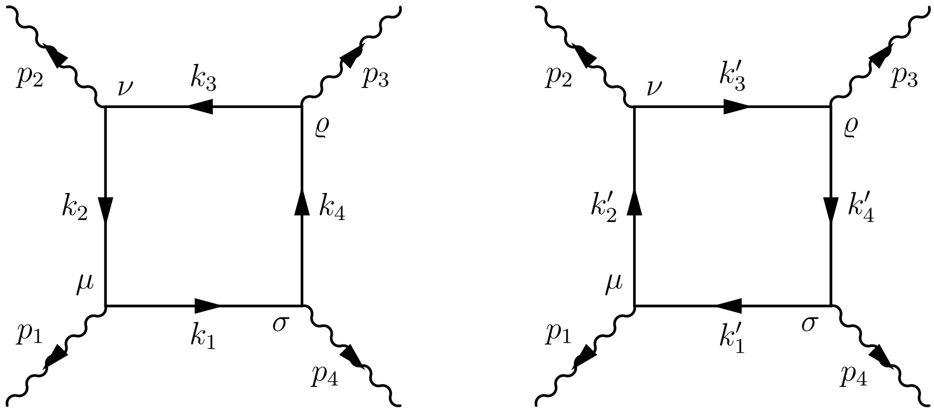

Figure 2.6: Box diagrams – the loop momentum circulating counter-clockwise and clockwise

where

$$\begin{aligned} k_1 &\equiv \ell \\ k_2 &\equiv \ell + p_1 \\ k_3 &\equiv \ell + p_1 + p_2 \\ k_4 &\equiv \ell + p_1 + p_2 + p_3 \end{aligned} \tag{2.61}$$

and

$$\begin{aligned} k'_1 &\equiv \ell \\ k'_2 &\equiv \ell - p_1 \\ k'_3 &\equiv \ell - p_1 - p_2 \\ k'_4 &\equiv \ell - p_1 - p_2 - p_3. \end{aligned} \tag{2.62}$$

Replacing $\ell \to -\ell$ in (2.60) and then using identity[5]

$$\mathrm{Tr}[\gamma_\alpha \gamma_\beta \cdots \gamma_\psi \gamma_\omega] = \mathrm{Tr}[\gamma_\omega \gamma_\psi \cdots \gamma_\beta \gamma_\alpha] \tag{2.63}$$

we obtain

$$T^{\mathrm{R}}_{\mu\nu\varrho\sigma}(p_1, p_2, p_3) = \int \mathrm{d}^4\ell \frac{\mathrm{Tr}[(\not{k}_1 - m)\gamma_\mu(\not{k}_2 - m)\gamma_\nu(\not{k}_3 - m)\gamma_\varrho(\not{k}_4 - m)\gamma_\sigma]}{\prod_{i=1}^4 (k_i^2 - m^2)}. \tag{2.64}$$

Imagine the expansion of the traces in (2.59) and (2.64) and compare the terms. Since $(-1)^n = 1$ for n even, we can see that the terms proportional to an even power of m are identical. On the other hand, the terms proportional to an odd power of m are zero because a trace of an odd number of gamma matrices is zero. Therefore

$$T_{\mu\nu\varrho\sigma}(p_1, p_2, p_3) = T^{\mathrm{R}}_{\mu\nu\varrho\sigma}(p_1, p_2, p_3). \tag{2.65}$$

Now we show that (2.53) can be actually reduced to the following form

$$\begin{aligned} \Gamma_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) = 2[&\Theta_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) \\ &+ \Theta_{\nu\varrho\mu\sigma}(p_2, p_3, p_1, p_4) \\ &+ \Theta_{\varrho\mu\nu\sigma}(p_3, p_1, p_2, p_4)]. \end{aligned} \tag{2.66}$$

[5]This identity can be proved via identity $C\gamma_\mu C^{-1} = -\gamma_\mu^{\mathrm{T}}$ and some basic linear algebra. The trace identity holds for the even and the odd number of gamma matrices.

For the sake of brevity, we use the tensor T instead of the tensor Θ (the tensors T and Θ are proportional to each other).

Let us compare the (321) permutation

$$
\begin{aligned}
&T_{\varrho\nu\mu\sigma}(p_3,p_2,p_1) = \\
&= \int \mathrm{d}^4\ell\, \mathrm{Tr}\left[\frac{1}{\not\ell - m}\gamma_\varrho \frac{1}{\not\ell + \not p_3 - m}\gamma_\nu \frac{1}{\not\ell + \not p_3 + \not p_2 - m}\gamma_\mu \frac{1}{\not\ell + \not p_3 + \not p_2 + \not p_1 - m}\gamma_\sigma\right]
\end{aligned}
\tag{2.67}
$$

with the (123) permutation reversed

$$
\begin{aligned}
&T^{\mathrm{R}}_{\mu\nu\varrho\sigma}(p_1,p_2,p_3) = \\
&= \int \mathrm{d}^4\ell\, \mathrm{Tr}\left[\frac{1}{\not\ell - \not p_1 - \not p_2 - \not p_3 - m}\gamma_\varrho \frac{1}{\not\ell - \not p_1 - \not p_2 - m}\gamma_\nu \frac{1}{\not\ell - \not p_1 - m}\gamma_\mu \frac{1}{\not\ell - m}\gamma_\sigma\right].
\end{aligned}
\tag{2.68}
$$

Shifting the loop momentum in the second expression $\ell \to \ell + p_1 + p_2 + p_3$ exactly reproduces the first one. Thanks to (2.65) we therefore have

$$
T_{\varrho\nu\mu\sigma}(p_3,p_2,p_1) = T_{\mu\nu\varrho\sigma}(p_1,p_2,p_3). \tag{2.69}
$$

Cyclic permutations

$$
\{p_1,\mu\} \to \{p_2,\nu\} \to \{p_3,\varrho\} \to \{p_1,\mu\} \tag{2.70}
$$

in (2.67) and (2.68) then yields the remaining two identities

$$
\begin{aligned}
T_{\mu\varrho\nu\sigma}(p_1,p_3,p_2) &= T_{\nu\varrho\mu\sigma}(p_2,p_3,p_1) \\
T_{\nu\mu\varrho\sigma}(p_2,p_1,p_3) &= T_{\varrho\mu\nu\sigma}(p_3,p_1,p_2).
\end{aligned}
\tag{2.71}
$$

These equalities and (2.53) implies (2.66).

Finally we are ready to prove the Ward identity (2.58). We have

$$
\begin{aligned}
&T_{\mu\nu\varrho\sigma}(p_1,p_2,p_3)p_4^\sigma = \\
&= \int \mathrm{d}^4\ell\, \mathrm{Tr}\left[\frac{1}{\not\ell - m}\gamma_\mu \frac{1}{\not\ell + \not p_1 - m}\gamma_\nu \frac{1}{\not\ell + \not p_1 + \not p_2 - m}\gamma_\varrho \frac{1}{\not\ell + \not p_1 + \not p_2 + \not p_3 - m}\not p_4\right] \\
&= \int \mathrm{d}^4\ell\, \mathrm{Tr}\left[\gamma_\mu \frac{1}{\not\ell + \not p_1 - m}\gamma_\nu \frac{1}{\not\ell + \not p_1 + \not p_2 - m}\gamma_\varrho \frac{1}{\not\ell + \not p_1 + \not p_2 + \not p_3 - m}\right] \\
&\quad - \int \mathrm{d}^4\ell\, \mathrm{Tr}\left[\frac{1}{\not\ell - m}\gamma_\mu \frac{1}{\not\ell + \not p_1 - m}\gamma_\nu \frac{1}{\not\ell + \not p_1 + \not p_2 - m}\gamma_\varrho\right],
\end{aligned}
\tag{2.72}
$$

where we used

$$
\not p_4 = (\not\ell - m) - (\not\ell + \not p_1 + \not p_2 + \not p_3 - m) \tag{2.73}
$$

and trace cyclicity. Again, using (2.70) we obtain the remaining two equalities

$$\begin{aligned} &T_{\nu\varrho\mu\sigma}(p_2,p_3,p_1)p_4^\sigma = \\ &= \int \mathrm{d}^4\ell\, \mathrm{Tr}\left[\gamma_\nu \frac{1}{\not\ell+\not p_2-m}\gamma_\varrho \frac{1}{\not\ell+\not p_2+\not p_3-m}\gamma_\mu \frac{1}{\not\ell+\not p_2+\not p_3+\not p_1-m}\right] \\ &\quad - \int \mathrm{d}^4\ell\, \mathrm{Tr}\left[\frac{1}{\not\ell-m}\gamma_\nu \frac{1}{\not\ell+\not p_2-m}\gamma_\varrho \frac{1}{\not\ell+\not p_2+\not p_3-m}\gamma_\mu\right] \end{aligned} \tag{2.74}$$

$$\begin{aligned} &T_{\varrho\mu\nu\sigma}(p_3,p_1,p_2)p_4^\sigma = \\ &= \int \mathrm{d}^4\ell\, \mathrm{Tr}\left[\gamma_\varrho \frac{1}{\not\ell+\not p_3-m}\gamma_\mu \frac{1}{\not\ell+\not p_3+\not p_1-m}\gamma_\nu \frac{1}{\not\ell+\not p_3+\not p_1+\not p_2-m}\right] \\ &\quad - \int \mathrm{d}^4\ell\, \mathrm{Tr}\left[\frac{1}{\not\ell-m}\gamma_\varrho \frac{1}{\not\ell+\not p_3-m}\gamma_\mu \frac{1}{\not\ell+\not p_3+\not p_1-m}\gamma_\nu\right]. \end{aligned} \tag{2.75}$$

The rest of the proof is now easy. If we shift the loop momentum $\ell \to \ell - p_1$ in the first term in (2.72), we can see that term is exactly canceled by the second term in (2.74). Similarly, the first term in (2.74) is canceled by the second term in (2.75) (after shifting $\ell \to \ell - p_2$). And finally, the first term in (2.75) is canceled by the second term in (2.72) (shifting $\ell \to \ell - p_3$). Therefore, if we sum all of the terms up, the result is zero.

2.2.3 Diagram parametrization

We are interested in the evaluation of the box diagram in Fig. 2.7 in the low energy limit and a subsequent calculation of the polarization tensor.

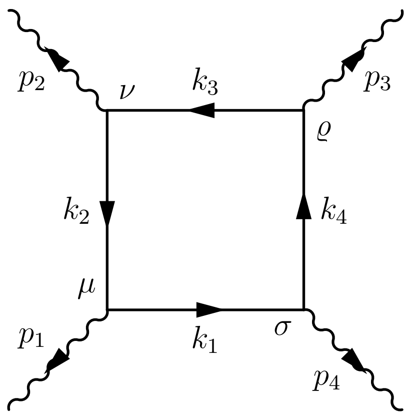

Figure 2.7: The fermion box diagram

The evaluation is carried out using dimensional regularization. The contribution of the diagram is obtained using the standard Feynman rules

$$\begin{aligned} &i\Theta_{\mu\nu\varrho\sigma}(p_1,p_2,p_3) \equiv \\ &\quad - e^4\mu^{4-D}\int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{\mathrm{Tr}[(\not k_1+m)\gamma_\mu(\not k_2+m)\gamma_\nu(\not k_3+m)\gamma_\varrho(\not k_4+m)\gamma_\sigma]}{\prod_{i=1}^4(k_i^2-m^2)}, \end{aligned} \tag{2.76}$$

where k_i are defined by (2.61) and μ is an auxiliary scale used in the regularization technique.

By employing the Feynman parametrization, we get

$$
\begin{aligned}
&i\Theta_{\mu\nu\varrho\sigma}(p_1,p_2,p_3) = \\
&-e^4\mu^{4-D}\int_{\mathrm{F}_3}\mathrm{d}X_{\mathrm{F}_3}\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{\mathrm{Tr}[(\not{k}_1+m)\gamma_\mu(\not{k}_2+m)\gamma_\nu(\not{k}_3+m)\gamma_\varrho(\not{k}_4+m)\gamma_\sigma]}{[(k_2^2-k_1^2)x+(k_3^2-k_1^2)y+(k_4^2-k_1^2)z+k_1^2-m^2]^4},
\end{aligned}
\tag{2.77}
$$

where

$$
\int_{\mathrm{F}_3}\mathrm{d}X_{\mathrm{F}_3}\equiv 6\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int_0^{1-x-y}\mathrm{d}z. \tag{2.78}
$$

Let us denote by R the inside of the bracket in the denominator. Application of (2.8) and (2.61) yields

$$
\begin{aligned}
R = {} & 2x(\ell\cdot p_1)+2y[(\ell\cdot p_1)+(\ell\cdot p_2)+(p_1\cdot p_2)] \\
& +2z[(\ell\cdot p_1)+(\ell\cdot p_2)+(\ell\cdot p_3)+(p_1\cdot p_2)+(p_1\cdot p_3)+(p_2\cdot p_3)] \\
& +\ell^2-m^2.
\end{aligned}
\tag{2.79}
$$

Introduce a shifted loop momentum

$$
\ell'\equiv\ell+S,\qquad S\equiv(x+y+z)p_1+(y+z)p_2+zp_3. \tag{2.80}
$$

It holds

$$
R=\ell'^2-C, \tag{2.81}
$$

where

$$
\begin{aligned}
C\equiv\ell'^2-R &= S^2-2y(p_1\cdot p_2)-2z[(p_1\cdot p_2)+(p_1\cdot p_3)+(p_2\cdot p_3)]+m^2 \\
&= 2[(x+y+z)(y+z)-y-z](p_1\cdot p_2) \\
&\quad +2[z(x+y+z)-z](p_1\cdot p_3) \\
&\quad +2[z(y+z)-z](p_2\cdot p_3) \\
&\quad +m^2.
\end{aligned}
\tag{2.82}
$$

We can write

$$
C=m^2+\sum_{i<j}^{3}f_{ij}(x,y,z)(p_i\cdot p_j), \tag{2.83}
$$

where the following functions have been defined

$$
\begin{aligned}
f_{12}(x,y,z)&\equiv 2(y+z)(x+y+z-1) \\
f_{13}(x,y,z)&\equiv 2z(x+y+z-1) \\
f_{23}(x,y,z)&\equiv 2z(y+z-1).
\end{aligned}
\tag{2.84}
$$

Or alternatively

$$
C=m^2(1+\lambda), \tag{2.85}
$$

where

$$
\lambda\equiv\sum_{i<j}^{3}f_{ij}(x,y,z)\frac{p_i\cdot p_j}{m^2}. \tag{2.86}
$$

By renaming the momentum $\ell' \to \ell$, we finally obtain

$$i\Theta_{\mu\nu\varrho\sigma}(p_1,p_2,p_3) = -e^4\mu^{4-D}\int_{\mathrm{F}_3}\mathrm{d}X_{\mathrm{F}_3}$$
$$\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{\mathrm{Tr}[(\not\ell+\not q_1+m)\gamma_\mu(\not\ell+\not q_2+m)\gamma_\nu(\not\ell+\not q_3+m)\gamma_\varrho(\not\ell+\not q_4+m)\gamma_\sigma]}{(\ell^2-C)^4}, \tag{2.87}$$

where we have defined

$$\begin{aligned}
q_1 &\equiv -(x+y+z)p_1-(y+z)p_2-zp_3\\
q_2 &\equiv -(x+y+z-1)p_1-(y+z)p_2-zp_3\\
q_3 &\equiv -(x+y+z-1)p_1-(y+z-1)p_2-zp_3\\
q_4 &\equiv -(x+y+z-1)p_1-(y+z-1)p_2-(z-1)p_3,
\end{aligned} \tag{2.88}$$

all generally

$$q_i = \sum_{j=1}^{3}\phi_{ij}(x,y,z)p_j, \tag{2.89}$$

where

$$\phi_{ij}(x,y,z) \equiv -\begin{pmatrix} x+y+z & y+z & z\\ x+y+z-1 & y+z & z\\ x+y+z-1 & y+z-1 & z\\ x+y+z-1 & y+z-1 & z-1\end{pmatrix}. \tag{2.90}$$

2.2.4 Diagram evaluation

We now break the trace in the numerator of our integral expression into individual terms and evaluate each term separately. The following master formula for loop integrations [22] is used

$$\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{(\ell^2)^r}{(\ell^2-C)^s} = \frac{i(-1)^{r-s}C^{r-s+D/2}}{(4\pi)^{D/2}}\frac{\Gamma\left(r+\frac{D}{2}\right)\Gamma\left(s-r-\frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)\Gamma(s)}. \tag{2.91}$$

The low energy limit is performed after the loop integrations (but before the integration over the Feynman parameters).

We begin with the $\ell\ell\ell\ell$ term

$$T_{\ell\ell\ell\ell} \equiv \mathrm{Tr}[\not\ell\gamma_\mu\not\ell\gamma_\nu\not\ell\gamma_\varrho\not\ell\gamma_\sigma] = \mathrm{Tr}[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\varrho\gamma_\delta\gamma_\sigma]\ell^\alpha\ell^\beta\ell^\gamma\ell^\delta \tag{2.92}$$

in the trace. Under the loop integration we may effectively set (the so-called symmetric integration)

$$\ell_\alpha\ell_\beta\ell_\gamma\ell_\delta \overset{\text{eff.}}{=} \frac{(\ell^2)^2}{D(D+2)}(g_{\alpha\beta}g_{\gamma\delta}+g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma}). \tag{2.93}$$

The trace was computed in `FeynCalc` and the result is

$$\tilde{T}_{\ell\ell\ell\ell} = \frac{4(D-2)(Dg_{\mu\sigma}g_{\nu\varrho}-Dg_{\mu\varrho}g_{\nu\sigma}+Dg_{\mu\nu}g_{\varrho\sigma}-4g_{\mu\varrho}g_{\nu\sigma})}{D(D+2)}, \tag{2.94}$$

where

$$(\ell^2)^2 \tilde{T}_{\ell\ell\ell\ell} \equiv T_{\ell\ell\ell\ell}. \tag{2.95}$$

Since $s - r = 4 - 2 = 2$, the corresponding loop integral is logarithmically divergent and thus we can not set $D = 4$. After setting $D = 4 - 2\varepsilon$, the trace can be expanded in powers of ε

$$\tilde{T}_{\ell\ell\ell\ell} = A + B\varepsilon + \mathcal{O}\left(\varepsilon^2\right), \tag{2.96}$$

where

$$\begin{aligned} A &\equiv \frac{4}{3}(g_{\mu\sigma}g_{\nu\rho} - 2g_{\mu\rho}g_{\nu\sigma} + g_{\mu\nu}g_{\rho\sigma}) \\ B &\equiv -\frac{2}{9}(4g_{\mu\sigma}g_{\nu\rho} - 5g_{\mu\rho}g_{\nu\sigma} + 4g_{\mu\nu}g_{\rho\sigma}). \end{aligned} \tag{2.97}$$

We are in fact interested in the following expression [see (2.87)]

$$i\bar{\Theta}^{\ell\ell\ell\ell}_{\mu\nu\varrho\sigma} \equiv -e^4\mu^{4-D}\tilde{T}_{\ell\ell\ell\ell}\int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{(\ell^2)^2}{(\ell^2 - C)^4}. \tag{2.98}$$

The bar over Θ means that we are ignoring integration over the Feynman paremeters $\int_{\mathrm{F}_3} \mathrm{d}X_{\mathrm{F}_3}$ for now. Via (2.87) and after expanding all the functions in powers of ε, we get [$\mathcal{O}(\varepsilon)$ terms are omitted]

$$\bar{\Theta}^{\ell\ell\ell\ell}_{\mu\nu\varrho\sigma} = -\frac{e^4}{16\pi^2}\left\{A\left[\Delta - \frac{5}{6} - \ln(1+\lambda)\right] + B\right\}, \tag{2.99}$$

where Δ is defined by

$$\Delta \equiv \frac{1}{\varepsilon} - \gamma_{\mathrm{E}} - \ln\left(\frac{m^2}{4\pi\mu^2}\right). \tag{2.100}$$

We also used the definition of C (2.85) and the expansion of Γ-function

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_{\mathrm{E}} + \mathcal{O}(\varepsilon). \tag{2.101}$$

Finally, let us consider the low energy limit. Suppose the photon energies are small compared to the mass m (i.e. $E_i \ll m$). This means

$$p_i \cdot p_j = E_i E_j(\cos\vartheta - 1) \ll m^2, \qquad i \neq j. \tag{2.102}$$

Scalar products are contained within the parameter λ (2.86). The low energy limit therefore means that the parameter λ is sufficiently small compared to unity. To perform the limit, we expand (2.99) in powers of λ to obtain

$$\begin{aligned} \bar{\Theta}^{\ell\ell\ell\ell}_{\mu\nu\varrho\sigma} =& \frac{e^4}{24\pi^2}[(3-2\Delta)g_{\mu\sigma}g_{\nu\rho} - (5-4\Delta)g_{\mu\rho}g_{\nu\sigma} + (3-2\Delta)g_{\mu\nu}g_{\rho\sigma}] \\ &+ \frac{e^4}{12\pi^2}(g_{\mu\sigma}g_{\nu\rho} - 2g_{\mu\rho}g_{\nu\sigma} + g_{\mu\nu}g_{\rho\sigma})(2\lambda - \lambda^2), \end{aligned} \tag{2.103}$$

where powers of λ higher than 2 have been omitted.

The terms proportional to higher powers of λ are not needed since they contain far too many momenta. It the end, we would like to match the calculated polarization tensor with the effective one (derived in the previous section). The effective polarization tensor is formed from terms that are constructed of four momenta. The parameter λ contains two momenta already, and that means that λ^2 contains four.

The calculation of the remaining terms is straightforward because the corresponding loop integrals are convergent for $D = 4$. The next terms we are going to work with are of type $\ell\ell mm$. There are six of them – we may collect all the traces into one term

$$\begin{aligned} T_{\ell\ell mm} \equiv m^2 (&\mathrm{Tr}[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\varrho\gamma_\sigma] \\ &+ \mathrm{Tr}[\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\beta\gamma_\varrho\gamma_\sigma] \\ &+ \mathrm{Tr}[\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\varrho\gamma_\beta\gamma_\sigma] \\ &+ \mathrm{Tr}[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\varrho\gamma_\sigma] \\ &+ \mathrm{Tr}[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\varrho\gamma_\beta\gamma_\sigma] \\ &+ \mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\varrho\gamma_\beta\gamma_\sigma])\ell^\alpha\ell^\beta. \end{aligned} \tag{2.104}$$

The symmetric integration

$$\ell_\alpha\ell_\beta \stackrel{\text{eff.}}{=} \frac{\ell^2}{4} g_{\alpha\beta} \tag{2.105}$$

and `FeynCalc` yields

$$\tilde{T}_{\ell\ell mm} = 4m^2(-g_{\mu\sigma}g_{\nu\rho} + 2g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma}), \tag{2.106}$$

where again

$$\ell^2\tilde{T}_{\ell\ell mm} \equiv T_{\ell\ell mm}. \tag{2.107}$$

So

$$i\bar{\Theta}^{\ell\ell mm}_{\mu\nu\varrho\sigma} \equiv -e^4\tilde{T}_{\ell\ell mm} \int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2 - C)^4}. \tag{2.108}$$

We then directly use (2.91) and the definition of C to get

$$\bar{\Theta}^{\ell\ell mm}_{\mu\nu\varrho\sigma} = \frac{e^4}{12\pi^2(1+\lambda)}(-g_{\mu\sigma}g_{\nu\rho} + 2g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma}). \tag{2.109}$$

By expanding in powers of λ we obtain

$$\bar{\Theta}^{\ell\ell mm}_{\mu\nu\varrho\sigma} = \frac{e^4}{12\pi^2}(-g_{\mu\sigma}g_{\nu\rho} + 2g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma})(1 - \lambda + \lambda^2). \tag{2.110}$$

The term of type $mmmm$ is the easiest to process of all of them. Analogously, we obtain for it

$$\bar{\Theta}^{mmmm}_{\mu\nu\varrho\sigma} \equiv -\frac{e^4}{24\pi^2(1+\lambda)^2}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\nu}g_{\rho\sigma}) \tag{2.111}$$

and then

$$\bar{\Theta}^{mmmm}_{\mu\nu\varrho\sigma} = -\frac{e^4}{24\pi^2}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\nu}g_{\rho\sigma})(1 - 2\lambda + 3\lambda^2). \tag{2.112}$$

The remaining terms are far more complicated. The next set of terms are terms of type $\ell\ell qq$. Again, there are six of them. All the traces combined can be written as

$$
\begin{aligned}
T_{\ell\ell qq} \equiv \mathrm{Tr}[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\varrho\gamma_\delta\gamma_\sigma(&\ell^\alpha\ell^\beta q_3^\gamma q_4^\delta \\
&+ \ell^\alpha q_2^\beta \ell^\gamma q_4^\delta \\
&+ \ell^\alpha q_2^\beta q_3^\gamma \ell^\delta \\
&+ q_1^\alpha \ell^\beta \ell^\gamma q_4^\delta \\
&+ q_1^\alpha \ell^\beta q_3^\gamma \ell^\delta \\
&+ q_1^\alpha q_2^\beta \ell^\gamma \ell^\delta). \qquad (2.113)
\end{aligned}
$$

Using the symmetric integration (2.105) and `FeynCalc`, the expression $\tilde{T}_{\ell\ell qq}$ is obtained, where

$$\ell^2\tilde{T}_{\ell\ell qq} \equiv T_{\ell\ell qq}. \qquad (2.114)$$

The expression $\tilde{T}_{\ell\ell qq}$ is shown in Fig. 2.8. The vectors q_i are given by (2.89).

$$
\begin{aligned}
-2\,(&q2^\mu\, q1^\nu\, g^{\rho\sigma} + q3^\mu\, q1^\nu\, g^{\rho\sigma} + q4^\mu\, q1^\nu\, g^{\rho\sigma} + q1^\mu\, q2^\nu\, g^{\rho\sigma} + \\
&q3^\mu\, q2^\nu\, g^{\rho\sigma} - q4^\mu\, q2^\nu\, g^{\rho\sigma} + q1^\mu\, q3^\nu\, g^{\rho\sigma} + q2^\mu\, q3^\nu\, g^{\rho\sigma} - \\
&q4^\mu\, q3^\nu\, g^{\rho\sigma} - q1^\mu\, q4^\nu\, g^{\rho\sigma} - q2^\mu\, q4^\nu\, g^{\rho\sigma} + q3^\mu\, q4^\nu\, g^{\rho\sigma} - \\
&g^{\mu\nu}\, q1\cdot q2\, g^{\rho\sigma} - g^{\mu\nu}\, q1\cdot q3\, g^{\rho\sigma} - g^{\mu\nu}\, q1\cdot q4\, g^{\rho\sigma} - \\
&g^{\mu\nu}\, q2\cdot q3\, g^{\rho\sigma} + g^{\mu\nu}\, q2\cdot q4\, g^{\rho\sigma} - g^{\mu\nu}\, q3\cdot q4\, g^{\rho\sigma} - \\
&q2^\mu\, g^{\nu\sigma}\, q1^\rho - q3^\mu\, g^{\nu\sigma}\, q1^\rho - q4^\mu\, g^{\nu\sigma}\, q1^\rho - g^{\mu\sigma}\, q2^\nu\, q1^\rho - \\
&g^{\mu\sigma}\, q3^\nu\, q1^\rho + g^{\mu\sigma}\, q4^\nu\, q1^\rho - q1^\mu\, g^{\nu\sigma}\, q2^\rho - q3^\mu\, g^{\nu\sigma}\, q2^\rho - \\
&q4^\mu\, g^{\nu\sigma}\, q2^\rho + g^{\mu\sigma}\, q1^\nu\, q2^\rho + g^{\mu\sigma}\, q3^\nu\, q2^\rho + g^{\mu\sigma}\, q4^\nu\, q2^\rho + \\
&q1^\mu\, g^{\nu\sigma}\, q3^\rho + q2^\mu\, g^{\nu\sigma}\, q3^\rho - q4^\mu\, g^{\nu\sigma}\, q3^\rho - g^{\mu\sigma}\, q1^\nu\, q3^\rho + \\
&g^{\mu\sigma}\, q2^\nu\, q3^\rho + g^{\mu\sigma}\, q4^\nu\, q3^\rho + q1^\mu\, g^{\nu\sigma}\, q4^\rho + q2^\mu\, g^{\nu\sigma}\, q4^\rho - \\
&q3^\mu\, g^{\nu\sigma}\, q4^\rho - g^{\mu\sigma}\, q1^\nu\, q4^\rho + g^{\mu\sigma}\, q2^\nu\, q4^\rho + g^{\mu\sigma}\, q3^\nu\, q4^\rho + \\
&q2^\mu\, g^{\nu\rho}\, q1^\sigma - q3^\mu\, g^{\nu\rho}\, q1^\sigma + q4^\mu\, g^{\nu\rho}\, q1^\sigma + g^{\mu\rho}\, q2^\nu\, q1^\sigma + \\
&g^{\mu\rho}\, q3^\nu\, q1^\sigma - g^{\mu\rho}\, q4^\nu\, q1^\sigma - g^{\mu\nu}\, q2^\rho\, q1^\sigma + g^{\mu\nu}\, q3^\rho\, q1^\sigma + \\
&g^{\mu\nu}\, q4^\rho\, q1^\sigma + q1^\mu\, g^{\nu\rho}\, q2^\sigma + q3^\mu\, g^{\nu\rho}\, q2^\sigma + q4^\mu\, g^{\nu\rho}\, q2^\sigma - \\
&g^{\mu\rho}\, q1^\nu\, q2^\sigma - g^{\mu\rho}\, q3^\nu\, q2^\sigma - g^{\mu\rho}\, q4^\nu\, q2^\sigma + g^{\mu\nu}\, q1^\rho\, q2^\sigma - \\
&g^{\mu\nu}\, q3^\rho\, q2^\sigma - g^{\mu\nu}\, q4^\rho\, q2^\sigma - q1^\mu\, g^{\nu\rho}\, q3^\sigma - q2^\mu\, g^{\nu\rho}\, q3^\sigma + \\
&q4^\mu\, g^{\nu\rho}\, q3^\sigma - g^{\mu\rho}\, q1^\nu\, q3^\sigma - g^{\mu\rho}\, q2^\nu\, q3^\sigma - g^{\mu\rho}\, q4^\nu\, q3^\sigma + \\
&g^{\mu\nu}\, q1^\rho\, q3^\sigma + g^{\mu\nu}\, q2^\rho\, q3^\sigma + g^{\mu\nu}\, q4^\rho\, q3^\sigma + q1^\mu\, g^{\nu\rho}\, q4^\sigma + \\
&q2^\mu\, g^{\nu\rho}\, q4^\sigma - q3^\mu\, g^{\nu\rho}\, q4^\sigma - g^{\mu\rho}\, q1^\nu\, q4^\sigma + g^{\mu\rho}\, q2^\nu\, q4^\sigma + \\
&g^{\mu\rho}\, q3^\nu\, q4^\sigma + g^{\mu\nu}\, q1^\rho\, q4^\sigma - g^{\mu\nu}\, q2^\rho\, q4^\sigma + g^{\mu\nu}\, q3^\rho\, q4^\sigma - \\
&g^{\mu\sigma}\, g^{\nu\rho}\, q1\cdot q2 + g^{\mu\rho}\, g^{\nu\sigma}\, q1\cdot q2 + g^{\mu\sigma}\, g^{\nu\rho}\, q1\cdot q3 + \\
&g^{\mu\rho}\, g^{\nu\sigma}\, q1\cdot q3 - g^{\mu\sigma}\, g^{\nu\rho}\, q1\cdot q4 + g^{\mu\rho}\, g^{\nu\sigma}\, q1\cdot q4 - \\
&g^{\mu\sigma}\, g^{\nu\rho}\, q2\cdot q3 + g^{\mu\rho}\, g^{\nu\sigma}\, q2\cdot q3 - g^{\mu\sigma}\, g^{\nu\rho}\, q2\cdot q4 + \\
&g^{\mu\rho}\, g^{\nu\sigma}\, q2\cdot q4 - g^{\mu\sigma}\, g^{\nu\rho}\, q3\cdot q4 + g^{\mu\rho}\, g^{\nu\sigma}\, q3\cdot q4)
\end{aligned}
$$

Figure 2.8: The expression $\tilde{T}_{\ell\ell qq}$

After substituting them into the expression, the expression gets much longer. As always, by applying the master formula (2.91), we find

$$\bar{\Theta}^{\ell\ell qq}_{\mu\nu\varrho\sigma} \equiv \frac{e^4}{48\pi^2(1+\lambda)}\tilde{T}_{\ell\ell qq}, \tag{2.115}$$

and by expanding in powers of λ

$$\bar{\Theta}^{\ell\ell qq}_{\mu\nu\varrho\sigma} = \frac{e^4}{48\pi^2}\tilde{T}_{\ell\ell qq}(1-\lambda). \tag{2.116}$$

It is sufficient to expand the expression only up to the first order in λ because we are already getting two momenta from the traces (and λ contains two momenta). The expression (2.115) can be found in the enclosed `Mathematica` notebook `spinor.nb` denoted by `llqq‘trace`[6].

The terms of type $qqmm$ are similar. For them, we get

$$\begin{aligned} T_{qqmm} \equiv m^2(&\operatorname{Tr}[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\varrho\gamma_\sigma]q_1^\alpha q_2^\beta \\ &+ \operatorname{Tr}[\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\beta\gamma_\varrho\gamma_\sigma]q_1^\alpha q_3^\beta \\ &+ \operatorname{Tr}[\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\varrho\gamma_\beta\gamma_\sigma]q_1^\alpha q_4^\beta \\ &+ \operatorname{Tr}[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\varrho\gamma_\sigma]q_2^\alpha q_3^\beta \\ &+ \operatorname{Tr}[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\varrho\gamma_\beta\gamma_\sigma]q_2^\alpha q_4^\beta \\ &+ \operatorname{Tr}[\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\varrho\gamma_\beta\gamma_\sigma]q_3^\alpha q_4^\beta). \end{aligned} \tag{2.117}$$

We can see this expression evaluated in Fig. 2.9. Then again

$$\bar{\Theta}^{mmqq}_{\mu\nu\varrho\sigma} \equiv -\frac{e^4}{96\pi^2 m^2(1+\lambda)^2}T_{qqmm} \tag{2.118}$$

and consequently

$$\bar{\Theta}^{qqmm}_{\mu\nu\varrho\sigma} = -\frac{e^4}{96\pi^2 m^2}T_{qqmm}(1-2\lambda). \tag{2.119}$$

The full form of (2.118) can be found in the file denoted by `qqmm‘trace`.

The last and the worst is the $qqqq$ term

$$T_{qqqq} \equiv \operatorname{Tr}[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\varrho\gamma_\delta\gamma_\sigma]q_1^\alpha q_2^\beta q_3^\gamma q_4^\delta. \tag{2.120}$$

The result is in Fig. 2.10. For the term, we have

$$\bar{\Theta}^{qqqq}_{\mu\nu\varrho\sigma} \equiv -\frac{e^4}{96\pi^2 m^4}T_{qqqq}. \tag{2.121}$$

Since we are already getting four momenta from the trace, we may directly set $\lambda = 0$ (that means we are expanding up to the zeroth order). Once again, the full form of the expression can be found in the file denoted by `qqqq`.

At last, we can determine the total contribution of the box graph

$$\begin{aligned} \Theta_{\mu\nu\varrho\sigma}(p_1,p_2,p_3) \equiv \int_{\mathrm{F}_3} \mathrm{d}X_{\mathrm{F}_3}(&\bar{\Theta}^{\ell\ell\ell\ell}_{\mu\nu\varrho\sigma} + \bar{\Theta}^{\ell\ell mm}_{\mu\nu\varrho\sigma} + \bar{\Theta}^{mmmm}_{\mu\nu\varrho\sigma} \\ &+ \bar{\Theta}^{\ell\ell qq}_{\mu\nu\varrho\sigma} + \bar{\Theta}^{qqmm}_{\mu\nu\varrho\sigma} + \bar{\Theta}^{qqqq}_{\mu\nu\varrho\sigma}). \end{aligned} \tag{2.122}$$

[6] Instead of the functions ϕ_{ij} [see (2.89)], temporary constants a_{ij} are used there (which are later replaced by ϕ_{ij}).

$$
\begin{aligned}
4\,(&q2^{\mu}\,q1^{\nu}\,g^{\rho\sigma} - q3^{\mu}\,q1^{\nu}\,g^{\rho\sigma} + q4^{\mu}\,q1^{\nu}\,g^{\rho\sigma} + q1^{\mu}\,q2^{\nu}\,g^{\rho\sigma} + \\
&q3^{\mu}\,q2^{\nu}\,g^{\rho\sigma} - q4^{\mu}\,q2^{\nu}\,g^{\rho\sigma} + q1^{\mu}\,q3^{\nu}\,g^{\rho\sigma} + q2^{\mu}\,q3^{\nu}\,g^{\rho\sigma} - \\
&q4^{\mu}\,q3^{\nu}\,g^{\rho\sigma} - q1^{\mu}\,q4^{\nu}\,g^{\rho\sigma} - q2^{\mu}\,q4^{\nu}\,g^{\rho\sigma} + q3^{\mu}\,q4^{\nu}\,g^{\rho\sigma} - \\
&g^{\mu\nu}\,q1\cdot q2\,g^{\rho\sigma} + g^{\mu\nu}\,q1\cdot q3\,g^{\rho\sigma} - g^{\mu\nu}\,q1\cdot q4\,g^{\rho\sigma} - \\
&g^{\mu\nu}\,q2\cdot q3\,g^{\rho\sigma} + g^{\mu\nu}\,q2\cdot q4\,g^{\rho\sigma} - g^{\mu\nu}\,q3\cdot q4\,g^{\rho\sigma} - \\
&q2^{\mu}\,g^{\nu\sigma}\,q1^{\rho} + q3^{\mu}\,g^{\nu\sigma}\,q1^{\rho} - q4^{\mu}\,g^{\nu\sigma}\,q1^{\rho} - g^{\mu\sigma}\,q2^{\nu}\,q1^{\rho} - \\
&g^{\mu\sigma}\,q3^{\nu}\,q1^{\rho} + g^{\mu\sigma}\,q4^{\nu}\,q1^{\rho} - q1^{\mu}\,g^{\nu\sigma}\,q2^{\rho} - q3^{\mu}\,g^{\nu\sigma}\,q2^{\rho} + \\
&q4^{\mu}\,g^{\nu\sigma}\,q2^{\rho} + g^{\mu\sigma}\,q1^{\nu}\,q2^{\rho} + g^{\mu\sigma}\,q3^{\nu}\,q2^{\rho} - g^{\mu\sigma}\,q4^{\nu}\,q2^{\rho} + \\
&q1^{\mu}\,g^{\nu\sigma}\,q3^{\rho} + q2^{\mu}\,g^{\nu\sigma}\,q3^{\rho} - q4^{\mu}\,g^{\nu\sigma}\,q3^{\rho} - g^{\mu\sigma}\,q1^{\nu}\,q3^{\rho} + \\
&g^{\mu\sigma}\,q2^{\nu}\,q3^{\rho} + g^{\mu\sigma}\,q4^{\nu}\,q3^{\rho} + q1^{\mu}\,g^{\nu\sigma}\,q4^{\rho} + q2^{\mu}\,g^{\nu\sigma}\,q4^{\rho} - \\
&q3^{\mu}\,g^{\nu\sigma}\,q4^{\rho} - g^{\mu\sigma}\,q1^{\nu}\,q4^{\rho} + g^{\mu\sigma}\,q2^{\nu}\,q4^{\rho} + g^{\mu\sigma}\,q3^{\nu}\,q4^{\rho} + \\
&q2^{\mu}\,g^{\nu\rho}\,q1^{\sigma} - q3^{\mu}\,g^{\nu\rho}\,q1^{\sigma} + q4^{\mu}\,g^{\nu\rho}\,q1^{\sigma} + g^{\mu\rho}\,q2^{\nu}\,q1^{\sigma} + \\
&g^{\mu\rho}\,q3^{\nu}\,q1^{\sigma} - g^{\mu\rho}\,q4^{\nu}\,q1^{\sigma} - g^{\mu\nu}\,q2^{\rho}\,q1^{\sigma} + g^{\mu\nu}\,q3^{\rho}\,q1^{\sigma} + \\
&g^{\mu\nu}\,q4^{\rho}\,q1^{\sigma} + q1^{\mu}\,g^{\nu\rho}\,q2^{\sigma} + q3^{\mu}\,g^{\nu\rho}\,q2^{\sigma} - q4^{\mu}\,g^{\nu\rho}\,q2^{\sigma} - \\
&g^{\mu\rho}\,q1^{\nu}\,q2^{\sigma} - g^{\mu\rho}\,q3^{\nu}\,q2^{\sigma} + g^{\mu\rho}\,q4^{\nu}\,q2^{\sigma} + g^{\mu\nu}\,q1^{\rho}\,q2^{\sigma} - \\
&g^{\mu\nu}\,q3^{\rho}\,q2^{\sigma} - g^{\mu\nu}\,q4^{\rho}\,q2^{\sigma} - q1^{\mu}\,g^{\nu\rho}\,q3^{\sigma} - q2^{\mu}\,g^{\nu\rho}\,q3^{\sigma} + \\
&q4^{\mu}\,g^{\nu\rho}\,q3^{\sigma} + g^{\mu\rho}\,q1^{\nu}\,q3^{\sigma} - g^{\mu\rho}\,q2^{\nu}\,q3^{\sigma} - g^{\mu\rho}\,q4^{\nu}\,q3^{\sigma} - \\
&g^{\mu\nu}\,q1^{\rho}\,q3^{\sigma} + g^{\mu\nu}\,q2^{\rho}\,q3^{\sigma} + g^{\mu\nu}\,q4^{\rho}\,q3^{\sigma} + q1^{\mu}\,g^{\nu\rho}\,q4^{\sigma} + \\
&q2^{\mu}\,g^{\nu\rho}\,q4^{\sigma} - q3^{\mu}\,g^{\nu\rho}\,q4^{\sigma} - g^{\mu\rho}\,q1^{\nu}\,q4^{\sigma} + g^{\mu\rho}\,q2^{\nu}\,q4^{\sigma} + \\
&g^{\mu\rho}\,q3^{\nu}\,q4^{\sigma} + g^{\mu\nu}\,q1^{\rho}\,q4^{\sigma} - g^{\mu\nu}\,q2^{\rho}\,q4^{\sigma} + g^{\mu\nu}\,q3^{\rho}\,q4^{\sigma} - \\
&g^{\mu\sigma}\,g^{\nu\rho}\,q1\cdot q2 + g^{\mu\rho}\,g^{\nu\sigma}\,q1\cdot q2 + g^{\mu\sigma}\,g^{\nu\rho}\,q1\cdot q3 - \\
&g^{\mu\rho}\,g^{\nu\sigma}\,q1\cdot q3 - g^{\mu\sigma}\,g^{\nu\rho}\,q1\cdot q4 + g^{\mu\rho}\,g^{\nu\sigma}\,q1\cdot q4 - \\
&g^{\mu\sigma}\,g^{\nu\rho}\,q2\cdot q3 + g^{\mu\rho}\,g^{\nu\sigma}\,q2\cdot q3 + g^{\mu\sigma}\,g^{\nu\rho}\,q2\cdot q4 - \\
&g^{\mu\rho}\,g^{\nu\sigma}\,q2\cdot q4 - g^{\mu\sigma}\,g^{\nu\rho}\,q3\cdot q4 + g^{\mu\rho}\,g^{\nu\sigma}\,q3\cdot q4)
\end{aligned}
$$

Figure 2.9: The expression T_{qqmm}

The integrand of course depends on the Feynman parameters x, y, z via the parameter λ and the vectors q_i – see (2.86) and (2.89). The final expression we got using `FeynCalc` is quite lengthy (it is denoted by `Theta`). To shorten and speed up the integration, we used the on-shell condition (2.8) to remove terms proportional to p_i^2 and also the transversality condition

$$p_i \cdot \varepsilon(p_i, \lambda_i) = 0, \qquad i = 1, 2, 3, 4, \tag{2.123}$$

which in our context translates to $p_1^{\mu} = p_2^{\nu} = p_3^{\varrho} = 0$ (effectively). Note that this "reduced" polarization tensor can not be used in formulae such as (2.42), but it can be still used for obtaining the invariant amplitude. This procedure of removing longitudinal terms is correct as long as we do the same with the effective tensor during the amplitude matching.

2.2.5 Polarization tensor

All changes when we use the symmetrization formula (2.53) to obtain the polarization tensor (see Fig. 2.11). It looks pretty good – most importantly, both the

$$
\begin{aligned}
4\,(&q4^{\mu}\, q3^{\nu}\, q2^{\rho}\, q1^{\sigma} - q3^{\mu}\, q4^{\nu}\, q2^{\rho}\, q1^{\sigma} + q4^{\mu}\, q2^{\nu}\, q3^{\rho}\, q1^{\sigma} + q2^{\mu}\, q4^{\nu}\, q3^{\rho}\, q1^{\sigma} + \\
&q3^{\mu}\, q2^{\nu}\, q4^{\rho}\, q1^{\sigma} + q2^{\mu}\, q3^{\nu}\, q4^{\rho}\, q1^{\sigma} - q4^{\mu}\, g^{\nu\rho}\, q2\cdot q3\, q1^{\sigma} + \\
&g^{\mu\rho}\, q4^{\nu}\, q2\cdot q3\, q1^{\sigma} - g^{\mu\nu}\, q4^{\rho}\, q2\cdot q3\, q1^{\sigma} + q3^{\mu}\, g^{\nu\rho}\, q2\cdot q4\, q1^{\sigma} - \\
&g^{\mu\rho}\, q3^{\nu}\, q2\cdot q4\, q1^{\sigma} - g^{\mu\nu}\, q3^{\rho}\, q2\cdot q4\, q1^{\sigma} - q2^{\mu}\, g^{\nu\rho}\, q3\cdot q4\, q1^{\sigma} - \\
&g^{\mu\rho}\, q2^{\nu}\, q3\cdot q4\, q1^{\sigma} + g^{\mu\nu}\, q2^{\rho}\, q3\cdot q4\, q1^{\sigma} - q4^{\mu}\, q3^{\nu}\, q1^{\rho}\, q2^{\sigma} + \\
&q3^{\mu}\, q4^{\nu}\, q1^{\rho}\, q2^{\sigma} - q4^{\mu}\, q1^{\nu}\, q3^{\rho}\, q2^{\sigma} + q1^{\mu}\, q4^{\nu}\, q3^{\rho}\, q2^{\sigma} - \\
&q3^{\mu}\, q1^{\nu}\, q4^{\rho}\, q2^{\sigma} + q1^{\mu}\, q3^{\nu}\, q4^{\rho}\, q2^{\sigma} - q4^{\mu}\, q2^{\nu}\, q1^{\rho}\, q3^{\sigma} - \\
&q2^{\mu}\, q4^{\nu}\, q1^{\rho}\, q3^{\sigma} + q4^{\mu}\, q1^{\nu}\, q2^{\rho}\, q3^{\sigma} - q1^{\mu}\, q4^{\nu}\, q2^{\rho}\, q3^{\sigma} + \\
&q2^{\mu}\, q1^{\nu}\, q4^{\rho}\, q3^{\sigma} + q1^{\mu}\, q2^{\nu}\, q4^{\rho}\, q3^{\sigma} + q3^{\mu}\, q2^{\nu}\, q1^{\rho}\, q4^{\sigma} + q2^{\mu}\, q3^{\nu}\, q1^{\rho}\, q4^{\sigma} - \\
&q3^{\mu}\, q1^{\nu}\, q2^{\rho}\, q4^{\sigma} + q1^{\mu}\, q3^{\nu}\, q2^{\rho}\, q4^{\sigma} + q2^{\mu}\, q1^{\nu}\, q3^{\rho}\, q4^{\sigma} + q1^{\mu}\, q2^{\nu}\, q3^{\rho}\, q4^{\sigma} + \\
&q4^{\mu}\, q3^{\nu}\, g^{\rho\sigma}\, q1\cdot q2 - q3^{\mu}\, q4^{\nu}\, g^{\rho\sigma}\, q1\cdot q2 + q4^{\mu}\, g^{\nu\sigma}\, q3^{\rho}\, q1\cdot q2 - \\
&g^{\mu\sigma}\, q4^{\nu}\, q3^{\rho}\, q1\cdot q2 + q3^{\mu}\, g^{\nu\sigma}\, q4^{\rho}\, q1\cdot q2 - g^{\mu\sigma}\, q3^{\nu}\, q4^{\rho}\, q1\cdot q2 - \\
&q4^{\mu}\, g^{\nu\rho}\, q3^{\sigma}\, q1\cdot q2 + g^{\mu\rho}\, q4^{\nu}\, q3^{\sigma}\, q1\cdot q2 - g^{\mu\nu}\, q4^{\rho}\, q3^{\sigma}\, q1\cdot q2 + \\
&q3^{\mu}\, g^{\nu\rho}\, q4^{\sigma}\, q1\cdot q2 - g^{\mu\rho}\, q3^{\nu}\, q4^{\sigma}\, q1\cdot q2 - g^{\mu\nu}\, q3^{\rho}\, q4^{\sigma}\, q1\cdot q2 + \\
&q4^{\mu}\, q2^{\nu}\, g^{\rho\sigma}\, q1\cdot q3 + q2^{\mu}\, q4^{\nu}\, g^{\rho\sigma}\, q1\cdot q3 - q4^{\mu}\, g^{\nu\sigma}\, q2^{\rho}\, q1\cdot q3 + \\
&g^{\mu\sigma}\, q4^{\nu}\, q2^{\rho}\, q1\cdot q3 - q2^{\mu}\, g^{\nu\sigma}\, q4^{\rho}\, q1\cdot q3 - g^{\mu\sigma}\, q2^{\nu}\, q4^{\rho}\, q1\cdot q3 + \\
&q4^{\mu}\, g^{\nu\rho}\, q2^{\sigma}\, q1\cdot q3 - g^{\mu\rho}\, q4^{\nu}\, q2^{\sigma}\, q1\cdot q3 + g^{\mu\nu}\, q4^{\rho}\, q2^{\sigma}\, q1\cdot q3 - \\
&q2^{\mu}\, g^{\nu\rho}\, q4^{\sigma}\, q1\cdot q3 - g^{\mu\rho}\, q2^{\nu}\, q4^{\sigma}\, q1\cdot q3 + g^{\mu\nu}\, q2^{\rho}\, q4^{\sigma}\, q1\cdot q3 - \\
&q3^{\mu}\, q2^{\nu}\, g^{\rho\sigma}\, q1\cdot q4 - q2^{\mu}\, q3^{\nu}\, g^{\rho\sigma}\, q1\cdot q4 + q3^{\mu}\, g^{\nu\sigma}\, q2^{\rho}\, q1\cdot q4 - \\
&g^{\mu\sigma}\, q3^{\nu}\, q2^{\rho}\, q1\cdot q4 - q2^{\mu}\, g^{\nu\sigma}\, q3^{\rho}\, q1\cdot q4 - g^{\mu\sigma}\, q2^{\nu}\, q3^{\rho}\, q1\cdot q4 - \\
&q3^{\mu}\, g^{\nu\rho}\, q2^{\sigma}\, q1\cdot q4 + g^{\mu\rho}\, q3^{\nu}\, q2^{\sigma}\, q1\cdot q4 + g^{\mu\nu}\, q3^{\rho}\, q2^{\sigma}\, q1\cdot q4 + \\
&q2^{\mu}\, g^{\nu\rho}\, q3^{\sigma}\, q1\cdot q4 + g^{\mu\rho}\, q2^{\nu}\, q3^{\sigma}\, q1\cdot q4 - g^{\mu\nu}\, q2^{\rho}\, q3^{\sigma}\, q1\cdot q4 - \\
&q4^{\mu}\, q1^{\nu}\, g^{\rho\sigma}\, q2\cdot q3 + q1^{\mu}\, q4^{\nu}\, g^{\rho\sigma}\, q2\cdot q3 + q4^{\mu}\, g^{\nu\sigma}\, q1^{\rho}\, q2\cdot q3 - \\
&g^{\mu\sigma}\, q4^{\nu}\, q1^{\rho}\, q2\cdot q3 - q1^{\mu}\, g^{\nu\sigma}\, q4^{\rho}\, q2\cdot q3 + g^{\mu\sigma}\, q1^{\nu}\, q4^{\rho}\, q2\cdot q3 - \\
&q1^{\mu}\, g^{\nu\rho}\, q4^{\sigma}\, q2\cdot q3 + g^{\mu\rho}\, q1^{\nu}\, q4^{\sigma}\, q2\cdot q3 - g^{\mu\nu}\, q1^{\rho}\, q4^{\sigma}\, q2\cdot q3 + \\
&g^{\mu\sigma}\, g^{\nu\rho}\, q1\cdot q4\, q2\cdot q3 - g^{\mu\rho}\, g^{\nu\sigma}\, q1\cdot q4\, q2\cdot q3 + g^{\mu\nu}\, g^{\rho\sigma}\, q1\cdot q4\, q2\cdot q3 + \\
&q3^{\mu}\, q1^{\nu}\, g^{\rho\sigma}\, q2\cdot q4 - q1^{\mu}\, q3^{\nu}\, g^{\rho\sigma}\, q2\cdot q4 - q3^{\mu}\, g^{\nu\sigma}\, q1^{\rho}\, q2\cdot q4 + \\
&g^{\mu\sigma}\, q3^{\nu}\, q1^{\rho}\, q2\cdot q4 - q1^{\mu}\, g^{\nu\sigma}\, q3^{\rho}\, q2\cdot q4 + g^{\mu\sigma}\, q1^{\nu}\, q3^{\rho}\, q2\cdot q4 + \\
&q1^{\mu}\, g^{\nu\rho}\, q3^{\sigma}\, q2\cdot q4 - g^{\mu\rho}\, q1^{\nu}\, q3^{\sigma}\, q2\cdot q4 + g^{\mu\nu}\, q1^{\rho}\, q3^{\sigma}\, q2\cdot q4 - \\
&g^{\mu\sigma}\, g^{\nu\rho}\, q1\cdot q3\, q2\cdot q4 + g^{\mu\rho}\, g^{\nu\sigma}\, q1\cdot q3\, q2\cdot q4 - g^{\mu\nu}\, g^{\rho\sigma}\, q1\cdot q3\, q2\cdot q4 - \\
&q2^{\mu}\, q1^{\nu}\, g^{\rho\sigma}\, q3\cdot q4 - q1^{\mu}\, q2^{\nu}\, g^{\rho\sigma}\, q3\cdot q4 + q2^{\mu}\, g^{\nu\sigma}\, q1^{\rho}\, q3\cdot q4 + \\
&g^{\mu\sigma}\, q2^{\nu}\, q1^{\rho}\, q3\cdot q4 + q1^{\mu}\, g^{\nu\sigma}\, q2^{\rho}\, q3\cdot q4 - g^{\mu\sigma}\, q1^{\nu}\, q2^{\rho}\, q3\cdot q4 - \\
&q1^{\mu}\, g^{\nu\rho}\, q2^{\sigma}\, q3\cdot q4 + g^{\mu\rho}\, q1^{\nu}\, q2^{\sigma}\, q3\cdot q4 - g^{\mu\nu}\, q1^{\rho}\, q2^{\sigma}\, q3\cdot q4 + \\
&g^{\mu\sigma}\, g^{\nu\rho}\, q1\cdot q2\, q3\cdot q4 - g^{\mu\rho}\, g^{\nu\sigma}\, q1\cdot q2\, q3\cdot q4 + g^{\mu\nu}\, g^{\rho\sigma}\, q1\cdot q2\, q3\cdot q4)
\end{aligned}
$$

Figure 2.10: The expression T_{qqqq}

UV divergent terms [terms proportional to Δ, see (2.100)] and the terms proportional to two momenta are gone. This means that our tensor is UV finite and has the proper structure that corresponds with the effective theory.

$$
\begin{aligned}
&\frac{1}{180\, m^4\, \pi^2} e^4 \,(14\, g^{\mu\sigma}\, \mathrm{p3}^{\nu}\, \mathrm{p1}^{\rho}\, \mathrm{p1}\cdot\mathrm{p2} - 3\, g^{\mu\sigma}\, \mathrm{p3}^{\nu}\, \mathrm{p2}^{\rho}\, \mathrm{p1}\cdot\mathrm{p2} - \\
&\quad 7\, g^{\mu\rho}\, \mathrm{p3}^{\nu}\, \mathrm{p1}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p2} + 7\, g^{\mu\rho}\, \mathrm{p3}^{\nu}\, \mathrm{p2}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p2} + \\
&\quad 7\, g^{\mu\rho}\, \mathrm{p3}^{\nu}\, \mathrm{p3}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p2} - 3\, g^{\mu\nu}\, \mathrm{p1}^{\rho}\, \mathrm{p3}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p2} - \\
&\quad 3\, g^{\mu\nu}\, \mathrm{p2}^{\rho}\, \mathrm{p3}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p2} - 14\, g^{\mu\sigma}\, g^{\nu\rho}\, \mathrm{p1}\cdot\mathrm{p3}\, \mathrm{p1}\cdot\mathrm{p2} + \\
&\quad 3\, g^{\mu\rho}\, g^{\nu\sigma}\, \mathrm{p1}\cdot\mathrm{p3}\, \mathrm{p1}\cdot\mathrm{p2} + 3\, g^{\mu\nu}\, g^{\rho\sigma}\, \mathrm{p1}\cdot\mathrm{p3}\, \mathrm{p1}\cdot\mathrm{p2} + \\
&\quad 3\, g^{\mu\sigma}\, g^{\nu\rho}\, \mathrm{p2}\cdot\mathrm{p3}\, \mathrm{p1}\cdot\mathrm{p2} - 14\, g^{\mu\rho}\, g^{\nu\sigma}\, \mathrm{p2}\cdot\mathrm{p3}\, \mathrm{p1}\cdot\mathrm{p2} + \\
&\quad 3\, g^{\mu\nu}\, g^{\rho\sigma}\, \mathrm{p2}\cdot\mathrm{p3}\, \mathrm{p1}\cdot\mathrm{p2} + 14\, g^{\mu\sigma}\, \mathrm{p1}^{\nu}\, \mathrm{p2}^{\rho}\, \mathrm{p1}\cdot\mathrm{p3} - \\
&\quad 3\, g^{\mu\sigma}\, \mathrm{p3}^{\nu}\, \mathrm{p2}^{\rho}\, \mathrm{p1}\cdot\mathrm{p3} - 7\, g^{\mu\nu}\, \mathrm{p2}^{\rho}\, \mathrm{p1}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p3} - \\
&\quad 3\, g^{\mu\rho}\, \mathrm{p1}^{\nu}\, \mathrm{p2}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p3} - 3\, g^{\mu\rho}\, \mathrm{p3}^{\nu}\, \mathrm{p2}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p3} + \\
&\quad 7\, g^{\mu\nu}\, \mathrm{p2}^{\rho}\, \mathrm{p2}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p3} + 7\, g^{\mu\nu}\, \mathrm{p2}^{\rho}\, \mathrm{p3}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p3} - \\
&\quad 14\, g^{\mu\sigma}\, \mathrm{p1}^{\nu}\, \mathrm{p1}^{\rho}\, \mathrm{p2}\cdot\mathrm{p3} + 7\, g^{\mu\rho}\, \mathrm{p1}^{\nu}\, \mathrm{p1}^{\sigma}\, \mathrm{p2}\cdot\mathrm{p3} + \\
&\quad 7\, g^{\mu\nu}\, \mathrm{p1}^{\rho}\, \mathrm{p1}^{\sigma}\, \mathrm{p2}\cdot\mathrm{p3} + 7\, g^{\mu\rho}\, \mathrm{p1}^{\nu}\, \mathrm{p2}^{\sigma}\, \mathrm{p2}\cdot\mathrm{p3} - \\
&\quad 7\, g^{\mu\nu}\, \mathrm{p1}^{\rho}\, \mathrm{p2}^{\sigma}\, \mathrm{p2}\cdot\mathrm{p3} - 7\, g^{\mu\rho}\, \mathrm{p1}^{\nu}\, \mathrm{p3}^{\sigma}\, \mathrm{p2}\cdot\mathrm{p3} + \\
&\quad 7\, g^{\mu\nu}\, \mathrm{p1}^{\rho}\, \mathrm{p3}^{\sigma}\, \mathrm{p2}\cdot\mathrm{p3} + 3\, g^{\mu\sigma}\, g^{\nu\rho}\, \mathrm{p1}\cdot\mathrm{p3}\, \mathrm{p2}\cdot\mathrm{p3} + \\
&\quad 3\, g^{\mu\rho}\, g^{\nu\sigma}\, \mathrm{p1}\cdot\mathrm{p3}\, \mathrm{p2}\cdot\mathrm{p3} - 14\, g^{\mu\nu}\, g^{\rho\sigma}\, \mathrm{p1}\cdot\mathrm{p3}\, \mathrm{p2}\cdot\mathrm{p3} + \\
&\quad \mathrm{p3}^{\mu}\, (-3\, g^{\nu\sigma}\, \mathrm{p1}^{\rho}\, \mathrm{p1}\cdot\mathrm{p2} + 14\, g^{\nu\sigma}\, \mathrm{p2}^{\rho}\, \mathrm{p1}\cdot\mathrm{p2} + \\
&\qquad 7\, g^{\nu\rho}\, \mathrm{p1}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p2} - 7\, g^{\nu\rho}\, \mathrm{p2}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p2} + 7\, g^{\nu\rho}\, \mathrm{p3}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p2} + \\
&\qquad \mathrm{p3}^{\nu}\, (3\, \mathrm{p2}^{\rho}\, \mathrm{p1}^{\sigma} + 3\, \mathrm{p1}^{\rho}\, \mathrm{p2}^{\sigma} - 14\, g^{\rho\sigma}\, \mathrm{p1}\cdot\mathrm{p2}) - \\
&\qquad 3\, g^{\nu\sigma}\, \mathrm{p1}^{\rho}\, \mathrm{p2}\cdot\mathrm{p3} - 3\, g^{\nu\rho}\, \mathrm{p1}^{\sigma}\, \mathrm{p2}\cdot\mathrm{p3} + \\
&\qquad \mathrm{p1}^{\nu}\, (3\, \mathrm{p1}^{\rho}\, \mathrm{p2}^{\sigma} - 7\, \mathrm{p2}^{\rho}\, (\mathrm{p1}^{\sigma} + \mathrm{p2}^{\sigma} + \mathrm{p3}^{\sigma}) + 14\, g^{\rho\sigma}\, \mathrm{p2}\cdot\mathrm{p3})) + \\
&\quad \mathrm{p2}^{\mu}\, (-14\, g^{\nu\sigma}\, \mathrm{p2}^{\rho}\, \mathrm{p1}\cdot\mathrm{p3} + 7\, g^{\nu\rho}\, \mathrm{p1}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p3} + \\
&\qquad 7\, g^{\nu\rho}\, \mathrm{p2}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p3} - 7\, g^{\nu\rho}\, \mathrm{p3}^{\sigma}\, \mathrm{p1}\cdot\mathrm{p3} + \\
&\qquad \mathrm{p3}^{\nu}\, (3\, \mathrm{p2}^{\rho}\, \mathrm{p1}^{\sigma} - 7\, \mathrm{p1}^{\rho}\, (\mathrm{p1}^{\sigma} + \mathrm{p2}^{\sigma} + \mathrm{p3}^{\sigma}) + 14\, g^{\rho\sigma}\, \mathrm{p1}\cdot\mathrm{p3}) + \\
&\qquad 14\, g^{\nu\sigma}\, \mathrm{p1}^{\rho}\, \mathrm{p2}\cdot\mathrm{p3} - 3\, g^{\nu\rho}\, \mathrm{p1}^{\sigma}\, \mathrm{p2}\cdot\mathrm{p3} + \\
&\qquad 3\, \mathrm{p1}^{\nu}\, (\mathrm{p1}^{\rho}\, \mathrm{p3}^{\sigma} + \mathrm{p2}^{\rho}\, \mathrm{p3}^{\sigma} - g^{\rho\sigma}\, (\mathrm{p1}\cdot\mathrm{p3} + \mathrm{p2}\cdot\mathrm{p3}))))
\end{aligned}
$$

Figure 2.11: The "reduced" polarization tensor for spinor QED

2.2.6 Amplitude matching

Now we are ready to enjoy the fruits of our hard labor. Let us compare the calculated spinor QED polarization tensor $\Gamma^{\text{sp.}}_{\mu\nu\varrho\sigma}$ (Fig. 2.11) and the effective one $\Gamma^{\text{eff.}}_{\mu\nu\varrho\sigma}$ (Fig. 2.4). Since we have thrown away longitudinal terms proportional to p_1^μ, p_2^ν or p_3^ϱ from the spinor QED tensor, we need to do the same with the effective one.

Basically, we need to solve the following equation for the coupling constants g_1, g_2

$$\Gamma^{\text{sp.}}_{\mu\nu\varrho\sigma} - \Gamma^{\text{eff.}}_{\mu\nu\varrho\sigma}(g_1, g_2) = 0. \tag{2.124}$$

First, compare only the terms with no metric tensors. Fig. 2.12 shows what we obtain on the left-hand-side. Is it self-evident that this is equal to zero if the constants g_1, g_2 solve the following linear system

$$\begin{aligned} e^4 + 1920\pi^2 m^4 (g_1 - g_2) &= 0 \\ 7e^4 - 5760\pi^2 m^4 g_2 &= 0. \end{aligned} \tag{2.125}$$

The solution is

$$\begin{aligned} g_1 &= \frac{e^4}{1440\pi^2 m^4} = \frac{\alpha^2}{90 m^4} \\ g_2 &= \frac{7e^4}{5760\pi^2 m^4} = \frac{7\alpha^2}{360 m^4}, \end{aligned} \tag{2.126}$$

where we have introduced the fine structure constant $\alpha = e^2/4\pi$.

Substituting the solution back into the left-hand-side of our equation (2.124) results in Fig. 2.13, which is equal to zero, thanks to the constraint (2.10) [which is a consequence of the on-shell conditions (2.8)].

$$\begin{aligned} \frac{1}{180\,\pi^2\,m^4}\Big(& \mathrm{p2}^\mu \big(3\,\mathrm{p1}^\nu\,\mathrm{p3}^\sigma\,(\mathrm{p1}^\rho + \mathrm{p2}^\rho)\left(e^4 + 1920\,\pi^2\,m^4\,(\mathrm{g1} - \mathrm{g2})\right) + \\ & \quad \mathrm{p3}^\nu \big(3\,\mathrm{p1}^\sigma\,\mathrm{p2}^\rho \left(e^4 + 1920\,\pi^2\,m^4\,(\mathrm{g1} - \mathrm{g2})\right) - \\ & \qquad \mathrm{p1}^\rho \left(7\,e^4 - 5760\,\pi^2\,\mathrm{g2}\,m^4\right)(\mathrm{p1}^\sigma + \mathrm{p2}^\sigma + \mathrm{p3}^\sigma)\big)\big) + \\ & \mathrm{p3}^\mu \big(3\,\mathrm{p3}^\nu \left(e^4 + 1920\,\pi^2\,m^4\,(\mathrm{g1} - \mathrm{g2})\right)(\mathrm{p1}^\sigma\,\mathrm{p2}^\rho + \mathrm{p1}^\rho\,\mathrm{p2}^\sigma) + \\ & \quad \mathrm{p1}^\nu \big(3\,\mathrm{p1}^\rho\,\mathrm{p2}^\sigma \left(e^4 + 1920\,\pi^2\,m^4\,(\mathrm{g1} - \mathrm{g2})\right) - \\ & \qquad \mathrm{p2}^\rho \left(7\,e^4 - 5760\,\pi^2\,\mathrm{g2}\,m^4\right)(\mathrm{p1}^\sigma + \mathrm{p2}^\sigma + \mathrm{p3}^\sigma)\big)\big)\Big) \end{aligned}$$

Figure 2.12: Matching the terms with no metric tensors

$$\begin{aligned} & \frac{1}{180\,\pi^2\,m^4}\,e^4\,(\mathrm{p1}\cdot\mathrm{p2} + \mathrm{p1}\cdot\mathrm{p3} + \mathrm{p2}\cdot\mathrm{p3}) \\ & (-7\,\mathrm{p1}^\rho\,\mathrm{p2}^\sigma\,g^{\mu\nu} + 7\,\mathrm{p1}^\nu\,\mathrm{p2}^\rho\,g^{\mu\sigma} - 7\,\mathrm{p1}^\sigma\,\mathrm{p2}^\rho\,g^{\mu\nu} + 7\,\mathrm{p1}^\nu\,\mathrm{p2}^\sigma\,g^{\mu\rho} - \\ & \quad 7\,g^{\mu\sigma}\,g^{\nu\rho}\,\mathrm{p1}\cdot\mathrm{p2} - 7\,g^{\mu\rho}\,g^{\nu\sigma}\,\mathrm{p1}\cdot\mathrm{p2} + 10\,g^{\mu\nu}\,g^{\rho\sigma}\,\mathrm{p1}\cdot\mathrm{p2} + \\ & \quad \mathrm{p3}^\mu\,(7\,\mathrm{p1}^\nu\,g^{\rho\sigma} + g^{\nu\sigma}\,(7\,\mathrm{p2}^\rho - 10\,\mathrm{p1}^\rho) + 7\,g^{\nu\rho}\,(\mathrm{p1}^\sigma - \mathrm{p2}^\sigma)) + \\ & \quad \mathrm{p2}^\mu\,(7\,(\mathrm{p1}^\rho\,g^{\nu\sigma} + \mathrm{p1}^\sigma\,g^{\nu\rho} + \mathrm{p3}^\nu\,g^{\rho\sigma} - \mathrm{p3}^\sigma\,g^{\nu\rho}) - 10\,\mathrm{p1}^\nu\,g^{\rho\sigma}) + \\ & \quad 7\,\mathrm{p1}^\rho\,\mathrm{p3}^\nu\,g^{\mu\sigma} + 7\,\mathrm{p1}^\rho\,\mathrm{p3}^\sigma\,g^{\mu\nu} - 7\,\mathrm{p1}^\sigma\,\mathrm{p3}^\nu\,g^{\mu\rho} - 7\,\mathrm{p1}^\nu\,\mathrm{p3}^\sigma\,g^{\mu\rho} - \\ & \quad 7\,g^{\mu\sigma}\,g^{\nu\rho}\,\mathrm{p1}\cdot\mathrm{p3} + 10\,g^{\mu\rho}\,g^{\nu\sigma}\,\mathrm{p1}\cdot\mathrm{p3} - 7\,g^{\mu\nu}\,g^{\rho\sigma}\,\mathrm{p1}\cdot\mathrm{p3} - \\ & \quad 10\,\mathrm{p2}^\rho\,\mathrm{p3}^\nu\,g^{\mu\sigma} + 7\,\mathrm{p2}^\sigma\,\mathrm{p3}^\nu\,g^{\mu\rho} + 7\,\mathrm{p2}^\rho\,\mathrm{p3}^\sigma\,g^{\mu\nu} + \\ & \quad 10\,g^{\mu\sigma}\,g^{\nu\rho}\,\mathrm{p2}\cdot\mathrm{p3} - 7\,g^{\mu\rho}\,g^{\nu\sigma}\,\mathrm{p2}\cdot\mathrm{p3} - 7\,g^{\mu\nu}\,g^{\rho\sigma}\,\mathrm{p2}\cdot\mathrm{p3}) \end{aligned}$$

Figure 2.13: The $p_4^2 = 0$ constraint is factorized

We may conclude that the effective Lagrangian of Euler-Heisenberg type for the case of spinor QED can be expressed as

$$\mathscr{L}_{\text{eff.}}^{\text{sp.}} = \frac{\alpha^2}{360 m^4}\left[4(F_{\mu\nu}F^{\mu\nu})^2 + 7(\star F_{\mu\nu}F^{\mu\nu})^2\right], \tag{2.127}$$

or

$$\mathscr{L}_{\text{eff.}}^{\text{sp.}} = \frac{2\alpha^2}{45m^4}\left[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 7(\mathbf{E}\cdot\mathbf{B})^2\right]. \tag{2.128}$$

One can also calculate the differential cross section using the formula (2.47)

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{\text{sp.}} = \frac{139\alpha^4}{32400\pi^2}\frac{\omega^6}{m^8}(3+\cos^2\theta)^2, \tag{2.129}$$

where $s = (2\omega)^2$. We can see that the Lagrangian is exactly replicated [see (1.1)] as well as the expression for the differential cross section [see (1.5)].

2.3 Case of scalar QED

Although many things are similar or even identical same, the case of scalar QED differs from the spinor case in some aspects. The interaction Lagrangian of scalar QED can be expressed as [23]

$$\mathscr{L}_{\text{int.}} = -ieA_\mu[\phi^\dagger(\partial^\mu\phi) - (\partial^\mu\phi^\dagger)\phi] + e^2A_\mu A^\mu\phi^\dagger\phi, \tag{2.130}$$

where ϕ is some charged scalar field, A_μ is again the electromagnetic field, and e is the coupling constant (charge).

We can split the Lagrangian into a trilinear and a quadrilinear part

$$\mathscr{L}_{\text{int.}} = \mathscr{L}_{\phi\phi\gamma} + \mathscr{L}_{\phi\phi\gamma\gamma}, \tag{2.131}$$

where

$$\begin{aligned}\mathscr{L}_{\phi\phi\gamma} &\equiv -ieA_\mu[\phi^\dagger(\partial^\mu\phi) - (\partial^\mu\phi^\dagger)\phi]\\ \mathscr{L}_{\phi\phi\gamma\gamma} &\equiv e^2A_\mu A^\mu\phi^\dagger\phi.\end{aligned} \tag{2.132}$$

From the structure of the Lagrangian, we can see that there are now three types of one-loop diagrams that contribute to our four-photon process. We can either use four trilinear vertices to build box diagrams, or two trilinear and one qudrilinear to build triangle diagrams. Bubble diagrams arise from two quadrilinear vertices (see Fig. 2.14).

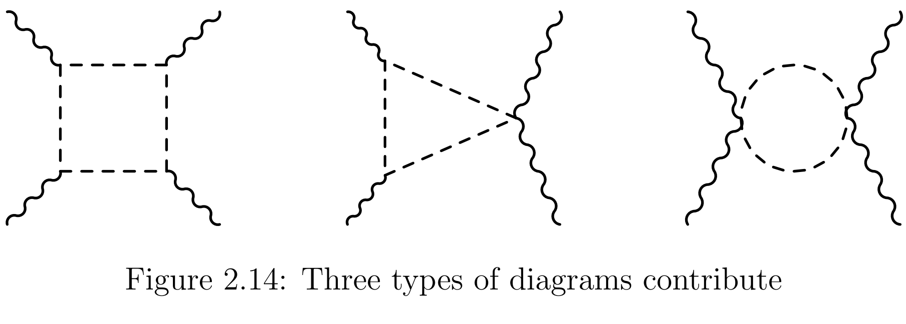

Figure 2.14: Three types of diagrams contribute

2.3.1 Box diagrams

The structure of the S-matrix element and the diagram parametrization is completely the same for the scalar box diagram as for the spinor one, so we can just use expressions from the previous section.

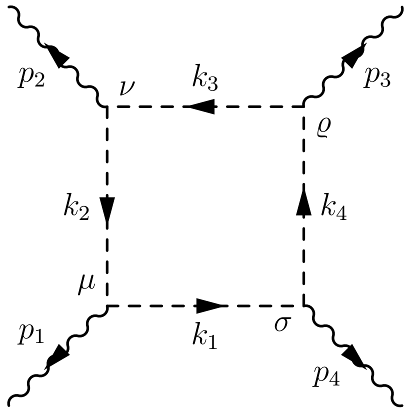

Figure 2.15: The scalar box diagram

Diagram evaluation

The diagram in question is shown in Fig. 2.15.

The expression for the diagram contribution is the only thing that is different. Using standard Feynman rules, we obtain

$$i\Theta^{\Box}_{\mu\nu\varrho\sigma}(p_1,p_2,p_3) \equiv \\ e^4\mu^{4-D}\int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{(k_1+k_2)_\mu(k_2+k_3)_\nu(k_3+k_4)_\varrho(k_4+k_1)_\sigma}{\prod_{i=1}^4(k_i^2-m^2)}, \tag{2.133}$$

where

$$\begin{aligned} k_1 &\equiv \ell \\ k_2 &\equiv \ell + p_1 \\ k_3 &\equiv \ell + p_1 + p_2 \\ k_4 &\equiv \ell + p_1 + p_2 + p_3. \end{aligned} \tag{2.134}$$

The Feynman parametrization and the loop momentum shift give

$$i\Theta^{\Box}_{\mu\nu\varrho\sigma}(p_1,p_2,p_3) = e^4\mu^{4-D}\int_{\mathrm{F}_3}\mathrm{d}X_{\mathrm{F}_3} \\ \int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{(2\ell+q_1+q_2)_\mu(2\ell+q_2+q_3)_\nu(2\ell+q_3+q_4)_\varrho(2\ell+q_4+q_1)_\sigma}{[\ell^2-m^2(1+\lambda)]^4}, \tag{2.135}$$

where

$$\int_{\mathrm{F}_3}\mathrm{d}X_{\mathrm{F}_3} \equiv 6\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int_0^{1-x-y}\mathrm{d}z, \tag{2.136}$$

and

$$\begin{aligned} \lambda &\equiv \sum_{i<j}^{3} f^{\Box}_{ij}(x,y,z)\frac{p_i\cdot p_j}{m^2} \\ f^{\Box}_{12}(x,y,z) &\equiv 2(y+z)(x+y+z-1) \\ f^{\Box}_{13}(x,y,z) &\equiv 2z(x+y+z-1) \\ f^{\Box}_{23}(x,y,z) &\equiv 2z(y+z-1), \end{aligned} \tag{2.137}$$

and

$$q_i \equiv \sum_{j=1}^{3} \phi^{\Box}_{ij}(x,y,z) p_j$$

$$\phi^{\Box}_{ij}(x,y,z) \equiv - \begin{pmatrix} x+y+z & y+z & z \\ x+y+z-1 & y+z & z \\ x+y+z-1 & y+z-1 & z \\ x+y+z-1 & y+z-1 & z-1 \end{pmatrix}. \tag{2.138}$$

Just as before, we break the numerator of the integrand into individual terms and deal with them one by one.

The first is the $\ell\ell\ell\ell$ term

$$T_{\Box\ell\ell\ell\ell} \equiv (2\ell)_\mu (2\ell)_\nu (2\ell)_\varrho (2\ell)_\sigma. \tag{2.139}$$

The symmetric integration (2.93) obviously yields

$$\tilde{T}_{\Box\ell\ell\ell\ell} = \frac{16(g_{\mu\nu}g_{\gamma\delta} + g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma})}{D(D+2)}, \tag{2.140}$$

where

$$\tilde{T}_{\Box\ell\ell\ell\ell} \equiv T_{\Box\ell\ell\ell\ell}(\ell^2)^2. \tag{2.141}$$

Exactly the same procedure [the master formula (2.91) and the expansion in powers of ε – check the previous section] eventually results in

$$\bar{\Theta}^{\Box\ell\ell\ell\ell}_{\mu\nu\varrho\sigma} = \frac{e^4}{24\pi^2}[\Delta - \ln(1+\lambda)](g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\nu}g_{\rho\sigma}), \tag{2.142}$$

where Δ is defined by (2.100), and the low energy limit leads to (again, we stop the expansion at λ^2 – we are getting two momenta per λ)

$$\bar{\Theta}^{\Box\ell\ell\ell\ell}_{\mu\nu\varrho\sigma} = \frac{e^4}{48\pi^2}(2\Delta - 2\lambda + \lambda^2)(g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\nu}g_{\rho\sigma}). \tag{2.143}$$

The remaining terms are much easier to deal with because they are UV finite. The next set of terms are terms of type $\ell\ell qq$

$$\begin{aligned} T_{\Box\ell\ell qq} \equiv\; & (q_1+q_2)_\mu (q_2+q_3)_\nu (2\ell)_\varrho (2\ell)_\sigma \\ & + (q_1+q_2)_\mu (2\ell)_\nu (q_3+q_4)_\varrho (2\ell)_\sigma \\ & + (2\ell)_\mu (q_2+q_3)_\nu (q_3+q_4)_\varrho (2\ell)_\sigma \\ & + (q_1+q_2)_\mu (2\ell)_\nu (2\ell)_\varrho (q_4+q_1)_\sigma \\ & + (2\ell)_\mu (q_2+q_3)_\nu (2\ell)_\varrho (q_4+q_1)_\sigma \\ & + (2\ell)_\mu (2\ell)_\nu (q_3+q_4)_\varrho (q_4+q_1)_\sigma \end{aligned} \tag{2.144}$$

The symmetric integration (2.105) brings us to

$$\begin{aligned} \tilde{T}_{\Box\ell\ell qq} =\; & (q_1+q_2)_\mu (q_2+q_3)_\nu g_{\varrho\sigma} \\ & + (q_1+q_2)_\mu (q_3+q_4)_\varrho g_{\nu\sigma} \\ & + (q_2+q_3)_\nu (q_3+q_4)_\varrho g_{\mu\sigma} \\ & + (q_1+q_2)_\mu (q_4+q_1)_\sigma g_{\nu\varrho} \\ & + (q_2+q_3)_\nu (q_4+q_1)_\sigma g_{\mu\varrho} \\ & + (q_3+q_4)_\varrho (q_4+q_1)_\sigma g_{\mu\nu}, \end{aligned} \tag{2.145}$$

where (as always)

$$\ell^2 \tilde{T}_{\Box\ell\ell qq} \equiv T_{\Box\ell\ell qq}, \tag{2.146}$$

then the loop integration master formula (2.91) to

$$\bar{\Theta}^{\Box\ell\ell qq}_{\mu\nu\varrho\sigma} \equiv -\frac{e^4}{48\pi^2 m^2 (1+\lambda)} \tilde{T}_{\Box\ell\ell qq}. \tag{2.147}$$

and the low energy limit to

$$\bar{\Theta}^{\Box\ell\ell qq}_{\mu\nu\varrho\sigma} = -\frac{e^4}{48\pi^2 m^2} \tilde{T}_{\Box\ell\ell qq}(1-\lambda). \tag{2.148}$$

The full expression (with the vectors q_i are substituted in) for (2.147) can be found inside `scalar.nb` denoted by `llqq‘term`.

The last term is of type $qqqq$

$$T_{\Box qqqq} \equiv (q_1+q_2)_\mu (q_2+q_3)_\nu (q_3+q_4)_\varrho (q_4+q_1)_\sigma, \tag{2.149}$$

for which we get (again, no λ here)

$$\bar{\Theta}^{\Box qqqq}_{\mu\nu\varrho\sigma} \equiv \frac{e^4}{96\pi^2 m^4} T_{\Box qqqq}. \tag{2.150}$$

The full form can be found within the file denoted by `qqqq`.

Now we just add all the terms up and integrate

$$\Theta^{\Box}_{\mu\nu\varrho\sigma}(p_1,p_2,p_3) \equiv \int_{\mathrm{F}_3} \mathrm{d}X_{\mathrm{F}_3} (\bar{\Theta}^{\Box\ell\ell\ell\ell}_{\mu\nu\varrho\sigma} + \bar{\Theta}^{\Box\ell\ell qq}_{\mu\nu\varrho\sigma} + \bar{\Theta}^{\Box qqqq}_{\mu\nu\varrho\sigma}). \tag{2.151}$$

The result is denoted by `Theta‘sq` in the file.

The total contribution from all of the box diagrams is then given by the symmetrization formula (2.53)

$$\Gamma^{\Box}_{\mu\nu\varrho\sigma}(p_1,p_2,p_3) = \mathrm{sym}_3\, \Theta^{\Box}_{\mu\nu\varrho|\sigma}(p_1,p_2,p_3). \tag{2.152}$$

2.3.2 Triangle diagrams

We leave the problem of the symmetrization to the end of this section – instead we directly start with the evaluation of one of the triangle diagrams.

Diagram evaluation

We are interested in the diagram in Fig. 2.16.

The relevant expression reads

$$i\Theta^{\triangle}_{\mu\nu\varrho\sigma}(p_1,p_2) \equiv -e^4 \mu^{4-D} \int \frac{\mathrm{d}^D\ell}{(2\pi)^D} \frac{(k_1+k_2)_\mu (k_2+k_3)_\nu g_{\varrho\sigma}}{\prod_{i=1}^{3}(k_i^2-m^2)}, \tag{2.153}$$

where

$$\begin{aligned} k_1 &\equiv \ell \\ k_2 &\equiv \ell + p_1 \\ k_3 &\equiv \ell + p_1 + p_2. \end{aligned} \tag{2.154}$$

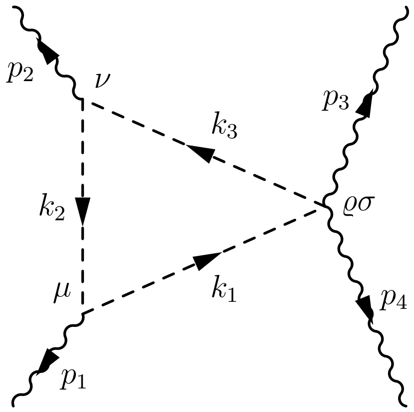

Figure 2.16: The scalar triangle diagram

The Feynman parametrization and the loop momentum shift results in

$$i\Theta^{\triangle}_{\mu\nu\varrho\sigma}(p_1,p_2) = -e^4\mu^{4-D}\int_{\mathrm{F}_2}\mathrm{d}X_{\mathrm{F}_2} \int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{(2\ell+q_1+q_2)_\mu(2\ell+q_2+q_3)_\nu g_{\varrho\sigma}}{[\ell^2-m^2(1+\lambda)]^3}, \tag{2.155}$$

where

$$\int_{\mathrm{F}_2}\mathrm{d}X_{\mathrm{F}_2} \equiv 2\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y, \tag{2.156}$$

and

$$\lambda \equiv f^{\triangle}_{12}(x,y)\frac{p_1\cdot p_2}{m^2}$$
$$f^{\triangle}_{12}(x,y) \equiv 2y(x+y-1), \tag{2.157}$$

and

$$q_i \equiv \sum_{j=1}^{2}\phi^{\triangle}_{ij}(x,y)p_j$$
$$\phi^{\triangle}_{ij}(x,y) \equiv -\begin{pmatrix} x+y & y \\ x+y-1 & y \\ x+y-1 & y-1 \end{pmatrix}. \tag{2.158}$$

The procedure we used for obtaining these results is standard (check the spinor QED section).

Again, let us break the numerator into separate terms. The $\ell\ell$ term is the first one

$$T_{\triangle\ell\ell} \equiv (2\ell)_\mu(2\ell)_\nu g_{\varrho\sigma}. \tag{2.159}$$

The symmetric integration

$$\ell_\alpha\ell_\beta \overset{\text{eff.}}{=} \frac{\ell^2}{D}g_{\alpha\beta} \tag{2.160}$$

yields

$$\tilde{T}_{\triangle\ell\ell} = \frac{4g_{\mu\nu}g_{\varrho\sigma}}{D}, \tag{2.161}$$

where

$$\ell^2 \tilde{T}_{\triangle \ell\ell} \equiv T_{\triangle \ell\ell}. \tag{2.162}$$

The master formula and the ε-expansion then provide

$$\bar{\Theta}^{\triangle \ell\ell}_{\mu\nu\varrho\sigma} \equiv -\frac{e^4}{16\pi^2}[\Delta - \ln(1+\lambda)] g_{\mu\nu} g_{\varrho\sigma}, \tag{2.163}$$

where Δ is defined by (2.100). The λ-expansion leaves us with

$$\bar{\Theta}^{\triangle \ell\ell}_{\mu\nu\varrho\sigma} = -\frac{e^4}{32\pi^2}(2\Delta - 2\lambda + \lambda^2) g_{\mu\nu} g_{\varrho\sigma}. \tag{2.164}$$

The second term is the qq term (which is UV finite)

$$T_{\triangle \mathrm{qq}} \equiv (q_1 + q_2)_\mu (q_2 + q_3)_\nu g_{\varrho\sigma}. \tag{2.165}$$

We obtain for it

$$\bar{\Theta}^{\triangle qq}_{\mu\nu\varrho\sigma} \equiv \frac{e^4}{32\pi^2 m^2 (1+\lambda)} T_{\triangle qq}, \tag{2.166}$$

and then

$$\bar{\Theta}^{\triangle qq}_{\mu\nu\varrho\sigma} \equiv \frac{e^4}{32\pi^2 m^2} T_{\triangle qq}(1-\lambda). \tag{2.167}$$

It is denoted `qq'term` in the file.

The last step is to add them together and integrate

$$\Theta^{\triangle}_{\mu\nu\varrho\sigma}(p_1, p_2) = \int_{\mathrm{F}_2} \mathrm{d}X_{\mathrm{F}_2} (\bar{\Theta}^{\triangle \ell\ell}_{\mu\nu\varrho\sigma} + \bar{\Theta}^{\triangle qq}_{\mu\nu\varrho\sigma}). \tag{2.168}$$

The result is denoted by `Theta'tr`.

Problem of symmetrization

To get the contribution to the total polarization tensor from all triangles, we need a formula similar to (2.53). Besides a tedious proof, we can try using an intuitive rule from the end of the section 2.2.1.

The structure of the relevant S-matrix element contains

$$a_1 a_2 a_3 a_4 \mathcal{T}[\mathscr{L}_{\phi\phi\gamma}(x_1) \mathscr{L}_{\phi\phi\gamma}(x_2) \mathscr{L}_{\phi\phi\gamma\gamma}(x_3)], \tag{2.169}$$

where $\mathscr{L}_{\phi\phi\gamma}$ and $\mathscr{L}_{\phi\phi\gamma\gamma}$ are given by (2.132), and $a_i \equiv a(p_i, \lambda_i)$. As always, there are 24 possible Wick contractions of the photon operators. Additionally, we can contract the scalar operators in 2 possible ways. Apart from that, there are also 3 ways how to assemble the actual time-ordered product – all of the following are valid

$$\begin{gathered} \mathcal{T}[\mathscr{L}_{\phi\phi\gamma}(x_1) \mathscr{L}_{\phi\phi\gamma}(x_2) \mathscr{L}_{\phi\phi\gamma\gamma}(x_3)] \\ \mathcal{T}[\mathscr{L}_{\phi\phi\gamma}(x_1) \mathscr{L}_{\phi\phi\gamma\gamma}(x_2) \mathscr{L}_{\phi\phi\gamma}(x_3)] \\ \mathcal{T}[\mathscr{L}_{\phi\phi\gamma\gamma}(x_1) \mathscr{L}_{\phi\phi\gamma}(x_2) \mathscr{L}_{\phi\phi\gamma}(x_3)]. \end{gathered} \tag{2.170}$$

These are $24 \times 3 \times 2 = 144$ contributing terms in total. Since our tensor $\Theta^{\triangle}$ is a rank-4 tensor, its total symmetrization contains exactly 24 terms – there are no more. In order to make these numbers match, we can theorize that each permutation in the symmetrization contributes 6 times (to get 144 terms). This argument leads to

$$\begin{aligned}\Gamma^{\triangle}_{\mu\nu\varrho\sigma}(p_1,p_2,p_3,p_4) &= \frac{6}{3!}\,\mathrm{sym}_4\,\Theta^{\triangle}_{\mu\nu\varrho\sigma}(p_1,p_2,p_3,p_4)\\ &= \mathrm{sym}_4\,\Theta^{\triangle}_{\mu\nu\varrho\sigma}(p_1,p_2,p_3,p_4),\end{aligned} \tag{2.171}$$

where the 1/3! prefactor comes from the Dyson expansion. Although $\Theta^{\triangle}$ is not an explicit function of p_3 or p_4, we can still use this formula. The conservation law (2.7) then can be used to get rid of the fourth momentum p_4.

2.3.3 Bubble diagrams

We again leave the problem of symmetrization to the end.

Diagram evaluation

This is the easiest diagram so far. It is displayed in Fig. 2.17.

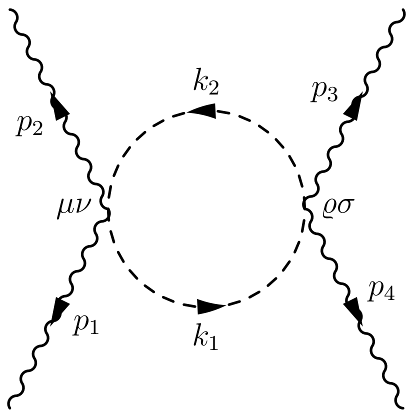

Figure 2.17: The scalar bubble diagram

Using standard Feynman rules, we get the relevant expression

$$i\Theta^{\bigcirc}_{\mu\nu\varrho\sigma}(p_1,p_2) \equiv e^4\mu^{4-D}\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{g_{\mu\nu}g_{\varrho\sigma}}{\prod_{i=1}^{2}(k_i^2-m^2)}, \tag{2.172}$$

where

$$\begin{aligned}k_1 &\equiv \ell\\ k_2 &\equiv \ell+p_1+p_2.\end{aligned} \tag{2.173}$$

The standard way of the Feynman parametrization and the loop momentum shift provide

$$i\Theta^{\bigcirc}_{\mu\nu\varrho\sigma}(p_1,p_2) = e^4\mu^{4-D}\int_{\mathrm{F}_1}\mathrm{d}X_{\mathrm{F}_1}\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{g_{\mu\nu}g_{\varrho\sigma}}{[\ell^2-m^2(1+\lambda)]^2}, \tag{2.174}$$

where

$$\int_{\mathrm{F}_1}\mathrm{d}X_{\mathrm{F}_1} \equiv \int_0^1\mathrm{d}x, \tag{2.175}$$

and

$$\lambda \equiv f_{12}^{\bigcirc}(x)\frac{p_1 \cdot p_2}{m^2}$$
$$f_{12}^{\bigcirc}(x) \equiv 2x(x-1). \tag{2.176}$$

We have only one simple term in this case

$$T_{\bigcirc} \equiv g_{\mu\nu}g_{\varrho\sigma}. \tag{2.177}$$

The standard way yields

$$\bar{\Theta}^{\bigcirc}_{\mu\nu\varrho\sigma} \equiv \frac{e^4}{16\pi^2}[\Delta - \ln(1+\lambda)]g_{\mu\nu}g_{\varrho\sigma}, \tag{2.178}$$

and

$$\bar{\Theta}^{\bigcirc}_{\mu\nu\varrho\sigma} = \frac{e^4}{32\pi^2}(2\Delta - 2\lambda + \lambda^2)g_{\mu\nu}g_{\varrho\sigma}. \tag{2.179}$$

The last step is the integration over the Feynman parameter

$$\Theta^{\bigcirc}_{\mu\nu\varrho\sigma}(p_1, p_2) \equiv \int_{\mathrm{F}_1} \mathrm{d}X_{\mathrm{F}_1}\; \bar{\Theta}^{\bigcirc}_{\mu\nu\varrho\sigma}. \tag{2.180}$$

Problem of symmetrization

We have no problem in this case. The relevant S-matrix element contains

$$a_1 a_2 a_3 a_4 \mathcal{T}[\mathscr{L}_{\phi\phi\gamma\gamma}(x_1)\mathscr{L}_{\phi\phi\gamma\gamma}(x_2)], \tag{2.181}$$

where $\mathscr{L}_{\phi\phi\gamma\gamma}$ is given by (2.132), and $a_i \equiv a(p_i, \lambda_i)$. There are 24 contractions of the photon operators and only 1 contraction of the scalar ones. The Dyson expansion contributes a prefactor of 1/2!. The correct formula can be immediately written down

$$\Gamma^{\bigcirc}_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4) = \frac{1}{2!}\,\mathrm{sym}_4\,\Theta^{\bigcirc}_{\mu\nu\varrho\sigma}(p_1, p_2, p_3, p_4). \tag{2.182}$$

Again, the conservation law (2.7) then eliminates the fourth momentum p_4.

2.3.4 Total polarization tensor

Now we can add up all the contributions to obtain the scalar QED polarization tensor

$$\Gamma^{\mathrm{sc.}}_{\mu\nu\varrho\sigma} \equiv \Gamma^{\square}_{\mu\nu\varrho\sigma} + \Gamma^{\triangle}_{\mu\nu\varrho\sigma} + \Gamma^{\bigcirc}_{\mu\nu\varrho\sigma}. \tag{2.183}$$

The result is displayed in Fig. 2.18. We have again thrown away the longitudinal terms proportional to p_1^{μ}, p_2^{ν} or p_3^{ϱ}.

Once again, we can see that the tensor has the proper structure (meaning it is UV finite and the terms proportional to two momenta vanished).

$$
\begin{aligned}
&\frac{1}{720\, m^4 \pi^2} \\
&e^4 \big(30\, g^{\mu\sigma} g^{\nu\rho}\, p1\cdot p2\, m^2 + 30\, g^{\mu\rho} g^{\nu\sigma}\, p1\cdot p2\, m^2 + 30\, g^{\mu\nu} g^{\rho\sigma}\, p1\cdot p2\, m^2 + \\
&\quad 30\, g^{\mu\sigma} g^{\nu\rho}\, p1\cdot p3\, m^2 + 30\, g^{\mu\rho} g^{\nu\sigma}\, p1\cdot p3\, m^2 + \\
&\quad 30\, g^{\mu\nu} g^{\rho\sigma}\, p1\cdot p3\, m^2 + 30\, g^{\mu\sigma} g^{\nu\rho}\, p2\cdot p3\, m^2 + \\
&\quad 30\, g^{\mu\rho} g^{\nu\sigma}\, p2\cdot p3\, m^2 + 30\, g^{\mu\nu} g^{\rho\sigma}\, p2\cdot p3\, m^2 + 4\, g^{\mu\sigma} g^{\nu\rho}\, p1\cdot p2^2 + \\
&\quad 4\, g^{\mu\rho} g^{\nu\sigma}\, p1\cdot p2^2 + 8\, g^{\mu\nu} g^{\rho\sigma}\, p1\cdot p2^2 + 4\, g^{\mu\sigma} g^{\nu\rho}\, p1\cdot p3^2 + \\
&\quad 8\, g^{\mu\rho} g^{\nu\sigma}\, p1\cdot p3^2 + 4\, g^{\mu\nu} g^{\rho\sigma}\, p1\cdot p3^2 + 8\, g^{\mu\sigma} g^{\nu\rho}\, p2\cdot p3^2 + \\
&\quad 4\, g^{\mu\rho} g^{\nu\sigma}\, p2\cdot p3^2 + 4\, g^{\mu\nu} g^{\rho\sigma}\, p2\cdot p3^2 + 2\, g^{\mu\sigma}\, p3^\nu\, p1^\rho\, p1\cdot p2 + \\
&\quad 6\, g^{\mu\sigma}\, p3^\nu\, p2^\rho\, p1\cdot p2 - g^{\mu\rho}\, p3^\nu\, p1^\sigma\, p1\cdot p2 - 4\, g^{\mu\rho}\, p1^\nu\, p2^\sigma\, p1\cdot p2 - \\
&\quad 3\, g^{\mu\rho}\, p3^\nu\, p2^\sigma\, p1\cdot p2 + g^{\mu\rho}\, p3^\nu\, p3^\sigma\, p1\cdot p2 + 2\, g^{\mu\nu}\, p1^\rho\, p3^\sigma\, p1\cdot p2 + \\
&\quad 2\, g^{\mu\nu}\, p2^\rho\, p3^\sigma\, p1\cdot p2 + 2\, g^{\mu\sigma}\, p1^\nu\, p2^\rho\, p1\cdot p3 + 6\, g^{\mu\sigma}\, p3^\nu\, p2^\rho\, p1\cdot p3 - \\
&\quad g^{\mu\nu}\, p2^\rho\, p1^\sigma\, p1\cdot p3 + 2\, g^{\mu\rho}\, p1^\nu\, p2^\sigma\, p1\cdot p3 + 2\, g^{\mu\rho}\, p3^\nu\, p2^\sigma\, p1\cdot p3 + \\
&\quad g^{\mu\nu}\, p2^\rho\, p2^\sigma\, p1\cdot p3 - 4\, g^{\mu\nu}\, p1^\rho\, p3^\sigma\, p1\cdot p3 - 3\, g^{\mu\nu}\, p2^\rho\, p3^\sigma\, p1\cdot p3 + \\
&\quad 6\, g^{\mu\sigma} g^{\nu\rho}\, p1\cdot p2\, p1\cdot p3 + 6\, g^{\mu\rho} g^{\nu\sigma}\, p1\cdot p2\, p1\cdot p3 + \\
&\quad 6\, g^{\mu\nu} g^{\rho\sigma}\, p1\cdot p2\, p1\cdot p3 - 2\, g^{\mu\sigma}\, p1^\nu\, p1^\rho\, p2\cdot p3 + g^{\mu\rho}\, p1^\nu\, p1^\sigma\, p2\cdot p3 + \\
&\quad g^{\mu\nu}\, p1^\rho\, p1^\sigma\, p2\cdot p3 - 3\, g^{\mu\rho}\, p1^\nu\, p2^\sigma\, p2\cdot p3 - 4\, g^{\mu\rho}\, p3^\nu\, p2^\sigma\, p2\cdot p3 - \\
&\quad g^{\mu\nu}\, p1^\rho\, p2^\sigma\, p2\cdot p3 - g^{\mu\rho}\, p1^\nu\, p3^\sigma\, p2\cdot p3 - 3\, g^{\mu\nu}\, p1^\rho\, p3^\sigma\, p2\cdot p3 - \\
&\quad 4\, g^{\mu\nu}\, p2^\rho\, p3^\sigma\, p2\cdot p3 + 6\, g^{\mu\sigma} g^{\nu\rho}\, p1\cdot p2\, p2\cdot p3 + \\
&\quad 6\, g^{\mu\rho} g^{\nu\sigma}\, p1\cdot p2\, p2\cdot p3 + 6\, g^{\mu\nu} g^{\rho\sigma}\, p1\cdot p2\, p2\cdot p3 + \\
&\quad 6\, g^{\mu\sigma} g^{\nu\rho}\, p1\cdot p3\, p2\cdot p3 + 6\, g^{\mu\rho} g^{\nu\sigma}\, p1\cdot p3\, p2\cdot p3 + \\
&\quad 6\, g^{\mu\nu} g^{\rho\sigma}\, p1\cdot p3\, p2\cdot p3 - p3^\mu\, (-6\, g^{\nu\sigma}\, p1^\rho\, p1\cdot p2 - \\
&\qquad 2\, g^{\nu\sigma}\, p2^\rho\, p1\cdot p2 + 3\, g^{\nu\rho}\, p1^\sigma\, p1\cdot p2 + g^{\nu\rho}\, p2^\sigma\, p1\cdot p2 - \\
&\qquad g^{\nu\rho}\, p3^\sigma\, p1\cdot p2 + 2\, p3^\nu\, (3\, p2^\rho\, p1^\sigma + 3\, p1^\rho\, p2^\sigma + g^{\rho\sigma}\, p1\cdot p2) + \\
&\qquad 4\, g^{\nu\rho}\, p1^\sigma\, p1\cdot p3 - 6\, g^{\nu\sigma}\, p1^\rho\, p2\cdot p3 - 2\, g^{\nu\rho}\, p1^\sigma\, p2\cdot p3 + \\
&\qquad p1^\nu\, (6\, p1^\rho\, p2^\sigma + p2^\rho\, (p1^\sigma + p2^\sigma + p3^\sigma) - 2\, g^{\rho\sigma}\, p2\cdot p3)) - \\
&\quad p2^\mu\, (4\, g^{\nu\rho}\, p1^\sigma\, p1\cdot p2 + 2\, g^{\nu\sigma}\, p2^\rho\, p1\cdot p3 + 3\, g^{\nu\rho}\, p1^\sigma\, p1\cdot p3 - \\
&\qquad g^{\nu\rho}\, p2^\sigma\, p1\cdot p3 + g^{\nu\rho}\, p3^\sigma\, p1\cdot p3 + \\
&\qquad p3^\nu\, (6\, p2^\rho\, p1^\sigma + p1^\rho\, (p1^\sigma + p2^\sigma + p3^\sigma) - 2\, g^{\rho\sigma}\, p1\cdot p3) - \\
&\qquad 2\, g^{\nu\sigma}\, p1^\rho\, p2\cdot p3 - 2\, g^{\nu\rho}\, p1^\sigma\, p2\cdot p3 + \\
&\qquad 6\, p1^\nu\, (p1^\rho\, p3^\sigma + p2^\rho\, p3^\sigma - g^{\rho\sigma}\, (p1\cdot p3 + p2\cdot p3)))\big)
\end{aligned}
$$

Figure 2.18: The "reduced" polarization tensor for scalar QED

2.3.5 Amplitude matching

We are finally ready to compare the scalar QED polarization tensor (Fig. 2.18) with the effective one (Fig. 2.4).

The fundamental equation reads

$$\Gamma^{\text{sc.}}_{\mu\nu\varrho\sigma} - \Gamma^{\text{eff.}}_{\mu\nu\varrho\sigma}(g_1, g_2) = 0. \tag{2.184}$$

Again, let us compare only the terms with no metric tensors. Fig. 2.19 shows what we obtain on the left-hand-side of our equation. Clearly, the linear system in this case is

$$
\begin{aligned}
e^4 - 3840\pi^2 m^4 (g_1 - g_2) &= 0 \\
e^4 - 23040\pi^2 m^4 g_2 &= 0,
\end{aligned} \tag{2.185}
$$

which has the following solution

$$
\begin{aligned}
g_1 &= \frac{7e^4}{23040\pi^2 m^4} = \frac{7\alpha^2}{1440 m^4} \\
g_2 &= \frac{e^4}{23040\pi^2 m^4} = \frac{\alpha^2}{1440 m^4}.
\end{aligned} \tag{2.186}
$$

Substituting the solution back into the left-hand-side of our equation leads to Fig. 2.20. Again, the constraint $p_4^2 = 0$ is factorized.

$$
\begin{aligned}
-\frac{1}{720\,\pi^2\, m^4}\Big(&\mathrm{p2}^\mu \big(6\,\mathrm{p1}^\nu\, \mathrm{p3}^\sigma\, (\mathrm{p1}^\rho + \mathrm{p2}^\rho)\big(e^4 - 3840\,\pi^2\, m^4\, (\mathrm{g1} - \mathrm{g2})\big) + \\
&\quad \mathrm{p3}^\nu \big(6\,\mathrm{p1}^\sigma\, \mathrm{p2}^\rho \big(e^4 - 3840\,\pi^2\, m^4\, (\mathrm{g1} - \mathrm{g2})\big) + \\
&\qquad \mathrm{p1}^\rho \big(e^4 - 23\,040\,\pi^2\, \mathrm{g2}\, m^4\big)(\mathrm{p1}^\sigma + \mathrm{p2}^\sigma + \mathrm{p3}^\sigma)\big)\big) + \\
&\mathrm{p3}^\mu \big(6\,\mathrm{p3}^\nu \big(e^4 - 3840\,\pi^2\, m^4\, (\mathrm{g1} - \mathrm{g2})\big)(\mathrm{p1}^\sigma\, \mathrm{p2}^\rho + \mathrm{p1}^\rho\, \mathrm{p2}^\sigma) + \\
&\quad \mathrm{p1}^\nu \big(6\,\mathrm{p1}^\rho\, \mathrm{p2}^\sigma \big(e^4 - 3840\,\pi^2\, m^4\, (\mathrm{g1} - \mathrm{g2})\big) + \\
&\qquad \mathrm{p2}^\rho \big(e^4 - 23\,040\,\pi^2\, \mathrm{g2}\, m^4\big)(\mathrm{p1}^\sigma + \mathrm{p2}^\sigma + \mathrm{p3}^\sigma)\big)\big)\Big)
\end{aligned}
$$

Figure 2.19: Matching the terms with no metric tensors

$$
\begin{aligned}
\frac{1}{720\,\pi^2\, m^4}\, e^4\, &(\mathrm{p1}\cdot\mathrm{p2} + \mathrm{p1}\cdot\mathrm{p3} + \mathrm{p2}\cdot\mathrm{p3})\big(30\, m^2\, g^{\mu\nu}\, g^{\rho\sigma} + g^{\mu\sigma} \\
&\big(3\, g^{\nu\rho}\big(10\, m^2 + \mathrm{p1}\cdot\mathrm{p2} + \mathrm{p1}\cdot\mathrm{p3} + \mathrm{p2}\cdot\mathrm{p3}\big) + \mathrm{p1}^\nu\, \mathrm{p2}^\rho + \mathrm{p3}^\nu\, (\mathrm{p1}^\rho + 5\,\mathrm{p2}^\rho)\big) + \\
&g^{\mu\rho}\big(3\, g^{\nu\sigma}\big(10\, m^2 + \mathrm{p1}\cdot\mathrm{p2} + \mathrm{p1}\cdot\mathrm{p3} + \mathrm{p2}\cdot\mathrm{p3}\big) - \mathrm{p3}^\nu\, (\mathrm{p1}^\sigma + 3\,\mathrm{p2}^\sigma) - \\
&\quad \mathrm{p1}^\nu\, (3\,\mathrm{p2}^\sigma + \mathrm{p3}^\sigma)\big) + 5\,\mathrm{p1}^\nu\, \mathrm{p2}^\mu\, g^{\rho\sigma} + \mathrm{p1}^\rho\, \mathrm{p2}^\mu\, g^{\nu\sigma} - \\
&3\,\mathrm{p1}^\sigma\, \mathrm{p2}^\mu\, g^{\nu\rho} - \mathrm{p1}^\sigma\, \mathrm{p2}^\rho\, g^{\mu\nu} - \mathrm{p1}^\rho\, \mathrm{p2}^\sigma\, g^{\mu\nu} + 3\, g^{\mu\nu}\, g^{\rho\sigma}\, \mathrm{p1}\cdot\mathrm{p2} + \\
&\mathrm{p1}^\nu\, \mathrm{p3}^\mu\, g^{\rho\sigma} + 5\,\mathrm{p1}^\rho\, \mathrm{p3}^\mu\, g^{\nu\sigma} - 3\,\mathrm{p1}^\sigma\, \mathrm{p3}^\mu\, g^{\nu\rho} - 3\,\mathrm{p1}^\rho\, \mathrm{p3}^\sigma\, g^{\mu\nu} + \\
&3\, g^{\mu\nu}\, g^{\rho\sigma}\, \mathrm{p1}\cdot\mathrm{p3} + \mathrm{p2}^\mu\, \mathrm{p3}^\nu\, g^{\rho\sigma} + \mathrm{p2}^\rho\, \mathrm{p3}^\mu\, g^{\nu\sigma} - \mathrm{p2}^\sigma\, \mathrm{p3}^\mu\, g^{\nu\rho} - \\
&\mathrm{p2}^\mu\, \mathrm{p3}^\sigma\, g^{\nu\rho} - 3\,\mathrm{p2}^\rho\, \mathrm{p3}^\sigma\, g^{\mu\nu} + 3\, g^{\mu\nu}\, g^{\rho\sigma}\, \mathrm{p2}\cdot\mathrm{p3}\big)
\end{aligned}
$$

Figure 2.20: The $p_4^2 = 0$ constraint is factorized

The effective Lagrangian of Euler-Heisenberg type for the case of scalar QED can be then expressed as

$$
\mathscr{L}_{\text{eff.}}^{\text{sc.}} = \frac{\alpha^2}{1440 m^4}\left[7(F_{\mu\nu}F^{\mu\nu})^2 + (\star F_{\mu\nu}F^{\mu\nu})^2\right], \tag{2.187}
$$

or

$$
\mathscr{L}_{\text{eff.}}^{\text{sc.}} = \frac{\alpha^2}{360 m^4}\left[7(\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E}\cdot\mathbf{B})^2\right]. \tag{2.188}
$$

This exact result can be found in [24]. Again, we can also calculate the differential cross section using the formula (2.47)

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{\mathrm{sc.}} = \frac{17\alpha^4}{64800\pi^2}\frac{\omega^6}{m^8}(3+\cos^2\theta)^2, \tag{2.189}$$

where $s = (2\omega)^2$.

2.4 Case of vector QED

The case of QED of charged vector bosons (vector QED as we call it) is really close to the case of scalar QED. There are some differences though. The interaction Lagrangian reads [22]

$$\mathscr{L}_{\mathrm{int.}} = \mathscr{L}_{WW\gamma} + \mathscr{L}_{WW\gamma\gamma}, \tag{2.190}$$

where

$$\begin{aligned} \mathscr{L}_{WW\gamma} &\equiv -ie(A_\mu W_\nu \overleftrightarrow{\partial^\mu} W^{\dagger\nu} + W_\mu W_\nu^\dagger \overleftrightarrow{\partial^\mu} A^\nu + W_\mu^\dagger A_\nu \overleftrightarrow{\partial^\mu} W^\nu) \\ \mathscr{L}_{WW\gamma\gamma} &\equiv -e^2(W_\mu W^{\dagger\mu} A_\nu A^\nu - W_\mu A^\mu W_\nu^\dagger A^\nu), \end{aligned} \tag{2.191}$$

and

$$f \overleftrightarrow{\partial^\mu} g \equiv f\partial^\mu g - g\partial^\mu f. \tag{2.192}$$

The Lagrangian consists of trilinear and quadrilinear vertices. The Lagrangian structure is, therefore, the same as in the case of scalar QED. This means that the case of vector QED is completely analogous to it. There are again three types of diagrams – boxes, triangles and bubbles (Fig. 2.21).

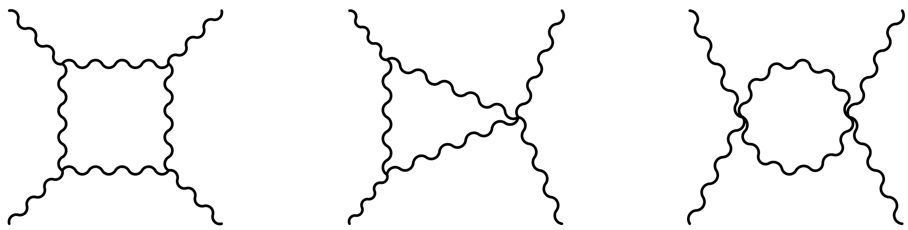

Figure 2.21: Three types of diagrams contribute

Since all the diagram parametrizations and the symmetrization formulae are exactly the same as in the previous section, we do not discuss them any longer.

The entire calculation is performed by using the canonical Proca propagator for the massive charged vector boson. If the vector QED is considered as a part of the standard electroweak theory, this corresponds to the so-called unitary gauge. Thus, in what follows, we will refer to this choice simply as unitary gauge. This is quite rare (this particular process probably has not been calculated using this gauge yet). Within the standard model, loop diagrams are usually processed using other gauges. In the unitary gauge, there are no auxiliary fields and there are no Faddeev-Popov ghosts. Only three types of diagrams contribute as we have stated before.

On the other hand, the expressions that appear during the calculation are ridiculously long (thousands of terms are common) and there are many of them.

Since there is nothing new about the method we demonstrated in the previous sections, we restrict ourselves to a brief overview and only discuss some subtleties and the used algorithm.

We start with the bubble since it is the easiest of the diagrams. But before that, let us introduce the following shorthand notation

$$\begin{aligned} V_{\alpha\beta\gamma}(a,b,c) &\equiv (a-b)_\gamma g_{\alpha\beta} + (b-c)_\alpha g_{\beta\gamma} + (c-a)_\beta g_{\gamma\alpha} \\ V_{\alpha\beta\gamma\delta} &\equiv g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\gamma}g_{\beta\delta} \\ P_{\alpha\beta}(a) &\equiv -g_{\alpha\beta} + \frac{a_\alpha a_\beta}{m^2}. \end{aligned} \tag{2.193}$$

2.4.1 Bubble diagrams

The diagram is shown in Fig. 2.22.

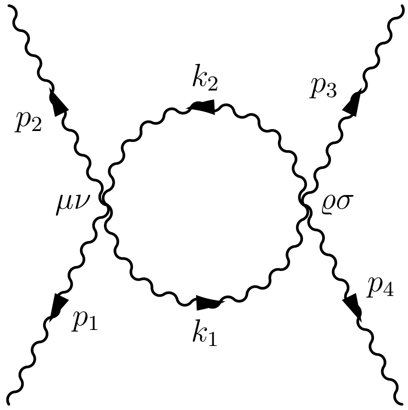

Figure 2.22: The vector bubble diagram

The standard Feynman rules lead to

$$i\Theta^{\bigcirc}_{\mu\nu\varrho\sigma}(p_1,p_2) \equiv e^4\mu^{4-D}\int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{M^{\bigcirc}_{\mu\nu\varrho\sigma}}{\prod_{i=1}^2(k_i^2-m^2)}, \tag{2.194}$$

where [see (2.193)]

$$\begin{aligned} M^{\bigcirc}_{\mu\nu\varrho\sigma} \equiv & P^{\beta_2\alpha_1}(k_1)V_{\mu\nu\alpha_1\alpha_2} \\ & P^{\alpha_2\beta_1}(k_2)V_{\varrho\sigma\beta_1\beta_2}. \end{aligned} \tag{2.195}$$

After the Feynman parametrization and the loop momentum shift, this becomes

$$i\Theta^{\bigcirc}_{\mu\nu\varrho\sigma}(p_1,p_2) \equiv e^4\mu^{4-D}\int_{\mathrm{F}_1}\mathrm{d}X_{\mathrm{F}_1}\int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{N^{\bigcirc}_{\mu\nu\varrho\sigma}}{[\ell^2-m^2(1+\lambda)]^2}, \tag{2.196}$$

where

$$\begin{aligned} N^{\bigcirc}_{\mu\nu\varrho\sigma} &\equiv P^{\beta_2\alpha_1}(\ell+q_1)V_{\mu\nu\alpha_1\alpha_2}P^{\alpha_2\beta_1}(\ell+q_2)V_{\varrho\sigma\beta_1\beta_2} \\ &= (g_{\mu\nu}g_{\alpha_1\alpha_2} - g_{\mu\alpha_1}g_{\nu\alpha_2})(g_{\varrho\sigma}g_{\beta_1\beta_2} - g_{\varrho\beta_1}g_{\sigma\beta_2}) \\ &\quad \left[-g_{\beta_2\alpha_1} + \frac{(\ell+q_1)^{\beta_2}(\ell+q_1)^{\alpha_1}}{m^2}\right]\left[-g_{\alpha_2\beta_1} + \frac{(\ell+q_2)^{\alpha_2}(\ell+q_2)^{\beta_1}}{m^2}\right] \end{aligned} \tag{2.197}$$

Our standard procedure of identifying non-vanishing terms in the numerator (terms containing even power of the loop momentum ℓ) does not work very well here. It would require expanding the entire expression, which is very tedious to do by hand (and it would get much worse later).

Fortunately, we can use `Mathematica` functions to implement some automation. We can rescale the loop momentum

$$\ell_\mu \to \xi \ell_\mu, \tag{2.198}$$

and then expand our expression in powers of ξ to pull out only the terms we want. Looking at the expression (2.197), it is clear that there are terms proportional to ξ^4 (terms containing four loop momenta), then ξ^2 (terms containing two loop momenta), and then ξ^0 (terms containing no loop momentum).

All these terms are UV divergent. From the terms ℓ^4 arise quartic divergences[7], from the terms ℓ^2 quadratic ones, and from the terms ℓ^0 logarithmic ones.

Since we now deal with more complex divergences, let us do a little adjustment to the master formula. We are interested in the following divergent integral

$$\mu^{4-D} \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{(\ell^2)^r \tilde{N}^{2r}_{\mu\nu\varrho\sigma}}{[\ell^2 - m^2(1+\lambda)]^s}, \tag{2.199}$$

where $\tilde{N}^{2r}_{\mu\nu\varrho\sigma}$ is the part of the numerator (in our case $N^{\bigcirc}_{\mu\nu\varrho\sigma}$) that contained $2r$ of loop momenta before this part was symmetrically integrated (in a general dimension), and all the loop momenta within were factored out. So, for instance, if the part proportional to four loop momenta was

$$\ell_\mu \ell_\nu \ell_\varrho \ell_\sigma \times [\text{something}], \tag{2.200}$$

$\tilde{N}^{4}_{\mu\nu\varrho\sigma}$ would be

$$\frac{g_{\mu\nu}g_{\varrho\sigma} + g_{\mu\varrho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\varrho}}{D(D+2)} \times [\text{something}]. \tag{2.201}$$

Clearly, $\tilde{N}^{2r}_{\mu\nu\varrho\sigma}$ is generally a function of the dimension.

Now set $D = 4 - 2\varepsilon$ and expand $\tilde{N}^{2r}_{\mu\nu\varrho\sigma}$ in powers of ε

$$\tilde{N}^{2r}_{\mu\nu\varrho\sigma} = A_N + B_N \varepsilon + \mathcal{O}\left(\varepsilon^2\right). \tag{2.202}$$

Define a rational function

$$R(s,r,\varepsilon) \equiv \frac{(r+1-\varepsilon)(r-\varepsilon)\dots(3-\varepsilon)(2-\varepsilon)}{(s-r-2+\varepsilon)(s-r-1+\varepsilon)\dots(\varepsilon-2)(\varepsilon-1)}. \tag{2.203}$$

It holds

$$\frac{\Gamma\left(r+2-\varepsilon\right)\Gamma(s-r-2\varepsilon)}{\Gamma(2-\varepsilon)} = R(s,r,\varepsilon)\Gamma(\varepsilon). \tag{2.204}$$

Expand this function in powers of ε

$$R(s,r,\varepsilon) = A_R + B_R \varepsilon + \mathcal{O}\left(\varepsilon^2\right). \tag{2.205}$$

[7] From now on, "the terms ℓ^{2n}" means "the terms proportional to ξ^{2n}".

Our integral now can be evaluated using the master formula (2.91)

$$\mu^{4-D}\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{(\ell^2)^r\tilde{N}^{2r}_{\mu\nu\varrho\sigma}}{[\ell^2-m^2(1+\lambda)]^s}$$
$$=\frac{i(-1)^{r-s}}{16\pi^2\Gamma(s)}[m^2(1+\lambda)]^{r-s+2}\left\{A_NA_R[\Delta-\ln(1+\lambda)]+B_RA_N+A_RB_N\right\}, \tag{2.206}$$

where we have set $C=m^2(1+\lambda)$, and

$$\Delta\equiv\frac{1}{\varepsilon}-\gamma_{\mathrm{E}}-\ln\left(\frac{m^2}{4\pi\mu^2}\right). \tag{2.207}$$

We also omitted $\mathcal{O}(\varepsilon)$ terms. This formula was actually used whenever we dealt with a divergent loop integral.

Let us return back to our original problem. It turns out, we need to categorize our terms in more detail (in order to do the λ-expansion properly). If we look at (2.197), we can see that there is no problem with the terms ℓ^4. These terms contain only the loop momenta and not the external momenta that are hidden inside the vectors q_i. In fact, they cannot contain them.

It is different for the terms ℓ^2. These could contain some external momenta, but they do not necessarily have to. For instance, we can just choose two ℓ from the first bracket and the metric tensor from the second one, which is a term with two ℓ and no external momenta.

The number of the external momenta contained within a term is actually related to some other characteristics of the term via the following self-evident formula

$$e+l-2n=d, \tag{2.208}$$

where e is the number of the external momenta, l is number of the loop momenta, n is the power of $1/m^2$ the term is proportional to, and d is the mass dimension of the entire expression. Consequently, we have

$$e=d-l+2n. \tag{2.209}$$

For instance, the mass dimension of our expression is 0. If we choose one ℓ from the first bracket and a second ℓ from the second one, we have $l=2$, $n=2$, $e=2$, and the formula holds.

We should have a good control of our terms now and should be able to summarize our general algorithm:

1. Categorize terms according to the number of the loop momenta.

2. Sub-categorize each categorized term according to the power of $1/m^2$ it is proportional to.

3. Perform the symmetric integration (depends on the number of the loop momenta, and is dimension-dependent for divergent terms).

4. Perform the loop integration using the master formula (2.91) or using (2.206).

5. Perform the λ-expansion up to the correct order (so there are four external momenta in the result).
6. Plug into the processed terms the expressions for q_i and λ.
7. Integrate over the Feynman parameters.
8. Use the symmetrization formulae and sum it all up.

Type of terms	*Divergence*	*# of ext. mom.*	*λ-expansion stops at*
ℓ^4	quartic	0	λ^2
$\ell^2 m^{-2}$	quadratic	0	λ^2
$\ell^2 m^{-4}$	quadratic	2	λ
$\ell^0 m^0$	logarithmic	0	λ^2
$\ell^0 m^{-2}$	logarithmic	2	λ^1
$\ell^0 m^{-4}$	logarithmic	4	λ^0

Table 2.1: The vector bubble terms summary

All the contributing terms are summarized in the Tab. 2.1. The full expressions can be found in the enclosed notebook `bubble.nb`. The notation should be self-explanatory.

2.4.2 Triangle diagrams

The triangle diagram is in the Fig. 2.23.

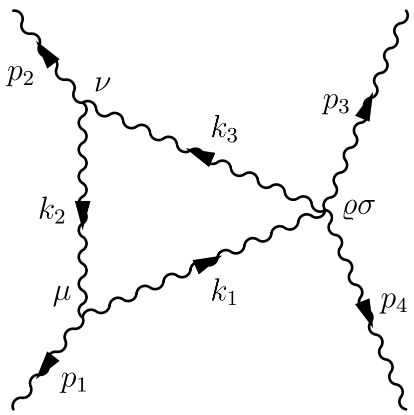

Figure 2.23: The vector triangle diagram

The relevant expression reads (after the Feynman parametrization and the loop momentum shift)

$$i\Theta^{\triangle}_{\mu\nu\varrho\sigma}(p_1,p_2) \equiv -e^4\mu^{4-D}\int_{\mathrm{F}_2}\mathrm{d}X_{\mathrm{F}_2}\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{N^{\triangle}_{\mu\nu\varrho\sigma}}{[\ell^2-m^2(1+\lambda)]^3}, \tag{2.210}$$

where

$$\begin{aligned} N^{\triangle}_{\mu\nu\varrho\sigma} \equiv\ & P^{\gamma_2\alpha_1}(\ell+q_1)V_{\mu\alpha_1\alpha_2}(p_1,\ell+q_1,-\ell-q_2) \\ & P^{\alpha_2\beta_1}(\ell+q_2)V_{\nu\beta_1\beta_2}(p_2,\ell+q_2,-\ell-q_3) \\ & P^{\beta_2\gamma_1}(\ell+q_3)V_{\varrho\sigma\gamma_1\gamma_2}. \end{aligned} \tag{2.211}$$

Since the mass dimension of the expression is 2 in this case, the formula (2.209) takes a form

$$e = 2 - l + 2n. \tag{2.212}$$

The contributing terms are shown in the Tab. 2.1 and the full expressions can be found in the enclosed notebook `triangle.nb`. We can see that the worst divergence vanishes algebraically (meaning the ℓ^8 terms are zero due to the algebraic structure of the expression).

Type of terms	*Divergence*	*# of ext. mom.*	*λ-expansion stops at*
ℓ^8	sextic	vanishes algebraically	
ℓ^6	quartic	0	λ^2
$\ell^4 m^{-2}$	quadratic	0	λ^2
$\ell^4 m^{-4}$	quadratic	2	λ^1
$\ell^4 m^{-6}$	quadratic	4	λ^0
$\ell^2 m^0$	logarithmic	0	λ^2
$\ell^2 m^{-2}$	logarithmic	2	λ^1
$\ell^2 m^{-4}$	logarithmic	4	λ^0
$\ell^0 m^0$	finite	2	λ^1
$\ell^0 m^{-2}$	finite	4	λ^0

Table 2.2: The vector triangle terms summary

Apart from that, the terms ℓ^6 do not contain any external momenta even though they could (again, this is due to the algebraic structure). That means we do not have to sub-categorize these terms according to the power of $1/m^2$.

It also means that we do not need the symmetric integration formula for six loop momenta (because there can be only four non-contracted loop momenta in the expression since there are only four Lorentz indices and there are no other momenta).

2.4.3 Box diagrams

The worst diagram is in Fig. 2.24.

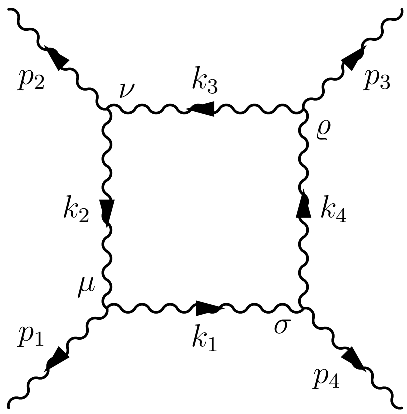

Figure 2.24: The vector box diagram

The relevant expression reads

$$i\Theta^{\Box}_{\mu\nu\varrho\sigma}(p_1,p_2,p_3) \equiv e^4\mu^{4-D}\int_{\mathrm{F}_3}\mathrm{d}X_{\mathrm{F}_3}\int\frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{N^{\Box}_{\mu\nu\varrho\sigma}}{[\ell^2-m^2(1+\lambda)]^4}, \qquad (2.213)$$

where

$$\begin{aligned} N^{\Box}_{\mu\nu\varrho\sigma} \equiv\ & P^{\delta_2\alpha_1}(\ell+q_1)V_{\mu\alpha_1\alpha_2}(p_1,\ell+q_1,-\ell-q_2) \\ & P^{\alpha_2\beta_1}(\ell+q_2)V_{\nu\beta_1\beta_2}(p_2,\ell+q_2,-\ell-q_3) \\ & P^{\beta_2\gamma_1}(\ell+q_3)V_{\varrho\gamma_1\gamma_2}(p_3,\ell+q_3,-\ell-q_4) \\ & P^{\gamma_2\delta_1}(\ell+q_4)V_{\sigma\delta_1\delta_2}(p_4,\ell+q_4,-\ell-q_1). \end{aligned} \qquad (2.214)$$

The formula (2.209) takes a form

$$e = 4 - l + 2n. \qquad (2.215)$$

The contributing terms are shown in the Tab. 2.1 and the full expressions can be found in the enclosed notebook `box.nb`. The worst two divergences vanish algebraically and the ℓ^8 terms do not contain any external momenta (so we do not need the symmetric integration formula for eight loop momenta).

Type of terms	*Divergence*	*# of ext. mom.*	*λ-expansion stops at*
ℓ^{12}	octic	vanishes algebraically	
ℓ^{10}	sextic	vanishes algebraically	
ℓ^8	quartic	0	λ^2
$\ell^6 m^{-2}$	quadratic	0	λ^2
$\ell^6 m^{-4}$	quadratic	2	λ^1
$\ell^6 m^{-6}$	quadratic	4	λ^0
$\ell^4 m^0$	logarithmic	0	λ^2
$\ell^4 m^{-2}$	logarithmic	2	λ^1
$\ell^4 m^{-4}$	logarithmic	4	λ^0
$\ell^2 m^0$	finite	2	λ^1
$\ell^2 m^{-2}$	finite	4	λ^0
$\ell^0 m^0$	finite	4	λ^0

Table 2.3: The vector box terms summary

The terms ℓ^6 unfortunately contain external momenta so the following formula has to used

$$\ell_\alpha\ell_\beta\ell_\gamma\ell_\delta\ell_\mu\ell_\nu \overset{\text{eff.}}{=} \frac{(\ell^2)^3}{D(D+2)(D+4)}(g_{\alpha\beta}g_{\gamma\delta}g_{\mu\nu}+\ldots), \qquad (2.216)$$

where the bracket contains 15 terms in total.

2.4.4 Total polarization tensor

At the end of the whole procedure, we got the vector QED polarization tensor

$$\Gamma^{\mathrm{sc.}}_{\mu\nu\varrho\sigma} \equiv \Gamma^{\Box}_{\mu\nu\varrho\sigma} + \Gamma^{\triangle}_{\mu\nu\varrho\sigma} + \Gamma^{\bigcirc}_{\mu\nu\varrho\sigma}. \qquad (2.217)$$

$$
-\frac{1}{80\, m^4 \pi^2}
$$

$$
\begin{aligned}
& e^4 \Big(-2\, g^{\mu\sigma}\, g^{\nu\rho}\, \mathrm{p1}\cdot\mathrm{p2}^2 - 54\, g^{\mu\rho}\, g^{\nu\sigma}\, \mathrm{p1}\cdot\mathrm{p2}^2 - 2\, g^{\mu\nu}\, g^{\rho\sigma}\, \mathrm{p1}\cdot\mathrm{p2}^2 - \\
&\quad 54\, g^{\mu\sigma}\, \mathrm{p1}^\nu\, \mathrm{p1}^\rho\, \mathrm{p1}\cdot\mathrm{p2} - 54\, g^{\mu\sigma}\, \mathrm{p3}^\nu\, \mathrm{p1}^\rho\, \mathrm{p1}\cdot\mathrm{p2} - \\
&\quad 2\, g^{\mu\sigma}\, \mathrm{p3}^\nu\, \mathrm{p2}^\rho\, \mathrm{p1}\cdot\mathrm{p2} + 27\, g^{\mu\rho}\, \mathrm{p1}^\nu\, \mathrm{p1}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} + \\
&\quad 27\, g^{\mu\rho}\, \mathrm{p3}^\nu\, \mathrm{p1}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} + 27\, g^{\mu\nu}\, \mathrm{p1}^\rho\, \mathrm{p1}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} + \\
&\quad 27\, g^{\mu\rho}\, \mathrm{p1}^\nu\, \mathrm{p2}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} - 27\, g^{\mu\rho}\, \mathrm{p3}^\nu\, \mathrm{p2}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} - \\
&\quad 27\, g^{\mu\nu}\, \mathrm{p1}^\rho\, \mathrm{p2}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} - 27\, g^{\mu\rho}\, \mathrm{p1}^\nu\, \mathrm{p3}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} - \\
&\quad 27\, g^{\mu\rho}\, \mathrm{p3}^\nu\, \mathrm{p3}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} + 25\, g^{\mu\nu}\, \mathrm{p1}^\rho\, \mathrm{p3}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} - \\
&\quad 2\, g^{\mu\nu}\, \mathrm{p2}^\rho\, \mathrm{p3}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} + 50\, g^{\mu\sigma}\, g^{\nu\rho}\, \mathrm{p1}\cdot\mathrm{p3}\, \mathrm{p1}\cdot\mathrm{p2} - \\
&\quad 54\, g^{\mu\rho}\, g^{\nu\sigma}\, \mathrm{p1}\cdot\mathrm{p3}\, \mathrm{p1}\cdot\mathrm{p2} - 54\, g^{\mu\nu}\, g^{\rho\sigma}\, \mathrm{p1}\cdot\mathrm{p3}\, \mathrm{p1}\cdot\mathrm{p2} - \\
&\quad 2\, g^{\mu\sigma}\, g^{\nu\rho}\, \mathrm{p1}\cdot\mathrm{p3}^2 - 2\, g^{\mu\rho}\, g^{\nu\sigma}\, \mathrm{p1}\cdot\mathrm{p3}^2 - 54\, g^{\mu\nu}\, g^{\rho\sigma}\, \mathrm{p1}\cdot\mathrm{p3}^2 - \\
&\quad 54\, g^{\mu\sigma}\, \mathrm{p1}^\nu\, \mathrm{p1}^\rho\, \mathrm{p1}\cdot\mathrm{p3} - 54\, g^{\mu\sigma}\, \mathrm{p1}^\nu\, \mathrm{p2}^\rho\, \mathrm{p1}\cdot\mathrm{p3} - \\
&\quad 2\, g^{\mu\sigma}\, \mathrm{p3}^\nu\, \mathrm{p2}^\rho\, \mathrm{p1}\cdot\mathrm{p3} + 27\, g^{\mu\rho}\, \mathrm{p1}^\nu\, \mathrm{p1}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} + \\
&\quad 27\, g^{\mu\nu}\, \mathrm{p1}^\rho\, \mathrm{p1}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} + 27\, g^{\mu\nu}\, \mathrm{p2}^\rho\, \mathrm{p1}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} + \\
&\quad 25\, g^{\mu\rho}\, \mathrm{p1}^\nu\, \mathrm{p2}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} - 2\, g^{\mu\rho}\, \mathrm{p3}^\nu\, \mathrm{p2}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} - \\
&\quad 27\, g^{\mu\nu}\, \mathrm{p1}^\rho\, \mathrm{p2}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} - 27\, g^{\mu\nu}\, \mathrm{p2}^\rho\, \mathrm{p2}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} - \\
&\quad 27\, g^{\mu\rho}\, \mathrm{p1}^\nu\, \mathrm{p3}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} + 27\, g^{\mu\nu}\, \mathrm{p1}^\rho\, \mathrm{p3}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} - \\
&\quad 27\, g^{\mu\nu}\, \mathrm{p2}^\rho\, \mathrm{p3}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} + \mathrm{p2}^\mu\, (54\, g^{\nu\sigma}\, \mathrm{p1}^\rho\, \mathrm{p1}\cdot\mathrm{p2} + \\
&\qquad 2\, g^{\nu\rho}\, \mathrm{p1}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} + 2\, \mathrm{p1}^\nu\, (\mathrm{p1}^\rho\, \mathrm{p3}^\sigma + \mathrm{p2}^\rho\, \mathrm{p3}^\sigma + g^{\rho\sigma}\, \mathrm{p1}\cdot\mathrm{p2}) + \\
&\qquad 54\, g^{\nu\sigma}\, \mathrm{p1}^\rho\, \mathrm{p1}\cdot\mathrm{p3} + 54\, g^{\nu\sigma}\, \mathrm{p2}^\rho\, \mathrm{p1}\cdot\mathrm{p3} - 25\, g^{\nu\rho}\, \mathrm{p1}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} - \\
&\qquad 27\, g^{\nu\rho}\, \mathrm{p2}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} + 27\, g^{\nu\rho}\, \mathrm{p3}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} + \mathrm{p3}^\nu \\
&\qquad\quad (2\, \mathrm{p2}^\rho\, \mathrm{p1}^\sigma + 27\, \mathrm{p1}^\rho\, (\mathrm{p1}^\sigma + \mathrm{p2}^\sigma + \mathrm{p3}^\sigma) - 54\, g^{\rho\sigma}\, \mathrm{p1}\cdot\mathrm{p3})) + \\
&\quad \mathrm{p3}^\mu\, (-54\, g^{\nu\sigma}\, \mathrm{p2}^\rho\, \mathrm{p1}\cdot\mathrm{p2} - 25\, g^{\nu\rho}\, \mathrm{p1}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} + \\
&\qquad 27\, g^{\nu\rho}\, \mathrm{p2}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} - 27\, g^{\nu\rho}\, \mathrm{p3}^\sigma\, \mathrm{p1}\cdot\mathrm{p2} + \\
&\qquad 2\, \mathrm{p3}^\nu\, (\mathrm{p2}^\rho\, \mathrm{p1}^\sigma + \mathrm{p1}^\rho\, \mathrm{p2}^\sigma + 27\, g^{\rho\sigma}\, \mathrm{p1}\cdot\mathrm{p2}) + \\
&\qquad 2\, g^{\nu\sigma}\, \mathrm{p1}^\rho\, \mathrm{p1}\cdot\mathrm{p3} + 2\, g^{\nu\rho}\, \mathrm{p1}^\sigma\, \mathrm{p1}\cdot\mathrm{p3} + \mathrm{p1}^\nu\, (2\, \mathrm{p1}^\rho\, \mathrm{p2}^\sigma + \\
&\qquad\quad 27\, \mathrm{p2}^\rho\, (\mathrm{p1}^\sigma + \mathrm{p2}^\sigma + \mathrm{p3}^\sigma) + 54\, g^{\rho\sigma}\, (\mathrm{p1}\cdot\mathrm{p2} + \mathrm{p1}\cdot\mathrm{p3})))\Big)
\end{aligned}
$$

Figure 2.25: The "reduced" polarization tensor for vector QED

The result is displayed in Fig. 2.25. Again, we have thrown away the longitudinal terms proportional to p_1^μ, p_2^ν or p_3^ϱ.

Clearly, the tensor has the proper structure (it is UV finite and the terms proportional to two momenta are not present)

2.4.5 Amplitude matching

Again, let us compare the vector QED polarization tensor (Fig. 2.25) with the effective one (Fig. 2.4).

The fundamental equation reads

$$
\Gamma^{\mathrm{vec.}}_{\mu\nu\varrho\sigma} - \Gamma^{\mathrm{eff.}}_{\mu\nu\varrho\sigma}(g_1, g_2) = 0. \tag{2.218}
$$

First, compare only the terms with no metric tensors. Fig. 2.26 shows what we obtain on the left-hand-side. The linear system in this case is

$$\begin{aligned} e^4 - 1280\pi^2 m^4(g_1 - g_2) = 0 \\ 27e^4 - 2560\pi^2 m^4 g_2 = 0, \end{aligned} \tag{2.219}$$

which has the following solution

$$\begin{aligned} g_1 &= \frac{29e^4}{2560\pi^2 m^4} = \frac{29\alpha^2}{160m^4} \\ g_2 &= \frac{27e^4}{2560\pi^2 m^4} = \frac{27\alpha^2}{160m^4}. \end{aligned} \tag{2.220}$$

Substituting the solution back into the left-hand-side of our equation leads to Fig. 2.27. Again, the constraint $p_4^2 = 0$ is factorized.

$$\begin{aligned} \frac{1}{80\,\pi^2\,m^4}\Big(&p2^\mu\,\big(2\,p1^\nu\,p3^\sigma\,(p1^\rho + p2^\rho)\,(e^4 - 1280\,\pi^2\,m^4\,(g1 - g2)) + \\ &\quad p3^\nu\,\big(2\,p1^\sigma\,p2^\rho\,(e^4 - 1280\,\pi^2\,m^4\,(g1 - g2)) + \\ &\qquad p1^\rho\,(27\,e^4 - 2560\,\pi^2\,g2\,m^4)\,(p1^\sigma + p2^\sigma + p3^\sigma)\big)\big) + \\ &p3^\mu\,\big(2\,p3^\nu\,(e^4 - 1280\,\pi^2\,m^4\,(g1 - g2))\,(p1^\sigma\,p2^\rho + p1^\rho\,p2^\sigma) + \\ &\quad p1^\nu\,\big(2\,p1^\rho\,p2^\sigma\,(e^4 - 1280\,\pi^2\,m^4\,(g1 - g2)) + \\ &\qquad p2^\rho\,(27\,e^4 - 2560\,\pi^2\,g2\,m^4)\,(p1^\sigma + p2^\sigma + p3^\sigma)\big)\big)\Big) \end{aligned}$$

Figure 2.26: Matching the terms with no metric tensors

$$\begin{aligned} &\frac{1}{80\,\pi^2\,m^4}\,e^4\,(p1\cdot p2 + p1\cdot p3 + p2\cdot p3) \\ &(-54\,p1^\nu\,p1^\rho\,g^{\mu\sigma} + 27\,p1^\rho\,p1^\sigma\,g^{\mu\nu} + 27\,p1^\nu\,p1^\sigma\,g^{\mu\rho} - \\ &\quad 27\,p1^\nu\,p2^\rho\,g^{\mu\sigma} + 27\,p1^\sigma\,p2^\rho\,g^{\mu\nu} + 25\,g^{\mu\sigma}\,g^{\nu\rho}\,p1\cdot p2 - \\ &\quad 27\,g^{\mu\rho}\,g^{\nu\sigma}\,p1\cdot p2 - 27\,g^{\mu\nu}\,g^{\rho\sigma}\,p1\cdot p2 + \\ &\quad p3^\mu\,(27\,p1^\nu\,g^{\rho\sigma} + 27\,g^{\nu\sigma}\,(p1^\rho - p2^\rho) + g^{\nu\rho}\,(27\,p2^\sigma - 25\,p1^\sigma)) + \\ &\quad p2^\mu\,(27\,p1^\nu\,g^{\rho\sigma} + 27\,p1^\rho\,g^{\nu\sigma} - 25\,p1^\sigma\,g^{\nu\rho} - 27\,p3^\nu\,g^{\rho\sigma} + \\ &\qquad 27\,p3^\sigma\,g^{\nu\rho}) - 27\,p1^\rho\,p3^\nu\,g^{\mu\sigma} + 27\,p1^\sigma\,p3^\nu\,g^{\mu\rho} + \\ &\quad 25\,g^{\mu\sigma}\,g^{\nu\rho}\,p1\cdot p3 - 27\,g^{\mu\rho}\,g^{\nu\sigma}\,p1\cdot p3 - 27\,g^{\mu\nu}\,g^{\rho\sigma}\,p1\cdot p3 + \\ &\quad 25\,p2^\rho\,p3^\nu\,g^{\mu\sigma} - 27\,p2^\sigma\,p3^\nu\,g^{\mu\rho} - 27\,p2^\rho\,p3^\sigma\,g^{\mu\nu} - \\ &\quad 25\,g^{\mu\sigma}\,g^{\nu\rho}\,p2\cdot p3 + 27\,g^{\mu\rho}\,g^{\nu\sigma}\,p2\cdot p3 + 27\,g^{\mu\nu}\,g^{\rho\sigma}\,p2\cdot p3) \end{aligned}$$

Figure 2.27: The $p_4^2 = 0$ constraint is factorized

The effective Lagrangian of Euler-Heisenberg type for the case of vector QED can be then expressed as

$$\mathscr{L}_{\text{eff.}}^{\text{vec.}} = \frac{\alpha^2}{160m^4}\left[29(F_{\mu\nu}F^{\mu\nu})^2 + 27(\star F_{\mu\nu}F^{\mu\nu})^2\right], \tag{2.221}$$

or

$$\mathscr{L}_{\text{eff.}}^{\text{vec.}} = \frac{\alpha^2}{40m^4}\left[29(\mathbf{E}^2 - \mathbf{B}^2)^2 + 108(\mathbf{E}\cdot\mathbf{B})^2\right]. \tag{2.222}$$

Precisely the same result can be found in [21], where it has been derived by completely different means, namely by using the functional methods. Again, one can also calculate the differential cross section using the formula (2.47)

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{\text{vec.}} = \frac{393\alpha^4}{800\pi^2}\frac{\omega^6}{m^8}(3+\cos^2\theta)^2, \tag{2.223}$$

where we used $s = (2\omega)^2$. The identical result is shown in [20].

3. Functional methods

In this chapter, as a counterpart to the previous one, we shall demonstrate the power of the functional methods on a "simple" example. We derive the Euler-Heisenberg Lagrangian for the case of spinor QED using the Fock-Schwinger proper time method. The calculation draws heavily on [23] and [25].

3.1 Preliminaries

The general theory of path integrals gives us the following formula for the effective Lagrangian[1]

$$\int \mathrm{d}^4x \; \mathscr{L}_{\text{eff.}} = -i \ln\det \bar{\mathscr{D}} \equiv -i \ln\det \left(\frac{\mathscr{D}}{\mathscr{D}|_{A_\mu=0}} \right), \tag{3.1}$$

where

$$\mathscr{D} \equiv i\partial\!\!\!/ - eA\!\!\!/ - m \tag{3.2}$$

is the Dirac operator. Both the logarithm and the determinant are understood in the operator sense.

The important observation is that it suffices to consider and calculate this determinant for the case of constant electromagnetic field (meaning constant $F_{\mu\nu}$). The reason is that the Lagrangian in question contains no derivatives of the electromagnetic tensor (see the beginning of the previous chapter).

We can write

$$\ln\det \bar{\mathscr{D}} = \mathrm{Tr} \ln \left(\frac{i\partial\!\!\!/ - eA\!\!\!/ - m}{i\partial\!\!\!/ - m} \right), \tag{3.3}$$

where we used the famous relation

$$\det \exp A = \exp \mathrm{Tr}\, A, \tag{3.4}$$

and the operator "Tr" denotes the trace both in the x-space (the position space) and in the space of Dirac indices.

Next, we perform a little trick

$$\begin{aligned} \ln\det \bar{\mathscr{D}} &= \mathrm{Tr} \ln \left[\frac{C(i\partial\!\!\!/ - eA\!\!\!/ - m)C^{-1}}{C(i\partial\!\!\!/ - m)C^{-1}} \right] \\ &= \mathrm{Tr} \ln \left[\frac{C(i\partial\!\!\!/ - eA\!\!\!/ - m)C^{-1}}{C(i\partial\!\!\!/ - m)C^{-1}} \right] \\ &= \mathrm{Tr} \ln \left[\frac{-(i\partial\!\!\!/ - eA\!\!\!/)^{\mathrm{T}} - m}{-(i\partial\!\!\!/)^{\mathrm{T}} - m} \right] \\ &= \mathrm{Tr} \ln \left(\frac{i\partial\!\!\!/ - eA\!\!\!/ + m}{i\partial\!\!\!/ + m} \right), \end{aligned} \tag{3.5}$$

[1] This can be proved using the generating functional [25]

$$Z(j) = \int \mathrm{d}\psi \mathrm{d}\bar{\psi} \mathrm{d}A_\mu \; \exp \left[i \int \mathrm{d}^4x \; (\mathscr{L}_{\text{QED}} + j^\mu A_\mu) \right],$$

where we "integrate out" the fermion degrees of freedom. This leads to a Berezin-Grassmann integral of a gaussian.

where we used the identity $C\gamma_\mu C^{-1} = -\gamma_\mu^{\mathrm{T}}$. This trick allows us to write

$$
\begin{aligned}
2 \ln \det \bar{\mathscr{D}} &= \mathrm{Tr} \ln \left(\frac{i\partial\!\!\!/ - eA\!\!\!/ - m}{i\partial\!\!\!/ - m} \right) + \mathrm{Tr} \ln \left(\frac{i\partial\!\!\!/ - eA\!\!\!/ + m}{i\partial\!\!\!/ + m} \right) \\
&= \mathrm{Tr} \ln \left[\frac{(i\partial\!\!\!/ - eA\!\!\!/)^2 - m^2}{(i\partial\!\!\!/)^2 - m^2} \right] . \qquad (3.6)
\end{aligned}
$$

Now we use the following integral formula

$$
\ln \frac{\alpha}{\beta} = \int_0^\infty \frac{\mathrm{d}s}{s} \left[\mathrm{e}^{is(\beta + i0)} - \mathrm{e}^{is(\alpha + i0)} \right] \qquad (3.7)
$$

to obtain

$$
2 \ln \det \bar{\mathscr{D}} = - \int_0^\infty \frac{\mathrm{d}s}{s} \, \mathrm{e}^{-is(m^2 - i0)} \int \mathrm{d}^4 x \, \mathrm{tr} \left[\langle x | \mathrm{e}^{is(i\partial\!\!\!/ - eA\!\!\!/)^2} | x \rangle - \langle x | \mathrm{e}^{is(i\partial\!\!\!/)^2} | x \rangle \right] . \qquad (3.8)
$$

The trace in the x-space is now written explicitly as $\int \mathrm{d}^4 x$, whereas the remaining trace in the space of Dirac indices is denoted by "tr".

Another needed identity reads

$$
\begin{aligned}
(i\partial\!\!\!/ - eA\!\!\!/)^2 &= (i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - \frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu} \\
&= (i\partial_\mu - eA_\mu)^2 - \frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu} , \qquad (3.9)
\end{aligned}
$$

where

$$
\sigma_{\mu\nu} \equiv \frac{i}{2} [\gamma_\mu, \gamma_\nu] . \qquad (3.10)
$$

This can be proved straightforwardly using

$$
\begin{aligned}
\gamma_\mu \gamma_\nu &= \frac{1}{2} \{\gamma_\mu, \gamma_\nu\} + \frac{1}{2} [\gamma_\mu, \gamma_\nu] \\
&= g_{\mu\nu} - i\sigma_{\mu\nu} . \qquad (3.11)
\end{aligned}
$$

Substituting (3.9) into (3.8) yields

$$
\begin{aligned}
2 \ln \det \bar{\mathscr{D}} = - \int_0^\infty \frac{\mathrm{d}s}{s} \, \mathrm{e}^{-is(m^2 - i0)} & \\
\Bigg\{ \left[\mathrm{tr} \exp \left(- \frac{ise}{2} \sigma_{\mu\nu} F^{\mu\nu} \right) \right] & \int \mathrm{d}^4 x \left[\langle x | \mathrm{e}^{is(i\partial_\mu - eA_\mu)^2} | x \rangle \right] \\
- \bigcirc_{A_\mu = 0} \Bigg\} & , \qquad (3.12)
\end{aligned}
$$

where

$$
\bigcirc_{A_\mu = 0} \equiv \left\{ \left[\mathrm{tr} \exp \left(- \frac{ise}{2} \sigma_{\mu\nu} F^{\mu\nu} \right) \right] \int \mathrm{d}^4 x \left[\langle x | \mathrm{e}^{is(i\partial_\mu - eA_\mu)^2} | x \rangle \right] \right\} \Bigg|_{A_\mu = 0} . \qquad (3.13)
$$

Note that we have just used the fact the electromagnetic field is constant which enabled us to pull the Dirac trace out of the integral.

3.2 Dirac trace

We are now ready to calculate the Dirac trace [the first part of the expression (3.12)]. But first, we need to treat the (constant) electromagnetic field in some way. We define two quantities a, b

$$
\begin{aligned}
a^2 - b^2 \equiv \mathbf{E}^2 - \mathbf{B}^2 &= -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2}\mathcal{F} \\
ab \equiv \mathbf{E} \cdot \mathbf{B} &= \frac{1}{4}(\star F_{\mu\nu} F^{\mu\nu}) = \frac{1}{4}\mathcal{G},
\end{aligned} \tag{3.14}
$$

and set

$$
F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & b & 0 \\ 0 & -b & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}. \tag{3.15}
$$

Two $\sigma_{\mu\nu}$ matrices are needed

$$
\sigma_{30} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \sigma_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.16}
$$

We used the standard representation of Dirac matrices to obtain these.

From this, we have

$$
\sigma_{\mu\nu} F^{\mu\nu} = \begin{pmatrix} 2b & 0 & 2ia & 0 \\ 0 & -2b & 0 & -2ia \\ 2ia & 0 & 2b & 0 \\ 0 & -2ia & 0 & -2b \end{pmatrix}. \tag{3.17}
$$

The matrix exponential was evaluated using `Mathematica`

$$
\exp\left(-\frac{ise}{2}\sigma_{\mu\nu} F^{\mu\nu}\right) = \begin{pmatrix} c_a\bar{z} & 0 & s_a\bar{z} & 0 \\ 0 & c_a z & 0 & -s_a z \\ s_a\bar{z} & 0 & c_a\bar{z} & 0 \\ 0 & -s_a z & 0 & c_a z \end{pmatrix}, \tag{3.18}
$$

where

$$
\begin{aligned}
c_a &\equiv \cosh(aes) \\
s_a &\equiv \sinh(aes) \\
z &\equiv \mathrm{e}^{iebs},
\end{aligned} \tag{3.19}
$$

from which we can easily see that

$$
\operatorname{tr}\exp\left(-\frac{ise}{2}\sigma_{\mu\nu} F^{\mu\nu}\right) = 4\cosh(aes)\cos(ebs). \tag{3.20}
$$

3.3 Spatial trace

Processing the second part of (3.12) (the part with $\int \mathrm{d}^4x$) is a bit harder. First, we define

$$p_\mu \equiv i\partial_\mu, \qquad [x_\mu, p_\nu] = -ig_{\mu\nu}. \tag{3.21}$$

We need a suitable electromagnetic potential for our constant electromagnetic tensor. The general solution for this problem is a linear function

$$A_\mu(x) = -\frac{1}{2}(F_{\mu\nu} + S_{\mu\nu})x^\nu + C_\mu, \tag{3.22}$$

where $F_{\mu\nu}$ is our constant electromagnetic tensor, $S_{\mu\nu}$ is any constant symmetric matrix, and C_μ is any constant vector. We set $C_\mu = 0$, and

$$S_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -a \\ 0 & 0 & b & 0 \\ 0 & b & 0 & 0 \\ -a & 0 & 0 & 0 \end{pmatrix}, \tag{3.23}$$

so

$$A_\mu(x) = \begin{pmatrix} 0 & -bx_2 & 0 & ax_0 \end{pmatrix}. \tag{3.24}$$

Now we define an auxiliary Hamiltonian

$$\begin{aligned} H &\equiv (i\partial_\mu - eA_\mu)^2 \\ &= p_0^2 - (p_1 + bx_2)^2 - p_2^2 - (p_3 - ax_0)^2. \end{aligned} \tag{3.25}$$

Using the BCH formula

$$\mathrm{e}^A B \mathrm{e}^{-A} = \sum_{n=0}^{\infty} \frac{C_n}{n!}, \qquad C_n \equiv [A, C_{n-1}], \qquad C_0 \equiv B, \tag{3.26}$$

the Hamiltonian can be recast into a separated form

$$H = H_a + H_b, \tag{3.27}$$

where

$$\begin{aligned} H_a &\equiv \mathrm{e}^{\frac{ip_0p_3}{ea}}(p_0^2 - e^2a^2x_0^2)\mathrm{e}^{-\frac{ip_0p_3}{ea}} \\ H_b &\equiv \mathrm{e}^{\frac{ip_1p_2}{eb}}(-p_2^2 - e^2b^2x_2^2)\mathrm{e}^{-\frac{ip_1p_2}{eb}}. \end{aligned} \tag{3.28}$$

For instance, BCH formula gives us [via (3.21)]

$$\begin{aligned} \mathrm{e}^{\frac{ip_0p_3}{ea}} x_0^2 \mathrm{e}^{-\frac{ip_0p_3}{ea}} &= x_0^2 + \left[\frac{ip_0p_3}{ea}, x_0^2\right] + \left[\frac{ip_0p_3}{ea}, \left[\frac{ip_0p_3}{ea}, x_0^2\right]\right] + 0 \\ &= x_0^2 - \frac{2x_0p_3}{ea} + \frac{p_3^2}{e^2a^2} \\ &= \left(x_0 - \frac{p_3}{ea}\right)^2, \end{aligned} \tag{3.29}$$

which then leads to

$$p_0^2 - (p_3 - ax_0)^2 \tag{3.30}$$

in (3.25). The other half is solved in the same way. The matrix element in (3.12) now can be written as

$$\langle x|\mathrm{e}^{isH}|x\rangle = \langle x_3x_0|\mathrm{e}^{isH_a}|x_0x_3\rangle\langle x_2x_1|\mathrm{e}^{isH_b}|x_1x_2\rangle. \tag{3.31}$$

Using

$$\exp\left(\mathrm{e}^{A}B\mathrm{e}^{-A}\right) = \mathrm{e}^{A}\mathrm{e}^{B}\mathrm{e}^{-A}, \tag{3.32}$$

we can write

$$\mathrm{e}^{isH_a} = \mathrm{e}^{\frac{ip_0p_3}{ea}}\mathrm{e}^{is(p_0^2-e^2a^2x_0^2)}\mathrm{e}^{-\frac{ip_0p_3}{ea}}. \tag{3.33}$$

Now, let us process the first element

$$\begin{aligned}\langle x_3x_0|\mathrm{e}^{isH_a}|x_0x_3\rangle = \int \frac{\mathrm{d}p_0\mathrm{d}p_3\mathrm{d}p_0'\mathrm{d}p_3'\mathrm{d}q_0\mathrm{d}q_3\mathrm{d}q_0'\mathrm{d}q_3'}{(2\pi)^8}\ &\langle x_3x_0|p_0p_3\rangle\\ &\langle p_3p_0|\mathrm{e}^{\frac{ip_0p_3}{ea}}|q_0q_3\rangle\\ &\langle q_3q_0|\mathrm{e}^{is(p_0^2-e^2a^2x_0^2)}|q_0'q_3'\rangle\\ &\langle q_3'q_0'|\mathrm{e}^{-\frac{ip_0p_3}{ea}}|p_0'p_3'\rangle\\ &\langle p_3'p_0'|x_0x_3\rangle\end{aligned} \tag{3.34}$$

where we used (3.33), and inserted eight relations of completeness. Using

$$\langle x|p\rangle = \mathrm{e}^{ipx} \tag{3.35}$$

on pairs $\langle x_0|p_0\rangle$, $\langle x_3|p_3\rangle$, $\langle p_0'|x_0\rangle$, $\langle p_3'|x_3\rangle$, we obtain

$$\begin{aligned}\langle x_3x_0|\mathrm{e}^{isH_a}|x_0x_3\rangle = \int \frac{\mathrm{d}p_0\mathrm{d}p_3\mathrm{d}p_0'\mathrm{d}p_3'\mathrm{d}q_0\mathrm{d}q_3\mathrm{d}q_0'\mathrm{d}q_3'}{(2\pi)^8}\ &\mathrm{e}^{ix_0(p_0-p_0')+ix_3(p_3-p_3')}\\ &\langle p_3p_0|\mathrm{e}^{\frac{ip_0p_3}{ea}}|q_0q_3\rangle\\ &\langle q_3q_0|\mathrm{e}^{is(p_0^2-e^2a^2x_0^2)}|q_0'q_3'\rangle\\ &\langle q_3'q_0'|\mathrm{e}^{-\frac{ip_0p_3}{ea}}|p_0'p_3'\rangle,\end{aligned} \tag{3.36}$$

and using

$$\begin{aligned}\langle p_3p_0|\mathrm{e}^{\frac{ip_0p_3}{ea}}|q_0q_3\rangle &= \mathrm{e}^{\frac{ip_0p_3}{ea}}(2\pi)^2\delta(p_0-q_0)\delta(p_3-q_3)\\ \langle q_3q_0|\mathrm{e}^{is(p_0^2-e^2a^2x_0^2)}|q_0'q_3'\rangle &= 2\pi\delta(q_3-q_3')\langle q_0|\mathrm{e}^{is(p_0^2-e^2a^2x_0^2)}|q_0'\rangle\\ \langle q_3'q_0'|\mathrm{e}^{-\frac{ip_0p_3}{ea}}|p_0'p_3'\rangle &= \mathrm{e}^{-\frac{ip_0p_3}{ea}}(2\pi)^2\delta(p_0'-q_0')\delta(p_3'-q_3'),\end{aligned} \tag{3.37}$$

we get

$$\langle x_3x_0|\mathrm{e}^{isH_a}|x_0x_3\rangle = \frac{ea}{4\pi^2}\int \mathrm{d}p_0\ \langle p_0|\mathrm{e}^{is(p_0^2-e^2a^2x_0^2)}|p_0\rangle. \tag{3.38}$$

We can calculate this element in an elegant way. Consider the Hamiltonian of the harmonic oscillator

$$H_{\mathrm{osc.}} \equiv \frac{p_0^2}{2} + \frac{\omega^2x_0^2}{2}. \tag{3.39}$$

It holds

$$H_{\mathrm{osc.}}|n\rangle = \omega\left(n+\frac{1}{2}\right)|n\rangle. \tag{3.40}$$

The following matrix element can be easily calculated

$$\frac{ea}{4\pi^2}\int \langle p_0|\mathrm{e}^{2isH_{\mathrm{osc.}}}|p_0\rangle. \tag{3.41}$$

Using the energy basis and (3.40), we get

$$\begin{aligned}\frac{ea}{4\pi^2}\int \langle p_0|\mathrm{e}^{2isH_{\mathrm{osc.}}}|p_0\rangle &= \frac{ea}{4\pi^2}\sum_{n=0}^{\infty}\int \mathrm{d}p_0\ |\langle p_0|n\rangle|^2 \exp\left[2is\omega\left(n+\frac{1}{2}\right)\right] \\ &= \frac{ea}{2\pi}\frac{\mathrm{e}^{is\omega}}{1-\mathrm{e}^{2is\omega}},\end{aligned} \tag{3.42}$$

where we used normalization of the oscillator wave functions

$$\int \mathrm{d}p_0\ |\langle p_0|n\rangle|^2 = 2\pi. \tag{3.43}$$

To recover the element (3.38), we need to substitute $\omega = iea$ in (3.42).

The result is

$$\langle x_3x_0|\mathrm{e}^{isH_a}|x_0x_3\rangle = \frac{ea}{4\pi\sinh(eas)}. \tag{3.44}$$

The second element can be processed analogously

$$\langle x_2x_1|\mathrm{e}^{isH_b}|x_1x_2\rangle = \frac{eb}{4\pi i\sin(ebs)}. \tag{3.45}$$

3.4 Final expansion

Now we can combine all the previous expressions. Substituting (3.44), (3.45), and (3.20) into (3.12), we obtain

$$-\,i\ln\det\bar{\mathscr{D}} = \int \mathrm{d}^4x\ \frac{1}{8\pi^2}\int_0^{\infty}\frac{\mathrm{d}s}{s}\ \mathrm{e}^{-is(m^2-i0)}\left[e^2ab\frac{\cosh(eas)\cos(ebs)}{\sinh(eas)\sin(ebs)}-\frac{1}{s^2}\right]. \tag{3.46}$$

If we compare this with (3.1), we can see that

$$\mathscr{L}_{\mathrm{eff.}} = \frac{1}{8\pi^2}\int_0^{\infty}\frac{\mathrm{d}s}{s}\ \mathrm{e}^{-is(m^2-i0)}\left[e^2ab\frac{\cosh(eas)\cos(ebs)}{\sinh(eas)\sin(ebs)}-\frac{1}{s^2}\right]. \tag{3.47}$$

This is the Euler-Heisenberg Lagrangian in all orders for spinor QED.

Let us expand the integrand in powers of e^2

$$\left[\frac{e^2}{3s}(a^2-b^2)-\frac{e^4s}{45}()+\mathcal{O}\left(e^6\right)\right]\mathrm{e}^{-is(m^2-i0)}, \tag{3.48}$$

and restore the original variables using (3.14)

$$\begin{aligned}a^4-2a^2b^2+b^4 &= \left(\mathbf{E}^2-\mathbf{B}^2\right)^2 \\ a^2b^2 &= \left(\mathbf{E}\cdot\mathbf{B}\right)^2.\end{aligned} \tag{3.49}$$

So

$$a^4 + 5a^2b^2 + b^4 = (a^4 - 2a^2b^2 + b^4) + 7(a^2b^2)$$
$$= \left(\mathbf{E}^2 - \mathbf{B}^2\right)^2 + 7\left(\mathbf{E}\cdot\mathbf{B}\right)^2. \tag{3.50}$$

We find that

$$\mathscr{L}_{\text{eff.}} = \frac{\alpha}{6}(\mathbf{E}^2 - \mathbf{B}^2)\int_0^\infty \frac{\mathrm{d}s}{s}\ \mathrm{e}^{-is(m^2-i0)}$$
$$+ \frac{2\alpha^2}{45m^4}\left[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 7(\mathbf{E}\cdot\mathbf{B})^2\right] + \cdots. \tag{3.51}$$

The first (divergent) term can be absorbed into the kinetic term $F_{\mu\nu}F^{\mu\nu}$ of the Maxwell Lagrangian, which leads to the charge renormalization.

The second term has been already integrated. We can clearly see that it is the same formula we have derived in the previous chapter using the direct amplitude matching.

Conclusion

In the first part of the thesis, we have successfully derived effective Lagrangians of the Euler-Heisenberg type in the lowest order for the case of spinor, scalar and vector QED. All the results coincide with those one we can find in the literature. Note however that most common way of evaluating the effective Euler-Heisenberg Lagrangian is based on functional methods, not on a direct diagrammatic calculation. The results are summarized in the following table

Version of QED	*coeff.* c_1	*coeff.* c_2	*coeff.* d_1	*coeff.* d_2	*coeff.* r
spinor	1/90	7/360	2/45	14/45	139/32400
scalar	7/1440	1/1440	7/360	1/90	17/64800
vector	29/160	27/160	29/40	27/10	393/800

We use the following notation

$$\begin{aligned}
\mathscr{L}_{\text{eff.}} &= \frac{\alpha^2}{m^4}\left[c_1(F_{\mu\nu}F^{\mu\nu})^2 + c_2(\star F_{\mu\nu}F^{\mu\nu})^2\right] \\
\mathscr{L}_{\text{eff.}} &= \frac{\alpha^2}{m^4}\left[d_1(\mathbf{E}^2-\mathbf{B}^2)^2 + d_2(\mathbf{E}\cdot\mathbf{B})^2\right] \\
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{r\alpha^4}{\pi^2}\frac{\omega^6}{m^8}(3+\cos^2\theta)^2.
\end{aligned}$$

In the second part, we have used functional methods to calculate the effective Lagrangian in the lowest order for the case of spinor QED. The calculated result match the one obtained via the diagrammatic methods.

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