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DOCTORAL THESIS

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Qualitative properties of solutions to equations of fluid mechanics

Department of Mathematical Analysis

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Finally I praise the Lord for I can ask myself the question of Paul the Apostle: “What do you have that you did not receive?” (1 Corinthians 4:7)

Ad maiorem Dei gloriam.

Declaration

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague, 22. 6. 2014

Mgr. Jakub Tichý

Název práce: Kvalitativní vlastnosti řešení rovnic mechaniky tekutin

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Katedra: Katedra matematické analýzy

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Abstrakt: Tato práce se zabývá hraniční regularitou slabých řešení nelineárních parciálních diferenciálních rovnic, které popisují nestlačitelné proudění jisté třídy zobecněných Newtonovských tekutin v omezených oblastech. Pohybová rovnice a rovnice kontinuity jsou doplněny hraničními podmínkami dokonalého skluzu. Pro stacionární zobecněný Stokesův systém v $\mathbb{R}^n$ s růstovými podmínkami popsanými pomocí $N-$funkce Φ je ukázána existence druhých derivací rychlosti a jejich regularita až do hranice. Pro stejný systém rovnic je dokázána integrovatelnost gtadientů rychlosti. L^q odhady jsou rovněž získané pro klasický evoluční Stokesův systém pomocí interpolačně-extrapolačních škál. Hölderovská spojitost gradientů rychlosti a tlaku je ukázána pro evoluční zobecněné Navierovy-Stokesovy rovnice v $\mathbb{R}^2$.

Klíčová slova: Zobecněné Stokesovy a Navier - Stokesovy rovnice, nestlačitelné tekutiny, hraniční podmínky dokonalého skluzu, regularita až do hranice

Title: Qualitative properties of solutions to equations of fluid mechanics

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Abstract: This thesis is devoted to the boundary regularity of weak solutions to the system of nonlinear partial differential equations describing incompressible flows of a certain class of generalized Newtonian fluids in bounded domains. Equations of motion and continuity equation are complemented with perfect slip boundary conditions. For stationary generalized Stokes system in $\mathbb{R}^n$ with growth condition described by N-function Φ the existence of the second derivatives of velocity and their regularity up to the boundary are shown. For the same system of equations integrability of velocity gradients is proven. L^q estimates are obtained also for classical evolutionary Stokes system via interpolation-extrapolation scales. Hölder continuity of velocity gradients and pressure is shown for evolutionary generalized Navier-Stokes equations in $\mathbb{R}^2$.

Keywords: Generalized Stokes and Navier - Stokes equations, incompressible fluids, perfect slip boundary conditions, regularity up to the boundary

Contents

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Generalized (Navier-) Stokes equations in Orlicz spaces

1.1 Derivation of the model

This thesis is concerned with the qualitative properties of weak solutions to the system of nonlinear partial differential equations describing incompressible flow of a certain class of generalized Newtonian fluids.

In this introductory chapter we mention shortly physical background of the equations. We precisely formulate the problem after a few historical remarks including commentary about constitutive relations and optimal choice of boundary conditions.

We start with the derivation of the differential form of the continuity equation and equation of motion. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $I = (0, T)$ a finite time interval. We fix a control volume $\mathcal{V}(0) \subset \Omega$ and define $\mathcal{V}(t)$ as a volume occupied by this fluid at the time $t \in I$. We assume $\mathcal{V}(t) \subset \overline{\mathcal{V}(t)} \subset \Omega$.

The continuity equation represents a mass balance which says that the mass m of any control volume of fluid $\mathcal{V}(t)$ is independent of the time t. Let ϱ denote the density and u stands for the velocity of the fluid. In this chapter we suppose that all quantities are sufficiently smooth such that all equations are well defined (for instance $\varrho, u \in \mathcal{C}^1(I \times \Omega)$).

The mass conservation can be formulated as follows

$$\frac{d}{dt} m(\mathcal{V}(t)) = \frac{d}{dt} \int_{\mathcal{V}(t)} \varrho(x, t) \, \mathrm{d}x = 0. \tag{1.1}$$

We would like to derive the differential form of (1.1). Since $\mathcal{V}(t)$ depends on t, we can't use the theorem about the differentiation of the integral with respect to the parameter. Instead so called transport theorem can be used. Precise formulation can be found for example in [40, Chapter III.10]. Thus,

$$\partial_t \varrho + \operatorname{div}(\varrho u) = 0. \tag{1.2}$$

If we consider incompressible flows of a homogeneous fluid, the density ϱ is constant and (1.2) reduces to

$$\operatorname{div} u = 0. \tag{1.3}$$

The equations of motion are derived from the balance of linear momentum which asserts that the total force on the control volume $\mathcal{V}(t)$ is equal to the rate which the linear momentum of the fluid in $\mathcal{V}(t)$ is increasing plus the rate of outflow of momentum across $\partial\mathcal{V}(t)$, c.f. [40, Chapter V.15]. In symbols,

$$\frac{d}{dt}\int_{\mathcal{V}(t)}(\varrho v)(x,t)\,\mathrm{d}x = \mathcal{F}(\mathcal{V}(t),t). \tag{1.4}$$

In order to derive the differential form of (1.4) we split the force $\mathcal{F}$ into a body force and a surface force

$$\mathcal{F}(\mathcal{V}(t),t) = \int_{\mathcal{V}(t)}\varrho f(x,t)\,\mathrm{d}x + \int_{\partial\mathcal{V}(t)} s(x,t,\nu(x))\,\mathrm{d}\sigma, \tag{1.5}$$

where f is the density of a body force, ν a unit outward normal and s is the stress vector expressing action of the fluid outside of $\mathcal{V}(t)$ at the time t on the control volume $\mathcal{V}(t)$. Cauchy theorem, one of the central results of the continuum mechanics states that $s(\nu)$ is linear in ν:

Theorem 1.1.1 (Cauchy) *If $s(x,\nu)$ is continuous in x, then there is a spatial tensor field $\mathcal{T}$ (called the Cauchy stress) such that $s(x,\nu) = T(x)\nu(x)$ for all $x \in \mathcal{V}$ and arbitrary ν.*

Sometimes the Cauchy theorem contains the claim that the Cauchy stress $\mathcal{T}$ is symmetric if and only if the balance of angular momentum is satisfied. The proof can be found in [40, Chapter V.14].

Using the transport theorem, Cauchy and Green theorems we obtain from (1.4) and (1.5) the equation of motion in the following form

$$\partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) = \varrho f + \operatorname{div}\mathcal{T}. \tag{1.6}$$

The equation (1.6) can be further specified. The relations between the Cauchy stress and other quantities describing the flows are characterized by so called rheological equations. We can consider the rheological equation of the form

$$\mathcal{T} = -p\mathbb{I} + \mathcal{T}',$$

where p is the pressure, $\mathbb{I}$ a unit tensor and $\mathcal{T}'$ represents friction forces which are consequence of viscosity. If we consider homogeneous incompressible fluids, ϱ is constant. Defining $\pi = p/\varrho$ and $\mathcal{S} = \frac{1}{\varrho}\mathcal{T}'$ we obtain

$$\partial_t u + \operatorname{div}(u \otimes u) = \varrho f - \nabla\pi + \operatorname{div}\mathcal{S}. \tag{1.7}$$

This formulation of equation of motion and its simplification is crucial for us in this thesis.

1.2 Constitutive relations

In the previous section we briefly mentioned the derivation of equations under interest. Now we would like to discuss how the stress tensor $\mathcal{S}$ can depend on other quantities. Generally we can expect the dependence between the stress tensor $\mathcal{S}$, the pressure π and the shear rate, which is represented by a symmetric part of the velocity gradient Du. These relations are called constitutive ones. We start with the linear dependence between $\mathcal{S}$ and Du and observe the historical evolution of the mathematical description of fluid mechanics with the emphasis on the constitutive relations. Information of this section were put together from [76, Section 1.2] and [15], where the interested reader can find more details.

I. Newton stated in [74]: "The resistance arising from the want of lubricity in parts of the fluid is, other things being equal, proportional to the velocity with which the parts of the fluid are separated from one another." It can be interpreted as to give rise to the linear relationship between the stress tensor $\mathcal{S}$ and the shear rate Du, in which the constant of the proportionality is the viscosity. In symbols,

$$\mathcal{S} = 2\mu_0 Du, \quad \mu_0 \in (0, \infty). \tag{1.8}$$

The mathematical description of the fluid motion came relatively late. In the year 1822 the French engineer C.M.L.H. Navier suggested a certain system of partial differential equations as a model describing flows of viscous incompressible fluids. However, his assumptions, under which he deduced the system from molecular physics, appeared to be unrealistic. Surprisingly, G. G. Stokes obtained in 1945 exactly the same system by more rigorous approach, i.e. the system (1.3) and (1.7) with the linear relation between $\mathcal{S}$ and Du (1.8).

Modern mathematical attempts to study this system go back to the twenties of the last century. Swedish mathematician and physicist C. W. Oseen [73] studied mostly the system with linearised convective term, but he was also the first one who proposed a weaker version of the formulation to the problem. French mathematician J. Leray followed Oseen's ideas and proved existence and uniqueness of a classical solution in the case when $\Omega = \mathbb{R}^2$ in [63]. However, he failed in the case when $\Omega = \mathbb{R}^3$ and therefore he proposed another approach, which is nowadays known as a weak formulation. J. Leray proved in [64] existence of such solutions for $\Omega = \mathbb{R}^3$. He was not able to decide, whether these solutions are unique and whether they are smooth if data are so.

After the second world war J. Leray didn't continue in work in mathematical fluid mechanics. A new generation represented by E. Hopf [45], O. A. Ladyzhenskaya [62] or J.-L. Lions [65] appeared. Previous results were extended to many other boundary value problems with similar results as for the Cauchy problem. In particular, in two space dimensions regularity and uniqueness was proven, in three space dimensions only the existence of weak solutions with partial results in the direction of regularity and uniqueness.

Up to this time we have mentioned only the linear relation between the stress tensor $\mathcal{S}$ and the shear rate Du. Fluids characterized by the linear dependence (1.8) are called Newtonian. In case of nonlinear relations we talk about non-Newtonian or generalized Newtonian fluids. O. A. Ladyzhenskaya was one of the first ones who suggested to study fluids described by the power-law relation instead of the linear one. On her lecture at International Mathematical Congress in 1966 she suggested among others to study the system (1.7) described by the power-law relation

$$\mathcal{S} = 2\mu_0(1 + |Du|^{p-2})Du, \quad \mu_0 \in (0,\infty),\ p \in [1,\infty) \tag{1.9}$$

with the growth $p = 4$. Later she extended this first results and presented in [59], [60], [61] and [62]. Similar system was considered by J.-L. Lions in [65]. Whereas O. A. Ladyzhenskaya derived non-linear dependence of $\mathcal{S}$ on Du with the help of kinetic theory, J.-L. Lions used p-Laplace operator. By combination of monotone operator theory together with compactness they showed existence of the weak solution of power-law model for certain values of p.

Lots of excellent mathematicians extended these results in many directions. Power-law model can be considered as simplest generalization of classical Newtonian fluid and various generalizations such as

$$\mathcal{S} = 2\mu_0(\kappa + |Du|^2)^{\frac{p-2}{2}}, \quad \mu_0, \kappa \in (0,\infty),\ p \in \mathbb{R}, \tag{1.10}$$

were studied in recent years.

Many results about existence of the weak solutions of (1.7) and (1.3) with some variant of (1.10) and their qualitative properties have been proven. To mention only a few of them, we can refer for example to [8, 9, 10, 11, 12, 31, 32, 34, 48, 49, 50, 51, 52, 67, 68, 69, 89].

Power-law model can be generalized by using the framework of $N-$function Φ as follows:

$$\mathcal{S} = \frac{\Phi'(|Du|)}{|Du|}Du \tag{1.11}$$

for given $N-$function Φ. One of the advantage of this approach is that the constitutive relation (1.11) allows to describe fluids with non-polynomial growth such as

$$\mathcal{S} = \mu_0(1 + |Du|^2)^{\frac{p-2}{2}} \ln(1 + |Du|)Du, \quad \mu_0 \in (0,\infty). \tag{1.12}$$

Note that the choice $\Phi(s) = \frac{1}{p}s^p$ describes the power law model. When we are dealing with (1.9), sometimes the different approach for $p > 2$ and $p < 2$ is needed. The setting (1.11) enables to work in a unified way in some cases. Nevertheless, sometimes we still need to distinguish between the case when Φ'' is almost increasing and almost increasing, which corresponds to $p > 2$ and $p < 2$. For more details, see Chapter 2.

The relation (1.11) can be slightly modified in order to catch fluids in which the experimental data are reflected by a convex function Φ with different polynomial upper and lower growth.

All the cases and examples of the constitutive relations mentioned above can be covered by the model of fluid with shear-dependent viscosity:

$$\mathcal{S} = \mu(|Du|)Du, \quad \mu : \mathbb{R}^+ \to \mathbb{R}^+. \tag{1.13}$$

The dominant departure from the Newtonian behaviour, captured by the constitutive relation (1.13), are effects such as shear thickening and shear thinning. It seems to be useful to describe by this model for example the behaviour of very dilute polymeric liquids or low molecular weights biological liquids.

Although we don't study more general constitutive relation than (1.13) in this thesis, for a sake of completeness we point out that recently there was developed theory for implicitly constituted incompressible fluid with response described by the implicit relation

$$G(\mathcal{S}, Du, \pi) = 0. \tag{1.14}$$

For more details and some consequences that come from this general viewpoint we refer to the original work [77, 78, 79]. In comparison with traditional models, in which $\mathcal{S}$ is a function of Du, the implicit equation (1.14) is capable of capturing several non-Newtonian phenomena, except shear thinning and shear thickening mentioned above also pressure thickening or various activation and deactivation criteria. A simple example that falls to the class given by (1.14) is following

$$G(\mathcal{S}, Du) = \mu(|Du|)(\tau + (|\mathcal{S}| - \tau)^+)D - (|\mathcal{S}| - \tau)^+\mathcal{S}, \tag{1.15}$$

where $x^+ = \max\{0, x\}$. One can easily observe that (1.15) is equivalent to the traditional description of fluid of Bingham or Herschel-Bulkley type:

$$|\mathcal{S}| \leq \tau \Leftrightarrow Du = 0, \quad |\mathcal{S}| > \tau \Leftrightarrow \mathcal{S} = \frac{\tau Du}{|Du|} + \mu(|Du|)Du.$$

For the existence theory of implicitly constituted fluids and some further information about the model see [14, 15, 16].

1.3 Boundary conditions

Although the system (1.3) with (1.7) together with the boundedness of the domain Ω represents a boundary value problem, up to now we haven't been spoken about boundary conditions. We discuss briefly the influence of the boundary and appropriate choice of the boundary conditions. This section was inspired by [70, Section 4], where interested reader can find more details.

We start with the simplest case when we don't need to deal with the influence of the boundary. Sometimes we are interested in the behaviour of fluid in the interior of the domain Ω and it is convenient to eliminate completely the presence of the boundary. It can be realized in two ways. First, assume that the fluid occupies the whole space, i.e. $\Omega = \mathbb{R}^n$, and velocity vanishes at $|x| \to +\infty$. Then we are interested in knowing the properties of the velocity u and pressure π of the governing equations at any instant of the time $t > 0$ and any position $x \in \mathbb{R}^n$. Second, assume that for a positive constant L the velocity u and pressure π are $L-$periodic at each direction x_i with zero mean values. Here $\Omega = (0, L)^n$ is a periodic cell. Advantage of this second case consist in working on a domain with a compact closure.

In most cases we can't neglect the presence of the boundary. Boundary conditions require an understanding of the nature of the bodies that are divided by the boundary. A variety of suggestions were put forward by the pioneers of the field, Bernoulli, DuBuat, Navier, Poisson, Grad, Stokes and others, as to the condition that ought to be applied on the boundary between an impervious solid and a liquid.

G. G. Stokes [84] knew from DuBuat's experiment with water flowing through a pipe that for small velocities the water near the inner surface of the pipe is at rest. In this case no-slip boundary conditions,

$$u = 0 \text{ at } \partial\Omega, \tag{1.16}$$

seems to be suitable. Nevertheless, Stokes believed that for the higher velocities the fluid is slipping at the boundary. The determination of appropriate boundary conditions was an open problem for him.

C. L. M. H. Navier [71] derived a slip boundary conditions which can be generalized to the condition

$$(1-\lambda)u\cdot\tau + \lambda[\mathcal{S}\nu]\cdot\tau = 0, \quad \lambda\in[0,1] \quad \text{at } \partial\Omega. \tag{1.17}$$

If the boundary is impermeable, the normal component of the velocity is equal to zero and therefore we add to (1.17) the relation

$$u\cdot\nu = 0 \text{ at } \partial\Omega. \tag{1.18}$$

The above mentioned boundary conditions (1.17) with (1.18), when $\lambda \in (0,1)$ are referred to as the slip boundary conditions or Navier boundary conditions. If $\lambda = 0$ we obtain classical no-slip boundary condition (1.16). If $\lambda = 1$ we talk about perfect slip boundary conditions. We point out that in this thesis we are mostly interested in perfect slip boundary conditions.

The parameter λ is usually assumed to be a constant but it could however be a function of the normal stresses and the shear rate. Then the Navier's boundary conditions can be generalized to

$$u\cdot\tau + \tilde{\lambda}(\mathcal{S}\nu\cdot\nu, |Du|)[\mathcal{S}\nu]\cdot\tau = 0 \quad \text{at } \partial\Omega.$$

Another boundary conditions that are sometimes used, especially when dealing with non-Newtonian fluids, are the threshold-slip conditions, which can be expressed as follows:

$$|\mathcal{S}\nu\cdot\tau| \le \alpha|\mathcal{S}\nu\cdot\nu| \Rightarrow u\cdot\tau = 0, \quad |\mathcal{S}\nu\cdot\tau| > \alpha|\mathcal{S}\nu\cdot\nu| \Rightarrow u\cdot\tau \neq 0, \quad -\gamma\frac{u\cdot\nu}{|u\cdot\nu|} = \mathcal{S}\nu\cdot\tau,$$

where α is a positive constant and $\gamma = \gamma(\mathcal{S}\nu\cdot\nu, u\cdot\tau, |Du|)$. The above mentioned conditions implies that fluid will not slip until the ration of the magnitude of shear stress and the magnitude of the normal stress exceeds a certain value. When it does exceed that value, it will slip and the slip velocity will depend on both the shear and normal stresses.

1.4 Formulation of the problem

After physical motivation and a few historical remarks we state the problem that is studied in following chapters. Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \ge 2$ be a bounded domain,

$I = (0, T)$ a finite time interval and $I \times \Omega$ a time space cylinder. We investigate qualitative properties of weak solutions to equations of flows of the incompressible generalized Newtonian fluids described by (1.7) and (1.3) complemented with perfect slip boundary conditions, i.e.

$$\partial_t u - \operatorname{div} \mathcal{S}(Du) + \operatorname{div}(u \otimes u) + \nabla \pi = f \qquad \text{in } I \times \Omega, \tag{1.19}$$

$$\operatorname{div} u = 0 \qquad \text{in } I \times \Omega, \tag{1.20}$$

$$u(0, \cdot) = u_0 \qquad \text{in } \Omega, \tag{1.21}$$

$$u \cdot \nu = 0, \quad [\mathcal{S}(Du)\nu] \cdot \tau = 0 \qquad \text{on } I \times \partial\Omega. \tag{1.22}$$

Recall that u is the velocity, π represents the pressure, f stands for the density of volume forces and $\mathcal{S}$ denotes the extra stress tensor. Du is the symmetric part of the velocity gradient, i.e. $Du = \frac{1}{2}[\nabla u + (\nabla u)^\top]$. . By ν we denote an outward normal vector and τ stands for any tangent vector to $\partial\Omega$.

Some chapters are devoted to the stationary variant of (1.19)–(1.22) without convective term:

$$-\operatorname{div} \mathcal{S}(Du) + \nabla \pi = f \qquad \text{in } \Omega, \tag{1.23}$$

$$\operatorname{div} u = 0 \qquad \text{in } \Omega, \tag{1.24}$$

$$u \cdot \nu = 0, \quad [\mathcal{S}(Du)\nu] \cdot \tau = 0 \qquad \text{on } I \times \partial\Omega. \tag{1.25}$$

To formulate the assumptions on the stress tensor $\mathcal{S}$ precisely we state some basic facts about N-functions. More information about $N-$functions can be found in Section A.2. We refer also to [55] or [80].

Definition 1.4.1 *A real function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called N-function if the derivative $\Phi'(s)$ exists and is right continuous for $s \geq 0$, positive for $s > 0$, non-decreasing, $\Phi'(0) = 0$ and $lim_{s \to \infty} \Phi'(s) = \infty$.*

Definition 1.4.2 *N-function Φ is said to satisfy the Δ_2-condition, denoted $\Phi \in \Delta_2$, if there exists a positive constant C, such that $\Phi(2s) \leq C\Phi(s)$ for $s > 0$. By $\Delta_2(\Phi)$ we denote the smallest such constant C.*

By $(\Phi')^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$ we denote the function

$$(\Phi')^{-1}(s) := \sup\{t \in \mathbb{R}^+ : \Phi'(t) \leq s\}.$$

The complementary function of Φ (which is again N-function) is defined as

$$\Phi^*(s) := \int_0^s (\Phi')^{-1}(t)\, \mathrm{d}t.$$

For a measurable function f we can define gauge norm as

$$\|f\|_\Phi := \inf \Big\{ \lambda > 0 : \int_\Omega \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\, \mathrm{d}x \leq 1 \Big\}.$$

The Orlicz space $L^\Phi(\Omega)$ is defined as the set $\{f : \|f\|_{\Phi,\Omega} < \infty\}$. We define

$$W^{1,\Phi}_\sigma(\Omega)^n = \{\varphi_i \in W^{1,\Phi}(\Omega),\, i = 1, \dots, n,\, \varphi \cdot \nu = 0 \,\text{on}\, \partial\Omega,\, \operatorname{div} \varphi = 0\}.$$

We consider the constitutive relation for $\mathcal{S}$ of the form (1.13)

$$\mathcal{S} = \mu(|Du|)Du,$$

where $\mu : [0,\infty) \mapsto [0,\infty)$ is generally non-constant function called generalized viscosity. Construct a scalar potential $\Phi : [0,\infty) \mapsto [0,\infty)$ to the stress tensor $\mathcal{S}$ as follows:

$$\mathcal{S}_{ij}(A) = \partial_{ij}\Phi(|A|) = \Phi'(|A|)\frac{A_{ij}}{|A|}, \quad \mu(|A|) = \frac{\Phi'(|A|)}{|A|} \quad \forall A \in \mathbb{R}^{n\times n}_{sym}. \tag{1.26}$$

By $f \sim g$ we mean that there are positive constants c and C such that $cf \le g \le Cf$. We require the following assumption to be fulfilled:

Assumption 1.4.3 *Suppose that $\Phi \in \mathcal{C}^{1,1}(0,\infty) \cap \mathcal{C}^1[0,\infty)$ is an N-function, $\Phi \in \Delta_2$, $\Phi^* \in \Delta_2$ and for $s > 0$*

$$\Phi'(s) \sim s\Phi''(s) \tag{1.27}$$

and $\Phi''(s)$ is almost monotone, i.e. there exists $C > 0$ such that for all $s \in (0,t]$ either $\Phi''(s) \le C\Phi''(t)$ (almost increasing) or $\Phi''(s) \ge C\Phi''(t)$ (almost decreasing).

Remark 1.4.4 *Every N-function Φ satisfying Δ_2-condition automatically satisfies*

$$\Phi(s) \sim s\Phi'(s).$$

The relation (1.26) and Assumption 1.4.3 give us non-standard $\Phi-$growth conditions, see [22, Lemma 21].

Corollary 1.4.5 *There are constants $C_1, C_2 > 0$ that for all $A, B \in \mathbb{R}^{n\times n}_{sym}$ holds*

$$\begin{aligned} \big(\mathcal{S}(A) - \mathcal{S}(B)\big)\cdot\big(A - B\big) &\ge C_1\Phi''(|A|+|B|)|A-B|^2, \\ |\mathcal{S}(A) - \mathcal{S}(B)| &\le C_2\Phi''(|A|+|B|)|A-B|. \end{aligned} \tag{1.28}$$

Example 1.4.6 *Let us mention that growth conditions* (1.28) *allow to consider models with a great deal of disparity, for example, power-law models*

$$\begin{aligned} \mathcal{S}(Du) &= \mu_0(1+|Du|^2)^{\frac{p-2}{2}}Du, \quad \Phi(|Du|) = \mu_0\int_0^{|Du|}(1+s^2)^{\frac{p-2}{2}}s\,\mathrm{d}s, \\ \mathcal{S}(Du) &= \mu_0(1+|Du|)^{p-2}Du, \quad \Phi(|Du|) = \mu_0\int_0^{|Du|}(1+s)^{p-2}s\,\mathrm{d}s, \end{aligned}$$

$\mu_0 \in \mathbb{R}^+$, $p \in (1,\infty)$. *Also the singular case*

$$\mathcal{S}(Du) = \mu_0|Du|^{p-2}Du, \quad \Phi(|Du|) = \mu_0\int_0^{|Du|} s^{p-1}\,\mathrm{d}s$$

is included.

We define function V and N-function Ψ which are very well suited for expressing differentiability properties of weak solutions. Definition of the function V in the framework of Orlicz spaces was first given in [22].

For given Φ we define the N-function Ψ by

$$\Psi'(s) := \sqrt{\Phi'(s)s}.$$

and we define $V(A)$ such that $\Psi(|A|)$ is a scalar potential to $V(A)$, i.e.

$$V_{ij}(A) := \partial_{ij}\Psi(|A|) = \Psi'(|A|)\frac{A_{ij}}{|A|} \quad \forall A \in \mathbb{R}^{n\times n}_{sym}.$$

It is shown in [22, Lemma 25] that

$$\Psi''(s) \sim \sqrt{\Phi''(s)}. \tag{1.29}$$

Example 1.4.7 *In the case of power-law models from Example* 1.4.6 *we have*

$$V(Du) = \mu_0(1+|Du|^2)^{\frac{p-2}{4}}Du, \quad \Psi(|Du|) = \mu_0\int_0^{|Du|}(1+s^2)^{\frac{p-2}{4}}s\,\mathrm{d}s,$$
$$V(Du) = \mu_0(1+|Du|)^{\frac{p-2}{2}}Du, \quad \Psi(|Du|) = \mu_0\int_0^{|Du|}(1+s)^{\frac{p-2}{2}}s\,\mathrm{d}s,$$

$\mu_0 \in \mathbb{R}^+$, $p \in (1,\infty)$. *Also the singular case*

$$V(|Du|) = \mu_0|Du|^{\frac{p-2}{2}}Du, \quad \Psi(|Du|) = \mu_0\int_0^{|Du|}s^{\frac{p}{2}}\,\mathrm{d}s$$

is included.

Some results demand to specify the shape of the domain Ω, thus we give the definition of axisymmetric domain in the same way as in [20, Definition-Lemma 1].

Definition 1.4.8 *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 2$. We say that Ω is axisymmetric if and only if there exists a nontrivial rigid motion R which is tangent to $\partial\Omega$; or equivalently, which satisfies for all $t \in \mathbb{R}$ $e^{tR}\Omega = \Omega$. Here e^{tR} is the isometry defined via $\frac{d}{dt}e^{tR}(x) = Re^{tR}(x)$.*

By rigid motions R we understand affine maps $R : \Omega \to \mathbb{R}^n$ whose linear part is antisymmetric. If we consider the most common dimensions $n = 2$ and $n = 3$ we can use a simpler definition. A domain in $\mathbb{R}^2$ is axisymmetric if it has a circular symmetry around some point. A domain in $\mathbb{R}^3$ is axisymmetric if it admits an axis of symmetry, i.e. the domain is preserved by a rotation of arbitrary angle around this axis. If the domain admits two nonparallel axes of symmetry, then it is spherically symmetric around some point.

1.5 Structure of the thesis

In this thesis we present results which from papers [54, 66, 85]. Unlike the articles we were able to bring forward more detailed proof in some parts. An example is Section 2.2 in Chapter 2 regarding difference quotient technique. In paper [54] this section is reduced to the short paragraph. We would like to point out that the notation from the three papers is unified.

We briefly describe the main result of each chapter and connection with the other chapters. Chapter 2 is concerned with the steady generalized Stokes system (1.23) and (1.24) complemented with perfect slip boundary conditions (1.25). The nonlinear elliptic operator satisfies non-standard $\Phi-$growth conditions described in Assumption 1.4.3. We show the existence of the second derivatives of the velocity and their regularity up to the boundary of Ω. These results can be also found in the paper [54].

Together with Václav Mácha we were able to extend the results from Chapter 2 and show higher integrability of the first gradient of weak solutions to the system (1.23)–(1.25). These results can be found in [66] and we bring them forward in Chapter 3. At this point we would like to accentuate that Chapter 3 consist of a joint work with Václav Mácha.

In Chapter 5 we study evolutionary generalized Navier–Stokes system (1.19)-(1.21) in two dimensions under perfect slip boundary conditions (1.22). The extra stress tensor $\mathcal{S}$ is assumed to possess $p-$potential structure with $p \geq 2$, therefore it is a special variant of more general setting of a framework of N-functions. Results from Chapter 2 together with Chapter 4 (which provides L^q theory for classical evolutionary Stokes system under perfect slip boundary conditions) allow us to show Hölder continuity of velocity gradients and pressure. Chapters 4 and 5 consist of the results from [85].

2

Differentiability of weak solutions to equations of steady flows

2.1 Main result

This chapter is concerned with steady flows of an incompressible fluid in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$ described by the system (1.23)–(1.25):

$$\begin{aligned} -\operatorname{div} \mathcal{S}(Du) + \nabla\pi &= f && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u \cdot \nu = 0, \quad [\mathcal{S}(Du)\nu] \cdot \tau &= 0 && \text{on } I \times \partial\Omega. \end{aligned}$$

Standard notation is used for Lebesgue spaces $(L^p(\Omega), \|\cdot\|_p)$, Sobolev spaces $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$, $1 \leq p \leq \infty$, $k \in \mathbb{N}$, Orlicz spaces $(L^\Phi(\Omega), \|\cdot\|_\Phi)$ and Orlicz-Sobolev spaces $(W^{1,\Phi}(\Omega), \|\cdot\|_{1,\Phi})$, Ω is a domain with $\mathcal{C}^3$ boundary.

We begin with the definition of the weak solution of the problem (1.23)–(1.25).

Definition 2.1.1 *We say that the function u is the weak solution to the problem* (1.23)–(1.25) *if $u \in W^{1,\Phi}_\sigma(\Omega)^n$ and*

$$\int_\Omega \mathcal{S}(Du) : D\varphi \,\mathrm{d}x = \int_\Omega f \cdot \varphi \,\mathrm{d}x$$

holds[1] *for all $\varphi \in W^{1,\Phi}_\sigma(\Omega)^n$.*

It is well known that the weak solution exists and is unique. It could be easily proven using the monotone operator theory.

Now we are ready to state the main theorem of this chapter.

Theorem 2.1.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded non-axisymmetric $\mathcal{C}^3$ domain, $f \in W^{1,\Phi^*}(\Omega)^n$ and suppose Assumption* 1.4.3 *is fulfilled. Let u be a weak solution to* (1.23)–(1.25)*. Then there exists constant C independent of u such that*

$$\int_\Omega |\nabla V(Du)|^2 \,\mathrm{d}x \leq C\Big(\int_\Omega \Phi^*(|f|)\,\mathrm{d}x + \int_\Omega \Phi^*(|\nabla f|)\,\mathrm{d}x\Big).$$

[1] As we use the notation $W^{1,\Phi}_\sigma(\Omega)^n$ for vector-valued functions with components in the function space $W^{1,\Phi}_\sigma(\Omega)$, we use analogically the notation $W^{1,\Phi}_\sigma(\Omega)^{n\times n}$ for tensor-valued functions.

If moreover $f = \operatorname{div} g$ *and* $g \in L^{(\Phi^*)^q}$, $q \in \left[1, \frac{n}{n-2}\right]$ *for* $n \geq 3$, $q \in [1, \infty)$ *for* $n = 2$, *then the corresponding pressure* π *satisfies*

$$\begin{aligned}\int_\Omega (\Phi^*(|\pi - \langle \pi \rangle_\Omega|))^q \, \mathrm{d}x \leq C \int_\Omega (\Phi^*(|g|))^q \, \mathrm{d}x \\ + C\Big(\int_\Omega \Phi^*(|\nabla g|) \, \mathrm{d}x + \int_\Omega \Phi^*(|\nabla^2 g|) \, \mathrm{d}x \Big)^q.\end{aligned} \tag{2.1}$$

Remark 2.1.3 (Assumptions on f and the domain Ω) *For a special choice of* Φ, *assumption on* f *could be weakened. For example if we would consider* Φ *such that* Φ'' *is bounded and decreasing (which corresponds to the power-law model with* $p < 2$ *and non-singular case), it is sufficient to take* $f \in L^{\Phi^*}(\Omega)^n$, *cf. [52].*

The assumption on the shape of Ω *is related to the boundary condition* $u \cdot \nu = 0$ *on* $\partial\Omega$. *In several parts of the proof we use a stronger version of Korn's inequality, see Lemma* A.4.2, *which is valid if the domain* Ω *is not axisymmetric (if we considered homogeneous Dirichlet boundary conditions, then an arbitrary shape of the domain would be admissible in the formulation of Korn's inequality).*

In this part we would like to mention the paper [29], where the author obtains, with a different method, result very similar to our results. C. Ebmeyer studies the problem (1.23)–(1.25) where the equation of motion contains the convective term $\operatorname{div}(u \otimes u)$ and $\Omega \subset \mathbb{R}^3$. He supposes that the tensor $\mathcal{S}$ has the p-potential structure and is interested in the case $p < 2$. The author obtains the regularity results in Sobolev spaces with fractional derivatives and in Nikolskiĭ's spaces. Among others, he shows $\int_\Omega (\kappa + |Du|)^{p-2} |\nabla Du|^2 \, \mathrm{d}x < \infty$, $\kappa \in \{0, 1\}$ for $p \in (\frac{9}{5}, 2)$ in the case of power-law Navier-Stokes system and for $p \in (1, 2)$ in the case of power-law Stokes system. He uses the fact that perfect slip boundary conditions allow to extend the solution beyond the flat boundary. Results are formulated for the flat boundary and, by the local change of coordinates for the general shape of the boundary.

In [12] the authors are concerned with the system (1.23) and (1.24) equipped with homogeneous Dirichlet boundary conditions. The extra stress tensor is given by a power-law ansatz with exponent $p \geq 2$. Among others they show that $V(Du) \in W^{1, \frac{2q}{p+q-2}}(\Omega)^{n \times n}$ for $q = \frac{np+2-p}{n-2}$, if $n \geq 3$ and for all $q < \infty$, if $n = 2$. In tangential directions they are able to improve regularity properties to $\int_\Omega |\partial_{\tau^\alpha} V(Du)|^2 \, \mathrm{d}x < \infty$, but in the normal direction there is a loss of regularity due to the absence of some special weighted version of Korn's inequality and the presence of pressure.

In [50, Theorem 3] the authors show a regularity result for non-circular domain in 2D and with an additional assumption in [50, Theorem 4] the same result is established for a circle.

The proof of Theorem 2.1.2 is divided into three main parts. In the first part, Section 2.2, we show that for the quadratic potential, i.e. Φ'' is bounded from below and from above (which corresponds to the case $p = 2$ in the power-law models), the solution u belongs to the space $W^{2,2}(\Omega)^n$. In the second part, Section 2.3, we introduce the regularized problem where instead of the generalized viscosity μ we consider truncated viscosity $\mu^\varepsilon = \min\big(\max(\mu(|Du^{e\varepsilon}|), \varepsilon), \frac{1}{\varepsilon}\big)$ for $\varepsilon \in (0,1)$. Using the fact that for the regularized problem $u^{e\varepsilon} \in W^{2,2}(\Omega)^n$, we show that the term $\int_\Omega \mu^\varepsilon(|Du^{e\varepsilon}|)|\partial_n Du^{e\varepsilon}|^2 \,\mathrm{d}x$ can be estimated by lower order terms and small same order terms, see Lemma 2.3.2. The main idea is to test the regularized version of (1.23) by second normal derivatives (up to some correction), which is possible due to perfect slip boundary conditions. Further we obtain similar result for the term $\int_\Omega \mu^\varepsilon(|Du^{e\varepsilon}|)|\partial_\alpha Du^{e\varepsilon}|^2 \,\mathrm{d}x$, $\alpha \in \{1,\ldots,n-1\}$, see Lemma 2.3.6. It can be done by taking tangent derivative of regularized version of (1.23) and testing by suitable function. We finish Section 2.3 by putting together estimates from Lemma 2.3.2 and Lemma 2.3.6, estimating lower order terms and absorbing small same order terms into the left hand side. Although we are using the function μ^ε in Section 2.3, due to the Assumption 1.4.3 and constitutive relation (1.13) we have $\mu(s) \sim \Phi''(s)$ and due to (1.29) we easily obtain the result in the terms of the function V^ε. In the third part, Section 2.4, we pass from the regularized problem to the original one.

2.2 Quadratic potential

In this section we will confine ourselves to the case Φ'' is bounded from below and from above. In the definition of the weak solution, Definition 2.1.1, the space $W^{1,\Phi}_\sigma(\Omega)^n$ reduces to $W^{1,2}_\sigma(\Omega)^n$.

Lemma 2.2.1 *Let $\Omega \subset \mathbb{R}^n$, $\Omega \in \mathcal{C}^3$, $f \in L^2(\Omega)^n$. Let Assumption* 1.4.3 *be fulfilled and $\Phi'' \in [c_3, c_4] \subset (0,\infty)$. Then for every weak solution to the problem* (1.23)–(1.25) *holds*

$$u \in W^{2,2}(\Omega)^n, \quad \pi \in W^{1,2}(\Omega).$$

Proof. We will follow the proof in [68, Section 3] where the authors are dealing with the evolutionary case in 3D under homogeneous Dirichlet boundary condition. The authors are interested in the power-law model for case $p \geq 2$. The proof is supposed to be divided to the interior regularity and the boundary regularity. We will focus only on the boundary regularity, because the interior regularity can be easily proved by the simple modification of the computation below. The proof consist of three steps. At first we focus on boundary regularity of u in tangent direction, later we obtain similar estimate for u in normal direction and at the end we reconstruct pressure π.

Step 1 *Boundary regularity in tangent direction*

In this step we work in the local system of coordinates. We suppose that in a neighborhood of $P \in \partial\Omega$ we can describe the boundary as a graph of a suitably smooth function. By Ω_{R_0} we understand an intersection of the domain Ω with a ball with radius R_0 and center P. Let $0 < r \leq R \leq \frac{R_0}{2}$. $T_\alpha : \Omega_R \to \Omega_{R_0}$ denotes the shift operator in tangent direction. δ^+_α stands for the differences, i.e. $\delta^+_\alpha g(y) = g(T_\alpha y) - g(y)$. For properties of differences and precise description of the boundary see Section A.1.

We start with the weak formulation

$$\int_\Omega \mathcal{S}(Du)(y) : D(\psi(y))\,\mathrm{d}y = \int_\Omega f(y)\cdot\psi(y)\,\mathrm{d}y, \tag{2.2}$$

which is fulfilled for $\psi \in W^{1,2}_\sigma(\Omega)^n$, $\operatorname{supp}\psi \subset \Omega_R$.

The goal is to derive the identity which contains differences of each term of the equation (2.2). To reach this, we need to test by $\varphi(T_\alpha^{-1}y)$ instead $\psi(y)$. This function doesn't belong to $W^{1,2}_\sigma(\Omega)^n$ because nor $\operatorname{div}\varphi(T_\alpha^{-1}y) \neq 0$ neither $\varphi(T_\alpha^{-1}y)\cdot\nu \neq 0$ on $\partial\Omega$. That's why we introduce the following correction:

$$\varphi_{cor}(y) := \varphi(T_\alpha^{-1}y) - (\varphi(T_\alpha^{-1}y)\cdot\delta_\alpha^-\nu(y))\nu(y) + z_c(y),$$

where $z_c(y)$ is the solution to the problem

$$\begin{aligned} \operatorname{div} z_c(y) &= \operatorname{div}[-\varphi(T^{-1}y) + (\varphi(T^{-1}y)\cdot\delta_\alpha^-\nu(y))\nu(y)] && \text{in } \Omega,\\ z_c(y) &= 0 && \text{on } \partial\Omega, \end{aligned}$$

$\operatorname{supp} z_c \subset \Omega_R$. On $\partial\Omega$ it holds

$$\int_{\partial\Omega} [-\varphi(T_\alpha^{-1}y) + (\varphi(T_\alpha^{-1}y)\cdot\delta_\alpha^-\nu(y))\nu(y)]\cdot\nu(y)\,\mathrm{d}y = 0.$$

Because of $z_c(y)$ we have $\operatorname{div}\varphi_{cor}(y) = 0$ and we can also easily see that $\varphi_{cor}(y)\cdot\nu(y) = 0$ on $\partial\Omega$. It means that $\varphi_{cor}(y)$ is good test function. Putting $\psi(y) := \varphi_{cor}(y)$ in (2.2) we obtain

$$\begin{aligned} &\int_\Omega \mathcal{S}(Du)(y) : D(\varphi(T_\alpha^{-1}y))\,\mathrm{d}y\\ &- \int_\Omega \mathcal{S}(Du)(y) : [(D(\varphi(T_\alpha^{-1}y))\delta_\alpha^-\nu(y)) \otimes \nu(y)]\,\mathrm{d}y\\ &- \int_\Omega \mathcal{S}(Du)(y) : [(\varphi(T_\alpha^{-1}y)D(\delta_\alpha^-\nu(y))) \otimes \nu(y)]\,\mathrm{d}y\\ &- \int_\Omega \mathcal{S}(Du)(y) : (\varphi(T_\alpha^{-1}y)\cdot\delta_\alpha^-\nu(y))D(\nu(y))\,\mathrm{d}y\\ &+ \int_\Omega \mathcal{S}(Du)(y) : D(z_c(y))\,\mathrm{d}y = \int_\Omega f(y)\varphi(T_\alpha^{-1}y)\,\mathrm{d}y\\ &- \int_\Omega f(y)(\varphi(T_\alpha^{-1}y)\cdot\delta_\alpha^-\nu(y))\nu(y)\,\mathrm{d}y + \int_\Omega f(y)z_c(y)\,\mathrm{d}y. \end{aligned} \tag{2.3}$$

In the first term of (2.3) there is the derivative of the composite function $D(\varphi(T_\alpha^{-1}(y)))$, but we need $D\varphi(T_\alpha^{-1}y)$. To reach this we apply Lemma A.1.3 which provides:

$$D(\varphi(T_\alpha^{-1}y)) = D\varphi(T_\alpha^{-1}y) + \partial_n\varphi(T_\alpha^{-1}y) \otimes_S (\delta_\alpha^-\nabla a).$$

In some terms we use the substitution $y = T_\alpha x$. From Section A.1 we know that $\mathrm{d}y = \mathrm{d}x$, because the Jacobian of the mapping T_α and T_α^{-1} is equal to 1. In (2.2) we put $\psi(y) := \varphi(x)$ and subtract (2.2) from the resulting equation (2.3). We get

$$
\begin{aligned}
0 = & \int_\Omega \delta_\alpha^+ \mathcal{S}(Du)(x) : D(\varphi(x)) \, \mathrm{d}x \\
& + \int_\Omega \mathcal{S}(Du)(T_\alpha x) : [\partial_n \varphi(x) \otimes_S (\delta_\alpha^- \nabla a)] \, \mathrm{d}x \\
& - \int_\Omega \mathcal{S}(Du)(y) : [(D(\varphi(T_\alpha^{-1} y)) \delta_\alpha^- \nu(y)) \otimes \nu(y)] \, \mathrm{d}y \\
& - \int_\Omega \mathcal{S}(Du)(y) : [(\varphi(T_\alpha^{-1} y) D(\delta_\alpha^- \nu(y))) \otimes \nu(y)] \, \mathrm{d}y \\
& - \int_\Omega \mathcal{S}(Du)(y) : (\varphi(T_\alpha^{-1} y) \cdot \delta_\alpha^- \nu(y)) D(\nu(y)) \, \mathrm{d}y \\
& + \int_\Omega \mathcal{S}(Du)(y) : D(z_c(y)) \, \mathrm{d}y - \int_\Omega \delta_\alpha^+ f(x) \varphi(x) \, \mathrm{d}x \\
& + \int_\Omega f(y) (\varphi(T_\alpha^{-1} y) \cdot \delta_\alpha^- \nu(y)) \nu(y) \, \mathrm{d}y - \int_\Omega f(y) z_c(y) \, \mathrm{d}y =: \sum_{i=1}^{9} \mathcal{A}_i
\end{aligned}
\tag{2.4}
$$

valid for the test function $\varphi \in W_\sigma^{1,2}(\Omega)^n$, $\operatorname{supp} \varphi \subset \Omega_R$.

We choose $\varphi = \varphi_a + \varphi_b + \varphi_c$ such that

$$
\varphi_a + \varphi_b = \frac{1}{h^2} \delta_\alpha^+ u(x) \xi^2(x) + \frac{1}{h^2} \nu_u(x) \xi^2(x), \tag{2.5}
$$

where $\nu_u(x) := (u(Tx) \cdot \delta_\alpha^+ \nu(x)) \nu(x)$, $h \in (0, \frac{R_0}{2})$ and the cut-off function $\xi(x) \in \mathcal{C}^\infty(B_R(P))$ is defined as follows:

$$
\xi(x) \begin{cases} = 1 & x \in B_r(P), \\ \in (0,1) & x \in B_R(P) \setminus B_r(P), \\ = 0 & x \in \mathbb{R}^n \setminus B_R(P). \end{cases} \tag{2.6}
$$

We can easily see that condition $(\varphi_a + \varphi_b) \cdot \nu = 0$ on $\partial\Omega$ holds and $\operatorname{supp}(\varphi_a + \varphi_b) \subset \Omega_R$. But generally it doesn't hold that $\operatorname{div}(\varphi_a + \varphi_b) = 0$. That's why the correction $\varphi_c \in W_\sigma^{1,2}(\Omega)^n$ with $\operatorname{supp} \varphi_c \subset \Omega_R$ is included in the test function φ. It is defined as a solution to the problem

$$
\begin{aligned}
\operatorname{div} \varphi_c &= \operatorname{div}(-\varphi_a - \varphi_b) && \text{in } \Omega, \\
\varphi_c &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

The compatibility condition is fulfilled, since

$$
\begin{aligned}
0 = & \int_{\partial\Omega} \varphi_c \cdot \nu \, \mathrm{d}\sigma = \int_\Omega \operatorname{div} \varphi_c \, \mathrm{d}x = \int_\Omega \operatorname{div}(-\varphi_a - \varphi_b) \, \mathrm{d}x = \\
= & - \int_{\partial\Omega} (\varphi_a + \varphi_b) \cdot \nu \, \mathrm{d}\sigma = 0.
\end{aligned}
$$

Bogovskiĭ's Lemma A.3.1 provides following estimate of z_c and φ_c:

$$
\|z_c\|_{1,r}^r \le C h^r \|\varphi\|_{1,r}^r, \tag{2.7}
$$

$$
\|\varphi_c\|_{1,r}^r \le C \| \operatorname{div}(\varphi_a + \varphi_b)\|_r^r. \tag{2.8}
$$

Using the definition of $\varphi_a + \varphi_b$ and the incompressibility conditions $\operatorname{div} u = 0$, (2.8) can be rewritten as

$$\|\varphi_c\|_{1,r}^r \leq \frac{C}{h^r}\|u\|_{1,r}^r(1 + \|\nabla\xi\|_\infty^r). \tag{2.9}$$

With the knowledge (2.5) and (2.9) we get the estimate of z_c:

$$\|z_c\|_{1,r}^r \leq C\Big(\Big\|\frac{\nabla\delta_\alpha^+ u}{h}\xi\Big\|_r^r + \|u\|_{1,r}^r(1 + \|\nabla\xi\|_\infty^r)\Big). \tag{2.10}$$

Let's estimate:

$$\mathcal{A}_1 = \int_{\Omega_R}\int_0^1 \partial_{ij}\mathcal{S}_{kl}(Du + \lambda\delta_\alpha^+ Du)\delta_\alpha^+ D_{ij}uD_{kl}\varphi \,\mathrm{d}\lambda\,\mathrm{d}x.$$

Using Lemma A.1.3 we get

$$\begin{aligned}
\mathcal{A}_1 &= \int_{\Omega_R}\int_0^1 \partial_{ij}\mathcal{S}_{kl}(Du + \lambda\delta_\alpha^+ Du)[D_{ij}\delta_\alpha^+ u \\
&\quad - \big((\partial_n u)(Tx) \otimes_S \delta_\alpha^+\nabla a\big)_{ij}]D_{kl}\varphi\,\mathrm{d}\lambda\,\mathrm{d}x \\
&= \int_{\Omega_R}\int_0^1 \partial_{ij}\mathcal{S}_{kl}(Du + \lambda\delta_\alpha^+ Du)D_{ij}(\delta_\alpha^+ u)D_{kl}\varphi\,\mathrm{d}\lambda\,\mathrm{d}x \\
&\quad - \int_{\Omega_R}\int_0^1 \partial_{ij}\mathcal{S}_{kl}(Du + \lambda\delta_\alpha^+ Du)\big((\partial_n u)(Tx) \otimes_S \delta_\alpha^+\nabla a\big)_{ij}D_{kl}\varphi\,\mathrm{d}\lambda\,\mathrm{d}x \\
&=: \mathcal{A}_{1.1} - \mathcal{A}_{1.2},
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{1.1} &= \frac{1}{h^2}\int_{\Omega_R}\int_0^1 \partial_{ij}\mathcal{S}_{kl}(Du + \lambda\delta_\alpha^+ Du)D_{ij}\delta_\alpha^+ u\big[D_{kl}\delta_\alpha^+ u\xi^2(x) + \\
&\quad + 2[\delta_\alpha^+ u]_k\xi\partial_l\xi + D_{kl}\nu_u\xi^2 + 2[\nu_u]_k\xi\partial_l\xi + D_{kl}\varphi_c\,\mathrm{d}x\big]\,\mathrm{d}\lambda\,\mathrm{d}x =: \sum_{i=1}^5 \mathcal{B}_i.
\end{aligned}$$

Due to growth conditions from Corollary 1.4.5 and Lemma A.4.1 we can estimate the term $\mathcal{B}_1$:

$$\begin{aligned}
\mathcal{B}_1 &\geq \frac{2C_1}{h^2}\int_{\Omega_R}|D\delta_\alpha^+ u|^2\xi^2\,\mathrm{d}x \\
&\geq 2C_1C_3\int_{\Omega_R}\Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2\xi^2\,\mathrm{d}x - 2C_1C_4\int_{\Omega_R}\Big|\frac{\delta_\alpha^+ u}{h}\Big|^2(|\nabla\xi|^2 + \xi^2)\,\mathrm{d}x.
\end{aligned}$$

On the second term $\mathcal{B}_2$ we apply $(1.28)_2$ and Cauchy inequality with $\varepsilon > 0$

$$\begin{aligned}
|\mathcal{B}_2| &\leq \frac{2C_2}{h^2}\int_{\Omega_R}|\nabla\delta_\alpha^+ u|\xi|\delta_\alpha^+ u||\nabla\xi|\,\mathrm{d}x \leq \varepsilon C_2^2\int_{\Omega_R}\Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2\xi^2\,\mathrm{d}x \\
&\quad + C\int_{\Omega_R}\Big|\frac{\delta_\alpha^+ u}{h}\Big|^2|\nabla\xi|^2\,\mathrm{d}x \leq C\Big(\varepsilon\int_{\Omega_R}\Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2\xi^2\,\mathrm{d}x + \|\nabla u\|_2^4 + \|\nabla\xi\|_\infty^4\Big).
\end{aligned}$$

In the third term $\mathcal{B}_3$ the following is used:

$$
\begin{aligned}
&D_{ij}(\nu_u(x)) = \partial_j\{u_k(Tx)[\delta_\alpha^+\nu]_k\nu_i(x)\} = \\
&= \partial_j u_k(Tx)[\delta_\alpha^+\nu]_k\nu_i(x) + u_k(Tx)\partial_j[\delta_\alpha^+\nu]\nu_i(x) + u_k(Tx)[\delta_\alpha^+\nu]\partial_j\nu_i(x).
\end{aligned}
$$

We use Lemma A.1.4 to estimate the modulus of gradient of the normal vector or the modulus of the difference of gradient gradient of the normal vector by the constant C_n.

$$
\begin{aligned}
|\mathcal{B}_3| &\le \frac{C_2}{h^2}\int_{\Omega_R} |D_{ij}\delta_\alpha^+ u D_{ij}\nu_u\xi^2|\,\mathrm{d}x \\
&\le C_2\int_{\Omega_R}\Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|(|\nabla u|C_n + |u|C_n + |u|C_n^2)\xi^2\,\mathrm{d}x \\
&\le C\Big(\varepsilon\int_{\Omega_R}\Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2\xi^2\,\mathrm{d}x + \|u\|_{1,2}^2\Big).
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{B}_4| &\le \frac{2C_2}{h^2}\int_{\Omega_R}|\nabla\delta_\alpha^+ u||\nu_u|\xi|\nabla\xi|\,\mathrm{d}x \\
&\le C_2^2\varepsilon\int_{\Omega_R}\Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2\xi^2\,\mathrm{d}x + C\int_{\Omega_R}|u|^2|\nabla\xi|^2\,\mathrm{d}x \\
&\le C\Big(\varepsilon\int_{\Omega_R}\Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2\xi^2\,\mathrm{d}x + \|u\|_2^4 + \|\nabla\xi\|_\infty^4\Big),
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{B}_5| &\le C_2\int_{\Omega_R}|D_{ij}\delta_\alpha^+ u D_{ij}\varphi_c|\,\mathrm{d}x \le \varepsilon C_2^2\int_{\Omega_R}\Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2\,\mathrm{d}x + C\int_{\Omega_R}h^2|\nabla\varphi_c|^2 \\
&\le C\Big(\varepsilon\int_{\Omega_R}\Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2\,\mathrm{d}x + \|u\|_{1,2}^2 + \|u\|_{1,2}^4 + \|\nabla\xi\|_\infty^4\Big).
\end{aligned} \tag{2.11}
$$

Corollary 1.4.5 and similar steps as before allow us to estimate the term $\mathcal{A}_{1.2}$.

$$
\begin{aligned}
|\mathcal{A}_{1.2}| &= |\int_{\Omega_R}\int_0^1\partial_{ij}\mathcal{S}_{kl}(Du + \lambda\delta_\alpha^+ Du)\big((\partial_n u)(Tx)\otimes_S\delta_\alpha^+\nabla a\big)_{ij}D_{kl}\varphi\,\mathrm{d}\lambda\,\mathrm{d}x| \\
&\le \frac{C_2}{h^2}\int_{\Omega_R}|\big((\partial_n u)(Tx)\otimes_S\delta_\alpha^+\nabla a\big)_{ij}\big[D_{ij}\delta_\alpha^+ u\xi^2 + 2[\delta_\alpha^+ u]_i\xi\partial_j\xi \\
&\quad + D_{ij}\nu_u\xi^2 + 2[\nu_u]_i\xi\partial_j\xi + D_{ji}\varphi_c\big]|\,\mathrm{d}x =: \sum_{i=6}^{10}\mathcal{B}_i.
\end{aligned}
$$

Applying Lemma A.1.4 and estimate (2.9) on correction φ_c leads to

$$
\begin{aligned}
|\mathcal{B}_6| + |\mathcal{B}_7| &\le C_2 \int_{\Omega_R} |\nabla u| C_n \Big(\Big| \frac{\nabla \delta_\alpha^+ u}{h} \Big| \xi^2 + \Big| \frac{\delta_\alpha^+ u}{h} \Big| \xi |\nabla \xi| \Big) \,\mathrm{d}x \\
&\le C_2^2 \varepsilon \int_{\Omega_R} \Big| \frac{\nabla \delta_\alpha^+ u}{h} \Big|^2 \xi^2 \,\mathrm{d}x + C \|\nabla u\|_2^2 \\
&+ C_2^2 C \int_{\Omega_R} \Big| \frac{\delta_\alpha^+ u}{h} \Big|^2 \xi^2 \,\mathrm{d}x + C \|\nabla \xi\|_\infty^2 \int_{\Omega_R} |\nabla u|^2 \,\mathrm{d}x \\
&\le C \Big(\varepsilon \int_{\Omega_R} \Big| \frac{\nabla \delta_\alpha^+ u}{h} \Big|^2 \xi^2 \,\mathrm{d}x + \|\nabla u\|_2^2 + \|\nabla u\|_2^4 + \|\nabla \xi\|_\infty^4 \Big),
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{B}_8| + |\mathcal{B}_9| &\le C_2 \int_{\Omega_R} |\nabla u| C_n [(|\nabla u| C_n + |u| C_n + |u| C_n^2) \xi^2 + |u| C_n \xi |\nabla \xi|] \,\mathrm{d}x \\
&\le C(\|u\|_{1,2}^2 + \|u\|_{1,2}^4 + \|\nabla \xi\|_\infty^4),
\end{aligned}
$$

On the term $\mathcal{B}_{10}$ we use estimate (2.9).

$$
\begin{aligned}
|\mathcal{B}_{10}| &\le C_2 \int_{\Omega_R} |\nabla u| C_n h |\nabla \varphi_c| \,\mathrm{d}x \le \frac{C_n^2}{2} \int_{\Omega_R} |\nabla u|^2 \,\mathrm{d}x + \frac{C_2^2 h^2}{2} \int_{\Omega_R} |\nabla \varphi_c|^2 \,\mathrm{d}x \\
&\le C(\|\nabla u\|_2^2 + \|u\|_{1,2}^4 + \|\nabla \xi\|_\infty^4),
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{A}_2| &= \Big| \int_{\Omega_R} \mathcal{S}_{kl}(Du)(Tx) \big((\partial_n \varphi)(x) \otimes_S \delta_\alpha^+ \nabla a \big)_{kl} \,\mathrm{d}x \Big| \\
&= \Big| \int_{\Omega_R} \int_0^1 \partial_{ij} \mathcal{S}_{kl}(\lambda Du(Tx)) (D_{ij} u)(Tx) \big((\partial_n \varphi)(x) \otimes_S \delta_\alpha^+ \nabla a \big)_{ij} \,\mathrm{d}\lambda \,\mathrm{d}x \Big| \\
&\le C_2 \int_{\Omega_R} |\nabla u| \Big| \nabla \Big(\frac{\delta_\alpha^+ u}{h} \xi^2 + \frac{\nu_u}{h} \xi^2 + h \varphi_c \Big) C_n \Big| \,\mathrm{d}x \\
&\le C_2 \int_{\Omega_R} |\nabla u| \Big(\Big| \frac{\nabla \delta_\alpha^+ u}{h} \Big| \xi^2 + \Big| \frac{\delta_\alpha^+ u}{h} \Big| 2 \xi \nabla \xi + |\nabla u| C_n \xi^2 + |u| C_n \xi^2 \\
&+ |u| C_n^2 \xi^2 + |u| C_n 2 \xi \nabla \xi + h \nabla \varphi_c \Big) C_n \,\mathrm{d}x \\
&\le C \Big(\varepsilon \int_{\Omega_R} \Big| \frac{\nabla \delta_\alpha^+ u}{h} \Big|^2 \xi^2 \,\mathrm{d}x + \|u\|_{1,2}^2 + \|u\|_{1,2}^4 + \|\nabla \xi\|_\infty^4 \Big),
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{A}_3| &\le C_2 \int_{\Omega_R} \Big| DuD \Big[\frac{1}{h^2} \delta_\alpha^- u \xi^2 (T^{-1} y) + \frac{1}{h^2} \nu_u (T^{-1} y) \xi^2 (T^{-1} y) + \varphi_c \Big] (\delta_\alpha^- \nu) \nu \Big| \,\mathrm{d}y \\
&\le C \Big(\varepsilon \int_{\Omega_R} \Big| \frac{\nabla \delta_\alpha^+ u}{h} \Big|^2 \xi^2 \,\mathrm{d}y + \|u\|_{1,2}^2 + \|u\|_{1,2}^4 + \|\nabla \xi\|_\infty^4 \Big),
\end{aligned}
$$

$$
\begin{aligned}
&|\mathcal{A}_4| + |\mathcal{A}_5| \le C_2 \int_{\Omega_R} \Big| (Du)(y) \Big[\frac{1}{h^2} \delta_\alpha^- u \xi^2 (T^{-1} y) + \frac{1}{h^2} \nu_u (T^{-1} y) \xi^2 (T^{-1} y) + \varphi_c \Big] \Big| \\
&\Big(|\nabla (\delta_\alpha^- \nu) \nu| + |(\delta_\alpha^- \nu) \nabla \nu| \Big) \,\mathrm{d}y \le C \Big(\varepsilon \int_{\Omega_R} \Big| \frac{\nabla \delta_\alpha^+ u}{h} \Big|^2 \xi^2 \,\mathrm{d}y + \|u\|_{1,2}^2 \\
&+ \|\nabla u\|_{1,2}^4 + \|\nabla \xi\|_\infty^4 \Big).
\end{aligned}
$$

In the next term we use the estimate (2.10) on z_c:

$$
\begin{aligned}
|\mathcal{A}_6| &\le C_2 \int_{\Omega_R} |Du||D(z_c)|\,\mathrm{d}y \le C\|\nabla u\|_2^2 + C_2^2 \varepsilon \|\nabla z_c\|_2^2 \\
&\le C\Big(\|\nabla u\|_2^2 + \varepsilon \int_{\Omega_R} \Big|\frac{\nabla \delta_\alpha^+ u}{h}\Big|^2 \xi^2\,\mathrm{d}y + \varepsilon \|u\|_{1,2}^2 \\
&+ \varepsilon \|u\|_{1,2}^4 + \varepsilon \|\nabla \xi\|_\infty^4 \Big).
\end{aligned}
$$

We express $\mathcal{A}_7$ in an alternative way

$$
\mathcal{A}_7 = \int_{\Omega_R} \delta_\alpha^+ f \cdot \varphi\,\mathrm{d}x = \int_{\Omega_R} f \cdot \delta_\alpha^- \varphi\,\mathrm{d}x.
$$

After substitution of the test function and addition and subtraction of the terms

$$
\frac{1}{h^2}\int_{\Omega_R} f_i [\delta_\alpha^- u]_i \xi^2\,\mathrm{d}x,\ \frac{1}{h^2}\int_{\Omega_R} f_i [\nu_u]_i \xi^2 (T^{-1}x)\,\mathrm{d}x
$$

we get

$$
\begin{aligned}
\mathcal{A}_7 &= \frac{1}{h^2}\int_{\Omega_R} f_i[\delta_\alpha^+ u - \delta_\alpha^- u]_i \xi^2\,\mathrm{d}x + \frac{1}{h^2}\int_{\Omega_R} f_i [\delta_\alpha^- u]_i [\xi^2 - \xi^2(T^{-1}x)]\,\mathrm{d}x \\
&- \frac{1}{h^2}\int_{\Omega_R} f_i [\delta_\alpha^+ \nu_u]_i \xi^2 (T^{-1}x)\,\mathrm{d}x - \frac{1}{h^2}\int_{\Omega_R} f_i[\nu_u]_i [\xi^2 - \xi^2(T^{-1}x)]\,\mathrm{d}x \\
&+ \int_{\Omega_R} f_i [\delta_\alpha^- \varphi_c]_i =: \sum_{i=11}^{15} \mathcal{B}_i.
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{B}_{11}| &\le C\int_{\Omega_R} |f|^2 \xi^2\,\mathrm{d}x + \varepsilon \int_{\Omega_R} \Big|\frac{\nabla \delta_\alpha^+ u}{h}\Big|^2 \xi^2\,\mathrm{d}x, \\
|\mathcal{B}_{12}| &\le C(\|f\|_2^2 + \|\nabla u\|_2^4 + \|\nabla \xi\|_\infty^4), \\
|\mathcal{B}_{13}| &\le \int_{\Omega_R} |f| C_n \Big|\frac{\delta_\alpha^+ u}{h}\Big| \xi^2 \le \varepsilon \int_{\Omega_R} \Big|\frac{\delta_\alpha^+ u}{h}\Big|^2 \xi^2\,\mathrm{d}x + C\int_{\Omega_R} |f|^2 \xi^2\,\mathrm{d}x, \\
|\mathcal{B}_{14}| &\le C(\|f\|_2^2 + \|u\|_2^4 + \|\nabla \xi\|_\infty^4), \\
|\mathcal{B}_{15}| &\le \frac{1}{2}\|f\|_2^2 + \frac{h^2}{2}\|\nabla \varphi_c\|_2^2 \le C(\|f\|_2^2 + \|u\|_{1,2}^2 + \|u\|_{1,2}^4 + \|\nabla \xi\|_\infty^4).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_8 &\le \int_{\Omega_R} |f| \Big(\Big|\frac{\delta_\alpha^- u}{h}\Big| \xi^2 + |u| C_n \xi^2 + \varphi_c \Big) C_n\,\mathrm{d}y \\
&\le C(\|f\|_2^2 + \|u\|_{1,2}^2 + \|u\|_{1,2}^4 + \|\nabla \xi\|_\infty^4).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_9 \le C\|f\|_2^2 + \varepsilon \|z_c\|_2^2 \le C\Big(\|f\|_2^2 + \varepsilon \int_{\Omega_R} \Big|\frac{\nabla \delta_\alpha^+ u}{h}\Big|^2 \xi^2\,\mathrm{d}y + \varepsilon \|u\|_{1,2}^2 \\
+ \varepsilon \|\nabla \xi\|_\infty^4 + \varepsilon \|u\|_{1,2}^4 \Big).
\end{aligned}
$$

Putting all estimates together we obtain

$$\int_{\Omega_R} \Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2 \xi^2 \,\mathrm{d}x \le \varepsilon C \int_{\Omega_R} \Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2 \xi^2 \,\mathrm{d}x + \varepsilon C' \int_{\Omega_R} \Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2 \,\mathrm{d}x \\ + C''(\|u\|_{1,2}^2 + \|u\|_{1,2}^4 + \|f\|_2^2) + C'''\|\nabla\xi\|_\infty^4. \tag{2.12}$$

We are able to choose $\varepsilon > 0$ as small as we want, therefore we pick the one in order to subsume the first term on the right hand side of (2.12) into the left hand side. We know from the first apriori estimate that $\|u\|_{1,2} < C$ and from the assumption $\|f\|_2 < C$. This gives the boundedness of the third term on the right hand side of (2.12).

We can see that the procedure fails because of the fact, that there is missing cut-off function $\xi(x)$ in the integral in the second term on right hand side in (2.12). But we can get rid of the uncomfortable term by virtue of the following lemma which can be found in [35].

Lemma 2.2.2 *Let $f : [a,b] \mapsto \mathbb{R}^+$ is bounded. Suppose there exists constants $A, B, \alpha > -1$ and $\varepsilon \in (0,1)$, that*

$$f(r) \le \varepsilon f(R) + A(R-r)^{-\alpha} + B \qquad \forall a \le r < R \le b.$$

Then there exists positive constant $c = c(\alpha, \varepsilon)$, that holds

$$f(r) \le c[A(R-r)^{-\alpha} + B] \qquad \forall a \le r < R \le b.$$

Fix $h \in (0, r)$. The term $\frac{\nabla\delta_\alpha^+ u}{h}$ has good sense on Ω_{R_0}. The cut-off function ξ satisfies $C'''\|\nabla\xi\|_\infty^4 \le A(R-r)^{-4}$ for some $A > 0$ independent of r, R. We rewrite the relation (2.12) to a more convenient form:

$$\int_{\Omega_r} \Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2 \,\mathrm{d}x \le \varepsilon \int_{\Omega_R} \Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2 \,\mathrm{d}x + A(R-r)^{-4} + B.$$

We use Lemma 2.2.2 for

$$f(r) := \int_{\Omega_r} \Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2 \,\mathrm{d}x,$$

As one can easily see, assumptions of the Lemma 2.2.2 are fulfilled. We obtain

$$\int_{\Omega_r} \Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2 \,\mathrm{d}x \le c[A(R-r)^{-4} + B] \quad \forall 0 < r < R \le \frac{R_0}{2},$$

where the constants A, B depends only on the norm of f in $L^2(\Omega)^n$ and on the norm of u in $W^{1,2}(\Omega)^n$. As well the whole right hand side of (2.12) now depends only on $\|f\|_2$ a $\|u\|_{1,2}$. Setting $r := \frac{R_0}{4}$ and $R = \frac{R_0}{2}$ we get

$$\int_{\Omega_{\frac{R_0}{4}}} \Big|\frac{\nabla\delta_\alpha^+ u}{h}\Big|^2 \,\mathrm{d}x \le C(\|f\|_2, \|u\|_{1,2}).$$

Due to (A.6) a (A.7) the previous estimate can be rewritten as

$$\sum_{i=1}^{2} \int_{\Omega_{\frac{R_0}{4}}} |\partial_i \partial_{\tau^\alpha} u|^2 \, \mathrm{d}x \le C(\|f\|_2, \|u\|_{1,2}). \tag{2.13}$$

Step 2 *Boundary regularity in normal direction*

We will follow [68], where the problem of normal direction is solved for the evolutionary variant of our problem in $\Omega \subset \mathbb{R}^3$. This computation for stationary problem in $\Omega \subset \mathbb{R}^2$ is also done in articles [50] and [51].

In order to show boundedness of $\nabla^2 u$ we need an estimate of type (2.13) in the normal direction which is locally x_n. After formal employment [2] of the operator curl

$$\operatorname{curl} g = \partial_\alpha g_j - \partial_j g_\alpha \quad \text{for } g : \Omega \mapsto \mathbb{R}^n, \ \alpha < j.$$

we get rid of the pressure from the equation (1.23). We obtain $\frac{n}{2}(n-1)$ equations in $W^{-1,2}(\Omega)^n$. Not all of them are useful for us. We put $j = n$ and $\alpha < n$, so

$$\sum_{k=1}^{n} (\partial_\alpha \partial_k \mathcal{S}_{nk} - \partial_n \partial_k \mathcal{S}_{\alpha k}) - \partial_\alpha f_n + \partial_n f_\alpha = 0, \quad \alpha = 1, \ldots, n-1. \tag{2.14}$$

Set $G_\alpha \equiv \partial_n \mathcal{S}_{\alpha n}$ for $\alpha \in \{1, \ldots, n-1\}$. It holds that

$$\|\xi G_\alpha\|_{-1,2} \le C \|\mathcal{S}_{\alpha n}\|_2 \le C\|Du\|_2 \le C.$$

Further, thanks to (1.28) we have for $k \in \{1, \ldots, n-1\}$

$$\|\partial_k(\xi G_\alpha)\|_{-1,2} \le C + \|\partial_k \mathcal{S}_{\alpha n}\|_2 \le C + C' \sum_{i=1}^{n} \|\partial_k \partial_i u\|_2.$$

From the equation (2.14) we obtain

$$\|\partial_n(\xi G_\alpha)\|_{-1,2} \le C + C' \sum_{i=1}^{n} \sum_{k=1}^{n-1} \|\partial_k \partial_i u\|_2.$$

Now we can use the Nečas' theorem on negative norms, see Theorem A.5.1, to get

$$\|\xi G_\alpha\|_2 \le C(\|\xi G_\alpha\|_{-1,2} + \|\nabla(\xi G_\alpha)\|_{-1,2}) \le C + C' \sum_{i=1}^{n} \sum_{k=1}^{n-1} \|\partial_k \partial_i u\|_2. \tag{2.15}$$

Recalling the definition of G_α we have $n-1$ equations for $\alpha \in \{1, \ldots, n-1\}$

$$G_\alpha = \sum_{i,j=1}^{n} \partial_{ij} \mathcal{S}_{\alpha n} \partial_n D_{ij} u. \tag{2.16}$$

[2]The process is based on testing the equation (1.23) by function $\operatorname{rot} g$. The result is the same, i.e. we get the equation (2.14) in distributional sense.

Symmetry of $\mathcal{S}$ and Du gives us $\sum_{j=1}^{n-1} \partial_{nj}\mathcal{S}_{\alpha n}\partial_n D_{nj}u = \sum_{i=1}^{n-1} \partial_{in}\mathcal{S}_{\alpha n}\partial_n D_{in}u$ and therefore from (2.16) we obtain

$$2\sum_{j=1}^{n-1} \partial_{nj}\mathcal{S}_{\alpha n}\partial_n D_{nj}u = G_\alpha - \sum_{i,j=1}^{n-1} \partial_{ij}\mathcal{S}_{\alpha n}\partial_n D_{ij}u - \partial_{nn}\mathcal{S}_{\alpha n}\partial_n D_{nn}u. \tag{2.17}$$

In the first and third term of (2.17) we use the definition of Du. Instead of the last term of (2.17) we can write $\sum_{j=1}^{n-1} \partial_{nn}\mathcal{S}_{\alpha n}\partial_n\partial_j u_j$, because $D_{nn}u = \partial_n u_n$ and the equation $\operatorname{div} u = 0$ gives us $\partial_n u_n = -\sum_{j=1}^{n-1} \partial_j u_j$. We have

$$\begin{aligned}\sum_{j=1}^{n-1} \partial_{nj}\mathcal{S}_{\alpha n}\partial_n^2 u_j = G_\alpha &- \frac{1}{2}\sum_{i,j=1}^{n-1} \partial_{ij}\mathcal{S}_{\alpha n}\partial_n(\partial_i u_j + \partial_j u_i) \\ &- \sum_{j=1}^{n-1} \partial_{nj}\mathcal{S}_{\alpha n}\partial_n\partial_j u_n + \sum_{j=1}^{n-1} \partial_{nn}\mathcal{S}_{\alpha n}\partial_n\partial_j u_j.\end{aligned} \tag{2.18}$$

Thanks to the Corollary 1.4.5 we know that the matrix $\mathbb{A} := (\partial_{nj}\mathcal{S}_{\alpha n})$ is regular. We can multiply (2.18) by $\xi^2\partial_n^2 u_\alpha$, where the cut-off function $\xi(x)$ is defined as in (2.6), sum over α and integrate over Ω. Using (2.15) we conclude that

$$\sum_{j=1}^{n-1} \|\xi\partial_n^2 u_j\|_2^2 \le C + C'\sum_{i=1}^{n}\sum_{k=1}^{n-1} \|\partial_i\partial_k u\|_2^2. \tag{2.19}$$

From the relation (2.19) using the definition of the tangent derivative and from relation (2.13) we derive

$$\begin{aligned}\sum_{j=1}^{n} \|\xi\partial_n^2 u_j\|_2^2 \le C + C'&\sum_{i,j=1}^{n}\sum_{\alpha=1}^{n-1} \|\partial_{\tau^\alpha}\partial_i u_j\|_2^2 + \\ &+ C'' \sup_{x'\in(-R_0,R_0)} \sum_{\alpha=1}^{n-1} |\partial_\alpha a(x')| \sum_{j=1}^{n} \|\xi\partial_n^2 u_j\|_2^2.\end{aligned} \tag{2.20}$$

If we choose R_0 in order to have

$$C'' \max_P \sup_{x'\in(-R_0,R_0)} \sum_{\alpha=1}^{n-1} |\partial_\alpha a(x')| \le \frac{1}{2},$$

we can move the last term in (2.20) to the left hand side and together with tangent direction and interior regularity we get $u \in W^{2,2}(\Omega)^n$. □

Step 3 *Reconstruction of the pressure*

In virtue of Lemma 2.2.1 we know that (1.23) holds almost everywhere. It helps us to reconstruct the pressure in a simple way. In the weak formulation in Definition 5.1.1 there is no pressure π, because we considered divergence-free test function. Now we are interested in the question whether there is $\pi \in \mathcal{D}'(\Omega)^n$ such that

$$\int_\Omega \mathcal{S}(Du) : D\varphi \,\mathrm{d}x + \langle \nabla\pi, \varphi\rangle = \langle f, \varphi\rangle$$

holds for all $\varphi \in \mathcal{D}(\Omega)^n$.

From De Rham's theorem, see Theorem A.5.2, we know that the right hand side of the equation

$$\nabla\pi = f + \operatorname{div}\mathcal{S} \tag{2.21}$$

can be written in the gradient form. The equation (2.21) holds almost everywhere and it's right hand side is in $L^2(\Omega)^n$. Thus, with the additional assumption $\int_\Omega \pi \,\mathrm{d}x = 0$ we get that there exists the pressure $\pi \in W^{1,2}(\Omega)^n$.

□

2.3 The regularized problem

In this section we are concerned with the regularized boundary value problem

$$-\operatorname{div}\mathcal{S}^\varepsilon(Du^{e\varepsilon}) + \nabla\pi^{e\varepsilon} = f^e \qquad \text{in } \Omega, \tag{2.22}$$

$$\operatorname{div} u^{e\varepsilon} = 0 \qquad \text{in } \Omega, \tag{2.23}$$

$$u^{e\varepsilon}\cdot\nu = 0, \quad [\mathcal{S}^\varepsilon(Du^{e\varepsilon})\nu]\cdot\tau^\alpha = 0 \qquad \text{on } \partial\Omega, \tag{2.24}$$

where the regularization of f is chosen in order to have $f^e \in \mathcal{C}^\infty(\bar{\Omega})^n$ and $f^e \to f$ in $W^{1,\Phi^*}(\Omega)^n$ as $e \to 0$ and

$$\begin{aligned} \mathcal{S}^\varepsilon &= \mu^\varepsilon(|Du^{e\varepsilon}|)Du^{e\varepsilon}, \\ \mu^\varepsilon &= \min\Big(\max(\mu(|Du^{e\varepsilon}|), \varepsilon), \frac{1}{\varepsilon}\Big), \quad \varepsilon \in (0,1). \end{aligned} \tag{2.25}$$

Scalar potential Φ^ε to $\mathcal{S}^\varepsilon$ is

$$\Phi^\varepsilon(s) := \int_0^s \mu^\varepsilon(t)t \,\mathrm{d}t.$$

As one can easily check, the Assumption 1.4.3 and therefore also growth conditions Corollary 1.4.5 hold if we replace Φ and $\mathcal{S}$ by Φ^ε and $\mathcal{S}^\varepsilon$.

Proposition 2.3.1 *If $\Phi \in \Delta_2$, $\Phi^* \in \Delta_2$ then also $\Phi^\varepsilon \in \Delta_2$, $(\Phi^\varepsilon)^* \in \Delta_2$ and $\Delta_2(\{\Phi^\varepsilon, (\Phi^\varepsilon)^*\})$ doesn't depend od ε.*

Proof. At first we consider only truncation of μ from below, i.e.

$$\mu^\varepsilon(s) := \max(\mu^\varepsilon(s), \varepsilon), \quad \Phi^\varepsilon(s) = \int_0^s \mu^\varepsilon(t)t \,\mathrm{d}t.$$

It is enough to show $\mu^\varepsilon(2s) \le C\mu^\varepsilon(s)$. We can distinguish four cases:

(i) $\mu(s) \geq \varepsilon$ and $\mu(2s) \geq \varepsilon$. Then $\mu^\varepsilon(s) = \mu(s)$, $\mu^\varepsilon(2s) = \mu(2s)$ and the claim holds, because $\Phi \in \Delta_2$.

(ii) $\mu(s) \geq \varepsilon$ and $\mu(2s) \leq \varepsilon$. Then $\mu^\varepsilon(s) = \mu(s) \geq \varepsilon$ and $\mu^\varepsilon(2s) = \varepsilon$. Trivially $\mu^\varepsilon(2s) \leq \mu^\varepsilon(s)$.

(iii) $\mu(s) \leq \varepsilon$ and $\mu(2s) \geq \varepsilon$. Then $\mu^\varepsilon(s) = \varepsilon$ and $\mu^\varepsilon(2s) = \mu(2s)$, $\mu(2s) \leq C\mu(s) \leq C\varepsilon \leq C\mu^\varepsilon(s)$ and the claim holds.

(iv) $\mu(s) \leq \varepsilon$ and $\mu(2s) \leq \varepsilon$. Then $\mu^\varepsilon(s) = \mu^\varepsilon(2s) = \varepsilon$ and the claim trivially holds.

The same is true for μ truncated from above, i.e. $\mu^\varepsilon(s) := \min(\mu(s), \frac{1}{\varepsilon})$. Putting these two considerations together we obtain the proof for the viscosity defined in (3.2). In the case of the complementary function one proceeds similarly. □

The first apriori estimate, i.e. testing by the weak solution, gives us

$$\int_\Omega \mathcal{S}^\varepsilon(Du^{e\varepsilon}) : Du^{e\varepsilon}\,\mathrm{d}x \leq \int_\Omega f^e \cdot u^{e\varepsilon}\,\mathrm{d}x. \tag{2.26}$$

Using Young's inequality (A.8) and Korn's inequality (A.20) on the right hand side, definition of function V^ε or the potential Φ^ε on the left hand side, the relation (2.26) can be rewritten in the form

$$C\int_\Omega |V^\varepsilon(Du^{e\varepsilon})|^2\,\mathrm{d}x \leq \int_\Omega \Phi^\varepsilon(|Du^{e\varepsilon}|)\,\mathrm{d}x \leq C\int_\Omega (\Phi^\varepsilon)^*(|f^e|)\,\mathrm{d}x =: \mathcal{M}_{ap}. \tag{2.27}$$

Now we fix a point P and work in the local system of coordinates for which $P = 0$. We work in $\Omega^P_{R_0}$, but as before, we will drop the index P. For simplicity we denote $r := \frac{R_0}{4}$. The following lemma shows us that the integral $\int_{\Omega_r} \mu^\varepsilon(|D(u^{e\varepsilon})|)|\partial_n D(u^{e\varepsilon})|^2\,\mathrm{d}x$ can be estimated by lower order terms and small terms of the same order which can be subsumed into the left hand side at the end.

Lemma 2.3.2 *Let $\Omega \subset \mathbb{R}^n$, $\Omega \in \mathcal{C}^3$ be a bounded non-axisymmetric domain. Let $u^{e\varepsilon}$ be the weak solution of the regularized problem* (2.22), (2.23) *and* (2.24). *Then there exist positive constants C and $\tilde{c}_i$, $i = 1, \ldots, 4$, independent of $u^{e\varepsilon}$ such that*

$$\begin{aligned}\int_{\Omega_r} \mu^\varepsilon(|Du^{e\varepsilon}|)|\partial_n Du^{e\varepsilon}|^2\,\mathrm{d}x \leq& C\Big(\int_{\Omega_{3r}} (\Phi^\varepsilon)^*(|f^e|)\,\mathrm{d}x \\ &+ \int_{\Omega_{3r}} (\Phi^\varepsilon)^*(|\nabla f^e|)\,\mathrm{d}x\Big) + \sum_{i=1}^4 \mathcal{M}_i,\end{aligned} \tag{2.28}$$

where

$$\mathcal{M}_1 = \tilde{c}_1 \sum_{i,j,k,l,m=1}^{n} \Big| \int_\Omega \mathcal{S}^\varepsilon_{ij} \partial_k \partial_l u^{e\varepsilon}_m \xi \, \mathrm{d}x \Big|, \quad \mathcal{M}_2 = \tilde{c}_2 \sum_{i,j,k,l=1}^{n} \Big| \int_\Omega \partial_i \mathcal{S}^\varepsilon_{jk} u^{e\varepsilon}_l \xi \, \mathrm{d}x \Big|,$$

$$\mathcal{M}_3 = \tilde{c}_3 \sum_{i,j,k,l,m=1}^{n} \Big| \int_\Omega \partial_i \mathcal{S}^\varepsilon_{jk} \partial_l u^{e\varepsilon}_m \xi \, \mathrm{d}x \Big|, \quad \mathcal{M}_4 = \tilde{c}_4 a_0 \int_{\Omega_{3r}} \mu^\varepsilon(|Du^{e\varepsilon}|) |\nabla^2 u^{e\varepsilon}|^2 \xi^2 \, \mathrm{d}x.$$

Constants $\tilde{c}_i$, $i = 1, 2, 3$, may depend on a_0 and c_0 defined in (A.1)*, but constant $\tilde{c}_4$ is an absolute one.*

Remark 2.3.3 *Terms $\mathcal{M}_i$, $i = 1, \ldots, 3$, are of lower order. Instead of the cut-off function ξ we should write $\tilde{\xi}$, where* $\operatorname{supp} \tilde{\xi} \subset \operatorname{supp} \xi$ *and in $\tilde{\xi}$ derivatives of ξ and a are included. Since this difference is not important, we write only ξ.*

The term $\mathcal{M}_4$ is of the same order as the left hand side of (2.28) *but it is also multiplied by a_0. We can pick this constant as small as we want what allows us to subsume $\mathcal{M}_4$ later into the left hand side of* (2.28)*.*

Proof of Lemma 2.3.2*.* From results of Section 2.2 we know that $u^{e\varepsilon} \in W^{2,2}(\Omega)^n$, $\mathcal{S}^\varepsilon(Du^{e\varepsilon}) \in W^{1,2}(\Omega)^{n \times n}$, $\pi^{e\varepsilon} \in W^{1,2}(\Omega)$. We can rewrite (2.22) into components, multiply by a suitable test function and integrate over Ω_{3r}. The test function has to belong at least to $L^2(\Omega)^n$ in order the integrals had sense.

$$- \sum_{k,l=1}^{n} \int_{\Omega_{3r}} \partial_l \mathcal{S}^\varepsilon_{kl} \varphi_k \, \mathrm{d}x + \sum_{k=1}^{n} \int_{\Omega_{3r}} \partial_k \pi^{e\varepsilon} \varphi_k \, \mathrm{d}x = \sum_{k=1}^{n} \int_{\Omega_{3r}} f^e_k \varphi_k \, \mathrm{d}x, \tag{2.29}$$

where $\varphi \in L^2(\Omega)^n$. We would like to use the second normal derivatives of the solution as a test function in (2.29). One can easily verify that this function is not divergence free and does not fulfill boundary conditions, so we would have to deal with terms containing the pressure. Instead we take as a test function

$$\varphi = \Big(\partial_n \Theta_1, \ldots, \partial_n \Theta_{n-1}, - \sum_{\alpha=1}^{n-1} \partial_\alpha \Theta_\alpha \Big), \tag{2.30}$$

where we denoted

$$\Theta_\alpha := \partial_\nu (u^{e\varepsilon} \cdot \tau^\alpha) \xi^2 - u^{e\varepsilon} \cdot (\partial_\nu \tau^\alpha + \partial_{\tau^\alpha} \nu) \xi^2, \quad \alpha = 1, \ldots, n-1. \tag{2.31}$$

The test function φ is constructed in order to fulfill $\operatorname{div} \varphi = 0$. It also has a useful property:

Proposition 2.3.4 $\Theta_\alpha = 0$ *for all $\alpha \in \{1, \ldots, n-1\}$ on $\partial\Omega$ in the sense of traces.*

Proof. We use two facts. First, $\partial_{\tau^\alpha}(u^{e\varepsilon} \cdot \nu) = 0$ on $\partial\Omega$, which is $u^{e\varepsilon} \cdot \partial_{\tau^\alpha} \nu = -\partial_{\tau^\alpha} u^{e\varepsilon} \cdot \nu$. Second, boundary conditions $(\mathcal{S}^\varepsilon \nu) \cdot \tau^\alpha = 0$ can be rewritten as $\partial_{\tau^\alpha} u^{e\varepsilon} \cdot \nu + \partial_\nu u^{e\varepsilon} \cdot \tau^\alpha = 0$ which gives us $\partial_\nu u^{e\varepsilon} \cdot \tau^\alpha = -\partial_{\tau^\alpha} u^{e\varepsilon} \cdot \nu$. Thus

$$\begin{aligned} \Theta_\alpha &= \partial_\nu u^{e\varepsilon} \tau^\alpha \xi^2 + u^{e\varepsilon} \partial_\nu \tau^\alpha \xi^2 - u^{e\varepsilon} \xi^2 \partial_\nu \tau^\alpha - u^{e\varepsilon} \xi^2 \partial_{\tau^\alpha} \nu \\ &= -\partial_{\tau^\alpha} u^{e\varepsilon} \nu \xi^2 - u^{e\varepsilon} \xi^2 \partial_{\tau^\alpha} \nu = u^{e\varepsilon} \partial_{\tau^\alpha} \nu \xi^2 - u^{e\varepsilon} \xi^2 \partial_{\tau^\alpha} \nu = 0. \end{aligned}$$

□

Proposition 2.3.4 helps us to get rid of terms in (2.29) containing pressure. In the case we are not on $\partial\Omega$, it is useful to write out Θ_α.

$$\begin{aligned}\Theta_\alpha = &\sum_{i,j=1}^{n} (\partial_i u_j^{e\varepsilon} \tau_j^\alpha \nu_i - u_i^{e\varepsilon} \partial_j \nu_i \tau_j^\alpha)\xi^2 = -\partial_n u_\alpha^{e\varepsilon}\xi^2 - \partial_\alpha a \partial_n u_n^{e\varepsilon}\xi^2 \\ &+ \sum_{\beta=1}^{n-1} \Big(\partial_\beta a (\partial_\beta u_\alpha^{e\varepsilon} + \partial_\alpha a \partial_\beta u_n^{e\varepsilon}) - \partial_\alpha \partial_\beta a u_\beta^{e\varepsilon}\Big)\xi^2,\end{aligned} \tag{2.32}$$

where we use only the definition of the normal, the normal derivative, the tangent and the tangent derivative.

Proposition 2.3.5 *Let φ be defined by* (2.30) *and* (2.31)*. Then*

$$\sum_{k=1}^{n} \int_{\Omega_{3r}} \partial_k \pi^{e\varepsilon} \varphi_k \,\mathrm{d}x = 0.$$

Proof. If $\pi^{e\varepsilon} \in W^{2,2}(\Omega)$, a straightforward computation gives us

$$\sum_{k=1}^{n} \int_{\Omega_{3r}} \partial_k \pi^{e\varepsilon} \varphi_k \,\mathrm{d}x = \sum_{\alpha=1}^{n-1} \int_{\Omega_{3r}} \partial_\alpha \pi^{e\varepsilon} \partial_n \Theta_\alpha \,\mathrm{d}x - \int_{\Omega_{3r}} \partial_n \pi^{e\varepsilon} \sum_{\alpha=1}^{n-1} \partial_\alpha \Theta_\alpha \,\mathrm{d}x = 0,$$

where we integrated by parts twice and use the fact that boundary integrals are equal to zero because $\Theta_\alpha = 0$ on $\partial\Omega$ due to Proposition 2.3.6. For $\pi^{e\varepsilon} \in W^{1,2}(\Omega)$ the statement follows from density of $W^{2,2}(\Omega)$ in $W^{1,2}(\Omega)$. □

For simplicity let us denote

$$\begin{aligned}\mathcal{A}_{kl} &:= -\int_{\Omega_{3r}} \partial_l \mathcal{S}_{kl}^\varepsilon \varphi_k \,\mathrm{d}x, \quad k,l = 1,\dots,n, \\ \mathcal{B}_{kl}^j &:= \int_{\Omega_{3r}} \partial_j \mathcal{S}_{kl}^\varepsilon \partial_j \partial_l u_k^{e\varepsilon} \xi^2 \,\mathrm{d}x, \quad j,k,l = 1,\dots,n.\end{aligned}$$

We put (2.30) into (2.29) and estimate terms $\mathcal{A}_{kl}$, $k,l = 1,\dots,n$. Our goal is to obtain terms $\mathcal{B}_{kl}^n$ on the left hand side of (2.29). It will be done in four steps. First we focus on $\mathcal{A}_{\alpha\beta}$ for fixed $\alpha,\beta \in \{1,\dots,n-1\}$. Later we estimate $\mathcal{A}_{\alpha n}$, $\mathcal{A}_{n\beta}$ and finally $\mathcal{A}_{nn}$.

In the first term we integrate by parts twice, use the fact that there are no boundary terms and apply (2.32).

$$\begin{aligned}\mathcal{A}_{\alpha\beta} &= \int_{\Omega_{3r}} \partial_\beta \mathcal{S}_{\alpha\beta}^\varepsilon \varphi_\alpha \,\mathrm{d}x = -\int_{\Omega_{3r}} \partial_\beta \mathcal{S}_{\alpha\beta}^\varepsilon \partial_n \Theta_\alpha \,\mathrm{d}x = -\int_{\Omega_{3r}} \partial_n \mathcal{S}_{\alpha\beta}^\varepsilon \partial_\beta \Theta_\alpha \,\mathrm{d}x \\ &= \int_{\Omega_{3r}} \partial_n \mathcal{S}_{\alpha\beta}^\varepsilon \partial_\beta \Big[\partial_\alpha a \partial_n u_n^{e\varepsilon}\xi^2 - \sum_{\beta=1}^{n-1} \Big(\partial_\beta a (\partial_\beta u_\alpha^{e\varepsilon} + \partial_\alpha a \partial_\beta u_n^{e\varepsilon}) - \partial_\alpha \partial_\beta a u_\beta^{e\varepsilon}\Big)\xi^2\Big] \\ &+ \int_{\Omega_{3r}} \partial_n \mathcal{S}_{\alpha\beta}^\varepsilon \partial_\beta \partial_n u_\alpha^{e\varepsilon} \xi^2 \,\mathrm{d}x \geq \mathcal{B}_{\alpha\beta}^n - \sum_{i=1}^{4} \mathcal{M}_i.\end{aligned}$$

In the second term $\mathcal{A}_{\alpha n}$ we use only (2.32) to get $\mathcal{B}^n_{\alpha n}$.

$$\begin{aligned}\mathcal{A}_{\alpha n} &= -\int_{\Omega_{3r}} \partial_n \mathcal{S}^\varepsilon_{\alpha n}\varphi_\alpha \,\mathrm{d}x = -\int_{\Omega_{3r}} \partial_n \mathcal{S}^\varepsilon_{\alpha n}\partial_n \Theta_\alpha \,\mathrm{d}x \\ &\geq \int_{\Omega_{3r}} \partial_n \mathcal{S}^\varepsilon_{\alpha n}\partial_n^2 u^{e\varepsilon}_\alpha \xi^2 \,\mathrm{d}x - \sum_{i=1}^4 \mathcal{M}_i = \mathcal{B}^n_{\alpha n} - \sum_{i=1}^4 \mathcal{M}_i.\end{aligned}$$

At the beginning of the extraction of the term $\mathcal{B}^n_{n\beta}$ from $\mathcal{A}_{n\beta}$ we use (2.32) and $\partial_n u^{e\varepsilon}_n = -\sum_{\alpha=1}^{n-1}\partial_\alpha u^{e\varepsilon}_\alpha$, which comes from the divergence-free constraint:

$$\begin{aligned}\mathcal{A}_{n\beta} &= -\int_{\Omega_{3r}} \partial_\beta \mathcal{S}^\varepsilon_{n\beta}\varphi_n \,\mathrm{d}x = \int_{\Omega_{3r}} \partial_\beta \mathcal{S}^\varepsilon_{n\beta}\sum_{\alpha=1}^{n-1}\partial_\alpha \Theta_\alpha \,\mathrm{d}x \\ &\geq -\int_{\Omega_{3r}} \partial_\beta \mathcal{S}^\varepsilon_{n\beta}\sum_{\alpha=1}^{n-1}\partial_\alpha\partial_n u^{e\varepsilon}_\alpha \xi^2 \,\mathrm{d}x - \sum_{i=1}^4 \mathcal{M}_i \\ &= \int_{\Omega_{3r}} \partial_\beta \mathcal{S}^\varepsilon_{n\beta}\partial_n^2 u^{e\varepsilon}_n \xi^2 \,\mathrm{d}x - \sum_{i=1}^4 \mathcal{M}_i = \tilde{\mathcal{B}} - \sum_{i=1}^4 \mathcal{M}_i.\end{aligned}$$

If we integrate by parts twice $\tilde{\mathcal{B}}$, we are done. At this moment there would appear boundary integrals. To avoid them we add and subtract some small terms (which could be included in $\mathcal{M}_4$) in order to have $(\mathcal{S}^\varepsilon\nu)\cdot\tau^\beta$ instead of $\mathcal{S}^\varepsilon_{n\beta}$ in $\tilde{\mathcal{B}}$. Writing out $(\mathcal{S}^\varepsilon\nu)\cdot\tau^\beta$ we get

$$\mathcal{S}^\varepsilon_{n\beta} = -(\mathcal{S}^\varepsilon\nu)\cdot\tau^\beta + \sum_{\alpha=1}^{n-1}\partial_\alpha a \mathcal{S}^\varepsilon_{\alpha\beta} + \partial_\alpha a\partial_\beta a \mathcal{S}^\varepsilon_{\alpha n} - \partial_\beta a \mathcal{S}^\varepsilon_{nn}.$$

Therefore

$$\begin{aligned}\tilde{\mathcal{B}} &= \int_{\Omega_{3r}} \partial_\beta \mathcal{S}^\varepsilon_{n\beta}\partial_n^2 u^{e\varepsilon}_n \xi^2 \,\mathrm{d}x \geq -\int_{\Omega_{3r}} \partial_\beta[(\mathcal{S}^\varepsilon\nu)\tau^\beta]\partial_n^2 u^{e\varepsilon}_n \xi^2 \,\mathrm{d}x - \sum_{i=1}^4 \mathcal{M}_i \\ &= -\int_{\Omega_{3r}} \partial_n[(\mathcal{S}^\varepsilon\nu)\tau^\beta]\partial_\beta\partial_n u^{e\varepsilon}_n \xi^2 \,\mathrm{d}x - \sum_{i=1}^4 \mathcal{M}_i \\ &\geq \int_{\Omega_{3r}} \partial_n \mathcal{S}^\varepsilon_{n\beta}\partial_n\partial_\beta u^{e\varepsilon}_n \xi^2 \,\mathrm{d}x - \sum_{i=1}^4 \mathcal{M}_i = \mathcal{B}^n_{n\beta} - \sum_{i=1}^4 \mathcal{M}_i.\end{aligned}$$

In the last term $\mathcal{A}_{nn}$ we use only (2.32) and the incompressibility condition $\partial_n u^{e\varepsilon}_n = -\sum_{\alpha=1}^{n-1}\partial_\alpha u^{e\varepsilon}_\alpha$.

$$\begin{aligned}\mathcal{A}_{nn} &= -\int_{\Omega_{3r}} \partial_n \mathcal{S}^\varepsilon_{nn}\varphi_n \,\mathrm{d}x \geq \int_{\Omega_{3r}} \partial_n \mathcal{S}^\varepsilon_{nn}\sum_{\alpha=1}^{n-1}\partial_\alpha\partial_n u^{e\varepsilon}_\alpha \xi^2 \,\mathrm{d}x - \sum_{i=1}^4 \mathcal{M}_i \\ &= \int_{\Omega_{3r}} \partial_n \mathcal{S}^\varepsilon_{nn}\partial_n^2 u^{e\varepsilon}_n \xi^2 \,\mathrm{d}x - \sum_{i=1}^4 \mathcal{M}_i = \mathcal{B}^n_{nn} - \sum_{i=1}^4 \mathcal{M}_i.\end{aligned}$$

$$\sum_{k=1}^{n}\int_{\Omega_{3r}} f_k^e\varphi_k\,\mathrm{d}x \le \int_{\Omega_{3r}} |\nabla f^e||\Theta_\alpha|\,\mathrm{d}x \\ \le C\Big(\int_{\Omega_{3r}} (\Phi^\varepsilon)^*(|\nabla f^e|)\,\mathrm{d}x + \int_{\Omega_{3r}} \Phi^\varepsilon(|Du^{e\varepsilon}|)\,\mathrm{d}x\Big). \tag{2.33}$$

Collecting all estimates and using the first apriori estimate (2.27) in (2.33) we obtain

$$\int_{\Omega_r} \mu^\varepsilon(|Du^{e\varepsilon}|)|\partial_n Du^{e\varepsilon}|^2\,\mathrm{d}x \le \sum_{k,l=1}^{n}\int_{\Omega_{3r}} \partial_n\mathcal{S}_{kl}^\varepsilon\partial_n\partial_l u_k^{e\varepsilon}\xi^2\,\mathrm{d}x = \sum_{k,l=1}^{n}\mathcal{B}_{kl}^n \\ \le C\Big(\int_{\Omega_{3r}} (\Phi^\varepsilon)^*(|f^e|)\,\mathrm{d}x + \int_{\Omega_{3r}} (\Phi^\varepsilon)^*(|\nabla f^e|)\,\mathrm{d}x\Big) + \sum_{i=1}^{4}\mathcal{M}_i,$$

which concludes the proof. □

Now we formulate lemma about boundedness of the term containing "tangential parts" of the second gradient.

Lemma 2.3.6 *Let $\Omega \subset \mathbb{R}^n$, $\Omega \in \mathcal{C}^3$ be a bounded non-axisymmetric domain. Let $u^{e\varepsilon}$ be the weak solution of the regularized problem* (2.22), (2.23) *and* (2.24)*. Then there exist positive constants C and $\tilde{c}_i$, $i = 1,\dots,5$, independent of $u^{e\varepsilon}$, such that for all $\alpha \in \{1,\dots,n-1\}$ holds*

$$\int_{\Omega_r} \mu^\varepsilon(|Du^{e\varepsilon}|)|\partial_\alpha Du^{e\varepsilon}|^2\,\mathrm{d}x \le C\Big(\int_{\Omega_{3r}} (\Phi^\varepsilon)^*(|f^e|)\,\mathrm{d}x \\ + \int_{\Omega_{3r}} (\Phi^\varepsilon)^*(|\nabla f^e|)\,\mathrm{d}x\Big) + \sum_{i=1}^{5}\mathcal{M}_i, \tag{2.34}$$

where $\mathcal{M}_i$ with $\tilde{c}_i$, $i = 1,\dots,4$ are defined in Lemma 2.3.2 *and the term $\mathcal{M}_5$ with absolute constant $\tilde{c}_5$ and small $\delta > 0$ is defined as*

$$\mathcal{M}_5 = \tilde{c}_5\delta\int_{\Omega} \mu^\varepsilon(|Du^{e\varepsilon}|)|\nabla^2 u^{e\varepsilon}|^2\,\mathrm{d}x.$$

Remark 2.3.7 *In contrary to Lemma* 2.3.2, *in the estimate* (2.34) *there appeared the term $\mathcal{M}_5$. It can be described as "small term of the same order as the left hand side of* (2.34)*". $\mathcal{M}_5$ comes from the usage of Bogovskiĭ's Lemma* A.3.3, *therefore it has bigger support than $\mathcal{M}_4$. Hence, we work on the whole Ω instead of Ω_{3r}. Smallness of $\mathcal{M}_5$ is provided by Young's inequality with $\delta > 0$, not by the presence of a_0.*

Proof. Let's take $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ such that $\operatorname{supp}\varphi \subset \overline{\Omega_{3r}}$, $\operatorname{div}\varphi = 0$ and $\varphi\cdot\nu = 0$ at $\partial\Omega$. We test (2.22) by $-\partial_{\tau^\alpha}\varphi$ and obtain

$$\int_{\Omega_{3r}} \operatorname{div}\mathcal{S}^\varepsilon(Du^{e\varepsilon})\cdot\partial_{\tau^\alpha}\varphi\,\mathrm{d}x - \int_{\Omega_{3r}} \nabla\pi^{e\varepsilon}\partial_{\tau^\alpha}\varphi\,\mathrm{d}x = -\int_{\Omega_{3r}} f^e\partial_{\tau^\alpha}\varphi\,\mathrm{d}x. \tag{2.35}$$

In order to let the boundary term vanish while integrating by parts in the first term of (2.35), we add and subtract

$$\int_{\Omega_{3r}} \operatorname{div} \mathcal{S}^\varepsilon(Du^{e\varepsilon}) \cdot \Big(\varphi \cdot \partial_{\tau^\alpha} \nu \frac{\nu}{|\nu|^2}\Big) \, \mathrm{d}x$$

in the equation (2.35). We get

$$\begin{aligned}\int_{\Omega_{3r}} \operatorname{div} \mathcal{S}^\varepsilon(Du^{e\varepsilon}) \cdot \Big(\partial_{\tau^\alpha}\varphi + \varphi \cdot \partial_{\tau^\alpha} \nu \frac{\nu}{|\nu|^2}\Big) \, \mathrm{d}x &= -\int_{\Omega_{3r}} \mathcal{S}^\varepsilon(Du^{e\varepsilon}) : \nabla \partial_{\tau^\alpha} \varphi \, \mathrm{d}x \\ - \int_{\Omega_{3r}} \mathcal{S}^\varepsilon(Du^{e\varepsilon}) : \nabla\Big(\varphi \cdot \partial_{\tau^\alpha} \nu \frac{\nu}{|\nu|^2}\Big) \, \mathrm{d}x &+ \int_{\partial\Omega_{3r}} \mathcal{S}^\varepsilon(Du^{e\varepsilon}) : (\psi \otimes \nu) \, \mathrm{d}\sigma,\end{aligned} \tag{2.36}$$

where $\psi := \partial_{\tau^\alpha}\varphi + \varphi \cdot \partial_{\tau^\alpha}\nu \frac{\nu}{|\nu|^2}$. We observe that $\psi \cdot \nu = 0$ on $\partial\Omega$, because $\psi \cdot \nu = \partial_{\tau^\alpha}\varphi \cdot \nu + \varphi \cdot \partial_{\tau^\alpha}\nu = \partial_{\tau^\alpha}(\varphi \cdot \nu) = 0$ on $\partial\Omega$. On $\partial\Omega_{3r} \setminus \partial\Omega$ it is clear. Therefore due to the condition $(\mathcal{S}^\varepsilon \nu) \cdot \tau^\alpha = 0$ at $\partial\Omega$ we realize that the boundary integral in (2.36) is equal to zero.

Now we would like to integrate by parts in tangent direction in the first term on the right hand side of (2.36). The tangent derivative doesn't commute with the gradient, but it holds that

$$\nabla \partial_{\tau^\alpha} \varphi = \partial_{\tau^\alpha} \nabla \varphi + \nabla \partial_\alpha a \otimes \partial_n \varphi. \tag{2.37}$$

With the help of this identity and Remark A.1.1 we obtain

$$\begin{aligned}-\int_{\Omega_{3r}} \mathcal{S}^\varepsilon(Du^{e\varepsilon}) : \nabla \partial_{\tau^\alpha} \varphi \, \mathrm{d}x &= \int_{\Omega_{3r}} \partial_{\tau^\alpha} \mathcal{S}^\varepsilon(Du^{e\varepsilon}) : \nabla \varphi \, \mathrm{d}x \\ &- \int_{\Omega_{3r}} \mathcal{S}^\varepsilon(Du^{e\varepsilon}) : \nabla \partial_\alpha a \otimes \partial_n \varphi \, \mathrm{d}x.\end{aligned} \tag{2.38}$$

We modify the term in (2.35) which contains the pressure. If $\pi^{e\varepsilon} \in W^{2,2}(\Omega)$, we can integrate by parts in tangent direction, use the similar identity as (2.37), integrate by parts in spatial direction, use the divergence-free constraint and the equation (2.22) to replace $\partial_n \pi^{e\varepsilon}$ by $f_n + [\operatorname{div} \mathcal{S}^\varepsilon(Du^{e\varepsilon})]_n$:

$$\begin{aligned}-\int_{\Omega_{3r}} \nabla \pi^{e\varepsilon} \partial_{\tau^\alpha} \varphi \, \mathrm{d}x = \int_{\Omega_{3r}} \partial_{\tau^\alpha} \nabla \pi^{e\varepsilon} \varphi \, \mathrm{d}x &= \int_{\Omega_{3r}} \nabla \partial_{\tau^\alpha} \pi^{e\varepsilon} \varphi \, \mathrm{d}x \\ - \int_{\Omega_{3r}} \nabla \partial_\alpha a \partial_n \pi^{e\varepsilon} \varphi \, \mathrm{d}x = -\int_{\Omega_{3r}} \nabla \partial_\alpha a &[f_n + [\operatorname{div} \mathcal{S}^\varepsilon(Du^{e\varepsilon})]_n] \varphi \, \mathrm{d}x.\end{aligned} \tag{2.39}$$

For $\pi^{e\varepsilon} \in W^{1,2}(\Omega)$ the statement follows from density of $W^{2,2}(\Omega)$ in $W^{1,2}(\Omega)$. Using modifications (2.36), (2.38) and (2.39) in (2.35) we obtain

$$
\begin{aligned}
&\int_{\Omega_{3r}} \partial_{\tau^\alpha}\mathcal{S}^\varepsilon(Du^{e\varepsilon}) : D\varphi \,\mathrm{d}x = \int_{\Omega_{3r}} \partial_{\tau^\alpha} f^e \cdot \varphi \,\mathrm{d}x \\
&+ \int_{\Omega_{3r}} \mathcal{S}^\varepsilon(Du^{e\varepsilon}) : \Big[\nabla\partial_\alpha a \otimes \partial_n\varphi + \nabla\Big(\varphi\cdot\partial_{\tau^\alpha}\nu\frac{\nu}{|\nu|^2}\Big)\Big]\,\mathrm{d}x \\
&+ \int_{\Omega_{3r}} \operatorname{div}\mathcal{S}^\varepsilon(Du^{e\varepsilon})\cdot(\varphi\cdot\partial_{\tau^\alpha}\nu)\frac{\nu}{|\nu|^2}\,\mathrm{d}x + \int_{\Omega_{3r}} \nabla\partial_\alpha a[f_n + (\operatorname{div}\mathcal{S}^\varepsilon(Du^{e\varepsilon}))_n]\varphi\,\mathrm{d}x.
\end{aligned} \tag{2.40}
$$

The identity (2.40) remains valid for $\varphi \in W^{1,2}_\sigma(\Omega)^n$, $\operatorname{supp}\varphi \subset \overline{\Omega_{3r}}$. As a test function φ we take

$$
\varphi = \partial_{\tau^\alpha}u^{e\varepsilon}\xi^2 + (u^{e\varepsilon}\cdot\partial_{\tau^\alpha}\nu)\frac{\nu}{|\nu|^2}\xi^2 + z = \varphi_a + \varphi_b + z, \tag{2.41}
$$

where z is the solution to

$$
\operatorname{div} z = \operatorname{div}(-\varphi_a - \varphi_b) \qquad \text{in } \Omega_{3r}, \tag{2.42}
$$

$$
z = 0 \qquad \text{on } \partial\Omega_{3r}. \tag{2.43}
$$

The role of z is to ensure that $\operatorname{div}\varphi = 0$. One easily checks that $\varphi\cdot\nu = 0$ on $\partial\Omega$:

$$
\varphi\cdot\nu = (\varphi_a+\varphi_b)\cdot\nu = (\partial_{\tau^\alpha}u^{e\varepsilon})\cdot\nu\xi^2 + (u^{e\varepsilon}\cdot\partial_{\tau^\alpha}\nu)\xi^2 = \partial_{\tau^\alpha}(u^{e\varepsilon}\cdot\nu)\xi^2 = 0.
$$

Therefore the compatibility condition holds

$$
0 = \int_{\partial\Omega} z\cdot\nu\,\mathrm{d}\sigma = \int_\Omega \operatorname{div} z\,\mathrm{d}x = \int_\Omega \operatorname{div}(-\varphi_a - \varphi_b)\,\mathrm{d}x = -\int_{\partial\Omega}(\varphi_a+\varphi_b)\cdot\nu\,\mathrm{d}\sigma = 0
$$

and z solving (5.31) and (5.32) exists by Bogovskiĭ's Lemma A.3.3 and has the following properties:

$$
\int_\Omega \Phi(|z|)\,\mathrm{d}x + \int_\Omega \Phi(|\nabla z|)\,\mathrm{d}x \le C\int_\Omega \Phi(|Du^{e\varepsilon}|)\,\mathrm{d}x, \tag{2.44}
$$

$$
\begin{aligned}
\int_\Omega \Phi(|\nabla^2 z|)\,\mathrm{d}x &\le C\Big(\int_\Omega \Phi(|Du^{e\varepsilon}|)\,\mathrm{d}x + \int_\Omega \Phi(|\nabla^2 u^{e\varepsilon}| + |\nabla u^{e\varepsilon}| + |u^{e\varepsilon}|)\,\mathrm{d}x\Big) \\
&\le C\Big(\int_\Omega \Phi(|Du^{e\varepsilon}|)\,\mathrm{d}x + \int_\Omega \Phi(|\nabla^2 u^{e\varepsilon}|)\,\mathrm{d}x\Big)
\end{aligned} \tag{2.45}
$$

for arbitrary N-function Φ. To derive these properties we used $\operatorname{div} u^{e\varepsilon} = 0$, Korn's inequality (A.20), Δ_2-condition and convexity of N-function Φ. Inserting (2.41) into (2.40) and using Young's inequality we obtain

$$
\begin{aligned}
\int_{\Omega_{3r}} \mu^\varepsilon(|Du^{e\varepsilon}|)|\partial_\alpha Du^{e\varepsilon}|^2\xi^2\,\mathrm{d}x + \mathcal{J}_1 &\le C\Big(\int_{\Omega_{3r}} (\Phi^\varepsilon)^*(|f^e|)\,\mathrm{d}x \\
&+ \int_{\Omega_{3r}} (\Phi^\varepsilon)^*(|\nabla f^e|)\,\mathrm{d}x + \mathcal{J}_2 + \mathcal{J}_3\Big) + \sum_{i=1}^4 \mathcal{M}_i,
\end{aligned}
$$

where

$$\mathcal{J}_1 = \int_{\Omega_{3r}} \partial_{\tau^\alpha} \mathcal{S}^\varepsilon(Du^{e\varepsilon}) : Dz \,\mathrm{d}x, \quad \mathcal{J}_2 = \int_{\Omega_{3r}} \Phi^\varepsilon(|z|) \,\mathrm{d}x,$$

$$\mathcal{J}_3 = \int_{\Omega_{3r}} |\mathcal{S}^\varepsilon(Du^{e\varepsilon})|(|\nabla z| + |z|) \,\mathrm{d}x.$$

The terms of the form $\int_{\Omega_{3r}} \operatorname{div} \mathcal{S}^\varepsilon(Du^{e\varepsilon}).c(\partial_\alpha a, \partial_\alpha\partial_\beta a)z \,\mathrm{d}x$ can be estimated as $\mathcal{J}_3$ after integration by parts and using facts that $\Omega \in \mathcal{C}^3$ and $z = 0$ at $\partial\Omega$. It remains to estimate terms containing Bogovskiĭ's correction z. The term $\mathcal{J}_2$ is estimated directly by (2.44). The term $\mathcal{J}_3$ is handled by Young's inequality and (2.44):

$$\mathcal{J}_3 \leq \int_{\Omega_{3r}} (\Phi^\varepsilon)'(|Du^{e\varepsilon}|)(|z| + |\nabla z|) \,\mathrm{d}x \leq C\Big(\int_{\Omega_{3r}} \Phi^\varepsilon(|Du^{e\varepsilon}|) \,\mathrm{d}x + \int_{\Omega_{3r}} \Phi^\varepsilon(|z|) \,\mathrm{d}x + \int_{\Omega_{3r}} \Phi^\varepsilon(|\nabla z|) \,\mathrm{d}x \Big) \leq \mathcal{M}_{ap},$$

where $\mathcal{M}_{ap}$ is the constant from the first apriori estimate, see (2.27). To estimate $\mathcal{J}_1$ the assumption on almost monotonicity of Φ'' is needed. For almost increasing $(\Phi^\varepsilon)''$ we move $\mathcal{J}_1$ to the right hand side and apply Young's inequality:

$$|\mathcal{J}_1| \leq \delta \int_{\Omega_{3r}} (\Phi^\varepsilon)''(|Du^{e\varepsilon}|)|\nabla^2 u^{e\varepsilon}|^2 \,\mathrm{d}x + c(\delta) \int_{\Omega_{3r}} (\Phi^\varepsilon)''(|Du^{e\varepsilon}|)|\nabla z|^2 \,\mathrm{d}x \leq \mathcal{M}_5 + \mathcal{J}_{1.1}.$$

In the term $\mathcal{J}_{1.1}$ we use the fact that $(\Phi^\varepsilon)''$ is almost increasing, the definition of shifted N-function (A.11),

$$\int_{\Omega_{3r}} |V^\varepsilon(Du^{e\varepsilon}) - V^\varepsilon(\langle Du^{e\varepsilon}\rangle)|^2 \,\mathrm{d}x \leq C \int_{\Omega_{3r}} |V^\varepsilon(Du^{e\varepsilon}) - \langle V^\varepsilon(Du^{e\varepsilon})\rangle|^2 \,\mathrm{d}x,$$

see [23, Lemma 2.8], shift change (A.12) and (2.44).

$$\mathcal{J}_{1.1} \leq C \int_{\Omega_{3r}} (\Phi^\varepsilon)''(|Du^{e\varepsilon}| + |\nabla z|)|\nabla z|^2 \,\mathrm{d}x \leq C \int_{\Omega_{3r}} \Phi^\varepsilon_{|Du^{e\varepsilon}|}(|\nabla z|) \,\mathrm{d}x$$
$$\leq C \int_{\Omega_{3r}} \Phi^\varepsilon_{|\langle Du^{e\varepsilon}\rangle|}(|\nabla z|) \,\mathrm{d}x + C \int_{\Omega_{3r}} |V^\varepsilon(Du^{e\varepsilon}) - V^\varepsilon(\langle Du^{e\varepsilon}\rangle)|^2 \,\mathrm{d}x \leq \mathcal{M}_{ap}.$$

If $(\Phi^\varepsilon)''$ is almost decreasing we integrate in $\mathcal{J}_1$ by parts using Lemma A.1.1 and get:

$$-\mathcal{J}_1 = \int_{\Omega_{3r}} \mathcal{S}^\varepsilon(Du^{e\varepsilon}) : \partial_{\tau^\alpha} Dz \,\mathrm{d}x.$$

Using Young's inequality with $\Phi^\varepsilon_{|Du^{e\varepsilon}|}$ and the fact that

$$(\Phi^\varepsilon_{|Du^{e\varepsilon}|})^*((\Phi^\varepsilon)'(|Du^{e\varepsilon}|)) \leq C\Phi^\varepsilon(|Du^{e\varepsilon}|)$$

we obtain

$$\mathcal{J}_1 \le C \int_{\Omega_{3r}} (\Phi^\varepsilon)'(|Du^{e\varepsilon}|)|\nabla^2 z| \,\mathrm{d}x \le C \int_{\Omega_{3r}} (\Phi^\varepsilon_{|Du^{e\varepsilon}|})^*((\Phi^\varepsilon)'(|Du^{e\varepsilon}|)) \,\mathrm{d}x$$
$$+C \int_{\Omega_{3r}} \Phi^\varepsilon_{|Du^{e\varepsilon}|}(|\nabla^2 z|) \,\mathrm{d}x \le \mathcal{M}_{ap} + C \int_{\Omega_{3r}} \Phi^\varepsilon_{|Du^{e\varepsilon}|}(|\nabla^2 z|) \,\mathrm{d}x = \mathcal{M}_{ap} + \mathcal{J}_{1.2}.$$

In the term $\mathcal{J}_{1.2}$ we apply shift change (A.12), (2.45), again shift change (A.12) and finally monotonicity of $(\Phi^\varepsilon)''$.

$$\begin{aligned}
\mathcal{J}_{1.2} &\le \delta \int_{\Omega_{3r}} \Phi^\varepsilon_{|\langle Du^{e\varepsilon}\rangle|}(|\nabla^2 z|) \,\mathrm{d}x + c(\delta) \int_{\Omega_{3r}} |V^\varepsilon(Du^{e\varepsilon})|^2 \,\mathrm{d}x \\
&\le \delta \int_{\Omega_{3r}} \Phi^\varepsilon_{|\langle Du^{e\varepsilon}\rangle|}(|\nabla^2 u|) \,\mathrm{d}x + \mathcal{M}_{ap} \le \delta \int_{\Omega_{3r}} \Phi^\varepsilon_{|Du^{e\varepsilon}|}(|\nabla^2 u|) \,\mathrm{d}x + \mathcal{M}_{ap} \\
&\le \delta \int_{\Omega_{3r}} (\Phi^\varepsilon)''(|Du^{e\varepsilon}| + |\nabla^2 u^{e\varepsilon}|)|\nabla^2 u^{e\varepsilon}|^2 \,\mathrm{d}x + \mathcal{M}_{ap} \\
&\le \delta \int_{\Omega_{3r}} (\Phi^\varepsilon)''(|Du^{e\varepsilon}|)|\nabla^2 u^{e\varepsilon}|^2 \,\mathrm{d}x + \mathcal{M}_{ap} = \mathcal{M}_5 + \mathcal{M}_{ap}.
\end{aligned}$$

□

The following lemma combines results from Lemma 2.3.2 and Lemma 2.3.6, estimates terms $\mathcal{M}_i$, $i = 1, 2, 3$, by $\mathcal{M}_{ap}$ and finally subsumes $\mathcal{M}_i$, $i = 4, 5$, to the left hand side.

Lemma 2.3.8 *Let $\Omega \subset \mathbb{R}^n$, $\Omega \in \mathcal{C}^3$ be a bounded non-axisymmetric domain. Let $u^{e\varepsilon}$ be the weak solution of the regularized problem* (2.22), (2.23) *and* (2.24). *Then there exists a positive constant C independent of $u^{e\varepsilon}$ such that*

$$\int_\Omega \mu^\varepsilon(|Du^{e\varepsilon}|)|\nabla^2 u^{e\varepsilon}|^2 \,\mathrm{d}x \le C\Big(\int_\Omega (\Phi^\varepsilon)^*(|f^e|) \,\mathrm{d}x + \int_\Omega (\Phi^\varepsilon)^*(|\nabla f^e|) \,\mathrm{d}x\Big). \quad (2.46)$$

Proof. We put together (2.28) and (2.34) and show that $\mathcal{M}_i$, $i = 1, \ldots, 3$ are estimated by $\mathcal{M}_{ap}$ and $\mathcal{M}_4$. Let us start with $\mathcal{M}_1$:

$$\mathcal{M}_1 \le \delta \int_{\Omega_{3r}} \mu^\varepsilon(|Du^{e\varepsilon}|)|\nabla^2 u^{e\varepsilon}|^2 \,\mathrm{d}x + c(\delta) \int_{\Omega_{3r}} \Phi^\varepsilon(|Du^{e\varepsilon}|) \,\mathrm{d}x \le \mathcal{M}_4 + \mathcal{M}_{ap},$$

where we used only Young's inequality with $\delta = a_0$. In $\mathcal{M}_2$ we integrate by parts

$$\mathcal{M}_2 \le C \int_{\Omega_{3r}} |\mathcal{S}^\varepsilon|(|\nabla u^{e\varepsilon}| + |u^{e\varepsilon}|) \,\mathrm{d}x + C \int_{\partial\Omega_{3r}} |\mathcal{S}^\varepsilon||u^{e\varepsilon}| \,\mathrm{d}\sigma = \mathcal{M}_{2.1} + \mathcal{M}_{2.2}.$$

To estimate $\mathcal{M}_{2.1}$ we use Young's inequality (A.8), relation (A.10) and Korn's inequality (A.20).

$$\mathcal{M}_{2.1} \le C \int_{\Omega_{3r}} \Phi^\varepsilon(|Du^{e\varepsilon}|) \,\mathrm{d}x + C \int_{\Omega_{3r}} \Phi^\varepsilon(|\nabla u^{e\varepsilon}|) \,\mathrm{d}x + C \int_{\Omega_{3r}} \Phi^\varepsilon(|u^{e\varepsilon}|) \,\mathrm{d}x \le \mathcal{M}_{ap}.$$

The boundary term $\mathcal{M}_{2.2}$ can be rewritten with the help of Young's inequality (A.8)

$$\mathcal{M}_{2.2} \le C \int_{\partial\Omega_{3r}} (\Phi^\varepsilon)'(|Du^{e\varepsilon}|)|u^{e\varepsilon}| \,\mathrm{d}\sigma \le C(\delta) \int_{\partial\Omega_{3r}} \Phi^\varepsilon(|u^{e\varepsilon}|) \,\mathrm{d}\sigma$$
$$+\delta \int_{\partial\Omega_{3r}} \Phi^\varepsilon(|Du^{e\varepsilon}|) \,\mathrm{d}\sigma = \mathcal{M}_{2.3} + \mathcal{M}_{2.4}.$$

To estimate $\mathcal{M}_{2.3}$ we show that $\Phi^\varepsilon(|u^{e\varepsilon}|) \in W^{1,1}(\Omega)$ and therefore $\Phi^\varepsilon(|u^{e\varepsilon}|)$ belongs to the space $L^1(\partial\Omega)$ by the trace theorem:

$$\begin{aligned}
\mathcal{M}_{2.3} = C \int_{\partial\Omega_{3r}} \Phi^\varepsilon(|u^{e\varepsilon}|) \,\mathrm{d}\sigma &\le C \int_{\Omega_{3r}} |\nabla \Phi^\varepsilon(|u^{e\varepsilon}|)| \,\mathrm{d}x + C \int_{\Omega_{3r}} \Phi^\varepsilon(|u^{e\varepsilon}|) \,\mathrm{d}x \\
&\le C \int_{\Omega_{3r}} (\Phi^\varepsilon)'(|u^{e\varepsilon}|)|\nabla u^{e\varepsilon}| \,\mathrm{d}x + C \int_{\Omega_{3r}} \Phi^\varepsilon(|u^{e\varepsilon}|) \,\mathrm{d}x \\
&\le C \int_{\Omega_{3r}} \Phi^\varepsilon(|\nabla u^{e\varepsilon}|) \,\mathrm{d}x + 2C \int_{\Omega_{3r}} \Phi^\varepsilon(|u^{e\varepsilon}|) \,\mathrm{d}x \le \mathcal{M}_{ap},
\end{aligned}$$

where we used Young's inequality (A.8) and Korn's inequality (A.20). To estimate $\mathcal{M}_{2.4}$ we start with $\Phi^\varepsilon(|\nabla u^{e\varepsilon}|) \in W^{1,1}(\Omega)$ and use the imbedding $W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$:

$$\begin{aligned}
\mathcal{M}_{2.4} &= C \int_{\partial\Omega_{3r}} \Phi^\varepsilon(|Du^{e\varepsilon}|) \,\mathrm{d}\sigma \\
&\le C \int_{\Omega_{3r}} |\nabla \Phi^\varepsilon(|Du^{e\varepsilon}|)| \,\mathrm{d}x + C \int_{\Omega_{3r}} \Phi^\varepsilon(|Du^{e\varepsilon}|) \,\mathrm{d}x \\
&\le C \int_{\Omega_{3r}} (\Phi^\varepsilon)''(|Du^{e\varepsilon}|)|Du^{e\varepsilon}||\nabla^2 u^{e\varepsilon}| \,\mathrm{d}x + C \int_{\Omega_{3r}} \Phi^\varepsilon(|Du^{e\varepsilon}|) \,\mathrm{d}x \\
&\le \delta \int_{\Omega_{3r}} (\Phi^\varepsilon)''(|Du^{e\varepsilon}|)|\nabla^2 u^{e\varepsilon}|^2 \,\mathrm{d}x + \mathcal{M}_{ap} = \mathcal{M}_4 + \mathcal{M}_{ap}.
\end{aligned}$$

In the term $\mathcal{M}_3$ almost monotonicity of $(\Phi^\varepsilon)''$ has to be used. For almost increasing $(\Phi^\varepsilon)''$ we use Young's inequality at first:

$$\begin{aligned}
\mathcal{M}_3 \le \int_{\Omega_{3r}} (\Phi^\varepsilon)''(|Du^{e\varepsilon}|)|\nabla^2 u^{e\varepsilon}||\nabla u^{e\varepsilon}| \,\mathrm{d}x &\le \delta \int_{\Omega_{3r}} (\Phi^\varepsilon)''(|Du^{e\varepsilon}|)|\nabla^2 u^{e\varepsilon}|^2 \,\mathrm{d}x \\
+ c(\delta) \int_{\Omega_{3r}} (\Phi^\varepsilon)''(|Du^{e\varepsilon}|)|\nabla u^{e\varepsilon}|^2 \,\mathrm{d}x &= \mathcal{M}_4 + \mathcal{M}_{3.1}.
\end{aligned}$$

For $\mathcal{M}_{3.1}$ we use that $(\Phi^\varepsilon)''$ is almost increasing and Korn's inequality.

$$\mathcal{M}_{3.1} \le C \int_{\Omega_{3r}} (\Phi^\varepsilon)''(|\nabla u^{e\varepsilon}|)|\nabla u^{e\varepsilon}|^2 \,\mathrm{d}x \le C \int_{\Omega_{3r}} \Phi^\varepsilon(|\nabla u^{e\varepsilon}|) \,\mathrm{d}x \le \mathcal{M}_{ap}.$$

In the case when $(\Phi^\varepsilon)''$ is almost decreasing, we integrate by parts in $\mathcal{M}_3$.

$$\begin{aligned}
\mathcal{M}_3 &\le C \int_{\Omega_{3r}} |\mathcal{S}^\varepsilon||\nabla^2 u^{e\varepsilon}|\xi \,\mathrm{d}x + C \int_{\Omega_{3r}} |\mathcal{S}^\varepsilon||\nabla u^{e\varepsilon}| \,\mathrm{d}x + C \int_{\partial\Omega_{3r}} |\mathcal{S}^\varepsilon||\nabla u^{e\varepsilon}|\xi \,\mathrm{d}\sigma \\
&= \mathcal{M}_{3.2} + \mathcal{M}_{3.3} + \mathcal{M}_{3.4}.
\end{aligned}$$

We treat $\mathcal{M}_{3.2}$ like $\mathcal{M}_1$, the term $\mathcal{M}_{3.3}$ is easily estimated with the help of Young's and Korn's inequalities, $\mathcal{M}_{3.4}$ (after Young's inequality) can be treated like $\mathcal{M}_{2.4}$ where we moreover use the fact that for almost decreasing $(\Phi^\varepsilon)''$ we can replace $(\Phi^\varepsilon)''(|\nabla u^{e\varepsilon}|)$ by $(\Phi^\varepsilon)''(|D(u^{e\varepsilon})|)$.

We put together (2.28) and (2.34), use estimates of $\mathcal{M}_i$, $i = 1, 2, 3$ and sum over all points P. We recall that P are divided into k groups and in each group the sets Ω^P_{3r} are mutually disjoint. We have

$$\int_\Omega \mu^\varepsilon(|Du^{e\varepsilon}|)|\nabla^2 u^{e\varepsilon}|^2 \,\mathrm{d}x \le k(\tilde{c}_4 a_0 + \tilde{c}_5\delta)\int_\Omega \mu^\varepsilon(|Du^{e\varepsilon}|)|\nabla^2 u^{e\varepsilon}|^2 \,\mathrm{d}x \\ + C\Big(\int_\Omega (\Phi^\varepsilon)^*(|f^e|)\,\mathrm{d}x + \int_\Omega (\Phi^\varepsilon)^*(|\nabla f^e|)\,\mathrm{d}x\Big), \tag{2.47}$$

If we choose a_0 and δ small enough, we can absorb the first integral on the right hand side of (2.47) into the left hand side. $\square$

Remark 2.3.9 *In two dimensions we can avoid the assumption on almost monotonicity of Φ''. Instead of Lemma 2.3.6 we would test (2.22) by*

$$\varphi = (-\partial_2[\partial_\tau(u^{e\varepsilon}\cdot\nu)\xi^2], \partial_1[\partial_\tau(u^{e\varepsilon}\cdot\nu)\xi^2]),$$

which is sufficient only in $\mathbb{R}^2$ to obtain all information in tangent direction. In $\mathcal{M}_3$ we would use the fact, that in boundary integral $\mathcal{M}_{3.4}$ (which comes after integration by parts) we are able to replace full gradient by the symmetric one. This works only in $\mathbb{R}^2$.

This technique concerning estimates in Orlicz setting was used first in [22], where one of the main features is that it handles the case of p-Laplacean for $1 < p < \infty$ in a unified way. It would be nice to avoid the assumption on almost monotonicity of Φ'' in the case where we work with symmetric gradients of velocity in n dimensions.

2.4 Limit passage

At first we fix $e > 0$. To pass with $\varepsilon \to 0$ in equations (2.22), (2.23) and (2.24) it is enough to have almost everywhere convergence of symmetric gradients, Lemma 2.4.1, and uniform integrability, Lemma 2.4.2.

Lemma 2.4.1 *Let $-1 < \beta < 0 < \alpha$, $c > 0$. We define $m(s) = cs^\alpha$ for $s \in (0,1)$, $m(s) = cs^\beta$ for $s \ge 1$. Let there exits $C > 0$ such that the sequence $\{A_k\}_{k=1}^\infty$, $A_k : \Omega \mapsto \mathbb{R}^{n\times n}$ fulfills*

$$\int_\Omega m^2(|A_k|)(|A_k|^2 + |\nabla A_k|^2)\,\mathrm{d}x \le C. \tag{2.48}$$

Then there exists a subsequence $\{A_{k_l}\}_{l=1}^\infty$ and $A : \Omega \mapsto \mathbb{R}^{n\times n}$ such that $A_{k_l} \to A$ a.e. in Ω as $l \to \infty$.

Proof. Let $\tilde{\Psi}$ be an N-function such that $m(|B|) = \tilde{\Psi}''(|B|)$ for all $B \in \mathbb{R}^{n\times n}$. We define

$$M_{ij}(B) := \partial_{ij}\tilde{\Psi}(|B|) = \tilde{\Psi}'(|B|)\frac{B_{ij}}{|B|} \quad \text{for } B \neq 0, \quad M_{ij}(B) = 0 \quad \text{for } B = 0.$$

M is Lipschitz mapping and also homeomorphism of $\mathbb{R}^{n\times n}$ onto $\mathbb{R}^{n\times n}$. From

$$\nabla_{A_k} M(A_k) = m(|A_k|)\frac{A_k}{|A_k|} \otimes \frac{A_k}{|A_k|} + \tilde{\Psi}'(|A_k|)\Big(\frac{Id}{|A_k|} - \frac{A_k \otimes A_k}{|A_k|^3}\Big)$$

we easily see that

$$|\nabla_{A_k} M(A_k)| \leq C\Big(m(|A_k|) + \frac{\tilde{\Psi}'(|A_k|)}{|A_k|}\Big).$$

Since $\frac{\tilde{\Psi}'(s)}{s} \leq Cm(s)$ for all $s > 0$, by simple computation we have

$$\begin{aligned} |\nabla M(A_k)| &\leq Cm(|A_k|)|\nabla A_k|, \\ |M(A_k)| &\leq Cm(|A_k|)|A_k|. \end{aligned}$$

The assumption (2.48) gives $M(A_k) \in W^{1,2}(\Omega)^{n\times n}$ uniformly in k, so there exists a subsequence k_l such that

$$\begin{aligned} M(A_{k_l}) &\rightharpoonup M^* \quad \text{in } W^{1,2}(\Omega)^{n\times n}, \\ M(A_{k_l}) &\to M^* \quad \text{in } L^2(\Omega)^{n\times n}, \\ A_{k_l} &\to M^{-1}(M^*) \quad \text{a.e. in } \Omega. \end{aligned}$$

Putting $M^{-1}(M^*) =: A$ we complete the proof. $\square$

Now we use Lemma 2.4.1, where instead of the general A_k we have $Du^{e\varepsilon}$. From Lemma A.2.3 we know that there exists $\alpha > 1, c > 0$ such that for all $s \in (0,1)$ holds $\mu(s) \geq cs^{2\alpha}$ and there exists $\beta \in (-\frac{1}{2}, 0), c' > 0$ such that for all $s > 1$ holds $\mu(s) \geq c's^{2\beta}$. It can be easily seen that function μ^ε also fulfills these conditions. So there exists function $m(s)$ defined in Lemma 2.4.1 satisfying $m^2(s) < \mu^\varepsilon(s)$ for all $s \in (0,\infty)$ and for all $\varepsilon \in (0,1)$.

The first apriori estimate (2.27) and Lemma 2.3.8 give

$$\int_\Omega m(|Du^{e\varepsilon}|)\big(|Du^{e\varepsilon}|^2 + |\nabla Du^{e\varepsilon}|^2\big)\,\mathrm{d}x \leq C,$$

therefore Lemma 2.4.1 provides existence of $A : \Omega \to \mathbb{R}^{n\times n}$ such that

$$Du^{e\varepsilon} \to A \quad \text{a.e. in } \Omega.$$

Moreover, if we use the definition of the N-function $\tilde{\Psi}$, using also Korn's inequality from the first apriori estimate (2.27) we have

$$\int_\Omega \tilde{\Psi}(|u^{e\varepsilon}|) + \int_\Omega \tilde{\Psi}(|\nabla u^{e\varepsilon}|)\,\mathrm{d}x \leq C.$$

Thus, there exists $u^e \in W^{1,\tilde{\Psi}}(\Omega)$, such that (up to a subsequence)

$$\begin{aligned}\nabla u^{e\varepsilon} &\rightharpoonup \nabla u^e \quad \text{in } L^{\tilde{\Psi}}(\Omega)^{n\times n},\\ u^{e\varepsilon} &\rightharpoonup u^e \quad \text{in } L^{\tilde{\Psi}}(\Omega)^n.\end{aligned}$$

Clearly $A = Du^e$ and $Du^{e\varepsilon} \to Du^e$ a.e.

The following lemma gives the uniform integrability.

Lemma 2.4.2 *Let $\int_\Omega \Phi^\varepsilon(|Du^{e\varepsilon}|)\,\mathrm{d}x \le C$. Then there exists $\delta > 0$ such that for all $\sigma \in (0,1)$ and for all $E \subset \Omega$ such that $|E| < \delta$ holds*

$$\int_E (\Phi^\varepsilon)'(|Du^{e\varepsilon}|)\,\mathrm{d}x < \sigma.$$

Proof. Let us denote $\phi(s) := \int_0^s \max(\mu(t),1)t\,\mathrm{d}t$. Then for all $s > 0$, $\varepsilon \in (0,1)$ holds $\phi(s) \ge \Phi^\varepsilon(s)$. Therefore

$$\frac{1}{(\Phi^\varepsilon)^{-1}(s)} \le \frac{1}{\phi^{-1}(s)}.$$

We note that $\|\chi_\Omega\|_{\Phi^\varepsilon} = \frac{1}{(\Phi^\varepsilon)^{-1}(|\Omega|^{-1})}$ by (A.9). Therefore

$$\|\chi_\Omega\|_{\Phi^\varepsilon} \le \frac{1}{(\phi)^{-1}(|\Omega|^{-1})}$$

and ϕ^{-1} is increasing function such that $\phi^{-1}(s) \to \infty$ as $s \to \infty$. Using Hölder's inequality (A.9) we obtain

$$\begin{aligned}&\int_E (\Phi^\varepsilon)'(|Du^{e\varepsilon}|)\,\mathrm{d}x \le 2\|\chi_E\|_{\Phi^\varepsilon}\|(\Phi^\varepsilon)'(|Du^{e\varepsilon}|)\|_{(\Phi^\varepsilon)^*}\\ &\le \frac{C}{\phi^{-1}(|E|^{-1})}\int_\Omega (\Phi^\varepsilon)^*((\Phi^\varepsilon)'(|Du^{e\varepsilon}|))\,\mathrm{d}x \le \frac{\tilde{C}}{\phi^{-1}(|E|^{-1})}\int_\Omega \Phi^\varepsilon(|Du^{e\varepsilon}|)\,\mathrm{d}x.\end{aligned} \tag{2.49}$$

The right hand side of (2.49) tends to zero as $|E| \to 0$. The constant $\tilde{C}$ depends on $\Delta_2(\{\Phi^\varepsilon, (\Phi^\varepsilon)^*\})$, but as we saw before, $\Delta_2(\{\Phi^\varepsilon, (\Phi^\varepsilon)^*\})$ doesn't depend on ε. □

Lemmata 2.4.1 and 2.4.2 allow us to pass to the limit $\varepsilon \to 0$ in the weak formulation of (2.22), (2.23) and (2.24). It remains to let $\varepsilon \to 0$ in (2.46). Since $\{V^\varepsilon(Du^{e\varepsilon})\}_\varepsilon$ is bounded in $W^{1,2}(\Omega)^{n\times n}$, it follows that up to a subsequence

$$\begin{aligned}V^\varepsilon(Du^{e\varepsilon}) &\rightharpoonup \chi \quad \text{in } W^{1,2}(\Omega)^{n\times n},\\ V^\varepsilon(Du^{e\varepsilon}) &\to \chi \quad \text{in } L^2(\Omega)^{n\times n}.\end{aligned}$$

To identify χ with $V(Du^e)$ we show $V^\varepsilon(Du^{e\varepsilon}) \to V(Du^e)$ a.e. in Ω. For that we need (besides almost everywhere convergence of symmetric gradients) locally uniform convergence of V^ε, which is provided by the following lemma.

Lemma 2.4.3 *Let* $K \subset\subset \mathbb{R}^{n\times n}$*, then* $V^\varepsilon \rightrightarrows V$ *on* K *as* $\varepsilon \to 0^+$.

Proof. We recall that $\psi^\varepsilon(|A|)$ is a potential to $V^\varepsilon(A)$, i.e. $V^\varepsilon(A) = (\Psi^\varepsilon)'(|A|)\frac{A}{|A|}$ and $(\Psi^\varepsilon)'(|A|) = \sqrt{|A|(\Phi^\varepsilon)'(|A|)} = \sqrt{|A|^2\mu^\varepsilon(|A|)}$. Therefore, $V^\varepsilon(A) = \sqrt{\mu^\varepsilon(|A|)}A$ and $|V(A) - V^\varepsilon(A)| = |\sqrt{\mu(|A|)} - \sqrt{\mu^\varepsilon(|A|)}||A|$.

We know that $\Phi'(s) = \mu(s)s$ and since $\Phi'(s)$ is bounded on $(0,1]$ we get that there exists $C > 0$ such that for all $s \in (0,1)$ and $\varepsilon \in (0,1)$ holds $\mu^\varepsilon(s) \le \max(\mu(s),1) \le \frac{C}{s}$. It follows that for any $\delta > 0$ we find $\sigma > 0$ such that for all $\varepsilon \in (0,1)$ holds $|V(A) - V^\varepsilon(A)| < \delta$ on $B_\sigma(0)$. Since $K \setminus B_\sigma(0)$ is compact, $\mu(|A|)$ attains there its maximum and minimum. Consequently, there is ε_0 such that for $\varepsilon \in (0,\varepsilon_0) : V^\varepsilon = V$ on $K \setminus B_\sigma(0)$. □

Using Lemma 2.4.3 we get $\chi = V(Du^e)$ and we can pass on the left hand side of (2.46) as $\varepsilon \to 0$ by weak lower semicontinuity of norms in $W^{1,2}(\Omega)^{n\times n}$.

Passing to the limit as $e \to 0$ in equations (2.22), (2.23), (2.24) and in (2.46) is easy, because the right hand side f was approximated in order to have $f^e \to f$ in $W^{1,\Phi^*}(\Omega)^n$.

To conclude the proof of Theorem 2.1.2 it remains to reconstruct the pressure and show the inequality (2.1). Using De Rham's theorem A.5.2 we know that there exist pressure π such that

$$\nabla\pi = \operatorname{div}\mathcal{S} + \operatorname{div} g$$

holds in the sense of distributions. We add the assumption $\int_\Omega \pi \,\mathrm{d}x = 0$. Since $g \in L^{(\Phi^*)^q}$ and $[\Phi^*(|S(Du)|)]^q \le C[\Phi^*(\Phi'(|Du|))]^q \le C(\Phi(|Du|))^q$, we obtain (2.1) by application of Nečas' Theorem (A.5.1) and imbedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$.

3

Integrability of weak solutions to equations of steady flows

3.1 Main theorem

In this chapter we show L^q theory for the system (1.23)–(1.25):

$$
\begin{aligned}
-\operatorname{div}\mathcal{S}(Du)+\nabla\pi &= f && \text{in } \Omega,\\
\operatorname{div} u &= 0 && \text{in } \Omega,\\
u\cdot\nu = 0, \quad [\mathcal{S}(Du)\nu]\cdot\tau &= 0 && \text{on } I\times\partial\Omega,
\end{aligned}
$$

where $f = \operatorname{div} F$. Since some results don't demand the assumption on almost monotonicity of the function Φ'' we exclude it from Assumption 1.4.3:

Assumption 3.1.1 *Suppose that $\Phi \in \mathcal{C}^{1,1}(0,\infty)\cap\mathcal{C}^1[0,\infty)$ is an N-function, $\Phi\in\Delta_2$, $\Phi^*\in\Delta_2$ and $\Phi'(s)\sim s\Phi''(s)$ holds for all $s>0$.*

We will use the space [1]

$$\varphi \in W^{1,\Phi}_\nu(\Omega)^n := \{\varphi_i \in W^{1,\Phi}, i=1,\dots,n;\ \varphi\cdot\nu = 0 \text{ on } \partial\Omega\}.$$

Unlike Definition 2.1.1 we consider the pressure in the definition of the weak solution:

Definition 3.1.2 *We say that the pair $(u,\pi)\in W^{1,\Phi}_\sigma(\Omega)^n\times L^{\Phi^*}(\Omega)$ is a weak solution to* (1.23)–(1.25) *if*

$$\int_\Omega \mathcal{S}(Du):D\varphi\,\mathrm{d}x - \int_\Omega \pi\operatorname{div}\varphi\,\mathrm{d}x = \int_\Omega F:\nabla\varphi\,\mathrm{d}x$$

holds for all $\varphi\in W^{1,\Phi}_\nu(\Omega)^n$.

Before formulating the main result we would like to mention some previous results which motivated us to our work. In [46] T. Iwaniec showed L^q theory result for linear problem based on local comparison with the solution to the problem with the zero right hand side. One year later in [47] he extended this result

[1] As we use the notation $W^{1,\Phi}_\sigma(\Omega)^n$ for vector-valued functions with components in the function space $W^{1,\Phi}_\sigma(\Omega)$, we use analogically the notation $W^{1,\Phi}_\sigma(\Omega)^{n\times n}$ for tensor-valued functions.

also for $p-$Laplace equations. Among lots of papers based on the comparison technique we mention especially [18]. The approach of L. Caffarelli and A. Peral presented in [18] will be used to prove our main result. In connection with Orlicz spaces we refer to [88], for results concerning the problem with growth described with variable exponent c.f. [42]. Calderón-Zygmund estimates based on the technique introduced by L. Caffarelli and I. Peral in [17] and [18] can be also found in Section 7 of [56] and in [57] where J. Kristensen and G. Mingione provided L^q estimates up to the boundary for Dirichlet problems involving general nonlinear elliptic systems. For local Calderón-Zygmund estimates for parabolic p-Laplacean systems we refer to [2]. To our knowledge the first result about L^q regularity for Stokes type system with growth described by N-function was given in [23]. L. Diening and P. Kaplický showed interior L^q regularity of generalized Stokes system in $\mathbb{R}^3$ under Assumption 1.4.3. The key part of the proof was Theorem 3.2, where for the problem with zero right hand side gradient of function $V(Du)$ is controlled by oscillations of $V(Du)$. Unlike the previous chapter, a different approach is used here. Instead of working on a general boundary from the beginning, we use flattening the boundary and reflection the solution beyond the boundary in a suitable way.

The main result of this paper concerns with higher integrability of the first gradient of solutions to (1.23) – (1.25). By $B_r(x_0)$ we denote a ball in $\mathbb{R}^n$ with a center x_0 and a diameter r, we also use an abbreviation $\Omega_r = \Omega \cap B_r(x_0)$.

Theorem 3.1.3 (Main Result) *Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^{2,1}$ domain and Assumption 3.1.1 be fulfilled. Then there exist $\lambda > 1$ depending only on the dimension and $r > 0$ such that for a weak solution u to (1.23)–(1.25) and for every $x_0 \in \partial\Omega$ it holds*

$$\big(\Phi^*(|F|) \in L^q(\Omega_{\lambda r})\big) \Rightarrow \big(\Phi(|Du|) \in L^q(\Omega_r)\big),$$

provided $q \in (1, \infty)$ for $n = 2$ and $q \in \left(1, \frac{n}{n-2}\right)$, resp. $q \in \left(1, \frac{n}{n-2} + \delta\right)$ for $n > 2$ and some $\delta > 0$ in case Φ'' is almost monotone.[2]

Moreover,

$$\fint_{\Omega_r} \Phi(|Du|)^q \,\mathrm{d}x \le c \left(\fint_{\Omega_{\lambda r}} \Phi^*(|F|)^q \,\mathrm{d}x + \fint_{\Omega_{\lambda r}} \Phi\left(|u|\right)^q \,\mathrm{d}x \right) + c \left(\fint_{\Omega_{\lambda r}} \Phi(|Du|) \,\mathrm{d}x \right)^q, \quad (3.1)$$

where c is independent of u, F, λ, r and x.

Theorem 3.1.3 provides a local regularity of solution near boundary. However, the interior regularity of solution was proven in [23] and thus one may easily derive global regularity of solution as well as global estimates in case Ω is a bounded domain.

The method of the proof is basically the same as in [23] and it is based on the approach published in [18]. The validity of two hypothesis (H1') and (H2)

[2] For the definition of almost monotonicity see Assumption 1.4.3.

from [18] has to be shown. We formulated these hypothesis for our problem in assumptions of Lemma A.5.4.

At first we study homogeneous system near the flat boundary and verify the hypothesis (H1'). Instead of working on the general smooth boundary like in Chapter 1, we use the special structure of perfect slip boundary conditions in order to extend the solution in a suitable way beyond the flat boundary. Finally, we flatten the general $\mathcal{C}^{2,1}$ boundary and we complete the proof of the main theorem by showing the validity of hypothesis (H2).

3.2 Approximative system on a flat boundary

In this section we put some ideas in the case when the boundary $\partial\Omega$ is flat. By Q we denote a cube in $\mathbb{R}^n$ with center x_0, sides parallel to the axis and one side equal to $2R$, i.e.

$$Q = Q(x_0, R) = \left\{ x \in \mathbb{R}^n ; \sup_i |x_i - (x_0)_i| < R \right\}.$$

For $s > 0$ the abbreviation sQ stands for a cube with the same center as Q and side $2sR$, i.e. $sQ = Q(x_0, sR)$. By $|Q|$ we mean the volume of Q. For $f \in L^1(Q)$ we define

$$\langle f \rangle_Q = \fint_Q f(x)\,\mathrm{d}x := \frac{1}{|Q|}\int_Q f(x)\,\mathrm{d}x.$$

Let $\Omega = \mathbb{R}^n_+ := \mathbb{R}^{n-1} \times \mathbb{R}_+$ and $x_0 \in \Omega$. We denote $(\frac{5}{3}Q)^+ = \frac{5}{3}Q \cap \mathbb{R}^n_+$ and $\Gamma_{(\frac{5}{3}Q)^+} = \partial(\frac{5}{3}Q)^+ \cap \{x; x_n = 0\}$. By e_i we denote the unit vector in the direction x_i, $i = 1, \dots, n$. Since the boundary is flat, $\tau^\alpha = e_\alpha$ for $\alpha = 1, \dots, n-1$ and $\nu = -e_n$, $\partial_\nu = \partial_n$ and $\partial_{\tau^\alpha} = \partial_\alpha$ for $\alpha = 1, \dots, n-1$. By x' we denote the first $n-1$ components of x, i.e. $x = (x', x_n)$.

Fix Q and consider the homogeneous system

$$\begin{aligned} -\operatorname{div}\mathcal{S}(D\overline{v}) + \nabla p = 0 \qquad & \text{in } \left(\frac{5}{3}Q\right)^+, \\ \operatorname{div}\overline{v} = 0 \qquad & \text{in } \left(\frac{5}{3}Q\right)^+, \\ \overline{v}\cdot\nu = 0, \quad [\mathcal{S}(D\overline{v})\nu]\cdot\tau = 0 \qquad & \text{on } \Gamma_{(\frac{5}{3}Q)^+}. \end{aligned} \tag{3.2}$$

Definition 3.2.1 *The function $\overline{v}$ is said to be a local weak solution to* (3.2)*, if $\overline{v} \in W^{1,\Phi}((\frac{5}{3}Q)^+)^n$ and the weak formulation*

$$\int_{(\frac{5}{3}Q)^+} \mathcal{S}(D\overline{v}) : D\varphi\,\mathrm{d}x = 0$$

holds for all $\varphi \in W^{1,\Phi}((\frac{5}{3}Q)^+)^n$ with $\operatorname{div}\varphi = 0$ in $(\frac{5}{3}Q)^+$, $\varphi\cdot\nu = 0$ on $\Gamma_{(\frac{5}{3}Q)^+}$, $\operatorname{supp}\varphi \subset\subset (\frac{5}{3}Q)$.

The aim of this section is to prove the following theorem.

Theorem 3.2.2 *Let $\overline{v} \in W^{1,\Phi}((\frac{5}{3}Q)^+)^n$ be a local weak solution to (3.2). Then there exists a constant C independent of $\overline{v}$ and R such that*

$$\left(\fint_{(\frac{4}{3}Q)^+} |V(D\overline{v})|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \leq C \left(\fint_{(\frac{5}{3}Q)^+} |V(D\overline{v})|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}, \tag{3.3}$$

for $q \in \left[2, \frac{2n}{n-2}\right]$ provided $n > 2$ and $q \in [2, \infty)$ for $n = 2$. In case Φ'' is almost monotone, $n > 2$, we can even allow $q = \frac{rn}{n-r}$ for some $r > 2$.

Proof. At first we extend the solution from $(\frac{5}{3}Q)^+$ to $\frac{5}{3}Q$. For $\alpha = 1, \dots, n-1$ define $\tilde{v}$ as follows

$$\tilde{v}_\alpha(x', x_n) = \begin{cases} \overline{v}_\alpha(x', x_n) & \text{for } x_n > 0, \\ \overline{v}_\alpha(x', -x_n) & \text{for } x_n < 0, \end{cases} \tag{3.4}$$

$$\tilde{v}_n(x', x_n) = \begin{cases} \overline{v}_n(x', x_n) & \text{for } x_n > 0, \\ -\overline{v}_n(x', -x_n) & \text{for } x_n < 0. \end{cases} \tag{3.5}$$

Using (3.4) and (3.5) we compute components of the symmetric gradient of $\tilde{v}$ for $x_n > 0$

$$\begin{aligned} D_{\alpha\alpha}\tilde{v}(x', -x_n) &= D_{\alpha\alpha}\tilde{v}(x', x_n), \\ D_{\alpha n}(x', -x_n) &= -D_{\alpha n}\tilde{v}(x', x_n), \\ D_{nn}\tilde{v}(x', -x_n) &= D_{nn}\tilde{v}(x', x_n). \end{aligned} \tag{3.6}$$

Note that for $\overline{v} \in W^{1,\Phi}_\sigma((\frac{5}{3}Q)^+)^n$ the extended solution $\tilde{v}$ belongs to $W^{1,\Phi}_\sigma(\frac{5}{3}Q)^n$ since $\tilde{v}$ is absolutely continuous on lines a.e. and the derivative of $\tilde{v}$ is in $L^\Phi(\frac{5}{3}Q)^{n\times n}$ pointwisely. For a test function $\varphi \in W^{1,\Phi}_\sigma(\frac{5}{3}Q)^n$ we define φ^+ by components

$$\begin{aligned} \varphi^+_\alpha &= \frac{1}{2}\big(\varphi_\alpha(x', x_n) + \varphi_\alpha(x', -x_n)\big), \quad \alpha = 1, \dots, n-1, \\ \varphi^+_n &= \frac{1}{2}\big(\varphi_n(x', x_n) - \varphi_n(x', -x_n)\big), \end{aligned}$$

and similarly

$$\begin{aligned} \varphi^-_\alpha &= \frac{1}{2}\big(\varphi_\alpha(x', x_n) - \varphi_\alpha(x', -x_n)\big), \quad \alpha = 1, \dots, n-1, \\ \varphi^-_n &= \frac{1}{2}\big(\varphi_n(x', x_n) + \varphi_n(x', -x_n)\big), \end{aligned}$$

One can easily check that $\operatorname{div}\varphi^+ = \operatorname{div}\varphi^- = 0$ holds in $\frac{5}{3}Q$. Thus,

$$\begin{aligned} \int_{\frac{5}{3}Q} \mathcal{S}(D\tilde{v}) : D\varphi \,\mathrm{d}x &= \int_{\frac{5}{3}Q} \mathcal{S}(D\tilde{v}) : D\varphi^+ \,\mathrm{d}x \\ &\quad + \int_{\frac{5}{3}Q} \mathcal{S}(D\tilde{v}) : D\varphi^- \,\mathrm{d}x = \mathcal{I}_1 + \mathcal{I}_2 = 0 \quad \forall \varphi \in W^{1,\Phi}_\sigma\Big(\frac{5}{3}Q\Big)^n, \end{aligned} \tag{3.7}$$

where $\mathcal{I}_1$ is equal to zero due to the equation (3.2) and $\mathcal{I}_2$ is equal to zero because of the symmetry of φ^-.

The proof of Theorem 3.2.2 is based on the following lemma which will be proven later.

Lemma 3.2.3 *Let $\tilde{v} \in W^{1,\Phi}(\frac{5}{3}Q)^n$ satisfies* (3.7). *Then there exists a positive constant C depending only on $\Delta_2(\{\Phi,\Phi^*\})$ and constants in* (1.28) *such that*

$$\fint_{\frac{4}{3}Q} |\nabla V(D\tilde{v})|^2 \,\mathrm{d}x \le \frac{C}{R^2}\Big(\fint_{\frac{5}{3}Q} |V(D\tilde{v})|^2 \,\mathrm{d}x\Big). \tag{3.8}$$

Moreover, if Φ'' is almost monotone, the estimate (3.8) *can be improved to*

$$\fint_{\frac{4}{3}Q} |\nabla V(D\tilde{v})|^2 \,\mathrm{d}x \le \frac{C}{R^2}\Big(\fint_{\frac{5}{3}Q} |V(D\tilde{v}) - \langle V(D\tilde{v})\rangle_{\frac{5}{3}Q}|^2 \,\mathrm{d}x\Big). \tag{3.9}$$

Application of Sobolev-Poincaré inequality (A.13) to the left hand side of (3.8) (after multiplication of R^2 and square root) leads to

$$\left(\fint_{\frac{4}{3}Q} |V(D\tilde{v}) - \langle V(D\tilde{v})\rangle_{\frac{4}{3}Q}|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\fint_{\frac{5}{3}Q} |V(D\tilde{v})|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \tag{3.10}$$

for $q \in [2,\infty)$ in case $n = 2$ and $q \in \big[2, \frac{2n}{n-2}\big]$ in case $n > 2$. For almost monotone Φ'' we can at first apply on (3.9) Sobolev-Poincaré inequality (A.13) and Reverse Hölder inequality (A.25) to obtain for some $r > 2$

$$\left(\fint_{\frac{4}{3}Q} |\nabla V(D\tilde{v})|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \le \left(\frac{C}{R^2}\fint_{\frac{5}{3}Q} |\nabla V(D\tilde{v}) - \langle \nabla V(D\tilde{v})\rangle_{\frac{5}{3}Q}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{3.11}$$

Consequently,

$$\left(\fint_{\frac{4}{3}Q} |V(D\tilde{v})|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\fint_{\frac{5}{3}Q} |V(D\tilde{v})|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \tag{3.12}$$

where q is the same as mentioned before. To conclude the proof it remains to go from $\tilde{v}$ to $\overline{v}$ and from $\frac{4}{3}Q$, resp. $\frac{5}{3}Q$ to $(\frac{4}{3}Q)^+$, resp. $(\frac{5}{3}Q)^+$. The integral on the left hand side of inequality (3.12) can be estimated from above by the integral over the smaller set $(\frac{4}{3}Q)^+$. The integral on the right hand side of (3.12) can be split into a integral over $(\frac{5}{3}Q)^+$ and integral over $\frac{5}{3}Q \setminus (\frac{5}{3}Q)^+$. Then we can use a fact that integral over $\frac{5}{3}Q \setminus (\frac{5}{3}Q)^+$ is proportional to the integral over $(\frac{5}{3}Q)^+$, where we moreover use that due to the definition of function V and (3.6) we have $|V(D\overline{v}(x', x_n))| = |V(D\tilde{v}(x', x_n))|$. $\square$

Proof of Lemma 3.2.3. The extension beyond the flat boundary reduces the problem to the interior regularity which is covered in [23, Theorem 3.2] in case $n = 3$. From this reason we would like to show only the generalization from $n = 3$ to higher dimensions. To underline the main idea of the proof we omit the technical steps concerning the regularization of the solution. We suppose that the solution is smooth enough to justify all calculations of this proof. At the end we briefly discuss how one should proceed in case the solution is not regular enough.

The test function in [23, Lemma 3.5] is constructed to take advantage of the operator curl, which is in $\mathbb{R}^3$ defined by $\operatorname{curl} g = (\partial_2 g_3 - \partial_3 g_2, \partial_3 g_1 - \partial_1 g_3, \partial_1 g_2 - \partial_2 g_1)$. Since we are not aware of any straightforward generalization of the curl operator to n dimensions, we use the language of exterior differential calculus to construct the right test function.

At first we state some notation. Although we denoted by $\{e_i, i = 1, \dots, n\}$ the orthonormal basis in $\mathbb{R}^n$ before, we need to distinguish vectors and forms now and therefore we use $\{\partial_i, i = 1, \dots, n\}$ to represent an orthonormal basis for vectors in $\mathbb{R}^n$, whereas $\{dx^i, i = 1, \dots, n\}$ denotes corresponding dual 1-form basis. In order to follow the standard notation, we use in this section the upper indices for components of vectors whereas the lower indices indicates components of form of any order. In the next section we won't work with such forms, therefore all indices will be the lower ones.

We will use so called musical isomorphisms $\sharp$ and $\flat$, where $\sharp$ raise the indices of a 1-form β to give the vector $\beta^\sharp$ whereas $\flat$ lowers the indices of a vector z to produces a 1-form $z^\flat$, i.e.

$$\beta = \sum_{i=1}^n \beta_i dx^i, \quad \beta^\sharp = \sum_{i=1}^n \beta^i \partial_i, \quad z = \sum_{i=1}^n z^i \partial_i, \quad z^\flat = \sum_{i=1}^n z_i dx^i.$$

By d we mean the exterior derivative and the symbol $\wedge$ denotes the wedge product. Let us denote

$$\begin{aligned} \widehat{dx^i} &:= dx^1 \cdots \wedge dx^{i-1} \wedge dx^{i+1} \cdots \wedge dx^n, \\ \widehat{dx^i \wedge dx^j} &:= dx^1 \cdots \wedge dx^{i-1} \wedge dx^{i+1} \cdots \wedge dx^{j-1} \wedge dx^{j+1} \cdots \wedge dx^n. \end{aligned}$$

The Hodge map $\star$ is linear isomorphism between the vector spaces of differential $k-$ and $(n-k)-$ forms. In Riemannian metric it holds

$$\star(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_n},$$

where $(i_1, \dots, i_n)$ is any even permutation of $(1, 2, \dots, n)$. Let $\xi \in \mathcal{C}_0^\infty(\frac{5}{3}Q)$ be a cut-off function with $\chi_{\frac{4}{3}Q} \le \xi \le \chi_{\frac{5}{3}Q}$ and $\|\nabla^j \xi\|_\infty \le C/R^j$ for $j = 1, 2$. Let $q : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function with $\nabla q = \langle \nabla \tilde{v} \rangle_{\frac{5}{3}Q}$. We test (3.7) by

$$\varphi = \big(\star d[\xi^2 \star d(\tilde{v} - q)^\flat] \big)^\sharp. \tag{3.13}$$

Note that the test function is well defined. $\flat$ converts the vector field $(\tilde{v} - q)$ into a 1-form $(\tilde{v} - q)^\flat$. d computes something like a curl but expressed as a 2-form $d(\tilde{v} - q)^\flat$. $\star$ turns this 2-form into a $(n-2)$-form. After multiplication by ξ^2 and application of the derivative d we obtain $(n-1)-$form and Hodge star $\star$ create $1-$form, which is by $\sharp$ converted to the vector.

Moreover, one can easily see that div $\varphi = 0$, since div $\varphi = \star d \star \varphi^\flat$ and $dd\gamma = 0$ for any differential form γ.

Let's see how (3.13) looks in components. For better lucidity we define $z = \tilde{v} - q$. At first we compute derivative of $z^\flat = \sum_{i=1}^n z_i dx^i$ and apply the Hodge map:

$$dz^\flat = \sum_{i,j=1}^n \partial_j z_i dx^j \wedge dx^i = \sum_{i<j} (\partial_i z_j - \partial_j z_i) dx^i \wedge dx^j,$$
$$\xi^2 \star dz^\flat = \sum_{i<j} \xi^2 (\partial_i z_j - \partial_j z_i)(-1)^{i+j-3} \widehat{dx^i \wedge dx^j},$$

where we used that $dx^i \wedge dx^j = -dx^j \wedge dx^i$ for $i \neq j$ and $dx^i \wedge dx^i = 0$. Further,

$$\begin{aligned} d(\xi^2 \star dz^\flat) =& \sum_{i<j} [\xi^2(\partial_i^2 z_j - \partial_i \partial_j z_i) + 2\xi \partial_i \xi (\partial_i z_j - \partial_j z_i)](-1)^{i+j-3} dx^i \wedge \widehat{dx^i \wedge dx^j} \\ &+ \sum_{i<j} [\xi^2(\partial_j \partial_i z_j - \partial_j^2 z_i) + 2\xi \partial_j \xi (\partial_i z_j - \partial_j z_i)](-1)^{i+j-3} dx^j \wedge \widehat{dx^i \wedge dx^j}. \end{aligned} \tag{3.14}$$

We can change the summation indices in the second sum in (3.14), move dx^i to the i^{th} position in the product and finally put these two sums together.

$$\begin{aligned} d(\xi^2 \star dz^\flat) =& \sum_{i<j} [\xi^2(\partial_i^2 z_j - \partial_i \partial_j z_i) + 2\xi \partial_i \xi (\partial_i z_j - \partial_j z_i)](-1)^{i+j-3+i-1} \widehat{dx^j} \\ &+ \sum_{i>j} [\xi^2(-\partial_i^2 z_j + \partial_i \partial_j z_i) + 2\xi \partial_i \xi (-\partial_i z_j + \partial_j z_i)](-1)^{i+j-3+i-2} \widehat{dx^j} \\ =& \sum_{i \neq j} [\xi^2(\partial_i^2 z_j - \partial_i \partial_j z_i) + 2\xi \partial_i \xi (\partial_i z_j - \partial_j z_i)](-1)^j \widehat{dx^j}. \end{aligned} \tag{3.15}$$

Thus, applying the Hodge star and going back from forms to vectors

$$[\star d(\xi^2 \star dz^\flat)]^\sharp = \sum_{i \neq j} [\xi^2(\partial_i^2 z^j - \partial_i \partial_j z^i) + 2\xi \partial_i \xi (\partial_i z^j - \partial_j z^i)](-1)^{j+j-1} \partial_j.$$

As one can easily check, z is divergence-free, therefore $\sum_{i \neq j} \partial_i \partial_j z_i = -\partial_j \partial_j z_j$ and we finally obtain

$$\varphi = \sum_{i,j=1}^n \Big(-\xi^2 \partial_i^2 (\tilde{v})^j + 2\xi \partial_i \xi [-\partial_i (\tilde{v} - q)^j + \partial_j (\tilde{v} - q)^i] \Big) \partial_j, \tag{3.16}$$

where we moreover used that q is a linear function, thus $\partial_i^2 q = 0$. Inserting (3.16) into (3.7) we get

$$
\begin{aligned}
-\fint_{\frac{5}{3}Q} [\mathcal{S}(D\tilde{v}) - \mathcal{S}(Dq)] : \nabla(\xi^2 \Delta \tilde{v}) \,\mathrm{d}x \\
+\fint_{\frac{5}{3}Q} [\mathcal{S}(D\tilde{v}) - \mathcal{S}(Dq)] : \nabla\big(2\xi\nabla\xi(\nabla\tilde{v} - \nabla q) - (\nabla\tilde{v} - \nabla q)^T\big) \,\mathrm{d}x \\
= \mathcal{J}_1 + \mathcal{J}_2 = 0. \quad (3.17)
\end{aligned}
$$

From this point we proceed almost in the same way as in [23, Proof of Lemma 3.5], where the authors L. Diening and P. Kaplický due to the regularization estimated more terms. For the sake of completeness we reproduce the computation also here.

Lets start with proving (3.9). We proceed in a different way when Φ'' is almost decreasing or almost increasing. At first let us assume that Φ'' is almost decreasing. After some manipulation involving integrating by parts in the first term we have

$$
\begin{aligned}
\mathcal{J}_1 = \fint_{\frac{5}{3}Q} \nabla\mathcal{S}(D\tilde{v})\xi^2\nabla^2\tilde{v}\,\mathrm{d}x - \fint_{\frac{5}{3}Q} [\mathcal{S}(D\tilde{v}) - \mathcal{S}(Dq)] \operatorname{div}(\nabla\xi^2 \otimes (\nabla\tilde{v} - \nabla q)) \,\mathrm{d}x \\
+\fint_{\frac{5}{3}Q} [\mathcal{S}(D\tilde{v}) - \mathcal{S}(Dq)]\nabla[(\nabla\tilde{v} - \nabla q)\nabla\xi^2] \,\mathrm{d}x \\
= \mathcal{J}_{1.1} + \mathcal{J}_{1.2} + \mathcal{J}_{1.3}. \quad (3.18)
\end{aligned}
$$

Assumption 3.1.1, symmetry of $\mathcal{S}$, the relation between Ψ'' and Φ'' (1.29) and the definition of the function V (5.38) are used to gain the following information from $\mathcal{J}_{1.1}$:

$$
\mathcal{J}_{1.1} \geq C\fint_{\frac{5}{3}Q} \Phi''(|D\tilde{v}|)\xi|\nabla D\tilde{v}|^2 \,\mathrm{d}x \geq C\fint_{\frac{4}{3}Q} |\nabla V(D\tilde{v})|^2 \,\mathrm{d}x. \quad (3.19)
$$

Notice that $\mathcal{J}_{1.2}$, $\mathcal{J}_{1.3}$ and the term $\mathcal{J}_2$ have similar structure and can be estimated together as follows

$$
\begin{aligned}
|\mathcal{J}_{1.2}| + |\mathcal{J}_{1.3}| + |\mathcal{J}_2| &\leq C\fint_{\frac{5}{3}Q} |\mathcal{S}(D\tilde{v}) - \mathcal{S}(Dq)|\Big(\frac{1}{R^2}|\nabla\tilde{v} - \nabla q)| + \frac{1}{R}\xi|\nabla^2\tilde{v}|\Big) \,\mathrm{d}x \\
&= \mathcal{J}_3 + \mathcal{J}_4.
\end{aligned}
\quad (3.20)
$$

In order to estimate $\mathcal{J}_3$ we use Young's inequality (A.8) together with (A.10), Korn's inequality (A.4.5) and Lemma A.2.6.

$$
\begin{aligned}
\mathcal{J}_3 \leq \frac{C}{R^2}\fint_{\frac{5}{3}Q} \Phi'_{|Dq|}(|D\tilde{v} - Dq|)|\nabla\tilde{v} - \nabla q| \,\mathrm{d}x \leq \frac{C}{R^2}\fint_{\frac{5}{3}Q} \Phi_{|Dq|}(|D\tilde{v} - Dq|) \,\mathrm{d}x \\
+\frac{C}{R^2}\fint_{\frac{5}{3}Q} \Phi_{|Dq|}(|\nabla\tilde{v} - \nabla q|) \,\mathrm{d}x \leq \frac{C}{R^2}\fint_{\frac{5}{3}Q} |V(D\tilde{v}) - V(Dq)|^2 \,\mathrm{d}x.
\end{aligned}
$$

In estimates of the last integral $\mathcal{J}_4$ the assumption on almost monotonicity of Φ'' will be needed. Using Assumption 3.1.1, the classical Young's inequality and Lemma A.2.6 we get

$$\begin{aligned}
\mathcal{J}_4 &\le \frac{C}{R} \fint_{\frac{5}{3}Q} \Phi''(|D\tilde{v}| + |Dq|)|D\tilde{v} - Dq|\xi|\nabla^2\tilde{v}| \,\mathrm{d}x \\
&\le \frac{C(\delta)}{R^2} \fint_{\frac{5}{3}Q} \Phi''(|D\tilde{v}| + |Dq|)|D\tilde{v} - Dq|^2 \,\mathrm{d}x \\
&+ \delta \fint_{\frac{5}{3}Q} \Phi''(|D\tilde{v}| + |Dq|)|\nabla^2\tilde{v}|^2\xi^2 \,\mathrm{d}x \\
&\le \frac{C}{R^2} \fint_{\frac{5}{3}Q} |V(D\tilde{v}) - V(Dq)|^2 \,\mathrm{d}x + \delta \fint_{\frac{5}{3}Q} \Phi''(|D\tilde{v}|)|\nabla^2\tilde{v}|^2\xi^2 \,\mathrm{d}x,
\end{aligned}$$

where in the last term we used the fact that Φ'' is almost decreasing. Since $\delta > 0$ can be chosen arbitrarily small, this term can be subsumed into (3.19).

We use that $Dq = \langle D\tilde{v} \rangle_{\frac{5}{3}Q}$ and apply Lemma A.2.7 to obtain

$$\fint_{\frac{5}{3}Q} |V(D\tilde{v}) - V(\langle D\tilde{v} \rangle_{\frac{5}{3}Q})|^2 \,\mathrm{d}x \le \fint_{\frac{5}{3}Q} |V(D\tilde{v}) - \langle V(D\tilde{v}) \rangle_{\frac{5}{3}Q}|^2 \,\mathrm{d}x,$$

Gathering estimates of $\mathcal{J}_1 - \mathcal{J}_4$ leads to the estimate (3.9) provided Φ'' is almost decreasing.

In case Φ'' is almost increasing, it suffices to estimate $\mathcal{J}_{1.2}$, $\mathcal{J}_{1.3}$ and $\mathcal{J}_2$ in a different way, other estimates remains the same. We integrate by parts in $\mathcal{J}_{1.2}$, $\mathcal{J}_{1.3}$ and $\mathcal{J}_2$ and obtain

$$\begin{aligned}
|\mathcal{J}_{1.2}| + |\mathcal{J}_{1.3}| + |\mathcal{J}_2| &\le \frac{C}{R} \fint_{\frac{5}{3}Q} |\nabla \mathcal{S}(D\tilde{v})||\nabla\tilde{v} - \nabla q|\xi \,\mathrm{d}x \\
&\le \delta \fint_{\frac{5}{3}Q} \Phi''(|D\tilde{v}|)|\nabla^2\tilde{v}|^2\xi^2 \,\mathrm{d}x \\
&+ \frac{C(\delta)}{R^2} \fint_{\frac{5}{3}Q} \Phi''(|D\tilde{v}|)|\nabla\tilde{v} - \nabla q|^2 \,\mathrm{d}x = \mathcal{J}_5 + \mathcal{J}_6,
\end{aligned}$$

where we moreover used the classical Young's inequality in the last step. The term $\mathcal{J}_5$ can be subsumed into (3.19). Since Φ'' is almost increasing, we can add $|\nabla\tilde{v} - \nabla q|$ to the argument of $\mathcal{J}_6$, use the definition of shifted N-functions (A.11) and apply Lemma A.2.6.

$$\begin{aligned}
\mathcal{J}_6 &\le \frac{C(\delta)}{R^2} \fint_{\frac{5}{3}Q} \Phi''(|D\tilde{v}| + |\nabla\tilde{v} - \nabla q|)|\nabla\tilde{v} - \nabla q)|^2 \,\mathrm{d}x \\
&\le \frac{C(\delta)}{R^2} \fint_{\frac{5}{3}Q} \Phi_{|D\tilde{v}|}(|\nabla\tilde{v} - \nabla q|) \,\mathrm{d}x \le \frac{C}{R^2} \fint_{\frac{5}{3}Q} |V(D\tilde{v}) - V(Dq)|^2 \,\mathrm{d}x.
\end{aligned}$$

From this point we proceed in the same way like for almost decreasing Φ''.

To prove (3.8) it is enough to focus on estimates of $\mathcal{J}_{1.2}$, $\mathcal{J}_{1.3}$ and $\mathcal{J}_2$ where the assumption of almost monotonicity was used. Considering the same test function and omitting the term $\mathcal{S}(Dq)$ in (3.17) we have

$$|\mathcal{J}_{1.2}|+|\mathcal{J}_{1.3}|+|\mathcal{J}_2| \le C\fint_{\frac{5}{3}Q} |\mathcal{S}(D\tilde{v})|\Big(\frac{1}{R^2}|\nabla\tilde{v}-\nabla q)|+\frac{1}{R}\xi|\nabla^2\tilde{v}|\Big)\,\mathrm{d}x = \mathcal{J}_7+\mathcal{J}_8.$$

In $\mathcal{J}_7$ we proceed like in $\mathcal{J}_3$. The situation is easier since we don't need to deal with shifted N-functions.

$$\begin{aligned}\mathcal{J}_7 &\le \frac{C}{R^2}\fint_{\frac{5}{3}Q}\Phi'(|D\tilde{v}|)|\nabla\tilde{v}-\nabla q|\,\mathrm{d}x \le \frac{C}{R^2}\fint_{\frac{5}{3}Q}\Phi(|D\tilde{v}|)\,\mathrm{d}x\\ &\quad+\frac{C}{R^2}\fint_{\frac{5}{3}Q}\Phi(|\nabla\tilde{v}-\nabla q|)\,\mathrm{d}x \le \frac{C}{R^2}\fint_{\frac{5}{3}Q}|V(D\tilde{v})|^2\,\mathrm{d}x.\end{aligned}$$

The term $\mathcal{J}_8$ can be handled in a similar way like $\mathcal{J}_4$:

$$\begin{aligned}\mathcal{J}_8 &\le \frac{C}{R}\fint_{\frac{5}{3}Q}\Phi''(|D\tilde{v}|)|D\tilde{v}|\xi|\nabla^2\tilde{v}|\,\mathrm{d}x\\ &\le \frac{C(\delta)}{R^2}\fint_{\frac{5}{3}Q}\Phi''(|D\tilde{v}|)|D\tilde{v}|^2\,\mathrm{d}x+\delta\fint_{\frac{5}{3}Q}\Phi''(|D\tilde{v}|)|\nabla^2\tilde{v}|^2\xi^2\,\mathrm{d}x\\ &\le \frac{C}{R^2}\fint_{\frac{5}{3}Q}|V(D\tilde{v})|^2\,\mathrm{d}x+\delta\fint_{\frac{5}{3}Q}\Phi''(|D\tilde{v}|)|\nabla^2\tilde{v}|^2\xi^2\,\mathrm{d}x,\end{aligned}$$

where the last term can be subsumed to (3.19). Thus, after similar steps as before, the estimate (3.8) holds.

Now we comment very briefly how to proceed in case we don't assume the smoothness of the solution in order to do above mentioned computations. We could consider the following truncation

$$(\Phi^\varepsilon)''(s)=\min\left(\max\left(\Phi''(s),\varepsilon\right),\frac{1}{\varepsilon}\right),\quad \varepsilon\in(0,1).$$

Except this "quadratic" approximation, which corresponds to the case $p=2$ for power-law models, we should moreover mollify boundary conditions on $\partial(\frac{5}{3}Q)^+\setminus\Gamma_{(\frac{5}{3}Q)^+}$. We would proceed in a similar way as in Chapter 2, including the limit passage. For the solution $\tilde{v}^\varepsilon$ of corresponding approximated problem would show $\tilde{v}^\varepsilon\in W^{2,2}((\frac{5}{3}Q)^+)$ via the difference quotient technique. After uniform estimates with respect to ε we would show almost everywhere convergence of symmetric velocity gradients and uniform integrability of $(\Phi^\varepsilon)'$ to pass from the approximated problem to the original one by Vitali's theorem. To pass with $\varepsilon\to 0$ in the estimates we would moreover show the uniform integrability of V^ε, where V^ε has the same relation to Φ^ε as V to Φ.

Another possibility would be the system of approximations considered in [23].

□

3.3 Flattening

In order to handle a general $\mathcal{C}^{2,1}$ non-flat boundary, we present its following description. Throughout this section, we assume that $x_0 \in \partial\Omega$ is fixed. The boundary can be understood on a neighborhood of the point x_0 as a graph of a function $a : \mathbb{R}^{n-1} \mapsto \mathbb{R}^n$, $a(0) = x_0$ such that $\partial_\alpha a(0) = e_\alpha$ for $\alpha = 1, \dots, n-1$. Using the function a we can describe the normal vector as [3] $\nu(x') = \partial_1 a \times \ldots \times \partial_{n-1} a(x')$. We introduce a function $H_{x_0} : \mathbb{R}^n \mapsto \mathbb{R}^n$ which is defined as

$$H_{x_0}(x) := a(x') - \nu(x')x_n.$$

We work with a cube $Q = Q(0, R)$, for which we denoted $Q^+ = Q \cap \mathbb{R}^n_+$. We also consider restrictions $H_{x_0,R}$ of the function H_{x_0} on a rectangle Q^+, i.e.:

$$H_{x_0,R}(x) = H_{x_0}(x)|_{Q^+}.$$

Since x_0 is fixed, we use H_R instead of $H_{x_0,R}$ throughout this chapter. It holds that $H_R(0) = x_0$ and $\nabla H_R(0) = I$ and smoothness of the boundary implies that $H_R \in \mathcal{C}^{1,1}$ and, consequently, $\nabla H_R(x) - \nabla H_R(0) = R\omega$ where ω is a function bounded independently of R. Similarly, also $\nabla H_R^{-1}(x) - \nabla H_R^{-1}(0) = R\omega$. Hereinafter, ω stands for a matrix valued function, ω' for a tensor of third order and ω'' for a real-valued function which express a perturbation arising from a curvature of the boundary. These functions may vary from line to line, however they are bounded independently of R.

The function H_R maps Q^+ into Ω for all $R \in (0, R_0)$. Furthermore, we set $y = H_R(x)$ and $\Gamma_R = \overline{H_R(Q^+)} \cap \partial\Omega$.

For a general function $f : H_R(Q^+) \mapsto \mathbb{R}$ we state a function $\overline{f} : Q^+ \mapsto \mathbb{R}$ defined as $\overline{f}(x) = f(H_R(x)) = f(y)$. It holds that

$$\nabla_y f = \nabla_x \overline{f} \nabla_x H_R^{-1} = \nabla_x \overline{f} + R \nabla_x \overline{f} \omega. \tag{3.21}$$

In case $f : H_R(Q^+) \mapsto \mathbb{R}^n$ it also holds

$$\begin{aligned} 2D_y f &= \left(\nabla_x \overline{f} \nabla_x H_R^{-1}\right) + \left(\nabla_x \overline{f} \nabla_x H_R^{-1}\right)^T = 2(D_x \overline{f} + Z_{\overline{f}}), && (3.22)\\ \operatorname{div}_y f &= \operatorname{Tr}\left(\nabla_x \overline{f} \nabla_x H_R^{-1}\right) = \operatorname{div}_x \overline{f} + \operatorname{Tr}(\nabla_x \overline{f}(\nabla_x H_R^{-1} - I)) && (3.23)\\ &= \operatorname{div}_x \overline{f} + R \operatorname{Tr}(\nabla_x \overline{f} \omega), \end{aligned}$$

where

$$\begin{aligned} Z_{\overline{f}} &= \frac{1}{2}\left(\nabla_x \overline{f}(\nabla_x H_R^{-1} - I) + (\nabla_x H_R^{-1} - I)^T (\nabla_x \overline{f})^T\right) && (3.24)\\ &= \frac{R}{2}\left(\nabla_x \overline{f} \omega + (\nabla_x \overline{f} \omega)^T\right). && (3.25) \end{aligned}$$

We consider a function $\pi_a := \pi - \pi_c$ where a constant π_c will be determined later. From the Definition 5.1.1 we have

$$\int_{H_R(Q^+)} \mathcal{S}(Du) : D\varphi \, \mathrm{d}y - \int_{H_R(Q^+)} \pi_a \operatorname{div} \varphi \, \mathrm{d}y = \int_{H_R(Q^+)} F : D\varphi \, \mathrm{d}y, \tag{3.26}$$

[3] Recall that by x' we denote the first $n-1$ coordinates of x, i.e. $x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$

whenever $\varphi \in W^{1,\Phi}_\nu(\Omega)^n$, $\varphi = 0$ on $\partial H_R(Q^+) \setminus \partial\Omega$. The equation (3.26) can be transformed using the function H_R into the following identity

$$\begin{aligned}\int_{Q^+} &\mathcal{S}(D\overline{u} + Z_{\overline{u}}) : (D\overline{\varphi} + Z_{\overline{\varphi}}) \det \nabla H_R \,\mathrm{d}x \\ &- \int_{Q^+} \overline{\pi}_a(\operatorname{div}\overline{\varphi} + \operatorname{Tr}(\nabla\overline{\varphi}(\nabla H_R^{-1} - I))) \det \nabla H_R \,\mathrm{d}x \\ &= \int_{Q^+} \overline{F} : (\nabla\overline{\varphi} + \nabla\overline{\varphi}(\nabla H_R^{-1} - I)) \det \nabla H_R \,\mathrm{d}x. \end{aligned} \tag{3.27}$$

which holds for all $\overline{\varphi} \in W^{1,\Phi}(Q^+)^n$ satisfying

$$\overline{\varphi} = 0 \text{ on } \partial Q^+ \setminus \Gamma_{Q^+}, \tag{3.28}$$

$$\overline{\varphi}\cdot\overline{\nu} = 0 \text{ on } \Gamma_{Q^+}, \tag{3.29}$$

where $\Gamma_{Q^+} = \partial Q^+ \cap \{x; x_n = 0\}$. On the flat boundary portion it is natural to work with the outward normal $\nu = -e_n$. Using the function H_R we can describe relation between $\overline{\nu}$ and $-e_n$ as $\overline{\nu} = -\nabla H_R e_n$. Thus, (3.29) can be rewritten as

$$\psi \cdot e_n = 0 \text{ on } \Gamma_{Q^+},$$

where $\psi(x) = (\nabla H_R)^T \varphi(H_R x) = (\nabla H_R)^T \overline{\varphi}$. In order to express (3.27) with the help of the test function ψ instead of $\overline{\varphi}$ we need to rewrite (3.21)–(3.23) for $f = \varphi$:

$$\begin{aligned}\nabla_y \varphi &= ((\nabla_x H_R)^T)^{-1} \nabla_x \psi (\nabla_x H_R)^{-1} + \nabla_x((\nabla_x H_R)^T)^{-1}\psi(\nabla_x H_R)^{-1} \\ &= \nabla_x \psi + R\omega\nabla_x\psi + \omega'\psi, \end{aligned} \tag{3.30}$$

$$D_y\varphi = D_x\psi + Z_\psi + \omega'\psi, \tag{3.31}$$

$$\operatorname{div}_y \varphi = \operatorname{div}_x \psi + R\operatorname{Tr}(\omega\nabla_x\psi) + \operatorname{Tr}(\omega'\psi), \tag{3.32}$$

where we used that second gradient of H_R is bounded independently of R. Denote $\nabla_x((\nabla_x H_R)^T)^{-1}$ by $\tilde\omega'$. Instead of $\tilde\omega'\psi\omega$ we can write $\omega'\psi$ for some another bounded third order tensor ω'.[4]

Whereas in (3.27) we expressed all terms using function H_R, after employment of (3.30)–(3.32) for a better lucidity we prefer to write all terms of transformed version of (3.27) with the help of general bounded functions ω, ω' and ω'' with bounds independent of R. We get:

$$\begin{aligned}\int_{Q^+} &\mathcal{S}(D\overline{u} + Z_{\overline{u}}) : (D\psi + Z_\psi + \omega'\psi)(1 + R\omega'') \,\mathrm{d}x \\ &- \int_{Q^+} \overline{\pi}_a\big(\operatorname{div}\psi + R\operatorname{Tr}(\omega\nabla\psi) + \operatorname{Tr}(\omega'\psi)\big)(1 + R\omega'') \,\mathrm{d}x \\ &= \int_{Q^+} \overline{F} : (\nabla\psi + R\omega\nabla\psi + \omega'\psi)(1 + R\omega'') \,\mathrm{d}x, \end{aligned} \tag{3.33}$$

where we used that $\det \nabla H_R = 1 + R\omega''$.

[4] In the same spirit we proceed in case of two bounded matrix functions ω_1 and ω_2. Instead of $\omega_1\psi\omega_2$ we write $\omega\psi$ for some another bounded matrix ω. The effect of this abbreviation is negligible in further estimates. Nevertheless, it affects tremendously the lucidity of calculations.

3.4 Comparison

In what follows, we verify that for a solution $\overline{u}$ there exists an approximative function $\overline{v}$, such that $D\overline{u}$ and $D\overline{v}$ satisfy assumptions of Lemma A.5.4.

Let $R_0 > 0$ be fixed and sufficiently small. Precise size of R_0 will be determined later. We denote $Q_0^+ = Q_0 \cap \mathbb{R}^n_+$, where $Q_0 = Q(0, R_0)$. Let the cube Q satisfy $Q \subset Q_0$ and $\frac{1}{4}Q \subset Q_0^+$. Denote $Q' = Q \cap \mathbb{R}^n_+$. Side of such Q is equal to $2R'$ with $R' \in (0, R_0)$. In what follows we assume that $\Gamma_{Q'} := \partial Q' \cap \{x; x_n = 0\} \neq \emptyset$. We define a function $\overline{u}_2$ as a function which satisfies the system of equation

$$\begin{aligned} \operatorname{div} \overline{u}_2 &= -R' \operatorname{Tr}(\nabla \overline{u} \omega) \text{ in } Q', && (3.34) \\ \overline{u}_2 \cdot e_n &= R'(\omega \overline{u}) \cdot e_n \text{ on } \Gamma_{Q'}. && (3.35) \end{aligned}$$

On the right hand side of (3.34) and (3.35) there appeared R', because we will apply flattening with the function $H^{-1}_{R'}$. The boundary condition (3.35) comes from the fact that we want

$$\overline{u}_2 \cdot e_n = \overline{u} \cdot e_n = (I - (\nabla H_{R'})^T)\overline{u} \cdot e_n + (\nabla H_{R'})^T \overline{u} \cdot e_n = R'(\omega \overline{u}) \cdot e_n \text{ on } \Gamma_{Q'},$$

because it holds $-(\nabla H_{R'})^T \overline{u} \cdot e_n = \overline{u} \cdot \overline{\nu} = 0$ and $(I - (\nabla H_{R'})^T) = R'\omega$. Using the fact that $\omega \in W^{2,\infty}(Q')$, Lemmata A.3.3, A.2.5 and A.4.5 we obtain

$$\begin{aligned} \int_{Q'} \Phi(|\nabla \overline{u}_2|) \,\mathrm{d}x &\leq C \left(\int_{Q'} \Phi(R'|\nabla \overline{u}|) \,\mathrm{d}x + \int_{Q'} \Phi(R'|\overline{u}|) \,\mathrm{d}x \right) \\ &\leq C(R')^\alpha \int_{Q'} \Phi(|D\overline{u}|) \,\mathrm{d}x + C \int_{Q'} \Phi\left(|\overline{u}|\right) \,\mathrm{d}x, \qquad (3.36) \end{aligned}$$

where $\alpha > 1$ comes from Lemma A.2.5. We set $\overline{u}_1 = \overline{u} - \overline{u}_2$ and from (3.36) we get

$$\int_{Q'} \Phi(|D\overline{u}_1|) \,\mathrm{d}x \leq C \int_{Q'} \Phi(|D\overline{u}|) \,\mathrm{d}x + C \int_{Q'} \Phi\left(|\overline{u}|\right) \,\mathrm{d}x. \qquad (3.37)$$

We consider a solution $\overline{v}$ to (3.2) in Q' such that

$$\begin{aligned} \overline{v} &= \overline{u}_1 \text{ on } \partial Q' \setminus \Gamma_{Q'}, && (3.38) \\ \overline{v} \cdot e_n = 0, \ \ [\mathcal{S}(D\overline{v})e_n] \cdot e_\alpha &= 0 \text{ on } \Gamma_{Q'}, && (3.39) \end{aligned}$$

where $\alpha = 1, \ldots, n-1$. Existence of such v can be shown by monotone operator theory. It is worth emphasizing that $\overline{u}_1 \cdot e_n = 0$ on $\Gamma_{Q'}$ and $\operatorname{div} \overline{u}_1 = 0$ on Q', therefore also $(\overline{u}_1 - \overline{v}) \cdot e_n = 0$ on $\Gamma_{Q'}$ and $\operatorname{div}(\overline{u}_1 - \overline{v}) = 0$ on Q'.

The integrability of approximative function (A.26) was verified in Theorem 3.2.2. The verification of (A.27) and (A.28) is presented in the following lemma.

Lemma 3.4.1 *Let $\overline{v}$ be the function constructed in* (3.38) *and* (3.39)*. Then there exists a positive constant C independent of $\overline{u}$, $\overline{v}$ and Q' such that*

$$\int_{Q'} |V(D\overline{v})|^2 \,\mathrm{d}x \leq C \int_{Q'} |V(D\overline{u})|^2 \,\mathrm{d}x + C \int_{Q'} \Phi\left(|\overline{u}|\right) \,\mathrm{d}x, \qquad (3.40)$$

Furthermore, for all δ there exists a positive constant C_δ independent of $\overline{v}$, $\overline{u}$ and Q' such that for some $\alpha > 1$ it holds

$$\int_{Q'} |V(D\overline{u}) - V(D\overline{v})|^2 \,\mathrm{d}x \le C_\delta \int_{Q'} \Phi^*(|F|)\,\mathrm{d}x \\ + \left(\delta + C(R')^\alpha\right) \int_{Q'} |V(D\overline{u})|^2 \,\mathrm{d}x + C\int_{Q'} \Phi\left(|\overline{u}|\right)\,\mathrm{d}x. \quad (3.41)$$

Proof. We choose a constant π_c such that, using considerations presented in [23, Section 4.2], we can derive that

$$\int_{Q'} \Phi^*(|\overline{\pi}_a|)\,\mathrm{d}x \le C\left(\int_{Q'} \Phi(|D\overline{u}|)\,\mathrm{d}x + \int_{Q'} \Phi^*(|\overline{F}|)\,\mathrm{d}x + \int_{Q'} \Phi\left(|\overline{u}|\right)\,\mathrm{d}x\right), \quad (3.42)$$

where we moreover employed similar steps as in (3.36).

Following [23], we test the weak formulation of (3.2) by a function $\overline{u}_1 - \overline{v}$ and obtain

$$\int_{Q'} \mathcal{S}(D\overline{v}) : D\overline{v}\,\mathrm{d}x = \int_{Q'} \mathcal{S}(D\overline{v}) : D\overline{u}_1\,\mathrm{d}x. \quad (3.43)$$

We point out that we decomposed $\overline{u}$ to $\overline{u}_1 + \overline{u}_2$ in order to $\overline{u}_1 \cdot e_n = 0$ on $\Gamma_{Q'}$ and $\operatorname{div} \overline{u}_1 = 0$ in Q' and due to the properties of the function $\overline{v}$ we also have $\operatorname{div}(\overline{u}_1 - \overline{v}) = 0$ in Q' and $(\overline{u}_1 - \overline{v})\cdot e_n = 0$ on $\Gamma_{Q'}$. Whereas the left hand side of (3.43) can be estimated from below by $\int_{Q'} |V(D\overline{v})|^2\,\mathrm{d}x$ due to Lemma A.2.6, we estimate the right hand side of (3.43) as follows

$$\int_{Q'} \mathcal{S}(D\overline{v}) : D\overline{u}_1\,\mathrm{d}x \le \delta\int_{Q'} \Phi(|D\overline{v}|)\,\mathrm{d}x + C_\delta \int_{Q'} \Phi(|D\overline{u}_1|)\,\mathrm{d}x \\ \le c\delta \int_{Q'} |V(D\overline{v})|^2\,\mathrm{d}x + C_\delta \int_{Q'} |V(D\overline{u})|^2\,\mathrm{d}x + C\int_{Q'} \Phi\left(|\overline{u}|\right)\,\mathrm{d}x,$$

where we used Young's inequality (A.8), (3.37) and Lemma A.2.6. Thus, for sufficiently small $\delta > 0$ we have

$$\int_{Q'} |V(D\overline{v})|^2\,\mathrm{d}x \le C\int_{Q'} |V(D\overline{u})|^2\,\mathrm{d}x + C\int_{Q'} \Phi\left(|\overline{u}|\right)\,\mathrm{d}x, \quad (3.44)$$

which proves (3.40).

To conclude the proof of Lemma 3.4.1, it remains to prove (3.41). The function $\overline{u}_1 - \overline{v}$ can be taken as a test function in (3.33). With the knowledge

$$\int_{Q'} S(D\overline{v}) : (D\overline{u}_1 - D\overline{v})\,\mathrm{d}x = 0$$

we derive

$$
\begin{aligned}
\int_{Q'} S(D\overline{u}+Z_{\overline{u}}) : (D\overline{u}_1 - D\overline{v} + Z_{\overline{u}_1-\overline{v}} + \omega'(\overline{u}_1-\overline{v}))(1+R'\omega'')\,\mathrm{d}x& \\
- \int_{Q'} S(D\overline{v}) : (D\overline{u}_1 - D\overline{v})\,\mathrm{d}x& \\
- \int_{Q'} \overline{\pi}_a\big(\operatorname{Tr}(\nabla\overline{u}_1 - \nabla\overline{v})R'\omega + \operatorname{Tr}(\omega'(\overline{u}_1-\overline{v}))\big)(1+R'\omega'')\,\mathrm{d}x& \\
= \int_{Q'} \overline{F} : \big(\nabla\overline{u}_1 - \nabla\overline{v} + R'\omega(\nabla\overline{u}_1 - \nabla\overline{v})\big)(1+R'\omega'')\,\mathrm{d}x& \\
+ \int_{Q'} \overline{F} : \big(\omega'(\overline{u}_1-\overline{v})\big)(1+R'\omega'')\,\mathrm{d}x =: \mathcal{I}_1 + \mathcal{I}_2.&
\end{aligned}
$$

We can rewrite this identity as follows

$$
\begin{aligned}
&\int_{Q'} \big(S(D\overline{u}_1) - S(D\overline{v})\big) : (D\overline{u}_1 - D\overline{v})\,\mathrm{d}x = \mathcal{I}_1 + \mathcal{I}_2 \\
&- \int_{Q'} S(D\overline{u}+Z_{\overline{u}}) : (D\overline{u}_1 - D\overline{v})R'\omega''\,\mathrm{d}x - \int_{Q'} S(D\overline{u}+Z_{\overline{u}}) : Z_{\overline{u}_1-\overline{v}}(1+R'\omega'')\,\mathrm{d}x \\
&\qquad - \int_{Q'} S(D\overline{u}+Z_{\overline{u}}) : (\omega'(u_1-\overline{v}))(1+R'\omega'')\,\mathrm{d}x \\
&\qquad + \int_{Q'} \big(S(D\overline{u}) - S(D\overline{u}+Z_{\overline{u}})\big) : (D\overline{u}_1 - D\overline{v})\,\mathrm{d}x \\
&+ \int_{Q'} \overline{\pi}_a \operatorname{Tr}\big((\nabla\overline{u}_1 - \nabla\overline{v})R'\omega\big)(1+R'\omega'')\,\mathrm{d}x + \int_{Q'} \overline{\pi}_a \operatorname{Tr}(\omega'(\overline{u}_1-\overline{v}))(1+R'\omega'')\,\mathrm{d}x \\
&\qquad + \int_{Q'} \big(S(D\overline{u}_1) - S(D\overline{u})\big) : (D\overline{u}_1 - D\overline{v})\,\mathrm{d}x = \sum_{i=1}^{9} \mathcal{I}_i. \qquad (3.45)
\end{aligned}
$$

The left hand side of (3.45) can be estimated from below due to Lemma A.2.6 as

$$
\begin{aligned}
\int_{Q'} \big(\mathcal{S}(D\overline{v}) - \mathcal{S}(D\overline{u}_1)\big) : (D\overline{v} - D\overline{u}_1)\ \mathrm{d}x &\geq C \int_{Q'} |V(D\overline{v}) - V(D\overline{u}_1)|^2\,\mathrm{d}x \\
&\geq C \int_{Q'} |V(D\overline{v}) - V(D\overline{u})|^2\,\mathrm{d}x - C \int_{Q'} |V(D\overline{u}) - V(D\overline{u}_1)|^2\,\mathrm{d}x \\
&= C \int_{Q'} |V(D\overline{v}) - V(D\overline{u})|^2\,\mathrm{d}x - \mathcal{I}_{10}. \qquad (3.46)
\end{aligned}
$$

Lemmata A.2.4, A.2.6 and the relation (3.36) yield

$$
\begin{aligned}
\mathcal{I}_{10} = C \int_{Q'} |V(D\overline{u}) - V(D\overline{u}_1)|^2\,\mathrm{d}x &\leq c \int_{Q'} \Phi_{|D\overline{u}|}(|D\overline{u} - D\overline{u}_1|)\,\mathrm{d}x \\
&\leq C_\delta \int_{Q'} \Phi(|D\overline{u}_2|)\,\mathrm{d}x + \delta \int_{Q'} |V(D\overline{u})|^2\,\mathrm{d}x \\
&\leq (C_\delta (R')^\alpha + \delta) \int_{Q'} |V(D\overline{u})|^2\,\mathrm{d}x + C \int_{Q'} \Phi\left(|\overline{u}|\right)\ \mathrm{d}x.
\end{aligned}
$$

Further, using Lemmata A.2.6, A.2.4 and (3.44), we have

$$
\begin{aligned}
|\mathcal{I}_9| &\le \int_{Q'} |S(D\overline{u}) - S(D\overline{u}_1)||D\overline{v} - D\overline{u}_1| \,\mathrm{d}x \le C_\delta \int_{Q'} \Phi'_{|D\overline{u}|}(|D\overline{u}_2|)|D\overline{v} - D\overline{u}_1| \,\mathrm{d}x \\
&\le C_\delta \int_{Q'} \Phi^*_{|D\overline{u}|}(\Phi'_{D\overline{u}}(|D\overline{u}_2|)) \,\mathrm{d}x + \delta \int_{Q'} \Phi_{|D\overline{u}|}(|D\overline{v} - D\overline{u}_1|) \,\mathrm{d}x \\
&\le C_\delta \int_{Q'} \Phi_{|D\overline{u}|}(|D\overline{u}_2|) \,\mathrm{d}x + \delta \int_{Q'} (|V(D\overline{u})|^2 + |V(D\overline{u}_1 - D\overline{v})|^2) \,\mathrm{d}x \\
&\le C_\delta \int_{Q'} \Phi(|D\overline{u}_2|) \,\mathrm{d}x + \delta \int_{Q'} |V(D\overline{u})|^2 \,\mathrm{d}x + \delta \int_{Q'} |V(D\overline{u}_1)|^2 \,\mathrm{d}x \\
&+ \delta \int_{Q'} |V(D\overline{v})|^2 \,\mathrm{d}x \le (C_\delta (R')^\alpha + \delta) \int_{Q'} |V(D\overline{u})|^2 \,\mathrm{d}x + C \int_{Q'} \Phi\left(|\overline{u}|\right) \,\mathrm{d}x.
\end{aligned}
$$

The term $\mathcal{I}_6$ can be estimated in the same way as term $\mathcal{I}_9$. Briefly

$$
\begin{aligned}
|\mathcal{I}_6| &\le C_\delta \int_{Q'} \Phi(R|(\omega\nabla\overline{u} + (\nabla\overline{u})^T\omega^T)|) \,\mathrm{d}x + \delta \int_{Q'} |V(D\overline{u})|^2 \,\mathrm{d}x \\
&+ \delta \int_{Q'} |V(D\overline{v})|^2 \,\mathrm{d}x + \delta \int_{Q'} |V(D\overline{u}_1)|^2 \,\mathrm{d}x \\
&\le (C(R')^\alpha + \delta) \int_{Q'} |V(D\overline{u})|^2 \,\mathrm{d}x + C \int_{Q'} \Phi\left(|\overline{u}|\right) \,\mathrm{d}x.
\end{aligned}
$$

The term $\mathcal{I}_7$ can be estimated using Young's inequality (A.8), Lemma A.2.5, Korn's inequality (A.24), (3.37), (3.44), (3.42) and Lemma A.2.6 as follows

$$
\begin{aligned}
|\mathcal{I}_7| &\le C \int_{Q'} |\overline{\pi}_a||R'(\nabla\overline{u}_1 - \nabla\overline{v})| \,\mathrm{d}x \\
&\le C_\delta \int_{Q'} \Phi(R'|\nabla\overline{u}_1 - \nabla\overline{v}|) \,\mathrm{d}x + \delta \int_{Q'} \Phi^*(|\overline{\pi}_a|) \,\mathrm{d}x \\
&\le (\delta + C(R')^\alpha) \int_{Q'} |V(D\overline{u})|^2 \,\mathrm{d}x + \delta \int_{Q'} \Phi^*(|F|) \,\mathrm{d}x + C \int_{Q'} \Phi\left(|\overline{u}|\right) \,\mathrm{d}x.
\end{aligned}
$$

Terms $\mathcal{I}_3$ and $\mathcal{I}_4$ can be estimated easily as follows

$$
|\mathcal{I}_3 + \mathcal{I}_4| \le C(R')^\alpha \int_{Q'} |V(D\overline{u})|^2 \,\mathrm{d}x + C \int_{Q'} \Phi\left(|\overline{u}|\right) \,\mathrm{d}x.
$$

In the same spirit as before

$$
\begin{aligned}
|\mathcal{I}_1| \le C \int_{Q'} |\overline{F}||\nabla\overline{u}_1 - \nabla\overline{v}| \,\mathrm{d}x &\le C_\delta \int_{Q'} \Phi^*(|\overline{F}|) \,\mathrm{d}x + \delta \int_{Q'} \Phi(|D\overline{u}_1 - D\overline{v}|) \,\mathrm{d}x \\
&\le C_\delta \int_{Q'} \Phi^*(|\overline{F}|) \,\mathrm{d}x + \delta \int_{Q'} |V(D\overline{u})|^2 \,\mathrm{d}x + C \int_{Q'} \Phi\left(|\overline{u}|\right) \,\mathrm{d}x.
\end{aligned}
$$

It remains to estimate terms $\mathcal{I}_2$, $\mathcal{I}_5$ and $\mathcal{I}_8$ which can be estimated in the same way using Young's inequality (A.8) and Poincaré inequality together with

Δ_2-condition and Lemma A.2.5:

$$|\mathcal{I}_1| + |\mathcal{I}_2| + |\mathcal{I}_8| \le C \int_{Q'} \Phi^*(|\overline{F}|)\,\mathrm{d}x + \delta \int_{Q'} \Phi^*(|\overline{\pi}_a|)\,\mathrm{d}x$$
$$+C(\delta+(R')^\alpha) \int_{Q'} |V(D\overline{u})|^2\,\mathrm{d}x + C\int_{Q'} \Phi\left(|\overline{u}|\right)\,\mathrm{d}x + C(R')^\alpha \int_{Q'} \Phi\left(\frac{\overline{u}_1 - \overline{v}}{R'}\right)\mathrm{d}x. \tag{3.47}$$

The term containing $\overline{\pi}_a$ can be estimated in the same spirit like in the term $\mathcal{I}_7$. The last term in (3.47), denoted by $\mathcal{I}_{11}$, is prepared for Poincaré inequality:

$$\mathcal{I}_{11} \le C(R')^\alpha \int_{Q'} \Phi(|D\overline{u}_1 - D\overline{v}|) \le C(R')^\alpha \left(\int_{Q'} |V(D\overline{u})|^2\,\mathrm{d}x + \int_{Q'} \Phi\left(|\overline{u}|\right)\,\mathrm{d}x \right),$$

where we moreover used Korn's inequality A.4.2 since $\overline{u}_1 - \overline{v} \in W^{1,\Phi}_\nu(Q')$ and Q' is not axisymmetric domain. The same steps like in $\mathcal{I}_9$ were also applied.

Putting these estimates into (3.46) and (3.45), we get (3.41). □

3.5 Proof of the main theorem

Proof of Theorem 3.1.3. Recall that Q' was defined as a rectangle $Q \cap \mathbb{R}^n_+$ with $Q \subset \Omega_0$, $\frac{1}{4}Q \subset Q_0^+$ and one side of Q is equal to $2R'$. In order to obtain results for a general $\mathcal{C}^{2,1}$ boundary, we are going to verify assumptions of Corollary A.5.8.

Let $x_0 \in \partial\Omega$ be fixed and let $H_{x_0,R_0}(Q_0^+) \subset \Omega$ be an image of Q_0^+ under the mapping H_{x_0,R_0}. Let R_0 be so small, such that for every rectangle $Q' \subset Q_0^+$ there exist cubes Q_a, Q_b, $x_1 \in \partial\Omega$ and $R' \in (0, R_0)$ such that

$$\Omega_{\frac{8}{9}Q_a} := \frac{8}{9}Q_a \cap \Omega \subset H_{x_0,R_0}(Q') \subset \frac{10}{9}Q_a \cap \Omega =: \Omega_{\frac{10}{9}Q_a}, \tag{3.48}$$
$$H_{x_0,R_0}\left(\frac{1}{2}Q'\right) \subset H_{x_1,R'}\left(\left(\frac{4}{3}Q_b\right)^+\right) \subset H_{x_1,R'}\left(\left(\frac{5}{3}Q_b\right)^+\right) \subset H_{x_0,R_0}(Q'). \tag{3.49}$$

This is possible, because $\nabla H_{R_0} - I = R_0\omega$, where ω is function bounded independently of R_0.

Let Q_k be a dyadic cube obtained from $\hat{Q}$, where $\hat{Q} \subset Q_0^+$ with $4\hat{Q} \subset Q_0$. If

$$H_{x_0,R_0}(4\tilde{Q}_k) \;\subset\; H_{x_0,R_0}(Q_0^+), \quad H_{x_0,R_0}(4\tilde{Q}_k) \cap \partial\Omega = \emptyset,$$

we use interior regularity result from [23]. Otherwise we proceed in the same spirit as in previous section where instead of Q' we consider $H^{-1}_{x_1,R'}(H_{x_0,R_0}(Q'))$. We point out that $H^{-1}_{x_1,R'}(H_{x_0,R_0}(Q'))$ doesn't need to be a rectangle, since in general $x_1 \ne x_0$. What is important for us is that

$$\Gamma_{H^{-1}_{x_1,R'}(H_{x_0,R_0}(Q'))} \subset \{x \in \mathbb{R}^n,\, x_n = 0\},$$

therefore results from previous section can be applied. Now we verity assumptions of Corollary A.5.8, i.e. the validity of inequalities (A.26) - (A.28) after transformation H_{x_0,R_0}.

From Theorem 3.2.2 we have

$$\left(\fint_{(\frac{4}{3}Q)^+} |V(D\overline{v})|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \le C \left(\fint_{(\frac{5}{3}Q)^+} |V(D\overline{v})|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}, \tag{3.50}$$

for $q \in \left[2, \frac{2n}{n-2}\right]$ provided $n > 2$ and $q \in [2, \infty)$ for $n = 2$. In case Φ'' is almost monotone, $n > 2$, we can even allow $q = \frac{rn}{n-r}$ for some $r > 2$. After substitution $y = H_{x_1,R'}(x)$ we get[5]

$$\left(\fint_{H_{x_1,R'}((\frac{4}{3}Q)^+)} |V(Dv + Z_v)|^q \,\mathrm{d}y \right)^{\frac{1}{q}} \le C \left(\fint_{H_{x_1,R'}((\frac{5}{3}Q)^+)} |V(Dv + Z_v)|^2 \,\mathrm{d}y \right)^{\frac{1}{2}}, \tag{3.51}$$

Using (3.49) we obtain from (3.51)

$$\left(\fint_{H_{x_0,R_0}(\frac{1}{2}Q')} |V(Dv + Z_v)|^q \,\mathrm{d}y \right)^{\frac{1}{q}} \le C \left(\fint_{H_{x_0,R_0}(Q')} |V(Dv + Z_v)|^2 \,\mathrm{d}y \right)^{\frac{1}{2}}, \tag{3.52}$$

which is the first of the three inequalities in Corollary A.5.8 where
$|w_a|^p := |V(Dv + Zv)|^2$ and $4\tilde{Q}_k \cap \mathcal{O} = Q'$. First outcome of Lemma 3.4.1 is

$$\begin{aligned} \fint_{H^{-1}_{x_1,R'}(H_{x_0,R_0}(Q'))} |V(D\overline{v})|^2 \,\mathrm{d}x \le C \fint_{H^{-1}_{x_1,R'}(H_{x_0,R_0}(Q'))} |V(D\overline{u})|^2 \,\mathrm{d}x \\ + C \fint_{H^{-1}_{x_1,R'}(H_{x_0,R_0}(Q'))} \Phi\left(|\overline{u}|\right) \,\mathrm{d}x, \end{aligned} \tag{3.53}$$

Application of the substitution $y = H_{x_1,R'}(x)$ leads to

$$\fint_{H_{x_0,R_0}(Q')} |V(Dv + Z_v)|^2 \,\mathrm{d}y \le C \fint_{H_{x_0,R_0}(Q')} |V(Du)|^2 \,\mathrm{d}y + C \fint_{H_{x_0,R_0}(Q')} \Phi\left(|u|\right) \,\mathrm{d}y, \tag{3.54}$$

where we estimated Z_u by Korn's inequality. This provides the second inequality in Corollary A.5.8 for $|w|^p = |V(Du)|^2$. The second outcome of Lemma 3.4.1 is

$$\begin{aligned} \fint_{H^{-1}_{x_1,R'}(H_{x_0,R_0}(Q'))} |V(D\overline{u}) - V(D\overline{v})|^2 \,\mathrm{d}x \le C_\delta \fint_{H^{-1}_{x_1,R'}(H_{x_0,R_0}(Q'))} \Phi^*(|F|) \,\mathrm{d}x \\ + \left(\delta + C(R')^\alpha\right) \int_{H^{-1}_{x_1,R'}(H_{x_0,R_0}(Q'))} |V(D\overline{u})|^2 \,\mathrm{d}x + C \fint_{H^{-1}_{x_1,R'}(H_{x_0,R_0}(Q'))} \Phi\left(|\overline{u}|\right) \,\mathrm{d}x. \end{aligned} \tag{3.55}$$

---

[5] Unlike in (3.24) we go from $\overline{v}$ to v, therefore we should consider ∇H_R instead of ∇H_R^{-1} in (3.24). Nevertheless, due to the properties of H_R, (3.25) remains true. We point out that Z_v is small due to R', not R_0.

After same steps as before we can estimate the left hand side of (3.55) as

$$\fint_{H_{x_0,R_0}(Q')} |V(Du+Z_u)-V(Dv+Z_v)|^2 \,\mathrm{d}y \geq \fint_{H_{x_0,R_0}(Q')} |V(Du)-V(Dv+Z_v)|^2 \,\mathrm{d}y - c\fint_{H_{x_0,R_0}(Q')} |V(Z_u) - V(Dv + Z_v)|^2 \,\mathrm{d}y. \quad (3.56)$$

The second term on the right hand side of (3.56) can be moved to the right hand side of transformed variant of (3.55) and estimated from above as

$$\fint_{H_{x_0,R_0}(Q')} |V(Z_u) - V(Dv + Z_v)|^2 \,\mathrm{d}y \leq C\fint_{H_{x_0,R_0}(Q')} |V(Du)|^2 \,\mathrm{d}y + C\fint_{H_{x_0,R_0}(Q')} \Phi(|u|) \,\mathrm{d}y.$$

where we used properties of function V, Korn's inequality on Z_u and (3.51). Altogether we have

$$\fint_{H_{x_0,R_0}(Q')} |V(Du) - V(Dv + Z_v)|^2 \,\mathrm{d}y \leq C_\delta \fint_{H_{x_0,R_0}(Q')} \Phi^*(|F|) \,\mathrm{d}y + \big(\delta + C(R')^\alpha\big)\fint_{H_{x_0,R_0}(Q')} |V(Du)|^2 \,\mathrm{d}y + C\fint_{H_{x_0,R_0}(Q')} \Phi\left(|u|\right) \,\mathrm{d}y, \quad (3.57)$$

which provides the third assumption of Corollary A.5.8 in case δ and R' in (3.57) are sufficiently small.

Let $\Phi^*(|F|) \in L^q(\Omega)$. Since $u \in W^{1,\Phi}(\Omega)$, it holds

$$\int_\Omega |\nabla\Phi(|u|)| \,\mathrm{d}x = \int_\Omega \Phi'(|u|)\,|\nabla u| \,\mathrm{d}x \leq C\int_\Omega \Phi^*\Phi'(|u|) \,\mathrm{d}x + C\int_\Omega \Phi(|\nabla u|) \,\mathrm{d}x \leq C. \quad (3.58)$$

Thus, from Orlicz–Sobolev embedding we know that $\Phi(|u|) \in L^{\frac{n}{n-1}}(\Omega)$. All assumptions of Corollary A.5.8 are met (with $g = \Phi(|u|)$ and $f = \Phi^*(|F|)+\Phi(|u|)$) and therefore we get $V(Du) \in L^{\tilde q}(H_{x_0,R_0}(\hat Q))^{n\times n}$ for $\tilde q = \min\{2q, \frac{2n}{n-1}\}$ and $\hat Q \subset Q_0^+$ with $4\hat Q \subset Q_0$. Consequently, $\Phi(|Du|) \in L^{\tilde q/2}(H_{x_0,R_0}(\hat Q))$. If $\frac{\tilde q}{2} = q$, we are done, otherwise, we use (3.58) on an N-function $\Psi := \Phi^{\frac{n}{n-1}}$ in order to get $\Phi(|u|) \in L^{\left(\frac{n}{n-1}\right)^2}(\Omega)$. We may again use Corollary A.5.8 with the same setting in order to get $V(Du) \in L^{\tilde q}(H_{x_0,R_0}(\hat Q))^{n\times n}$ for $\tilde q = \min\{2q, \frac{2n^2}{(n-1)^2}\}$. Again, if $\tilde q = 2q$ we are done, otherwise we iterate this process till $\tilde q = 2q$.

The estimate (3.1) follows easily from (A.29). To avoid the formulation of (3.1) with the mapping H_{x_0,R_0} we moreover use that there is some $r > 0$ and $\lambda > 1$ depending only on dimension such that for each $x_0 \in \partial\Omega$ it holds

$$\Omega_r := B_r(x_0) \cap \Omega \subset H_{x_0,R_0}(\hat Q) \subset H_{x_0,R_0}((4\hat Q)^+) \subset B_{\lambda r}(x_0) \cap \Omega =: \Omega_{\lambda r}.$$

□

4

L^q theory for classical Stokes system

4.1 Introduction

In this chapter we collect facts about L^q theory for the Stokes system

$$\begin{aligned} \partial_t u - \Delta u + \nabla \pi &= f && \text{in } I \times \Omega, \\ \operatorname{div} u &= 0 && \text{in } I \times \Omega, \\ u(0,\cdot) &= u_0 && \text{on } \Omega, \end{aligned} \tag{4.1}$$

equipped with the perfect slip boundary conditions

$$u \cdot \nu = 0, \quad [(Du)\nu] \cdot \tau = 0 \quad \text{on}\, I \times \partial\Omega. \tag{4.2}$$

Let E be a Banach space and $\alpha \in (0,1)$, $p, q \in [1,\infty)$, $s \in \mathbb{R}$. Recall that we use standard notation[1] for Lebesgue spaces $L^q(\Omega)$, Sobolev-Slobodeckiĭ spaces $W^{s,q}(\Omega)$, Bochner spaces $L^q(I,E)$ and $W^{\alpha,q}(I,E)$. By $H^s_q(\Omega)$ we mean Bessel potential spaces and $B^s_{p,q}(\Omega)$ are Besov spaces. BUC stands for bounded and uniformly continuous functions. Since the domain Ω is in our case at least $\mathcal{C}^{2,1}$, we can define $L^q_\sigma(\Omega)$ and $W^{1,q}_\sigma(\Omega)$ as follows:

$$\begin{aligned} L^q_\sigma(\Omega) &= \{\varphi \in L^q(\Omega),\, \operatorname{div}\varphi = 0 \,\text{in}\, \Omega,\, \varphi\cdot\nu = 0 \,\text{on}\, \partial\Omega\}, \\ W^{1,q}_\sigma(\Omega) &= \{\varphi \in W^{1,q}(\Omega),\, \operatorname{div}\varphi = 0,\, \text{in}\, \Omega,\; \varphi\cdot\nu = 0 \,\text{on}\, \partial\Omega\}. \end{aligned}$$

Set $W^{-1,p'}_\sigma(\Omega) := (W^{1,p}_\sigma(\Omega))'$.

Let P denote the projection operator from $L^q(\Omega)$ to $L^q_\sigma(\Omega)$ associated with the Helmholtz decomposition. By $Bu = 0$ we mean that (4.2) holds in the sense of traces. Using the projection P we shall define the Stokes operator $\mathbb{A}$ by $\mathbb{A}u = -P\Delta u$ for $u \in \mathcal{D}(\mathbb{A})$, where

$$\mathcal{D}(\mathbb{A}) = L^q_\sigma(\Omega) \cap H^2_{q,B}(\Omega), \quad H^2_{q,B}(\Omega) := \{u \in H^2_q(\Omega),\; Bu = 0,\; \text{on } \partial\Omega\}.$$

Applying the Helmholtz projection P to (4.1) with (4.2), we eliminate the pressure from equations and with the help of the newly established notation the Stokes system reduces to

$$\begin{aligned} \partial_t u + \mathbb{A}u = Pf, \quad \operatorname{div} u = 0 \quad \text{in}\, I \times \Omega, \\ u(0,\cdot) = u_0 \quad \text{on}\, \Omega, \quad Bu = 0 \quad \text{on}\, I \times \partial\Omega. \end{aligned} \tag{4.3}$$

[1] In this chapter we don't use different notation for scalar, vector-valued or tensor-valued functions.

4.2 Dynamical systems

Before we study properties of the Stokes operator $\mathbb{A}$ we establish some notation. Let E and F be reflexive Banach spaces. Although it is not necessary to have reflexive spaces in all definitions, for convenience we assume it. By $\mathcal{L}(E,F)$ we mean the Banach space of all bounded linear operators from E to F and $\mathcal{L}(E) := \mathcal{L}(E,E)$. If E is a linear subspace of F and the natural injection $i : x \mapsto x$ belongs to $\mathcal{L}(E,F)$, we write $E \hookrightarrow F$. In the case E is also dense in F, it will be denoted by $E \overset{d}{\hookrightarrow} F$. Furthermore, $\mathcal{L}is(E,F)$ consists of all topological linear isomorphisms from E onto F. We also write $E \doteq F$ if $E \hookrightarrow F$ and $F \hookrightarrow E$, i.e. E equals F with equivalent norms. The bilinear map $\langle \cdot, \cdot \rangle : E' \times E \to \mathbb{R}$ is the duality pairing between E' and E.

A Banach space E is said to be of class $\mathcal{HT}$, if the Hilbert transform is bounded on $L^p(\mathbb{R}, E)$ for some (and then for all) $p \in (1,\infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R}, E)$, the Schwartz space of rapidly decreasing $E-$valued functions, is defined by $Hf := \frac{1}{\pi}PV(\frac{1}{t}) * f$. It is well known theorem that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces, where the UMD stands for the property of unconditional martingale differences. Note that all closed subspaces of $L^q(\Omega)$ are UMD spaces provided $q \in (1,\infty)$.

If A is a linear operator with domain, denoted by $\mathrm{dom}(A)$, in a locally convex space X and range in a locally convex space Y, we write $A : \mathrm{dom}A \subset X \to Y$. If X and Y are normed vector spaces, $\mathcal{D}(A) := (\mathrm{dom}(A), \|\cdot\|_A)$, where $\|x\|_A := \|Ax\|_Y + \|x\|_Y$ for $x \in \mathrm{dom}(A)$, is the graph norm of A. If X and Y are Banach spaces, $\mathcal{D}(A)$ is a Banach space if and only if A is closed.

We follow the dynamical-system-type approach where the basic idea is to interpret the partial differential system as an ordinary differential equation in an infinite-dimensional Banach space. Consider a linear operator A and a boundary operator B which are general, but fixed. In order to reformulate initial-boundary value problems of the form (4.3) as an initial value problem for an ordinary differential equation in E_0:

$$\dot{u} + Au = f(t,u), \quad t > 0, \quad u(0) = u_0,$$

we have to choose our basic space E_0 in which we want to analyze the problem and impose certain minimal requirements for the operator A. E_0 will be embedded to a Banach space of distributions and we define

$$\mathrm{dom}(A) := \{u \in E_0, Au \in E_0 \text{ and } Bu = 0\}.$$

Observe that the distributions in $\mathrm{dom}(A)$ has to be regular enough to admit traces. For the linear operator A, we request that

(i) A is closed and densely defined in E_0,

(ii) A has nonempty resolvent set.

Then, denoting by E_1 the domain of A, endowed with its graph norm, we see that $E_1 \overset{d}{\hookrightarrow} E_0$, i.e. (E_0, E_1) is a densely injected Banach couple. It can be shown that the resolvent set of $-A$ consist of a half-plane $\{\lambda \in \mathbb{C}; \mathrm{Re}\lambda \geq \omega\}$ for some $\omega \in \mathbb{R}$ and that there exists a constant κ such that

$$|\lambda| \|u\|_{E_0} \leq \kappa \|(\lambda + A)u\|_{E_0}, \quad u \in E_1, \quad \mathrm{Re}\lambda \geq \omega. \tag{4.4}$$

The conditions (i) and (ii) together with (4.4) is equivalent to the assertion that $-A$ generates a strongly continuous analytic semigroup $\{e^{-At}; t \geq 0\}$ on E_0. By $\mathcal{H}(E_1, E_0)$ we denote the set of all $A \in \mathcal{L}(E_1, E_0)$ such that $-A$, considered as a linear (possibly unbounded) operator in E_0 with a domain E_1, is the infinitesimal generator of a strongly continuous analytic semigroup on E_0. It holds

$$\mathcal{H}(E_1, E_0) = \bigcup_{\kappa \geq 1, \omega > 0} \mathcal{H}(E_1, E_0, \kappa, \omega),$$

where $A \in \mathcal{H}(E_1, E_0, \kappa, \omega)$ if $\omega + A \in \mathcal{L}is(E_1, E_0)$ and

$$\kappa^{-1} \leq \frac{\|(\lambda + A)u\|_{E_0}}{|\lambda| \|u\|_{E_0} + \|u\|_{E_1}} \leq \kappa, \quad \mathrm{Re}\lambda \geq \omega, \quad u \in E_1.$$

By $\sigma(A)$ we mean the spectrum of A and $\varrho(A)$ denotes the resolvent set. A linear operator A in E is said to be of positive type if it belongs to $\mathcal{P}(E) := \bigcup_{K>1} P_K(E)$. $A \in P_K(E)$ if it is closed, densely defined, $\mathbb{R}^+ \subset \varrho(-A)$ and $(1+s)\|(s+A)^{-1}\|_{\mathcal{L}(E)} \leq K$ for $s \in \mathbb{R}^+$, where $K \geq 1$.

We say that a linear operator A in E is of type (E, K, ϑ), denoted by $A \in \mathcal{P}(E, K, \vartheta)$, if it is densely defined and if

$$\Sigma_\vartheta := \{|\arg z| \leq \vartheta\} \cup \{0\} \subset \varrho(-A) \quad \text{and} \quad (1+|\lambda|)\|(\lambda+A)^{-1}\|_{\mathcal{L}(E)} \leq K, \quad \lambda \in \Sigma_\vartheta.$$

Put $\mathcal{P}(E, \vartheta) := \bigcup_{K>1} \mathcal{P}(E, K, \vartheta)$.

A linear operator A in E is said to have bounded imaginary powers, in symbols $A \in \mathcal{BIP}(E)$, if $A \in \mathcal{P}(E)$ and there exist $\epsilon > 0$ and $K \geq 1$ such that $A^{is} \in \mathcal{L}(E)$ and $\|A^{is}\|_{\mathcal{L}(E)} \leq K$ for $s \in (-\epsilon, \epsilon)$. We write $A \in \mathcal{BIP}(E, K, \theta)$, if $A \in \mathcal{BIP}(E)$ and if there are constants $K \geq 1$ and $\theta \geq 0$ such that $\|A^{is}\|_{\mathcal{L}(E)} \leq K e^{\theta |s|}$ for all $s \in \mathbb{R}$. It holds that

$$\mathcal{BIP}(E) := \bigcup_{K \geq 1, \theta \geq 0} \mathcal{BIP}(E, K, \theta).$$

We introduce an interpolation-extrapolation scale which is essential in the proof of Theorem 4.3.8. Let $p, q \in (1, \infty)$, $\theta \in (0,1)$ and $[\cdot,\cdot]_\theta$ denotes the complex and $(\cdot,\cdot)_{\theta,q}$ the real interpolation functor. Let $A \in \mathcal{H}(E_1, E_0)$. Then we denote by $[(E_\alpha, A_\alpha); \alpha \in \mathbb{R}]$ the interpolation-extrapolation scale generated by (E, A) and $[\cdot,\cdot]_\theta$ or $(\cdot,\cdot)_{\theta,q}$, where we set $E_k := \mathcal{D}(A^k)$ for $k \in \mathbb{N}$ with $k \geq 2$. Also set $E^\sharp := E'$ and $A^\sharp := A'$, where A' is the dual of A in E in the sense of unbounded linear operators. Finally let $E_k^\sharp := \mathcal{D}((A^\sharp)^k)$ for $k \in \mathbb{N}$. Then we define E_{-k} for $k \in \mathbb{N}$ by $E_{-k} := (E_k^\sharp)'$. We put $E_{k+\theta} := [E_k, E_{k+1}]_\theta$ (and similarly for the real interpolation functor). If $\alpha \geq 0$ we denote by A_α the maximal restriction of A to E_α whose domain equals $\{u \in E_\alpha \cap E_1;\ Au \in E_\alpha\}$. If $\alpha < 0$ then A_α is the closure of A in E_α.

For the dual interpolation functor $(\cdot,\cdot)^\sharp_\theta$ (which is equal to $[\cdot,\cdot]_\theta$ for the complex interpolation and $(\cdot,\cdot)_{\theta,q'}$ for real interpolation) we abbreviate the interpolation-extrapolation scale generated by $(E^\sharp, A^\sharp)$ and $(\cdot,\cdot)^\sharp_\theta$, by $[(E^\sharp_\alpha, A^\sharp_\alpha); \alpha \in \mathbb{R}]$ and call it interpolation-extrapolation scale dual to $[(E_\alpha, A_\alpha); \alpha \in \mathbb{R}]$.
It holds $(E_{-\alpha})' \doteq E^\sharp_\alpha$ and $(A_{-\alpha})' = A^\sharp_\alpha$. For more details see [3, Section V.2].

4.3 L^q theory for Stokes system

Now we are ready to mention some basic properties of the Stokes operator $\mathbb{A}$. From [81] we know that $\mathbb{A} \in \mathcal{H}(L^q_\sigma(\Omega) \cap H^2_{q,B}(\Omega), L^q_\sigma(\Omega))$. This also tells us that $\mathbb{A} \in \mathcal{P}(L^q_\sigma(\Omega), \omega)$ for some $\omega \in [0, \pi/2)$ (see [30, Theorem II.4.6]). R. Shimada later showed in [82] the L^q-maximal regularity for $\mathbb{A}$. In [1, Theorem 1] H. Abels and Y. Terasawa proved:

Proposition 4.3.1 *Let $q \in (1,\infty)$, $n \geq 2$, $r \in (n,\infty]$ such that $q, q' \leq r$. Let $\Omega \subset \mathbb{R}^n$ be a domain with $W^{2-\frac{1}{r},r}-$boundary and $\vartheta \in (0,\pi)$. Then there is some $R > 0$ such that $(\lambda + \mathbb{A})^{-1}$ exists and*

$$(1 + |\lambda|)\|(\lambda + \mathbb{A})^{-1}\|_{\mathcal{L}(L^q(\Omega))} \leq C$$

for all $\lambda \in \Sigma_\vartheta$ with $|\lambda| \geq R$. Moreover,

$$\left\| \int_{\Gamma_R} h(-\lambda)(\lambda + \mathbb{A})^{-1}\,\mathrm{d}\lambda \right\|_{\mathcal{L}(L^q(\Omega))} \leq C\|h\|_{L^\infty(\Sigma_{\pi-\vartheta})}$$

for every $h \in H^\infty(\vartheta)$, where $\Gamma = \partial\Sigma_\vartheta$, $\Gamma_R = \Gamma \setminus \overline{B_R(0)}$ and $H^\infty(\vartheta)$ denotes the Banach algebra of all bounded holomorphic functions $h : \Sigma_{\pi-\vartheta} \to \mathbb{C}$. In particular, for every $\omega \in \mathbb{R}$ and $\vartheta' \in (0,\vartheta]$ such that $\omega + \Sigma_{\vartheta'} \subset \varrho(-\mathbb{A})$ the shifted Stokes operator $\omega + \mathbb{A}$ admits a bounded $H^\infty-$calculus with respect to ϑ', i.e.,

$$h(\omega + \mathbb{A}) := \frac{1}{2\pi i}\int_\Gamma h(-\lambda)(\lambda + \omega + \mathbb{A})^{-1}\,\mathrm{d}\lambda$$

is a bounded operator satisfying

$$\|h(\omega + \mathbb{A})\|_{\mathcal{L}(L^q(\Omega))} \leq C\|h\|_{L^\infty(\Sigma_{\pi-\vartheta})}$$

for all $h \in H^\infty(\vartheta')$.

Note that the class of operators with a bounded $H^\infty-$calculus is a subclass of the operators which have $\mathcal{BIP}$, therefore these operators admit all important properties which have operators with bounded imaginary powers. For another properties of a bounded $H^\infty-$calculus we refer for example to [19, Section 2, Subsection 2.4], [41] or [58].

From the result of Y. Shibata and R. Shimada in [81] follows that $\omega + \Sigma_{\vartheta'} \subset \varrho(-\mathbb{A})$ even for $\omega = 0$ provided the domain Ω is bounded and non-axisymmetric (see Definition 1.4.8). Thus, Proposition 4.3.1 and [81, Theorem 1.3] gives $\mathbb{A} \in \mathcal{BIP}$. Let $\mathbb{E}_\alpha$ be interpolation-extrapolation scale and $\mathbb{A}_\alpha$ be the realization of $\mathbb{A}$ on $\mathbb{E}_\alpha$ for $\alpha \geq -1$. From [83, Section 2.2] we know that $\mathbb{A}_\alpha \in \mathcal{H}(\mathbb{E}_{\alpha+1}, \mathbb{E}_\alpha)$ for $\alpha \geq -1$. Steiger in [83] provides the characterization of spaces $\mathbb{E}_\alpha$:

Proposition 4.3.2 *[83, Corollary 2.6] Set*

$$s_\alpha := \{-2+1/q, -1+1/q, 1/q, 1+1/q\}$$

and $F^s_q(\Omega) := H^s_p(\Omega)$ *for the complex interpolation functor and* $F^s_q(\Omega) := B^s_{q,q}(\Omega)$ *for the real interpolation functor. Define*

$$F^s_{q,B}(\Omega) := \begin{cases} \{u \in F^s_q(\Omega),\ Bu = 0 \ on\, \partial\Omega\}, & s \in (1+1/q, 2], \\ \{u \in F^s_q(\Omega),\ u\cdot\nu = 0 \ on\, \partial\Omega\}, & s \in [1/q, 1+1/q), \\ F^s_q(\Omega), & s \in [0, 1/q), \\ \big(F^{-s}_{q',B}(\Omega)\big)', & s \in [-2, 0) \setminus s_\alpha \end{cases} \tag{4.5}$$

and

$$F^s_{q,B,\sigma}(\Omega) := \begin{cases} F^s_{q,B}(\Omega) \cap L^q_\sigma(\Omega), & s \in [0,2] \setminus s_\alpha, \\ \big(F^{-s}_{q',B,\sigma}(\Omega)\big)', & s \in [-2,0) \setminus s_\alpha. \end{cases} \tag{4.6}$$

Then $\mathbb{E}_\alpha \doteq F^{2\alpha}_{q,B,\sigma}(\Omega)$ *for* $2\alpha \in [-2,2] \setminus s_\alpha$.

This gives

$$\mathbb{A}_\alpha \in \mathcal{H}(F^{2\alpha+2}_{q,B,\sigma}(\Omega), F^{2\alpha}_{q,B,\sigma}(\Omega)), \quad 2\alpha \in [-2,2] \setminus s_\alpha. \tag{4.7}$$

Remark 4.3.3 *[83, Remark 2.3c] The Helmholtz projection* P *enjoys following continuity properties:*

$$P \in \mathcal{L}(F^s_{q,B}(\Omega)) \cap \mathcal{L}(F^s_{q,B}(\Omega), F^s_{q,B,\sigma}(\Omega)), \quad s \in (-1+1/q, 1+1/q) \setminus s_\alpha. \tag{4.8}$$

We will use the fact, that the property of bounded imaginary powers can be carried over the interpolation-extrapolation scales:

Proposition 4.3.4 *[3, Proposition V.1.5.5] Let* $A \in \mathcal{P}(E)$ *and let* $[(E_\alpha, A_\alpha);\ \alpha \in (-n, \infty)]$ *be the interpolation-extrapolation scale generated by* (E, A) *and an exact functor. If* $A \in \mathcal{BIP}(E, M, \sigma)$ *then* $A_\alpha \in \mathcal{BIP}(E_\alpha, M, \sigma)$.

The reiteration property will be needed.

Proposition 4.3.5 *[3, Theorem V.1.5.4] Suppose that* $A \in \mathcal{BIP}(E)$. *Then the interpolation-extrapolation scale* $[(E_\alpha, A_\alpha); \alpha \in [-n, \infty)]$ *generated by* (E, A) *and complex interpolation functor possesses the reiteration property*

$$[E_\alpha, E_\beta]_\eta \doteq E_{(1-\eta)\alpha+\eta\beta}, \quad -n \le \alpha \le \beta < \infty, \quad \eta \in (0,1).$$

Let us define the maximal L^q-regularity for an operator A (compare [3, Section III.1, Subsection 1.5 and Section III.4, Remark 4.10.9.c])

Definition 4.3.6 *Let* $A \in \mathcal{H}(E_1, E_0)$ *and* $q \in (1, \infty)$. *We say that*

$$\big(L^q(I, E_0), L^q(I, E_1) \cap W^{1,q}(I, E_0)\big)$$

is a pair of maximal regularity for A *(or* A *has maximal regularity), if for* $u_0 \in (E_0, E_1)_{1-1/q,q}$ *and* $f \in L^q(I, E_0)$ *there exists a unique solution* $u \in L^q(I, E_1) \cap W^{1,q}(I, E_0)$ *of* (4.3)*, and*

$$\|\partial_t u\|_{L^q(I,E_0)} + \|u\|_{L^q(I,E_0)} + \|Au\|_{L^q(I,E_0)} \le C\Big(\|f\|_{L^q(I,E_0)} + \|u_0\|_{(E_0,E_1)_{1-1/q,q}}\Big). \tag{4.9}$$

Further we mention the relation between maximal regularity and negative infinitesimal generators of a bounded analytic semigroup.

Proposition 4.3.7 *[3, Theorem III.4.10.7] Suppose that E_0 is a UMD space, $A \in \mathcal{H}(E_1, E_0)$ and there are constants $M > 0$, $\vartheta \in (0, \pi/2)$ such that $\Sigma_\vartheta \subset \varrho(-A)$ and for $\lambda \in \Sigma_\vartheta$ and $j = 0, 1$ holds*

$$\|A\|_{\mathcal{L}(E_1,E_0)} + (1 + |\lambda|)^{1-j}\|(\lambda + A)^{-1}\|_{\mathcal{L}(E_0,E_j)} \le M$$

and suppose that there exist constants $N \ge 1$ and $\theta \in [0, \pi/2)$ such that $A \in \mathcal{BIP}(E_0, N, \theta)$. Then A has maximal regularity and the estimate (4.9) holds uniformly with respect to T.

The main result of this section is the following theorem, in which we use the abbreviation from Proposition 4.3.2. In particular, $B^{1-2/q}_{q,q,B,\sigma}(\Omega) = F^s_{q,B,\sigma}(\Omega)$ for $s = 1 - 2/q$ and the real interpolation functor.

Theorem 4.3.8 *Let $\Omega \subset \mathbb{R}^n$ be a bounded non-axisymmetric $\mathcal{C}^{2,1}$ domain, $q \in [2, \infty)$, $f \in L^q(I, W^{-1,q}_\sigma(\Omega))$, $u_0 \in B^{1-2/q}_{q,q,B,\sigma}(\Omega)$ then there exists a constant $C > 0$ and the unique weak solution to (4.3) satisfying*

$$\|\nabla u\|_{L^q(I\times\Omega)} + \|u\|_{BUC(I,B^{1-2/q}_{q,q,B,\sigma}(\Omega))} \le C\Big(\|f\|_{L^q(I,W^{-1,q}_\sigma(\Omega))} + \|u_0\|_{B^{1-2/q}_{q,q,B,\sigma}(\Omega)}\Big).$$

The constant C is independent of T, u, f and u_0.

Proof. We consider the system (4.3) instead of (4.1) with (4.2). Since for UMD space E, E' is one as well and for an interpolation couple of UMD spaces the interpolation spaces are also UMD (see [3, Theorem III.4.5.2]), $\mathbb{E}_{-1/2}$ is a UMD space. Proposition 4.3.4 gives us $\mathbb{A}_{-1/2}$ has $\mathcal{BIP}$. Together with (4.7), [3, Corollary I.1.4.3] and [81, Theorem 1.3] we can see that assumptions of Proposition 4.3.7 are fulfilled for $\mathbb{A}_{-1/2}$. Therefore we obtain (4.9) for $\mathbb{A}_{-1/2}$ and $E_0 = \mathbb{E}_{-1/2}$:

$$\begin{aligned} \|\partial_t u\|_{L^q(I,\mathbb{E}_{-1/2})} + \|u\|_{L^q(I,\mathbb{E}_{-1/2})} + \|\mathbb{A}_{-1/2}u\|_{L^q(I,\mathbb{E}_{-1/2})} \\ \le C\Big(\|f\|_{L^q(I,\mathbb{E}_{-1/2})} + \|u_0\|_{(\mathbb{E}_{-1/2},\mathbb{E}_{1/2})_{1-1/q,q}}\Big). \end{aligned} \tag{4.10}$$

It remains to determine the spaces in (4.10). For the space of initial condition u_0 we get by Proposition 4.3.2 for the complex interpolation functor

$$u_0 \in (H^{-1}_{q,B,\sigma}(\Omega), H^1_{q,B,\sigma}(\Omega))_{1-1/q,q}.$$

This space equals (with equivalent norms) to $B^{1-2/q}_{q,q,B,\sigma}(\Omega)$ since for $q \ge 2$

$$\begin{aligned} B^{1-2/q}_{q,q,B,\sigma}(\Omega) &\doteq (L^q_\sigma(\Omega), H^1_{q,B,\sigma}(\Omega))_{1-2/q,q} \\ &\doteq ([H^{-1}_{q,B,\sigma}(\Omega), H^1_{q,B,\sigma}(\Omega)]_{1/2}, H^1_{q,B,\sigma}(\Omega))_{1-2/q,q} \\ &\doteq (H^{-1}_{q,B,\sigma}(\Omega), H^1_{q,B,\sigma}(\Omega))_{1-1/q,q}, \end{aligned} \tag{4.11}$$

where we used Proposition 4.3.5. The similar interpolation of the solenoidal functions in case of Dirichlet boundary conditions is done in [4, Proof of Lemma 9.1]. From the embedding [3, Theorem V.4.10.2]

$$L^q(I, E_1) \cap W^{1,q}(I, E_0) \hookrightarrow BUC(I, (E_0, E_1)_{1-1/q,q}),$$

we obtain $u \in BUC(I, B^{1-2/q}_{q,q,B,\sigma}(\Omega))$. Due to $\|u\|_{\mathbb{E}_{1/2}} = \|\mathbb{A}_{-1/2}u\|_{\mathbb{E}_{-1/2}}$ (which follows from [3, Corollary V.1.3.9 and Theorem V.1.5.4]) and $\mathbb{E}_{1/2} \doteq W^{1,q}_{\sigma}(\Omega)$ we have boundedness of ∇u in $L^q(I \times \Omega)$. It remains to find the space for f. By Proposition 4.3.2

$$f \in L^q(I, W^{-1,q}_{\sigma}(\Omega)),$$

since $H^s_q(\Omega) \doteq W^{s,q}(\Omega)$ for $s \in \mathbb{Z}$.

□

Remark 4.3.9 *In case of homogeneous Dirichlet boundary conditions we are able to obtain the same result like in Theorem* 4.3.8. *We can omit the assumption on the shape of the domain* Ω. *The results needed in proof of Theorem* 4.3.8 *are for the homogeneous Dirichlet conditions more extended. It is well known that the Stokes operator has bounded imaginary powers provided* Ω *is whole space, half-space, bounded domain or exterior domain with sufficiently smooth boundary, c.f. [36] and [38].* L^q *estimates based on the theory developed by G. Dore and A. Venni in [28] can be found in [37] and [39].*

4.4 Interpolation

Without loss of generality we may assume that there exists a symmetric tensor $G \in L^q(Q)$, such that the weak formulation of the right hand side of (4.1) can be written in the form

$$\int_I\int_\Omega G : D\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_I \langle f, \varphi\rangle \,\mathrm{d}t \quad \forall \varphi \in L^{q'}(I, W^{1,q'}_{\sigma}(\Omega)). \tag{4.12}$$

To prove it, we proceed in the same way like in [52, Proof of Proposition 2.1, Step 1] where the authors are dealing with periodic boundary conditions. Consider the Stokes system which can be formulated in the weak form for a. a. $t \in I$ as follows

$$\int_\Omega Dw(t) : D\varphi \,\mathrm{d}x = \langle f(t), \varphi\rangle \quad \forall \varphi \in W^{1,q'}_{\sigma}(\Omega). \tag{4.13}$$

As $f \in L^q(I, W^{-1,q}_{\sigma}(\Omega))$, there exists a solution $w(t) \in W^{1,q}_{\sigma}(\Omega)$ of (4.13) enjoying the estimate

$$\|w(t)\|_{W^{1,q}(\Omega)} \le C\|f\|_{W^{-1,q}_{\sigma}(\Omega)}$$

with the positive constant C independent of t. Consequently, w can be constructed such that $w \in L^q(I, W^{1,q}_{\sigma}(\Omega))$ and

$$\|w\|_{L^q(I,W^{1,q}(\Omega))} \le C\|f\|_{L^q(I,W^{-1,q}_{\sigma}(\Omega))}.$$

Defining $G = Dw$ we conclude (4.12) from (4.13) by density arguments. Therefore for all $f \in L^q(I, W_\sigma^{-1,q}(\Omega))$ there exists $G \in L^q(I \times \Omega)$ such that (4.12) and following estimate

$$\|G\|_{L^q(I\times\Omega)} \le C\|f\|_{L^q(I,W_\sigma^{-1,q}(\Omega))}$$

holds. We would like to point out that the perfect slip boundary conditions are hidden in the weak formulation. If G is smooth enough then it holds

$$\int_I \langle f, \varphi\rangle \,\mathrm{d}t = -\int_I\int_\Omega \operatorname{div} G\cdot\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_I\int_{\partial\Omega} (G\nu)\tau(\varphi\cdot\tau)\,\mathrm{d}\sigma\,\mathrm{d}t \ \forall \varphi \in L^{q'}(I, W_\sigma^{1,q'}(\Omega)).$$

The Stokes system (4.1) with (4.2) can be formulated in the weak form as follows

$$\int_I \langle \partial_t u, \varphi\rangle \,\mathrm{d}t + \int_I\int_\Omega Du : D\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_I\int_\Omega G : D\varphi \,\mathrm{d}x\,\mathrm{d}t \quad \forall \varphi \in L^{q'}(I, W_\sigma^{1,q'}(\Omega)). \tag{4.14}$$

Introducing the solution operator $S : (G, u_0) \mapsto Du$, we conclude first from the existence theory, that S is continuous from $L^2(Q) \times L^2_\sigma(\Omega)$ to $L^2(I \times \Omega)$ with the norm less or equal to 1. By Theorem 4.3.8 we know that S is continuous from $L^{q_1}(I \times \Omega) \times B^{1-2/q_1}_{q_1,q_1,B,\sigma}(\Omega)$ to $L^{q_1}(I \times \Omega)$ with norm estimated by $C_q > 1$. Since $S(G, u_0) = S(G, 0) + S(0, u_0)$, Riesz-Thorin theorem and the real interpolation method implies following assertion, see for example [13, Theorem 5.2.1 and Theorem 6.4.5].

Lemma 4.4.1 *Let Ω be a bounded non-axisymmetric $\mathcal{C}^{2,1}$ domain and $q_1 > 2$. There exist constant $C > 0$ and $K := C_{q_1}^{q_1/(q_1-2)}$ such that for every $q \in (2, q_1)$, arbitrary $G \in L^q(I, L^q_\sigma(\Omega))$, $u_0 \in B^{1-2/q}_{q,q,B,\sigma}(\Omega)$ there exists a unique solution u of* (4.14) *satisfying*

$$\|Du\|_{L^q(I\times\Omega)} \le K^{1-\frac{2}{q}}\Big(\|G\|_{L^q(I\times\Omega)} + C\|u_0\|_{B^{1-2/q}_{q,q,B,\sigma}(\Omega)}\Big).$$

4.5 L^q theory for generalized Stokes system

For $q > 2$ small enough Lemma 4.4.1 allows us to prove the L^q theory for a generalized Stokes system, where the Stokes operator is replaced by a general elliptic operator with bounded measurable coefficients. More precisely, let
$0 < \gamma_1 \le \gamma_2$ and suppose that the coefficient matrix $\mathbb{M} \in L^\infty(I \times \Omega)$ is symmetric in the sense $M_{ij}^{kl} = M_{kl}^{ij} = M_{kl}^{ji}$ for $i, j, k, l = 1, 2$ and fulfils for all $B \in \mathbb{R}^{2\times 2}$, $x \in \Omega$ and $t \in I$

$$\gamma_1|B|^2 \le \mathbb{M}(t, x) : B \otimes B \le \gamma_2|B|^2.$$

We consider the following system

$$\int_I \langle \partial_t u, \varphi\rangle \,\mathrm{d}t + \int_I\int_\Omega \mathbb{M} : Du \otimes D\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_I\int_\Omega G : D\varphi \,\mathrm{d}x\,\mathrm{d}t \qquad \forall \varphi \in L^{q'}(I, W_\sigma^{1,q'}(\Omega)). \tag{4.15}$$

Following lemma states the L^q theory result.

Lemma 4.5.1 *Let* Ω *be a bounded non-axisymmetric* $\mathcal{C}^{2,1}$ *domain and* $q > 2$. *There exist constants* $K, L > 0$ *such that if* $q \in [2, 2 + L\frac{\gamma_1}{\gamma_2})$, $G \in L^q(Q)$ *and* $u_0 \in B^{1-2/q}_{q,q,B,\sigma}(\Omega)$ *then the unique weak solution* $u \in L^q(I, W^{1,q}_\sigma(\Omega))$ *of* (4.15) *satisfies*

$$\|Du\|_{L^q(I\times\Omega)} + \gamma_2^{-\frac{1}{q}}\|u\|_{BUC(I,B^{1-2/q}_{q,q,B,\sigma}(\Omega))} \le \frac{K}{\gamma_1}\Big(\|G\|_{L^q(I\times\Omega)} + \gamma_2^{1-\frac{1}{q}}\|u_0\|_{B^{1-2/q}_{q,q,B,\sigma}(\Omega)}\Big).$$

Proof. We omit the proof. It can be found in [51, Proposition 2.1] for periodic boundary conditions or in [49, Proposition 2.1] for homogeneous Dirichlet boundary conditions. The only generalization consists of including perfect slip boundary conditions. L^q theory result for classical Stokes system with perfect slip boundary conditions is needed, but it is shown in Lemma 4.4.1. □

We also use the L^q theory for stationary variant of the system (4.15). For symmetric coefficient matrix $\mathbb{M} \in L^\infty(\Omega)$ fulfilling for all $B \in \mathbb{R}^{2\times 2}$ and $x \in \Omega$ $\gamma_1|B|^2 \le \mathbb{M}(x) : B \otimes B \le \gamma_2|B|^2$, $0 < \gamma_1 \le \gamma_2$ we investigate the problem

$$\int_\Omega \mathbb{M} : Du \otimes D\varphi \,\mathrm{d}x = \int_\Omega G : D\varphi \,\mathrm{d}x \quad \forall \varphi \in W^{1,q'}_\sigma(\Omega). \tag{4.16}$$

It holds:

Lemma 4.5.2 *Let* Ω *be a bounded non-axisymmetric* $\mathcal{C}^{2,1}$ *domain. Then there are constants* $K, L > 0$ *such that if* $q \in [2, 2 + L\frac{\gamma_1}{\gamma_2})$ *and* $G \in L^q(\Omega)$, *then the unique weak solution of* (4.16) *satisfies*

$$\|Du\|_{L^q(\Omega)} \le \frac{K}{\gamma_1}\|G\|_{L^q(\Omega)}.$$

Proof. See [51, Lemma 2.6] for no slip boundary conditions. For perfect slip boundary conditions we would proceed analogically. □

5

Full regularity of weak solutions to equations of evolutionary planar flows

5.1 Main theorem

In this chapter we investigate flows of incompressible shear-thickening fluids, which in evolutionary case are governed by the system (1.19)–(1.22):

$$
\begin{aligned}
\partial_t u - \operatorname{div} \mathcal{S}(Du) + \operatorname{div}(u \otimes u) + \nabla \pi &= f && \text{in } I \times \Omega,\\
\operatorname{div} u &= 0 && \text{in } I \times \Omega,\\
u(0,\cdot) &= u_0 && \text{in } \Omega,\\
u \cdot \nu = 0, \quad [\mathcal{S}(Du)\nu] \cdot \tau &= 0 && \text{on } I \times \partial\Omega.
\end{aligned}
$$

We restrict ourselves to the case of planar flows, i.e. $\Omega \subset \mathbb{R}^2$. Instead of the general formulation in the framework of N-functions we assume that the extra stress tensor $\mathcal{S}$ possess $p-$potential structure with $p \geq 2$. More precisely, we can construct scalar potential $\Phi : [0,\infty) \mapsto [0,\infty)$ to the stress tensor $\mathcal{S}$, i.e.

$$
\mathcal{S}(A) = \partial_A \Phi(|A|) = \Phi'(|A|)\frac{A}{|A|} \quad \forall A \in \mathbb{R}^{2\times 2}_{sym},
$$

such that $\Phi \in \mathcal{C}^{1,1}((0,\infty)) \cap \mathcal{C}^1([0,\infty))$, $\Phi(0) = 0$ and there exist $p \in [2,\infty)$ and $0 < C_1 \leq C_2$ such that for all $A, B \in \mathbb{R}^{2\times 2}_{sym}$

$$
C_1(1+|A|^2)^{\frac{p-2}{2}}|B|^2 \leq \partial^2_A \Phi(|A|) : B \otimes B \leq C_2(1+|A|^2)^{\frac{p-2}{2}}|B|^2. \tag{5.1}
$$

In analysis of equations of fluid motions the question of Hölder continuity of velocity gradients is an important issue. For instance, in optimal control problems, global regularity results that guarantee boundedness of velocity gradients are needed in order to establish the existence of the weak solution for adjoint equation to the original problem and for linearised models. These results are closely related to the regularity of the coefficients in the main part of the associated differential operators and enable to derive corresponding optimality conditions, as is done for example in [87]. For optimal control of flows with shear dependent viscosities in the stationary case where the author is dealing with the lack of the regularity result we refer to [6] and [7].

Hölder continuity of velocity gradients is also important when one studies exponential attractors. With such a regularity it is possible to show the differentiability of the solution operator with respect to the initial condition, which is the key technical step in the method of Lyapunov exponents. Differentiability of the solution is equivalent to the linearisation of the equation around particular solution which is used to study infinitesimal volume elements and leads to sharp dimension estimates of the global attractor. This is done for example in [53].

We closely follow [49], where P. Kaplický shows Hölder continuity of velocity gradients and pressure for (1.19)–(1.21) with $p \in [2, 4)$ under no slip boundary conditions. Based on the same structure of the proof and using the results from [54] we extend the result to perfect slip boundary conditions and $p \in [2, \infty)$. Although some steps of the proof in [49] can be easily modified, we have to overcome a new difficulties connected to the another type of boundary conditions. Particularly, the L^p theory result for the Stokes problem equipped with perfect slip boundary conditions has to be established (this is done in previous chapter). Keeping at our disposal results from Chapter 2, we are able to cover the case $p \geq 4$. From the point of application it would be very interesting to obtain also the result for the case $p \in (1, 2)$ for perfect slip or homogeneous Dirichlet boundary condition.

The idea of the proof goes back to [72], where the authors show that every weak solution u of $\partial_t u - \operatorname{div}(\mathcal{S}(\nabla u)) = 0$ in Q has locally Hölder continuous gradient in case that $\Omega \subset \mathbb{R}^2$ and $p = 2$. This result was extended in [33] to the case $p \in (1, 2)$. Regularity of $\partial_t u$ is shown first and after moving $\partial_t u$ to the right hand side the stationary L^q theory is applied.

In the case of generalized Newtonian fluids this method was modified in [52], where the authors consider the shear-thinning fluid model with periodic boundary conditions. In contrary to [72] the regularity of $\partial_t u$ and ∇u had to be obtained at once. The authors showed that velocity gradients are Hölder continuous for $p \in (4/3, 2]$. These results were extended to electro-rheological fluids and non-zero initial condition in [21].

Among many works concerning regularity theory for generalized Newtonian fluids we would like to mention two papers dealing with the stationary case. In [51] the stationary version of (1.19)–(1.21) under homogeneous Dirichlet boundary conditions is considered. The same authors later in [50] studied the problem equipped with non-homogeneous Dirichlet boundary conditions with two types of restriction on boundary data and perfect slip boundary conditions.

We begin with the definition of the weak solution to the problem (1.19)–(1.22).

Definition 5.1.1 *Let $f \in L^{p'}(I, W^{-1,p'}_\sigma(\Omega))$, $p \in [2, \infty)$ and $u_0 \in L^2(\Omega)$. We say that the function $u : I \times \Omega \mapsto \mathbb{R}^2$ is a weak solution to the problem* (1.19)–(1.22), *if $u \in L^\infty(I, L^2(\Omega)) \cap L^p(I, W^{1,p}_\sigma(\Omega))$, $\partial_t u \in L^{p'}(I, W^{-1,p'}_\sigma(\Omega))$, $u(0, \cdot) = u_0$ in $L^2(\Omega)$ and weak formulation*

$$\int_I \langle \partial_t u, \varphi \rangle \,\mathrm{d}t + \int_I \int_\Omega \mathcal{S}(Du) : D\varphi \,\mathrm{d}x \,\mathrm{d}t + \int_I \int_\Omega (u \cdot \nabla) u \varphi \,\mathrm{d}x \,\mathrm{d}t = \int_I \langle f, \varphi \rangle \,\mathrm{d}t$$

holds for all $\varphi \in L^p(I, W^{1,p}_\sigma(\Omega))$.

If we studied also the case $p \in (1,2)$, we would have to consider only test functions from the space of smooth functions which are regular enough. It is well known that the weak solution exists and is unique. It could be easily proven using the monotone operator theory. See for example [67, Chapter 5] for periodic boundary conditions.

Now we formulate the main results of this paper.

Theorem 5.1.2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded non-circular $\mathcal{C}^3$ domain and* (5.1) *holds for some $p \in [2,\infty)$. Let $u_0 \in W^{2+\beta,2}(\Omega)$ for $\beta \in (0,1/4)$, $\operatorname{div} u_0 = 0$, $f \in L^\infty(I, L^{q_0}(\Omega))$ and $\partial_t f \in L^{q_0}(I, W_\sigma^{-1,q_0'}(\Omega))$ for some $q_0 > 2$. Then there exists a unique solution (u,π) of* (1.19)–(1.22)*, such that for some $\alpha > 0$*

$$\nabla u, \pi \in \mathcal{C}^{0,\alpha}(\overline{I \times \Omega}).$$

Remark 5.1.3 *It would be very interesting to obtain the same result as in Theorem* 5.1.2 *also for the Navier's boundary condition. In several parts of the proof of Theorem* 5.1.2 *we apply results from [54] that are formulated only for perfect slip boundary conditions. We don't know how to generalize these results also for partial slip boundary conditions.*

The proof of the main theorem is divided into two parts. At first, the main theorem is proven in the case of quadratic growth, i.e. $p = 2$. Further we introduce the quadratic approximation of the stress tensor $\mathcal{S}(Du)$ which is done by the truncation of the generalized viscosity from above, i.e. $\mu^\varepsilon(|Du^\varepsilon|) := \min\{\mu(|Du|), 1/\varepsilon\}$ for $\varepsilon \in (0,1)$. We prove the main result for the approximated problem and pass from the approximated problem to the original one at the end.

5.2 Quadratic potential

In this section we prove Theorem 5.1.2 for $p = 2$.

Step 1 *recalls apriori estimates from the existence theory.*

For $f \in W^{1,2}(I, W_\sigma^{-1,2}(\Omega))$ with $f(0) \in L^2(\Omega)$ and $u_0 \in W^{2,2}(\Omega) \cap W_\sigma^{1,2}(\Omega)$ we know the existence of a unique weak solution of (1.19)–(1.22) fulfilling

$$\begin{aligned} u &\in L^\infty(I, L^2(\Omega)) \cap L^2(I, W_\sigma^{1,2}(\Omega)),\\ \partial_t u &\in L^\infty(I, L^2(\Omega)) \cap L^2(I, W_\sigma^{1,2}(\Omega)),\\ \pi &\in L^2(I, L^2(\Omega)). \end{aligned} \tag{5.2}$$

It can be shown using Galerkin approximation. Let $\{\omega^k\}_{k=1}^\infty$ be the orthogonal basis of $L^2_\sigma(\Omega)$ and $W_\sigma^{1,2}(\Omega)$ consisting of eigenvectors of the Stokes operator with perfect slip boundary conditions. Such basis can be easily constructed provided Ω is non-circular domain. Set $H^n = \operatorname{span}\{\omega_1, \ldots, \omega^N\}$ and define the continuous projection $P^N : L^2_\sigma(\Omega) \to H^N$ as follows:

$$P^N u = \sum_{k=1}^N (u, \omega^k)\omega^k.$$

Define $u^N(t,x) = \sum_{k=1}^N c_k^N(t)\omega^k$ where $c_k^N(t)$ solves the Galerkin system

$$\begin{aligned}\langle \partial_t u^N(t), \omega^k\rangle + \int_\Omega \mathcal{S}(Du^N) : D(\omega^k)\,\mathrm{d}x + \int_\Omega (u^n \otimes u^n) : \nabla\omega^k\,\mathrm{d}x = \langle f, w^k\rangle,\\ u^N(0) = u_0^N = P^N u_0, \quad 1 \le k \le N.\end{aligned} \tag{5.3}$$

After multiplying the Galerkin system (5.3) by $c_k^N(t)$, summing up, using Gronwall's and Korn's inequalities we derive the following apriori estimate

$$\sup_{t\in I} \|u^N(t)\|_2^2 + \int_I \|u^N(\tau)\|_{1,2}^2\,\mathrm{d}\tau \le C.$$

Further we apply the time derivative to (5.3), multiply it by $\partial_t c_k^N(t)$ and sum up. Unlike the previous apriori estimates, before using Gronwall's inequality, the boundednes of $\|\partial_t u^N(0)\|_2^2$ needs to be shown. This can be done easily, since $P^N : W^{2,2}(\Omega)\cap W_\sigma^{1,2}(\Omega) \to H^N$ is bounded uniformly with respect to N (c. f. [67, Lemma 4.26] in case of periodic boundary conditions), we can use (5.3). Thus, after Gronwall's inequality we have

$$\sup_{t\in I} \|\partial_t u^N(t)\|_2^2 + \int_I \|\partial_t u^N(\tau)\|_{1,2}^2\,\mathrm{d}\tau \le C.$$

Passing to the limit with $N \to \infty$ (where we use the Aubin-Lions' lemma to obtain the strong convergence of u^N in $L^2(I, L^4(\Omega))$ and Minty's trick to identify the limit of $\mathcal{S}(Du^N)$ with $\mathcal{S}(Du)$) we get the first two relations in (5.2).

Since $\partial_t u$, $\operatorname{div}\mathcal{S}(Du)$, $\operatorname{div}(u\otimes u)$ and f lie in $L^2(I, W_\sigma^{-1,2}(\Omega))$, we can reconstruct the pressure π at almost every time level via De Rham's theorem (Theorem A.5.2) and Nečas' theorem on negative norms (Theorem A.5.1) and obtain $\pi \in L^2(\Omega)$ for almost every $t \in I$.

Step 2 *improves the regularity in space.*

If we additionally assume $f \in L^\infty(I, L^2(\Omega))$ we are able to show that

$$u \in L^\infty(I, W^{2,2}(\Omega)), \quad \pi \in L^\infty(I, W^{1,2}(\Omega)). \tag{5.4}$$

From Step 1 we know that $\partial_t u$ is regular enough in order to move it to the right hand side of (1.19). At almost every time level $t \in I$ we can use the stationary theory. Boundary regularity in tangent direction is based on the difference quotient technique. In normal direction near the boundary the main tools are the operator curl and Nečas' theorem on negative norms. For details see Section 2.2. The information about the pressure comes from the fact that the right hand side of $\nabla\pi = f + \operatorname{div}\mathcal{S} - \operatorname{div}(u\otimes u) - \partial_t u$ is in $L^2(\Omega)$ for a. a. $t \in I$. Adding the assumption $\int_\Omega \pi\,\mathrm{d}x = 0$ we get by Poincaré inequality the existence of $\pi \in W^{1,2}(\Omega)$ at almost every time level $t \in I$ together with a bound independent of t.

Step 3 *improves the regularity in time using L^p theory for Stokes system.*

If we moreover suppose that $f \in L^{q_1}(I, W_\sigma^{-1,q_1'}(\Omega))$ for some $q_1 > 2$ and $u_0 \in W^{2+\beta,2}(\Omega)$ for $\beta \in (0, 1/4)$ we are able to prove the existence of $q_2 > 2$ such that the unique weak solution satisfies for all $q \in (2, q_2)$

$$\partial_t u \in L^q(I, W_\sigma^{1,q}(\Omega)) \cap BUC(I, B_{q,q,B,\sigma}^{1-2/q}(\Omega)). \tag{5.5}$$

Denoting $w := \partial_t u$ and $\tau := \partial_t \pi$ in the sense of distributions, we observe from (1.19) that (w, τ) solves

$$\int_I \langle \partial_t w, \varphi \rangle \, \mathrm{d}t + \int_Q \partial^2_{Du} \Phi(|Du|) : Dw \otimes D\varphi \, \mathrm{d}x \, \mathrm{d}t = \int_I \langle \partial_t (f - (u \cdot \nabla) u), \varphi \rangle \, \mathrm{d}t \tag{5.6}$$

for all $\varphi \in L^q(I, W_\sigma^{1,q}(\Omega))$. It is easy to see that $\partial_t (u \cdot \nabla u) \in L^s(I, W^{-1,s}(\Omega))$ for all $s \in [1, 4]$.

In order to obtain (5.5) as a result of application of Lemma 4.5.1 for the system (5.6) we need to ensure that $\|\partial_t u(0)\|_{B_{q,q,B,\sigma}^{1-2/q}(\Omega)}$ is bounded. Let $\beta \in (0, 1/4)$ and $\varphi \in W^{-\beta,2}(\Omega)$ with $\|\varphi\|_{W^{-\beta,2}(\Omega)} \le 1$ be arbitrary. We recall that the Helmholtz projection P enjoys the continuity properties as mentioned in Remark 4.3.3 Thus,

$$\begin{aligned} |\langle \partial_t u(0), \varphi \rangle| = |\langle \partial_t u(0), P\varphi \rangle| &\le |\langle \operatorname{div} S(Du_0) + (u_0 \cdot \nabla) u_0 - f(0), P\varphi \rangle| \\ &\le C(\|u_0\|_{W^{2+\beta,2}(\Omega)} + \|u_0\|^2_{W^{2,2}(\Omega)} + \|f(0)\|_{W^{\beta,2}(\Omega)}) \le C. \end{aligned} \tag{5.7}$$

Since $W^{\beta,2}(\Omega) \hookrightarrow B_{q,q}^{1-2/q}(\Omega)$ if q is close enough to 2 we obtain

$$\|\partial_t u(0)\|_{B_{q,q,B,\sigma}^{1-2/q}(\Omega)} \le C$$

for all $q \in (2, q_2)$ where q_2 is sufficiently close to 2.

Moving second term on the left hand side of (5.6) to the right hand side and taking supremum over $\varphi \in L^{q'}(I, W_\sigma^{1,q'}(\Omega))$ with norm less or equal to 1 we obtain

$$\partial_{tt} u \in L^q(I, W_\sigma^{-1,q}(\Omega)). \tag{5.8}$$

Step 4 *gives $u \in L^\infty(I, W^{2,q}(\Omega))$ due to the stationary theory.*

Previous step shows us that $\partial_t u \in L^\infty(I, L^q(\Omega))$ for some $q > 2$. Therefore we are able to move $\partial_t u$ to the right hand side of (1.19) and apply the stationary result [50, Theorem 3] for $p = 2$ which tells us that there exists a positive ε, such that $u \in W^{2,2+\varepsilon}(\Omega)$ and $\pi \in W^{1,2+\varepsilon}(\Omega)$ for a. a. $t \in I$.

Step 5 *improves the regularity of π in time.*

There exists a $q > 2$ such that for all $s \in (0, \frac{1}{2})$

$$\pi \in W^{s,q}(I, L^q(\Omega)). \tag{5.9}$$

We closely follow the proof of [49, Lemma 3.4]. For a function $g(t)$ defined on the time interval I and $(t_1,t_2)\subset I$ set $\delta^t g := g(t_2)-g(t_1)$. The idea of the proof is based on subtracting the equation (1.19) in the time t_2 from the same equation in time t_1 which leads to

$$\int_\Omega \delta^t\pi \operatorname{div}\varphi \,\mathrm{d}x = \int_\Omega [\delta^t(\partial_t u - f)\varphi - \delta^t(u\otimes u - \mathcal{S}(Du))D\varphi]\,\mathrm{d}x \quad \text{for a.a. } t_1,t_2\in I, \tag{5.10}$$

which holds for all $\varphi\in W^{1,2}(\Omega)$ with $\varphi\cdot\nu=0$ on $\partial\Omega$. From (5.5) and (5.8) one may easily show by interpolation that for all $s\in(0,1/2)$ there exists $q>2$ such that $u\in W^{s,q}(I,W^{1,q}(\Omega))$ and $\partial_t u\in W^{s,q}(I,L^q(\Omega))$. Together with the assumptions on the right hand side f we can notice that (5.10) holds also for all $\varphi\in W^{1,q'}(\Omega)$ with $\varphi=0$ at $\partial\Omega$. Consider the problem

$$\begin{aligned} \operatorname{div}\varphi^t &= \delta^t\pi|\delta^t\pi|^{q-2} - \frac{1}{|\Omega|}\int_\Omega \delta^t\pi|\delta^t\pi|^{q-2}\,\mathrm{d}x \quad \text{in } \Omega,\\ \varphi^t &= 0 \quad \text{at } \partial\Omega. \end{aligned} \tag{5.11}$$

The right hand side of (5.11) has zero mean value over Ω and belongs to $L^{q'}(\Omega)$ due to (5.4), therefore Bogovskiĭ's Lemma A.3.1 guaranties the existence of φ^t satisfying the estimate $\|\varphi^t\|_{1,q'}\le C\|\delta^t\pi\|_q^{q-1}$. Taking φ^t as a test function in (5.10) leads to

$$\|\delta^t\pi\|_q^q \le \varepsilon\|\delta^t\pi\|_q^q + C_\varepsilon(\|\delta^t\partial_t u\|_q^q + \|\delta^t f\|_{-1,q}^q + \|\delta^t\nabla u\|_q^q). \tag{5.12}$$

Dividing (5.12) by $|t_2-t_1|^{1+sq}$ and integrating twice over I gives

$$\|\pi\|^q_{W^{s,q}(I,L^q(\Omega))} = \int_I\int_I \frac{\|\delta^t\pi\|_q^q}{|t_2-t_1|^{1+sq}}\,\mathrm{d}t_1\,\mathrm{d}t_2 \le C,$$

which concludes the proof of (5.9).

Step 6 *summarizes the result of this section and uses imbedding theorems to complete the proof.*

Up to now we have shown

$$u\in L^\infty(I,W^{2,q}(\Omega))\cap W^{1,q}(I,L^q(\Omega)), \quad \pi\in L^\infty(I,W^{1,q}(\Omega))\cap W^{s,q}(I,L^q(\Omega)).$$

As we are in two dimensions, $q>2$, $s\in(\frac{1}{q},\frac{1}{2})$, following imbeddings hold

$$L^\infty(I,W^{1,q}(\Omega))\hookrightarrow L^\infty(I,\mathcal{C}^{0,1-\frac{2}{q}}(\overline{\Omega})), \tag{5.13}$$

$$W^{1,q}(I,L^q(\Omega))\hookrightarrow \mathcal{C}^{1-\frac{1}{q}}(\overline{I},L^q(\Omega)), \tag{5.14}$$

$$W^{s,q}(I,L^q(\Omega))\hookrightarrow \mathcal{C}^{s-\frac{1}{q}}(\overline{I},L^q(\Omega)). \tag{5.15}$$

Now we are ready to apply

Lemma 5.2.1 *[49, Lemma 2.6] Let $\Omega\subset\mathbb{R}^2$ be a bounded $\mathcal{C}^2$ domain. Let $f\in L^\infty(I,\mathcal{C}^{0,\alpha}(\overline{\Omega}))$ and $f\in\mathcal{C}^{0,\beta}(\overline{I},L^s(\Omega))$ for some $\alpha,\beta\in(0,1)$ and $s>1$. Then $f\in\mathcal{C}^{0,\gamma}(\overline{I\times\Omega})$ with $\gamma=\min\{\alpha,\frac{\alpha\beta s}{\alpha s+2}\}$.*

Using (5.13) and (5.14) together with Lemma 5.2.1 we obtain $\nabla u\in\mathcal{C}^{0,\alpha}(\overline{I\times\Omega})$ for certain $\alpha>0$. (5.13), (5.15) with Lemma 5.2.1 gives us $\pi\in\mathcal{C}^{0,\alpha}(\overline{I\times\Omega})$ for some $\alpha>0$, which concludes the proof of main results for $p=2$. □

5.3 Super-quadratic potential

In this section we prove Theorem 5.1.2 for $p > 2$. The proof consists of several steps.

Step 1 *introduces quadratic approximations.*

In a similar way like in Section 2.3 of Chapter 2 we are concerned with the regularized problem

$$\begin{aligned} \partial_t u^\varepsilon - \operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon) + (u^\varepsilon \cdot \nabla)u^\varepsilon + \nabla \pi^\varepsilon = f, \quad \operatorname{div} u^\varepsilon = 0 \text{ in } I \times \Omega, \\ u^\varepsilon(0, \cdot) = u_0 \text{ in } \Omega, \end{aligned} \tag{5.16}$$

where we consider quadratic approximation $\mathcal{S}^\varepsilon$ of $\mathcal{S}$ defined for $\varepsilon \in (0,1)$ by the truncation of the viscosity μ from above:

$$\mu^\varepsilon(|Du^\varepsilon|) := \min\Big\{\mu(|Du|), \frac{1}{\varepsilon}\Big\}, \quad \mathcal{S}^\varepsilon(Du^\varepsilon) := \mu^\varepsilon(|Du^\varepsilon|)Du^\varepsilon. \tag{5.17}$$

Scalar potential Φ^ε to $\mathcal{S}^\varepsilon(Du^\varepsilon)$ can be constructed in the following way

$$\Phi^\varepsilon(s) := \int_0^s \mu^\varepsilon(t)t \,\mathrm{d}t$$

and satisfies growth conditions (5.1) for $p = 2$, i.e. there exists $C_1 > 0$ and $C(\varepsilon)$ such that for all $A, B \in \mathbb{R}^{2\times 2}_{sym}$

$$C_1|B|^2 \leq \partial^2_A \Phi^\varepsilon(|A|) : B \otimes B \leq C(\varepsilon)|B|^2. \tag{5.18}$$

The approximation (5.17) guarantees that for a fixed $\varepsilon \in (0,1)$ the results of the previous section holds for u^ε and π^ε solving (5.16) equipped with the perfect slip boundary conditions.

Step 2 *gives growth conditions dependent on ε.*

Due to the results of the previous section we are able to use techniques which enable us to gain uniform estimates with respect to ε. At first we need a growth estimates of Φ^ε with precise dependence on ε. In other words, the constant $C(\varepsilon)$ in the estimate (5.18) needs to be specified. To this purpose we define the function ϑ^ε by $\vartheta^\varepsilon(s) := \min\{(1+s^2)^{\frac{1}{2}}, \frac{1}{\varepsilon}\}$. Now, there exist constants $0 < C_3 \leq C_4$ such that for all $\varepsilon \in (0,1)$ and $A, B \in \mathbb{R}^{2\times 2}_{sym}$

$$C_3\vartheta^\varepsilon(|A|)^{p-2}|B|^2 \leq \partial^2_A \Phi^\varepsilon(|A|) : B \otimes B \leq C_4\vartheta^\varepsilon(|A|)^{p-2}|B|^2. \tag{5.19}$$

As a corollary of (5.19) following estimates can be derived (see [68, Lemma 2.22] for the proof.)

$$C\vartheta^\varepsilon(|A|)^{p-2}|A|^2 \leq \mathcal{S}^\varepsilon(A) : A, \tag{5.20}$$

$$C|\mathcal{S}^\varepsilon(A)| \leq \vartheta^\varepsilon(|A|)^{p-2}|A|. \tag{5.21}$$

The lower estimate in (5.20) can be done independent of ε, since (5.18) holds:

$$C_5|A|^2 \le \mathcal{S}^\varepsilon(A) : A. \tag{5.22}$$

At this point we would like to emphasize that from now all constants in following steps are independent of ε.

Step 3 *provides* $L^\infty(I, L^2(\Omega)) \cap L^2(I, W^{1,2}(\Omega))$ *estimates of* u^ε *and* $\partial_t u^\varepsilon$.

We recall estimates from the previous section which hold also for the approximated problem since the lower bound in (5.22) is independent on ε.

$$\|u^\varepsilon\|_{L^\infty(I,L^2(\Omega))} + \|\nabla u^\varepsilon\|_{L^2(Q)} \le C, \tag{5.23}$$

$$\|\partial_t u^\varepsilon\|^2_{L^\infty(I,L^2(\Omega))} + \|\nabla \partial_t u^\varepsilon\|_{L^2(Q)} \le C. \tag{5.24}$$

The relation (5.23) is an apriori estimate obtained by taking solution as a test function (at the level of Galerkin approximation). Roughly speaking, the estimate (5.24) is performed by taking time derivative of the equation (5.16) and testing by time derivative of u^ε. More precisely, it is not applied directly to the equation (5.16), but still to the Galerkin system. In order to estimate the time derivative of the Galerkin approximation of u^ε at the time $t = 0$ we proceed in the same way like in (5.7).

Note that (5.23) and (5.24) give $u^\varepsilon \in L^\infty(I, W^{1,2}(\Omega))$:

$$\begin{aligned}\|\nabla u^\varepsilon(s,\cdot)\|_2^2 - \|\nabla u^\varepsilon(0,\cdot)\|_2^2 &= \int_\Omega \int_0^s \partial_t |\nabla u^\varepsilon(t,\cdot)|^2 \,\mathrm{d}t\,\mathrm{d}x \\ &\le 2\|\nabla u^\varepsilon\|_{L^2(I\times\Omega)} \|\partial_t \nabla u^\varepsilon\|_{L^2(I\times\Omega)} \le C.\end{aligned}$$

Step 4 *describes the boundary* $\partial\Omega$.

In order to discuss boundary regularity in following steps, we need a suitable description of the boundary $\partial\Omega$. Let us denote $x = (x_1, x_2)$. We suppose that $\Omega \in \mathcal{C}^3$, therefore there exists $c_0 > 0$ such that for all $a_0 > 0$ there exists n_0 points $P \in \partial\Omega$, $r > 0$ and open smooth set $\Omega_0 \subset\subset \Omega$ that we have

$$\Omega \subset \Omega_0 \cup \bigcup_P B_r(P)$$

and for each point $P \in \partial\Omega$ there exists local system of coordinates for which $P = 0$ and the boundary $\partial\Omega$ is locally described by $\mathcal{C}^3$ mapping a_P that for $x_1 \in (-3r, 3r)$ fulfils

$$\begin{aligned} x \in \partial\Omega \Leftrightarrow x_2 = a_P(x_1), \quad & B_{3r}(P) \cap \Omega = \{x \in B_r(P) \,\text{and}\, x_2 > a_P(x_1)\} =: \Omega^P_{3r}, \\ \partial_1 a_P(0) = 0, \quad & |\partial_1 a_P(x_1)| \le a_0, \quad |\partial_1^2 a_P(x_1)| + |\partial_1^3 a_P(x_1)| \le c_0. \end{aligned}$$

Points P can be divided into k groups such that in each group Ω^P_{3r} are disjoint and k depends only on dimension n. Let the cut-off function $\xi_P(x) \in \mathcal{C}^\infty(B_{3r}(P))$ and reaches values

$$\xi_P(x) \begin{cases} = 1 & x \in B_r(P), \\ \in (0,1) & x \in B_{2r}(P) \setminus B_r(P), \\ = 0 & x \in \mathbb{R}^2 \setminus B_{2r}(P). \end{cases}$$

Next, we assume that we work in the coordinate system corresponding to P. Particularly, $P = 0$. Let us fix P and drop for simplicity the index P. The tangent vector and the outer normal vector to $\partial\Omega$ are defined as

$$\tau = \big(1, \partial_1 a(x_1)\big), \quad \nu = \big(\partial_1 a(x_1), -1\big),$$

tangent and normal derivatives as

$$\partial_{\tau^\alpha} = \partial_1 + \partial_1 a(x_1)\partial_2, \quad \partial_\nu = -\partial_2 + \partial_1 a(x_1)\partial_1.$$

Step 5 *gives* $u^\varepsilon \in L^\infty(I, W^{2,2}(\Omega))$ *uniformly in* $\varepsilon \in (0,1)$.

From Step 3 we obtained that $\partial_t u^\varepsilon \in L^\infty(I, L^2(\Omega))$, therefore we can fix $t \in I$, move $\partial_t u^\varepsilon$ to the right hand side of (5.16) and at almost every time level consider the stationary problem

$$\begin{aligned} -\operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon) + (u^\varepsilon \cdot \nabla)u^\varepsilon + \nabla\pi^\varepsilon = h, \quad \operatorname{div} u^\varepsilon = 0 \text{ in } \Omega, \\ u^\varepsilon \cdot \nu = 0, \ [\mathcal{S}^\varepsilon(Du^\varepsilon)\nu] \cdot \tau = 0 \text{ at } \partial\Omega, \end{aligned} \tag{5.25}$$

where $h := f - \partial_t u^\varepsilon \in L^2(\Omega)$. Section 5.2 provides $u^\varepsilon \in W^{2,2}(\Omega)$, $\mathcal{S}^\varepsilon(Du^\varepsilon) \in W^{1,2}(\Omega)$ and $\pi^\varepsilon \in W^{1,2}(\Omega)$. Thus we can multiply (5.25) by a suitable test function which is at least in $L^2(\Omega)$ and integrate over Ω. We focus only on the boundary regularity and work in the local system of coordinates. Following [54, Lemma 4.2, Remark 4.9] we choose as a test function $\varphi = (\varphi_1, \varphi_2)$

$$\begin{aligned} \varphi &= (\partial_2\Theta, -\partial_1\Theta), \\ \Theta : &= \partial_\nu(u^\varepsilon \cdot \tau)\xi^2 - u^\varepsilon \cdot (\partial_\nu\tau + \partial_{\tau^\alpha}\nu)\xi^2 - \partial_{\tau^\alpha}(u^\varepsilon \cdot \nu)\xi^2. \end{aligned}$$

This test function is constructed in order to get rid of the pressure π^ε and to obtain optimal information from the elliptic term. These most difficult estimates, in which we extract from $-\int_\Omega \operatorname{div} \mathcal{S}^\varepsilon(Du^\varepsilon) \cdot \varphi \,\mathrm{d}x$ boundedness of the term $\int_\Omega \mu^\varepsilon(|Du^\varepsilon|)|\nabla^2 u^\varepsilon|^2 \,\mathrm{d}x$, are done in Chapter 2, Section 2.3, therefore we omit the calculations. It remains to estimate the convective term and the right hand side of (5.25). We start with the convective term.

$$\begin{aligned} \int_\Omega [(u^\varepsilon \cdot \nabla)u^\varepsilon \cdot \varphi \,\mathrm{d}x &= \int_\Omega \partial_2\Theta(u_1^\varepsilon\partial_1 u_1^\varepsilon + u_2^\varepsilon\partial_2 u_1^\varepsilon) - \partial_1\Theta(u_1^\varepsilon\partial_1 u_2^\varepsilon + u_2^\varepsilon\partial_2 u_2^\varepsilon)] \,\mathrm{d}x \\ &= \int_\Omega \Theta(-u_1^\varepsilon\partial_2\partial_1 u_1^\varepsilon - u_2^\varepsilon\partial_2^2 u_1^\varepsilon + u_1^\varepsilon\partial_1^2 u_2^\varepsilon + u_2^\varepsilon\partial_1\partial_2 u_2^\varepsilon) \,\mathrm{d}x = \mathcal{J}_1, \end{aligned}$$

where we used the fact that there arise no boundary terms while integrating by parts since $\Theta = 0$ at $\partial\Omega$. Four terms in $\mathcal{J}_1$ vanished due to $\partial_1 u_1^\varepsilon = -\partial_2 u_2^\varepsilon$. Now we put together terms containing u_1^ε and integrate by parts in the direction x_1, in other terms we integrate by parts in direction x_2. We get

$$\begin{aligned} \mathcal{J}_1 = \int_\Omega \Theta(\partial_1 u_1^\varepsilon\partial_2 u_1^\varepsilon - \partial_1 u_1^\varepsilon\partial_1 u_2^\varepsilon + \partial_2 u_2^\varepsilon\partial_2 u_1^\varepsilon - \partial_2 u_2^\varepsilon\partial_1 u_2^\varepsilon) \,\mathrm{d}x \\ + \int_\Omega [\partial_1\Theta(u_1^\varepsilon\partial_2 u_1^\varepsilon - u_1^\varepsilon\partial_1 u_2^\varepsilon) + \partial_2\Theta(u_2^\varepsilon\partial_2 u_1^\varepsilon - u_2^\varepsilon\partial_1 u_2^\varepsilon)] \,\mathrm{d}x = \mathcal{J}_2 + \mathcal{J}_3. \end{aligned}$$

One can easily see that $\mathcal{J}_2 = 0$ since $\operatorname{div} u^\varepsilon = 0$. Using

$$\Theta = (-\partial_2 u_1^\varepsilon + \partial_1 u_2^\varepsilon - 2a'' u_1^\varepsilon)\xi^2$$

we write out $\mathcal{J}_3$ in the following way

$$\begin{aligned}\mathcal{J}_3 = \int_\Omega &[(-\partial_1\partial_2 u_1^\varepsilon + \partial_1^2 u_2^\varepsilon)(u_1^\varepsilon \partial_2 u_1^\varepsilon - u_1^\varepsilon \partial_1 u_2^\varepsilon) \\ &\qquad + (-\partial_2^2 u_1^\varepsilon + \partial_1\partial_2 u_2^\varepsilon)(u_2^\varepsilon \partial_2 u_1^\varepsilon - u_2^\varepsilon \partial_1 u_2^\varepsilon)]\xi^2 \,\mathrm{d}x \\ + \int_\Omega &[(-2a''' u_1^\varepsilon \xi^2 - 2a''\partial_1 u_1^\varepsilon \xi^2 + (-\partial_2 u_1^\varepsilon + \partial_1 u_2^\varepsilon - 2a'' u_1^\varepsilon)2\xi\partial_1\xi)(u_1^\varepsilon \partial_2 u_1^\varepsilon - u_1^\varepsilon \partial_1 u_2^\varepsilon) \\ + (-2a''\partial_2 u_1^\varepsilon \xi^2 &+ (-\partial_2 u_1^\varepsilon + \partial_1 u_2^\varepsilon - 2a'' u_1^\varepsilon)2\xi\partial_2\xi)(u_2^\varepsilon \partial_2 u_1^\varepsilon - u_2^\varepsilon \partial_1 u_2^\varepsilon)] \,\mathrm{d}x = \mathcal{J}_4 + \mathcal{J}_5.\end{aligned}$$

The term $\mathcal{J}_4$ can be rewritten in the following way:

$$\begin{aligned}\mathcal{J}_4 &= -\frac{1}{2}\int_\Omega [u_1 \partial_1(\partial_1 u_2^\varepsilon - \partial_2 u_1^\varepsilon)^2 + u_2^\varepsilon \partial_2(\partial_1 u_2^\varepsilon - \partial_2 u_1^\varepsilon)^2]\xi^2 \,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega (\partial_1 u_1^\varepsilon + \partial_2 u_2^\varepsilon)(\partial_1 u_2^\varepsilon - \partial_2 u_1^\varepsilon)^2 \xi^2 \,\mathrm{d}x + \int_\Omega (u_1^\varepsilon \xi\partial_1\xi + u_2^\varepsilon \xi \partial_2 \xi)(\partial_1 u_2^\varepsilon - \partial_2 u_1^\varepsilon)^2 \,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} (u_1^\varepsilon \nu_1 + u_2^\varepsilon \nu_2)(\partial_1 u_2^\varepsilon - \partial_2 u_1^\varepsilon)^2 \xi^2 \,\mathrm{d}x = \mathcal{J}_6 + \mathcal{J}_7 + \mathcal{J}_8.\end{aligned}$$

One can see that $\mathcal{J}_6 = 0$, since $\operatorname{div} u^\varepsilon = 0$ and $\mathcal{J}_8 = 0$, because $u^\varepsilon \cdot \nu = 0$ at $\partial\Omega$. Thus,

$$|\mathcal{J}_5| + |\mathcal{J}_7| \le C \int_\Omega (|u^\varepsilon||\nabla u^\varepsilon|^2 + |u^\varepsilon|^2 |\nabla u^\varepsilon|) \,\mathrm{d}x. \tag{5.26}$$

Using Hölder and Young inequalities, $\|\cdot\|_4^2 \le C\|\cdot\|_{1,2}\|\cdot\|_2$ and the information $u^\varepsilon \in L^\infty(I, W^{1,2}(\Omega))$ we continue estimating (5.26):

$$C(\|u^\varepsilon\|_2 \|\nabla u^\varepsilon\|_4^2 + \|u^\varepsilon\|_4^2 \|\nabla u^\varepsilon\|_2) \le \varepsilon\|\nabla^2 u^\varepsilon\|_2^2 + C\|u\|_{1,2}^2 + C\|\nabla u^\varepsilon\|_2^2 \|u^\varepsilon\|_2^2.$$

The right hand side of (5.25) is estimated easily:

$$\Big|\int_\Omega h \cdot \varphi \,\mathrm{d}x\Big| \le \int_\Omega |h|(|\nabla^2 u^\varepsilon| + |\nabla u^\varepsilon| + |u^\varepsilon|) \,\mathrm{d}x \le C\|h\|_2^2 + \varepsilon\|\nabla^2 u^\varepsilon\|_2^2 + C\|u\|_{1,2}^2.$$

Since $\mu^\varepsilon(|Du^\varepsilon|) > 1$ and $\varepsilon > 0$ can be chosen arbitrarily small, we obtain

$$\|\nabla^2 u^\varepsilon\|_2^2 \le \int_\Omega \mu^\varepsilon(|Du^\varepsilon|)|\nabla^2 u^\varepsilon|^2 \,\mathrm{d}x \le C, \tag{5.27}$$

where C doesn't depend on ε and $t \in I$, therefore we have

$$u^\varepsilon \in L^\infty(I, W^{2,2}(\Omega)). \tag{5.28}$$

Step 6 *improves information about $\partial_t u^\varepsilon$.*

In the same spirit as in Step 3 in Section 5.2 we denote $w := \partial_t u^\varepsilon$ and $\tau := \partial_t \pi^\varepsilon$ in the sense of distributions, which solves (5.6) where Φ is replaced by Φ^ε. The right hand side of (5.6) is bounded uniformly with respect to $\varepsilon \in (0,1)$ in $L^{q_0}(I, W_\sigma^{-1,q_0'}(\Omega))$ for some $q_0 > 2$, since from (5.23), (5.24) and (5.28) we have $\partial_t[(u^\varepsilon \cdot \nabla)u^\varepsilon] \in L^s(I, W^{-1,s}(\Omega))$ for all $s \in [1,4]$.

Set $V_\varepsilon := \sup_Q |\vartheta^\varepsilon(|Du^\varepsilon|)|$. From (5.19) we have for all $t \in I$, $x \in \Omega$, for all $\varepsilon \in (0,1)$ and $A, B \in \mathbb{R}^{2\times 2}_{sym}$

$$c|B|^2 \le \partial_A^2 \Phi^\varepsilon(|A|) : B \otimes B \le C V_\varepsilon^{p-2}(|A|)|B|^2.$$

From Lemma 4.5.1 we have the existence of positive constants K and L such that for all $q \in (2, q_2]$, where $q_2 := 2 + L/V_\varepsilon^{p-2}$ holds

$$\begin{aligned} &\|\nabla w\|_{L^q(I\times\Omega)} + V_\varepsilon^{\frac{2-p}{q}} \|w\|_{BUC(I, B^{1-2/q}_{q,q,B,\sigma}(\Omega))} \\ &\le K\Big(\|f\|_{L^q(I, W^{-1,q'}(\Omega))} + V_\varepsilon^{(p-2)(1-1/q)} \|\partial_t u_0\|_{B^{1-2/q}_{q,q,B,\sigma}(\Omega)}\Big). \end{aligned} \tag{5.29}$$

Without loss of generality we may assume that $q_2 < q_0$. Thus, after estimating last norm on the right hand side of (5.29) in the same way like in Step 3 in Section 5.2 we have

$$\|\partial_t u^\varepsilon\|_{BUC(I, B^{1-2/q}_{q,q,B,\sigma}(\Omega))} \le C\Big(V_\varepsilon^{\frac{p-2}{q}} + V_\varepsilon^{p-2}\Big) \le C V_\varepsilon^{p-2}.$$

Step 7 *improves information about $\nabla^2 u^\varepsilon$.*

In this step we obtain better space regularity. Up to now we have $\vartheta^\varepsilon \in L^\infty(I, W^{1,2}(\Omega))$. We are going to show that $\vartheta^\varepsilon \in L^\infty(I, W^{1,q}(\Omega))$ for some $q > 2$.

We omit estimates of $\nabla^2 u^\varepsilon$ in the interior of Ω and we focus on estimates near the boundary. We start with the tangential direction. Localizing the problem, we work in Ω^P_{3r}, where the boundary is locally described by the $\mathcal{C}^3$ mapping a_p. For simplicity we drop the index P.

We multiply (5.25) by $-\partial_{\tau^\alpha}\varphi\xi$, integrate over Ω_{3r} and after similar steps as in [54, Lemma 4.6] we derive the identity

$$\begin{aligned} \int_{\Omega_{3r}} \partial_{\tau^\alpha} \mathcal{S}^\varepsilon(Du^\varepsilon) : D\varphi\xi \,\mathrm{d}x &= -\int_{\Omega_{3r}} h\cdot \partial_{\tau^\alpha}(\varphi\xi)\,\mathrm{d}x + \int_\Omega (u^\varepsilon\cdot\nabla)u^\varepsilon \partial_{\tau^\alpha}\varphi\xi\,\mathrm{d}x \\ &+ \int_{\Omega_{3r}} \mathcal{S}^\varepsilon(Du^\varepsilon) : \Big[\partial_{\tau^\alpha}\varphi\otimes\nabla\xi - \nabla\varphi\partial_{\tau^\alpha}\xi + (\partial_1^2 a, 0)\otimes\partial_2\varphi\xi \\ +\nabla\Big(\varphi\cdot\partial_{\tau^\alpha}\nu\frac{\nu}{|\nu|^2}\xi\Big)\Big]\,\mathrm{d}x &+ \int_{\Omega_{3r}} \operatorname{div}\mathcal{S}^\varepsilon(Du^\varepsilon)\cdot[(\varphi\cdot\partial_{\tau^\alpha}\nu)\frac{\nu}{|\nu|^2}\xi - \varphi\partial_{\tau^\alpha}\xi]\,\mathrm{d}x \\ &+ \int_{\Omega_{3r}} \partial_1^2 a[h_2 + (\operatorname{div}\mathcal{S}^\varepsilon(Du^\varepsilon))_2 - (u^\varepsilon\cdot\nabla u^\varepsilon)_2]\varphi_1\xi\,\mathrm{d}x \\ &+ \int_{\Omega_{3r}} [h_1 + \partial_1 a h_2 + \operatorname{div}\mathcal{S}^\varepsilon(Du^\varepsilon)_1 + \partial_1 a \operatorname{div}\mathcal{S}^\varepsilon(Du^\varepsilon)_2 \\ &+(u^\varepsilon\cdot\nabla u^\varepsilon)_1 + \partial_1 a(u^\varepsilon\cdot\nabla u^\varepsilon)_2]\varphi\nabla\xi\,\mathrm{d}x \end{aligned} \tag{5.30}$$

for all $\varphi \in W^{1,q'}_\sigma(\Omega)$, $\operatorname{supp}\varphi \subset \overline{\Omega_{3r}}$. Terms on the right hand side of (5.30) comes at first from the fact that we add and subtract some lower order terms in order to let the boundary term vanish while integrating by parts. Second, tangent derivative doesn't commute with the gradient and we use $\nabla\partial_{\tau^\alpha}\varphi = \partial_{\tau^\alpha}\nabla\varphi + (\partial_1^2 a, 0)\otimes\partial_2\varphi$. Third, we use the equation (5.25) and replace $\partial_2\pi^\varepsilon$ by $h_2 + (\operatorname{div}\mathcal{S}^\varepsilon(Du^\varepsilon))_2 + (u^\varepsilon\cdot\nabla u^\varepsilon)_2$ and similarly for $\partial_{\tau^\alpha}\pi^\varepsilon$.

We denote $w := \partial_{\tau^\alpha}u^\varepsilon\xi - (0, \partial_1^2 a u_1^\varepsilon)\xi + z$, where z is the solution of

$$\begin{aligned} \operatorname{div} z &= -\partial_{\tau^\alpha}u^\varepsilon\cdot\nabla\xi - \partial_1^2 a u_1^\varepsilon\partial_2\xi && \text{in } \Omega_{3r}, && (5.31)\\ z &= 0 && \text{on } \partial\Omega_{3r}. && (5.32)\end{aligned}$$

The right hand side of (5.31) was obtained from $\operatorname{div}\big(-\partial_{\tau^\alpha}u^\varepsilon\xi + (0,\partial_1^2 a u_1^\varepsilon)\xi\big)$ using the fact that $\operatorname{div} u^\varepsilon = 0$. The role of z is to ensure that $\operatorname{div} w = 0$. On $\partial\Omega$ it holds $w\cdot\nu = 0$ since

$$w\cdot\nu = [\partial_{\tau^\alpha}u^\varepsilon\cdot\nu + \partial_1^2 a u_1^\varepsilon]\xi + z\cdot\nu = \partial_{\tau^\alpha}(u^\varepsilon\cdot\nu)\xi = 0.$$

Thus, the compatibility condition holds

$$\begin{aligned}\int_{\partial\Omega} z\cdot\nu\,\mathrm{d}\sigma = \int_\Omega \operatorname{div} z\,\mathrm{d}x &= \int_\Omega \operatorname{div}(-\partial_{\tau^\alpha}u^\varepsilon\xi + (0,\partial_1^2 a u_1^\varepsilon)\xi)\,\mathrm{d}x\\ &= -\int_{\partial\Omega}\partial_{\tau^\alpha}(u^\varepsilon\cdot\nu)\xi\,\mathrm{d}\sigma = 0\end{aligned}$$

and z solving (5.31) and (5.32) exists by Bogovskiĭ's Lemma and enjoys the estimate $\|z\|_{1,q} \le C\|\nabla u^\varepsilon\|_q$ for some $C > 0$.

Using the definition of w we get from (5.30)

$$\int_\Omega \partial_{Du^\varepsilon}\mathcal{S}^\varepsilon(Du^\varepsilon) : Dw\otimes D\varphi\,\mathrm{d}x = \langle g,\varphi\rangle \quad \forall\varphi\in W^{1,q'}_\sigma(\Omega),$$

with

$$\begin{aligned}\langle g,\varphi\rangle = \text{RHS of}\,(5.30) + \int_\Omega \partial_{Du^\varepsilon}\mathcal{S}^\varepsilon(Du^\varepsilon) : [Dz + \partial_{\tau^\alpha}u^\varepsilon\otimes\nabla\xi +\\ (\partial_1^2 a, 0)\otimes_S\partial_2 u^\varepsilon\xi - D\big((0,\partial_1^2 a, 0)\xi\big)]D\varphi\,\mathrm{d}x.\end{aligned}$$

Due to the assumption on f and results from Step 4 we have $\|g\|_{-1,q_2'} \le CV_\varepsilon^{p-2}$ and after application of Lemma 4.5.2 we obtain

$$\|\nabla\partial_{\tau^\alpha}u^\varepsilon\xi\|_{L^q(\Omega)} \le CV_\varepsilon^{p-2}. \tag{5.33}$$

We recall that q depends on ε by the relation $q\in(2, 2 + L/V_\varepsilon^{p-2}]$. In order to control whole $\nabla^2 u^\varepsilon$ we need an estimate of type (5.33) in the normal direction which is locally x_2. Since $\partial_2^2 u_2^\varepsilon$ can be expressed from the condition $\operatorname{div} u^\varepsilon = 0$, we focus on $\partial_2^2 u_1^\varepsilon$. Following [51, Theorem 3.19] we can extract the desired estimate from the equation (5.25) after employment of the operator curl. Let us shorten

$\mathcal{S}^\varepsilon(Du^\varepsilon)$ to $\mathcal{S}^\varepsilon$ and $\vartheta^\varepsilon(|Du^\varepsilon|)$ to ϑ^ε. Denoting $G := \partial_2\mathcal{S}^\varepsilon_{12}$ we have due to (5.21) and (5.19)

$$\begin{aligned}\|\xi G\|_{-1,q} &\le \|\mathcal{S}^\varepsilon_{12}\|_q \le \|(\vartheta^\varepsilon)^{p-2}Du^\varepsilon\|_q,\\ \|\partial_1(\xi G)\|_{-1,q} &\le C\|(\vartheta^\varepsilon)^{p-2}Du^\varepsilon\|_q + C'\|(\vartheta^\varepsilon)^{p-2}\partial_1\nabla u^\varepsilon\|_q.\end{aligned}$$

From the equation (5.25) after application of curl we have

$$\begin{aligned}\|\partial_2(\xi G)\|_{-1,q} &\le C(\|\partial_1(\mathcal{S}^\varepsilon_{21}+\mathcal{S}^\varepsilon_{22}-\mathcal{S}^\varepsilon_{11})\|_q + \|f\|_q + \|u^\varepsilon\cdot\nabla u^\varepsilon\|_q + \|\partial_t u^\varepsilon\|_q)\\ &\le C\big\{\|(\vartheta^\varepsilon)^{p-2}Du^\varepsilon\|_q + \|(\vartheta^\varepsilon)^{p-2}\partial_1\nabla u^\varepsilon\|_q + V_\varepsilon^{p-2}+1\big\} := H.\end{aligned}$$

Nečas' theorem on negative norms A.5.1 gives

$$\|\xi G\|_q \le C(\|\xi G\|_{-1,q} + \|\nabla(\xi G)\|_{-1,q}) \le H.$$

From definition of G and symmetry of Du we obtain

$$\partial_{12}\mathcal{S}^\varepsilon_{12}\partial_2 Du^\varepsilon_{12} = \frac{G}{2} - \frac{1}{2}\partial_{11}\mathcal{S}^\varepsilon_{12}\partial_2 Du^\varepsilon_{11} - \frac{1}{2}\partial_{22}\mathcal{S}^\varepsilon_{12}\partial_2 Du^\varepsilon_{22}.$$

Using $\partial_{12}\mathcal{S}^\varepsilon_{12} \ge C\vartheta^{\varepsilon p-2}$ and the condition $\operatorname{div} u^\varepsilon = 0$ we get that

$$\|\xi(\vartheta^\varepsilon)^{p-2}\partial_2^2 u_1^\varepsilon\|_q \le H.$$

Hence,

$$\begin{aligned}\|\xi(\vartheta^\varepsilon)^{p-2}\nabla^2 u^\varepsilon\|_q \le{}& C\|\xi G\|_q + \|\xi(\vartheta^\varepsilon)^{p-2}\nabla\partial_{\tau^\alpha}u^\varepsilon\|_q\\ &+ \tilde{C}\sup_{x_1\in(-3r,3r)}|\partial_1 a|\|\xi(\vartheta^\varepsilon)^{p-2}\nabla^2 u^\varepsilon\|_q, \quad (5.34)\end{aligned}$$

where $\tilde{C}$ is absolute constant. Since we can choose r sufficiently small in order to $\tilde{C}\max_{P\in\partial\Omega}\sup_{x_1\in(-3r,3r)}|\partial_1 a| \le 1/2$, the last term (5.34) can be absorbed into the left hand side. We have

$$\|\xi(\vartheta^\varepsilon)^{p-2}\nabla^2 u^\varepsilon\|_{q_2} \le CV_\varepsilon^{p-2}V_\varepsilon^{p-2}. \quad (5.35)$$

From (5.27) the boundedness of the term $\int_\Omega \mu^\varepsilon(|Du^\varepsilon|)|\nabla^2 u^\varepsilon|^2\,\mathrm{d}x$ is obtained, in other words $\|(\vartheta^\varepsilon)^{\frac{p-2}{2}}\nabla^2 u^\varepsilon\|_2 \le C$. Interpolation of this result with (5.35) gives for $q\in(2,q_2)$

$$\|(\vartheta^\varepsilon)^{\frac{p-2}{2}}\nabla^2 u^\varepsilon\|_q \le CV_\varepsilon^{\beta 2(p-2)}, \quad (5.36)$$

where $1/q = \beta/q_2 + (1-\beta)/2$. Since it holds $\|(\vartheta^\varepsilon)^{p/2}\|_{1,q} \le \|(\vartheta^\varepsilon)^{p/2}\|_q + \|(\vartheta^\varepsilon)^{\frac{p-2}{2}}\nabla^2 u^\varepsilon\|_q$, we want to use the following lemma for $f = (\vartheta^\varepsilon)^{p/2}$.

Lemma 5.3.1 *Let $\Omega\subset\mathbb{R}^2$ be a bounded $\mathcal{C}^2$ domain and $f\in W^{1,q}(\Omega)$ for some $q>2$. Then $f\in\mathcal{C}(\overline{\Omega})$ and there is $C>0$ independent of q such that*

$$\sup_\Omega |f| \le C\Big(\frac{q-1}{q-2}\Big)^{1-1/q}\|f\|_{1,q}. \quad (5.37)$$

Proof. Follows from the proof of [90, Theorem 2.4.1]. The result holds also for $\Omega \subset \mathbb{R}^n$, with $q > n$ and $q - n$ instead of $q - 2$ in the denominator of (5.37). $\square$

Because $\frac{q-1}{q-2} \leq CV^{p-2}$, we obtain

$$V_\varepsilon^{\frac{p}{2}} \leq CV^{(p-2)(1-\frac{1}{q})} V_\varepsilon^{\beta 2(p-2)}. \tag{5.38}$$

Note that $(p-2)(1-1/q) \to p/2 - 1$ as $q \to 2$ and the exponent containing the interpolation parameter β can be made arbitrarily small, therefore we can rewrite (5.38) as $V_\varepsilon \leq \hat{C}$ for suitable $\hat{C} > 0$. This together with (5.36) gives

$$\sup_{t\in I} \|\nabla^2 u^\varepsilon\|_q \leq C.$$

Step 8 *passes from the regularized problem to the original one.*

In the previous step we showed $V_\varepsilon \leq \hat{C}$, where $V_\varepsilon = \sup_{I\times\Omega} |\vartheta_\varepsilon(|Du^\varepsilon|)|$. Since $\vartheta^\varepsilon(s) = \min\{(1+s^2)^{\frac{1}{2}}, \frac{1}{\varepsilon}\} \leq \frac{1}{\varepsilon}$, it is sufficient to choose ε in order to have $\hat{C} \leq \frac{1}{\varepsilon}$. Thus, $u^\varepsilon = u$ is the solution of the original problem (1.19)–(1.22) and it holds that $\sup_{I\times\Omega}(1+|Du|^2)^{1/2} \leq C$ which leads to $\sup_{t\in I} \|\nabla^2 u\|_q \leq C$.

Since we passed from the regularized problem to the original one, the regularity of pressure π which we proved in Section 5.2 for quadratic potential holds also for the super-quadratic case.

$\square$

6
Conclusion

In presented work some results known for homogeneous Dirichlet boundary conditions were extended also for perfect slip boundary conditions. In Chapter 2, which is concerned with stationary generalized Stokes system, we were able to obtain optimal regularity results which are not known in case of homogeneous Dirichlet boundary conditions. Structure of the perfect slip boundary conditions allowed us to gain the information about normal part of the gradient of the function V, in case of homogeneous Dirichlet conditions there is a loss of regularity due to the absence of some special weighted version of Korn's inequality and the presence of pressure. The proof of the main theorem of Chapter 2 is more technical since we need to overcome new difficulties that bring perfect slip boundary conditions.

In Chapter 3 we showed L^q estimates up to the boundary. The proof is based on the paper [23], where the interior estimates are obtained. Here we consider different method than in Chapter 2. Instead of working directly on a general boundary, we extended a solution beyond a flat boundary and then went from the flat boundary to a general one.

In Chapter 4 we collect some known results about the classical Stokes operator which allow us to get an L^q theory result using the semigroup approach and interpolation-extrapolation scales. These results together with the results from Chapter 2 were applied in Chapter 5 to show Hölder continuity of solutions to the evolutionary generalized Navier-Stokes equations in $\mathbb{R}^2$.

Finally, let us mention some open problems which were mentioned only marginally or remained unsolved in spite of several attempts.

In particular, all the results obtained for perfect slip boundary conditions would be more interesting in case of Navier's slip boundary conditions that are from the point of application more useful. Although some parts of the proofs can be generalized easily, some technical steps relies on the structure of the perfect slip boundary conditions. Up to now, we don't know how to overcome these difficulties.

Results of Chapters 2 and 3 can be considered as a starting point in achieving BMO estimates for generalized Stokes system. In [24] BMO estimates for $p-$Laplace operator were obtained. These results were later extended in [25] for generalized Stokes system in the interior of $\Omega \subset \mathbb{R}^2$. Keeping at our disposal results from Chapters 2 and 3, we could try to extend these result up to the boundary for perfect slip boundary conditions.

The question of Hölder continuity of velocity gradients and pressure in case of evolutionary planar flow would be interesting from the point of application in case of shear thinning fluids, i.e. the case $p \le 2$.

Also for Dirichlet boundary conditions there is still space for new results. For instance optimal regularity for generalized stationary Stokes problem (Chapter 2) would allow us to cover the case $p \in [4, \infty)$ in case of Hölder continuity of velocity gradients and pressure.

A Appendix

A.1 Description of the boundary and differences

In the following we use the notation $x = (x', x_n)$. Suppose[1] that $\Omega \in \mathcal{C}^3$, therefore there exists $c_0 > 0$ such that for all $a_0 > 0$ there exists n_0 points $P \in \partial\Omega$, $R_0 > 0$ and open smooth set $\Omega_0 \subset\subset \Omega$ that we have

$$\Omega \subset \Omega_0 \cup \bigcup_P B_{\frac{R_0}{4}}(P)$$

and for each point $P \in \partial\Omega$ there exists local system of coordinates for which $P = 0$ and the boundary $\partial\Omega$ is locally described by $\mathcal{C}^3$ mapping a_P that for $x' \in (-R_0, R_0)$ and $\alpha, \beta, \gamma \in \{1, \dots, n-1\}$ fulfils

$$\begin{aligned}
&x \in \partial\Omega \Leftrightarrow x_n = a_P(x'),\\
&\qquad \Omega^P_{R_0} := B_{R_0}(P) \cap \Omega = \{(x', x_n) \in B_{R_0}(P) \text{ and } x_n > a_P(x')\},\\
&\partial_\alpha a_P(0) = 0, \quad |\partial_\alpha a_P(x')| \le a_0, \quad |\partial_\alpha\partial_\beta a_P(x')| + |\partial_\alpha\partial_\beta\partial_\gamma a_P(x')| \le c_0.
\end{aligned} \tag{A.1}$$

Points P can be divided into k groups such that in each group $\Omega^P_{R_0}$ are disjoint and k depends only on dimension n. By $\nabla a_P(x')$ we denote the vector $(\partial_1 a_P, \dots \partial_{n-1} a_P, 0)$.

Let the cut-off function $\xi_P(x) \in \mathcal{C}^\infty(B_{R_0}(P))$ be defined via

$$\xi_P(x) \begin{cases} = 1 & x \in B_{\frac{R_0}{4}}(P),\\ \in (0,1) & x \in B_{\frac{R_0}{2}}(P) \setminus B_{\frac{R_0}{4}}(P),\\ = 0 & x \in \mathbb{R}^n \setminus B_{\frac{R_0}{2}}(P). \end{cases}$$

Let us fix P and drop for simplicity the index P. Next, assume that we work in the coordinate system anchored at P, i.e. $P = 0$. The tangent vector in the α direction and the outer normal vector to $\partial\Omega$ are defined as

$$\begin{aligned}
\tau^\alpha &= \big(0, \dots, 0, 1, 0, \dots, 0, \partial_\alpha a(x')\big), \quad \alpha = 1, \dots, n-1,\\
\nu &= \big(\partial_1 a(x'), \dots, \partial_{n-1} a(x'), -1\big),
\end{aligned} \tag{A.2}$$

[1] In some parts of the thesis lower regularity of the boundary is required, but the description would be analogical.

tangent and normal derivatives as

$$\begin{aligned} \partial_{\tau^\alpha} &= \partial_\alpha + \partial_\alpha a(x')\partial_n, \quad \alpha = 1,\dots,n-1, \\ \partial_\nu &= \sum_{\alpha=1}^{n-1} \partial_\alpha a(x')\partial_\alpha - \partial_n. \end{aligned} \tag{A.3}$$

We need to work with τ^α, ν, ∂_{τ^α} and ∂_ν not only on $\partial\Omega$, but on the whole Ω_{R_0}. We can notice that (A.2) and (A.3) actually define τ^α, ν, ∂_{τ^α} and ∂_ν on Ω_{R_0}. Next, we assume that u is sufficiently smooth. It is easy to see, that $\partial_{\tau^\alpha}(u\cdot\nu) = 0$ on $\partial\Omega$.

Remark A.1.1 *There arises no boundary term from the tangent integration by parts, because if* $\operatorname{supp} g \subset B_{\frac{R_0}{2}}$ *or* $\operatorname{supp} f \subset B_{\frac{R_0}{2}}$ *then*

$$\begin{aligned} \int_\Omega (\partial_{\tau^\alpha} f) g \,\mathrm{d}x = \int_\Omega (\nabla f)\tau^\alpha g \,\mathrm{d}x &= -\int_\Omega f\nabla(\tau^\alpha g) + \int_{\partial\Omega} f\tau^\alpha g\nu \,\mathrm{d}x \\ &= -\int_\Omega f(\partial_{\tau^\alpha} g + g\nabla\tau^\alpha)\,\mathrm{d}x. \end{aligned}$$

Let e^α, $\alpha = 1,\dots,n-1$ be the basis of the coordinate system in $\mathbb{R}^{n-1}$. For $h \in (0, \frac{R_0}{2})$ we define the mapping $T_\alpha : \Omega_{\frac{R_0}{2}} \mapsto \Omega_{R_0}$:

$$T_\alpha : x \mapsto (x' + he^\alpha, x_n + a(x' + he^\alpha) - a(x')) = (y', y_n). \tag{A.4}$$

Then the inverse mapping T_α^{-1} is given by

$$T_\alpha^{-1} : y \mapsto (y' - he^\alpha, y_n + a(y' - he^\alpha) - a(y')) = (x', x_n).$$

One easily checks that both matrices $\big(\partial_j (T_\alpha)_i(x)\big)_{i,j=1}^n$ and $\big(\partial_j (T_\alpha)_i^{-1}(y)\big)_{i,j=1}^n$ have determinant equal to 1. Put

$$\begin{aligned} \delta_\alpha^+ g(x) &:= g(T_\alpha x) - g(x), \\ \delta_\alpha^- g(x) &:= g(x) - g(T_\alpha^{-1} x). \end{aligned}$$

For the tangential derivative of any vector function g holds in $x \in \Omega_{\frac{R_0}{2}}$

$$\partial_{\tau^\alpha} g(x) = \lim_{h\to 0} \frac{\delta_\alpha^+ g(x)}{h}.$$

The following lemma describes the relation between difference and gradient, respectively difference and the tangential derivative.

Lemma A.1.2 *[68, Section 3, 3.17-3.20] Let* $p > 1$. *Then for all* $g \in W_\nu^{1,p}(\Omega)^n$ *it holds*

$$\int_{\Omega_{\frac{R_0}{2}}} \Big|\frac{\delta_\alpha^+ g}{h}\Big|^p \,\mathrm{d}x \le c(a)\|\nabla g\|_p^p. \tag{A.5}$$

If $g \in L^p(\Omega)$ *and if for all* $h \in (0, \frac{R_0}{2})$ *holds*

$$\int_{\Omega_{\frac{R_0}{2}}} \left|\frac{\delta_\alpha^+ g}{h}\right|^p \mathrm{d}x \le C, \tag{A.6}$$

then $\partial_{\tau^\alpha} g$ *exists in the sense of distribution*

$$\int_{\Omega_{\frac{R_0}{2}}} |\partial_{\tau^\alpha} g|^p \,\mathrm{d}x \le C. \tag{A.7}$$

Lemma A.1.3 *[12, Lemma 2.1] Let* $\operatorname{supp} u \subset \operatorname{supp} \xi$. *Then*

$$D(u(T_\alpha x)) = (D(u))(T_\alpha x) + (\partial_n u)(T_\alpha x) \otimes_S \delta_\alpha^+ \nabla a,$$

by the symbol $u \otimes_S \nu$ *we understand* $\frac{1}{2}(u \otimes \nu + (u \otimes \nu)^T)$.

Lemma A.1.4 *Let* $a(x')$ *be a* $\mathcal{C}^3$ *mapping locally describing* $\partial\Omega$ *and* $x' \in (-\frac{R_0}{2}, \frac{R_0}{2})$. *Then there exists* C_n *such that for all* $h \in (0, \frac{R_0}{2})$ *it holds:*

$$\|a(x')\|_{3,\infty} + \left\|\frac{\delta_\alpha^\pm a(x')}{h}\right\|_{2,\infty} \le C_n,$$

Proof. Clear from (A.5) and properties of the function $a(x')$. □

A.2 N-functions

In this section we focus on some basic properties of $N-$functions.

Definition A.2.1 *A real function* $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is called N-function if the derivative* $\Phi'(s)$ *exists and is right continuous for* $s \ge 0$, *positive for* $s > 0$, *non-decreasing,* $\Phi'(0) = 0$ *and* $lim_{s\to\infty}\Phi'(s) = \infty$.

Definition A.2.2 *N-function* Φ *is said to satisfy the* Δ_2-*condition, denoted* $\Phi \in \Delta_2$, *if there exists a positive constant* C, *such that* $\Phi(2s) \le C\Phi(s)$ *for* $s > 0$. *By* $\Delta_2(\Phi)$ *we denote the smallest such constant* C.

By $(\Phi')^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$ we denote the function

$$(\Phi')^{-1}(s) := \sup\{t \in \mathbb{R}^+ : \Phi'(t) \le s\}.$$

The complementary function of Φ is defined as

$$\Phi^*(s) := \int_0^s (\Phi')^{-1}(t)\,\mathrm{d}t.$$

It is again an N-function and for all $\delta > 0$ there exists $c(\delta) > 0$ such that for all $s, t \ge 0$ holds so called Young's inequality

$$st \le \delta\Phi(s) + c(\delta)\Phi^*(t). \tag{A.8}$$

For a measurable function f we can define gauge norm as

$$\|f\|_\Phi := \inf\Big\{\lambda > 0 : \int_\Omega \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\,\mathrm{d}x \le 1\Big\}.$$

The Orlicz space $L^\Phi(\Omega)$ is defined as the set $\{f : \|f\|_{\Phi,\Omega} < \infty\}$.[2]

Let $f \in L^\Phi(\Omega)$, $g \in L^{\Phi^*}(\Omega)$. Then $fg \in L^1(\Omega)$ and

$$\int_\Omega |fg|\,\mathrm{d}x \le 2\|f\|_\Phi \|g\|_{\Phi^*}.$$

In particular, for $f = \chi_\Omega$ where χ_Ω is the characteristic function of Ω

$$\int_\Omega |g|\,\mathrm{d}x \le \frac{2}{\Phi^{-1}(|\Omega|^{-1})}\|g\|_{\Phi^*}. \tag{A.9}$$

It holds

$$\Phi^*(\Phi'(s)) \sim \Phi(s). \tag{A.10}$$

Lemma A.2.3 *[80, Section 2.3, Corollary 5]. If $\Phi \in \Delta_2$ is an $N-$function, then $\Phi(s) \le Cs^\alpha$, $s > s_0$ for some $C > 0$ and $\alpha > 1$. Its complementary function Φ^* satisfies $\Phi^*(t) \ge C't^\beta$, $t > t_0 > 0$ for some $C' > 0$ and $\beta > 1$.*

For $a \ge 0$ we define shifted N-function Φ_a by

$$\Phi_a'(s) := \Phi'(a+s)\frac{s}{a+s}. \tag{A.11}$$

This basically states that $c\Phi_a''(s) \le \Phi''(a+s) \le C\Phi_a''(s)$ for some $C, c > 0$. Moreover, $\Phi_a \in \Delta_2$ and $\Phi_a^* \in \Delta_2$ uniformly in a, see [22, Appendix].

Lemma A.2.4 (Shift change) *[26, Lemma 5.15] Let Φ fulfils Assumption* 3.1.1. *Then for any $\delta > 0$ there exists $c(\delta) > 1$ such that for all $A, B \in \mathbb{R}^{n\times n}$ and $s \ge 0$*

$$\Phi_{|A|}(s) \le c(\delta)\Phi_{|B|}(s) + \delta|V(A) - V(B)|^2. \tag{A.12}$$

Lemma A.2.5 *[22, Lemma 31] Let Φ be an N-function with $\Delta_2(\{\Phi^*, \Phi\}) < \infty$. Then there exist $\delta > 0$, $c > 0$ which depend only on $\Delta_2(\{\Phi^*, \Phi\})$ such that for all $t > 0$ and all $s \in [0, 1]$*

$$\Phi_a(st) \le cs^{1+\delta}\Phi_a(t).$$

Lemma A.2.6 *[23, Lemma 2.4] Let Φ satisfy Assumption* 3.1.1 *and V be defined as in* (5.38)*. Then for all $P, Q \in \mathbb{R}^{n\times n}$ we have*

$$(A(P)-A(Q)) : (P-Q) \sim |V(P)-V(Q)|^2 \sim \Phi_{|P|}(|P-Q|) \sim \Phi''(|P|+|Q|)|P-Q|^2$$

and

$$|A(P) - A(Q)| \le C\Phi_{|P|}(|P-Q|).$$

[2] In our case Φ always fulfils Δ_2-condition, therefore the following sets coincide: $\{f; \exists\lambda > 0 : \int_\Omega \Phi\big(\frac{|f(x)|}{\lambda}\big)\,\mathrm{d}x < \infty\}$, $\{f; \forall\lambda > 0 : \int_\Omega \Phi\big(\frac{|f(x)|}{\lambda}\big)\,\mathrm{d}x < \infty\}$, $\{f; \int_\Omega \Phi(|f|)\,\mathrm{d}x < \infty\}$.

By Q we denote a cube in $\mathbb{R}^n$ with center x_0, sides parallel to the axis and one side equal to $2R$, i.e.

$$Q = Q(x_0, R) = \left\{ x \in \mathbb{R}^n;\ \sup_i |x_i - (x_0)_i| < R \right\}.$$

For $s > 0$ the abbreviation sQ stands a cube with the same center as Q and one side $2sR$, i.e. $sQ = Q(x_0, sR)$. By $|Q|$ we mean the volume of Q. For $f \in L^1(Q)$ we define

$$\langle f \rangle_Q = \fint_Q f(x)\,\mathrm{d}x := \frac{1}{|Q|} \int_Q f(x)\,\mathrm{d}x.$$

Lemma A.2.7 *[23, Lemma 2.7] For all $A \in L^\Phi(Q)^{n\times n}$ it holds*

$$\fint_Q |V(A) - V(\langle A\rangle_Q)|^2\,\mathrm{d}x \sim \fint_Q |V(A) - \langle V(A)\rangle_Q|^2\,\mathrm{d}x.$$

Lemma A.2.8 (Sobolev - Poincaré) *[22, Lemma 7] Let Φ be an N-function, $\Delta_2(\{\Phi^*, \Phi\}) < \infty$. Then there exist $\theta \in (0, 1)$ and $c > 0$ such that the following holds. If $Q \subset \mathbb{R}^n$ is some cube and $f \in W^{1,\Phi}(Q)$, then*[3]

$$\fint_Q \Phi\left(\frac{|f - \langle f\rangle_Q|}{\operatorname{diam} Q}\right)\mathrm{d}x \le c \left(\fint_Q \Phi^\theta(|\nabla f|)\,\mathrm{d}x\right)^{\frac{1}{\theta}}. \tag{A.13}$$

A.3 Bogovskiĭ-type lemmata

Lemma A.3.1 (Bogovskiĭ's Lemma for Sobolev spaces) *[5, Lemma 3.3, Corollary 3.4] Let $\Omega \subset \mathbb{R}^n$ be a bounded $\mathcal{C}^{0,1}$ domain. Let $r \in (1, \infty)$ and $g \in W^{1,r}(\Omega)^n$ fulfils the compatibility condition*

$$\int_{\partial\Omega} g \cdot \nu\,\mathrm{d}x = 0.$$

Then there exists $u \in W_0^{1,r}(\Omega)^n$ solving

$$\begin{aligned} \operatorname{div} u &= \operatorname{div} g \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Moreover, there exists $C > 0$ such that

$$\|u\|_r \le C\|g\|_r.$$

[3] If we have a cube Q with one side equal to $2R$, the general definition $\operatorname{diam} Q = \sup\{|x - y|,\ x, y \in Q\}$ leads to $\operatorname{diam} R = 2\sqrt{n}R$.

Lemma A.3.2 (Bogovskiĭ's Lemma for Orlicz spaces, case 1) *Let $\Omega \subset \mathbb{R}^n$ be a bounded $\mathcal{C}^{0,1}$ domain. Let Φ be N-function with $\Phi \in \Delta_2$ and $\Phi^* \in \Delta_2$, $g \in L_0^\Phi(\Omega) := \{g \in L^\Phi(\Omega); \int_\Omega g \,\mathrm{d}x = 0\}$. Then there exists $z \in W_0^{1,\Phi}(\Omega)^n$ solving*

$$\begin{aligned} \operatorname{div} z &= g \quad \text{in } \Omega, && \text{(A.14)}\\ z &= 0 \quad \text{on } \partial\Omega. && \text{(A.15)} \end{aligned}$$

Moreover, there exists $C > 0$ depending only on $\Delta_2(\Phi)$ and $\Delta_2(\Phi^*)$ such that

$$\int_\Omega \Phi(|z|)\,\mathrm{d}x + \int_\Omega \Phi(|\nabla z|)\,\mathrm{d}x \le C \int_\Omega \Phi(|g|)\,\mathrm{d}x.$$

Let $\Omega \in \mathcal{C}^{1,1}$. There is a constant $C > 0$ depending only on $\Delta_2(\{\Phi, \Phi^*\})$ such that if additionally $g \in W^{1,\Phi}(\Omega)$, then there is a solution $z \in W^{2,\Phi}(\Omega)^n \cap W_0^{1,\Phi}(\Omega)^n$ solving (A.17), (A.18) *and the following estimate holds*

$$\sum_{j=0}^{2} \int_\Omega \Phi(|\nabla^j z|)\,\mathrm{d}x \le C \sum_{j=0}^{1} \int_\Omega \Phi(|\nabla^j g|)\,\mathrm{d}x. \tag{A.16}$$

If, moreover, for some $r > 0$, $\operatorname{supp} g \subset \overline{\Omega_r}$, *we can assume that* $\operatorname{supp} z \subset \overline{\Omega_{2r}}$.

Proof. The first part of the theorem follows from [27, Theorem 6.6]. Now we proceed to the situation when $g \in W^{1,\Phi}(\Omega)^n$. It consist of 5 steps.

Step 1

We notice that from the fact that Φ and Φ^* satisfy Δ_2 condition follows the existence of $1 < p < q$ such that $\Phi(t) = t^p h(t^{q-p})$ and h is pseudoconcave, i.e. there is a $C > 0$ such that $h(\lambda t) \le C \max(1, \lambda) h(t)$, c.f. [75, Section 5]. Constants p and q are determined by $\Delta_2(\{\Phi, \Phi^*\})$.

According to [75, Section 5] a bounded linear operator $T \in \mathcal{L}(L^p(\Omega), L^p(\Omega)) \cap \mathcal{L}(L^q(\Omega), L^q(\Omega))$ with norm bounded by $M > 0$ belongs to $\mathcal{L}(L^\Phi(\Omega), L^\Phi(\Omega))$ and there is a $C > 0$ depending only on p and q such that the following modular estimate holds:

$$\int_\Omega \Phi\Big(\frac{|Tg|}{M}\Big)\,\mathrm{d}x \le C \int_\Omega \Phi(|g|)\,\mathrm{d}x.$$

The proof of this estimate is based on L-functional and can be directly repeated also for $T \in \mathcal{L}(L^p(\Omega)^{n+1}, L^p(\Omega)) \cap \mathcal{L}(L^q(\Omega)^{n+1}, L^q(\Omega))$. The corresponding estimate then looks as follows

$$\int_\Omega \Phi\Big(\frac{|Tg|}{M}\Big)\,\mathrm{d}x \le C \sum_{k=1}^{n} \int_\Omega \Phi(|g_k|)\,\mathrm{d}x.$$

Step 2

Now, we define the canonical linear isometry of Sobolev spaces

$$\mathcal{J} : W^{1,r}(\Omega) \to L^r(\Omega)^{n+1}, \quad \mathcal{J}g = (g, \partial_1 g, \dots, \partial_n g).$$

If $\Omega \in \mathcal{C}^{1,1}$ the subspace $\mathcal{J}(W^{1,r}(\Omega))$ of $L^r(\Omega)^{n+1}$ is closed and complemented. This is equivalent to the continuity of the solution operator for the problem

$$-\Delta u + u = f + \operatorname{div} F \quad \text{in } \Omega$$

with homogeneous Neumann boundary condition from $L^r(\Omega)^{n+1}$ to $W^{1,r}(\Omega)$. The result can be found for $r \geq 2$ in [86, Theorem 3.16], for $r < 2$ it follows by the duality argument. The required projection is then the solution operator.

The subspace $W^{1,r}(\Omega) \cap L_0^r(\Omega)$ is complemented in $W^{1,r}(\Omega)$. The projection is defined as $P(g) = g - \int_\Omega g \,\mathrm{d}x$ and it is continuous by embedding theorem. Consequently, there is a projection $Q : L^r(\Omega)^{n+1} \overset{onto}{\to} \mathcal{J}(W^{1,r}(\Omega) \cap L_0^r(\Omega))$ which is continuous for all $r > 1$, Q is independent of r.

Step 3

In [5, Corollary 3.8] it is possible to find that the problem (A.17), (A.18) is solvable if $g \in W^{1,r}(\Omega) \cap L_0^r(\Omega)$ for $r > 1$. In fact analyzing the proof itself it is possible to construct a continuous solution operator $R : W^{1,r}(\Omega) \cap L_0^r(\Omega) \to W^{2,r}(\Omega)^n \cap W_0^{1,r}(\Omega)^n$. This operator is independent of $r > 1$.

Step 4

Next, we identify $W^{1,r}(\Omega) \cap L_0^r(\Omega)$ with $\mathcal{J}(W^{1,r}(\Omega) \cap L_0^r(\Omega))$ and fix $\alpha \in (\mathbb{N} \cup \{0\})^n$ with $|\alpha| \leq 2$, $k \in \{1, \dots, n\}$. We define $S : L^r(\Omega)^{n+1} \to L^r(\Omega)$ by $S(G) = D^\alpha[R_k \circ \mathcal{J}_{-1} \circ P(G)]$. This mapping is continuous for any $r > 1$ by Step 2 and Step 3. It is possible to interpolate it by Step 1. Restricting the operator S to $\mathcal{J}(W^{1,\Phi}(\Omega) \cap L_0^\Phi(\Omega))$ and defining $g = \mathcal{J}_{-1}(G)$ we obtain for $g \in W^{1,\Phi}(\Omega) \cap L_0^\Phi(\Omega)$ the modular estimate

$$\int_\Omega \Phi(|D^\alpha[R_k(g)]|) \,\mathrm{d}x \leq C\Big(\int_\Omega \Phi(|g|) \,\mathrm{d}x + \sum_{k=1}^n \int_\Omega \Phi(|\partial_k g|) \,\mathrm{d}x \Big).$$

Since α was arbitrary, the estimate (A.16) follows by properties of Φ.

Step 5

It remains to find z with $\operatorname{supp} z \subset \overline{\Omega_{2r}}$ for g with $\operatorname{supp} g \subset \overline{\Omega_r}$. We find a solution to (A.17) and (A.18), take a smooth cut-off function η such that $\chi_{\Omega_r} \leq \eta \leq \chi_{\Omega_{2r}}$. Compute $\operatorname{div}(z\eta) = (\operatorname{div} z)\eta + z \cdot \nabla\eta = g\eta + z \cdot \nabla\eta$. It is enough to find a correction $v \in W_0^{2,\Phi}(\Omega_{2r})^n$, the solution of the problem $\operatorname{div} v = z \cdot \nabla\eta$. Since $z \cdot \nabla\eta \in W_0^{1,\Phi}(\Omega_{2r}) \cap L_0^\Phi(\Omega_{2r})$, this is possible by the same methods used to prove (A.16). The solution operator R from Step 3 satisfies $R : W_0^{1,r}(\Omega_{2r}) \cap L_0^r(\Omega_{2r}) \to W_0^{2,r}(\Omega_{2r})^n$, cf. [34, Theorem III.3.2].

$\square$

Lemma A.3.3 (Bogovskiĭ's Lemma for Orlicz spaces, case 2) *Let $\Omega \subset \mathbb{R}^n$ be a rectangle. Let Φ be an N-function with $\Delta_2(\{\Phi^*, \Phi\}) < \infty$, $g \in L^\Phi(\Omega)$, $h \in W^{1,\Phi}(\Omega)^n$ and Γ be one side of Ω. Then there exists a linear mapping $B : (g, h) \to z$, where $z \in W^{1,\Phi}(\Omega)^n$ solves*

$$\operatorname{div} z = g \quad \text{in } \Omega, \tag{A.17}$$

$$z \cdot \nu = h \cdot \nu \quad \text{on } \Gamma, \tag{A.18}$$

Moreover,

$$\int_\Omega \Phi(|\nabla z|)\,\mathrm{d}x \le C \left(\int_\Omega \Phi(|g|)\,\mathrm{d}x + \int_\Omega \Phi(|\nabla h|)\,\mathrm{d}x \right), \tag{A.19}$$

where the positive constant C depends only on $\Delta_2(\{\Phi, \Phi^*\})$.

Proof. Without loss of generality, we may assume that Γ is a part of a hyperplane $\{x; x_n = 0\}$. It is enough to consider equation

$$\begin{aligned} \operatorname{div} \tilde{z} &= g - \operatorname{div} h - \fint_\Omega (g - \operatorname{div} h)\,\mathrm{d}x \text{ in } \Omega, \\ \tilde{z} &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Furthermore, we define an affine function $b : \Omega \mapsto \mathbb{R}^3$ as follows

$$b_i(x) = \begin{cases} 0 & \text{for } i \in \{1, \dots, n-1\} \text{ and } x \in \Omega, \\ 0 & \text{for } i = n \text{ and } x \in \Gamma, \\ x_n \fint_\Omega (g - \operatorname{div} h)\,\mathrm{d}x & \text{for } i = n \text{ and } x \in \Omega. \end{cases}$$

Then $z = \tilde{z} + h + b$ solves (A.17) and (A.18). According to [27, Theorem 6.6] there exists a positive constant c independent of diamΩ such that

$$\int_\Omega \Phi(|\nabla \tilde{z}|)\,\mathrm{d}x \le c \int_\Omega \Phi \left(\left| g - \operatorname{div} h - \fint_\Omega (g - \operatorname{div} h)\,\mathrm{d}x \right| \right) \mathrm{d}x.$$

The estimate (A.19) follows easily. □

A.4 Korn-type Inequalities

Lemma A.4.1 *[68, Lemma 6.5] Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^1$ domain, $u \in W^{1,2}(\Omega)^n$ and $\xi \in \mathcal{D}(\Omega)$. Then there are positive constants C_3 and C_4 such that*

$$C_3 \int_\Omega |\nabla u|^2 \xi^2\,\mathrm{d}x \le \int_\Omega |Du|^2 \xi^2\,\mathrm{d}x + C_4 \int_\Omega |u|^2 (|\nabla \xi|^2 + \xi^2)\,\mathrm{d}x.$$

Lemma A.4.2 (Korn's inequality for Orlicz spaces, case 1) *Let $\Omega \subset \mathbb{R}^n$ be a bounded non-axisymmetric $\mathcal{C}^{0,1}$ domain. Let Φ be N-function with $\Delta_2(\{\Phi^*, \Phi\}) < \infty$. Then for all $u \in W^{1,\Phi}_\nu(\Omega)^n$ it holds that*

$$\int_\Omega \Phi(|u|)\,\mathrm{d}x + \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x \le C \int_\Omega \Phi(|Du|)\,\mathrm{d}x, \tag{A.20}$$

where $C = C(\Omega, \Delta_2(\{\Phi^*, \Phi\}))$.

For the proof of this version of Korn's inequality two propositions will be needed.

Proposition A.4.3 *Let $q \in (1, 1+\varepsilon)$ for some suitably small $\varepsilon > 0$. Then function $h(s)$ defined as $h(s) := \Phi(s^{\frac{1}{q}})$ is convex.*

Proof.

$$\begin{aligned} h''(s) &= \Big(\Phi'(s^{\frac{1}{q}})\frac{1}{q}s^{\frac{1}{q}-1}\Big)' = \Phi''(s^{\frac{1}{q}})\Big(\frac{1}{q}s^{\frac{1}{q}-1}\Big)^2 + \Phi'(s^{\frac{1}{q}})\frac{1}{q}\Big(\frac{1}{q}-1\Big)s^{\frac{1}{q}-2} \\ &\geq C\Phi''(s^{\frac{1}{q}})\big(s^{\frac{1}{q}-1}\big)^2\frac{1}{q}\Big(\frac{2}{q}-1\Big) > 0, \end{aligned}$$

where we used $\Phi''(s^{\frac{1}{q}})s^{\frac{1}{q}} \leq C\Phi'(s^{\frac{1}{q}})$. □

Proposition A.4.4 *[34, Exercise 4.5] Let Ω be a bounded $\mathcal{C}^1$ domain, $u \in W^{1,q}(\Omega)$, $1 \leq q < \infty$ and $u \cdot \nu = 0$ at $\partial\Omega$. Then*

$$\|u\|_q \leq C\|\nabla u\|_q, \quad C \leq \delta(\Omega)(|q-2| + n + 1).$$

Proof of Lemma A.4.2. Using Korn's inequality [27, Theorem 6.13], which tells us

$$\int_\Omega \Phi(|\nabla u - \langle\nabla u\rangle|)\,\mathrm{d}x \leq C\int_\Omega \Phi(|Du - \langle Du\rangle|)\,\mathrm{d}x,$$

and convexity of Φ we obtain

$$\begin{aligned} \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x &\leq \int_\Omega \Phi(|\nabla u - \langle\nabla u\rangle|)\,\mathrm{d}x + \int_\Omega \Phi(|\langle\nabla u\rangle|)\,\mathrm{d}x \\ &\leq C\int_\Omega \Phi(|Du - \langle Du\rangle|)\,\mathrm{d}x + \int_\Omega \Phi(|\langle\nabla u\rangle|)\,\mathrm{d}x \\ &\leq C\int_\Omega \Phi(|Du|)\,\mathrm{d}x + \int_\Omega \Phi(|\langle\nabla u\rangle|)\,\mathrm{d}x. \end{aligned} \tag{A.21}$$

We also used convexity of Φ, Δ_2-condition and Jensen's inequality in the last estimate in (A.21). It remains to estimate the last term in (A.21).

$$\begin{aligned} \int_\Omega \Phi(|\langle\nabla u\rangle|)\,\mathrm{d}x &\leq \int_\Omega \Phi\Big(\frac{1}{|\Omega|}\int_\Omega |\nabla u|\,\mathrm{d}x\Big)\,\mathrm{d}x \\ &\leq C(\Omega, \Delta_2(\Phi))\int_\Omega \Phi\Big(\Big[\int_\Omega |\nabla u|^q\,\mathrm{d}x\Big]^{\frac{1}{q}}\Big)\,\mathrm{d}x \\ &\leq C\int_\Omega \Phi\Big(\Big[\int_\Omega |Du|^q\,\mathrm{d}x\Big]^{\frac{1}{q}}\Big)\,\mathrm{d}x =: \mathcal{J}, \end{aligned}$$

where we used Hölder's inequality, Δ_2-condition and Korn's inequality for Lebesgue spaces (see cf. [43] and [44]).

To estimate the term $\mathcal{J}$ we use the definition of the function $h(s)$, Jensen's inequaliy and the fact, that $h(t^q) = \Phi(t)$ and therefore $t = \Phi\big[h_{-1}^{\frac{1}{q}}(t)\big]$, which follows from implication below

$$t = \Phi\Phi_{-1}(t) = h\big(\Phi_{-1}^q(t)\big) \Rightarrow h_{-1}(t) = \Phi_{-1}^q(t) \Rightarrow \Phi\big[h_{-1}^{\frac{1}{q}}(t)\big] = t.$$

Thus,

$$\begin{aligned}
\mathcal{J} &= C\int_\Omega \Phi\Big(\Big[h_{-1}h\int_\Omega |Du|^q\,\mathrm{d}x\Big]^{\frac{1}{q}}\Big)\,\mathrm{d}x \\
&\le C\int_\Omega \Phi\Big(\Big[h_{-1}\Big(\int_\Omega h(|Du|^q)\,\mathrm{d}x\Big)\Big]^{\frac{1}{q}}\Big)\,\mathrm{d}x \\
&= C\int_\Omega \Phi\Big(\Big[h_{-1}\Big(\int_\Omega \Phi(|Du|)\,\mathrm{d}x\Big)\Big]^{\frac{1}{q}}\Big)\,\mathrm{d}x \\
&= C\int_\Omega\int_\Omega \Phi(|Du|)\,\mathrm{d}x\,\mathrm{d}x \le C(\Omega)\int_\Omega \Phi(|Du|)\,\mathrm{d}x.
\end{aligned}$$

The first term on the left hand side of (A.20) can be estimated after adding and subtracting the average of u, Poincaré inequality [27, Theorem 6.5] and Δ_2-condition. To estimate $\int_\Omega \Phi(|\langle u\rangle|)$ we use Proposition A.4.4 and then follow in the same way like in the estimate of $\int_\Omega \Phi(|\langle \nabla u\rangle|)$. □

This version of Korn's inequality differs from more standard versions which have an additional term on the right hand side. If we considered boundary conditions $u = 0$ on $\partial\Omega$, we could admit an arbitrary shape of the domain Ω. Only because of the boundary conditions $u\cdot\nu = 0$ on $\partial\Omega$ the restriction on the shape of the domain Ω is necessary. We need to know that $Du = 0$ a.e. in Ω and $u\cdot\nu = 0$ on $\partial\Omega$ together imply $u = 0$ a.e. in Ω. It holds when Ω is not axisymmetric (see for example [43, 44] if $n = 3$). The constant in (A.20) could be used to quantify the deviation of Ω from axisymmetry. In [20, Theorem 3] the authors obtained fully explicit upper bounds for the constant C in terms of the geometrical information about Ω in the case when there is an L^2-norm instead of the modular norm in (A.20).

Lemma A.4.5 (Korn's inequalities for Orlicz spaces, case 2) *Let Φ be an $N-$ function with $\Delta_2(\{\Phi,\Phi^*\}) < \infty$. There exists a positive constant C such that for any cube $Q\subset\mathbb{R}^n$ and function $u\in W^{1,\Phi}(Q)^n$ it holds that*

$$\int_Q \Phi(|\nabla u|)\,\mathrm{d}x \le C\left(\int_Q \Phi(|Du|)\,\mathrm{d}x + \int_Q \Phi\left(\frac{|u|}{\operatorname{diam} Q}\right)\,\mathrm{d}x\right), \tag{A.22}$$

$$\fint_Q \Phi_a(|\nabla u - \langle\nabla u\rangle_Q|)\,\mathrm{d}x \le C\fint_Q \Phi_a(|Du - \langle Du\rangle_Q|)\,\mathrm{d}x, \tag{A.23}$$

where a is a positive constant or $|Du|$. Moreover, if $u|_{\partial Q} = 0$, it holds that

$$\int_Q \Phi(|\nabla u|)\,\mathrm{d}x \le c\int_Q \Phi(|Du|)\,\mathrm{d}x. \tag{A.24}$$

Proof. The inequality (A.22) folows from [27]. Namely, one should focus on Lemma 5.17, Proposition 6.1 and Theorem 6.13 given there. The inequality (A.23) for $a = |Du|$ is proven in [23, Lemma 2.9] and in [27]. For the proof of (A.24) see Theorem 6.10 in [27]. □

A.5 Miscellaneous

In the following two theorems we deal with the standard Sobolev space $W^{m,r}(\Omega)$ defined for $m \in \mathbb{N}$ and $r \in (1, \infty)$ as follows

$$W^{m,r}(\Omega) = \{f \in L^r(\Omega);\ \forall k \in \mathbb{N}^n,\ 1 \leq |k| \leq m,\ \partial_k f \in L^r(\Omega)\}.$$

By $W_0^{m,r}(\Omega)$ we mean the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{m,r}(\Omega)$. The dual space of $W_0^{m,r}(\Omega)$ is denoted by $W^{-m,r'}(\Omega)$, and this extends the definition of $W^{m,r}(\Omega)$ to all negative integer values of m.

Theorem A.5.1 (Nečas' theorem on negative norms) *[5, Theorem 2.3] Let $\Omega \subset \mathbb{R}^n$ be a bounded $\mathcal{C}^{0,1}$ domain, $m \in \mathbb{Z}$ and $r \in (1, \infty)$. Then there exists a positive constant C depending only on Ω, m and r such that for all $f \in W^{m,r}(\Omega)^n$ it holds*

$$\|f\|_{m,r} \leq C\big(\|\nabla f\|_{m-1,r} + \|f\|_{m-1,r}\big).$$

Theorem A.5.2 (De Rham-type theorem) *[5, Theorem 2.8] Let $\Omega \subset \mathbb{R}^n$ be a bounded $\mathcal{C}^{0,1}$ domain, m a non-negative integer, and $r \in (1, \infty)$. Let $f \in W^{-m,r}(\Omega)^n$ satisfy*

$$\langle f, \varphi \rangle = 0, \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega)^n \text{ with } \operatorname{div} \varphi = 0.$$

Then there exists $\pi \in W^{-m+1,r}(\Omega)$ such that $f = \nabla \pi$. If in addition the domain Ω is connected, then π is defined uniquely, up to an additive constant, by f and there exists a positive constant C independent of f, such that the following estimate holds:

$$\|\pi - \langle \pi \rangle_\Omega\|_{W^{-m+1,r}(\Omega)^n} \leq C\|f\|_{W^{-m,r}(\Omega)^n}.$$

We would like to remark that classical De Rham's theorem is usually formulated for distributions, c.f. [5, Theorem 2.1]. Our version can be considered as a combination of the classical version with Nečas' theorem.

Lemma A.5.3 (Reverse Hölder inequality) *[35, Proposition V.1.1] Let Q_0 be a cube in $\mathbb{R}^n$. Suppose*

$$\fint_{Q'} f^p \, \mathrm{d}x \leq K \left(\fint_{2Q'} f \, \mathrm{d}x \right)^p + \theta \fint_{2Q'} f^p \, \mathrm{d}x + \fint_{2Q'} g^p \, \mathrm{d}x$$

holds for any $Q' = Q(x_0, R') \subset Q_0$ with $x_0 \in Q_0$ and $R' < \min\{\frac{1}{2}\operatorname{dist}(x_0, \partial Q_0), R_0\}$, where $R_0 > 0$, $K > 1$, $\theta \in [0, 1)$ are given constants. Then there exists $q > p$ such that $f \in L^q_{loc}(Q_0)$ and

$$\left(\fint_Q f^q \, \mathrm{d}x \right)^{\frac{1}{q}} \leq C \left(\fint_{2Q} f^p \, \mathrm{d}x \right)^{\frac{1}{p}} + C \left(\fint_{2Q} g^q \, \mathrm{d}x \right)^{\frac{1}{q}} \tag{A.25}$$

holds for $2Q \subset Q_0$, where side of Q is less than $2R_0$ and C, ε are positive constants depending only on K, θ, p, n.

For a dyadic sub-cube Q_k of Q we denote its predecessor by $\tilde{Q}_k$.

Lemma A.5.4 *Let $\mathcal{O} \subset \mathbb{R}^n$, $1 \le p < q < s < \infty$, $f \in L^{q/p}(\mathcal{O})$, $g \in L^{q/p}(\mathcal{O})$ and $w \in L^p(\mathcal{O})^n$. Further, let $Q \subset \mathcal{O}$ be a cube and Q_k be dyadic cubes obtained from Q. Then there exists $\varepsilon > 0$ independent of Q and $\mathcal{O}$ such that the following implication holds:*

If for every dyadic sub-cube $Q_k \subset Q$ there exists $w_a \in L^p(4\tilde{Q}_k \cap \mathcal{O})^n$ with following properties:

$$\begin{aligned}
\left(\fint_{2\tilde{Q}_k \cap \mathcal{O}} |w_a|^s \, \mathrm{d}x \right)^{\frac{1}{s}} &\le \frac{C}{2} \left(\fint_{4\tilde{Q}_k \cap \mathcal{O}} |w_a|^p \, \mathrm{d}x \right)^{\frac{1}{p}}, && \text{(A.26)} \\
\fint_{4\tilde{Q}_k \cap \mathcal{O}} |w_a|^p \, \mathrm{d}x &\le C \fint_{4\tilde{Q}_k \cap \mathcal{O}} |w|^p \, \mathrm{d}x + C \fint_{4\tilde{Q}_k \cap \mathcal{O}} |g| \, \mathrm{d}x, && \text{(A.27)} \\
\fint_{4\tilde{Q}_k \cap \mathcal{O}} |w - w_a|^p \, \mathrm{d}x &\le \varepsilon \fint_{4\tilde{Q}_k \cap \mathcal{O}} |w|^p \, \mathrm{d}x + C \fint_{4\tilde{Q}_k \cap \mathcal{O}} |f| \, \mathrm{d}x, && \text{(A.28)}
\end{aligned}$$

then $w \in L^q(Q)^n$. Positive constants C and ε are independent of Q_k, w_a and w. Furthermore, there exists a positive constant c independent of f, g and w such that

$$\fint_Q |w|^q \, \mathrm{d}x \le c \left(\fint_{4Q \cap \mathcal{O}} |f|^{\frac{q}{p}} \, \mathrm{d}x + \fint_{4Q \cap \mathcal{O}} |g|^{\frac{q}{p}} \, \mathrm{d}x + \left(\fint_{4Q \cap \mathcal{O}} |w|^p \, \mathrm{d}x \right)^{\frac{q}{p}} \right). \quad \text{(A.29)}$$

The proof itself is based on nonlinear Calderón–Zygmund theory and the considerations presented in [18]. L. A . Caffarelli and I. Peral proved Lemma A.5.4 in [18, Theorem A] in case $f, g = 0$. This lemma was later used by L. Diening and P. Kaplický in [23] for $f \neq 0$, however, authors did not provide any proof.

Throughout the proof we suppose that the functions w_a, w, f and g are extended by zero outside the domain $\mathcal{O}$. Since volume of $4\tilde{Q}_k \cap \mathcal{O}$ is proportional to $4\tilde{Q}_k$, the estimates (A.26), (A.27) and (A.28) still hold true for slightly changed constants when we replace $4\tilde{Q}_k \cap \mathcal{O}$ with $4\tilde{Q}_k$ and $2\tilde{Q}_k \cap \mathcal{O}$ with $2\tilde{Q}_k$.

We introduce Hardy-Littlewood maximal operator

$$\begin{aligned}
M(f)(x) = \sup \Big\{ \fint_P |f(y)| \, \mathrm{d}y, P \subset 4Q \text{ is a cube containing } x \\
\text{with sides parallel to axes} \Big\},
\end{aligned}$$

which satisfies the weak type $(1,1)$ inequality on $4Q$ and strong (r,r) estimate on $4Q$ if $r \in (1,\infty)$. In order to prove Lemma A.5.4 we present the following observation

Lemma A.5.5 *There exists $K_0 > 2^{n(p+1)}$ such that for all $K > K_0$ and for every $\delta \in (0,1)$ there exists $L \in \left(0, \frac{K_0}{2}\right)$ and $\varepsilon > 0$ such that for every $\lambda > 0$, for $A = \{x \in Q, M(|w|^p) > K\lambda, M(|f|) + M(|g|) \le L\lambda\}$ and $B = \{x \in Q, M(|w|^p) > \lambda\}$ it holds, that if* (A.26)*,* (A.27) *and* (A.28) *hold with ε, then following implication is true*

$$|Q_k \cap A| > (\delta + C_B K^{-s/p})|Q_k| \Rightarrow \tilde{Q}_k \subset B,$$

where C_B is a constant coming from (A.26), (A.27) *and from strong type* (r,r) *estimate for Hardy-Littlewood maximal operator.*

Proof. We proceed in a similar way like in [18]. We suppose, for contradiction, that $|Q_k \cap A| > (\delta + C_B K^{-s/p})|Q_k|$ and it is not true that $\tilde{Q}_k \subset B$ for K_0 arbitrarily large and $K > K_0$, $\delta \in (0,1)$ and ε, L arbitrarily small. Thus there are points $x_0 \in \tilde{Q}_k$ and $x_1 \in (Q_k \cap A) \subset \tilde{Q}_k$ such that

$$M(|w|^p)(x_0) \le \lambda \text{ and } M(|f|)(x_1) + M(|g|)(x_1) \le L\lambda. \tag{A.30}$$

Then $\fint_{4\tilde{Q}_k} |w_a|^p \,\mathrm{d}x \le C\lambda$ due to (A.27) and (A.30). From (A.26), (A.28) and (A.30) we get

$$\fint_{2\tilde{Q}_k} |w_a|^s \,\mathrm{d}x \le C'\lambda^{s/p}, \qquad \fint_{4\tilde{Q}_k} |w - w_a|^p \,\mathrm{d}x \le (\varepsilon + L)C\lambda. \tag{A.31}$$

We define an operator M^* as follows:

$$M^*(f)(x) = \sup\left\{\fint_P f(y)\,\mathrm{d}y, P \text{ is a cube containig } x, P \subset 2\tilde{Q}_k\right\}. \tag{A.32}$$

Due to (A.30), it holds for every $x \in Q_k$ that $M(|w|^p)(x) \le \max\{M^*(|w|^p), 2^n\lambda\}$. For $K > 2^n$ it follows that

$$M(|w|^p) > K\lambda \Rightarrow M^*(|w|^p) > K\lambda. \tag{A.33}$$

We use (A.31), Tchebyshev inequality and a strong type $\left(\frac{s}{p}, \frac{s}{p}\right)$ estimate for M^* in order to obtain the following estimate

$$\begin{aligned} &\left|\left\{x \in Q_k, M^*(|w_a|^p) > \frac{K}{2^{p+1}}\lambda\right\}\right| = \left|\left\{x \in Q_k, M^*(|w_a|^p)^{s/p} > \left(\frac{K}{2^{p+1}}\lambda\right)^{s/p}\right\}\right| \\ &\le 2^{s+s/p}(\lambda K)^{-s/p}\||w_a|^p\|_{s/p}^{s/p} = 2^{s+s/p}(\lambda K)^{-s/p}|2\tilde{Q}_k|\fint_{2\tilde{Q}_k} |w_a|^s \,\mathrm{d}x \le C_B K^{-s/p}|Q_k|. \end{aligned} \tag{A.34}$$

Using weak type $(1,1)$ estimate and (A.31), we get

$$\begin{aligned} &\left|\left\{x \in Q_k, M^*(|w - w_a|^p) > \frac{K}{2^{p+1}}\lambda, M(|f|) + M(|g|) \le L\lambda\right\}\right| \\ &\qquad \le C\frac{2^{p+1}}{K\lambda}\int_{4\tilde{Q}_k} |w - w_a|^p \,\mathrm{d}x \le C\frac{2^{p+1}}{K}(\varepsilon + L)|Q_k|. \end{aligned} \tag{A.35}$$

Due to (A.33)

$$\begin{aligned} &|\{x \in Q_k, M(|w|^p) > K\lambda, M(|f|) + M(|g|) \le L\lambda\}| \\ &\qquad \le |\{x \in Q_k, M^*(|w|^p) > K\lambda, M(|f|) + M(|g|) \le L\lambda\}| \end{aligned}$$

and, further, using (A.34) and (A.35)

$$
\begin{aligned}
&|\{x \in Q_k, M^*(|w|^p) > K\lambda, M(|f|)+M(|g|) \le L\lambda\}| \\
&\le \left|\left\{x \in Q_k, M^*(|w-w_a|^p) + M^*(|w_a|^p) > \frac{K}{2^p}\lambda, M(|f|)+M(|g|) \le L\lambda\right\}\right| \\
&\le \left|\left\{x \in Q_k, M^*(|w-w_a|^p) > \frac{K}{2^{p+1}}\lambda, M(|f|)+M(|g|) \le L\lambda\right\}\right| \\
&+ \left|\left\{x \in Q_k, M^*(|w_a|^p) \ge \frac{K}{2^{p+1}}\lambda\right\}\right| \le C\frac{2^{p+1}}{K}(\varepsilon + L)|Q_k| + C_B K^{-s/p}|Q_k|.
\end{aligned}
$$

By a suitable choice of constants L and ε we get the contradiction with the very first assumption of this proof. □

We will need the following consequence of Calderón-Zygmund decomposition proved in [18, Lemma 1.2]:

Lemma A.5.6 *Let Q be a bounded cube in $\mathbb{R}^n$. Assume that A and B are measurable sets, $A \subset B \subset Q$ and that there exists a $\delta > 0$ such that*

(i) $|A| < \delta|Q|$ and

(ii) for each dyadic sub-cube Q_k such that $|A \cap Q_k| > \delta|Q_k|$, its predecessor $\tilde{Q}_k$ is contained in B.

Then $|A| < \delta|B|$.

Corollary A.5.7 *Let A and B be defined as in Lemma* A.5.5. *Let $K > 1$ be so large and $\delta \in (0,1)$ so small, that $K^{\frac{q}{p}}C_K < 1$ where $C_K := (\delta + C_B K^{-\frac{s}{p}})$, moreover, let $\lambda_0 > 0$ be so large that*

$$\frac{C_1}{\lambda_0}\int_Q |w|^p = \delta|Q|, \tag{A.36}$$

where C_1 depends only on the dimension. Then for all $\lambda > \lambda_0$ holds $|A| < C_K|B|$.

Proof. From (A.36) follows for all $\lambda > \lambda_0$

$$A \le |\{x \in Q, M(|w|^p) > K\lambda\}| \le \frac{C_1}{K\lambda}\int_Q |w|^p < \delta|Q|, \tag{A.37}$$

which provides the first assumption of Lemma A.5.6. The second one is given by Lemma A.5.5. Hence, we conclude the proof by application of Lemma A.5.6. □

Proof of Lemma A.5.4. We set $h = M(|f|) + M(|g|)$ and $l = M(|w^p|)$. By μ_h we denote the distribution function of a function h:

$$\mu_h(s) = |\{x \in Q : |h(x)| > s\}|, \quad s \ge 0.$$

Suppose that constants K, δ and λ_0 are fixed as in Corollary A.5.7. Therefore for any $\lambda > \lambda_0$ one may derive

$$\mu_l(K\lambda) - \mu_h(L\lambda) \le C_K\mu_l(\lambda). \tag{A.38}$$

By arguments of a measure theory it is true that $l \in L^{q/p}(Q)$ if and only if

$$\sum_{k=1}^{\infty} K^{k\frac{q}{p}} \mu_l(K^k \lambda) < \infty.$$

Setting $\lambda = K^{k-1}\lambda_0$ in (A.38) we get $\mu_l(K^k\lambda_0) \le C_K \mu_l(K^{k-1}\lambda_0) + \mu_h(LK^{k-1}\lambda_0)$ for every $k \in \mathbb{N}$. Thus,

$$\mu_l(K^k\lambda_0) \le C_K^k \mu_l(\lambda_0) + \sum_{j=0}^{k-1} C_K^j \mu_h(LK^{k-1-j}\lambda_0).$$

Consequently,

$$\begin{aligned} \sum_{k=1}^{\infty} K^{k\frac{q}{p}} \mu_l(K^k\lambda_0) &\le \mu_l(\lambda_0) \sum_{k=1}^{\infty} K^{k\frac{q}{p}} C_K^k + \sum_{k=1}^{\infty} K^{k\frac{q}{p}} \sum_{j=0}^{k-1} C_K^j \mu_h(LK^{k-1-j}\lambda_0) \\ &\le \mu_l(\lambda_0) \sum_{k=1}^{\infty} \left(K^{q/p} C_K\right)^k + \sum_{k=1}^{\infty} K^{k\frac{q}{p}} \sum_{j=0}^{k-1} C_K^j \mu_h(LK^{k-1-j}\lambda_0). \end{aligned} \tag{A.39}$$

We point out that δ and K were chosen in order to $K^{q/p}C_K < 1$. Thus,

$$\mu_l(\lambda_0) \sum_{k=1}^{\infty} \left(K^{q/p} C_K\right)^k < \infty.$$

It holds that

$$\begin{aligned} \sum_{k=1}^{\infty} K^{k\frac{q}{p}} \sum_{j=0}^{k-1} C_K^j \mu_h(LK^{k-1-j}\lambda_0) &= \sum_{j=0}^{\infty} C_K^j \sum_{k=j+1}^{\infty} K^{k\frac{q}{p}} \mu_h(LK^{k-1-j}\lambda_0) \\ &= \sum_{j=0}^{\infty} \mu_h(LK^j\lambda_0) \sum_{i=0}^{\infty} (K^{\frac{q}{p}})^{j+i+1} C_K^i \\ &= \sum_{j=0}^{\infty} (K^{q/p})^j \mu_h(LK^j\lambda_0) K^{\frac{q}{p}} \sum_{i=0}^{\infty} \left(K^{q/p} C_K\right)^i \\ &= \frac{K^{q/p}}{1 - K^{q/p} C_K} \sum_{j=0}^{\infty} (K^{q/p})^j \mu_h(K^j L\lambda_0) < \infty, \end{aligned}$$

provided $h \in L^{\frac{q}{p}}(Q)$. Thus from (A.39) we have $l \in L^{\frac{q}{p}}(Q)$ and, consequently, $w \in L^q(Q)^n$.

Further,

$$\begin{aligned} \int_Q |w|^q \, \mathrm{d}x = \int_Q (|w|^p)^{q/p} \, \mathrm{d}x \le \int_Q (M(|w|^p))^{q/p} \, \mathrm{d}x &= \int_0^{\infty} \underbrace{\frac{q}{p} t^{\frac{q}{p}-1} \mu_l(t)}_{=:\nu(t)} \, \mathrm{d}t \\ &= \int_0^{K\lambda_0} \nu(t) \, \mathrm{d}t + \int_{K\lambda_0}^{\infty} \nu(t) \, \mathrm{d}t = \mathcal{J}_1 + \mathcal{J}_2. \end{aligned} \tag{A.40}$$

To estimate $\mathcal{J}_1$ we use weak 1-1 inequality for $|w|^p$. We get

$$\mathcal{J}_1 = \int_0^{K\lambda_0} \frac{q}{p} t^{\frac{q}{p}-1} \mu_l(t)\,\mathrm{d}t \le C \int_0^{K\lambda_0} \frac{q}{p} t^{\frac{q}{p}-1} \frac{1}{t} \int_{4Q} |w|^p \,\mathrm{d}x\,\mathrm{d}t \le C \lambda_0^{\frac{q}{p}-1} \int_{4Q} |w|^p \,\mathrm{d}x.$$

In $\mathcal{J}_2$ we use the substitution $t = Ks$ at first, further we increase the domain of integration and apply (A.38).

$$\begin{aligned} \mathcal{J}_2 = \int_{\lambda_0}^{\infty} \frac{q}{p} (Ks)^{\frac{q}{p}-1} \mu_l(Ks) K \,\mathrm{d}s &\le K^{\frac{q}{p}} \int_0^{\infty} \frac{q}{p} s^{\frac{q}{p}-1} \left(C_K \mu_l(s) + \mu_h(Ls) \right) \mathrm{d}s \\ &\le K^{\frac{q}{p}} C_K \int_0^{\infty} \frac{q}{p} s^{\frac{q}{p}-1} \mu_l(s)\,\mathrm{d}s + C \int_Q \left(M(f)^{\frac{q}{p}} + M(g)^{\frac{q}{p}} \right) \mathrm{d}x. \end{aligned} \tag{A.41}$$

The first term on the right hand side of (A.41) can be subsumed in the term $\int_0^\infty \nu(t)\,\mathrm{d}t$ in (A.40), because $K^{\frac{q}{p}} C_K < 1$. We put estimates of $\mathcal{J}_1$ and $\mathcal{J}_2$ into (A.40) and, after application of a strong-type $\left(\frac{q}{p}, \frac{q}{p}\right)$ estimate for the maximal operator, we get

$$\int_Q |w|^q \,\mathrm{d}x \le C \int_{4Q} \left(|f|^{\frac{q}{p}} + |g|^{\frac{q}{p}} \right) \mathrm{d}x + C \lambda_0^{\frac{q}{p}-1} \int_{4Q} |w|^p \,\mathrm{d}x, \tag{A.42}$$

From (A.37) we know that $\lambda_0 = \frac{C}{\delta} ⨍_Q |w|^p \,\mathrm{d}x$. Dividing (A.42) by $|Q|$ leads to

$$⨍_Q |w|^q \,\mathrm{d}x \le C \left(⨍_{4Q} |f|^{\frac{q}{p}} \,\mathrm{d}x + ⨍_{4Q} |g|^{\frac{q}{p}} \,\mathrm{d}x + \left(⨍_{4Q} |w|^p \,\mathrm{d}x \right)^{\frac{q}{p}} \right),$$

which concludes the proof. $\square$

We would like to apply the Calderón-Zygmund theory not only on cubes, but on a image of diadic subdivision of Q under sufficiently regular mapping T. Therefore we formulate following corollary.

Corollary A.5.8 *Let $Q \subset \mathcal{O}$ be a cube and Q_k diadic subcubes obtained from Q. Let $T : Q \to \mathbb{R}^n$ be a bi-Lipschitz mapping and let assumptions of Lemma* A.5.4 *hold on $T(4\tilde{Q}_k \cap \mathcal{O})$, resp. $T(2\tilde{Q}_k \cap \mathcal{O})$ instead of $4\tilde{Q}_k \cap \mathcal{O}$, resp. $2\tilde{Q}_k \cap \mathcal{O}$. Then the claim of Lemma* A.5.4 *holds for $T(Q)$, resp. $T(4Q \cap \mathcal{O})$.*

Proof. The corollary follows easily from the substitution $y = Tx$ and properties of the mapping T. $\square$

List of Notations

Ω	bounded domain	8
∂_{τ^α}	derivative in tangent direction	88
∂_ν	derivative in normal direction	88
$\delta_\alpha^\pm g(x)$	difference quotient in space	88
$\delta^t g$	difference quotient in time	76
D	symmetric part of the gradient, i.e. $Df = \frac{1}{2}[\nabla f + (\nabla f)^\top]$	9
a_P	mapping locally describing the boundary $\partial\Omega$	87
$\xi_P(x)$	cut-off function	87
Q	cube in $\mathbb{R}^n$ with sides parallel to the axis	43
$\fint_Q f(x)\,\mathrm{d}x$	avarage integral over Q	43
$\langle f\rangle_Q$	avarage integral over Q	43
$\partial\Omega$	boundary of the domain Ω	87
$\mathcal{S}$	extra stress tensor	10
$\mathcal{S}^\varepsilon$	regularized extra stress tensor with scalar potential Φ^ε	25
μ	generalized viscosity	10
μ^ε	truncated generalized viscosity	25
Φ	N-function, scalar potential to $\mathcal{S}$	10
Φ^*	complementary function of Φ	89
Δ_2	N-function Φ satisfying delta two condition	89
$\Delta_2(\Phi)$	the smallest constant such that Φ satisfies Δ_2-condition	89

V	function expressing differentiability properties with a scalar potential Ψ	11
P	Helmholtz projection	61
$B_r(x_0)$	ball with the diameter r and the center at x_0	87
$\mathbb{A}$	Stokes operator	61
$M(f)(x)$	Hardy-Littlewood maximal operator	98
$M^*(f)(x)$	restricted Hardy-Littlewood maximal operator	99
$[\cdot,\cdot]_\theta$	complex interpolation functor	63
$(\cdot,\cdot)_{\theta,q}$	real interpolation functor	63
$[(E_\alpha, A_\alpha); \alpha \in \mathbb{R}]$	interpolation-extrapolation scale	63
$\mathcal{D}(A)$	domain of the operator A	62
$\sigma(A)$	spectrum of the operator A	63
$\varrho(A)$	resolvent set of the operator A	63
$\mathcal{H}(E_1, E_0)$	infinitesimal generators of a strongly continuous analytic semigroup on E_0	63
Ω_r	intersection of the ball $B_r(x_0)$ with Ω	87
$\mathcal{P}(E)$	positive operators	63
$\mathcal{BIP}(E)$	operators with bounded imaginary powers	63
BUC	bounded uniformly continuous functions	61
$\mathcal{C}^k$	k-times continuously differentiable functions	10
$\mathcal{C}^{k,\alpha}$	Hölder continuous functions	10
$L^p(\Omega)$	Lebesgue spaces	13
$L^q_\sigma(\Omega)$	functions form Lebesgue spaces which are divergence-free and have zero normal component on $\partial\Omega$	61
$L^\Phi(\Omega)$	Orlicz spaces	13
$W^{k,p}(\Omega)$	Sobolev spaces	13
$W^{s,q}(\Omega)$	Sobolev-Slobodeckiĭ spaces	61
I	finite time interval	9

$W^{1,\Phi}_\sigma(\Omega)$	functions from Orlicz spaces which are divergence-free and have zero normal component on $\partial\Omega$ ... 10
$W^{1,\Phi}_\nu(\Omega)^n$	functions from Orlicz spaces which have zero normal component on $\partial\Omega$ ... 41
$B^s_{p,q}(\Omega)$	Besov spaces ... 61
$H^s_q(\Omega)$	Bessel potential spaces ... 61
$L^q(I,E)$	Bochner spaces ... 61
$W^{\alpha,q}(I,E)$	Bochner spaces ... 61
$\langle\cdot,\cdot\rangle$	duality pairing ... 62
$I\times\Omega$	time space cylinder ... 9
χ_Ω	characteristic function of Ω ... 38
ν	outward normal vector to $\partial\Omega$ ... 9
τ	tangent vector to $\partial\Omega$ ... 9

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