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Munich, 26.08.2014

Report on the doctoral thesis of Mr. Jakub Tichý

Dear Dean Professor Jan Kratochvíl,

this is my report on the doctoral thesis of Mr. Jakub Tichý with the title *"Qualitative properties of solutions to equations of fluid mechanics"*.

In his thesis Mr. Tichý studies the global regularity of weak solutions of the generalized (Navier-)Stokes equations complemented with perfect slip boundary conditions. Different from the classical Navier-Stokes equations the extra stress S depends in a non-nonlinear way on the shear rate Du, which is the symmetric part of the velocity gradient. A very common model in this direction is the so called power law model, where $S = S(Du)$ behaves like $(\kappa + |Du|)^{p-2}Du$ with $1 < p < \infty$ and $\kappa \geq 0$. In his Mr. Tichý goes one step further and works in the more general setting of Orlicz functions, where $|S(Du)|$ is just a monotone function of $|Du|$. This generalization is more suitable to fit $S(Du)$ with the data from the experiments. Due to the monotonicity one can write $S(Du)$ as the variation of a scalar potential $\Phi(|Du|)$, where Φ is a suitable Orlicz function. The conditions on Φ in the thesis are quite standard and ensure the strict monotonicity of the induced operator $-\mathrm{div}(S(Du))$ in the Sobolev-Orlicz space $W^{1,\Phi}$. Only in a few places of the thesis it is necessary to assume that Φ'' is monotone, which corresponds to a pure shear thinning or pure shear thickening situation. Mr. Tichý explains and develops the model of the generalized (Navier-)Stokes system in Chapter 1.

The thesis contains three main results each with a different focus. The first main result in presented in Chapter 2. There he studies the global regularity of the stationary generalized Stokes system with perfect slip boundary conditions, i.e.

$$\begin{aligned} -\mathrm{div} S(Du) + \nabla\pi &= f && \text{in } \Omega, \\ \mathrm{div} u &= 0 && \text{in } \Omega, \\ u \cdot \nu = 0, \quad [S(Du)\nu] \cdot \tau &= 0 && \text{on } \partial\Omega, \end{aligned}$$

see (1.23)–(1.25) in the thesis. It has been known for a long time (by means of the difference quotient technique) that the interior regularity can be best described by $V(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$, where $|V(Du)|^2 = S(Du) : Du$ and $V(Du) = |V(Du)|\frac{Du}{|Du|}$. However, the regularity up to the boundary is still subject to current research. It is still unknown under which boundary conditions it is possible to obtain $V(Du) \in W^{1,2}(\Omega)$ (globally) even for smooth boundary and data. It is for example yet unkown, if this regularity is available for homogeneous Dirichlet boundary condition. Let us recall that in his thesis Mr. Tichý studies the case of perfect slip boundary conditions. In a certain sense this situation is nicer than the one of Dirichlet boundary conditions. Indeed, in the case of a flat boundary it is possible to use a reflection method to reduce the question of boundary regularity to the one of interior regularity. This approach has been used by Ebmeyer in [29] to obtain $V(Du) \in W^{1,2}$ for perfect slip at least in the setting of a power law model. Mr. Tichý extends this results to the setting of Orlicz growth, which is not a straight forward task. In Chapter 2 he does not use the reflection method, but develops a different technique. His steps are the following: First, derive information on the tangential derivatives by a subtle modification of the difference quotient technique in tangential direction. Second, use the pointwise information of the equation together with the boundary conditions to gain estimates for the normal derivatives. Many difficiculties aries in these steps due to the general growth in combination with the difference quotient technique. Moreover, many standard tools of the L^p-setting like Korn's inequality and the Bogovskiĭ operator require extra attention in this setting (see also the Appendix). The approach by Mr. Tichý looks at the first sight more complicated than the reflection method. However, his method seems to have a better potential in the context of other boundary conditions and is therefore of independent interest. The conditions on the boundary namely $\partial\Omega \in C^3$ in this chapter are comparable to the ones of Ebmeyer, namely $\partial\Omega \in W^{3,\infty}$.

The second main result in contained in Chapter 3. It concerns the higher integrability of the gradients in direct relation to the integrability of the force term. Such estimates are known from the context of the p-Laplacian and called *non-linear Calderón-Zymgund estimates* (which is an area of very active reseach). The goal is to provide optimal integrability results, in the sense that the force term F (given as $f = -\mathrm{div} F$) and the extra stress and the pressure share exactly the same higher integrability. In the setting of the p-Laplacian such estimates are indeed possible. However, the reduced interior regularity of the generalized Stokes does now allow to obtain this result in full generality but the power of the higher integrability result is bounded by the interior regularity of the system with (at least locally) $f = 0$. The corresponding interior higher integrability results have been obtained in a joint paper of the supervisor and myself and were based on the interior $V(Du) \in W^{1,2}_{\mathrm{loc}}$ estimates. The new boundary estimates of Chapter 2 are the motivation to extend the non-linear Calderón-Zygmund estimates to the global situation. Since Mr. Tichý requires a few additional oscillation estimate on $V(Du)$ locally at the boundary, he derives these estimates by means of the already mentioned reflection method. The final main result on higher integrablity are in fact optimal in the sense above. (The estimate on the pressure is not included in the statement, but it is an immediate consequence of the obtained main result and the negative norm theorem.) These nice results are very useful for example in order to derive higher integrability for the system with convection. Let me also mention that as a by product he shows $V(Du) \in W^{1,2+\varepsilon}$ for some $\varepsilon > 0$ (if $\varPhi''$ is monotone), see (3.11). Note that in the two dimensional setting this implies $V(Du) \in C^{0,\beta}$ and hence $Du \in C^{0,\alpha}$ for some $\alpha, \beta > 0$. It seems to me that his results could be used to extend the interior BMO-estimates of the two-dimensional setting in [24] to global BMO-estimates for perfect slip. Mr. Tichý seems aware of this, since he makes a corresponding remark in his conclusion of the thesis.

The third main result is contained in Chapter 5. In this part Mr. Tichý studies the parabolic version of the system in two space dimensions including convection. He is able to prove global $C^{0,\alpha}$-regularity of Du and π with respect to space and time (under suitable conditions on the data). The result are restricted to the super-quadratic case of a power potential. In the stationary situation this Hölder continuity follows from $V(Du) \in W^{1,2+\varepsilon}$ as mentioned above. However, this approach cannot be used in the parabolic setting, since $V(Du) \in W^{1,2+\varepsilon}(I \times \Omega)$ would NOT imply Hölder continuity (the space+time dimension is three and therefore to big). For this reason Mr. Tichý uses a different technique, which is based on a comparison approach to the system with quadratic growth. By cutting the extra stress growth at some parameter $1/\varepsilon$ he obtains a system with quadratic growth, where the ellipticity constants depend on ε. A using the L^q theory on linear parabolic system with perfect slip, he is able to derive certain estimates that allow to control $\|V(Du)\|_\infty$ by a $c\,\|V(Du)\|_\infty^\theta$ for some $\theta \in (0,1)$ independently of ε. This is the crucial step to get L^∞-estimates for Du and then Hölder estimates for Du and π. An important step in this proof is the crucial L^q estimates for the linear parabolic system with perfect slip. These advanced and new estimates are derived in Chapter 4 and will be very useful to other researchers.

Mr. Tichý has written an very strong thesis containing many interesting results. The structure and form of the thesis is excellent and well organized. Indeed, he summarizes his results in the introduction and at the beginning of each chapter in a very clever way. This makes the results very easy to access for the reader. In his thesis he shows great ability to master many different techniques, since his results are based on many different methods. His results are of strong interest to the community and have lead as I understand already to three good publications (two accepted and one submitted). Moreover, in his conclusions he states already further ideas for interesting follow up projects. Overall, this thesis clearly proves that the author is able to create new interesting scientific work. I strongly recommend to accept the doctoral thesis.

Sincerely,

Prof. Dr. Lars Diening