Charles University in Prague
Faculty of Mathematics and Physics

BACHELOR THESIS

Karel Král

Viditelnostní grafy

Department of Applied Mathematics

Supervisor of the bachelor thesis:	doc. RNDr. Pavel Valtr, Dr.
Study programme:	Computer Science
Specialization:	General Computer Science

Prague 2014

I would like to thank my supervisor Pavel Valtr for all the support and advice.

I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Coll., the Copyright Act, as amended, in particular the fact that the Charles University in Prague has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 paragraph 1 of the Copyright Act.

In .... date .... signature of the author

Název práce: Viditelnostní grafy

Autor: Karel Král

Katedra: Katedra aplikované matematiky

Vedoucí bakalářské práce: doc. RNDr. Pavel Valtr, Dr., KAM

Abstrakt: V předložené práci se zabýváme viditelnostními grafy, se zaměřením na domněnku ,,velká přímka či velká klika." Pro danou množinu bodů P v rovině řekneme, že se dva body vidí, právě když otevřená úsečka mezi nimi neobsahuje žádný bod z P. Vrcholy viditelnostního grafu jsou body z P a dva body jsou spojeny hranou, právě když na sebe vidí. Kára a spol. [35] vyslovili domněnku, že každá dost velká konečná množina bodů obsahuje buď ℓ bodů na jedné přímce nebo její viditelnostní graf má klikovost aspoň k. V práci zobecňujeme domněnku na širší třídu grafů a tím poskytujeme alternativní důkaz pro $k = \ell = 4$. Dále shrneme dosavadní související poznatky. Zesílíme pozorování o výskytu Hamiltonovy kružnice ve viditelnostních grafech. Charakterizujeme asymptotické chování hranové barevnosti viditelnostních grafů. Ukážeme, že pro daná n, ℓ, k lze počítačově rozhodnout, zda původní domněnka platí. Zároveň provedeme počítačové experimenty jak pro zobecněnou, tak pro původní domněnku.

Klíčová slova: viditelnostní graf, rovina, množina bodů

Title: Visibility Graphs

Author: Karel Král

Department: Department of Applied Mathematics

Supervisor: doc. RNDr. Pavel Valtr, Dr., KAM

Abstract: In the thesis we study visibility graphs focusing on the Big Line Big Clique conjecture. For a given finite point set P in real plane we say that two points see each other if and only if the open line segment between them contains no point from P. Points from P are vertices of the visibility graph, and two points are connected by an edge if and only if they see each other. Kára et al. [35] conjectured that for every finite big enough point set there are at least ℓ collinear points, or the clique number of its visibility graph is at least k. In the thesis we generalize the conjecture, and thus provide an alternative proof for $k = \ell = 4$. We also review related known results. We strengthen an observation about occurrence of a Hamiltonian cycle in visibility graphs. We characterize the asymptotic behavior of the edge chromatic number of visibility graphs. We show that for given n, ℓ, k the original conjecture is decidable by a computer. We also provide computer experiments both for the generalized and for the original conjecture.

Keywords: visibility graph, plane, point set

Contents

Introduction

Visibility graphs of various kinds are studied both in computational geometry and computer science. Vertices of a visibility graph correspond to some geometric objects, and two vertices are connected by an edge if they "see" each other. For instance there are various classes of visibility graphs of polygons. Visibility graphs of a set of points and some obstacles are used for example in robot motion planning to find Euclidean shortest paths. We do not investigate these classes here. For more examples and references see Develin et al. [15] for theoretical point of view or de Berg et al. [7] for applications in path planning.

We consider a special class of visibility graphs called *point visibility graphs* where the set of vertices corresponds to a set of points P in real plane, and two vertices are connected by an edge if no other point from P lies on the open segment between them. We often use the word point (resp. edge) instead of point (resp. line segment).

We concentrate on the area of results that are related to the Big Line Big Clique conjecture of Kára et al. [35]. They conjectured that for every k, ℓ there is an integer n such that every finite visibility graph with at least n vertices contains a K_k, or it contains ℓ collinear points.

Outline

Chapter 1 serves as a brief introduction to some selected parts of discrete geometry and introduces some results about point line incidences with and without bounded number of collinear points. We begin with the famous Szemerédi-Trotter theorem. We review the problem of how many lines can meet a point set P at exactly ℓ points if no $\ell + 1$ points in the set are collinear. This problem is called Sylvester's Orchard problem for $\ell = 3$. We also mention bounds on the number of points in general position in sets with no ℓ collinear points. Reader familiar with these classical results may feel free to skip this chapter.

We study some properties of point visibility graphs in Chapter 2. Two easy but useful techniques – minimizing the distance and line sweeping – are discussed in the first part. Using line sweeping we show pancyclicity of visibility graphs. We make an observation about the infinite case. Line sweeping and minimizing distance inspired the generalization of the Big Line Big Clique conjecture discussed in Chapter 3. Later we review some results about chromatic number of point visibility graphs. Characterization of two and three colorable graphs by Kára et al. [35]. Using a result of Beck [6] we are able to characterize asymptotic behavior of edge chromatic number of visibility graphs.

Chapter 3 mentions the history and the state of art knowledge about the Big Line Big Clique conjecture. We also define much wider class of graphs and discuss the conjecture for those providing alternative proofs of small cases. Using SAT solvers we provide computer experiments suggesting better bound for $k = \ell = 4$. We disprove the generalized conjecture for $k = 6$ and $\ell = 4$.

The notion of blocking and being blocked by another point set is reviewed in Chapter 4. Such results are vital because they capture the nature of visibility graphs. We present bounds of size of point sets needed to block.

The main result of Chapter 5 is that for given parameters n, ℓ, k the Big Line Big Clique conjecture is decidable by a computer. We briefly review the background of existential theory of reals and computer experiments.

At the end of the thesis we mention possible ways of further research together with a brief summary of our results.

1. Point Sets in the Plane

In this chapter we review some known results dealing with point sets in real or projective plane and lines spanned by them. We use results and definitions from this chapter throughout the thesis. Let us call an *incidence* the point – line pair where the point lies on the line. In the first part we present the powerful framework of counting point-line incidences. Many problems in combinatorial geometry can be elegantly solved using this theory.

In the second section we remind the famous Sylvester's orchard problem which asks what is the maximum number of collinear point triples when no four-tuple is collinear. Sylvester [56] proves that this number is quadratic in the number of points. The question may be generalized asking how many collinear k-tuples can be in a point sets with no collinear r-tuple. It is still an open problem whether there is a point set with quadratic number of collinear k-tuples without a collinear $k+1$-tuple. Solymosi and Stojaković [53] give almost quadratic construction with $n^{2-\varepsilon}$ collinear k-tuples without collinear $k+1$-tuple.

Finally we recall bounds by Payne and Wood [44] of the maximal size of an independent subset of a point set with bounded number of collinear points.

1.1 Lines Intersecting Many Points

First of all we remind the Crossing lemma, bounding the minimum number of crossings in a drawing of a graph. As shown by Székely [57] this provides easy and elegant proofs of lots of bounds involving incidences or distance problems.

We remind the famous Crossing lemma and the Szemerédi–Trotter theorem, bounding the number of point line incidences, and its modification bounding the number of lines containing many points. We use these results later when recalling the proof of Beck's theorem implying a linear lover bound on edge chromatic number of visibility graph.

1.1.1 Crossing Number Inequality

Let us consider an undirected graph $G = (V, E)$. A *drawing* of G is placing each of its vertices to a different point of real plane $\mathbb{R}^2$ and for every edge uv placing a simple curve connecting representations of its vertices. Furthermore we allow only such drawings that no curve passes through a vertex other than its endpoints.

For a particular drawing D of the graph G we define $\mathrm{cr}(D)$ to be the number of pairs of edges which are crossing. When three edges cross at one point we take this as three individual crossings. On the other hand for an edge pair crossing in more points we count only one crossing.

Definition. Let G be an undirected simple graph. Let the crossing number, denoted by $\mathrm{cr}(G)$, be the minimum of $\mathrm{cr}(D)$ over all drawings D of the graph G.

Let us mention that by definition a graph is planar if and only if its crossing number is zero.

Lemma 1 (Crossing lemma, Leighton [37]). *Let $G = (V, E)$ be an undirected graph with at least $|E| \geq 4|V|$ edges then*

$$\operatorname{cr}(G) \geq \frac{|E|^3}{64|V|^2}$$

holds.

The Crossing lemma was conjectured by Erdős and Guy [24] [30] and proved by Leighton [37] and then independently by Ajtai, Chvátal, Newborn, and Szemerédi [3].

Proof. The graph G is planar if and only if $\operatorname{cr}(G) = 0$. For a finite, connected, plane graph with f faces Euler's formula $|V| - |E| + f = 2$ holds.

By double-counting we get $3f \leq 2|E|$ as there are two faces touching each edge and every edge is surrounded by at least three edges. Plugging this inequality to Euler's formula gives us $|E| < 3|V|$.

For use of probabilistic method we want

$$\operatorname{cr}(G) > |E| - 3|V|.$$

For a contradiction suppose that this does not hold, we take drawing of G realizing $\operatorname{cr}(G) \leq |E| - 3|V|$. Now for every crossing we remove arbitrarily one of the crossing edges. The new graph is plane and has at least $|E| - (|E| - 3|V|) = 3|V|$ edges which is a contradiction.

We take $G' = (V', E')$ to be a random vertex induced subgraph where we take each vertex independently randomly with probability p to be chosen later. We know that $\operatorname{cr}(G') > |E'| - 3|V'|$. By linearity of expectation the expected value of the right hand side is $p^2|E| - 3p|V|$. For the left hand side we have $p^4 \operatorname{cr}(G) \geq \operatorname{cr}(G')$ because every crossing stays with probability p^4. There might be a better drawing of the graph chosen by chance, but that is the right side inequality, so

$$p^4 \operatorname{cr}(G) \geq p^2|E| - 3p|V|$$

holds. Now we choose probability $0 \leq p = \frac{4|V|}{|E|} \leq 1$ and we get the desired bound. $\square$

1.1.2 Szemerédi–Trotter Theorem

Let us recall that an incidence is a point line pair where the point p lies on the line l, we write $p \in l$. We would like to bound the maximal number of incidences between a set of n points and a set of m lines.

Definition. Let P be a set of points in $\mathbb{R}^2$, and let L be a set of lines in $\mathbb{R}^2$. We define $I(P, L) = \{(p, l) \in P \times L \mid p \in l\}$ to be the set of incidences between P and L.

Theorem 2 (Szemerédi–Trotter [57]). *Let P be a finite set of points in $\mathbb{R}^2$, and let L be a finite set of lines in $\mathbb{R}^2$. We have $|I(P, L)| = \mathcal{O}((|P||L|)^{2/3} + |P| + |L|)$.*

Proof. Without loss of generality we may assume that every line in L meets a point in P. Given a set of points P and a set of lines L we define the graph $G = (P, E)$ and its drawing in such a way that P is the vertex set, and two vertices $u, v \in P$ are connected by an edge, drawn as a straight line segment, uv if and only if these are consecutive on a line $l \in L$.

There are no more than $\binom{|L|}{2} \leq |L|^2$ line crossings, so we have $\text{cr}(G) \leq |L|^2$. Number of point line incidences along one line is one more than the number of edges along that line. Number of incidences is then $|I(P, L)| \leq |E| + |L|$. The Crossing lemma gives us either $4|P| \geq |I(P, L)| - |L|$ or $|L|^2 \geq \text{cr}(G) \geq \frac{(|I(P,L)|-|L|)^3}{|P|^2}$ which concludes the bound.

□

To see that the bound in the Szemerédi–Trotter theorem is tight up to a multiplicative constant we use simple construction mentioned by Dvir [17]. Let us denote $[n]$ the set $1, 2, \ldots, n$. We take the line set L to be the set of lines of the form $\{(x, y) \in \mathbb{R}^2 \mid y = ax + b\}$ for $a \in [M], b \in [M^2]$ and the point set $P = \{(x, y) \in \mathbb{R}^2 \mid x \in [M], y \in [2M^2]\}$. Sizes of both of these sets are of the order M^3. Notice that each line $l \in L$ meets P in at least M points as $y = ax + b \leq 2M^2$ for every $a, x \in [M]$ and $b \in [M^2]$.

This theorem also gives us an interesting corollary showing that not many lines may contain many points.

Corollary 3 (Szemerédi and Trotter [57])**.** *Let $2 \leq k \leq \sqrt{n}$, P be a finite set of points, and let L be a finite set of lines both in $\mathbb{R}^2$. The number of lines containing at least k lines is bounded by $|L_k| = \mathcal{O}\left(\frac{|P|^2}{k^3} + \frac{|P|}{k}\right)$.*

Proof. Let us consider the same graph G as in the proof of the Szemerédi–Trotter theorem, but only with lines containing at least k points in P. The graph G has at least $(k-1)|L_k|$ edges. By the Crossing lemma we have either $|L_k|^2 > \frac{|E|^3}{64|V|^2} \geq \frac{(|L_k|(k-1))^3}{64|P|^2}$ or $|L_k|(k-1) < 4|P|$ which gives us the desired bound.

□

This bound is asymptotically tight for instance for points of the $\sqrt{n} \times \sqrt{n}$ grid as shown by Beck [6].

1.2 Orchard Problem

As opposite to the corollary from the previous section when investigating the Big Line Big Clique Problem we are interested in point sets with no more than ℓ collinear points. First problem of this kind is called Sylvester's Orchard Problem [56] asking for the function bounding the maximum number of collinear triples in a set of n points with no four of them collinear. Figure 1.1 of Burr et al. [10] shows an example with nine points no four of them collinear.

Let us note that we often use l for a straight line and ℓ for the maximum number of collinear points throughout the thesis.

Definition (Solymosi, Stojaković [53])**.** For a finite set of points P in $\mathbb{R}^2$ and $k \geq 2$ let $t_k(P)$ be the number of lines meeting P in exactly k points. Let $T_r(P)$

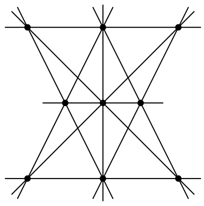

Figure 1.1: Nine points forming ten collinear triples without four collinear points.

be the number of lines meeting P in at least r points. For $r > k$ and n we define

$$t_k^{(r)}(n) := \max_{\substack{|P|=n \\ T_r(P)=0}} t_k(P).$$

Theorem 4 (Sylvester [56]). $t_3^{(4)}(n) > \frac{1}{8}n^2 + \mathcal{O}(n)$.

Here we present the simplified version of Sylvester's proof by Stone as shown in Graham [28].

Proof. There is no straight line meeting the cubic curve $y = x^3$ in at least four points. Let us denote the point $P(x) := (x, x^3)$. The idea of this proof is that points $P(x_1), P(x_2), P(x_3)$ are collinear if and only if $x_1 + x_2 + x_3 = 0$.

If $n = 2m+1$ we set the point set to $P(-m), P(-m+1), \ldots, P(0), \ldots, P(m)$ these points form at least $m^2/2$ collinear triples. For n even we omit the point $P(-m)$. This gives $t_3^{(4)}(n) \geq n^2/8$.

□

Burr, Grünbaum and Sloane [10] give a construction based on Weierstrass's elliptic functions with $n^2/6 - n/2 + 1$ collinear triples. Füredi and Palásti give a construction using hypocycloids with same number of collinear triples. Green and Tao [29] prove that the bound of Burr, Grünbaum, and Sloane is actually tight. This also implies that Figure 1.1 shows an optimal arrangement.

Given this we cannot successfully use Turán's theorem as these constructions have too many edges missing.

1.3 Orchard Problem for Higher ℓ

Now we want a point set P of the size n without $\ell + 1$ collinear points but with as many collinear ℓ-tuples as possible. Erdős [53] asked this question with $\ell = 4$: *Is it possible that a planar point set contains many collinear four-tuples, but it contains no five points on a line?* Erdős also conjectured that $t_k^{(r)}(n) = o(n^2)$ later and it was one of his favorite problems in geometry, see Erdős [20] or Solymosi, Stojaković [53] for more references. It is still not known if this conjecture holds, but if so it is tight.

Theorem 5 (Solymosi, Stojaković [53]). *For $k \geq 4$ even and $\varepsilon > 0$ there is a positive integer n_0 such that for $n > n_0$ we have $t_k^{(k+1)}(n) > n^{2-\varepsilon}$.*

Proof. We are going to find lattice points on $k/2$ concentric high-dimensional spheres such that there are a lot of lines through k of them. It is clear that no line can intersect $k/2$ spheres in more than k points.

For $r > 0$ and a positive integer d let us denote by $B_d(r)$ and $S_d(r)$ the closed ball in $\mathbb{R}^d$ of radius r and the sphere in $\mathbb{R}^d$ of radius r respectively. For a set $S \subseteq \mathbb{R}^d$ we denote the number of lattice points by $N(S) := |S \cap \mathbb{Z}^d|$.

Walfisz [60] proves that for a large enough r_0 there are

$$\begin{aligned} N(B_d(r_0)) &= (1+o(1))V(B_d(r_0)) \\ &= (1+o(1))\frac{\pi^{d/2}}{\Gamma((d+2)/2)}r_0^d \\ &\geq c_1(d)r^d \end{aligned}$$

where $c_1(d)$ is a constant depending only on d.

For every lattice point in $B_d(r_0)$ the square of its distance from the origin is at most r_0^2. Since these squares are integers by pigeonhole principle we get that there is a number r such that $N(S_d(r)) \geq \frac{N(B_d(r_0))}{r_0^2} \geq \frac{c_1(d)r_0^d}{r_0^2} = c_1(d)r_0^{d-2}$.

There are

$$\binom{N(S_d(r))}{2} \geq \binom{c_1(d)r_0^{d-2}}{2} \geq c_2(d)r_0^{2d-4}$$

tuples of lattice point in $S_d(r)$ where $c_2(d)$ is a constant dependent only on d. For every two lattice points in $S_d(r)$ their distance is smaller than $2r_0$ and the square of it is an integer. By pigeonhole principle we have at least

$$\frac{c_2(d)r_0^{2d-4}}{4r^2} \geq \frac{c_2(d)r_0^{2d-6}}{4}$$

pairs of points with same distance ℓ.

Let $p_1, q_1 \in \mathbb{Z}^d \cap S_d(r)$ be two points at distance $d(p_1, q_1) = \ell$ we define points $p_2, \ldots, p_{k/2}$ on the line p_1q_1 such that the distance $d(p_1, p_i) = i \cdot \ell$ and $d(q_1, p_i) = (i+1)\ell$. We define points $q_2, \ldots, q_{k/2}$ in a similar way such that the distance $d(q_1, q_i) = i \cdot \ell$ and $d(p_1, q_i) = (i+1)\ell$.

All points $p_i, q_i \in \mathbb{Z}^d$ due to definition and the fact that $p_1, q_1 \in \mathbb{Z}^d$. Defining $r_i := \sqrt{r^2 + i(i-1)\ell^2}$ we see that $p_i, q_i \in S_d(r_i)$. The r_i defined this way are the same for different p_1, q_1 and p_1', q_1'.

We let $P := \mathbb{Z}^d \cap \left(\cup_{i=1}^{k/2} S_d(r_i)\right)$ and $n := |P|$. By definition $P \subset B_d(r_{k/2})$, so we have $n \leq N(B_d(r_{k/2})) = (1+o(1))V(B_d(r_{k/2})) = c_1(d)r_{k/2}^d$. Having $\ell \leq 2r$ we bound $n \leq c_1(d)(\sqrt{r^2 + k/2(k/2-1)4r^2})^d \leq c_1(d)(k^2+1)^{d/2}r^d \leq c_3(d,k)r^d \leq c_3(d,k)r_0^d$ where $c(d,k)$ is a constant depending only on d, k.

The point set P is a subset of the union of $k/2$ spheres, so there are no $k+1$ collinear points. Every pair $p_1, q_1 \in \mathbb{Z}^d \cap S_d(r)$ with distance $d(p_1, q_1) = \ell$ defines one line through k points thus we have

$$t_k(P) \geq \frac{c_2(d)}{4}r_0^{2d-6} \geq \frac{c_2(d)}{4c_3(d,k)^{\frac{2d-6}{d}}}n^{\frac{2d-6}{d}} \geq c_4(d,k)n^{\frac{2d-6}{d}},$$

where $c_4(d,k)$ is a constant depending only on d, k.

To obtain a two-dimensional set we project P to an arbitrary two-dimensional plane in $\mathbb{R}^d$ along vector v choosing v such that no two points project to the same point, and every three non-collinear points are mapped to three non-collinear points.

□

The projection at the end of the previous proof can be used to see that the Big Line Big Clique problem is not more interesting in more dimensions.

Solymosi and Stojaković [53] also prove a theorem similar to Theorem 5 saying: for every integer $k \geq 4$, there is a positive integer n_0 such that for $n > n_0$ we have $t_k^{k+1}(n) > n^{2-\frac{2\log(4k+9)}{\log n}}$ where log means base two logarithm. We omitted it because the idea is the same, but the proof requires more precise calculations.

When k is odd another sphere can be used in that way that only one of the k tuple lies on it. We omit this proof as the idea is almost the same. The complete proof for k odd can be found in Solymosi Stojaković [53].

1.4 General Position Subsets

We say that a point set in the plane is in general position if it contains no three collinear points. It is easy to see that the vertex set of any clique subgraph of a point visibility graph forms a set in general position. But it is not necessarily so the other way — there might be a set of points in general position having another point on the line-segment between the two points of that set. Nevertheless the knowledge of this result appears to be interesting for us.

Erdős [21] asked for bounds of how big is the largest general position subset of a finite point set where at most ℓ of them are collinear. For every n point set P in the plane with no more than $\ell \geq 3$ collinear points we want to say that it contains at least $f(n, \ell)$ points in general position. We distinguish two cases: the case $\ell \leq \mathcal{O}(\sqrt{n})$ and the case $\ell \leq o(\sqrt{n})$ both of them give different estimates of $f(n, \ell)$.

1.4.1 At Most $\ell \leq \mathcal{O}(\sqrt{n})$ Collinear Points

When the maximal number of collinear points is $\ell \leq \mathcal{O}(\sqrt{n})$ we can find at least $f(n, \ell) \geq \Omega(\sqrt{\frac{n}{\ln \ell}})$ points in general position. To prove this we need a useful lemma bounding the number of collinear triples in a point set with no $\ell + 1$ collinear points.

Lemma 6 (Payne, Wood [44])**.** *Let P be a set of n points in the plane with at most ℓ of them collinear. The number of collinear triples in P is at most $c(n^2 \ln n + \ell^2 n)$ for a constant c.*

Proof. Let s_i denote the number of lines meeting P in exactly i points. By the Corollary 3 we have $\sum_{j \geq i} s_j \leq c(\frac{n^2}{i^3} + \frac{n}{i})$. The number of collinear triples is

$$\sum_{i=2}^{\ell} \binom{i}{3} s_i \leq \sum_{i=2}^{\ell} i^2 \sum_{j=i}^{\ell} s_j \leq \sum_{i=2}^{\ell} i^2 c \left(\frac{n^2}{i^3} + \frac{n}{i} \right) \leq c(n^2 \ln n + \ell^2 n).$$

□

In order to apply Lemma 6 it is useful to define a 3-uniform hypergraph with vertex set P and collinear triples of vertices as edges.

Lemma 7 (Spencer [54]). *Let H be an r-uniform hypergraph with n vertices and m edges. If $m < n/r$ then $\alpha(H) > n/2$. If $m \geq n/r$ then*

$$\alpha(H) > \frac{r-1}{r^{r/(r-1)}} \frac{n}{(m/n)^{1/(r-1)}}.$$

We now show the promised result of Payne and Wood [44].

Theorem 8 (Payne, Wood [44]). *If $\ell \leq \mathcal{O}(\sqrt{n})$ then $f(n, \ell) \geq \Omega(\sqrt{\frac{n}{\ln \ell}})$.*

Proof. Let us denote a 3-uniform hypergraph $H(P) = (P, E)$ where $u, v, w \in E$ for $u, v, w \in P$ if and only if points u, v, w are collinear. By Lemma 6 $H(P)$ satisfy $|E|/|P| \leq cn \ln n$ for a constant c. Now we use Lemma 7 with $r = 3$. If $|E| < |P|/3$ than $\alpha(H(P)) > n/2$. Otherwise

$$\alpha(H(P)) > \frac{2n}{3^{3/2}(m/n)^{1/2}} \geq \frac{2n}{3^{3/2}\sqrt{cn \ln \ell}} = \frac{2}{3\sqrt{3c}} \sqrt{\frac{n}{\ln \ell}}.$$

□

1.4.2 At Most $\ell \leq o(\sqrt{n})$ Collinear Points

Assuming there are slightly asymptotically less collinear points we can show even better estimates of the number of points in general position.

Lemma 9 (Sudakov [55]). *Let H be a 3-uniform hypergraph on n vertices with m edges. Let $t \geq \sqrt{m/n}$ and suppose there exists $\varepsilon > 0$ such that the number of edges containing any fixed pair of vertices of H is at most $t^{1-\varepsilon}$ then $\alpha(H) \geq \Omega_\varepsilon(\frac{n}{t}\sqrt{\ln t})$.*

Theorem 10 (Payne, Wood [44]). *Fix constants $\varepsilon > 0$ and $d > 0$. Let P be a point set with at most ℓ collinear points where $3 \leq \ell \leq (dn)^{(1-\varepsilon)/2}$ where $n := |P|$ then P contains a set of $\Omega(\sqrt{n \log_\ell n})$ points in general position.*

Proof. Let $m = |E(H(P))|$ by Lemma 6 for a constant $c \geq 1$

$$m \leq c\ell^2 n + cn^2 \ln \ell < cdn^2 + cn^2 \ln \ell \leq (d+1)cn^2 \ln \ell.$$

Let $t := \sqrt{(d+1)cn \ln \ell}$ so $t \geq \sqrt{m/n}$. Each pair of vertices of H is in less than ℓ edges and

$$\ell \leq (dn)^{(1-\varepsilon)/2} < ((d+1)cn \ln \ell)^{(1-\varepsilon)/2} = t^{1-\varepsilon}.$$

Assumptions of Lemma 9 are satisfied, so H contains an independent set of size $\Omega(\frac{n}{t}\sqrt{\ln t})$. Moreover,

$$\begin{aligned}
\frac{n}{t}\sqrt{\ln t} &= \sqrt{\frac{n}{(d+1)c \ln \ell}} \sqrt{\ln \sqrt{(d+1)cn \ln \ell}} \\
&\geq \sqrt{\frac{n}{(d+1)c \ln \ell}} \sqrt{\frac{\ln \ell}{2}} \\
&= \sqrt{\frac{1}{2(d+1)c}} \sqrt{\frac{n \ln n}{\ln \ell}} \\
&= \Omega(\sqrt{n \log_\ell n}).
\end{aligned}$$

Thus P contains a subset of $\Omega(\sqrt{n \log_\ell n})$ points in general position.

□

2. Some Properties of Point Visibility Graphs

For a point set $P \subseteq \mathbb{R}^2$ in real plane we say that two distinct points $p, q \in P$ are *visible* (or *see* each other) with respect to P if $P \cap \overline{pq} = \{p, q\}$ where $\overline{pq}$ is the closed line segment between points p, q. The *point visibility graph* $\nu(P)$ of P has the vertex set P and two distinct points are connected by an edge if and only if they are visible with respect to P.

One example of a point visibility graph is shown by Figure 2.1. As we do not consider other than point visibility graphs we may use just visibility graph. We call an edge the line segment between two visible points.

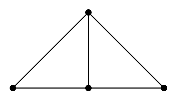

Figure 2.1: An illustration of a visibility graph with four vertices and five edges.

We review some general results about point visibility graphs that might be useful when investigating the Big Line Big Clique conjecture. First we review some observations where we can use minimality of distances and line sweeping to prove them. We also strengthen an observation showing the existence of a Hamiltonian cycle. Later we look at both vertex and edge chromatic number.

2.1 Geometric Properties Of Visibility Graphs

Using geometrical nature of visibility graphs we can easily deduce their diameter and even prove that visibility graphs are *pancyclic*, i.e., contain cycles of all lengths between three and the number of vertices.

More applications where minimality is necessary appear later in this thesis. We do not include such applications in this section, but rather state and prove them where used.

2.1.1 Minimizing the Distance

The fact that in a finite point set P there are two points $p, q \in P$ with minimum distance is often enough to prove interesting results. Moreover this is often necessary and most theorems assume that the point set P is finite. Indeed this is sound as there are often counterexamples with countably many points where no two points with minimum distance exist.

Observation 11 (Kára et al. [35]). *Let $P \subset \mathbb{R}^2$ be a finite point set. Then the diameter of the visibility graph $\nu(P)$ is*

$$\begin{cases} 1 & \text{if } P \text{ is in general position,} \\ |P| - 1 & \text{if all points of } P \text{ are collinear,} \\ 2 & \text{otherwise.} \end{cases}$$

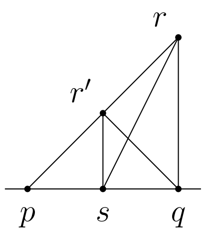

Figure 2.2: An example of minimizing distance.

Proof. Diameter of visibility graph is one by the definition of visibility graphs if and only if P is in general position. When all points are collinear the visibility graph is a path and has the diameter $|P| - 1$.

Consider a finite point set P not in general position and not all of them collinear. There are two non-visible points $p, q \in P$. We choose $r \in P$ such that it does not lie on the line pq and the distance to it is minimized. Such r exists because not all points are collinear and P is finite. There is no other r' blocking visibility of p and r or q and r as such r' would be closer to the line pq as in Figure 2.2.

□

The last proof actually says that if not all of finitely many points are collinear then for every line l there is a point $r \in P$ (r' in Figure 2.2) which sees all points on that line l.

A lemma by Ghosh and Roy [27] says that if not all of the points are collinear then a breadth-first search from any point p has only three levels – the root r, the points visible from r and the rest. This directly follows from the previous observation. Moreover we can see that the second layer itself induces a connected subgraph. This can be proved by taking points visible from r in an angular order and observing that two consecutive points must see each other. Indeed points r', s, q in Figure 2.2 form a connected subgraph.

Payne et al. [45] investigated both vertex and edge connectivity of point visibility graphs. Let P be a finite point set not all of them collinear then every minimum edge cut in the point visibility graph $\nu(P)$ is the set of edges incident to a vertex of minimum degree. When no ℓ points are collinear in a finite point set P then the point visibility graph $\nu(P)$ has vertex connectivity at least $\frac{n-1}{\ell-1}$. For proofs of those theorems and some other related results see Payne et al. [45].

Now we proceed to an easy but useful observation by Ghosh and Roy [27] which allows us to use lower bound on degree of a vertex without minimizing the distance to a straight line.

Observation 12 (Ghosh and Roy [27])**.** *Let P be a finite point set not all of them collinear. Then for any straight line l all points not incident with l have degree at least $|P \cap l|$.*

Proof. We take an arbitrary point $r \notin l$, and we assume that it does not see a point $p \in l$. There must be a point r' blocking visibility of r, p. We take the r' closest to the point r. Now r sees a point p on the line l or the closest blocker (Figure 2.2) thus it has at least $|l \cap P|$ neighbors.

□

2.1.2 Line Sweeping

Line Sweeping is a useful technique not just for geometric algorithms, but it can be used to provide easy proofs as well. Ghosh et al. [27] prove that every finite point visibility graph is either a path, or contains a Hamiltonian cycle. We use sweeping to provide a different proof of a stronger result that every finite visibility graph is either a path, or it is pancyclic. Actually we prove a bit stronger result – a finite visibility graph is a path, or for every its edge e and every $3 \le k \le |P|$ there is a C_k containing e and not intersecting itself. Typically there are many such cycles.

Maybe most applications of line sweeping consider a finite set of points in plane such that no two points have the same x coordinate. Then we "sweep" by line collinear to y axis, move it from $-\infty$ to ∞ and do something with objects it intersects. Algorithmically it is often enough to consider just lines $\{(x_i, t) \mid t \in \mathbb{R}\}$ where x_i are sorted x coordinates of points.

When the given point set contains two points with same x coordinates we rotate the whole set around the origin by an angle α. As the considered set is finite and there are infinitely many angles α we can choose α arbitrarily small such that points in the rotated set have distinct x coordinates.

In our proof we do not sweep as usually, but instead we sweep with a ray (half-line) rotating around its initial point.

Observation 13. *Let P be a finite point set such that not all points are collinear. For every edge e of the visibility graph $\nu(P)$ and every $3 \le k \le |P|$ there is a cycle C_k containing e and not intersecting itself.*

Proof. In the first part of this proof we show how to find a Hamiltonian cycle. In the second part we generalize to shorter cycles. Finding a Hamiltonian cycle is equivalent to finding an ordering of points in P such that no other point lies between neither two consecutive ones nor between the first and the last one.

Let e be an edge of $\nu(P)$, and let $a, b \in P$ be corresponding endpoints. We consider the ray $\vec{r} = \{a + t(b-a) \mid t \in [0, \infty)\}$. Let $Q := P \cap \vec{r}$ be ordered by the distance from a, so we have $q_0 = a < q_1 = b < \dots$ Points in Q precede all other points $\forall q \in Q, \forall p \in P \setminus Q : q < p$.

Rotating $\vec{r}$ around a we say that $\vec{r}$ meets point p with the angle α if we need to rotate the ray $\vec{r}$ by α to the right to meet the point p. We say for $p, q \in P \setminus Q$ that $p < q$ if and only if $\vec{r}$ meets p with a strictly smaller angle than q, or meeting angles are the same and q is closer to the point a.

It is easily seen that the defined total ordering of points in P defines a Hamiltonian cycle in $\nu(P)$ which does not intersect itself.

The only thing left is to prove the existence of shorter cycles. We do this by taking away points from P in a specific order. If there is a point w on the line ab distinct from a, b we remove the point w that is farthest from a. If there is no point on the line ab left we remove an arbitrary vertex of the convex hull of P distinct from points a, b. We get a smaller point set with either just two points a, b or with a point outside the line ab. No removed vertex of the convex hull may block.

□

Here we also used minimality although it is not so obvious. At first sight this observation seems possible to generalize to infinite sets. The visibility graph of the

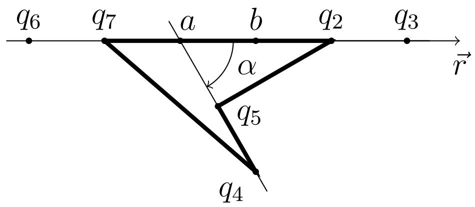

Figure 2.3: An eight point arrangement with the ordering and a C_6.

rational lattice $\nu(\mathbb{Q}^2)$ is indeed empty. So we cannot expect infinite point visibility graphs to have a Hamiltonian path. Strictly speaking the last observation still holds for $\mathbb{Q}^2$ as we quantified that for every edge there is a cycle of length k.

On the other hand when we require that in an infinite point set no ℓ points lie on one line there needs not to be a C_3 as shown by Pór and Wood [48] with $\ell = 4$ (described in Theorem 40, in Section 3.3 of this thesis). When no ℓ points are collinear there must be some cycles longer than ℓ.

Observation 14. *Let P be an infinite point set with no ℓ of them collinear. For every $k \geq 3$ we have a C_c subgraph of $\nu(P)$ where $k \leq c \leq k + \ell - 2$.*

Proof. A well known result in discrete geometry, the Erdős–Szekeres theorem, states that for every k there is a minimum integer $ES(k)$ such that all point sets in general position contain a subset of k points in convex position. Abel et al. [1] generalized the Erdős–Szekeres theorem for point sets with bounded collinearity. Note that we do not need finite point sets as we can always pick a big enough finite subset.

For k given we choose points in convex position $p_1, p_2, \ldots, p_k$ in this order. There might be at most $\ell - 2$ points between p_i and p_{i+1}. When there are points between p_i and p_{i+1} we change p_{i+1} to be the point on the line segment $p_i p_{i+1}$ which is the closest to p_i. We get another point set in convex position. Repeating this process we get a point set in convex position with no points between p_i and p_{i+1}.

The only thing we do not control is the number of points between p_1 and p_k. As there are no ℓ collinear points by the assumption we get a cycle C_c as subgraph in the visibility graph $\nu(P)$ where $k \leq c \leq k + \ell - 2$. □

2.2 Chromatic Number

For a graph G we say that a function $c\colon V(G) \to [q]$ is a *proper vertex coloring* if $c(u) \neq c(v)$ holds for every edge $uv \in E(G)$. The *vertex chromatic number* of a graph G denoted by $\chi(G)$ is the minimum q such that there is a proper vertex coloring of G. Similarly for a graph G we say that a function $c'\colon E(G) \to [q']$ is a *proper edge coloring* if $c(e) \neq c(f)$ holds for every two incident edges $e, f \in E(G), |e \cap f| = 1$. The *edge chromatic number* is the minimum number of colors such that there is a proper edge coloring of G. Let us write $\chi'(G)$ for the edge chromatic number.

The chromatic number is one of the most basic graph properties. We review what is known about the vertex chromatic number of visibility graphs. Pfender [46] shows there are point visibility graphs with arbitrary vertex chromatic number and with no K_7 subgraph. We characterize the asymptotic behavior of edge chromatic number at the end of this section.

2.2.1 Vertex Chromatic Number

Observation 15 (Kára et al. [35]). *If P can be covered by k lines then the chromatic number of its visibility graph is at most $\chi(\nu(P)) \leq 2k$.*

Proof. Points on each line induce a path in the visibility graph. Each point of P lies on a covering line, so we choose for each point an arbitrary covering line on which it lies. It now suffices to color points assigned to one line alternatively by two colors and to use different colors for different lines.

□

We have already bounded the size of a maximal clique by twice the number of lines to cover P. A simple observation of Kára et al. [35] is that if there is a vertex p of degree d we can cover P by d lines passing through p. This follows immediately by definition as p sees exactly d points and other points are blocked by those.

Visibility Graphs of Chromatic Number Two

We present a characterization of Kára et al. [35] of point visibility graphs with chromatic number two.

Theorem 16 (Kára et al. [35]). *For the visibility graph $\nu(P)$ of any finite point set P the following are equivalent:*

(a) chromatic number $\chi(\nu(P)) \leq 2$,

(b) all points of P are collinear,

(c) $\nu(P)$ does not contain K_3.

Proof. Visibility graph of a collinear set of points is a path, and we can color it by two colors thus (b) implies (a). It is trivial that (a) implies (c). It remains to prove that (c) implies (b). Assume not all points lye on the same line. Let us take three points u, v, w not collinear and such that the triangle uvw is of minimal area. There is no point blocking visibility between them else we get new triangle of smaller area, which is a contradiction. Points u, v, w induce K_3 in visibility graph thus the chromatic number is strictly greater than two.

□

Visibility Graphs of Chromatic Number Three

We use a special case of a theorem by Develin et al. [15] to see that a point visibility graph is either plane, or contains a K_4. To characterize point visibility graphs with chromatic number three we first review a characterization of plane visibility graphs by Eppstein [18]. Kára et al. [35] combine these results to a complete characterization of point visibility graphs with chromatic number three.

Lemma 17 (Develin et al. [15]). *Let P be a finite point set. Assume that the visibility graph $\nu(P)$ contains two edges ab and uv that cross each other. Under these assumptions points a, b are part of a K_4.*

Develin et al. [15] prove this for a wider class of visibility graphs, but this special case is enough for us.

Proof. Let us take two points u, v such that the edge uv crosses the edge ab, and u, v are the closest such points to the line ab. There are such uv by assumptions.

For a contradiction let us assume that the point a does not see the point u. Then there is a point p closer to the line ab which blocks visibility of au. Moreover the line segment pv crosses the line segment ab. By minimality p cannot see v, so there must be another blocker q blocking pv (we choose the q closest to v). We can see that the point q must be in the same half plane of the line ab as v otherwise we would choose points qv instead of uv. But now instead of points uv we can choose two points c, d from the line segment pq such that the edge cd crosses the edge ab. Points cd are closer to the line ab than points uv – we have a contradiction.

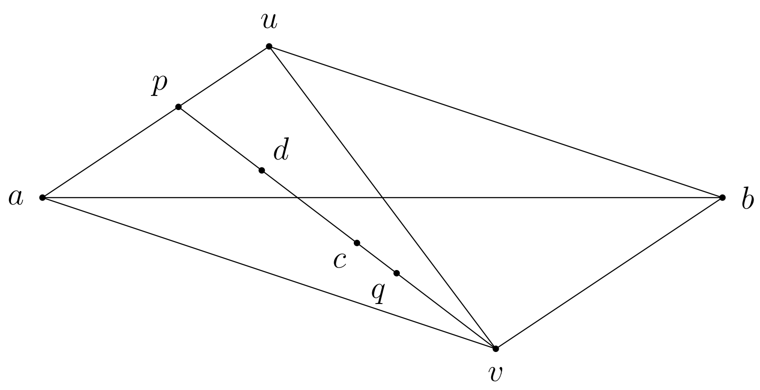

Figure 2.4: The contradiction situation.

By symmetry every two points from $\{a, b, c, d\}$ can see each other.

□

Eppstein [18] characterized those plane graphs where between each pair of vertices there is a line segment consisting of one or more edges. Those graphs are plane visibility graphs. Here we provide a proof of Ghosh and Roy [27] as it contains less case analysis.

Lemma 18 (Ghosh, Roy [27]). *Let P be a finite point set and l a line meeting P at $k \geq 4$ points. If the visibility graph $\nu(P)$ is planar we have $|P| \leq k + \left\lfloor \frac{2k-5}{k-3} \right\rfloor$.*

Proof. We have at least $k-1$ edges along the line l. By Observation 12 we have at least $(|P|-k)k$ other edges. By Euler's formula we have $|E(H)| \leq |V(H)|-6$ for every planar graph H. We have $(k-1)+(|P|-k)k \leq 3|P|-6$ so $(|P|-k) \leq \frac{2k-5}{k-3}$. As $|P| - k$ is integer we get the desired bound.

□

Corollary 19 (Ghosh, Roy [27]). *Let P be a finite point set with a line l that meets P in at least five points. There are six infinite families of such sets P with planar point visibility graph $\nu(P)$.*

Proof. By Lemma 18 we have that at most two points of P that are not incident with the line l.

1. There is no point outside of the line l (see Figure 2.8 (d)).
2. There is only one point outside of the line l (see Figure 2.8 (b)).
3. There are two points outside of the line l both adjacent to each other and all other points (see Figure 2.6).
4. There are two points outside of the line l not adjacent to each other but adjacent to all other points (see Figure 2.8 (c)).
5. There are two points p_i, p_j outside of the line l adjacent to each other and p_j blocks visibility of p_j to a point p_k from the line (see Figure 2.5). Note that this graph is planar but no visibility embedding of it is plane. Ghosh and Roy [27] also distinguish two cases when p_k is an endpoint of the line l.

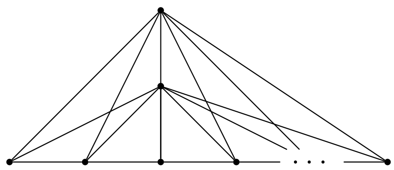

Figure 2.5: Planar visibility graph family with no plane visibility embedding.

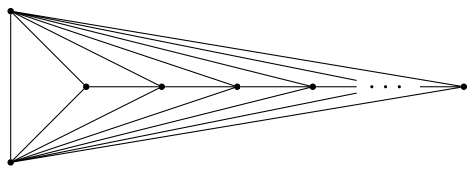

Figure 2.6: Planar visibility graph family with a K_4 subgraph.

□

We need to deal with point visibility graphs with no five points collinear to finish the characterization. Although it is possible we do not want to do case analysis for big point sets. If there are four collinear points by Lemma 18 we have at most seven vertices. With this and the next lemma we are left to deal with at most eight vertices.

Lemma 20 (Ghosh, Roy [27]). *Assume we have a finite point set P with no four points collinear and with a planar visibility graph $\nu(P)$ then $|P| \leq 8$ holds.*

Proof. Assume that P has at least five vertices. We have a line l through three points of P otherwise we would have a K_5 subgraph, and thus $\nu(P)$ could not be planar. Let us create the set P by adding vertices. We start with the three points $p_1, p_2, p_3 \in l$. We cannot add p_4 and p_5 on the line l, so each of them adds at least three edges by Observation 12.

Adding the point p_i adds at least $\lceil \frac{i-1}{2} \rceil$ new edges because no four points are collinear, thus p_i sees at least that much other points. By assumption $\nu(P)$ is planar, so by Euler's formula we have $2+2\cdot 3+\sum_{i=6}^{n}\left\lceil \frac{i-1}{2}\right\rceil \leq 3n-6$ which holds for $n \leq 8$.

$\square$

In the previous proof we saw that if no four points are collinear the visibility graph has minimal degree at least $\lceil \frac{n-1}{2} \rceil$. This is just a corollary of an observation of Payne et al. [45], Ghosh and Roy [27]. For every point set P of size n with no $\ell+1$ collinear points the minimal degree of a point visibility graph $\nu(P)$ is at least $\lceil \frac{n-1}{\ell-1} \rceil$. The proof goes immediately by definition as every point sees at least $\lceil \frac{n-1}{\ell-1} \rceil$ others.

Case analysis of visibility graphs with at most eight vertices gives us special cases given by Ghosh and Roy [27] (Figure 2.7 and Figure 2.8 (a)). Every case in Figure 2.7 contains a K_4.

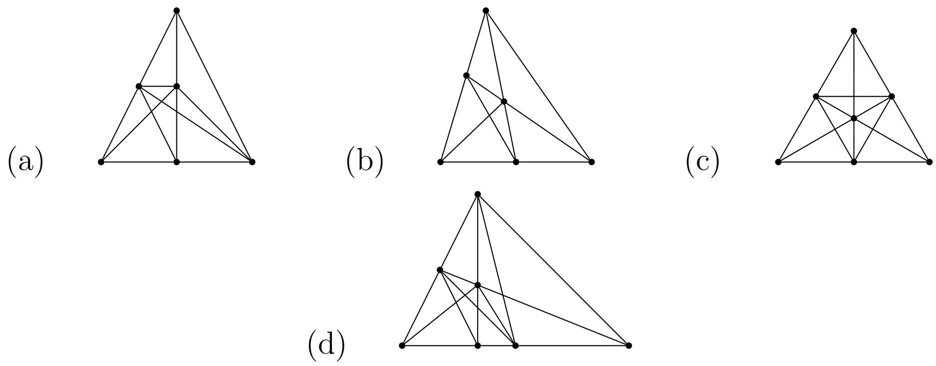

Figure 2.7: Special cases of planar visibility graphs.

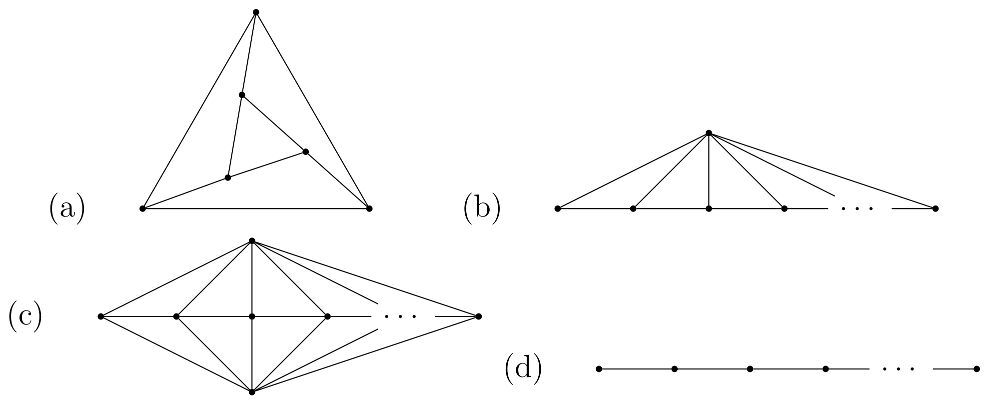

Figure 2.8: One special case and all infinite families of visibility graphs with chromatic number at most three.

Theorem 21 (Kára et al. [35])**.** *For a finite point set P the following are equivalent:*

(i) $\chi(\nu(P)) \leq 3$,

(ii) P is one of the point sets in Figure 2.8,

(iii) $\nu(P)$ has no K_4 subgraph.

Proof. By definition (i) implies (iii). Graphs in Figure 2.8 are easy to properly color by three colors thus (ii) implies (i). We only need that (iii) implies (ii) now. We saw all planar visibility graphs in Figures 2.5, 2.6, 2.7, and 2.8 and just those in Figure 2.8 contain no four clique. On the other hand if a point visibility graph is not plane, i.e., contains two crossing edges, we have a K_4 subgraph by Develin et al. [15] or its special case Lemma 17.

□

2.2.2 Chromatic versus Clique Number

Kára, Pór, and Wood [35] asked if the chromatic number of a visibility graph can be bounded by a function of its clique number. In the previous subsection we saw a characterization of visibility graphs with $\omega(\nu(P)) \in \{2, 3\}$ where such bound is easy. On the other hand Pfender [46] managed to construct visibility graphs with arbitrarily large chromatic number and no K_7 subgraph.

Lemma 22 (Pfender [46]). *For M large enough there is a set of prime numbers $\{p_{ij} \mid 1 \leq i < j \leq n\}$ satisfying:*

1. $2^M < p_{ij} < 2^{M+1}$.
2. *For* $1 \leq k \leq n$ *we define* $P_k = 2^{n_k} \prod_{i=1}^{k-1} p_{ik} \prod_{j=k+1}^{n} p_{kj}$ *where we choose such* $n_k \in \mathbb{Z}$ *that* $\lfloor \log_2 P_k \rfloor = nM + 2k$. *Then* p_{kl} *is the only number in* $\{p_{ij} \mid 1 \leq i < j \leq n\}$ *dividing* $P_\ell - P_k$ *for* $1 \leq k < \ell \leq n$.

Proof. Finsler [25] proves there are more than $2^M/(3(M+1)\ln 2) > 2n^3$ primes between 2^M and 2^{M+1}. We pick p_{ij} in lexicographical order, i.e., in the order $p_{12}, p_{13}, p_{14}, \ldots, p_{23}, \ldots, p_{(n-1)n}$ satisfying:

1. $2^M < p_{ij} < 2^{M+1}$ is a prime,
2. no p_{ij} is chosen twice,
3. p_{ij} does not divide $P_k - P_\ell$ for all $1 \leq \ell < k < i$,
4. and when $j = n$ no p_{kl} divides $P_i - P_r$ for $\{k, l\} \neq \{i, r\}$.

We are about to pick p_{ij}. The first case is $j < n$. We selected no more than $\binom{n}{2}$ primes before and each $P_k - P_\ell$ has at most n prime divisors greater than 2^M. Thus only $\binom{n}{2} + n\binom{n}{2} < n^3$ choices are taken, and we can pick p_{ij} satisfying all conditions 1. – 4.

Second case is picking p_{in}. We saw we can pick p_{in} satisfying conditions 1. – 3. Suppose for some $\{k, \ell\} \neq \{i, r\}$ that $p_{k\ell}$ divides $P_i - P_r$. By definition of P_i all $p_{i\ell}$ divide P_i, so if $k = i$ we would have also that $p_{i\ell}$ divides P_r too, and thus $r = \ell$ gives a contradiction. We proceed similarly to prove $\ell \neq i$.

We pick another p'_{in}. If $p_{k\ell}$ divides $P'_i - P_r$ then $p_{k\ell}$ divides $P'_i - P_i = (p'_{in} - p_{in})P_i/p_{in}$, so $p_{k\ell}$ divides $p'_{in} - p_{in}$. But this leads to a contradiction as $|p'_{in} - p_{in}| < 2^M < p_{k\ell}$. We see that each $p_{k\ell}$ can block at most one choice of p_{ij}, so we can always find p_{ij} satisfying all conditions 1. – 4.

□

Theorem 23 (Pfender [46]). *For every graph G there is a point set $X \subset \mathbb{R}^2$ such that the subgraph of $\mathcal{V}(X \cup \mathbb{Z}^2)$ induced by X is isomorphic to G.*

Proof. We have $n_k > 0$ as $\prod_{i=1}^{k-1} \prod_{j=k+1}^{n} p_{kj} < 2^{(n-1)(M+1)} < 2^{nM}$ so $P_k \in \mathbb{Z}$.

Let our set of points $X = \{x_i \mid 1 \leq i \leq n\}$ be defined as

$$x_i = \left(2^{-nM} P_i, i \frac{\prod_{k<j}(P_j - P_k)}{\prod_{kj \in E(G)} p_{kj}} \right).$$

We want this point set X to be in general position. For $1 \leq i < \ell \leq n$ we denote the slope of line through points x_i, x_ℓ by

$$m_{i\ell} = \frac{\ell - i}{P_\ell - P_i} \frac{2^{nM} \prod_{k<j}(P_j - P_k)}{\prod_{kj \in E(G)} p_{kj}}.$$

Having $2^{nM+2i+1} \leq P_{i+1} - P_i < 2^{nM+2i+3}$ it holds that $m_{i(i+1)} > m_{(i+1)(i+2)}$ thus $m_{i\ell} > m_{ik}$ for all $i < \ell < k$. Slopes of lines connecting points are distinct, so there are no three collinear points in X.

We are left to prove that there is a lattice point between x_i and x_j if and only if $ij \notin E(G)$. As $2^{2j} \leq 2^{-nM} P_j < 2^{2j+1}$ for every j there is an s such that $2^{-nM} P_i < s < 2^{-nM} P_\ell < 2^{2n+1}$. Let $z_{i\ell}^s = (s, y_{i\ell}^s)$ be a point on the line segment from x_i to x_ℓ. We have

$$y_{i\ell}^s = i \frac{\prod_{k<j}(P_j - P_k)}{\prod_{kj \in E(G)} p_{kj}} + (s - 2^{-nM} P_i) m_{i\ell} =$$

$$= i \frac{\prod_{k<j}(P_j - P_k)}{\prod_{kj \in E(G)} p_{kj}} + s \frac{\ell - i}{P_\ell - P_i} \frac{2^{nM} \prod_{k<j}(P_j - P_k)}{\prod_{kj \in E(G)} p_{kj}} + P_i \frac{\ell - i}{P_\ell - P_i} \frac{\prod_{k<j}(P_j - P_k)}{\prod_{kj \in E(G)} p_{kj}}.$$

Given that p_{kj} divides $P_j - P_k$ the first summand is an integer. Moreover $p_{i\ell}$ divides P_i, so the third summand is also an integer.

If $i\ell \notin E(G)$ the also the second summand is an integer which means there is a lattice point blocking visibility of x_i and x_ℓ.

If $i\ell \in E(G)$ we have $p_{i\ell} > 2^M > \max\{\ell - i, s\}$, so $p_{i\ell}$ divides neither s nor $\ell - i$. Obviously $p_{i\ell}$ does not divide 2^{nM}, and it divides no other $P_j - P_k$ than $P_\ell - P_i$ leaving $z_{i\ell}^s \notin \mathbb{Z}^2$ for all s possible. □

Proposition 24 (Kára et al. [35]). *Chromatic number of the visibility graph of integer lattice is $\chi(\nu(\{(x, y) \mid x, y \in \mathbb{Z}\})) = 4$.*

It is easy to four color the square lattice. On the other hand there is a K_4 in the visibility graph $\nu(\{0, 1\} \times \{0, 1\})$.

Corollary 25 (Pfender [46]). *For every $k \in \mathbb{N}$ there is a point set $P \subset \mathbb{R}^2$ such that $\chi(\nu(P)) \geq k$ and $\omega(\nu(P)) \leq 6$.*

Proof. We use Theorem 23 with Mycielski graph [41] ($\chi(M) = k, \omega(M) = 2$). □

2.2.3 Edge Chromatic Number

Edge chromatic number behaves more predictably – it is either two or asymptotically linear in the number of vertices.

Theorem 26 (Beck [6])**.** *Let $P \subset \mathbb{R}^2$ be a finite point set, and let L be the set of lines containing at least two points from P. There is either a line containing $\Omega(n)$ points or $|L| = \Omega(n^2)$.*

Proof. We partition lines to sets $L_j \subseteq L$ of lines containing at least 2^j and less than 2^{j+1} points from P ($\forall l \in L_j : 2^j \leq |l \cap P| < 2^{j+1}$). Corollary 3 bounds $|L_j| \leq \mathcal{O}\left(\frac{|P|^2}{2^{3j}} + \frac{|P|}{2^j}\right)$.

We want to say that there are not enough lines with $C \leq 2^j \leq |P|/C$ points for a constant C. There are at most $2^{2(j+1)}$ pairs of points on each line in L_j. Thus summing all pairs of points on all lines with $C \leq 2^j \leq |P|/C$ points for big enough C gives us only $|P|^2/100$ pairs instead of $\binom{|P|}{2}$.

Either there is a line with at least $|P|/C$ points, and we are done. Or there are $\Omega(|P|^2)$ pairs of points on lines containing at most C points each, and we have $|P|^2/C^2$ lines.

□

Corollary 27. *Let P be a finite point set not all of them on one line then there is a vertex of at least linear degree in the visibility graph $\nu(P)$.*

Proof. By Beck's theorem we have either a line with linearly many points or quadratic number of edges. Let C be as in Beck's theorem.

If there is a line l with at least $|P|/C$ points by assumptions there is a point p not on l, and by Observation 12 it has degree at least $|P|/C$ which is linear.

If there are at least $|P|^2/C^2$ lines there must be a point incident with at least linearly many of them.

□

The corollary tells us there are two possibilities – all points lie on the same line, and we have edge chromatic number two $\chi'(\nu(P)) = 2$, or there is a vertex of linear degree, and we have $\chi'(\nu(P)) = \Omega(|P|)$. On the other hand by Vizing's theorem [59] we have $\chi'(G) \leq \Delta(G) + 1$.

3. Big Line Big Clique Conjecture

Conjecture 28 (Kára et al. [35]). *For every $k \geq 2$ and $\ell \geq 2$ there is an integer n such that every finite set of at least n points in the plane $\mathbb{R}^2$ contains ℓ points on one line or k pairwise visible points.*

Note that classical the Ramsey theorem [49] tells us that for every k, ℓ there is an n such that in every graph on at least n vertices there is a clique K_k or an independent set $\overline{K_\ell}$ as subgraph. Finding an independent set is not enough for us as this does not imply that all of those points are on one line. It could even happen that there is a set in general position $I \subseteq P$ inducing an independent set in the visibility graph $\nu(P)$ which is blocked by some other points in P. We investigate such blocking sets and asymptotic of their size in Chapter 4.

3.1 History

The Big Line Big Clique Conjecture first appears in Kára et al. [35] where it is proved for $k \leq 4$ and arbitrary ℓ. Addario-Berry et al. [2] prove the conjecture with $k = 5$ and $\ell = 4$ later.

Both Abel et al. [1] and Barát et al. [5] prove the conjecture for $k = 5$ and $\ell \in \mathbb{N}$ by finding five points $Q = \{p_1, p_2, \ldots, p_5\} \subseteq P$ in convex position with no other point $p \in P$ in $\mathrm{conv}(Q)$.

3.2 Proof of the Big Line Big Clique conjecture for $k = 5$ and $\ell \in \mathbb{N}$

Both Abel et al. [1] and Barát et al. [5] find empty pentagons in large enough point sets. Here we discuss the proof of Barát et al. [5] as it is asymptotically optimal. Indeed a square grid of size $(\ell - 1)(\ell - 1)$ contains no K_5 and no ℓ collinear points.

Theorem 29 (Barát et al. [5]). *Let P be a finite point set in plane containing at least $328\ell^2$ points then there are ℓ collinear points or an empty pentagon.*

The last theorem is in fact strengthening the result of Harborth [31] that there is an empty pentagon in every finite set in general position containing at least ten points. Later Nicolás [42] and Gerken [26] independently prove that a finite big enough point set in general position contains an empty hexagon. It is open if this can be also strengthened for point sets with bounded collinearity or not.

On the other hand Horton [32] constructed arbitrarily large point sets in general position with no empty heptagon.

We need a bit more precise definitions in this section. We use definitions of Barát et al. [5]. A point set P is in a *weakly convex position* if every point of P lies on the boundary of the convex hull $\mathrm{conv}(P)$. The point $p \in P$ is a *corner*

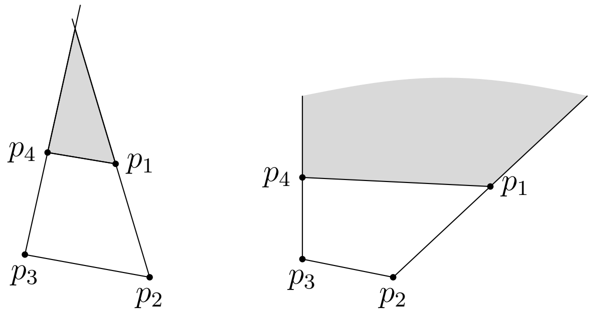

Figure 3.1: Shaded regions represent a bounded and an unbounded 4-sector $S(p_1, p_2, p_3, p_4)$.

if $\mathrm{conv}(P \setminus \{p\}) \neq \mathrm{conv}(P)$. We say that a point set P is in a *strictly convex position* if every point of P is a corner.

For a point set P we define *convex layers* $L_1, L_2, \ldots, L_r \subseteq P$ recursively: let L_i be the set of points of P on the boundary of $P \setminus \bigcup_{j=1}^{i-1} L_j$ and we take the index r such that L_r is the last such nonempty set. We thus have each L_i nonempty and in weakly convex position and P is the disjoint union $P = \dot{\bigcup} L_i$. The *edges of a layer* are line segments connecting two consecutive points of a layer.

Let P be a point set and let Q be its strictly convex subset $Q \subseteq P$ of five (resp. m) points with no other points in the convex hull of Q ($P \cap \mathrm{conv}(Q) = Q$) we call the set Q an *empty pentagon* (resp. an *empty m-gon*). It is easy to see that such a Q forms a clique in the visibility graph $\nu(P)$.

Let $p_1p_2p_3p_4$ be a strictly convex quadrilateral with points numbered in the clockwise order we define the *4-sector* $S(p_1, p_2, p_3, p_4)$ to be the set of all points q such that $qp_1p_2p_3p_4$ is a strictly convex pentagon. The 4-sector is an intersection of three open half-planes and the closure of it is denoted by square brackets $S[p_1p_2p_3p_4]$. Figure 3.1 shows both bounded and unbounded 4-sectors and also shows that the order of arguments matters.

If the finite point set P contains no empty pentagon then for every empty quadrilateral we have $P \cap S(p_1p_2p_3p_4) = \emptyset$. If this did not hold we would select the point $q \in P \cap S(p_1p_2p_3p_4)$ closest to the line p_1p_4 forming an empty pentagon.

3.2.1 8ℓ Points in Convex Position

Theorem 30 (Barát et al. [5])**.** *A point set containing 8ℓ points in weakly convex position contains also an empty pentagon or ℓ collinear points.*

Let P be a finite set that contains 8ℓ points in weakly convex position. We could set P' to be the inclusion minimal subset of P such that P' contains 8ℓ points in weakly convex position but not so for any its proper subset $P'' \subsetneq P'$. It is enough for us to consider just P', and find an empty pentagon E in it as E will be also an empty pentagon of P.

Let A be the set of at least 8ℓ points in weakly convex position. It is easy to see that A are exactly those points of P that are on the boundary of the convex hull $\mathrm{conv}(P)$. So A is the first convex layer of P. We set B to be the second convex layer of P. The set A is at least weakly convex 9-gon otherwise we would have ℓ points on one line. The set B is not empty otherwise we could just select five pairwise visible points from A, and get an empty pentagon.

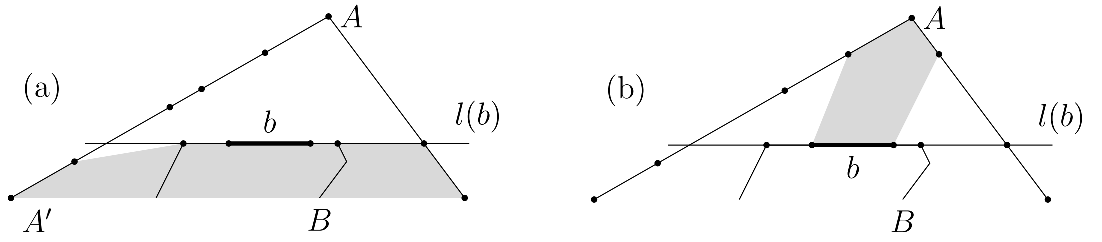

Figure 3.2: (a) If $|A \cap b^+| \leq |B \cap l(b)|$ then A is not minimal. The convex hull $\mathrm{conv}(A')$ is shaded. (b) If b^+ contains three non-collinear points there is an empty pentagon (shaded).

For an edge e of A or B we define e^+ to be the open half-plane containing no point from B which is determined by the line containing e. For convenience we call $l(e)$ the line containing the line segment e.

Observation 31 (Barát et al. [5])**.** *For every edge b of the layer B we have $|A \cap b^+| > |B \cap l(b)|$. The same holds for $b_1, b_2, \ldots, b_j$ edges of B, and we have $|A \cap \bigcup_{i=1}^{j} b_i^+| > |B \cap \bigcup_{i=1}^{j} l(b_i)|$.*

Proof. If $|A \cap b^+| \leq |B \cap l(b)|$ we might remove vertices from $A \cap b^+$ and replace those with vertices from $B \cap l(b)$ getting A' a set of at least 8ℓ points in weakly convex position with $\mathrm{conv}(A') \subsetneq \mathrm{conv}(A)$ a contradiction with minimality of A. This can be seen in Figure 3.2 (a).

The second claim follows from the minimality by a similar argument.

□

Observation 32 (Barát et al. [5])**.** *For every edge e of the layer B all vertices of $A \cap e^+$ are collinear, or we could find an empty pentagon.*

Proof. By the previous observation (Observation 31) we have at least three points in $A \cap e^+$. When the set $A \cap e^+$ is not collinear we can find an empty pentagon as shown by Figure 3.2 (b).

□

Lemma 33 (Barát et al. [5])**.** *We have $2|B| \geq |A|$ or ℓ collinear points or an empty pentagon.*

Proof. We assume that there are no ℓ collinear points thus A has at least nine corners because it has at least 8ℓ points. We already observed that B is not empty, or we could just take an empty pentagon containing points from A. Moreover B can not be covered by a line as there would be at least four corners strictly on one side of that line which would create an empty weakly convex pentagon with one point in B implying that there is a strictly convex empty pentagon.

We call a side the set of edges between two consecutive corners. We just saw that B has at least three corners and thus at least three sides. Taking $b_1, \ldots, b_k$ one edge of B from each side, and applying Observation 32 each set $A \cap b_i$ is collinear thus $|A| \leq \sum_{i=1}^{k} |A \cap b_i| < k\ell$, so we have $k \geq 9$. The layer B has at least nine corners which means there is a point $z \in P$ in the interior of the convex hull of B.

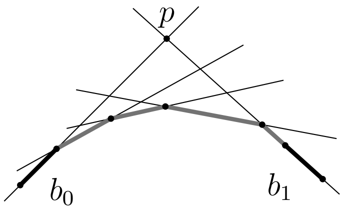

Figure 3.3: Edges in E_X have to be consecutive.

Assume we have an edge xy of A such that the closed triangle $\Delta[x, y, z]$ has an empty intersection with B. This means there is an edge $x'y'$ of B crossing the triangle thus the 4-sector $S(x', x, y, y')$ is not empty, and as we already saw we can find an empty pentagon. A point in B can be in at most two such triangles implying that $2|B| \geq |A|$.

□

We thus have at least 4ℓ points in the layer B.

Lemma 34 (Barát et al. [5])**.** *Let X be a nonempty set of points we call E_X the set of edges of B such that $X \subseteq b^+$ for every $b \in E_X$. Edges in E_X are consecutive, and we have $E_X \subsetneq B$.*

Proof. First if we have a point of X in the convex hull of B ($X \cap \mathrm{conv}(B) \neq \emptyset$) we have $E_X = \emptyset$ and the lemma holds. Let $x \in X$ be a point such that $x \notin \mathrm{conv}(B)$, and let y be an arbitrary point in the interior of $\mathrm{conv}(B)$ such that it is not collinear with any tuple of points from $B \cup \{x\}$. The line xy intersects exactly two edges b_0, b_1 of B one of which such that $x \notin b_0^+$ and one $x \in b^+$ thus not all edges of B are in E_X.

When E_X contains just one edge the lemma holds. Let $b_0, b_1 \in E_X$ be two edges that are not consecutive. If these edges lie on the same line $l(b_0) = l(b_1)$ the edges between them are clearly also in E_X. We can not have $l(b_0)$ and $l(b_1)$ parallel as we would have $b_0^+ \cap b_1^+ = \emptyset$. Without loss of generality let b_0 be on the left, b_1 on the right and the intersection $p := l(b_0) \cap l(b_1)$ above B as in Figure 3.3. We clearly have $p \in b_0^+ \cap b_1^+$ and for a b the next edge of B between b_0 and b_1 we have also $b^+ \supseteq b_0^+ \cap b_1^+$ thus $b \in E_X$. Iterating this we have that edges in E_X are consecutive.

□

By Observations 31 and 32 we have an edge a of the layer A such that $|A \cap l(a)| \geq 3$, we name vertices along the line in clockwise order $\{v_1, \ldots, v_k\} = A \cap l(a)$. We have $k < \ell$, or we would have ℓ collinear points.

Lemma 35 (Barát et al. [5])**.** *Let $\{v_1, \ldots, v_k\}$ be as above. There is an edge b of the layer B such that $\{v_1, v_2, v_3\} \subseteq b^+$ or $\{v_{k-2}, v_{k-1}, v_k\} \subseteq b^+$.*

Proof. As v_2 is not in $\mathrm{conv}(B)$ we have an edge b of B with $v_2 \in b^+$. By Observations 31 and 32 all points of $A \cap b^+$ are collinear and there are at least three such points. Thus if $v_1 \in b^+$ then also $v_3 \in b^+$ and we are done. Otherwise the line $l(b)$ intersects the line $l(a)$ between points v_1 and v_2 leaving other points in b^+. By Observations 31 and 32 we have at least three such points all of them collinear as required.

□

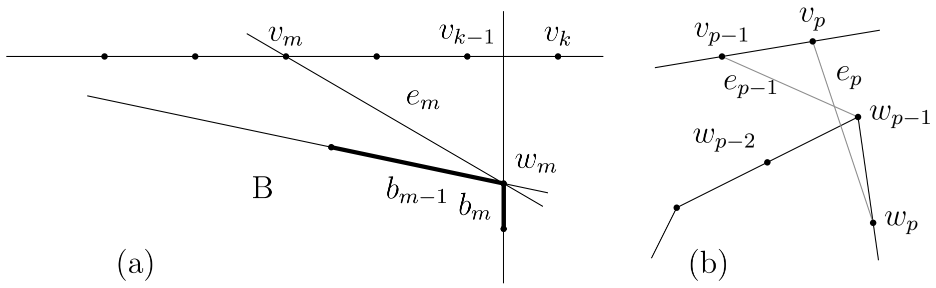

Figure 3.4: (a) If the edge e_m is good we have $B \subseteq e_m^{\ominus}$. (b) If the edge e_{p-1} is the last good edge we have $B \subseteq e_{p-1}^{\ominus}$.

In the rest of this section we also follow the approach of Barát et al. [5]. Let a be an edge of the layer A, and let $v_1, v_2, v_3 \in A \cap l(a)$ be points as above. By Lemma 35 without loss of generality there is an edge b of B such that $\{v_1, v_2, v_3\} \subseteq b^+$. By Lemma 33 such edges are consecutive. Let b_1 be the first such edge in clockwise order.

The condition $|b^+ \cap A \cap l(a)| \geq 3$ can not hold for all edges of B otherwise we would have $l(a) \cap A = A$ and thus many collinear points. Let us denote the endpoints of b_1 by w_1 and w_2 in clockwise order. Let $w_3, \ldots, w_{m+1}$ and $b_i = w_i w_{i+1}$ be subsequent edges and their endpoints in clockwise order where $|A \cap l(a) \cap b_{m-1}^+| \geq 3$ but $|A \cap l(a) \cap b_m^+| \leq 1$. Observation 31 implies that $m \leq |B \cap \bigcup_{i=1}^{m-1} l(b_i)| < |A \cap \bigcup_{i=1}^{m-1} b_i^+| \leq k$. We define $e_i := v_i w_i$ for $i = 1, \ldots, m$. And call e_i^- the open half-plane determined by $l(e_i)$ that contains v_1 and e_1^- the open half-plane that does not contain v_2.

Let the number j denote the minimum index such that the closed half-plane $e_j^{\ominus}$ contains B. We have that $w_2 \in e_1^+$, so we have $j \neq 1$. First of all we need to prove that the number j is well-defined. We call an edge e_i *good* if w_i is the closest point of $l(e_i) \cap \operatorname{conv}(B)$ to the point v_i. There are two cases, first suppose that the edge e_m is good so $v_m \in b_{m-1}^+$. The index m was defined in such a way that $|A \cap l(a) \cap b_{m-1}^+| \geq 3$ but $|A \cap l(a) \cap b_m^+| \leq 1$ and because $m < k$ we have $v_m \in b_m^{\ominus}$. This implies that $B \subseteq e_m^{\ominus}$ (Figure 3.4), so the index j is well-defined. The second case is that the edge e_m is not good. Edges e_1 and e_2 are both good by the choice of the point b_1. Let p be the minimal index such that e_p is not good. We have that $3 \leq p \leq m$. We have $w_{p-2} \in e_{p-1}^-$ as the edge e_{p-1} is good and $w_p \in e_{p-1}^-$ as the edge e_p is not good. This implies that $B \subseteq e_{p-1}^{\ominus}$ (Figure 3.4) and that the index $j = p-1$, and thus it is well-defined.

Let j be the minimum index such that the closed half-plane $e_j^{\ominus}$ contains B as in the last paragraph. We denote the quadrilaterals $Q_i := w_i v_i v_{i+1} w_{i+1}$ for $i = 1, \ldots, j-1$. Suppose that Q_h is not convex and h is the minimal such index. There are two possibilities. Assume that we have $v_h \in b_h^{\ominus}$, so we would have $B \subseteq e_h^{\ominus}$ since the edge e_h is good and thus contradicting the minimality of the index j (Figure 3.5). The second possibility is that $v_{h+1} \in b_h^{\ominus}$ so $A \cap \bigcup_{i=1}^{h} b_i^+ = \{v_1, \ldots, v_h\}$ which is in a contradiction with Observation 31 since $|B \cap \bigcup_{i=1}^{h} l(b_i)| \geq h+1$ (Figure 3.5). Thus we have that quadrilaterals Q_i for $i = 1, \ldots, j-1$ are strictly convex.

We define the 4-sectors $S_i := S[w_i, v_i, v_{i+1}, w_{i+1}]$ to be the closed 4-sectors of quadrilaterals Q_i for $i = 1, \ldots, j-1$. By definition we have $S_i \cap B = B \cap e_i^{\oplus} \cap e_{i+1}^{\ominus}$.

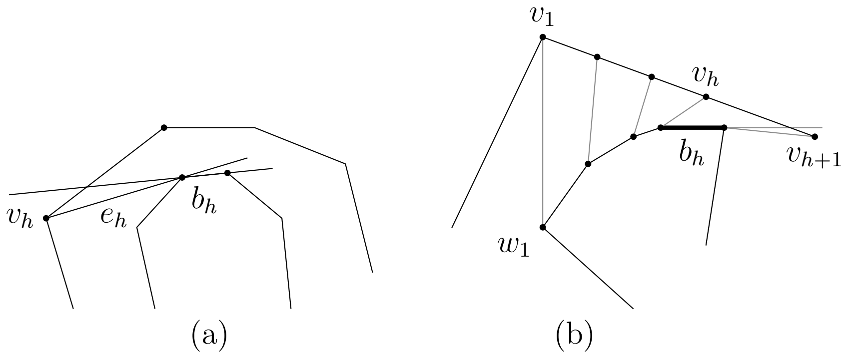

Figure 3.5: (a) If $v_h \in b_h^\ominus$ we have $B \subseteq e_h^\ominus$. (b) If $v_{h+1} \in b_h^\ominus$ we have $A \cap \bigcup_{i=1}^{h} b_i^+ = \{v_1, \dots, v_h\}$.

For every point $x \in B \cap e_1^\oplus$ we have $x \in e_j^\ominus$ since $B \in e_j^\ominus$. We set the number h to be the minimal index such that $x \in e_{h+1}^\ominus$. If $h = 0$ holds we have $x \in l(e_1) \cap B \subseteq S_1$. Otherwise $x \notin e_h^\ominus$ so $x \in e_h^\oplus$ and thus $x \in S_h$. We finally have $B \cap e_1^\oplus \subseteq \bigcup_{i=1}^{j-1} S_i$.

Every quadrilateral Q_i is empty as they lie between two layers thus no S_i contains a point in its interior, and thus all points from $B \cap e_1^\oplus$ lie on the lines $l(e_1), \dots, l(e_j)$. We have that $|B \cap l(e_i)| \leq 2$ for $i = 2, \dots, j-1$ because the layer B is in weakly convex position. Because we have $A \cap b_{m-1}^+ \neq A \cap b_m^+$ the point w_m is a corner. This implies that $B \cap l(e_j) \subseteq \{w_j, \dots, w_m\}$ and thus $|B \cap l(e_j)| \leq m - j + 1$ holds. Since we have $m < \ell$ and also $j < \ell$ we add bounds of each $l(e_i)$ and we get $|B \cap e_1^\oplus| \leq (\ell - 2) + 2(j-2) + (m - j + 1) < 3\ell$. As we have $|B| \geq 4\ell$ by Lemma 34 we have that $B \not\subseteq e_1^\oplus$ and thus $|B \cap l(e_1)| \leq 2$ must hold. Hence $|B \cap e_1^\oplus| \leq 2(j-1) + (m - j + 1) < 2\ell$ also holds.

We now want to bound the size of the rest of the layer B that is the number of points in the set $B \cap e_1^-$. We denote $v_0, v_{-1}, v_{-2}, \dots$ and $w_0, w_{-1}, w_{-2}, \dots$ vertices in the layer A and B proceeding anticlockwise from the vertex v_1 respectively the vertex w_1. We let $b_0 := w_0 w_1$. We have that $v_1 \in b_0^+$ as $B \not\subseteq e_1^\oplus$ (Figure 3.6). We know that the edge b_1 is the first in clockwise order with $\{v_1, v_2, v_3\} \subseteq b_0^+$ thus neither $v_2 \in b_0^+$ nor $v_1 \in b_0^+$. We thus have $\{v_1, v_0, v_{-1}\} \subseteq b_0^+$ by Observation 31. By Observation 32 we have that neither $v_0 \in b_1^+$ nor $v_{-1} \in b_1^+$, so the edge b_0 is the first edge with $\{v_1, v_0, v_{-1}\} \subseteq b_0^+$ in anticlockwise order, and by Lemma 35 we have that these edges are consecutive in the layer B. We may make the same argument that started at b_1 and went clockwise, and start it at b_0 and go anticlockwise. The edge e_1 will remain the same as the starting points v_1 and w_1 remain unchanged. We will again cover $B \cap e_1^\ominus$ with 4-sectors and show that $|B \cap e_1^\ominus| < 2\ell$. This with the previous paragraph implies that $|B| < 4\ell$ which is in contradiction with Lemma 34. This finishes the proof of Theorem 30.

3.2.2 Finding 8ℓ Points in Weakly Convex Position

Let P be a finite point set with at least 328ℓ points no ℓ of them collinear. For the sake of contradiction let us assume that there is no empty pentagon in P. We name $L_1, L_2, \dots, L_r$ convex layers of P where L_1 is the outermost layer, and L_r the innermost layer. For each layer $|L_i| < 8\ell$ holds by Theorem 29. We divide layers into three groups. Layers $L_{r-\ell+1}, \dots, L_r$ are called *inner layers*, and

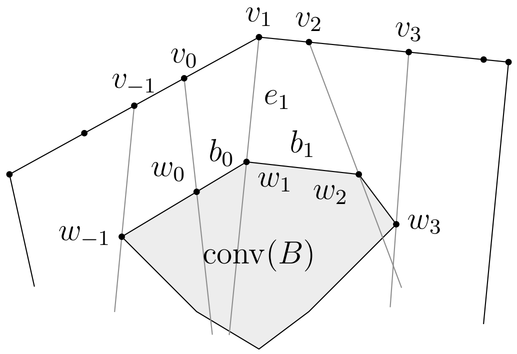

Figure 3.6: The union of closed sectors S_i covers the conv(B).

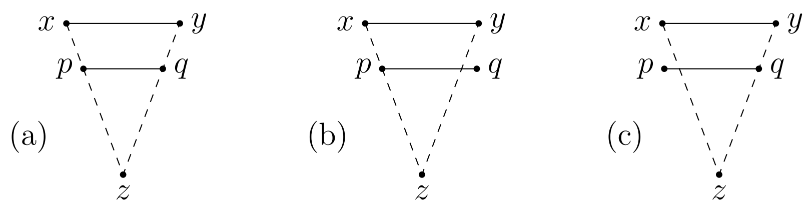

Figure 3.7: (a) Double-aligned. (b) Left-aligned. (c) Right-aligned.

layers $L_1, \dots, L_a$ are called *outer layers* where a is the minimum integer such that $|\bigcup_{i=1}^{a} L_i| \geq 64\ell(\ell-1)$. The remaining layers are called *middle layers*.

Barát et al. [5] show that if there are too many middle layers then outer layers contain less points than in definition. This argument with Theorem 29 bound the number of points in middle layers giving a contradiction with the assumed size of P.

Abel et al. [1] define for a fixed point $z \in L_r$ that an edge xy of L_i is *empty* if the open triangle $\Delta(x, y, z)$ contains no points of L_{i+1}.

Lemma 36 (Abel et al. [1]). *If a layer L_i contains an empty edge for an integer $i \in \{1, \dots, r-\ell+1\}$ then P contains an empty pentagon or ℓ collinear points.*

This lemma appears implicitly in the paper of Abel et al. [1].

Proof. For a contradiction we suppose that the set P contains no empty pentagon. Let $z \in L_r$, and suppose there is an empty edge xy in an L_i where $i \in \{1, \dots, r-\ell+1\}$. We call pq the edge of L_{i+1} that crosses the triangle $\Delta(x, y, z)$. We call the edge pq a *follower* of xy in this proof.

For an empty edge xy and its follower pq we have that the quadrilateral $pxyq$ is empty because $x, y \in L_i$ and $p, q \in L_{i+1}$. Moreover if the edge pq is not empty we have $L_{i+2} \cap \Delta(p, q, z) \neq \emptyset$. This means we have nonempty 4-sector $S(p, x, y, q)$, and thus we have an empty pentagon. We thus have that pq is empty.

For an empty edge xy and its follower qp we say that

(a) *double-aligned* if $p \in l(xz)$ and $q \in l(yz)$ as in Figure 3.7 (a).

(b) *left-aligned* if $p \in l(xz)$ and $q \notin l(yz)$ as in Figure 3.7 (b).

(c) *right-aligned* if $p \notin l(xz)$ and $q \in l(yz)$ as in Figure 3.7 (c).

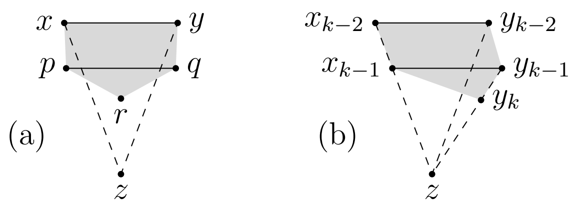

Figure 3.8: (a) Shaded empty pentagon when edges are not aligned. (b) Shaded empty pentagon formed by last left-aligned, its follower, and one point from the next follower.

If pq is a follower of an empty edge xy and pq is neither double-aligned nor left-aligned nor right-aligned then we have nonempty 4-sector $S(p,x,y,q)$ and empty quadrilateral $pxyq$ thus an empty pentagon.

Let x_1y_1 be an empty edge of the layer L_i for an $i \in \{1,\ldots,r-\ell+1\}$. Without loss of generality we may assume that $i = r-\ell+1$ because there are followers of x_1y_1 in the layer L_{i+1} and so on. Let x_1y_1 be an empty edge of the layer $L_{r-\ell+1}$. We number x_iy_i the follower of $x_{i-1}y_{i-1}$ for $i = 2,3,\ldots,\ell-1$.

There is an integer j such that the edge x_jy_j is not doubly-aligned otherwise we would have both $\{x_1,x_2,\ldots,x_{\ell-2},z\}$ and $\{y_1,y_2,\ldots,y_{\ell-2},z\}$ collinear. This would imply that either $\{x_1,x_2,\ldots,x_{\ell-1},z\}$ or $\{y_1,y_2,\ldots,y_{\ell-2},z\}$ are collinear as $x_{\ell-1}y_{\ell-1}$ must be either doubly-aligned or left-aligned or right-aligned.

We choose j to be the minimum integer from $\{2,\ldots,\ell-2\}$ such that x_jy_j is not doubly-aligned. Such j exists by the previous paragraph. Without loss of generality x_jy_j is left-aligned. Not all x_ky_k for $k \in \{j+1,\ldots,\ell-1\}$ may be left-aligned otherwise we would have $\{x_1,\ldots,x_{\ell-1},z\}$ collinear. We take k the minimum integer in $\{j+1,\ldots,\ell-1\}$ such that x_ky_k is not left-aligned. We thus have $x_{k-1}y_{k-1}$ left-aligned and x_ky_k not left-aligned. Points $x_{j-2}y_{j-2}y_{j-1}y_jx_{j-1}$ form an empty pentagon as in Figure 3.8. This concludes the proof.

□

We consider only points in middle layers L_i for $i \in \{a+1,\ldots,r-\ell\}$ for now. Let v be a point in L_i, and let x denote the closest point in $vz \cap \mathrm{conv}(L_{i+1})$. We define a *right child* of v to be the point in L_{i+1} immidiately clockwise from x. Similarly a *left child* of v is the point in L_{i+1} immidiately anticlockwise from x. When $x \in P$ we do not call it neither left nor right child. There is an example of a right and a left child in Figure 3.9 (a).

The sequence $v_1,\ldots,v_t$ of points in middle layers is called a *right chain* (resp. a *left chain*) if v_{i+1} is the right (resp. the left) child of v_i. A *subchain* is a chain contained in another chain. We say that a chain is *maximal* if it is not contained in a strictly bigger chain.

Lemma 37 (Barát et al. [5]). *Every point of every middle layer is contained in one and only one maximal right chain and one and only one maximal left chain. The number of maximal left chains is equal to the number of maximal right chains which is* $|L_{r-\ell}| \leq 8\ell - 1$.

Proof. If a point x is the right child of both points u and v then the edge (or edges in the segment uv if u and v are not adjacent) is empty by the definition of right childs. By Lemma 36 we have an empty pentagon or ℓ collinear points.

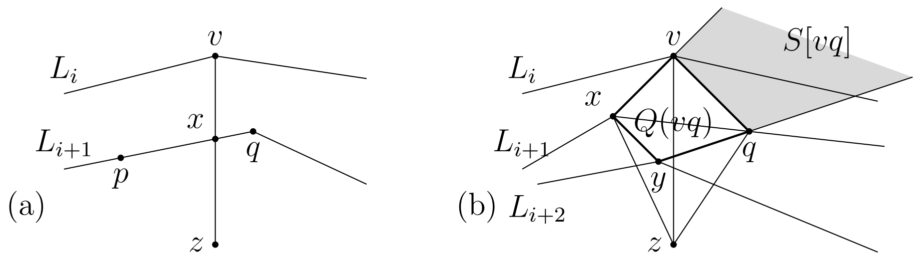

Figure 3.9: (a) The point p is the left and the point q is the right child of the vertex v. (b) Two points vq define the quadrilateral $Q(vq)$ and the sector $S[vq]$.

Moreover by the construction each point of a middle layer has both a left and a right child. Thus maximal chains do not intersect one another and each contains a point in the layer $L_{r-\ell}$.

□

Let V be a chain. The edges of the chain V are the closed line segments between consecutive vertices. We say that the chain V *wraps around* if every ray from the point z meets at least two edges of V. In other words edges of V are covering an angle of at least 4π around the point z.

Lemma 38 (Barát et al. [5])**.** *If there are at least $r - \ell - a \geq 32\ell$ middle layers then there is a chain with at most 32ℓ vertices that wraps around.*

Proof. Let $V = v_1, \ldots, v_t$ be a right chain with $v_1 \in L_{a+1}$. We may assume that $t = 32\ell$ as $r - \ell - a \geq 32\ell$. By Lemma 37 there are at most $8\ell - 1$ left chains and each point v_i lies in a left chain. By Dirichlet's principle there is a left chain U meeting V in at least five points. We select such left chain U that U and V are meeting in exactly five points $p_1, \ldots, p_5$ where p_1 is the first point of U and p_5 is the last point of U.

By the definition right chains advance clockwise and left chains anticlockwise. Edges of U and V between p_1 and p_2 thus cover an angle of 2π around z. The same holds for p_i and p_{i+1} with $i = 2, \ldots, 4$. Together edges from U and V cover an angle of at least 8π thus at least one of them is wrapping around. Both layers U and V have at most t vertices as they lie in layers L_i for $i \in \{a+1, \ldots, a+t\}$.

□

Let v be a vertex of a middle layer L_i, and let q be its right child. Let x be the point in the layer L_{i+1} that is anticlockwise from q. Note that x is either the left child of v, or it lies on the line segment vz. We choose y to be the point in the open triangle $\Delta(x, q, z)$ that is closest to xq. There is such y as otherwise xq would be an empty edge. We set $Q(vq) := vxyq$ to be the quadrilateral asociated with vq as in Figure 3.9 (b). By the construction $Q(vq)$ is strictly convex.

Since x and q are adjacent vertices in the layer L_i and y is the closest point to xq we have that the closed triangle $\Delta[x, q, y]$ is empty. The closed triangle $\Delta[v, q, x]$ is empty too as it can not contain a point neither from L_i nor from L_{i+1}. This implyes that $Q(vq)$ is an empty quadrilateral. We let $S[vq]$ be the closed 4-sector $S[v, x, y, q]$ which must be empty as its quadrilateral $Q[vq]$ is empty, and we assume that there is no empty pentagon in P.

Let $V = v_1, \dots, v_t$ be a chain, and let us call its edges $e_i = v_i v_{i+1}$. We define $e_i^{\oplus}$ to be the closed half-plane not containing the point z and determined by the line containing e_i. We denote $Q(e_i) = v_i x_i y_i v_{i+1}$ and we let c_i to denote the edge between x_i and v_i. Let d_i be the opposite edge $y_i v_{i+1}$ in the quadrilateral $Q(e_i)$. Let us denote $c_i^{\oplus}$ to be the half-plane containing the edge d_i and determined by the line containing c_i. Similarly let $d_i^{\oplus}$ be the half-plane containing the edge c_i and determined by the line containing d_i. We set the closed 4-sector defined by $Q(e_i)$ to be $S[e_i] = c_i^{\oplus} \cap d_i^{\oplus} \cap e_i^{\oplus}$.

Lemma 39 (Barát et al. [5])**.** *If $V = v_1, \dots, v_t$ wraps around then the corresponding closed 4-sectors $S[e_i]$ cover all points of outer layers L_i for $i = 1, \dots, a$.*

Proof. Let u be a fixed point in $\bigcup_{i=1}^{a} L_i$. Without loss of generality we may assume that V is a right chain and the point u is above the point z, i.e. the x-coordinate of u is the same as the x-coordinate of z and the y-coordinate of u is strictly bigger than the y-coordinate of z. We take the ray h contained in the line $l(uz)$ that starts in z and does not contain u. Edges of V intersect the ray h at least twice as V wraps around. Thus there are two edges e_j and e_k which are not consecutive with $j < k$ and there is also an edge e_p intersecting the line segment uz with $j < p < k$.

We have that u lies in $e_p^+ \cap e_j^- \cap e_k^-$. We set $\tilde{V}$ to be the maximal subchain of V containing e_p where for each edge $e \in \tilde{V}$ we have $u \in e^+$. We let e_m and e_n be the first and the last edge in $\tilde{V}$. We know that e_j and e_k are not in the subchain $\tilde{V}$ and $j < m \le n < k$ thus $u \in e_{m-1}^{\ominus} \cap e_m^+$. The point v_m lies to the left of the line $l(uz)$ since $j < m \le p$ implying that u and v_{m+1} lie on the same side of the line $l(c_m)$ and $u \in c_m^{\oplus}$. Furthermore $u \in e_n^+ \cap e_{n+1}^{\ominus}$ holds and v_{n+1} lies to the right of the line $l(uz)$ since $p \le n < k$. This implyes that $u \in d_n^{\oplus}$.

We have that $u \in e_i^+ \cap e_{i+1}^+$ for $m \le i \le n-1$. The point y_i either precedes x_{i+1} in L_{i+2} or $y_i = x_{i+1}$ thus the point u can not be in both d_i^- and c_{i+1}^-. We have to prove that $u \in S[e_i] = c_i^{\oplus} \cap d_i^{\oplus} \cap e_i^{\oplus}$. It is enough to prove that $u \in S[e_i] = c_i^{\oplus} \cap d_i^{\oplus}$ for an $i \in \{m, \dots, n\}$. Set the integer q to be the minimal index such that $u \in d_q^{\oplus}$. Since $u \in d_n^{\oplus}$ there is such q. Then we have either $q = m$ or $u \in d_{q-1}^-$, so in any case $u \in c_q^{\oplus}$ holds therefore $u \in S[e_q]$ holds too.

□

By Lemma 38 if we have at least 32ℓ middle layers then there is a chain $V = v_1, \dots, v_t$ with $t = 32\ell$ that wraps around. By Lemma 39 all points of outer layers are covered by closed 4-sectors of V. We assumed that there are no empty pentagons, so every point of an outer layer must lie on a line $l(c_i)$ or $l(d_i)$ bounding the sector $S[e_i]$. Thus we have at most $2t(\ell - 3) = 64\ell(\ell - 3)$ points in outer layers as no ℓ lines are collinear. By definition there were at least $64\ell(\ell - 1)$ points in outer layers, so there are strictly less than 32ℓ middle layers.

By Theorem 30 we have no 8ℓ points in weakly convex position, so there are strictly less than $32\ell \cdot 8\ell = 256\ell^2$. At the beginning of the proof we defined outer layers in a way that there were no more than $64\ell^2$ points and inner layers contained no more than $8\ell^2$ points. Summing everything we get thet P contains strictly less than $328\ell^2$ points – a contradiction with assumptions concluding the proof of Theorem 29.

3.3 False for Infinite Point Sets

The Big Line Big Clique conjecture does not hold for infinite sets by the result of Pór and Wood [48]. This implies that we cannot use an infinitary compactness argument to prove it.

Theorem 40 (Pór, Wood [48])**.** *There is a point set $P \subset \mathbb{R}^2$ of countable size that has no three collinear points and no three mutually visible points.*

Proof. Let $x_1, x_2, x_3 \in \mathbb{R}^2$ be three points in general position. Given points $x_1, \ldots, x_{k-1}$ we want to add a point x_k. There is a line meeting exactly two of points $\{x_1, \ldots, x_{k-1}\}$ as proved by the Sylvester-Gallai theorem [39]. We choose such two points x_i, x_j with $i < j$ that there is no other point on the line $x_i x_j$ and first j is minimized and then i is minimized. Finally we insert the point x_k on the line segment $x_i x_j$ at such position that the triple (x_i, x_k, x_j) is the only collinear triple containing x_k.

By construction there are no four collinear points in the set $\{x_i \mid i \in \mathbb{N}\}$. The key observation is that if x_i, x_j are visible where $i < j$ there is a third point $x_{i'}$ such that $x_{i'}$ lies on the line $x_i x_j$ (otherwise we would add a point between x_i, x_j later in the construction). Moreover x_j lies on the line segment $x_i x_{i'}$, and $i' < j$ holds.

Let us suppose that there are three pairwise visible points x_i, x_j, x_k with indexes $i < j < k$. By the observation in the previous paragraph we have points $x_{i'}$ and $x_{j'}$ such that x_k lies on the line segments $x_i x_{i'}$ and $x_j x_{j'}$. This is a contradiction because x_k belongs only to one collinear triple in the set $\{x_i \mid 1 \leq i \leq k\}$.

□

3.4 Ordered Set Representation

As we have seen some properties of visibility graphs can be proved using just the sweep line technique. It is natural to ask if the Big Line Big Clique conjecture is provable in this way. In this section we provide a proof of the Big Line Big Clique for $\ell = 4$ and $k = 4$ for a class of graphs of which visibility graphs are a strict subset. For this class of graphs the Big Line Big Clique conjecture does not hold for $\ell = 4, k = 6$.

3.4.1 Definition of the Ordered Set Representation

When we are line-sweeping a point set P we obtain an ordered set of points $p_1, p_2, \ldots, p_{|P|}$ and two points $p_i, p_j, i < j$ do not see each other if there is a point $p_k, i < k < j$ on the line $p_i p_j$. A line l induces a set $Q \subseteq P$ of points lying on l. On the other hand if two point sets induced by lines share at least two points they are equal, as no more than one line goes through two distinct points.

Definition. An *ordered set representation* is a tuple $M = (n, L)$ where $n \in \mathbb{N}$, and L is a family of subsets of $[n]$ such that $\forall i, j \in [n], i \neq j \colon \exists l \in L \colon i, j \in l$ and $\forall l_0, l_1 \in L \colon |l_0 \cap l_1| \leq 1$. Numbers $1, \ldots, n$ are called points, and elements of L are called lines.

The definition is indeed sound as setting $L = \{[n]\}$ gives an ordered set representation. As there is always one and only one line containing two points we may call $l(i, j)$ the line containing both points i, j.

Definition. We say that two points $i < j$ see each other with respect to an ordered set representation M if and only if there is no k such that $i < k < j$ and $l(i, k) = l(k, j)$. For an ordered set representation M we define the visibility graph of M as $\nu(M) = ([n], E)$ where there is an edge $ij \in E$ if and only if i, j see each other with respect to M.

We call a graph G *ordered set representable*, or *OSR*, if there is an ordered set representation M such that the visibility graph $\nu(M)$ of M is isomorphic to G. It is easy to see that for instance empty graph is not ordered set representable. On the other hand a point visibility graph is ordered set representable. We can see this by line-sweeping the set P and setting L to be the family of all lines through at least two points of the set P.

3.4.2 $k = 4$ or $\ell = 4$ Holds for OSR Graphs

In this section we show that Big Line Big Clique conjecture holds for $k = 4$ and $\ell = 4$ even with ordered set representable graphs. This also provides a different purely combinatorial proof for point visibility graphs.

Theorem 41 (Turán [58]). *Let $G = (V, E)$ be a graph with n vertices that does not contain K_{r+1} as a subgraph then $|E| \leq (1 - 1/r)n^2/2$ holds.*

Theorem 42. *There is an integer n_0 such that for every $n \geq n_0$ every ordered set representable graph with a set representation $M = (n, L)$ contains a K_4 as a subgraph or $\exists l \in L\colon |l| \geq 4$.*

Proof. Let us assume that there is no line $l \in L\colon |l| \geq 4$ and the graph $\nu(M)$ contains no clique K_4 as subgraph. It follows from Turán's theorem that there are at least

$$\binom{n}{2} - \left(1 - \frac{1}{r}\right)\frac{n^2}{2} = \frac{n(n-r)}{2r}$$

missing edges.

One line $l \in L$ of size three blocks exactly one edge of $\nu(M)$ and a line of size two blocks no edge. Assuming there is no bigger line we must have at least as many lines of size three as there are missing edges in $\nu(M)$.

The point $i \in [n]$ may block at most $\min(i-1, n-i)$ edges as if i blocks visibility of $a < b \in [n]$ we have $a < i < b$. It follows that there are at most

$$2\sum_{i=1}^{\lceil n/2\rceil - 1}(i-1) = \begin{cases} 2\binom{m}{2} & \text{if } n = 2m \\ 2\binom{m}{2} + m & \text{if } n = 2m+1 \end{cases}$$

edges blocked. This is not enough for us yet. We can assume $n = 2m$. There have to be at least $\frac{n(n-2r)}{8r}$ edges missing in the first half of points, the same holds for the second half of points. Edge in one half can be blocked just by points in that half. Totally there are at least $\frac{n(n-2r)}{4r}$ edges missing in both halves.

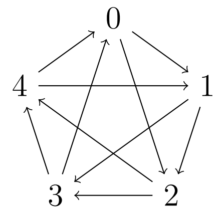

Figure 3.10: Points from equivalence class A modulo five block points from another class B if and only if there is an arrow from A to B.

If b blocks the edge ac then we have $a < b < c$, and we are sure that c is not blocking visibility of ad or bd for a $d > c$ or there would be a big line. In other words if ac is blocked by b then c can block visibility of neither a nor b. We thus have to subtract these impossible lines from the number of edges needing blocking.

$$\frac{n(n-r)}{2r} - \left(\frac{(n-2)n}{4} - 4\frac{n(n-2r)}{8r}\right) = n\left(n\left(\frac{1}{r} - \frac{1}{4}\right) - 1\right)$$
$$\frac{n(n-3)}{6} - \left(\frac{(n-2)n}{4} - 4\frac{n(n-6)}{24}\right) = \frac{n(n-12)}{12}$$

We can see that for $n = 14$ and $k - 1 = r = 3$ the formula is positive, so there are more edges to block than we possibly could block which is a contradiction.

□

3.4.3 $k = 6$ or $\ell = 4$ Is False for OSR Graphs

We saw that a big enough ordered set representable graph contains a K_4 subgraph or a line of size four. On the other hand here we show that complete five-partite graphs are ordered set representable without using a line bigger than three.

Theorem 43. *There is a set representation M such that $\forall l \in L\colon |l| \leq 3$ and $\nu(M)$ does not contain K_6 subgraph.*

Proof. We divide points to equivalence classes modulo five, and we represent complete five-partite graphs. We block points of modulo class j by points $j-1 \pmod 5$ and $j-2 \pmod 5$ as depicted on the Figure 3.10.

We still need to tell which points block which pair of points. Without loss of generality we may assume the number of points is $n = 5m$. We are trying to block visibility of pairs of points where $i \equiv j \pmod 5$. Let us denote sets of blockers $B_b = \{i \in [n] \mid i \equiv b - 1 \pmod 5 \vee i \equiv b - 2 \pmod 5\}$ for $b = 0, 1, \ldots 4$. To block visibility of pairs of points $\{(i,j) \mid i,j \in [n], i \equiv j \pmod 5\}$ in lexicographic order we use points from $B_b, b \equiv i \pmod 5$ bigger than i in increasing order. This is depicted for $n = 35$ and $j = 1$ in the Table 3.1.

There are enough points to block because if we have points $i, j \in [n]$ and $i < j, i \equiv j \pmod 5$ we also have the point $j - 1$ and $j - 2$ for blocking.

It remains to prove that no two lines coincide at two points, i.e., there is no line of size at least four. When we have two lines $a < b < c$ and $d < e < f$ we want to distinguish which pairs of points coincide. This is done by writing which point in the first line is not present in the second one and vice versa. For example

	6	11	16	21	26	31
1	4	5	9	10	14	15
6		9	10	14	15	19
11			14	15	19	20
16				19	20	24
21					24	25
26						29

Table 3.1: Rows represent lesser and columns higher points of blocked pairs.

having lines $a < b < c$ and $d < a < b$ we call this case $(3, 1)$. There are nine such cases in total.

It is easy to see that there are no two lines of size three with the lowest and the highest point in common as we block each edge just once – the case $(2, 2)$. There are no two lines with lower two points in common as we use different points to block visibility of the lower one – the case $(3, 3)$. The same follows for the upper two points – the case $(1, 1)$.

If there were two lines of cases $(1, 3)$ or $(3, 1)$ this would mean lines $a < b < c$ and $b < c < d$ or $a < b < c$ and $d < a < b$ meaning that the point b blocks visibility of the point a and vice versa. This cannot happen as in the Figure 3.10 we do not have arrows in both directions, and we block just edges inside one equivalence class.

In all of other cases $(1, 2), (2, 1), (2, 3), (3, 2)$ the middle point from one line and the first or the last point from the other line coincide. But by construction we block just edges in one equivalence class and by points from other equivalence classes.

□

3.4.4 Computer Experiments

In combinatorics solutions of small cases can give us a valuable insight. In such cases we often want to run some computer experiments. In this part we describe computer experiments that we used for ordered set representations. In Chapter 5 we describe computer experiments for point visibility graphs.

Formulating SAT Formula

Boolean satisfiability problem, or *SAT*, is the problem of determining if boolean variables of a given boolean formula φ can be assigned values to satisfy it. SAT solver programs mostly assume that the input formula is in conjunctive normal form, e.g. $(x_1 \vee x_2) \wedge (\neg x_1)$ which we can satisfy by setting $x_1 =$ False and $x_2 =$ True.

For given n, k, ℓ there is a SAT formula deciding if there exists an OSR graph of size n without K_k as subgraph and without a line of size at least ℓ. This can be seen as a simple generalization of the next part where we consider just $\ell = 4$ for efficiency reasons.

We number all lines (subsets of $[n]$) of size three, and for each line we have a boolean variable. We do not consider lines of size two as those are blocking no edge. We also allow no line bigger than three. In later text we designate the variables by $x_{abc}, a < b < c \in [n]$ for easier reading.

We need to express that there are no two lines with intersection of size at least two. To do this we use that at most one of x_{abc} and x_{def} where $|\{a, b, c\} \cap \{d, e, f\}| = 2$ can be true:

$$\bigwedge_{|\{a,b,c\}\cap\{d,e,f\}|=2} (\neg x_{abc} \vee \neg x_{def}).$$

To express that the edge ac is blocked we use $N(a, c) := \bigvee_{b>a \wedge b<c} x_{abc}$. For a k element subset K we write disjunction over all tuples $\{a, b\} \in K$ of the prior clause: $\bigvee_{\{a,b\}\in K} N(a, b)$. We need this for every k element subset of $[n]$. So we have another $\binom{n}{k}$ clauses.

Experimental Results

We just generate the formula then we let a fast solver to decide that formula for us. We use freely available SAT solvers *Glucose* [4] and *PicoSAT* [8]. For our experiments we used personal computers and servers kamna and kamenozrout at the Department of Applied Mathematics, Charles University.

The program `sat.cpp` is a straightforward generator of formulae in the DIMACS format [14]. Many SAT solvers use this format, so it is easy to run these experiments with multiple solvers. The program `paint.c` expects numbers where negative number means that the triple denoted by the number is not selected and positive means that it is. It outputs the graph in the MetaPost format. Both programs were written to be run just once and by an informed user. There are some limitations and not validated inputs.

We compile the program `sat.cpp` by the command `g++ sat.cpp -o sat.out` and the command `gcc -std=c99 paint.c -o paint.out` is used to compile the program `paint.c`. To decide the existence of an OSR graph for given parameters the number of vertices and size of the forbidden clique we run `./sat.out 9 4 | picosat` where `picosat` is a SAT solver that accepts the DIMACS format. To get a MetaPost file run `./sat.out 9 4 | picosat | sed '/s SATISFIABLE/d' | sed '/c.*/d' | tr -d "v" | ./paint.out 9`. Forbidden assignments can be written in the `forbidden.txt` file in the same form as for the program `paint.out`.

Due to computer results an ordered set representable graph without a K_4 subgraph and without a line of size four can have at most nine vertices. Moreover there are only three such graphs (see the Figure 3.11) each of them with exactly two possible ordered set representations listed in Attachments as outputs of the SAT solver.

Thanks to the nature of the problem and fast implementation of SAT solvers we were able to find rather big ordered representable graphs. The Figure 3.12 shows the largest ordered set representable graph found with no $\ell = 4$ or K_5. We do not know if there are any bigger, or if this is the only such graph on 42 vertices. Because of the running time I think this might be close to the largest such graph.

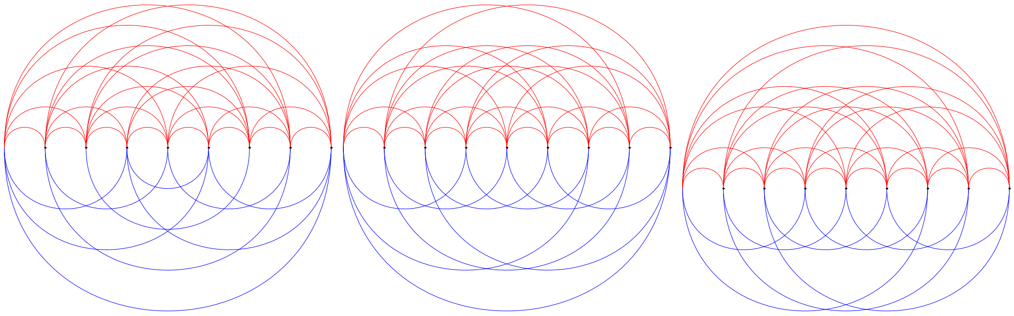

Figure 3.11: Only three OSR graphs on nine vertices without K_4 and without a line of size four. Edges are depicted as upper arcs in red, blocked (i.e. missing) edges as lower arcs in blue.

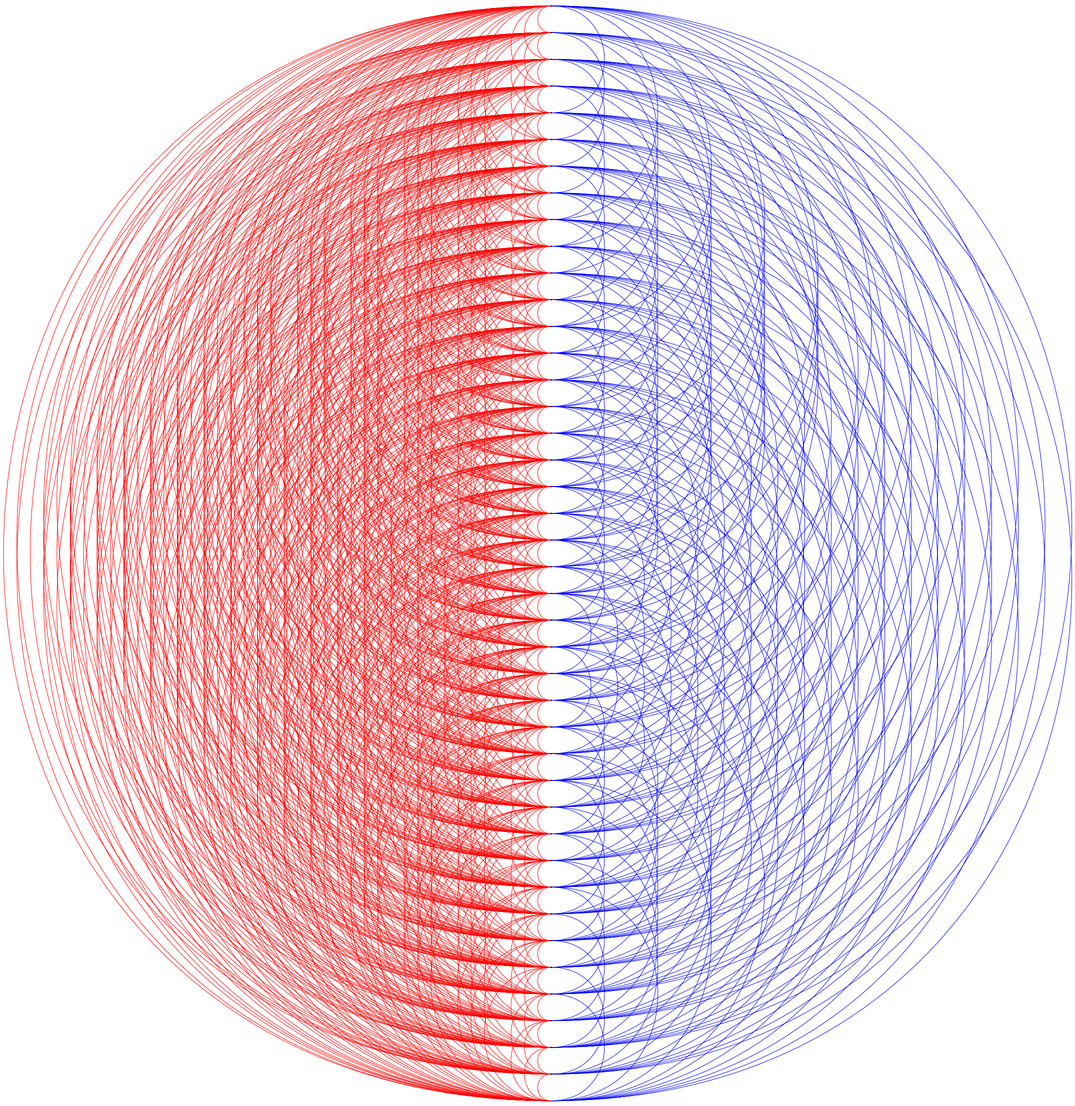

Figure 3.12: An OSR graph on 42 vertices without K_5 and without a line of size four. Edges are depicted as arcs on the left in red, blocked (i.e. missing) edges on the right in blue.

4. Blocking Visibility

For a finite set of points P in real plane (or, more generally, in $\mathbb{R}^d$) we say that a set of points Q is a *visibility blocking set* if Q is disjoint from P and for every $p_0, p_1 \in P$ there is at least one point $q \in Q$ lying on the line segment p_0p_1, alternatively we could say that P is an independent set in the visibility graph of the union $P \cup Q$.

If points in P are collinear then there is a visibility blocking set of size $|P|-1$. The question is what is the smallest size of a blocking set for P having no three points collinear. We let

$$\begin{aligned} b(P) &:= \min\{|Q| \mid Q \text{ a visibility-blocking set for } P\}, \\ b(n) &:= \min\{b(P) \mid P \subset \mathbb{R}^2 \text{ with no three points collinear}, |P| = n\}, \end{aligned}$$

and we would like to estimate the asymptotic behavior of $b(n)$.

4.1 Lower Bound

Matoušek [38] observes that the blocking set Q has the size at least as big as the number of edges in a triangulation of P. Dumitrescu et al. [16] use this observation in combination with bound of Kostochka and Kratochvíl [36] to provide a slightly better constant.

Observation 44 (Matoušek [38])**.** *Let P be an n-point planar set without three collinear vertices and with* $\mathrm{conv}(P)$ *having p vertices then $b(P) \geq 3n - p - 3$.*

Proof. An arbitrary triangulation of P has $3n - p - 3$ edges by Euler's formula. Each point of a visibility-blocking set covers at most one edge of the triangulation. □

Superlinear bounds are known only for point sets in convex position. For points in convex position the $\Omega(n \log n)$ bound of Kostochka and Kratochvíl [36] is known. For vertices of a convex n-gon even greater bound of $\Omega(n^2)$ can be given, as shown by Poonen and Rubinstein [47].

Theorem 45 (Kostochka and Kratochvíl [36])**.** *For every n-point planar set P in strictly convex position, we have*

$$b(n) \geq \begin{cases} n \sum_{k=1}^{m} 1/k & \text{for } n = 2m+1, \\ 1 + n \sum_{k=1}^{m-1} 1/k & \text{for } n = 2m. \end{cases}$$

Thus $b(P) \geq \frac{1}{2} n \log n$.

Proof. Let $p_1, p_2, \ldots, p_n$ be the points numbered as they appear around the circumference of the $\mathrm{conv}(P)$. We define the *length* $\ell(p_i, p_j)$ of the line segment p_i, p_j where $i < j$ as the number of convex hull edges between p_i, p_j, i.e., $\min(j - i, n + i - j)$.

The key observation is that for every blocking set Q and every $q \in Q$ let ℓ be the length of the shortest segment p_i, p_j that q is blocking then q is incident to at most ℓ segments (Figure 4.1).

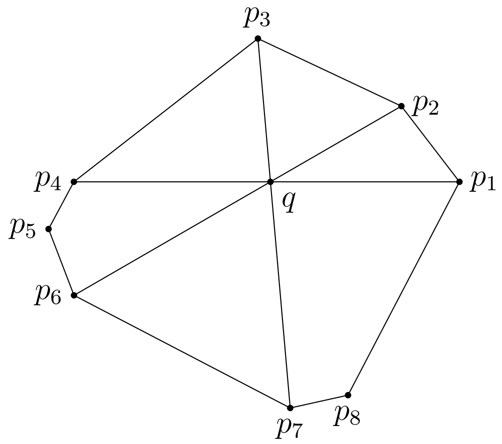

Figure 4.1: The point q lies on the segment p_1p_4 with $\ell(p_1, p_4) = 3$ and blocks three segments.

We assign the weight $1/\ell(p_i, p_j)$ to the segment p_i, p_j. No point in Q is incident to segments with the sum of their weights greater than one. The theorem follows by summing all weights.

$\square$

Dumitrescu et al. [16] improve the constant of Observation 44 using the previous theorem for convex sets.

Theorem 46 (Dumitrescu [16])**.** *Let P be an n-point planar set in general position then every blocking set has size at least* $(25/8 - o(1))n - 3$.

Proof. We can assume $n \geq 10$. Let P' denote the set of vertices of the convex hull of the point set P and $p := |P'|$. We distinguish two cases depending on p. First assume $p \geq \frac{25}{2}\frac{n}{\log n}$ we can see that $\log p \geq (\log n)/2$ and by Theorem 45 we have $b(P) \geq b(P') \geq \frac{1}{2}\frac{25}{2}\frac{n}{\log n}\frac{\log n}{2} = \frac{25}{8}n$.

Let $p \leq \frac{25}{8}\frac{n}{\log n}$, and for simplicity let $n = 8k + 2$ for a natural number k. We choose a point p_0 in P', and number the rest of points in clockwise order of visibility from p_0. We denote sets of ten points $P_i := \{p_0, p_{8i-7}, p_{8i-6}, \ldots, p_{8i+1}\}$ for $i = 1, 2, \ldots k$. Each two consecutive sectors have two points in common. An example of one such sector is in Figure 4.2.

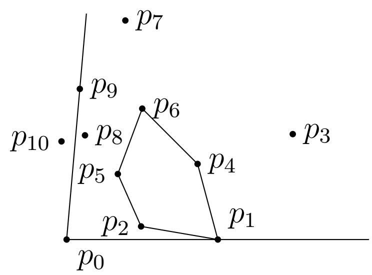

Figure 4.2: An illustration of one sector with an empty pentagon $p_1p_2p_5p_6p_4$.

Harborth [31] proves that every set of ten points in general position contains a 5-hole, i.e., five points with empty convex hull. For each i we denote Q_i these sets of five points with empty inside. To cover diagonals of a Q_i we need at least three points. We choose edges of convex hull of every Q_i, and extend those edges to a triangulation. As interiors of sets Q_i are pairwise disjoint, we can see that for each sector we need one extra blocking point more than the number of edges in the triangulation. We need at least $k = \lfloor n/8 \rfloor$ more points in total. This gives us that the set P needs at least $3n - p - 3 + k = \left(\frac{25}{8} - o(1)\right) n - 3$ blocking points. $\square$

4.2 Upper Bound

Surprisingly the best upper bound known is for special blocking sets containing all midpoints. For a point set P, let $\mu(P)$ be the cardinality of the set of all midpoints $\{\frac{1}{2}(p+q) \mid p, q \in P, p \neq q\}$ of all pairs of points of P. This set is always blocking, so we have $b(n) \leq \min \mu(P)$ where the minimum is over all point sets in general position of size n.

Erdős, Fishburn, and Füredi [22] studied the problem of estimating $\min \mu(P)$ over all P in convex position, and established the upper and lower bounds of $0.40n^2$ and $0.45n^2$ respectively.

Pach [43] (based on Erdős et al. [23]) provided a construction of a point set P and its midpoint-blocking set Q implying the upper bound $\mu(n) \leq ne^{C\sqrt{\log n}}$ for a constant C.

Theorem 47 (Pach [43])**.** *There is a constant C such that for n big enough $b(n) \leq \mu(n) \leq ne^{C\sqrt{\log n}}$ holds.*

Proof. For simplicity let us consider $n = \left\lfloor 2^{d(d-2)}/d \right\rfloor$ for an integer d greater or equal to four. We take the lattice $L = \left\{0, 1, \dots 2^d\right\}^d$ with intersections with spheres $S_k = \{x \in L \mid \|x\|^2 = k\}$. We can see that $L = \cup_{k=0}^{d(2^d)^2} S_k$. By pigeonhole principle there is a k for which $|S_k| \geq \frac{|L|}{d(2^d)^2} \geq n$ points, let us denote P an arbitrary such set.

As $|\{\frac{1}{2}(p+q) \mid p, q \in P, p \neq q\}| = |\{p + q \mid p, q \in P, p \neq q\}| = |P + P|$ where $p + q$ are points of positive integer coordinates strictly less than 2^{d+1}. We thus have $\mu(P) = |P + P| \leq (2^{d+1})^d < n2^{8\sqrt{\log n}}$.

By the argument we already saw, we can find a linear projection to a plane where no two points from P coincide, and as P lies on a sphere no three elements of the projected P are collinear. This argument can be extended for general values of n.

$\square$

Pach [43] also proves that the midpoint-blocking number is asymptotically greater than n, i.e., $\lim_{n\to\infty} \mu(n)/n = \infty$. It is still not known if the same holds for general blocking sets.

5. Existential Theory of the Reals

Satisfiability is a large branch of computer science with many important applications. Perhaps the best known is the boolean satisfiability problem which we already saw in Chapter 3.

The boolean satisfiability problem is not the only problem of its kind. A whole branch of decision problems is called *satisfiability modulo theories*, or *SMT*, where the problem is to determine if a logic formula is satisfiable with respect to some background theories. Examples of theories used are the theory of real numbers, the theory of integers or theories of data structures such as arrays and bit vectors.

SMT has wide range of applications involving theorem proving, robotics vision, computer design, etc. Many software verification and test case generation programs use SMT solvers internally. SMT is used even for planning. Design of complex systems such as cars or trains often relies on some invariants that can be checked by SMT. Scheduling of tasks to machines with some constraints can be also solved using SMT. Often it is vital to use more SMTs (for example the theory of reals and the theory of integers) to solve one problem. For more applications of SMT see de Moura and Bjørner [9].

Here we discuss the existential theory of real numbers as it has direct application to geometric problems. Canny [11] proves that any sentence in the existential theory of the reals can be decided in PSPACE. By a result of Ghosh and Roy [27] the problem of deciding whether a given graph is a visibility graph of a point set is decidable by the existential theory of the reals. We show that the same holds even for deciding the Big Line Big Clique conjecture for given parameters n, k, ℓ. Using an idea of Schaefer [50] we are able to construct such a formula that neither the number of variables, nor the number of polynomials, nor their degrees depend on the parameter ℓ. We also try this formula with available solvers in Section 5.4.

5.1 Introduction

Informally the task of the existential theory of reals is to decide if a logical formula consisting of inequalities involving real valued variables is satisfiable. For example we might want to decide if variables $x_1, \ldots, x_n$ could be assigned real values to satisfy a given formula, e.g. $(x_1 < 5) \wedge (x_2 + x_5 = 7 \vee x_3 x_4 = 0) \wedge \ldots$

More formally the first-order *existential theory of the real numbers* is the set of all true sentences of the form $(\exists x_1, \ldots, x_n)\varphi(x_1, \ldots, x_n)$ where φ is a quantifier free $(\vee, \wedge, \neg)-$Boolean formula over the signature $(0, 1, +, *, <, \leq, =)$ and the sentence is interpreted over real numbers.

It is known that the theory of reals can be decided in PSPACE. This result is due to Canny [11]. As proposed by Schaefer [50] we can introduce a new complexity class $\exists\mathbb{R}$ of problems with the same complexity as deciding the existential theory of reals. Note that Schaefer [50] defines two classes with and without equality and proves with Štefankovič [51] that these classes are the same, although the two classes algebraically differ ($x^2 = 2$ defines irrational number which is impossible without equality).

We say that a problem is $\exists\mathbb{R}$-*complete* if it can be solved using the existential theory of reals, and every problem in $\exists\mathbb{R}$ can be reduced to it by a polynomial reduction. Stretchability of simple pseudoline arrangements is the first problem known to be $\exists\mathbb{R}$-complete. It is quite easy to see that this is decidable by the existential theory of the reals. Mnev [40, 52, 34] proves the other side. Later other problems proved to be $\exists\mathbb{R}$-complete; for references see Schaefer [50].

5.2 Recognition and Reconstruction

A complete characterization of the class of visibility graphs still remains open. One natural question is to determine the visibility graph of a given point set. The reverse question is to decide whether a given graph is the visibility graph of a point set. We are going to see that both these problems are decidable by a computer.

5.2.1 Visibility Graph Reconstruction

Let us begin with the simpler question – the problem of computing the visibility graph $\nu(P)$ of a point set P.

Algorithm mentioned by Ghosh and Roy [27]: Let P be a given point set of size n. For every $p \in P$ we order the rest of points in angular order around p. The angular order gives us a partition of $P \setminus \{p\} = P_1 \dot\cup \dots \dot\cup P_k$ to sets of points lying on the same line going through p ($q_1, q_2 \in P_i$ are collinear with p but not so for $q_0 \in P_i, q_1 \in P_j, i \neq j$). Taking one such set P_i at most one point at each side closest to p might be visible from p. Chazelle et al. [12] found an algorithm to construct angular orders of all points in $\mathcal{O}(n^2)$ time, so we could improve time complexity of constructing the graph $\nu(P)$ to $\mathcal{O}(n^2)$ using their approach.

5.2.2 Visibility Graph Recognition

We would like to decide if a given undirected graph G is the visibility graph of a point set P. Ghosh and Roy [27] prove that this recognition problem lies in PSPACE.

Theorem 48 (Ghosh, Roy [27])**.** *The recognition problem for point visibility graphs lies in PSPACE.*

Proof. For a given graph $G = (V, E)$ it is sufficient to construct a formula in the existential theory of reals which is polynomial in the size of G and which is true if and only if G is the visibility graph of a set of points P.

Suppose $(v_i, v_j) \notin E(G)$. This means that for every point set P with the $\nu(P)$ isomorphic to G there is a blocker, say p_k, on the line segment joining p_i and p_j. Let the coordinates of the points p_i, p_j, p_k be $(x_i, y_i), (x_j, y_j), (x_k, y_k)$ respectively. So we define:

$$N(i,j,k) := (0 < t_{i,j,k} < 1) \wedge ((x_k - x_i) = t_{i,j,k}(x_j - x_i)) \wedge ((y_k - y_i) = t_{i,j,k}(y_j - y_i)).$$

We need at least one blocker, so we have $\left(\bigvee_{k \neq i, k \neq j} N(i,j,k)\right)$.

Suppose $(v_i, v_j) \in E(G)$. This means that for every point set P which visibility graph is G there is no blocker on the line segment joining p_i and p_j. So either p_k forms a triangle with p_i and p_j, or p_k lies on the line passing through p_i and p_j but not between p_i and p_j. With quantifier-free formulas we can define the predicate $\overline{\text{collinear}}(x_1, y_1, x_2, y_2, x_3, y_3)$ expressing that the three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are not collinear. Indeed the area of the triangle $(x_1, y_1)(x_2, y_2)(x_3, y_3)$ can be computed using determinant. It is enough for us that the determinant is nonzero thus we can safely omit the one half and absolute value, and write $(x_2 - x_1)(y_3 - y_1) \neq (y_2 - y_1)(x_3 - x_1)$.

As the result of the previous paragraph we define:

$$E(i, j, k) = \overline{\text{collinear}}(x_i, y_i, x_j, y_j, x_k, y_k)\vee$$

$$\vee\left((t_{i,j,k} < 0 \vee 1 < t_{i,j,k}) \wedge (x_k - x_i = t_{i,j,k}(x_j - x_i)) \wedge (y_k - y_i = t_{i,j,k}(y_j - y_i))\right).$$

We can have no blocker, so we have $\left(\bigwedge_{k\neq i, k\neq j} E(i, j, k)\right)$.

For every triple of vertices v_i, v_j, v_k we add a $t = t_{i,j,k}$ to the existential part and corresponding quantifier free formula. The final formula is:

$$\exists x_1 \exists y_1 \ldots \exists x_n \exists y_n \exists t_{1,2,3}, \ldots \exists t_{n-2,n-1,n} P(x_1, y_1, \ldots, x_n, y_n, t_{1,2,3}, \ldots, t_{n-2,n-1,n})$$

where P is conjunction of E resp. N which is of size $\mathcal{O}(n^3)$.

□

We could use an equisatisfiable formula in conjunctive normal form instead of the formula given in the proof. This could be done by adding new variables and only linearly increasing the size.

5.3 Deciding the Big Line Big Clique Conjecture for Given n, ℓ, k

The existential theory of reals is strong enough to even decide the Big Line Big Clique conjecture for given parameters n, ℓ, k. We give a deciding formula of size depending only on n and k but not on ℓ.

5.3.1 No k-Clique

Using a quantifier-free formula, we can define that a k-tuple of vertices $p_1, \ldots, p_k$ is not a clique. For every tuple $p_i, p_j, 1 \leq i < j \leq k$ we need a blocking point (possibly not one of the k-tuple). For every k-tuple T we thus define $\overline{C(T)} := \bigvee_{1\leq i<j\leq k} \bigvee_{m\in[n], m\neq i, m\neq j} N(i, j, m)$. We need no k-tuple to form a clique thus we have $\bigwedge_{k\text{-tuple } T} \overline{C(T)}$.

For just one k-tuple we need a formula with $\mathcal{O}(k^2 n)$ polynomials. Thus to forbid all cliques of size k we use a formula with $\mathcal{O}(k^2 n\binom{n}{k})$ polynomials.

5.3.2 No ℓ Collinear Points

We can define that no more than ℓ points lie on the same line with a formula of size not depending on ℓ. Schaefer [50] used a predicate $\text{atmost}_a(z_1, \ldots, z_b)$ which guarantees that no more than a of variables z_i are greater than zero. Here we present a slightly modified version:

$$\sum_{i\in[b]} z_i < a \wedge \bigwedge_{i\in[b]} \left(\left(-\frac{a+3}{3b(b-a)} < z_i < 0 \right) \vee \left(1 + \frac{1}{3b} < z_i \right) \right)$$

where $0 < a < b$. Options "at most zero" of the variables z_i or "at most b" of the b variables z_i make little use anyway.

If strictly more than a of the z_i are positive, the total sum is at least

$$\sum_{i\in[b]} z_i \geq (a+1)\left(1+\frac{1}{3b}\right) - (b-(a+1))\left(\frac{a+3}{3b(b-a)}\right) \geq a+1-\frac{2}{3b} > a$$

as there are at most $b-(a+1)$ negative z_i.

On the other hand given any subset of the z_i of size at most a, we can assign each z_i in the set the value $1+(a+1)/(3ab)$ and every other z_i gets the value $-(a+2)/(3b(b-a))$, so that

$$\sum_{i\in[b]} z_i \leq a\left(1+\frac{a+1}{3ab}\right) - (b-a)\left(\frac{a+2}{3b(b-a)}\right) = a - \frac{1}{3b} < a.$$

For every two points $p_i, p_j \in P, i < j$ we want no more than $\ell-2$ points on the same line. We use $z_{i,j,m} > 0$ to indicate that a point p_m may lie on the line p_ip_j. Let $A(i,j) := \text{atmost}_{\ell-2}(z_{i,j,1}, \ldots, z_{i,j,i-1}, z_{i,j,i+1}, \ldots, z_{i,j,j-1}, z_{i,j,j+1}, \ldots, z_{i,j,n})$. Finally we write $A(i,j) \wedge \bigwedge \left(z_{i,j,m} > 0 \vee \overline{\text{collinear}}(x_i, y_i, x_j, y_j, x_m, y_m)\right)$ for the point tuple p_i, p_j.

The predicate atmost is of linear size and so is the last big conjunction. We use these for every two points in P, so we create a formula of size $\mathcal{O}(n^3)$.

5.3.3 Final Formula

We already gave a description of quantifier free formulae forbidding big line and big clique. We just concatenate those, and we close this formula by putting $\exists x$ before it for every free variable x.

Formulae forbidding a clique of size k use $\mathcal{O}(k^2 n\binom{n}{k})$ polynomials and formulae forbidding ℓ collinear points have $\mathcal{O}(n^3)$ polynomials. Using both formulae and considering only interesting cases where $k \geq 2$ we get the total number of polynomials $\mathcal{O}(k^2 n\binom{n}{k})$. Moreover we only use $\mathcal{O}(n^3)$ variables and polynomials of degree at most two. The biggest constant used is $3n\ell$. It thus follows that deciding the Big Line Big Clique conjecture for any given n, ℓ, k is in PSPACE.

5.4 Computer Experiments

If there is a solution we can rotate it to get distinct x-coordinates, and thus there is a solution with $x_i < x_{i+1}$ for all indexes i. To further improve the length of the used formula by a multiplicative constant we add the condition $x_i < x_{i+1}$ for all $1 \leq i < n$ and thus a point p_j can block points $p_i, p_k, i < k$ only if $i < j < k$.

The program `smt.cpp` straightforwardly generates an input in the SMT2 format [13] which seems to be the most popular input format amongst moder solvers. The program expects three arguments `n l k` that are numbers where $n \leq \ell$ and $n \leq k$. This program is written to be run just once and by an informed user. There might be some inputs which are not validated.

Unlike SAT solvers there are many formats used in SMT. Moreover there are not so many solvers that are able to solve nonlinear formulas.

We compile the program with the command `g++ smt.cpp -o smt.out`. To run tests we use `./smt.out 5 4 3 | ./z3 -in -smt2` with the `z3` program available from `http://z3.codeplex.com/`.

The solver did not finished with parameters $n = 6, \ell = k = 4$ although there is a solution in Figure 2.8. The solver is able to solve formulae with $n = 5$. The solver did not finished with $n = 6$.

For our experiments we use mainly personal computers. For parameters $n = 6, \ell = k = 4$ we use the server kamenozrout at the Department of Applied Mathematics, Charles University. The solver we use is z3 [33]. Due to the size of used formulae and the fact that the solver has to deal with non-linear rational arithmetics we are not able to produce any non-trivial results.

I hope that thanks to the development in computer science and hardware it would be possible to get either some larger counterexamples or computer evidence for $\ell = 4$ and $k = 6$ someday.

Conclusion

In this chapter we briefly summarize new results that appear in this thesis, and mention some possible directions for further research.

We make two observations about properties of point visibility graphs in Chapter 2. First we prove that point visibility graphs are pancyclic in Observation 13. We also discuss the infinite case. Then we give a characterization of edge chromatic number of point visibility graphs in Corollary 27. Techniques used in these proofs inspired the definition of ordered set representable graphs which gave an alternative proof for the Big Line Big Clique conjecture with $\ell = k = 4$ in Theorem 42. In the last chapter we deduce that the Big Line Big Clique conjecture is decidable by a computer for given parameters n, ℓ, k.

It is still an open question if there are asymptotically n^2 lines with $|P \cap l| = \ell$ where $|P| = n$ and no $\ell + 1$ lines in P are collinear.

It might be interesting to characterize which properties of visibility graphs can be proved using just minimizing distance and line sweeping. We discuss some such results in Section 2.1. Many results concerning visibility graphs use these properties implicitly.

The asymptotic behavior of the size of blocking set is still open. We reminded a linear lower bound and a superlinear upper bound in Chapter 4.

We know that the problem of recognition of visibility graphs lies in PSPACE. I do not know if the recognition problem is in $\exists\mathbb{R}$-complete or in another class.

The Big Line Big Clique conjecture still remains open. We provided an alternative proof for the case $k = \ell = 4$. On the other hand we also provided Theorem 43 which proves that ordered set representable graphs cannot help with open cases. Nevertheless it would be nice to decide remaining cases.

With the fast development of both hardware and computer science it would be possible to try to run provided computer experiments in the horizon of several years. Especially the experiment provided in Section 5.4 could bring interesting results as SMT solvers are still young compared to SAT solvers.

Bibliography

[1] Abel, Zachary, Ballinger, Brad, Bose, Prosenjit, Collette, Sébastien, Dujmović, Vida, Hurtado, Ferran, Kominers, Scott D., Langerman, Stefan, Pór, Attila, Wood, David R. *Every large point set contains many collinear points or an empty pentagon.* In: Graphs and Combinatorics. 2011, vol. 27, no. 1, pp. 47–60. ISSN 0911-0119. `arXiv:0904.0262 [math.CO]`.

[2] Addario-Berry, Luigi, Fernandes, Cristina, Kohayakawa, Yoshiharu, de Pina, Jos Coelho, Wakabayashi, Yoshiko. *On a geometric Ramsey-style problem* [online]. 2007 [cit. 2014-04-19].

`http://crm.umontreal.ca/cal/en/mois200708.html`

[3] Ajtaj, Miklós, Chvátal, Václav, Newborn, Monroe M., Szemerédi, Endre. *Crossing-free subgraphs.* In: Annals of Discrete Mathematics. 1982, vol. 12, pp. 9–12. ISSN 01675060.

[4] Audemard, Gilles, Simon, Laurent. *GLUCOSE 2.1: Aggressive – but reactive – clause database management, dynamic restarts.* Pragmatics of SAT (Workshop of SAT'12). 2012.

[5] Barát, János, Dujmović, Vida, Joret, Gwenaël, Payne, Michael S., Scharf, Ludmila, Schymura, Daria, Valtr, Pavel, Wood, David R. *Empty pentagons in point sets with collinearities.* `arxiv:1207.3633 [math.CO]`, 2012.

[6] Beck, József. *On the lattice property of the plane and some problems of the plane and some problems of Dirac, Motzkin, and Erdos in combinatorial geometry.* In: Combinatorica. 1983, vol. 3, no. 3, pp. 281–297. ISSN 0209-9683.

[7] de Berg, Mark, van Kreveld, Marc, Overmars, Mark, Schwarzkopf, Otfried. *Chapter 15: Visibility graphs, computational geometry.* 2nd ed., Springer-Verlag, 2000, pp. 307–317, ISBN 3-540-65620-0.

[8] Biere, Armin. *PicoSAT essentials.* In: Journal on Satisfiability, Boolean Modeling and Computation (JSAT), 2008 vol. 4, pp. 75–97, Delft University. ISSN 1574-0617.

[9] Bjørner, Nikolaj, de Moura, Leonardo. *Satisfiability modulo theories: introduction and applications.* In: Commun. ACM. 2011, vol. 54, no. 9, pp. 69–77. ISSN 0001-0782.

[10] Burr, Stefan A., Grünbaum, Branko, Sloane, Neil J. A. *The Orchard problem.* In: Geometriae Dedicata. 1974, vol. 2, no. 4, pp. 397–424. ISSN 0046-5755.

[11] Canny, John. *Some algebraic and geometric computations in PSPACE.* Proceedings of the 20th Annual ACM Symposium on Theory of Computing. 1988, pp. 460–467.

[12] Chazelle, Bernard, Guibas, Leo J., Lee, Der-Tsai. *The power of geometric duality.* In: BIT. 1985, vol. 25, pp. 76–90. ISSN 0006-3835.

[13] Cok, David R. *The SMT-LIBv2 language and tools: a tutorial* [online]. 2013 [cit. 2014-04-20].

http://www.grammatech.com/resource/smt/SMTLIBTutorial.pdf

[14] *DIMACS format* [online]. 1993 [cit. 2014-04-20].

http://www.satcompetition.org/2009/format-benchmarks2009.html

[15] Develin, Mike, Hartke, Stephen, Moulton, David P. *A general notion of visibility graphs.* In: Discrete Comput. Geom. 2003, vol. 29, no. 4, pp. 511–524. ISSN 0179-5376.

[16] Dumitrescu, Adrian, Pach, János, Tóth, Géza. *A note on blocking visibility between points.* In: Geombinatorics. 2009, vol. 19, no. 1, pp. 67–73. ISSN 1065-7371. www.cs.uwm.edu/faculty/ad/blocking.pdf

[17] Dvir, Zeev. *Incidence theorems and their applications.* arXiv:1208.5073 [math.CO].

[18] Eppstein, David. *Dilation-free planar graphs* [online]. 1997 [cit. 2014-04-19]. http://www.ics.uci.edu/~eppstein/junkyard/dilation-free/

[19] Erdős, Paul. *Problems and results in combinatorial geometry.* In: Proceedings of the New York Academy of Sciences. 1985, vol. 440, pp. 1–11. ISSN 1749-6632.

[20] Erdős, Paul. *Research problems.* In: Periodica Mathematica Hungarica. 1984, vol. 15, pp. 101–103. ISSN 0031-5303.

[21] Erdős, Paul. *Some old and new problems in combinatorial geometry.* In: SIAM Applications of discrete mathematics. 1988, pp. 32–37.

[22] Erdős, Paul, Fishburn, Peter, Füredi, Zoltán: *Midpoints of diagonals of convex n-gons.* In: SIAM J. Discrete Math. 1991, vol. 4, no. 3, pp. 329–341. ISSN 0895-4801.

[23] Erdős, Paul, Füredi, Zoltán, Pach, János, Ruzsa, Imre Z. *The grid revisited.* In: Discrete Math. 1993, vol. 111, no. 1–3, pp. 189–196. ISSN 0012-365X.

[24] Erdős, Paul, Guy, Richard K. *Crossing number problems.* In: American Mathematical Monthly. 1973, vol. 80, pp. 52–58. ISSN 0002-9890.

[25] Finsler, Paul. *Über die Primzahlen zwischen n und $2n$, in Festschrift zum 60. Geburtstag von Prof. Dr. Andreas Speiser.* Zürich: Füssli, 1945, pp. 118–122.

[26] Gerken, Tobias. *Empty convex hexagons in planar point sets.* In: Discrete Comput. Geom. 2008, vol. 39, pp. 239–272. ISSN 0179-5376.

[27] Ghosh, Subir Kumar, Roy, Bodhayan. *Some results on point visibility graphs.* `arXiv:1209.2308 [cs.CG]`.

[28] Graham, Ronald L., Grötschel, Martin, Lovász, László. *Handbook of combinatorics.* Elsevier, 1995. ISBN 0444823468, 9780444823465.

[29] Green, Ben, Tao, Terence. *On sets defining few ordinary lines.* In: Discrete Comput. Geom. 2013, vol. 50, pp. 409–468. DOI 10.1007/s00454-013-9518-9.

[30] Guy, Richard K. *Crossing numbers of graphs.* Graph Theory and its Applications, Lecture Notes in Mathematics Vol. 303, Springer-Verlag, pp. 111–124.

[31] Harborth, Heiko. *Konvexe Fünfecke in ebenen Punktmengen.* In: Elem. Math. 1978, vol. 33, pp. 116–118. ISSN 0002-9890.

[32] Horton, Joseph D., *Sets with no empty convex 7-gons.* In: Canad. Math. Bull. 1983, vol. 26, pp. 482–484. ISSN 0008-4395.

[33] Jovanović, Dejan, de Moura, Leonardo. *Solving non-linear arithmetic.* Technical Report MSR-TR-2012-20, Microsoft Research, 2012.

[34] Richter-Gebert, Jürgen. *Mnëv's universality theorem revisited.* In: Sém. Lothar. Combin. 1995, vol. 34. ISSN 1286-4889.

[35] Kára, Jan, Pór, Attila, Wood, David R. *On the chromatic number of the visibility graph of a set of points in the plane.* `arXiv:math/0510467 [math.CO]`.

[36] Kostochka, Alexandr, Kratochvíl, Jan. *Covering and coloring polygon-circle graphs.* In: Discrete Math. 1997, 163, pp. 299–305. ISSN 0012-365X.

[37] Leighton, Frank T. *Complexity issues in VLSI.* Cambridge, Ma.: M.I.T. Press, 1983. ISBN 9780262621786.

[38] Matoušek, Jiří. *Blocking visibility for points in general position.* In: Discrete Comput. Geom. 2009, vol. 42, pp. 219–223. DOI 10.1007/s00454-009-9185-z.

[39] Melchior, Eberhard. *Über Vielseite der Projektive Ebene.* In: Deutsche Mathematik. 1941, vol. 5, pp. 461–475.

[40] Mnev, Nikolai E. *The universality theorems on the classification problem of configuration varieties and convex polytopes varieties.* In Topology and geometry—Rohlin Seminar, Berlin: Springer, 1988, vol. 1346 of Lecture Notes in Math., pages 527–543. ISBN 978-3-540-50237-1.

[41] Mycielski, Jan. *Sur le coloriage des graphs.* In: Colloq. Math. 1955, vol. 3, pp. 161–162. ISSN 1730-6302.

[42] Nicolás, Carlos M. *The empty hexagon theorem.* In: Discrete & Computational Geometry. 2007, vol. 38, pp. 389–397. ISSN 0179-5376.

[43] Pach, János. *Midpoints of segments induced by a point set.* In: Geombinatorics. 2003, vol. 13, no. 2, pp. 98–105. ISSN 1065-7371.

[44] Payne, Michael S., Wood, David R. *On the general position subset selection problem.* `arXiv:1208.5289v2 [math.CO]`.

[45] Payne, Michael S., Pór, Attila, Valtr, Pavel, Wood, David R. *On the connectivity of visibility graphs.* In: Discrete & Computational Geometry. 2012, vol. 48, no. 3, pp. 669–681. ISSN 0179-5376.

[46] Pfender, Florian. *On visibility graphs of point sets in the plane.* In: Twentieth Anniversary Volume: Discrete & Computational Geometry. 2008, vol. 39, pp. 455–459. ISBN 978-0-387-87363-3.

[47] Poonen, Bjorn, Rubinstein, Michael. *The number of intersection points made by the diagonals of a regular polygon.* In: SIAM J. Discrete Math. 1998, vol. 11, no. 1, pp. 135–156. ISSN 0895-4801.

[48] Pór, Attila, Wood, David R. *The Big-Line-Big-Clique conjecture is false for infinite point sets.* `arXiv:1008.2988 [math.CO]`.

[49] Ramsey, Frank P. *On a problem of formal logic.* In: Proceedings of the London Mathematical Society. 1930, vol. 30, pp. 264–286, DOI:10.1112/plms/s2-30.1.264.

[50] Schaefer, Marcus. *Complexity of some geometric and topological problems.* Springer 2010. ISBN 978-3-642-11804-3.

[51] Schaefer, Marcus, Štefankovič, Daniel. *Fixed points, nash equilibria, and the existential theory of the reals.* Unpublished manuscript.

[52] Shor, Peter W. *Stretchability of pseudolines is NP-hard.* In: Applied geometry and discrete mathematics. 1991, vol. 4 of DIMACS Ser. Discrete Math. Theoret. Comput. Sci., pp. 531–554. Amer. Math. Soc., Providence, RI. ISSN 1052-1798.

[53] Solymosi, Josef, Stojaković, Miloš. *Many collinear k-tuples with no $k+1$ collinear points.* EuroCG 2012, Assisi, Italy, March 19–21, 2012. `arXiv:1107.0327 [math.CO]`.

[54] Spencer, Joel. *Turán's theorem for k-graphs.* In: Discrete Math. 1972, vol. 2, pp. 183–186. DOI: 10.1016/0012-365X(72)90084-2.

[55] Sudakov, Benny. *Large K_r-free subgraphs in K_s-free graphs and some other Ramsey-type problems.* In: Random Structures Algorithms. 2005, vol. 26, no. 3, pp. 253–265. DOI: 10.1002/rsa.20035.

[56] Sylvester, James J. *Mathematical question 2571.* Educational Times, February 1868.

[57] Székely, László A. *Crossing numbers and hard Erdős problems in discrete geometry.* In: Comb. Probab. Comput. September 1997, vol. 6, no. 3, pp. 353–358. ISSN 0963-5483.

[58] Turán, Paul. *On an extremal problem in graph theory.* In: Matematikai és Fizikai Lapok (in Hungarian). 1941, vol. 48, pp. 436–452.

[59] Vizing, Vadim G. *On an estimate of the chromatic class of a p-graph* (In Russian). In: Diskret. Analiz. 1964, vol. 3, pp. 25–30.

[60] Walfisz, Arnold. *Gitterpunkte in mehrdimensionalen Kugeln*. Monografie Matematyczne. Vol. 33. Warsaw: Państwowe Wydawnictwo Naukowe, 1957.

List of Tables

List of Abbreviations

1. OSR graph G is a graph for which there is an Ordered Set Representation $(|V(G)|, L)$ such that the given graph G is isomorphic to the visibility graph $\nu((|V(G)|, L))$ as defined in Definition 3.4.1 and Definition 3.4.1.

2. SMT stands for Satisfiability Modulo Theories, the branch of satisfiability theory, see Chapter 5.

3. SMT2 is a popular input format of SMT formulae.

Attachments

1. SAT experiments:

   (a) `sat.cpp` is a program that generates input in the DIMACS format for a SAT solver.

   (b) `paint.c` is a program that gets numbers of edges, and paints the OSR graph in a MetaPost file.

   (c) Files `91.txt, 92.txt, 93.txt, 94.txt, 95.txt, 96.txt` containing lexicographically numbered triples of vertices. Positive ones are exactly those which are in the L of the Ordered Set Representation. These are the only ones with nine vertices without K_4 in the visibility graph or $\ell = 4$. There are no such OSR graphs with ten vertices.

   (d) File `42.txt` contains the biggest OSR graph without $k = 5$ and $\ell = 4$ found. The format is the same as the format of previous files.

2. File `smt.cpp` is a program that generates an input for a solver in the SMT2 format.