Report on the thesis

Qualitative properties of solutions to equations of fluid dynamics

written by

Mgr. Jakub Tichý

The thesis deals with regularity properties of solutions to the equations describing a motion of a class of incompressible homogeneous non-Newtonian fluids in bounded domains subjected to the perfect slip boundary conditions. The author focuses on two main difficulties, namely, to include very general constitutive assumptions on the Cauchy stress tensor, that allow one to consider also the nonstandard growth conditions, and to prove all results and estimates up to the boundary. It should be mentioned here that while the standard growth conditions and the regularity results in the interior of the domain are well known, the extension of such results up to the boundary and the ability to handle the nontrivial constitutive assumptions form a very nontrivial task of the mathematical analysis of nonlinear partial differential equations.

The thesis is divided into several chapters. In Chapter 1, the model of non-Newtonian fluid is introduced together with the reasonable constitutive assumptions on the Cauchy stress tensor. Since the growth conditions in the Cauchy stress are described by a general N function Φ, the basic notation and the properties of the Orlicz spaces are recalled there.

Chapter 2 deals with the first main result of the thesis. The estimates of the second derivatives of the velocity field and the estimates of the gradient of the pressure as well are established for steady problem in suitable function spaces provided that the Ω is a $\mathcal{C}^3$ domain that is not axially symmetric, Φ and the conjugate Φ^* satisfy Δ_2 condition and Φ'' is almost decreasing/increasing. The method consists of an introduction of the quadratic approximation for which the theory is proved in the full rigour by the method of the generalized difference quotients in the tangential directions and the proper use of the equation for estimating the normal derivative. Then the a priori uniform estimates are shown which allows one to take the limit from the quadratic approximation to the general case. This chapter surely deserves the attention because of the very innovative procedure in the choice of proper test functions that keep all necessary compatibility conditions due to the boundary conditions and the divergence free constraint on one hand and on the other do not disturb essentially the key estimates.

In Chapter 3 the better integrability of the symmetric gradient is proved for steady problem. The method is based on the comparison (Campanato technique) method of the original problem and the problem with zero right hand side for which the theory is known. In this section, although I believe the results from Chapter 2 can be used, all the estimates are proved once again by flattering the boundary. Contrary to Chapter 2, the estimates near the flat boundary can be proved relatively easily but their connection to the non-flat case is indeed very technical and difficult.

Chapter 4 mostly recall the theory for the linear evolutionary Stokes-like problems and the interpolation technique and finally in Chapter 5 the whole circle is closed to prove the existence of a $\mathcal{C}^{1,\alpha}$ solution to the general evolutionary problem in dimension two by the comparison with the linear problem.

The thesis summarizes and extends the already published results of the author (and co-authors) and extends the mathematical theory of the considered model in a significant way. It is very well written with minimum number of typos and missprints. Moreover, although the boundary regularity is one of the most technical discipline in the theory of partial differential equations, the thesis is, according to my opinion, written in very elegant and "understandable" way. On the other hand in some cases some essential steps are done very quickly or without any comments, which might be for the reader not being familiar with the regularity techniques very difficult. Also in the Appendix I believe the author should be more careful and also at some places more precise.

It is evident from the thesis that the author has deep insight in the regularity theory of the partial differential equations and is able to introduce very innovative procedures or extensions of the known methods. Therefore, I **strongly** recommend the thesis to be accepted as the Ph.D. thesis and I believe that it is just the beginning of the successful scientific career of the author.

Questions that should be answered during the defense:

- In Chapter 2 in the final estimate. It is not clear to me how the cut-off function ξ appeared in the term on the right hand side of (2.20) if it is not in (2.19). Moreover, since we do not know a priori that the right hand side of (2.20) is finite (we do not control normal derivative yet), it is not clear to me how the last term can be absorbed by the left hand side rigorously.

- The assumptions on the regularity of the boundary are not unified and also very strong, namely $\mathcal{C}^3$ or $\mathcal{C}^{2,1}$. First, I believe that in all parts $\mathcal{C}^{2,1}$ regularity is enough. Second I would really appreciate if the author would explain where exactly such regularity is needed because in the standard elliptic or parabolic case the $\mathcal{C}^{1,1}$ regularity would be enough.

- In the whole thesis the author considers only the non-axisymmetric domains. Is such an assumption really needed? Or can we overcome it by introducing a different concept of a solution? Moreover, in the last part (the evolutionary case), the assumption on non-symetricity of the domain seems to be irrelevant due to the presence of the time derivative.

- Would it be possible to avoid the presence of the "non-natural" term involving ∇f in the statement of Theorem 2.1.2?

Weak points & further comments:

- The thesis is supposed to be in the branch of mathematical modelling. Therefore I would expect that the introductory part will contain more details and will be written more carefully. For example in (1.2) the unknown u appear without saying what it is. In addition, the use of the transport theorem can be done on the level of very weak assumption, much weaker than smoothness or continuity. Since the thesis deals with general constitutive laws, I would recall more theory how such constitutive laws can look like and also refer to some models that are used in praxis. Finally, I would at least recall some experiments, where the Navier slip or perfect slip are observed in the reality.

- Δ_2 condition is usually formulated as $\Phi(2s) \leq C_1\Phi(s) + C_2$ for $s \geq 1$ contrary to Definition 1.4.2. Also the complementary function Φ^* is usually defined via the Legendre transformation $\Phi^*(t) := \sup_s(st - \Phi(s))$.

- In (1.26) there appear the notation $\partial_{ij} := \partial_{A_{ij}}$ without any explanation and even worse it is used in the whole thesis.

- Sometimes writing sums and indexes would be very useful, e.g., in (2.3) in the second term, it is not clear a priori what is the object in $[\ldots]$.
- Sometimes when referring to some result I would appreciate also the page or the number of the theorem in the source, e.g., to find the proof of Lemma 2.2.2 in [35] without any a priori knowledge of the book is a real nightmare. The same holds true also for the Korn inequality in [43] and [44].
- On page 24, after (2.18) it is claimed that $\mathbb{A}$ is regular thanks to Corollary 1.4.5. But it does not say anything concerning the regularity of $\mathbb{A}$ and this point surely deserves some attention.
- According to the notation of $\Gamma_{\ldots}$ the only solution to (3.2) is zero. I think that the boundary condition are assumed only on the part of the boundary, where $x_n = 0$.
- I really do not understand why "musical isomorphisms" on page 46 are introduced and the same for the wedge-product. It is just used to find the proper choice of φ in (3.16). Moreover, the notion for $\#$ is little bit inconsistent because it creates a differential operator (as φ in (3.16)) and not a test function!
- I would suggest to move Chapter 4 into the appendix. On one hand the reader familiar with the semigroup theory and the interpolation techniques does not find there any novelty. On the other hand the reader not familiar with such tools will be completely lost due to the missing details. In addition this is the only place in the thesis where the Besov and the Bessel spaces are used without any explanation why, if the whole theory is then applied to the standard Sobolev spaces.
- The definition (4.5) is little bit confusing. I would appreciate if in the first case the function space $F^s_{q,B}$ would contain only the function having zero component of the trace equal to zero. It is then corrected in (4.6) but still the first and the second line in (4.5) seems on the first sight incompatible.
- ν defined in (A.2) is not the unit vector and therefore it should not be called the outer normal vector.

- In the definition of the mapping T_α I would appreciate also the symbol h since it differs with h.

- I believe that (A.5) holds also for $p = 1$. Moreover, $c(a)$ should be rather $c(\Omega)$.

- The Young inequality in the form (A.8) for $\delta \in (0,1)$ holds only if Φ^* satisfies the Δ_2 condition.

- In Lemmata A.2.4–A.2.6 I should explicitly recall Assumption 3.1.1.

- The statement of Lemma A.3.1 holds true only if g has zero normal component on the boundary. Moreover, I believe that instead of the statement the author wanted to say $\|u\|_{1,r} \leq C\|g\|_{1,r}$.

- In the proof of Proposition A.4.3, the opposite inequality to $\Phi''(s)s \leq C\Phi(s)$ must be used! Moreover, such an assumptions then must be necessarily in Lemma A.4.2 since Proposition A.4.3 is used in the proof. However, I believe that Lemma A.4.2 holds true even without such an assumption just by using the standard contradiction argument, the compact embedding and (A.22). In addition, I would unify the assumptions on the domain in Appendix to $\mathcal{C}^{0,1}$ instead of sometimes used $\mathcal{C}^1$.

- In Lemma A.5.3 the condition $p > 1$ must be assumed.

- The last chapter has in the title "full regularity" but only the Hölder continuity of the first derivatives is proved there.

In Bonn, August 22, 2014 RNDr. Miroslav Bulíček, Ph.D.