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# Essays in Heterogeneous Learning

**Anna Bogomolova**

Dissertation

Prague, May 2014

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**Abstract**

Essays in Heterogeneous Learning

by

Anna Bogomolova

My dissertation makes a contribution to the field of heterogeneous adaptive learning in macroeconomic models. This contribution is presented in the form of three research papers that constitute different chapters of my thesis.

In the first chapter of my dissertation, "E-stability That Does Imply Learnability", I provide criteria and sufficient conditions for the stability of a structurally heterogeneous economy under the heterogeneous learning of agents, extending the results of Honkapohja and Mitra [36], Bogomolova and Kolyuzhnov [5], and Kolyuzhnov [40]. I provide general criteria (in terms of the corresponding Jacobian matrices) for stability under heterogeneous mixed RLS/SG learning for four classes of models: models without lags and with lags of the endogenous variable and with $t$- or $t-1$- dating of expectations, and provide sufficient conditions for stability for some simpler cases, where simplifications include either the diagonal structure of the shock process behaviour or the heterogeneous RLS learning. I also provide sufficient conditions for stability in terms of the structural heterogeneity independent of heterogeneity in learning ($\delta$-stability) in terms of $E$-stability of a suitably defined aggregate economy for all four classes of models considered. In addition, I have found a very useful criterion for stabilty for all types of models in the general (non-diagonal) shock process case under mixed RLS/SG learning with equal degrees of inertia for each type of learning algorithm in terms of the stability of a suitably defined average economy with two agents.

In the second chapter, "Heterogeneous Learning: Beyond The Aggregate Economy Sufficient Conditions for Stability", I extend the results of the first chapter and of Honkapohja and Mitra [36], Bogomolova and Kolyuzhnov [5], and Kolyuzhnov [40]. Using the alternative definition of the $D$-stability approach, I provide alternative (to criteria written in terms of the corresponding Jacobian matrices in Kolyuzhnov [40] and in the first chapter of my thesis) general criteria for stability under mixed RLS/SG learning for the four classes of models considered and alternative sufficient conditions for stability for some simpler cases. This approach also allows me to provide criteria for $\delta$-stability for univariate models without lags of the endogenous variable under mixed RLS/SG learning in economically meaningful terms, such as the "same sign" conditions and $E$-stability of a suitably defined average economy and its subeconomies, and to provide quite weak sufficient conditions for $\delta$-stability for univariate models with a lag of the endogenous variable using the same economic terms. Using the characteristic equation approach, I provide quite strong, economically tractable, necessary conditions that can be used as an easy quick test for non-$\delta$-stability.

The fundamental nature of the approach adopted in the papers presented in the first two chapters of my thesis allows one to apply its results to a vast majority of the existing and prospective linear and linearized economic models (including estimated DSGE models) with the adaptive learning of agents.

The third chapter of my dissertation is presented by the paper "Optimal Monetary Policy Rules: The Problem of Stability Under Heterogeneous Learning" (a joint work with Dmitri Kolyuzhnov). In this paper, we extend the analysis of optimal monetary policy rules in terms of the stability of the economy, started by Evans and Honkapohja [26], to the case of heterogeneous learning, using the results on $\delta$-stability derived in Bogomolova and Kolyuzhnov [5], and Kolyuzhnov [40], which can be derived as special cases of the results presented in the first two chapters of the thesis.

## Abstrakt

Eseje o heterogenním učení

Anna Bogomolova

Moje dizertační práce je příspěvek v oblasti heterogenního adaptivního učení v makroekonomických modelech. Tento příspěvek je prezentován ve formě tří výzkumných prací, které představují kapitoly mé dizertace.

V první kapitole mé dizertační práce, „E-stabilita implikující naučitelnost“, stanovuji kritéria a postačující podmínky pro stabilitu strukturálně heterogenní ekonomiky s heterogenně učícími se agenty, a rozšiřuji tak výsledky Honkapohji a Mitri [36], Bogomolové a Kolyuzhnova [5], a Kolyuzhnova [40]. Stanovuji obecná kritéria (na základě Jacobiho matic) pro stabilitu při heterogenním smíšeném RLS/SG učení pro čtyři třídy modelů: modelů bez zpožděných a se zpožděnými endogenními proměnnými a s $t$- nebo $t-1$-časováním očekávání, a stanovuji postačující podmínky pro stabilitu u některých jednodušších případů, kde zjednodušení zahrnuje buď diagonální strukturu chování šokového procesu, nebo heterogenní RLS učení. Stanovuji také postačující podmínky pro stabilitu, pokud jde o strukturální heterogenitu nezávislou na heterogenitě v učení ($\delta$-stabilita) na základě $E$-stability vhodně definované agregátní ekonomiky pro všechny čtyři uvažované typy modelů. Navíc jsem nalezla velmi užitečné kritérium pro stabilitu všech typů modelů v případě obecného (ne-diagonálního) procesu pro šoky, při smíšeném RLS/SG učení se stejným stupněm inercie pro každý typ učícího algoritmu, na základě stability vhodně definované průměrné ekonomiky se dvěma agenty.

Ve druhé kapitole, “Heterogenní učení: za postačujícími podmínkami stability pro agregátní ekonomiku”, rozšiřuji výsledky první kapitoly, Honkapohji a Mitri [36], Bogomolové a Kolyuzhnova [5], a Kolyuzhnova [40]. Při použití alternativní definice $D$-stability, stanovuji alternativní (ke kritériím zapsaným na základě odpovídajících Jacobiho matic v Kolyuzhnovi [40] a v první kapitole mé práce) obecná kritéria pro stabilitu při smíšeném RLS/SG učení pro čtyři uvažované kategorie modelů a alternativní postačující podmínky pro stabilitu pro některé jednodušší případy. Tento přístup mi také umožňuje stanovit kritéria pro $\delta$-stabilitu pro modely s jednou proměnnou a bez zpožděných endogenních proměnných při heterogenním smíšeném RLS/SG učení na základě ekonomicky interpretovatelných podmínek „stejného znaku“ a $E$-stability vhodně definované agregátní ekonomiky a její podekonomik, a stanovit poměrně slabé postačující podmínky pro $\delta$-stabilitu pro modely s jednou proměnnou a se zpozděnými endogenními proměnnými za použití stejných ekonomických konceptů.Použitím přístupu charakteristické rovnice, stanovuji poměrně silné, ekonomicky interpretovatelné, nezbytné podmínky, které mohou být použity jako jednoduchý, rychlý test $\delta$-nestability.

Základní podstata přístupu použitého ve článcích prezentovaných v prvních dvou kapitolách mé dizertační práce umožňuje aplikaci jejich výsledků na velkou většinu existujících a budoucích lineárních a linearizovaných ekonomických modelů (včetně odhadnutých DSGE modelů) s adaptivně učícími se agenty.

Třetí kapitolu mé dizertační práce představuje článek "Optimální měnově-politická pravidla: problém stability při heterogenním učení" (společná práce s Dmitrim Kolyuzhnovem). V tomto článku rozšiřujeme analýzu optimálních měnově-politických pravidel pokud jde o stabilitu ekonomiky, započatou Evansem and Honkapohjou [26], na případ heterogenního učení, použitím výsledků ohledně $\delta$-stability odvozených v Bogomolové a Kolyuzhnovi [5], a Kolyuzhnovi [40], které mohou být odvozené jako speciální případy výsledků prezentovaných v prvních dvou kapitolách práce.

To Dmitri, Egor, Adelaida, and Artur

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# Chapter 1

# E-stability That Does Imply Learnability

# E-stability That Does Imply Learnability*

Anna Bogomolova†

CERGE–EI‡
Politických vězňů 7, 111 21 Praha 1,
Czech Republic

**Abstract**

I provide criteria and sufficient conditions for the stability of a structurally heterogeneous economy under the heterogeneous learning of agents, extending the results of Honkapohja and Mitra [36], Bogomolova and Kolyuzhnov [5], and Kolyuzhnov [40]. I provide general criteria (in terms of the corresponding Jacobian matrices) for stability under heterogeneous mixed RLS/SG learning for four classes of models: models without lags and with lags of the endogenous variable and with the $t$- or $t-1$-dating of expectations, and sufficient conditions for the stability in some simpler cases, where simplifications include either the diagonal structure of the shock process behaviour or heterogeneous RLS learning. I also provide sufficient conditions for stability in terms of the structural heterogeneity independent of heterogeneity in learning ($\delta$-stability) in terms of $E$-stability of a suitably defined aggregate economy for all four classes of models considered. In addition, I have found a very useful criterion for the stabilty for all types of models in the general (non-diagonal) shock process case under mixed RLS/SG learning with equal degrees of inertia for each type of learning algorithm in terms of the stability of a suitably defined average economy with two agents. The fundamental nature of the approach adopted in the paper allows one to apply the results to a vast majority of the existing and prospective linear and linearized economic models (including estimated DSGE models) with the adaptive learning of agents.

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† *Anna.Bogomolova@cerge-ei.cz.*

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## 1.1 Introduction

Contemporary macroeconomic models include expectations that influence the dynamics of endogenous variables. The main question that always arises in this respect is how to model these expectations. Historically, moving from naive (static) and adaptive expectations to rational expectations, the most widely used expectation formation function implied, until some time ago, was the rational expectations (RE) of agents. However, it has been pointed out that this assumption does not always produce appropriate results in terms of simulated model data behaviour. One of the reasons for that is the serious restrictions imposed on the knowledge of agents under this assumption. One argument against the RE assumption was formulated by Sargent [50]: If economists (who are naturally assumed to know economics better than other agents) themselves do not know the exact economic models and have to estimate the model parameters econometrically, then we may think that economic agents behave no better themselves. Thus, it makes sense to consider agents as econometricians or statisticians who update their beliefs (loosely speaking, regression coefficients) as a new data point arrives, thus trying to learn the underlying true economic model better as new information arrives. This approach is a specific form of bounded rationality and represents adaptive learning — namely, adaptive econometric learning.

Adaptive econometric learning has become widely used in the literature, see e.g., Bray [7]; Bray and Savin [8]; Fourgeaud, Gourieroux, and Pradel [28]; Marcet and Sargent [43]; Evans and Honkapohja [19, 20, 21]; Cho and Sargent [14]; Marimon [44]; Giannitsarou [31]; Adam [1]; Honkapohja and Mitra [36]; Carceles-Poveda and Giannitsarou [11]; Cho and Kasa [13]; Kolyuzhnov, Bogomolova and Slobodyan [41], a useful monograph by Evans and Honkapohja [24], and many others. Adaptive learning in macroeconomics plays several roles. First, it can be used as a testing procedure for the validity of the RE hypothesis; second, it can be used as a selection device for a model with multiple RE equilibria; third, the dynamics generated by learning may resemble the actual data behaviour (see e.g. the escape dynamics papers by Cho, Williams and Sargent [15]; Sargent and Williams [51]; Kolyuzhnov, Bogomolova and Slobodyan [41]; Slobodyan, Bogomolova and Kolyuzhnov [52]; Sargent, Williams and Zha [49]; and Cho and Kasa [13]); and fourth, the learning algorithm may serve as a method for calculating an RE equilibrium (REE).

Despite its growing popularity, this approach to modelling expectations has nevetherless some pitfalls. Usually when one applies the adaptive learning scheme as a form of expectation formation, one assumes a homogeneous type of learning, that is, that there is some representative agent that uses some particular type of learning algorithm. The most widely used learning algorithms are recursive least squares (RLS) and stochastic gradient (SG). They differ only in one respect: the RLS algorithm updates the second moments matrix, while the SG algorithm keeps it fixed[1]. The structure of a learning algorithm assumes that the current parameter (a belief, loosely speaking a regression coefficient) equals the previous value of this parameter plus a gain coefficient sequence[2] multiplied by the error correction function that depends on the most recent forecast error. The main question that arises with homogeneous learning is whether the stability results under homogeneous learning follow from this homogeneity, that is, whether the stability results remain valid if one uses e.g. some mixture of different algorithms or different speeds of updating information, different starting values, which would be, in fact, checking the representative agent hypothesis. Among the papers that consider this question are Giannitsarou [31], who assumes that agents are homogeneous in all respects but in the way they learn; Honkapohja and Mitra [36], who consider a structurally heterogeneous economy meaning that besides heterogeneity in learning, agents may also differ in structural parameters such as technologies and preferences, etc.; and Bogomolova and Kolyuzhnov [5] and Kolyuzhnov [40], who consider conditions for stability of a structurally heterogeneous forward–looking model with one lead in expectations and with the diagonal structure of shocks — conditions independent of heterogeneity in learning.

The learning heterogeneity in these papers comes from the different type of learning algorithm used by agents: RLS or SG, where the first allows us to model "more sophisticated" agents; the different speeds of reaction to innovation by different agents (usually expressed as positive multipliers before a decreasing sequence of gain coefficients common for all agents in the beliefs updating mechanism, called degrees of inertia); different initial perceptions reflected in different starting points for algorithms; and different shares of

[1] One more type of econometric learning is Bayesian learning. I also stress that in this paper I consider only the econometric type of adaptive learning. The discussion on other types of adaptive learning approaches: the generalized expectation function approach (in nonstochastic models) considered, e.g., by Fuchs [29]; Fuchs and Laroque [30]; Grandmont [32, 33]; and Grandmont and Laroque [34]) and the computational intelligence approach in the form of classifier systems, neural networks and genetic algorithms considered e.g., by Arifovic [2]; Kirman and Vriend [39]; and Cho and Sargent [14]), can be found in Evans and Honkapohja [23, pp. 464-465].

[2] This gain sequence is usually assumed to be decreasing in time, though constant gains (the so called perpetual learning) are also sometimes considered. The constant gain learning discounts the past by assigning more weight to more recent data and makes sense when agents are assumed to suspect the world around them to be non-stationary and expect sudden breaks in data.

agents using a particular type of learning algorithm. All of the above mentioned learning heterogeneity characteristics can be expressed by a type of adaptive learning when one type of agents use RLS and the other one uses SG, the so called heterogeneous mixed RLS/SG learning.

In my paper, I, following Bogomolova and Kolyuzhnov [5] and Kolyuzhnov [40], solve the following open question posed by Honkapohja and Mitra [36]: to find conditions for the stability of a structurally heterogeneous economy under mixed RLS/SG learning with (possibly) different degrees of inertia in terms of structural heterogeneity only, independent of the heterogeneity in learning.

Though Honkapohja and Mitra [36] have formulated a general criterion for such stability and have been able to solve for sufficient conditions for the case of a univariate model (a model with one endogenous variable), they did not derive the conditions (necessary, and/or sufficient) in terms of the model structure only, independent of the learning characteristics, for the general forward-looking (multivariate) case. In turn, Bogomolova and Kolyuzhnov [5] and Kolyuzhnov [40] consider conditions for stability irrespective of the heterogeneity in learning. However, they consider only a forward-looking model with one lead and without lags of the endogenous variable and the diagonal environment case that means the diagonal structure of the AR (1) coefficients matrix in the shock process. It leaves aside many economic models. For example, it leaves aside the DSGE models with an endogenous variable lag. I resolve this issue in my paper.

I extend the results of Honkapohja and Mitra [36], Bogomolova and Kolyuzhnov [5], and Kolyuzhnov [40]. I provide the general criteria for stability under heterogeneous mixed RLS/SG learning for four classes of models: models without lags and with lags of the endogenous variable and with the $t$- or $t-1$-dating of expectations, and sufficient conditions for stability in some simpler cases, where simplifications include either the diagonal structure of the shock process behaviour or heterogeneous RLS learning. I also want to stress a very useful criterion I obtain for the stabilty of all types of models in the general (non-diagonal) shock process case under mixed RLS/SG learning with equal degrees of inertia for each type of learning algorithm in terms of the stability of a suitably defined average economy with two agents.

Essentially, it turns out that all stability conditions written in terms of the stability of the corresponding Jacobian matrices require $D$-stability of some matrix (matrices)

$\Omega$. Thus, all Jacobians look like $D\Omega$, where $D$ is a positive diagonal matrix. Among the mathematical approaches to $D-$stability (studied, for example, in Johnson [37]) highlighted by Bogomolova and Kolyuzhnov [5] and Kolyuzhnov [40] are the ones based on the Lyapunov Theorem[3], on the negative diagonal dominance[4], on an alternative definition of $D$-stability[5], on the characteristic equation, and on the Routh-Hurwitz conditions[6].

In the work by Bogomolova and Kolyuzhnov [5] and by Kolyuzhnov [40], the negative diagonal dominance approach, based on the MacKenzie Theorem[7], turns out to be useful in deriving a sufficient condition for the stability of a forward-looking model with one lead and without lags of the endogenous variable and with the diagonal structure of shocks, irrespective of learning heterogeneity. I also follow this approach in the current paper, and it also allows me to find sufficient conditions for stability irrespective of the heterogeneity in learning in terms of aggregate economies but for all model classes considered. I also have to redefine the concept of $\delta-$stability introduced in Kolyuzhnov [40] that assumed stability independent of all types of learning characteristis: I find conditions for stability independent of the degrees of inertia and different initial perceptions; thus, my definition does not include different shares of agents using a particular type of learning algorithm. In this paper, using the negative diagonal dominance approach, I derive sufficient conditions for $\delta$-stability (in my definition) in some simpler cases, where simplifications include either the diagonal structure of shocks or heterogeneous RLS learning for all types of models considered. These results are written in terms of $E$-stability of suitably defined aggregate economies. The results that are based on the alternative definition of $D$-stability, the necessary conditions based on the characteristic equation approach in terms of the "same" sign conditions, and the $E$-stability of a suitably defined average economy and its subeconomies are considered in a companion paper (presented in Chapter 2).

The fundamental nature of the approach adopted here allows one to apply its results to a vast majority of the existing and prospective linear and linearized economic models with the adaptive learning of agents. For example, the models considered include (estimated) DSGE models with an introduced learning of agents. In this sense, the results derived could be very helpful in terms of checking the robustness of a particular DSGE

[3]See Theorem A.2 in Appendix A.3.
[4]See Theorem A.4 in Appendix A.4.
[5]See Theorem A.6 in Appendix A.6.
[6]See Theorem A.5 in Appendix A.5.
[7]See Theorem A.4 in Appendix A.4.

model[8] to an expectation-formation hypothesis (usually, the RE hypothesis) and checking the validity of the representative agent assumption.

The rest of the paper is organized as follows. In Section 2, I present the four classes of structurally heterogeneous models with the expectations of agents and describe the REE in each of them. In Section 3, I subsequently apply the assumption of heterogeneous adaptive learning to each model class; I apply to each model the general results of the stochastic approximation literature on the convergence of models under learning written as stochastic recursive algorithms in order to provide criteria and sufficient conditions for the stability of the REE in the general and simpler cases; and I formulate the concepts of heterogeneous expectational stability and of $\delta$-stability. In Section 4, I provide a useful stability criterion for all types of models considered in the general (non-diagonal) shock process case under mixed RLS/SG learning with equal degrees of inertia for each type of learning algorithm in terms of the stability of a suitably defined average economy with two agents. In Section 5, I use the negative diagonal dominance approach to provide sufficient conditions for $\delta-$stability in terms of the $E$-stability of a suitably defined aggregate economy for all four classes of models considered. Section 6 concludes the paper.

## 1.2 The model classes setup. The PLM, the $T$-map, and the MSV REE

### 1.2.1 The general setup of structurally heterogeneous linear models with expectations

I consider models from the general setup of Evans and Honkapohja [24, ch. 8, p. 173, eq. (8.1)] extended to allow for a heterogeneous structure and for expectations formed at time $t$. The general class of structurally heterogeneous linear models with $S$

[8] A typical DSGE model in structural form looks like

$$A_0 \begin{bmatrix} y_{t-1} \\ w_{t-1} \end{bmatrix} + A_1 \begin{bmatrix} y_t \\ w_t \end{bmatrix} + A_2 E_t y_{t+1} + B_0 \varepsilon_t = const.$$

After the estimation (for example, by DYNARE [38]), the solution of the model under rational expectations is given by

$$\begin{bmatrix} y_t \\ w_t \end{bmatrix} = \mu + T \begin{bmatrix} y_{t-1} \\ w_{t-1} \end{bmatrix} + R\varepsilon_t.$$

See, for example, Slobodyan and Wouters [53].

types of agents with different forecasts can be presented by

$$y_t = \alpha + \sum_{i=1}^{d} L_i y_{t-i} + \sum_{h=1}^{S} \sum_{b=0}^{m} \sum_{f=b}^{n} A_{bf}^h \hat{E}_{t-b}^h y_{t-b+f} + Bw_t + \zeta\varepsilon_t, \; A_{00}^h \equiv 0, \quad (1.1)$$

$$w_t = Fw_{t-1} + v_t, \quad (1.2)$$

where $y_t$ is an $n \times 1$ vector of endogenous variables, $w_t$ is a $k \times 1$ vector of exogenous variables; $v_t$ and $\varepsilon_t$ are vectors of (independent) white noise shocks; $\hat{E}_{t-b}^h y_{t-b+f}$ are (in general, non-rational) expectations of the vector of endogenous variables by agent $h$; and $L_i$, $A_{bf}^h$, $B$, and $\zeta$ are conformable matrices. It is also assumed that $F$ (a $k \times k$ matrix) is such that $w_t$ follows a stationary VAR(1) process, with $M_w = \lim_{t\to\infty} w_t w_t'$ being a positive definite matrix.

The model presented above is a linear (or linearized) model describing the whole economy written in a reduced form, that is, it corresponds to the intertemporal equilibrium of the dynamic model. In this model, the expectations of endogenous variables formed by different agent types linearly influence the current values of these variables.

Structural heterogeneity of the model, similarly to the original setup of Honkapohja and Mitra [36], is expressed through matrices $A_{bf}^h$, which are assumed to incorporate the mass $\zeta_h$ of each agent type with $\sum \zeta_h = 1$. That is, $A_{bf}^h = \zeta_h \cdot \tilde{A}_{bf}^h$, where $\tilde{A}_{bf}^h$'s are defined as describing how agents of type $h$ respond to their own forecasts. So $\tilde{A}_{bf}^h$'s contain the structural parameters characterizing a given economy, such as the basic characteristics of agents, like those describing their preferences, technology, and endowments. Subscript $b$ shows the lag in time from the current time $t$ for the information set, which is used to calculate conditional expectation $\hat{E}_{t-b}^h$, while $f$ shows how the predicted variable is far in time from the information set used to form its conditional expectation, the minimal value of $f$ is equal to $b$ as at time $t$ all the previous lags of the endogenous variable are known. Structural heterogeneity means that all $\tilde{A}_{bf}^h$'s are different for different types of agents. When $\tilde{A}_{bf}^h = A_{bf}$ for all $h$, the economy is structurally homogenous.

Though the discussion is easily extendable to this general case, in order to simplify the exposition of the results, I restrict my analysis to the examples mostly used in the literature: $d = 0$, $m = 1, n$ – any, $A_{0f}^h \equiv 0$ (Model I); $d = 1$, $m = 1, n = 2$, $A_{0f}^h \equiv 0$ (Model II), $d = 0$, $m = 0, n$ – any (Model III); and $d = 1$, $m = 0, n = 1$ (Model IV).

### 1.2.2 The class of structurally heterogeneous models without endogenous variable lags and with $t-1$-dating of expectations (Model I)

The first class of models considered is the class of structurally heterogeneous models without endogenous variable lags and with $t-1$-dating of expectations. Hereafter, I will refer to this formulation as Model I.

$$y_t = \alpha + \sum_{h=1}^{S} A_0^h \hat{E}_{t-1}^h y_t + \sum_{h=1}^{S} A_1^h \hat{E}_{t-1}^h y_{t+1} + ... + \sum_{h=1}^{S} A_\tau^h \hat{E}_{t-1}^h y_{t+\tau} + Bw_t + \zeta\varepsilon_t, \quad (1.3)$$
$$\text{and } (1.2),$$

where the definitions of variables and matrices are the same as for the general class of structurally heterogeneous linear models with $S$ types of agents with different forecasts above.

Agents of each type $h$ are assumed to form their expectations $\hat{E}_{t-1}^h y_{t+r}$, $r = 0, 1, ..., \tau$, about the endogenous variables, believing (perceiving) that the economic system follows the model called the agents' **perceived law of motion (PLM)**

$$y_t = a_{h,t-1} + b_{h,t-1} w_{t-1}.$$

Note that I consider here and through the rest of the paper only such PLMs that correspond to the fundamental or **minimal state variable (MSV) rational expectations equilibrium (REE)** solution[9].

The forecasts of each agent type $h$ based on this PLM can be written as follows

$$\begin{aligned} \hat{E}_{t-1}^h y_t &= a_{h,t-1} + b_{h,t-1} w_{t-1} \quad (1.4) \\ \hat{E}_{t-1}^h y_{t+1} &= a_{h,t-1} + b_{h,t-1} F w_{t-1} \\ \hat{E}_{t-1}^h y_{t+\tau} &= a_{h,t-1} + b_{h,t-1} F^\tau w_{t-1}. \end{aligned}$$

After plugging the forecasts of each agent (1.4) into the reduced form (1.3), one obtains the **actual law of motion (ALM)** of the model, given the PLM. The corresponding mapping

[9] The concept of the MSV solution for linear rational expectations models was introduced by McCallum [45]. As is defined in Evans and Honkapohja [24, ch.8, p. 176], this is a solution that depends linearly on a set of variables and is such that there does not exist a solution that depends linearly on a smaller set of variables.

from the parameters of the PLM to the parameters of the ALM (called the $T$-**map**) is

$$T\begin{bmatrix} a_{1,t} \\ b_{1,t} \\ \vdots \\ a_{S,t} \\ b_{S,t} \end{bmatrix} = \begin{bmatrix} \left[\alpha + \sum_{h=1}^{S}\left[A_0^h a_{h,t} + A_1^h a_{h,t} + ... + A_\tau^h a_{h,t}\right]\right]' \\ \left[\left(\sum_{h=1}^{S} A_0^h b_{h,t}\right) + \left(\sum_{h=1}^{S} A_1^h b_{h,t}\right) F + ... + \left(\sum_{h=1}^{S} A_\tau^h b_{h,t}\right) F^\tau + BF\right]' \end{bmatrix} \equiv \begin{bmatrix} T_a(\Phi_t) \\ T_b(\Phi_t) \end{bmatrix}. \tag{1.5}$$

Thus, the MSV REE solution can be found as

$$T\begin{bmatrix} \bar{a} \\ \bar{b} \\ \vdots \\ \bar{a} \\ \bar{b} \end{bmatrix} = \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix}.$$

Similar procedures are then applied to each class of models considered.

### 1.2.3 The class of structurally heterogeneous models with one lag of the endogenous variable, $t-1$-dating of expectations, and one forward-looking term in expectations (Model II)

The second class of models considered (hereafter Model II) is a class of structurally heterogeneous models with one lag of the endogenous variable, $t-1$-dating of expectations, and one forward-looking term in expectations:

$$\begin{aligned} y_t &= \alpha + Ly_{t-1} + \sum_{h=1}^{S} A_0^h \hat{E}_{t-1}^h y_t + \sum_{h=1}^{S} A_1^h \hat{E}_{t-1}^h y_{t+1} + Bw_t + \zeta\varepsilon_t, \qquad (1.6) \\ & \textit{and } (1.2), \end{aligned}$$

where the definitions of the variables and the matrices are as before defined to be the same as for the general class of structurally heterogeneous linear models with $S$ types of agents with different forecasts above.

Each agent type $h$ forms its expectations $\hat{E}_{t-1}^h y_t$, $\hat{E}_{t-1}^h y_{t+1}$ using a PLM (corresponding to the MSV solution $y_t = a + by_{t-1} + cw_t + Bv_t + \zeta\varepsilon_t$) that looks like

$$y_t = a_{h,t-1} + b_{h,t-1}y_{t-1} + c_{h,t-1}w_{t-1}.$$

The forecast functions based on this PLM are, in turn, given by

$$\begin{aligned} \hat{E}^h_{t-1} y_t &= a_{h,t-1} + b_{h,t-1} y_{t-1} + c_{h,t-1} w_{t-1} \\ \hat{E}^h_{t-1} y_{t+1} &= a_{h,t-1} + b_{h,t-1} \hat{E}^h_{t-1} y_t + c_{h,t-1} \hat{E}^h_{t-1} (F w_{t-1} + v_t) = \\ &= (I_n + b_{h,t-1}) a_{h,t-1} + b^2_{h,t-1} y_{t-1} + (b_{h,t-1} c_{h,t-1} + c_{h,t-1} F) w_{t-1}. \end{aligned} \quad (1.7)$$

The ALM, derived after plugging the forecasts (1.7) of agents into the reduced form of the model (1.6), is

$$\begin{aligned} y_t &= \alpha + L y_{t-1} + \sum_{h=1}^{S} A^h_0 [a_{h,t-1} + b_{h,t-1} y_{t-1} + c_{h,t-1} w_{t-1}] + \\ &+ \sum_{h=1}^{S} A^h_1 [(I_n + b_{h,t-1}) a_{h,t-1} + b^2_{h,t-1} y_{t-1} + (b_{h,t-1} c_{h,t-1} + c_{h,t-1} F) w_{t-1}] + B w_t + \zeta \varepsilon_t. \end{aligned}$$

Thus, the corresponding $T$-map is given by

$$T \begin{bmatrix} a_{1,t} \\ b_{1,t} \\ c_{1,t} \\ \vdots \\ a_{S,t} \\ b_{S,t} \\ c_{S,t} \end{bmatrix} = \begin{bmatrix} \left[\alpha + \sum_{h=1}^{S} \left[A^h_0 a_{h,t} + A^h_1 a_{h,t} + A^h_1 b_{h,t} a_{h,t}\right]\right]' \\ \left[L + \left(\sum_{h=1}^{S} A^h_0 b_{h,t}\right) + \left(\sum_{h=1}^{S} A^h_1 b^2_{h,t}\right)\right]' \\ \left[\left(\sum_{h=1}^{S} A^h_0 c_{h,t}\right) + \left(\sum_{h=1}^{S} A^h_1 b_{h,t} c_{h,t}\right) + \left(\sum_{h=1}^{S} A^h_1 c_{h,t} F\right) + BF\right]' \end{bmatrix} \equiv \begin{bmatrix} T_a(\Phi_t) \\ T_b(\Phi_t) \\ T_c(\Phi_t) \end{bmatrix}. \quad (1.8)$$

As a result, the MSV REE of Model II can be derived from the following system of equations

$$\begin{aligned} \alpha + \sum_{h=1}^{S} \left(A^h_0 + A^h_1 + A^h_1 \bar{b}\right) \bar{a} &= \bar{a} \\ L + \sum_{h=1}^{S} A^h_0 \bar{b} + \left(\sum_{h=1}^{S} A^h_1\right) \bar{b}^2 &= \bar{b} \\ \left(\sum_{h=1}^{S} A^h_0 + A^h_1 \bar{b}\right) \bar{c} + \sum_{h=1}^{S} A^h_1 \bar{c} F + BF &= \bar{c}. \end{aligned} \quad (1.9)$$

### 1.2.4 The class of structurally heterogeneous models without lags of the endogenous variable and with $t$-dating of expectations (Model III)

The third class of models considered (hereafter Model III) is the class of structurally heterogeneous models without lags of the endogenous variable and with $t$-dating of expectations:

$$\begin{aligned} y_t &= \alpha + \sum_{h=1}^{S} A_1^h \hat{E}_t^h y_{t+1} + ... + \sum_{h=1}^{S} A_\tau^h \hat{E}_t^h y_{t+\tau} + B w_t + \zeta \varepsilon_t, \qquad (1.10) \\ & \quad \textit{and } (1.2), \end{aligned}$$

where the variables and the matrices are defined as before.

The PLM (corresponding to the MSV solution of this Model) of agent type $h$ has the form

$$y_t = a_{h,t} + b_{h,t} w_t,$$

and the forecast functions are presented by

$$\begin{aligned} \hat{E}_t^h y_{t+1} &= a_{h,t} + b_{h,t} F w_t \qquad (1.11) \\ \hat{E}_t^h y_{t+\tau} &= a_{h,t} + b_{h,t} F^\tau w_t. \end{aligned}$$

After plugging them into the reduced form (1.10), one obtains the corresponding $T$-map:

$$T \begin{bmatrix} a_{1,t} \\ b_{1,t} \\ \vdots \\ a_{S,t} \\ b_{S,t} \end{bmatrix} = \begin{bmatrix} \left[\alpha + \sum_{h=1}^{S} \left[A_1^h a_{h,t} + ... + A_\tau^h a_{h,t}\right]\right]' \\ \left[\left(\sum_{h=1}^{S} A_1^h b_{h,t}\right) F + ... + \left(\sum_{h=1}^{S} A_\tau^h b_{h,t}\right) F^\tau + B\right]' \end{bmatrix} \equiv \begin{bmatrix} T_a(\Phi_t) \\ T_b(\Phi_t) \end{bmatrix}. \qquad (1.12)$$

The MSV REE is then defined as usual.

### 1.2.5 The class of structurally heterogeneous models with a lagged endogenous variable, $t$-dating of expectations, the $(1, y'_{t-1}; w'_t)$ information set, and one forward-looking term in expectations (Model IV)

The fourth (the last one in this paper) class of models considered (hereafter Model IV) is the class of structurally heterogeneous models with a lagged endogenous variable, $t$-dating of expectations, the $(1, y'_{t-1}; w'_t)$ information set[10], and one forward-looking term

[10]In order to keep the presentation of results concise I do not consider the case of the $(1, y'_t; w'_t)$ information set (considered, for example, in Evans and Honkapohja [22] for a structurally homogeneous economy

in expectations:

$$y_t = \alpha + Ly_{t-1} + \sum_{h=1}^{S} A_1^h \hat{E}_t^h y_{t+1} + Bw_t + \zeta\varepsilon_t, \quad (1.13)$$
$$and\ (1.2)$$

with the same definition for the variables and the matrices as above.

The PLM (corresponding to the MSV solution for Model IV) of agent type $h$ has the form

$$y_t = a_{h,t} + b_{h,t} y_{t-1} + c_{h,t} w_t,$$

and the corresponding forecast functions based on this PLM are given by

$$\begin{aligned} \hat{E}_t^h y_{t+1} &= a_{h,t} + b_{h,t}(a_{h,t} + b_{h,t} y_{t-1} + c_{h,t} w_t) + c_{h,t} F w_t = \\ &= (I_n + b_{h,t})\, a_{h,t} + b_{h,t}^2 y_{t-1} + (b_{h,t} c_{h,t} + c_{h,t} F) w_t. \end{aligned} \quad (1.14)$$

The ALM, derived by plugging the forecasts (1.14) of agents into the model's reduced form (1.13), is

$$\begin{aligned} y_t &= \alpha + Ly_{t-1} + \sum_{h=1}^{S} A_1^h ((I_n + b_{h,t})\, a_{h,t} + b_{h,t}^2 y_{t-1} + (b_{h,t} c_{h,t} + c_{h,t} F) w_t) + Bw_t + \zeta\varepsilon_t = \\ &= \alpha + \sum_{h=1}^{S} \left( A_1^h (I_n + b_{h,t})\, a_{h,t} \right) + \left( \sum_{h=1}^{S} \left( A_1^h b_{h,t}^2 \right) + L \right) y_{t-1} + \\ &\quad + \left( \sum_{h=1}^{S} \left( A_1^h b_{h,t} c_{h,t} \right) + \sum_{h=1}^{S} \left( A_1^h c_{h,t} F \right) + B \right) w_t + \zeta\varepsilon_t. \end{aligned}$$

Finally, the $T$-map is presented by

$$T \begin{bmatrix} a_{1,t} \\ b_{1,t} \\ c_{1,t} \\ \vdots \\ a_{S,t} \\ b_{S,t} \\ c_{S,t} \end{bmatrix} = \begin{bmatrix} \left[ \alpha + \sum_{h=1}^{S} \left( A_1^h (I_n + b_{h,t})\, a_{h,t} \right) \right]' \\ \left[ \sum_{h=1}^{S} \left( A_1^h b_{h,t}^2 \right) + L \right]' \\ \left[ \sum_{h=1}^{S} \left( A_1^h b_{h,t} c_{h,t} \right) + \sum_{h=1}^{S} \left( A_1^h c_{h,t} F \right) + B \right]' \end{bmatrix} \equiv \begin{bmatrix} T_a(\Phi_t) \\ T_b(\Phi_t) \\ T_c(\Phi_t) \end{bmatrix}, \quad (1.15)$$

under homogeneous learning) in this paper. Instead, I consider a realistic situation when the value of the endogenous variable at time $t$ cannot be used to predict the future value of this variable since it is not known yet. It allows me to avoid simultaneity between $y_t$ and $\hat{E}_t^h y_{t+1}$. The case of the $(1, y_t'; w_t')$ information set clearly falls under this paper's technical constructions with some modifications and is a matter for my future research.

and the MSV REE of Model IV is given by

$$\begin{aligned} \left(I_n - \sum_{h=1}^{S} A_1^h \bar{b} - \sum_{h=1}^{S} A_1^h\right) \bar{a} &= \alpha \\ \sum_{h=1}^{S} A_1^h \bar{b}^2 - \bar{b} + L &= 0 \\ \left(I_n - \sum_{h=1}^{S} A_1^h \bar{b}\right) \bar{c} - \sum_{h=1}^{S} A_1^h \bar{c} F &= B. \end{aligned} \tag{1.16}$$

## 1.3 Heterogeneous adaptive learning, the SRA, the associated ODE, and the criteria for stability under heterogeneous learning for various classes of models

### 1.3.1 Heterogeneous adaptive learning and the general setup of a stochastic recursive algorithm and the associated ODE

In all classes of structurally heterogeneous linear models with the expectations presented above, it is assumed that agents use the adaptive learning procedure to form and update their forecast functions. They use the so-called heterogeneous **mixed RLS/SG learning**, when a part of agents, $h = \overline{1, S_0}$, is assumed to use the RLS learning algorithm, while others, $h = \overline{S_0 + 1, S}$, are assumed to use the SG learning algorithm. Heterogeneity in learning comes in the form of different types of learning algorithms used by agents (RLS and SG), different speeds of reacting to innovations, different initial perceptions (presented by different starting values for learning algorithms for each agent), and different shares of agents using a particualr type of learning algorithm. Different speeds of reacting to innovations (or different degrees of responsiveness to the updating function) are presented by different degrees of inertia $\delta_h > 0$, which, in the formulation of Giannitsarou [31], are constant coefficients before the deterministic decreasing gain sequence in the learning algorithm, which is common for all agents.[11]:

$$\alpha_{h,t} = \delta_h \alpha_t,$$

where $\alpha_t$ is a deterministic, decreasing and positive gain sequence that satisfies the usual conditions:

[11] Honkapohja and Mitra [36] use the generalized form of degrees of inertia (see Honkapohja and Mitra [36, Ch.3]). For the ease of the exposition, I prefer to stick to the definition by Giannitsarou [31] though the results derived are easily extendable to the generalized formulation.

**Assumption A** $\sum_{t=1}^{\infty}\alpha_t=\infty$ *and* $\sum_{t=1}^{\infty}\alpha_t^2<\infty$, *and* $\limsup_{t\to\infty}\left[\left(\frac{1}{\alpha_{t+1}}\right)-\left(\frac{1}{\alpha_t}\right)\right]<\infty$ *(an additional technical assumption).*

These conditions on $\alpha_t$ are standard and always assumed to hold in order to guarantee convergence to a REE of the model under learning written in the form of a **stochastic recursive algorithm (SRA)**. All classes of models considered in this paper (as well as the majority of economic models under learning) can be written in a standard form of the SRA, the convergence properties of which can be studied using the stochastic approximation techniques developed in Benveniste, Métivier and Priouret [3], adapted and presented for economic (in particular, linear) models under learning, for example, by Evans and Honkapohja [22] and Evans and Honkapohja [24]. According to this approach, the majority of linear economic models under learning can be written in the form of a SRA that has the following representation.

$$\theta_t=\theta_{t-1}+\alpha_t H\left(\theta_{t-1},X_t\right)+\alpha_t^2\rho_t\left(\theta_{t-1},X_t\right),$$

$$X_t=A(\theta_{t-1})X_{t-1}+B(\theta_{t-1})W_t, \tag{1.17}$$

where $\theta_t$ is a vector of recursively updated parameters (called **beliefs**), which in typical adaptive learning algorithms (RLS, SG) includes the regression coefficients and elements of the second moments matrix. The second equation gives the law of motion for the state in the model, where $W_t$ is a random disturbance term.

Under the regularity conditions on $H\left(\theta_{t-1},X_t\right)$ and $\rho_t\left(\theta_{t-1},X_t\right)$, under Assumption A on $\alpha_t$, and under assumptions on the properties of the law of motion for the state (1.17) (specified in Evans and Honkapohja [24, pp.124-125] and in Evans and Honkapohja [22, pp. 26-27] and presented here for the reader's convenience in Appendix A.1), conditions for the convergence of $\theta_t$ to an equilibrium $\bar{\theta}$ (that in this case is considered to be the MSV REE) are determined by the conditions for stability of the **associated ODE:**

$$\frac{d\theta}{d\tau}=h(\theta),\text{where } h(\theta)=\lim_{t\to\infty}EH\left(\theta,X_t(\theta)\right). \tag{1.18}$$

After writing the model in the standard form of an SRA and deriving the associated ODE, one may start using the local stability properties of this ODE as the local stability properties under learning of the model's equilibrium (in this case the MSV REE).

Due to different time-dating and the corresponding differences in updating algorithms, it is convenient to consider adaptive learning and the corresponding SRAs for the classes of models grouped by the dating of expectations: models with $t-1$-dating of expectations (Model I and Model II) and models with $t$-dating of expectations (Model III and Model IV).

### 1.3.2 Heterogeneous adaptive learning in models with $t-1$-dating of expectations with information available up to $t-1$ (Models I and II)

First, I consider heterogeneous adaptive learning in models with $t-1$-dating of expectations with information available up to $t-1$[12] (that is, in Models I and II). After denoting $z_t' = (1, w_t')$ for Model I (similarly in Model II, when a lag is included, $z_t' = (1, y_t', w_t')$ ) and $\Phi_{h,t}' = (a_{h,t}, b_{h,t})$ (if a lag is included, then $\Phi_{h,t}' = (a_{h,t}, b_{h,t}, c_{h,t})$), $\Phi_t' = (\Phi_{1,t}', ..., \Phi_{S,t}')$, the formal representation of the learning algorithms in these classes of models can be written as follows.

RLS: for $h = \overline{1, S_0}$

$$\Phi_{h,t+1} = \Phi_{h,t} + \alpha_{h,t+1} R_{h,t+1}^{-1} z_t \left(y_{t+1} - \Phi_{h,t}' z_t\right)' \tag{1.19a}$$

$$R_{h,t+1} = R_{h,t} + \alpha_{h,t+1} \left(z_t z_t' - R_{h,t}\right) \tag{1.19b}$$

SG: for $h = \overline{S_0 + 1, S}$

$$\Phi_{h,t+1} = \Phi_{h,t} + \alpha_{h,t+1} z_t \left(y_{t+1} - \Phi_{h,t}' z_t\right)'. \tag{1.20}$$

Agents use $\Phi_{h,t-1}' = (a_{h,t-1}, b_{h,t-1})$ in Model I (or $\Phi_{h,t-1}' = (a_{h,t-1}, b_{h,t-1}, c_{h,t-1})$ in Model II when a lag is included) and $z_{t-1}$ to make their forecasts $\hat{E}_{t-1}^h y_t$ and $\hat{E}_{t-1}^h y_{t+1}$. The actual law of motion will be

$$y_t = T(\Phi_{t-1})' z_{t-1} + B\nu_t + \zeta\varepsilon_t,$$

where $T(\Phi_{t-1})'$ is defined in (1.5) for Model I and in (1.8) for Model II above.

To convert the system into **the standard form of an SRA**, I make a change in the timing of the system for $R_{h,t}$. I set $S_{h,t-1} = R_{h,t}$. Thus, the beliefs updating algorithm of Models I and II will have the following SRA representation

RLS: for $h = \overline{1, S_0}$

$$\Phi_{h,t+1} = \Phi_{h,t} + \alpha_{t+1} \frac{\alpha_{h,t+1}}{\alpha_{t+1}} \left(S_{h,t}^{-1} z_t z_t' \left[T(\Phi_t) - \Phi_{h,t}\right] + S_{h,t}^{-1} z_t \left(B\nu_{t+1} + \zeta\varepsilon_{t+1}\right)'\right)$$

[12] Similar technical constructions can be found in Evans and Honkapohja [24, ch 10.2.2] but for a structurally homogeneous economy under homogeneous learning.

$S_{h,t+1} = S_{h,t} + \alpha_{t+1}\left(\delta_h\right)\left(z_{t+1}z'_{t+1} - S_{h,t}\right) + \alpha^2_{t+1}\left(\frac{\alpha_{t+2} - \alpha_{t+1}}{\alpha^2_{t+1}}\right)\left(\delta_h\right)\left(z_{t+1}z'_{t+1} - S_{h,t}\right)$

SG: for $h = \overline{S_0 + 1, S}$

$\Phi_{h,t+1} = \Phi_{h,t} + \alpha_{t+1}\frac{\alpha_{h,t+1}}{\alpha_{t+1}} z_t z'_t \left[T(\Phi_t) - \Phi_{h,t}\right] + \alpha_{t+1}\frac{\alpha_{h,t+1}}{\alpha_{t+1}} z_t \left(B\nu_{t+1} + \zeta\varepsilon_{t+1}\right)'.$

For the case without lags (Model I), **the law of motion for the state** can be written as

$$X_t = AX_{t-1} + BW_t,$$

where

$$X'_t = \left(1, w'_t, w'_{t-1}, \nu'_t, \varepsilon'_t\right),$$

$$W'_t = (1, \nu'_t, \varepsilon'_t),$$

and where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & F & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

For the case with a lag and one forward-looking term in expectations (Model II), **the law of motion for the state** can be written as

$$X_t = A(\theta_{t-1})X_{t-1} + B(\theta_{t-1})W_t,$$

where

$$\theta'_t = \left(\theta'_{1,t}, ..., \theta'_{S,t}\right),$$

where

$$\begin{aligned} \theta'_{h,t} &= \left(vec\left(\Phi'_{h,t}\right)', vec(S_{h,t})'\right), h = \overline{1, S_0}, \\ \theta'_{h,t} &= vec\left(\Phi'_{h,t}\right), h = \overline{S_0 + 1, S}, \end{aligned}$$

$$X'_t = \left(1, y'_t, w'_t, y_{t-1}, w'_{t-1}, \nu'_t, \varepsilon'_t\right),$$

$$W'_t = (1, \nu'_t, \varepsilon'_t),$$

and where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ T_a'(\theta_{t-1}) & T_b'(\theta_{t-1}) & T_c'(\theta_{t-1}) & 0 & 0 & 0 & 0 \\ 0 & 0 & F & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & B & \zeta \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

**The associated ODEs** for these SRAs (for the proof see Appendix A.7.1) are given by

$$\frac{d\Phi_h}{d\tau} = \delta_h \left( T(\Phi) - \Phi_h \right), h = \overline{1, S_0} \tag{1.21a}$$

$$\frac{d\Phi_h}{d\tau} = \delta_h M_z \left( T(\Phi) - \Phi_h \right), h = \overline{S_0 + 1, S}. \tag{1.21b}$$

### 1.3.3 General criteria for stability under heterogeneous learning for Models I and II

The next step is to take the derivatives of the $T$-maps: (1.5) for Model I and (1.8) for Model II, and to compose the Jacobians for the right-hand side of the associated ODEs (1.21).

For the SRA of Model I, the system of the associated ODEs (linear by the setup) after dropping the inessential constant terms is written by components as

$$\begin{pmatrix} \dot{a}_1 \\ \vdots \\ \dot{a}_S \end{pmatrix} = D_1 \Omega \begin{pmatrix} a_1 \\ \vdots \\ a_S \end{pmatrix}, \text{ where } D_1 = \begin{pmatrix} \delta_1 I_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_n \end{pmatrix},$$

$$\Omega = \begin{pmatrix} A_0^1 + A_1^1 ... + A_\tau^1 - I_n & \cdots & A_0^S + A_1^S ... + A_\tau^S \\ \vdots & \ddots & \vdots \\ A_0^1 + A_1^1 ... + A_\tau^1 & \cdots & A_0^S + A_1^S ... + A_\tau^S - I_n \end{pmatrix}$$

$$\begin{pmatrix} vec\dot{b}_1 \\ \vdots \\ vec\dot{b}_S \end{pmatrix} = D_w \Omega_F \begin{pmatrix} vec b_1 \\ \vdots \\ vec b_S \end{pmatrix}, \text{ where } D_w = \begin{pmatrix} D_{w1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{wS} \end{pmatrix},$$

$$\begin{array}{l} D_{wh} = \delta_h I_{nk}, h = \overline{1, S_0} \\ D_{wh} = \delta_h \left( M_w \otimes I_n \right), h = \overline{S_0 + 1, S} \end{array},$$

$$\Omega_F = \begin{pmatrix} F'^{\tau}\otimes A_{\tau}^{1}+...+F'\otimes A_{1}^{1}+I_k\otimes A_{0}^{1}\text{-}I_{nk} & \cdots & F'^{\tau}\otimes A_{\tau}^{S}+...+F'\otimes A_{1}^{S}+I_k\otimes A_{0}^{S} \\ \vdots & \ddots & \vdots \\ F'^{\tau}\otimes A_{\tau}^{1}+...+F'\otimes A_{1}^{1}+I_k\otimes A_{0}^{1} & \cdots & F'^{\tau}\otimes A_{\tau}^{S}+...+F'\otimes A_{1}^{S}+I_k\otimes A_{0}^{S}\text{-}I_{nk} \end{pmatrix}.$$

For the SRA of Model II, the system of the associated ODEs linearized around the MSV REE given by (1.9) after dropping the inessential constant terms is written by components as

$$\begin{pmatrix} \dot{a}_1 \\ vec\dot{b}_1 \\ vec\dot{c}_1 \\ \vdots \\ \dot{a}_S \\ vec\dot{b}_S \\ vec\dot{c}_S \end{pmatrix} = D_{1yw}\Omega_{1bF} \begin{pmatrix} a_1 \\ vecb_1 \\ vecc_1 \\ \vdots \\ a_S \\ vecb_S \\ vecc_S \end{pmatrix}, \text{ where } D_{1yw} = \begin{pmatrix} D_{1yw1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{1ywS} \end{pmatrix},$$

$$D_{1ywh} = \delta_h I_{n+n^2+nk}, h = \overline{1, S_0}$$

$$D_{1ywh} = \delta_h \left(M_{1yw} \otimes I_n\right), h = \overline{S_0+1, S}$$

$$\Omega_{1bF} = \begin{bmatrix} R^1 - I_{n+n^2+nk} & R^1 & \cdots & R^1 \\ R^2 & R^2 - I_{n+n^2+nk} & \cdots & R^2 \\ \vdots & \vdots & \ddots & \vdots \\ R^S & R^S & \cdots & R^S - I_{n+n^2+nk} \end{bmatrix}$$

$$R^h = \begin{bmatrix} A_1^h + \left(A_0^h + A_1^h\bar{b}\right) & \bar{a}'\otimes A_1^h & 0 \\ 0 & \bar{b}'\otimes A_1^h + I_n\otimes\left(A_0^h + A_1^h\bar{b}\right) & 0 \\ \vdots & \bar{c}'\otimes A_1^h & F'\otimes A_1^h + I_k\otimes\left(A_0^h + A_1^h\bar{b}\right) \end{bmatrix}.$$

Using the above, I derive general criteria for stability under mixed RLS/SG learning (à la Honkapohja and Mitra [36, Prop. 5]) for Models I and II.

**Criterion 1.1** *In economy (1.3) and (1.2),* Model I, *mixed RLS/SG learning converges globally (almost surely) to the minimal state variable (MSV) solution if and only if the corresponding matrices $D_1\Omega$ and $D_w\Omega_F$ have eigenvalues with negative real parts.*

**Proof.** See Appendix A.7.3. □

**Criterion 1.2** *In economy (1.6) and (1.2),* Model II, *in which all roots of $\bar{b}$ defined in (1.9) lie inside the unit circle, mixed RLS/SG learning converges (almost surely) to the* *minimal state variable (MSV) solution if and only if the corresponding matrix $D_{1yw}\Omega_{1bF}$ has eigenvalues with negative real parts.*

**Proof.** See Appendix A.7.4. □

Note that these conditions are written in terms of mixture of structural and learning heterogeneity. In order to be able to write down economically meaningful conditions in terms of structural heterogeneity only, one has to consider several simplifications of the general setup. For example, it is easy to obtain more pleasant sufficient conditions for stabiltiy of the MSV REE of Model II under heterogeneous RLS learning, which allows for a further elaboration of sufficient conditions.

**Corollary 1.3** *(Sufficient conditions for stabiltiy of the MSV REE of Model II under heterogeneous RLS learning). In economy (1.6) and (1.2),* Model II*, in which all roots of $\bar{b}$ defined in (1.9) lie inside the unit circle, heterogeneous RLS learning converges (almost surely) to the minimal state variable (MSV) solution if the corresponding matrices $D_1\Omega$, $D_y\Omega_b$, and $D_w\Omega_F$ (below) have eigenvalues with negative real parts; thus, the MSV REE is a locally stable point of the following system*

$$\begin{pmatrix} \dot{a}_1 \\ \vdots \\ \dot{a}_S \end{pmatrix} = D_1\Omega \begin{pmatrix} a_1 \\ \vdots \\ a_S \end{pmatrix}, \text{ where } D_1 = \begin{pmatrix} \delta_1 I_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_n \end{pmatrix},$$

$$\Omega = \begin{pmatrix} A_0^1 + A_1^1 + A_1^1\bar{b} - I_n & \cdots & A_0^S + A_1^S + A_1^S\bar{b} \\ \vdots & \ddots & \vdots \\ A_0^1 + A_1^1 + A_1^1\bar{b} & \cdots & A_0^S + A_1^S + A_1^S\bar{b} - I_n \end{pmatrix}$$

$$\begin{pmatrix} vec\dot{b}_1 \\ \vdots \\ vec\dot{b}_S \end{pmatrix} = D_y\Omega_b \begin{pmatrix} vecb_1 \\ \vdots \\ vecb_S \end{pmatrix}, \text{ where } D_y = \begin{pmatrix} \delta_1 I_{n^2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_{n^2} \end{pmatrix},$$

$$\Omega_b = \begin{pmatrix} \bar{b}' \otimes A_1^1 + I_n \otimes \left(A_0^1 + A_1^1\bar{b}\right) - I_{n^2} & \cdots & \bar{b}' \otimes A_1^S + I_n \otimes \left(A_0^S + A_1^S\bar{b}\right) \\ \vdots & \ddots & \vdots \\ \bar{b}' \otimes A_1^1 + I_n \otimes \left(A_0^1 + A_1^1\bar{b}\right) & \cdots & \bar{b}' \otimes A_1^S + I_n \otimes \left(A_0^S + A_1^S\bar{b}\right) - I_{n^2} \end{pmatrix},$$

$$\begin{pmatrix} vec\dot{c}_1 \\ \vdots \\ vec\dot{c}_S \end{pmatrix} = D_w\Omega_F \begin{pmatrix} vecc_1 \\ \vdots \\ vecc_S \end{pmatrix}, \text{where } D_w = \begin{pmatrix} \delta_1 I_{nk} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_{nk} \end{pmatrix},$$

$$\Omega_F = \begin{pmatrix} F' \otimes A_1^1 + I_k \otimes \left(A_0^1 + A_1^1\bar{b}\right) - I_{nk} & \cdots & F' \otimes A_1^S + I_k \otimes \left(A_0^S + A_1^S\bar{b}\right) \\ \vdots & \ddots & \vdots \\ F' \otimes A_1^1 + I_k \otimes \left(A_0^1 + A_1^1\bar{b}\right) & \cdots & F' \otimes A_1^S + I_k \otimes \left(A_0^S + A_1^S\bar{b}\right) - I_{nk} \end{pmatrix}.$$

**Proof.** See Appendix A.7.5. □

### 1.3.4 Conditions for stability in the diagonal environment case for Models I and II

Another simplification that can be used to derive economically meaningful conditions in terms of structural heterogeneity only, is to assume a diagonal structure of the shocks process. Now, I will obtain a criterion for the following diagonal environment case

$$F = diag(\rho_1, ..., \rho_k), M_w = \lim_{t\to\infty} w_t w_t' = diag\left(\tfrac{\sigma_1^2}{1-\rho_1^2}, ..., \tfrac{\sigma_k^2}{1-\rho_k^2}\right). \tag{1.22}$$

**For Model I** in the "diagonal" environment, the problem of finding conditions for the stability of both $D_1\Omega$ and $D_w\Omega_F$ under any (possibly different) degrees of inertia of agents, $\delta > 0$, is simplified to finding stability conditions for $D_1\Omega$ and $D_1\Omega_{\rho_l}$,where $\Omega_{\rho_l}$ is obtained from $\Omega$ by substituting all of $A_0^h + A_1^h .. + A_\tau^h$ for $A_0^h + \rho_l A_1^h .. + \rho_l^\tau A_\tau^h$, where $|\rho_l| < 1$ as $w_t$ follows a stationary VAR(1) process, by the setup of the model.

$$D_1 = \begin{pmatrix} \delta_1 I_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_n \end{pmatrix}, \tag{1.23}$$

$$\Omega_{\rho_l} = \begin{pmatrix} A_0^1 + \rho_l A_1^1 + ... + \rho_l^\tau A_\tau^1 - I_n & \cdots & A_0^S + \rho_l A_1^S + ... + \rho_l^\tau A_\tau^S \\ \vdots & \ddots & \vdots \\ A_0^1 + \rho_l A_1^1 + ... + \rho_l^\tau A_\tau^1 & \cdots & A_0^S + \rho_l A_1^S + ... + \rho_l^\tau A_\tau^S - I_n \end{pmatrix}, \forall l = 0, ..., k, (\rho_0 = 1). \tag{1.24}$$

**Proposition 1.4** *(A criterion for the stability of* Model I *under mixed RLS/SG learning for the diagonal environment case under any (possibly different) degrees of inertia of agents, $\delta > 0$). In the structurally heterogeneous economy (1.3), (1.2), and (1.22), mixed RLS/SG learning (2.8), (1.20), and (1.4) converges globally (almost surely) to an MSV REE solution for any (possibly different) degrees of inertia of agents, $\delta > 0$, if and only* *if matrices $D_1\Omega_{\rho l}$ are stable for any $\delta > 0$, where $D_1$ and $\Omega_{\rho_l}$ are defined in (1.23) and (1.24), respectively.*

**Proof.** See Appendix A.7.6. $\square$

**For Model II with heterogeneous RLS learning** in the "diagonal" environment, the problem of finding conditions for stability for $D_1\Omega$, $D_y\Omega_b$, and $D_w\Omega_F$ under any (possibly different) degrees of inertia of agents, $\delta > 0$, is simplified to finding stability conditions for $D_y\Omega_b$ and $D_1\Omega_{\rho_l}$, where $\Omega_{\rho_l}$ is given by

$$\Omega_{\rho_l} = \begin{pmatrix} A_0^1 + \rho_l A_1^1 + A_1^1\bar{b} - I_n & \cdots & A_0^S + \rho_l A_1^S + A_1^S\bar{b} \\ \vdots & \ddots & \vdots \\ A_0^1 + \rho_l A_1^1 + A_1^1\bar{b} & \cdots & A_0^S + \rho_l A_1^S + A_1^S\bar{b} - I_n \end{pmatrix}, \forall l = 0, ..., k, (\rho_0 = 1), \tag{1.25}$$

where $|\rho_l| < 1$ as $w_t$ follows a stationary VAR(1) process, by the setup of the model.

**Proposition 1.5** *(Sufficient conditions for the stability of* Model II *under heterogeneous RLS learning for the diagonal environment case under any (possibly different) degrees of inertia of agents, $\delta > 0$). In the structurally heterogeneous economy (1.6), (1.2), and (1.22), in which all roots of $\bar{b}$ defined in (1.9) lie inside the unit circle, heterogeneous RLS learning (2.8), (1.20), and (1.7) converges (almost surely) to an MSV REE solution for any (possibly different) degrees of inertia of agents, $\delta > 0$, if matrices $D_y\Omega_b$* and *$D_1\Omega_{\rho_l}$ are stable for any $\delta > 0$, where $D_1$ and $\Omega_{\rho_l}$ are defined in (1.23) and (1.25), respectively.*

**Proof.** See Appendix A.7.7. $\square$

### 1.3.5 Adaptive learning in models with $t$-dating of expectations (Models III and IV)

A similar approach to writing down the corresponding SRAs and the associated ODEs can be applied to models with $t$-dating of expectations (that is, **Models III and IV**)[13] under heterogeneous mixed RLS/SG learning.

After denoting $z_t' = (1, w_t')$ for Model III ($z_t' = (1, y_{t-1}', w_t')$ for Model IV) and $\Phi_{h,t}' = (a_{h,t}, b_{h,t})$ (if a lag is included, then $\Phi_{h,t}' = (a_{h,t}, b_{h,t}, c_{h,t})$ ), $\Phi_t' = (\Phi_{1,t}', ..., \Phi_{S,t}')$, the formal presentation of the learning algorithms in this model can be written as follows.

[13]Similar technical constructions can be found in Evans and Honkapohja [24, ch. 10.5] but for a structurally homogeneous economy under homogeneous learning.

RLS: for $h = \overline{1, S_0}$

$$\Phi_{h,t+1} = \Phi_{h,t} + \alpha_{h,t+1} R_{h,t+1}^{-1} z_t \left(y_t - \Phi_{h,t}' z_t\right)' \tag{1.26a}$$

$$R_{h,t+1} = R_{h,t} + \alpha_{h,t+1} \left(z_t z_t' - R_{h,t}\right) \tag{1.26b}$$

SG: for $h = \overline{S_0 + 1, S}$

$$\Phi_{h,t+1} = \Phi_{h,t} + \alpha_{h,t+1} z_t \left(y_t - \Phi_{h,t}' z_t\right)'. \tag{1.27}$$

Agents use $\Phi_{h,t}' = (a_{h,t}, b_{h,t})$ (or $\Phi_{h,t}' = (a_{h,t}, b_{h,t}, c_{h,t})$ when a lag is included) and $z_t$ to make their forecasts $\hat{E}_t^h y_{t+\tau}$. The actual law of motion will be

$$y_t = T(\Phi_t)' z_t + \zeta \varepsilon_t,$$

where $T(\Phi_{t-1})'$ is defined in (1.12) for Model III and in (1.15) for Model IV above.

To convert the system into the **standard form of an SRA**, again, as for the first group of models, a change is made in the timing of the system for $R_{h,t}$. I set $S_{h,t-1} = R_{h,t}$. Thus, the beliefs updating algorithm of Models III and IV will have the following SRA representation

RLS: for $h = \overline{1, S_0}$

$$\Phi_{h,t+1} = \Phi_{h,t} + \alpha_{t+1} \frac{\alpha_{h,t+1}}{\alpha_{t+1}} \left(S_{h,t}^{-1} z_t z_t' \left[T(\Phi_t) - \Phi_{h,t}\right] + S_{h,t}^{-1} z_t \left(\zeta \varepsilon_t\right)'\right)$$

$$S_{h,t+1} = S_{h,t} + \alpha_{t+1} \left(\delta_h\right) \left(z_{t+1} z_{t+1}' - S_{h,t}\right) + \alpha_{t+1}^2 \left(\frac{\alpha_{t+2} - \alpha_{t+1}}{\alpha_{t+1}^2}\right) \left(\delta_h\right) \left(z_{t+1} z_{t+1}' - S_{h,t}\right)$$

SG: for $h = \overline{S_0 + 1, S}$

$$\Phi_{h,t+1} = \Phi_{h,t} + \alpha_{t+1} \frac{\alpha_{h,t+1}}{\alpha_{t+1}} z_t z_t' \left[T(\Phi_t) - \Phi_{h,t}\right] + \alpha_{t+1} \frac{\alpha_{h,t+1}}{\alpha_{t+1}} z_t \left(\zeta \varepsilon_t\right)'.$$

For the case without lags (**Model III**), **the law of motion for the state** can be written as

$$X_t = A X_{t-1} + B W_t,$$

where

$$X_t' = \left(1, w_t', w_{t-1}', \varepsilon_{t-1}'\right),$$

$$W_t' = (1, \varepsilon_{t-1}', \nu_t'),$$

and where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & F & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & I & 0 \end{pmatrix}.$$

For the case with a lag and one forward-looking term in expectations, the $(1, y'_{t-1}, w'_t)$ information set (**Model IV**), **the law of motion for the state** can be written as

$$X_t = A(\theta_{t-1})X_{t-1} + B(\theta_{t-1})W_t,$$

where

$$\theta'_t = \left(\theta'_{1,t}, ..., \theta'_{S,t}\right),$$

where

$$\begin{aligned} \theta'_{h,t} &= \left(vec\left(\Phi'_{h,t}\right)', vec(S_{h,t})'\right), h = \overline{1, S_0}, \\ \theta'_{h,t} &= vec\left(\Phi'_{h,t}\right), h = \overline{S_0 + 1, S}, \end{aligned}$$

$$X'_t = \left(1, y'_{t-1}, w'_t, y'_{t-2}, w'_{t-1}, \varepsilon'_{t-1}\right),$$

$$W'_t = (1, \varepsilon'_{t-1}, \nu'_t),$$

and where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ T'_a(\theta_{t-1}) & T'_b(\theta_{t-1}) & T'_c(\theta_{t-1}) & 0 & 0 & 0 \\ 0 & 0 & F & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \end{pmatrix}.$$

**The associated ODEs** for these SRAs (for proof see Appendix A.7.2) again look like (1.21).

### 1.3.6 General criteria for stability under heterogeneous learning for Models III and IV

Again, the next step is to take derivatives of the $T$-maps: (1.12) for Model III and (1.15) for Model IV, and to compose the Jacobians for the right-hand side of the associated ODEs (1.21).

For the SRA of Model III, the system of the associated ODEs (linear by the setup), after dropping the inessential constant terms, is written by components as

$$\begin{pmatrix} \dot{a}_1 \\ \vdots \\ \dot{a}_S \end{pmatrix} = D_1\Omega \begin{pmatrix} a_1 \\ \vdots \\ a_S \end{pmatrix}, \text{ where } D_1 = \begin{pmatrix} \delta_1 I_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_n \end{pmatrix},$$

$$\Omega=\begin{pmatrix} A_1^1...+A_\tau^1-I_n & \cdots & A_1^S...+A_\tau^S \\ \vdots & \ddots & \vdots \\ A_1^1...+A_\tau^1 & \cdots & A_1^S...+A_\tau^S-I_n \end{pmatrix}$$

$$\begin{pmatrix} vec\dot{b}_1 \\ \vdots \\ vec\dot{b}_S \end{pmatrix}=D_w\Omega_F\begin{pmatrix} vecb_1 \\ \vdots \\ vecb_S \end{pmatrix}, \text{ where } D_w=\begin{pmatrix} D_{w1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{wS} \end{pmatrix},$$

$$\begin{array}{l} D_{wh}=\delta_h I_{nk}, h=\overline{1,S_0} \\ D_{wh}=\delta_h\left(M_w\otimes I_n\right), h=\overline{S_0+1,S} \end{array},$$

$$\Omega_F=\begin{pmatrix} F^\tau\otimes A_\tau^1+...+F'\otimes A_1^1-I_{nk} & \cdots & F^\tau\otimes A_\tau^S+...+F'\otimes A_1^S \\ \vdots & \ddots & \vdots \\ F^\tau\otimes A_\tau^1+...+F'\otimes A_1^1 & \cdots & F^\tau\otimes A_\tau^S+...+F'\otimes A_1^S-I_{nk} \end{pmatrix}.$$

For the SRA of Model IV, the system of associated ODEs linearized around the MSV REE given by (1.16), after dropping the inessential constant terms, is written by components as

$$\begin{pmatrix} \dot{a}_1 \\ vec\dot{b}_1 \\ vec\dot{c}_1 \\ \vdots \\ \dot{a}_S \\ vec\dot{b}_S \\ vec\dot{c}_S \end{pmatrix}=D_{1yw}\Omega_{1bF}\begin{pmatrix} a_1 \\ vecb_1 \\ vecc_1 \\ \vdots \\ a_S \\ vecb_S \\ vecc_S \end{pmatrix}, \text{ where } D_{1yw}=\begin{pmatrix} D_{1yw1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{1ywS} \end{pmatrix},$$

$$D_{1ywh}=\delta_h I_{n+n^2+nk}, h=\overline{1,S_0}$$

$$D_{1ywh}=\delta_h\left(M_{1yw}\otimes I_n\right), h=\overline{S_0+1,S}$$

$$\Omega_{1bF}=\begin{bmatrix} R^1-I_{n+n^2+nk} & R^1 & \cdots & R^1 \\ R^2 & R^2-I_{n+n^2+nk} & \cdots & R^2 \\ \vdots & \vdots & \ddots & \vdots \\ R^S & R^S & \cdots & R^S-I_{n+n^2+nk} \end{bmatrix}$$

$$R^h=\begin{bmatrix} A_1^h+A_1^h\bar{b} & \bar{a}'\otimes A_1^h & 0 \\ 0 & \bar{b}'\otimes A_1^h+I_n\otimes\left(A_1^h\bar{b}\right) & 0 \\ \vdots & \bar{c}'\otimes A_1^h & F'\otimes A_1^h+I_k\otimes\left(A_1^h\bar{b}\right) \end{bmatrix}.$$

The steps below are similar to those for Models I and II. Using the above, the general criteria for stability under mixed RLS/SG learning (à la Honkapohja and Mitra [36, Prop. 5]) for Models III and IV may be written as follows.

**Criterion 1.6** *In economy (1.10) and (1.2),* Model III*, mixed RLS/SG learning converges globally (almost surely) to the minimal state variable (MSV) solution if and only if the corresponding matrices $D_1\Omega$ and $D_w\Omega_F$ have eigenvalues with negative real parts.*

**Proof.** See Appendix A.7.8. □

**Criterion 1.7** *In economy (1.13) and (1.2),* Model IV*, in which all roots of $\bar{b}$ defined in (1.16) lie inside the unit circle, mixed RLS/SG learning converges (almost surely) to the minimal state variable (MSV) solution if and only if the corresponding matrix $D_{1yw}\Omega_{1bF}$ has eigenvalues with negative real parts.*

**Proof.** See Appendix A.7.9. □

Again, as in the case of the other model with a lag (Model II), it is easy to obtain more handy sufficient conditions for stability of the MSV REE of Model IV under heterogeneous RLS learning, which allows for a further elaboration of sufficient conditions.

**Corollary 1.8** *(Sufficient conditions for stability of the MSV REE of Model IV under heterogeneous RLS learning) In economy (1.13) and (1.2),* Model IV*, in which all roots of $\bar{b}$ defined in (1.16) lie inside the unit circle, heterogeneous RLS learning converges (almost surely) to the minimal state variable (MSV) solution if and only if the corresponding matrices $D_1\Omega$, $D_y\Omega_b$, and $D_w\Omega_F$ (below) have eigenvalues with negative real parts; thus, MSV REE is a locally stable point of the following system:*

$$\begin{pmatrix} \dot{a}_1 \\ \vdots \\ \dot{a}_S \end{pmatrix} = D_1\Omega \begin{pmatrix} a_1 \\ \vdots \\ a_S \end{pmatrix}, \text{ where } D_1 = \begin{pmatrix} \delta_1 I_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_n \end{pmatrix},$$

$$\Omega = \begin{pmatrix} A_1^1 + A_1^1\bar{b} - I_n & \cdots & A_1^S + A_1^S\bar{b} \\ \vdots & \ddots & \vdots \\ A_1^1 + A_1^1\bar{b} & \cdots & A_1^S + A_1^S\bar{b} - I_n \end{pmatrix}$$

$$\begin{pmatrix} vec\dot{b}_1 \\ \vdots \\ vec\dot{b}_S \end{pmatrix} = D_y\Omega_b \begin{pmatrix} vecb_1 \\ \vdots \\ vecb_S \end{pmatrix}, \text{ where } D_y = \begin{pmatrix} \delta_1 I_{n^2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_{n^2} \end{pmatrix},$$

$$\Omega_b = \begin{pmatrix} \bar{b}' \otimes A_1^1 + I_n \otimes \left(A_1^1\bar{b}\right) - I_{n^2} & \cdots & \bar{b}' \otimes A_1^S + I_n \otimes \left(A_1^S\bar{b}\right) \\ \vdots & \ddots & \vdots \\ \bar{b}' \otimes A_1^1 + I_n \otimes \left(A_1^1\bar{b}\right) & \cdots & \bar{b}' \otimes A_1^S + I_n \otimes \left(A_1^S\bar{b}\right) - I_{n^2} \end{pmatrix},$$

$$\begin{pmatrix} vec\dot{c}_1 \\ \vdots \\ vec\dot{c}_S \end{pmatrix} = D_w \Omega_F \begin{pmatrix} vecc_1 \\ \vdots \\ vecc_S \end{pmatrix} \text{,where } D_w = \begin{pmatrix} \delta_1 I_{nk} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_{nk} \end{pmatrix},$$

$$\Omega_F = \begin{pmatrix} F' \otimes A_1^1 + I_k \otimes \left(A_1^1 \bar{b}\right) - I_{nk} & \cdots & F' \otimes A_1^S + I_k \otimes \left(A_1^S \bar{b}\right) \\ \vdots & \ddots & \vdots \\ F' \otimes A_1^1 + I_k \otimes \left(A_1^1 \bar{b}\right) & \cdots & F' \otimes A_1^S + I_k \otimes \left(A_1^S \bar{b}\right) - I_{nk} \end{pmatrix}.$$

**Proof.** See Appendix A.7.10. □

### 1.3.7 Conditions for stability in the diagonal environment case for Models III and IV

The diagonal shock process simplification for Models III and IV yields results similar to the ones for Models I and II.

For **Model III** in the "diagonal" environment, the problem of finding conditions for the stability of both $D_1\Omega$ and $D_w\Omega_F$ under any (possibly different) degrees of inertia of agents, $\delta > 0$, is simplified to finding stability conditions for $D_1\Omega$ and $D_1\Omega_{\rho_l}$, where $\Omega_{\rho_l}$ is obtained from $\Omega$ by substituting all of $A_1^h .. + A_\tau^h$ with $\rho_l A_1^h .. + \rho_l^\tau A_\tau^h$, where $|\rho_l| < 1$ as $w_t$ follows a stationary VAR(1) process, by the setup of the model.

$$\Omega_{\rho_l} = \begin{pmatrix} \rho_l A_1^1 + ... + \rho_l^\tau A_\tau^1 - I_n & \cdots & \rho_l A_1^S + ... + \rho_l^\tau A_\tau^S \\ \vdots & \ddots & \vdots \\ \rho_l A_1^1 + ... + \rho_l^\tau A_\tau^1 & \cdots & \rho_l A_1^S + ... + \rho_l^\tau A_\tau^S - I_n \end{pmatrix}, \forall l = 0, ..., k, (\rho_0 = 1). \tag{1.28}$$

**Proposition 1.9** *(A criterion for the stability of* Model III *under mixed RLS/SG learning for the diagonal environment case under any (possibly different) degrees of inertia of agents, $\delta > 0$). In the structurally heterogeneous economy (1.10), (1.2), and (1.22), mixed RLS/SG learning (2.10), (1.27), and (1.11) converges globally (almost surely) to an MSV REE solution for any (possibly different) degrees of inertia of agents, $\delta > 0$, if and only if matrices $D_1\Omega_{\rho l}$ are stable for any $\delta > 0$, where $D_1$ and $\Omega_{\rho_l}$ are defined in (1.23) and (1.28), respectively.*

**Proof.** See Appendix A.7.11. □

For **Model IV with heterogeneous RLS learning** in the "diagonal" environment, the problem of finding conditions for the stability of $D_1\Omega$, $D_y\Omega_b$, *and* $D_w\Omega_F$ under any (possibly different) degrees of inertia of agents, $\delta > 0$, is simplified to finding stability conditions for $D_y\Omega_b$ and $D_1\Omega_{\rho_l}$, where $\Omega_{\rho_l}$ is given by

$$\Omega_{\rho_l} = \begin{pmatrix} \rho_l A_1^1 + A_1^1\bar{b} - I_n & \cdots & \rho_l A_1^S + A_1^S\bar{b} \\ \vdots & \ddots & \vdots \\ \rho_l A_1^1 + A_1^1\bar{b} & \cdots & \rho_l A_1^S + A_1^S\bar{b} - I_n \end{pmatrix}, \forall l = 0, ..., k, (\rho_0 = 1), \qquad (1.29)$$

where $|\rho_l| < 1$ as $w_t$ follows a stationary VAR(1) process, by the setup of the model.

**Proposition 1.10** *(Sufficient conditions for the stability of* Model IV *under heterogeneous RLS learning for the diagonal environment case under any (possibly different) degrees of inertia of agents, $\delta > 0$). In the structurally heterogeneous economy (1.13), (1.2), and (1.22), heterogeneous RLS learning (2.10), (1.27), and (1.14) converges (almost surely) to an MSV REE solution for any (possibly different) degrees of inertia of agents, $\delta > 0$, if matrices* $D_y\Omega_b$ and $D_1\Omega_{\rho_l}$ *are stable for any $\delta > 0$, where $D_1$ and $\Omega_{\rho_l}$ are defined in (1.23) and (1.29), respectively.*

**Proof.** See Appendix A.7.12. □

### 1.3.8 The concepts of $\delta$-stability and heterogeneous expectational stability

I will refer to the stability of a REE under ODE (1.21) as **heterogeneous expectational ($HE-$) stability** (or stability in heterogeneous expectations). I also *redefine* the concept of **$\delta$-stability** in the following way.

**Definition 1.1** ***$\delta$-stability*** *is the stability of a REE under heterogeneous (either RLS, SG, or mixed RLS/SG) learning for any positive values of degrees of inertia and for any starting values; that is, it is the stability of the system under heterogeneous learning that is provided by structural heterogeneity only and is independent of the heterogeneity in learning mentioned above.*

In this sense, the general stability criteria and the conditions mentioned above refer to $HE$-stability as they depend on $\delta$s. It is also clear that all necessary and sufficient conditions for $HE$-stability of all types of the models considered have the same algebraic representation.

We have to have stability of a matrix

$$\Omega_{KR} = \begin{bmatrix} \delta_1\left(R^1 - I\right) & \cdots & \delta_1 R^{S_0} & \delta_1 R^{S_0+1} & \cdots & \delta_1 R^S \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \delta_{S_0} R^1 & \cdots & \delta_{S_0}\left(R^{S_0} - I\right) & \delta_{S_0} R^{S_0+1} & \cdots & \delta_{S_0} R^S \\ \delta_{S_0+1} K R^1 & \cdots & \delta_{S_0+1} K R^{S_0} & \delta_{S_0+1}\left(K R^{S_0+1} - K\right) & \cdots & \delta_{S_0+1} K R^S \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \delta_S K R^1 & \cdots & \delta_S K R^{S_0} & \delta_S K R^{S_0+1} & \cdots & \delta_S\left(K R^S - K\right) \end{bmatrix},$$

where matrices $K$ and $R^h$ for each model are defined as follows.

**Model I**

$$K = \begin{bmatrix} 1 & 0 \\ 0 & M_w \end{bmatrix}, R^h = \begin{bmatrix} 1 & 0 \\ 0 & F'^{\tau} \end{bmatrix} \otimes A_\tau^h + ... + \begin{bmatrix} 1 & 0 \\ 0 & F' \end{bmatrix} \otimes A_1^h + I_{k+1} \otimes A_0^h.$$

**Model II**

$$K = \left(M_{1yw} \otimes I_n\right), M_{1yw} = \begin{bmatrix} M_{1y} & 0 \\ 0 & M_w \end{bmatrix},$$

$$R^h = \begin{bmatrix} A_1^h + \left(A_0^h + A_1^h \bar{b}\right) & \bar{a}' \otimes A_1^h & 0 \\ 0 & \bar{b}' \otimes A_1^h + I_n \otimes \left(A_0^h + A_1^h \bar{b}\right) & 0 \\ \vdots & \bar{c}' \otimes A_1^h & F' \otimes A_1^h + I_k \otimes \left(A_0^h + A_1^h \bar{b}\right) \end{bmatrix}.$$

**Model III**

$$K = \begin{bmatrix} 1 & 0 \\ 0 & M_w \end{bmatrix}, R^h = \begin{bmatrix} 1 & 0 \\ 0 & F'^{\tau} \end{bmatrix} \otimes A_\tau^h + ... + \begin{bmatrix} 1 & 0 \\ 0 & F' \end{bmatrix} \otimes A_1^h.$$

**Model IV**

$$K = \left(M_{1yw} \otimes I_n\right), M_{1yw} = \begin{bmatrix} M_{1y} & 0 \\ 0 & M_w \end{bmatrix},$$

$$R^h = \begin{bmatrix} A_1^h + A_1^h \bar{b} & \bar{a}' \otimes A_1^h & 0 \\ 0 & \bar{b}' \otimes A_1^h + I_n \otimes \left(A_1^h \bar{b}\right) & 0 \\ \vdots & \bar{c}' \otimes A_1^h & F' \otimes A_1^h + I_k \otimes \left(A_1^h \bar{b}\right) \end{bmatrix}.$$

## 1.4 Stability under mixed RLS/SG learning with equal degrees of inertia for each type of learning algorithm

It is possible to obtain a criterion for $HE$-stability for all types of models in terms of an average economy in case $\delta_i = \delta$, $\forall i$. It looks like a very strong result meaning that stability issues for a vast class of structurally heterogeneous models can be substituted with just one type of a structurally heterogeneous economy with heterogeneous learning of two agents. This criterion serves as an initial check for the possibility of $\delta$-stability in an economy since it provides a very general necessary condition for it. If the economy is not stable for equal $\delta$s, then it is not $\delta$-stable. To write down this criterion, let us first make the following natural definition.

**Definition 1.2** *For general model setup (1.1) and (1.2) with mixed RLS/SG learning with equal degrees of inertia for each type of learning algorithm,* $\delta_i = \delta^{RLS}$, $\forall i = \overline{1, S_0}$, $\delta_i = \delta^{SG}$, $\forall i = \overline{S_0+1, S}$, ***the average economy*** *with mixed RLS/SG learning with equal degrees of inertia for each type of learning algorithm,* $\delta^{RLS}$ *for RLS and* $\delta^{SG}$ *for SG, is defined as*

$$y_t = \alpha + \sum_{i=1}^{d} L_i y_{t-i} + \sum_{b=0}^{m} \sum_{f=b}^{n} \left( \sum_{h=1}^{S_0} A_{bf}^h \right) \hat{E}_{t-b}^{RLS} y_{t-b+f} + \tag{1.30}$$

$$+ \sum_{b=0}^{m} \sum_{f=b}^{n} \left( \sum_{h=S_0+1}^{S} A_{bf}^h \right) \hat{E}_{t-b}^{SG} y_{t-b+f} + Bw_t + \zeta \varepsilon_t,$$

*and (1.2),* $A_{00}^h \equiv 0$,

*where the agent with coefficients* $\left( \sum_{h=1}^{S_0} A_{bf}^h \right)$ *learns by RLS, while the agent with coefficients* $\left( \sum_{h=S_0+1}^{S} A_{bf}^h \right)$ *learns by SG.*

Now, I can formulate the criterion.

**Proposition 1.11** *(The criterion for stability under mixed RLS/SG learning with equal degrees of inertia of agents for each type of learning algorithm,* $\delta > 0$*). In the structurally heterogeneous economy (1.3) and (1.2), Model I ((1.10) and (1.2), Model III) or (1.6) and (1.2), Model II ((1.13) and (1.2), Model IV), in which all roots of* $\bar{b}$ *defined in (1.9) for Model II and in (1.16) for Model IV lie inside the unit circle, mixed RLS/SG learning with equal degrees of inertia of agents for each type of learning algorithm,* $\delta_i = \delta^{RLS}$, $\forall i = 1, ..., S_0$, $\delta_i = \delta^{SG}$, $\forall i = S_0 + 1, ..., S$, *converges globally, for Models I and III, or locally, for Models II and IV, (almost surely) to an MSV REE, if and only if the REE is a locally asymptotically stable fixed point of the corresponding average economy (1.30) under mixed RLS/SG learning of two agents with equal degrees of inertia for each type of learning algorithm,* $\delta^{RLS}$ *for RLS and* $\delta^{SG}$ *for SG.*

**Proof.** See Appendix A.7.13. □

## 1.5 Aggregate Economy Sufficient Conditions for $\delta$–stability

It turns out that for models without lags of the endogenous variable (Model I and III), it is possible to derive economically tractable sufficient conditions for $\delta$-stability in terms of an aggregate economy constructed for the original one. It is possible to derive these conditions for the general mixed RLS/SG learning but with the diagonal structure of the shock process (1.22), diagonal $F$ and consequently, $M$.

For the general non-diagonal case ($F$-any) it is possible to obtain economically tractable sufficient conditions for $\delta$-stability for all types of models (Models I, II, III, and IV) but in terms of aggregate economies constructed for the associated set of models that have equivalent stability properties of the MSV REE of the original model. It is possible to derive these conditions for the heterogeneous RLS learning case.

### 1.5.1 Aggregation for models without lags of endogenous variables under general heterogeneous mixed RLS/SG learning in the diagonal environment case

First, note that the stability properties of the MSV REE under mixed RLS/SG learning of **Model I** in the diagonal environment case are equivalent to the ones of the MSV REE of ***the set of the associated current value expectations models***

$$\begin{aligned} y_t &= \alpha + \sum_{h=1}^{S} (A_0^h + \rho_l A_1^h + ... + \rho_l^\tau A_\tau^h) \hat{E}_{t-1}^h y_t + B w_t + \zeta \varepsilon_t, \\ & \textit{and } (1.2). \end{aligned} \tag{1.31}$$

Similarly, stability properties of the MSV REE under mixed RLS/SG learning of **Model III** in the diagonal environment case is equivalent to the ones of the MSV REE of ***the set of the associated current value expectations models***

$$\begin{aligned} y_t &= \alpha + \sum_{h=1}^{S} (\rho_l A_1^h + ... + \rho_l^\tau A_\tau^h) \hat{E}_{t-1}^h y_t + B w_t + \zeta \varepsilon_t, \\ & \textit{and } (1.2). \end{aligned} \tag{1.32}$$

To proceed with aggregation, I start with the original Model I and employ the same approach to aggregation used by Bogomolova and Kolyuzhnov [5] and Kolyuzhnov

[40][14].

Given the weights of aggregation across endogenous variables $\psi_i > 0$, $\sum\limits_{i=1}^{n} \psi_i = 1$, and across agent types $\phi_h > 0$, $\sum\limits_{h=1}^{S} \phi_h = 1$ (and denoting $a_{ij}^h$ the element in the $i^{\text{th}}$ row and $j^{\text{th}}$ column of matrix $A_h$), I aggregate the economy in the following way.

$$y_t^{AG} = \sum_i \psi_i y_{it} = \sum_i \psi_i \alpha_i + \sum_h S\phi_h \sum_i \psi_i \sum_j a_{0ij}^h \hat{E}_{t-1}^h y_{jt} + ...+$$
$$+ \sum_h S\phi_h \sum_i \psi_i \sum_j a_{\tau ij}^h \hat{E}_{t-1}^h y_{jt+\tau} + \left(\sum_i \psi_i B^i\right) w_t + \left(\sum_i \psi_i \varsigma^i\right) \varepsilon_t =$$
$$= \sum_i \psi_i \alpha_i + \beta_0^{AG}(\psi,\phi)\, \hat{E}_{t-1}^{AG}\left(y_t^{AG}\right) + ... + \beta_\tau^{AG}(\psi,\phi)\, \hat{E}_{t-1}^{AG}\left(y_{t+\tau}^{AG}\right) + \left(\sum_i \psi_i B^i\right) w_t +$$
$$+ \left(\sum_i \psi_i \varsigma^i\right) \varepsilon_t, \text{ where}$$

$$\beta_r^{AG}(\psi,\phi) = S \sum_h \phi_h \sum_i \psi_i \sum_j a_{rij}^h, r = 0, 1, ..., \tau \tag{1.33}$$

$$\hat{E}_{t-1}^{AG}\left(y_{t+r}^{AG}\right) = \left(\beta_r^{AG}(\psi,\phi)\right)^{-1} \sum_{h=1}^{S} S\phi_h \sum_i \psi_i \sum_j a_{rij}^h \hat{E}_{t-1}^h y_{jt+r}, \tag{1.34}$$

and $B^i$ and $\varsigma^i$denote the $i^{th}$ row of $B$ and $\varsigma$, respectively. So, using the derivations above, I formulate the following definition.

**Definition 1.3** *Given the weights of aggregation across endogenous variables $\psi_i > 0$, $\sum\limits_{i=1}^{n} \psi_i = 1$, and across agent types $\phi_h > 0$, $\sum\limits_{h=1}^{S} \phi_h = 1$,* ***the aggregate economy*** *for the economy described by (1.3) and (1.2),* Model I, *is defined as*

$$y_t^{AG} = \sum_i \psi_i \alpha_i + \beta_0^{AG}(\psi,\phi)\, \hat{E}_{t-1}^{AG}\left(y_t^{AG}\right) + ... + \beta_\tau^{AG}(\psi,\phi)\, \hat{E}_{t-1}^{AG}\left(y_{t+\tau}^{AG}\right) +$$
$$+ \left(\sum_i \psi_i B^i\right) w_t + \left(\sum_i \psi_i \varsigma^i\right) \varepsilon_t,$$

*(1.2),*

*where $\beta_r^{AG}(\psi,\phi)$ and $\hat{E}_t^{AG}\left(y_{t+r}^{AG}\right)$ , $r = 0, 1, ..., \tau$ are defined in (1.33) and (1.34), respectively.*

It turns out that it is also useful (the reason for it will become clear later) to consider an economy that bounds above all possible economies with all possible combinations of signs of $a_{ij}^h$ aggregated using weights $\psi$ and $\phi$. This is the original aggregate

[14]For the idea and discussion of such aggregation, please see Bogomolova and Kolyuzhnov [5] and Kolyuzhnov [40].

model written in absolute values. When all elements of the model, $a^h_{ij}$, endogenous variables, and their expectations are positive, this limiting model exactly coincides with the model considered. So, this is an attainable supremum. Thus, I have the following limiting aggregate model:

$$y_t^{AG} = \sum_i \psi_i y_{it} \leq y_t^{AG\,\mathrm{mod}} = \sum_i \psi_i \left| y_{it} \right| \leq$$
$$\leq \sum_i \psi_i \left| \alpha_i \right| + \beta_0^{AG\,\mathrm{mod}} \left( \psi, \phi \right) \hat{E}_{t-1}^{AG\,\mathrm{mod}} \left( y_t^{AG\,\mathrm{mod}} \right) + ... +$$
$$+ \beta_\tau^{AG\,\mathrm{mod}} \left( \psi, \phi \right) \hat{E}_{t-1}^{AG\,\mathrm{mod}} \left( y_{t+\tau}^{AG\,\mathrm{mod}} \right) + \left| \left( \sum_i \psi_i B^i \right) w_t \right| + \left| \left( \sum_i \psi_i \varsigma^i \right) \varepsilon_t \right|, \text{ where}$$

$$\beta_r^{AG\,\mathrm{mod}} \left( \psi, \phi \right) = S \sum_h \phi_h \sum_i \psi_i \sum_j \left| a^h_{rij} \right|, r = 0, 1, ..., \tau \tag{1.35}$$

$$\hat{E}_t^{AG\,\mathrm{mod}} \left( y_{t+r}^{AG\,\mathrm{mod}} \right) = \left( \beta_r^{AG\,\mathrm{mod}} \left( \psi, \phi \right) \right)^{-1} \sum_{h=1}^S S \phi_h \sum_i \psi_i \sum_j \left| a^h_{rij} \right| \hat{E}_t^h \left| y_{jt+r} \right|. \tag{1.36}$$

**Definition 1.4** *Given the weights of aggregation across endogenous variables* $\psi_i > 0$, $\sum_{i=1}^n \psi_i = 1$, *and across agent types* $\phi_h > 0$, $\sum_{h=1}^S \phi_h = 1$, ***the limiting aggregate economy*** *for an economy described by (1.3) and (1.2),* Model I, *is defined as*

$$y_t^{AG\,\mathrm{mod}} = \sum_i \psi_i \left| \alpha_i \right| + \beta_0^{AG\,\mathrm{mod}} \left( \psi, \phi \right) \hat{E}_{t-1}^{AG\,\mathrm{mod}} \left( y_t^{AG\,\mathrm{mod}} \right) + ... +$$
$$+ \beta_\tau^{AG\,\mathrm{mod}} \left( \psi, \phi \right) \hat{E}_{t-1}^{AG\,\mathrm{mod}} \left( y_{t+\tau}^{AG\,\mathrm{mod}} \right) + \left| \left( \sum_i \psi_i B^i \right) w_t \right| + \left| \left( \sum_i \psi_i \varsigma^i \right) \varepsilon_t \right|,$$

*(1.2),*

*where* $\beta_r^{AG\,\mathrm{mod}} \left( \psi, \phi \right)$ *and* $\hat{E}_t^{AG\,\mathrm{mod}} \left( y_{t+r}^{AG\,\mathrm{mod}} \right)$, $r = 0, 1, ..., \tau$, *are defined in (1.35) and (1.36), respectively.*

**Remark 1.1** *If this limiting aggregate economy is E–stable, then all corresponding aggregate economies with various combinations of signs of* $a^h_{ij}$ *are E–stable.*

The same aggregation techniques may be applied to Model III and to the set of associated current value expectations models corresponding to Models I and III. I present the final results in the following definitions.

**Definition 1.5** *Given the weights of aggregation across endogenous variables* $\psi_i > 0$, $\sum_{i=1}^n \psi_i = 1$, *and across agent types* $\phi_h > 0$, $\sum_{h=1}^S \phi_h = 1$, ***the limiting aggregate***

***economy*** *for an economy described by (1.10) and (1.2),* Model III*, is defined as*

$$\begin{aligned} y_t^{AG\bmod} &= \sum_i \psi_i |\alpha_i| + \beta_1^{AG\bmod}(\psi,\phi)\,\hat{E}_t^{AG\bmod}\left(y_{t+1}^{AG\bmod}\right) + ... + \\ &+\beta_\tau^{AG\bmod}(\psi,\phi)\,\hat{E}_t^{AG\bmod}\left(y_{t+\tau}^{AG\bmod}\right) + \left|\left(\sum_i \psi_i B^i\right) w_t\right| + \left|\left(\sum_i \psi_i \varsigma^i\right)\varepsilon_t\right|, \\ &\text{(1.2)}, \end{aligned}$$

*where* $\beta_r^{AG\bmod}(\psi,\phi)$ *and* $\hat{E}_t^{AG\bmod}\left(y_{t+r}^{AG\bmod}\right), r = 0,1,...,\tau$ *are defined as*

$$\beta_r^{AG\bmod}(\psi,\phi) = S\sum_h \phi_h \sum_i \psi_i \sum_j \left|a_{rij}^h\right|, r = 1,...,\tau \tag{1.37}$$

$$\hat{E}_t^{AG\bmod}\left(y_{t+r}^{AG\bmod}\right) = \left(\beta_r^{AG\bmod}(\psi,\phi)\right)^{-1}\sum_{h=1}^{S} S\phi_h \sum_i \psi_i \sum_j \left|a_{rij}^h\right| \hat{E}_t^h |y_{jt+r}|. \tag{1.38}$$

**Definition 1.6** *Given the weights of aggregation across endogenous variables* $\psi_i > 0$, $\sum_{i=1}^{n}\psi_i = 1$*, and across agent types* $\phi_h > 0$, $\sum_{h=1}^{S}\phi_h = 1$*,* ***the limiting aggregate economies*** *for the set of associated current value expectations models corresponding to* Model I *(*Model III*) described by (1.3) and (1.2) ((1.10) and (1.2)) is defined as*

$$\begin{aligned} y_t^{AG\bmod} &= \sum_i \psi_i |\alpha_i| + \beta_l^{AG\bmod}(\psi,\phi)\,\hat{E}_{l\ t-1}^{AG\bmod}\left(y_t^{AG\bmod}\right) + \\ &+\left|\left(\sum_i \psi_i B^i\right) w_t\right| + \left|\left(\sum_i \psi_i \varsigma^i\right)\varepsilon_t\right|, \\ &\text{(1.2)}, \end{aligned} \tag{1.39}$$

*where* $\beta_l^{AG\bmod}(\psi,\phi)$ *and* $\hat{E}_{l\ t-1}^{AG\bmod}\left(y_t^{AG\bmod}\right)$*, are defined as*

*for Model I:*

$$\beta_l^{AG\bmod}(\psi,\phi) = S\sum_h \phi_h \sum_i \psi_i \sum_j \left|a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h\right| \tag{1.40}$$

$$\begin{aligned} \hat{E}_{l\ t-1}^{AG\bmod}\left(y_t^{AG\bmod}\right) &= \left(\beta_l^{AG\bmod}(\psi,\phi)\right)^{-1} \times \\ &\times \sum_{h=1}^{S} S\phi_h \sum_i \psi_i \sum_j \left|a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h\right| \hat{E}_{t-1}^h |y_{jt}|. \end{aligned} \tag{1.41}$$

*for Model III:* $a_{0ij}^h = 0$ *above.*

The structure of this limiting aggregate coefficient $\beta_l^{AG\bmod}(\psi,\phi)$ is as follows. $\sum_i \psi_i \left|a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h\right|$ (in the case of Model III, $a_{0ij}^h = 0$) is the coefficient before the expectation of endogenous variable $j$ in the aggregate economy composed of one single agent type $h$. Notice that this coefficient is calculated for the expectation of endogenous variable $j$, that enters the aggregate product with coefficient $\psi_j$. So for each $l = 0, 1, ..., k$, following Kolyuzhnov [40], I may name the ratio $\left(\sum_i \psi_i \left|a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho_l^\tau a^h_{\tau ij}\right|\right)/\psi_j$ **the endogenous variable** $j$ **"own" expectations relative coefficient.** By looking at the values of these coefficients I will be able to judge the weight a particular agent type has in the economy in terms of the aggregate $\beta$-coefficient. The next proposition that follows from the criteria for the $\delta$-stability of models without lags under mixed RLS/SG learning in the diagonal environment (1.22) (Propositions 1.4 and 1.9) and the McKenzie Theorem (1960) (see the Appendix) is formulated in terms of these relative coefficients and stresses the fact that weights of agents in calculating aggregate expectations have to be put into accordance with this economic intuition in order to have stability under heterogeneous learning.

**Proposition 1.12** *In the case of the diagonal environment (1.22), if for each economy from the set of associated current value expectations models corresponding to Model I (III) there exists at least one pair of vectors of weights for the aggregation of endogenous variables $\psi$ and weights $\phi$ for the aggregation of agents such that for each agent, every weighted endogenous variable's "own" expectations relative coefficient corresponding to the limiting aggregate economy (1.39) and (1.2) is less than the weight of the agent used in calculating aggregate expectations, i.e.*

$$\sum_i \psi_i \left(\left|a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho_l^\tau a^h_{\tau ij}\right|\right)/\psi_j < \phi_h, \; (\sum_i \psi_i \left(\left|\rho_l a^h_{1ij} + ... + \rho_l^\tau a^h_{\tau ij}\right|\right)/\psi_j < \phi_h) \forall j, \forall h, \forall l,$$

*then the economy described by (1.3) and (1.2),* Model I, *((1.10) and (1.2),* Model III*) is $\delta$-stable under mixed RLS/SG learning.*

**Proof.** See Appendix A.7.14. □

The results for Models I and III and the derivations may be rewritten not in terms of the aggregate economy to the associated models, but in terms of the aggregate model to the original one. In this sense, these results largely resemble the ones derived in Bogomolova and Kolyuzhnov [5] and Kolyuzhnov [40] except for the weighting of expectations of different leads of expectations.

The procedure is as follows. Let us return to the definition of the aggregate economy for the original Model I. It is also possible to simplify matters even further and to construct the weighted coefficient for $\beta_\tau^{AG\,\mathrm{mod}}$ and the corresponding one-expectation model in the form

$$y_t^{AG\,\mathrm{mod}} = \sum_i \psi_i \left|\alpha_i\right| + \beta_{weighted}^{AG\,\mathrm{mod}}\left(\psi,\phi\right) \hat{E}_{weighted t-1}^{AG\,\mathrm{mod}}\left(y_t^{AG\,\mathrm{mod}}\right) + \left|\left(\sum_i \psi_i B^i\right) w_t\right| + \left|\left(\sum_i \psi_i \varsigma^i\right) \varepsilon_t\right|,$$

where the weights for summing the coefficients before expectations of different leads of $y$ can naturally be taken to be the autocorrelation coefficients of $w$ or 1 (no discounting); that is, $\rho_l$, $l = 0, 1, ..., k$. Thus, I may formulate the following definition.

**Definition 1.7** *Given the weights of aggregation across endogenous variables* $\psi_i > 0$, $\sum\limits_{i=1}^{n} \psi_i = 1$, *and across agent types* $\phi_h > 0$, $\sum\limits_{h=1}^{S} \phi_h = 1$, ***the corresponding set of the current value aggregate expectation models f or the limiting aggregate economy*** *for an economy described by (1.3) and (1.2),* Model I, *is defined as*

$$y_t^{AG\,\mathrm{mod}} = \sum_i \psi_i \left|\alpha_i\right| + \beta_{weighted\ l}^{AG\,\mathrm{mod}}\ \left(\psi,\phi\right) \hat{E}_{weighted\ l\ t-1}^{AG\,\mathrm{mod}}\left(y_t^{AG\,\mathrm{mod}}\right) + \tag{1.42}$$
$$+ \left|\left(\sum_i \psi_i B^i\right) w_t\right| + \left|\left(\sum_i \psi_i \varsigma^i\right) \varepsilon_t\right|$$

*and (1.2),*

*where* $\beta_{weighted\ l}^{AG\,\mathrm{mod}}\left(\psi,\phi\right)$ *and* $\hat{E}_{weighted\ l\ t-1}^{AG\,\mathrm{mod}}\left(y_t^{AG\,\mathrm{mod}}\right)$, $l = 0, 1, ..., l$ *are defined as*

$$\beta_{weighted\ l}^{AG\,\mathrm{mod}}\left(\psi,\phi\right) = \beta_0^{AG\,\mathrm{mod}}\left(\psi,\phi\right) + \left|\rho_l\right| \beta_0^{AG\,\mathrm{mod}}\left(\psi,\phi\right) + ... + \left|\rho_l\right|^\tau \beta_\tau^{AG\,\mathrm{mod}}\left(\psi,\phi\right), l = 0, 1, ..., k$$

$$\hat{E}_{weighted\ l\ t-1}^{AG\,\mathrm{mod}}\left(y_t^{AG\,\mathrm{mod}}\right) = \left(\beta_{weighted\ l}^{AG\,\mathrm{mod}}\left(\psi,\phi\right)\right)^{-1} \sum_{r=0}^{\tau} \beta_r^{AG\,\mathrm{mod}}\left(\psi,\phi\right) \hat{E}_{t-1}^{AG\,\mathrm{mod}}\left(y_{t+r}^{AG\,\mathrm{mod}}\right).$$

The structure of this weighted limiting aggregate coefficient $\beta_{weighted\ l}^{AG\,\mathrm{mod}}$ is as follows. $\sum_i \psi_i \left|a_{0ij}^h\right| + \left|a_{0ij}^h\right| \sum_i \psi_i \left|a_{1ij}^h\right| + ... + \left|\rho_l\right|^\tau \sum_i \psi_i \left|a_{\tau ij}^h\right|$ is the coefficient before the expectation of endogenous variable $j$ in the aggregate economy composed of one single agent type $h$. Notice that this coefficient is calculated for the expectation of endogenous variable $j$, that enters the aggregate product with coefficient $\psi_j$. So for each $l = 0, 1, ..., k$, following Kolyuzhnov [40], I may interpret the ratio

$$\left(\sum_i \psi_i \left|a_{0ij}^h\right| + \left|\rho_l\right| \sum_i \psi_i \left|a_{1ij}^h\right| + ... + \left|\rho_l\right|^\tau \sum_i \psi_i \left|a_{\tau ij}^h\right|\right) / \psi_j$$

as the **weighted endogenous variable** $j$ **"own" expectations relative coefficient.** The next proposition is formulated in terms of these relative coefficients and stresses the fact that the weights of agents in calculating aggregate expectations have to be put into accordance with this economic intuition in order to have stability under heterogeneous learning.

**Proposition 1.13** *In the case of the diagonal environment (1.22), if for each economy from the corresponding set of current value aggregate expectation models for the limiting aggregate economy for an economy described by (1.3) and (1.2),* Model I*, there exists at least one pair of vectors of weights* $\psi$ *for the aggregation of endogenous variables and weights* $\phi$ *for the aggregation of agents such that for each agent every weighted endogenous variable's "own" expectations relative coefficient corresponding to the limiting aggregate economy (1.42) and (1.2) is less than the weight of the agent used in calculating aggregate expectations, i.e.*

$$\sum_i \psi_i \left( \left| a_{0ij}^h \right| + |\rho_l| \left| a_{1ij}^h \right| + ... + |\rho_l^\tau| \left| a_{\tau ij}^h \right| \right) / \psi_j < \phi_h \forall j, \forall h, \forall l,$$

*then the economy described by (1.3) and (1.2),* Model I*, is* $\delta$*–stable under mixed RLS/SG learning.*

**Proof.** Follows directly from Proposition 1.12 □

It is clear that this sufficient condition is stronger than the previous one and that the condition for $l = 0$ alone is sufficient for the result to hold true. For Model III, one has to set $a_{0ij}^h$ to zero everywhere in Definition 1.7 and Proposition 1.13 above.

### 1.5.2 Aggregation for models in the general (non-diagonal) case under heterogeneous RLS learning

In the general, non-diagonal case, sufficient stability conditions for the MSV REE of Model I and Model III under heterogeneous RLS learning may again be written in terms of the stability of models from the *corresponding set of current value aggregate expectation models for* ***the limiting aggregate economy*** of Model I and Model III, but now the definition of this set has to be extended to account for the non-diagonal structure of matrix $F$.

**Definition 1.8** *Given the weights of aggregation across endogenous variables* $\psi_i > 0$, $\sum_{i=1}^{n} \psi_i = 1$*, and across agent types* $\phi_h > 0$, $\sum_{h=1}^{S} \phi_h = 1$*,* ***the corresponding set***

***of current value aggregate expectation models for the limiting aggregate economy*** *for an economy described by (1.3) and (1.2),* Model I*, ((1.10) and (1.2),* Model III*) is defined as*

$$y_t^{AG\,\mathrm{mod}} = \sum_i \psi_i \left|\alpha_i\right| + \beta_{weighted\ l}^{AG\,\mathrm{mod}}\left(\psi,\phi\right)\hat{E}_{weighted\ l\ t-1}^{AG\,\mathrm{mod}}\left(y_t^{AG\,\mathrm{mod}}\right) + + \left|\left(\sum_i \psi_i B^i\right) w_t\right| + \left|\left(\sum_i \psi_i \varsigma^i\right)\varepsilon_t\right| \qquad (1.43)$$

*and (1.2),*

*where* $\beta_{weighted\ l}^{AG\,\mathrm{mod}}\left(\psi,\phi\right)$ *and* $\hat{E}_{weighted\ l\ t-1}^{AG\,\mathrm{mod}}\left(y_t^{AG\,\mathrm{mod}}\right)$ $, l = 0, 1, ..., k$ *are defined as*
*for Model I:*

$$\beta_{weighted\ l}^{AG\,\mathrm{mod}}\left(\psi,\phi\right) = \beta_0^{AG\,\mathrm{mod}}\left(\psi,\phi\right) + \rho_f\beta_1^{AG\,\mathrm{mod}}\left(\psi,\phi\right) + ... + \rho_f\beta_\tau^{AG\,\mathrm{mod}}\left(\psi,\phi\right), l = 0, 1, ..., k,$$

$$\hat{E}_{weighted\ l\ t-1}^{AG\,\mathrm{mod}}\left(y_t^{AG\,\mathrm{mod}}\right) = \left(\beta_{weighted\ l}^{AG\,\mathrm{mod}}\left(\psi,\phi\right)\right)^{-1}\sum_{r=0}^{\tau}\beta_r^{AG\,\mathrm{mod}}\left(\psi,\phi\right)\hat{E}_{t-1}^{AG\,\mathrm{mod}}\left(y_{t+r}^{AG\,\mathrm{mod}}\right);$$

*for Model III:*

$$\beta_{weighted\ l}^{AG\,\mathrm{mod}}\left(\psi,\phi\right) = \rho_f\beta_1^{AG\,\mathrm{mod}}\left(\psi,\phi\right) + ... + \rho_f\beta_\tau^{AG\,\mathrm{mod}}\left(\psi,\phi\right), l = 0, 1, ..., k,$$

$$\hat{E}_{weighted\ l\ t-1}^{AG\,\mathrm{mod}}\left(y_t^{AG\,\mathrm{mod}}\right) = \left(\beta_{weighted\ l}^{AG\,\mathrm{mod}}\left(\psi,\phi\right)\right)^{-1}\sum_{r=1}^{\tau}\beta_r^{AG\,\mathrm{mod}}\left(\psi,\phi\right)\hat{E}_{t}^{AG\,\mathrm{mod}}\left(y_{t+r}^{AG\,\mathrm{mod}}\right),$$

*where* $\rho_f = \sum_{r=1}^{k}\left|f_{lr}\right|$ *or* $\sum_{r=1}^{k}\left|f_{rl}\right|.$

Again, the index

$$\left(\sum_i \psi_i \left|a_{0ij}^h\right| + \rho_f \sum_i \psi_i \left|a_{1ij}^h\right| + ... + \rho_f \sum_i \psi_i \left|a_{\tau ij}^h\right|\right)/\psi_j$$

(for Model III $a_{0ij}^h = 0$) may be called, as before, **the weighted endogenous variable** $j$ **"own" expectations relative coefficient** and has the same meaning as before**.**

As for Models II and IV, complications arise due to the presence of one endogenous variable lag in the model. To alleviate the complications and to return the discussion to the "unlagged" structure of Bogomolova and Kolyuzhnov [5], Kolyuzhnov [40], and Honkapohja and Mitra (2006), I construct an economy without a lag corresponding to the model considered that has the same asymptotic behaviour around the REE. I call this model (by analogy to the associated ODE) the associated "unlagged" economy. With respect to Models II and IV, ***the associated "unlagged" model corresponding to Models II and IV*** is defined in the following proposition.

**Proposition 1.14** ***The associated "unlagged" model corresponding to Model II (IV)*** *(that is, the model that has the same asymptotic behaviour as Model II (IV) where the component for the lag coefficient is fixed at the MSV REE value) is the model: for Model II*

$$y_t = \alpha + \sum_{h=1}^{S} \left(A_0^h + A_1^h \bar{b}\right) \hat{E}_{t-1}^h y_t + \sum_{h=1}^{S} A_1^h \hat{E}_{t-1}^h y_{t+1} + Bw_t + \zeta\varepsilon_t,$$

*and (1.2),*

*for Model IV*

$$y_t = \alpha + \sum_{h=1}^{S} \left(A_1^h \bar{b}\right) \hat{E}_{t-1}^h y_t + \sum_{h=1}^{S} A_1^h \hat{E}_{t-1}^h y_{t+1} + BF^{-1} w_t + \zeta\varepsilon_t,$$

*and (1.2),*

*where $\bar{b}$ is defined in (1.9) for Model II and in (1.16) for Model IV.*

**Proof.** Follows directly from a comparison of the associated ODEs.

Now it is possible to employ the same aggregating procedures as for Model I and Model III to obtain the aggregate economy stability result.

**Definition 1.9** *Given the weights of aggregation across endogenous variables $\psi_i > 0$, $\sum_{i=1}^{n} \psi_i = 1$, and across agent types $\phi_h > 0$, $\sum_{h=1}^{S} \phi_h = 1$,* ***the limiting aggregate economy for the associated "unlagged" economy of the economy*** *described by (1.6) and (1.2),* Model II, *((1.13) and (1.2),* Model IV*) is defined as for Model II:*

$$y_t^{AG\,\mathrm{mod}} = \sum_i \psi_i \left|\alpha_i\right| + \sum_{r=0}^{1} \beta_r^{AG\,\mathrm{mod}}(\psi,\phi)\, \hat{E}_{t-1}^{AG\,\mathrm{mod}}\left(y_{t+r}^{AG\,\mathrm{mod}}\right) + \left|\left(\sum_i \psi_i B^i\right) w_t\right| + \left|\left(\sum_i \psi_i \varsigma^i\right)\varepsilon_t\right|$$

*and (1.2);*

*for Model IV:*

$$y_t^{AG\,\mathrm{mod}} = \sum_i \psi_i \left|\alpha_i\right| + \sum_{r=0}^{1} \beta_r^{AG\,\mathrm{mod}}(\psi,\phi)\, \hat{E}_{t-1}^{AG\,\mathrm{mod}}\left(y_{t+r}^{AG\,\mathrm{mod}}\right) + \left|\left(\sum_i \psi_i \left(BF^{-1}\right)^i\right) w_t\right| + \left|\left(\sum_i \psi_i \varsigma^i\right)\varepsilon_t\right|$$

*and (1.2),*

*where $\beta_r^{AG\,\mathrm{mod}}(\psi,\phi)$ and $\hat{E}_{t-1}^{AG\,\mathrm{mod}}\left(y_{t+r}^{AG\,\mathrm{mod}}\right)$, $r = 0, 1$ are defined as*

$$\beta_0^{AG\,\mathrm{mod}}(\psi,\phi) = S \sum_h \phi_h \sum_i \psi_i \sum_j \left| a_{0ij}^h + \left(A_1^h \bar{b}\right)_{ij} \right|$$

$$\beta_1^{AG\,\mathrm{mod}}(\psi,\phi) = S\sum_h \phi_h \sum_i \psi_i \sum_j \left| a_{1ij}^h \right|$$

$$\hat{E}_{t-1}^{AG\,\mathrm{mod}}\left(y_t^{AG\,\mathrm{mod}}\right) = \left(\beta_0^{AG\,\mathrm{mod}}(\psi,\phi)\right)^{-1} \sum_{h=1}^{S} S\phi_h \sum_i \psi_i \sum_j \left| a_{0ij}^h + \left(A_1^h \bar{b}\right)_{ij} \right| \hat{E}_{t-1}^h \left| y_{jt} \right|$$

$$\hat{E}_{t-1}^{AG\,\mathrm{mod}}\left(y_{t+1}^{AG\,\mathrm{mod}}\right) = \left(\beta_1^{AG\,\mathrm{mod}}(\psi,\phi)\right)^{-1} \sum_{h=1}^{S} S\phi_h \sum_i \psi_i \sum_j \left| a_{1ij}^h \right| \hat{E}_{t-1}^h \left| y_{jt+1} \right|.$$

*For Model IV:* $a_{0ij}^h \equiv 0.$

***The corresponding sets of current value aggregate expectation models for these limiting aggregate economies*** are given by Definition 1.10.

**Definition 1.10** *Given the weights of aggregation across endogenous variables* $\psi_i > 0$, $\sum_{i=1}^{n} \psi_i = 1$, *and across agent types* $\phi_h > 0$, $\sum_{h=1}^{S} \phi_h = 1$, ***the corresponding set of current value aggregate expectation models for the limiting aggregate economy for an associated "unlagged" economy*** *of the economy described by (1.6) and (1.2),* Model II, *((1.13) and (1.2),* Model IV*) is defined as*
*for Model II:*

$$y_t^{AG\,\mathrm{mod}} = \sum_i \psi_i |\alpha_i| + \beta_{weighted\ p}^{AG\,\mathrm{mod}}(\psi,\phi)\, \hat{E}_{weighted\ p\ t-1}^{AG\,\mathrm{mod}}\left(y_t^{AG\,\mathrm{mod}}\right) + \left|\left(\sum_i \psi_i B^i\right) w_t\right| + \left|\left(\sum_i \psi_i \varsigma^i\right)\varepsilon_t\right| \quad (1.44)$$

*and (1.2);*

*for Model IV:*

$$y_t^{AG\,\mathrm{mod}} = \sum_i \psi_i |\alpha_i| + \beta_{weighted\ l}^{AG\,\mathrm{mod}}(\psi,\phi)\, \hat{E}_{weighted\ l\ t-1}^{AG\,\mathrm{mod}}\left(y_t^{AG\,\mathrm{mod}}\right) + \left|\left(\sum_i \psi_i \left(BF^{-1}\right)^i\right) w_t\right| + \left|\left(\sum_i \psi_i \varsigma^i\right)\varepsilon_t\right|, \quad (1.45)$$

*and (1.2),*

*where* $p = l$ *or* $q$, $\beta_{weighted\ p}^{AG\,\mathrm{mod}}(\psi,\phi)$ *and* $\hat{E}_{weighted\ p\ t-1}^{AG\,\mathrm{mod}}\left(y_t^{AG\,\mathrm{mod}}\right)$, *are defined as*

$$\beta_{weighted\ l}^{AG\,\mathrm{mod}}(\psi,\phi) = \beta_0^{AG\,\mathrm{mod}}(\psi,\phi) + \rho_f \beta_1^{AG\,\mathrm{mod}}(\psi,\phi),\ l = 0,1,...,k,$$

*and*

$$\beta_{weighted\ q}^{AG\,\mathrm{mod}}(\psi,\phi) = \beta_0^{AG\,\mathrm{mod}}(\psi,\phi) + \rho_q \beta_1^{AG\,\mathrm{mod}}(\psi,\phi),\ q = 1,...,n,$$

$$\hat{E}^{AG\,\mathrm{mod}}_{weighted\ p\ t-1}\left(y_t^{AG\,\mathrm{mod}}\right) = \left(\beta^{AG\,\mathrm{mod}}_{weighted\ p}\left(\psi,\phi\right)\right)^{-1}\sum_{r=0}^{1}\beta_r^{AG\,\mathrm{mod}}\left(\psi,\phi\right)\hat{E}^{AG\,\mathrm{mod}}_{t-1}\left(y^{AG\,\mathrm{mod}}_{t+r}\right),$$

*where* $\rho_f = \sum_{r=1}^{k}|f_{lr}|$ *or* $\sum_{r=1}^{k}|f_{rl}|, \rho_q = \sum_{r=1}^{n}\left|\bar{b}_{qr}\right|$ *or* $\sum_{r=1}^{n}\left|\bar{b}_{rq}\right|.$

The second part of the definition using $\sum_{r=1}^{n}\left|\bar{b}_{qr}\right|$ as weights for the leads in expectations reflects the dependence of $y_t$ on $y_{t-1}$ at the REE for endogenous variable $j$.

The structure of these weighted limiting aggregate coefficients $\beta^{AG\,\mathrm{mod}}_{weighted\ p}$, similarly to the ones of Model I and Model III, allows for the ratios $\sum_i \psi_i\left(\rho_f\left|a^h_{1ij}\right| + \left|a^h_{0ij} + \left(A^h_1\bar{b}\right)_{ij}\right|\right)/\psi_j$ and $\sum_i \psi_i\left(\rho_q\left|a^h_{1ij}\right| + \left|a^h_{0ij} + \left(A^h_1\bar{b}\right)_{ij}\right|\right)/\psi_j$ (for Model IV $a^h_{0ij} \equiv 0$) to be interpreted as the **weighted endogenous variable** $j$ **"own" expectations relative coefficients.** Again, as above, the existence of a similar correspondence between the values of these coefficients and the weight of a particular agent type that follows from the criteria (Criteria 1.1 and 1.6) for the stability of models without lags of the endogenous variables or from the sufficient conditions (Corollaries 1.3 and 1.8) for the stability of models with lags of the endogenous variable, with both model types being under heterogeneous RLS learning in the general (non-diagonal) environment, and from the McKenzie Theorem (1960), the existense of this correspondence allows for the $\delta$-stability of the original economies.

**Proposition 1.15** *In the general (non-diagonal) case, if for all economies from the corresponding set of the current value aggregate expectation models for the limiting aggregate economy for the economy described by (1.3) and (1.2),* Model I *((1.10) and (1.2),* Model III*) under heterogeneous RLS learning, there exists at least one pair of vectors of weights* $\psi$ *for the aggregation of endogenous variables and weights* $\phi$ *for the aggregation of agents such that for each agent, every weighted endogenous variable's "own" expectations relative coefficient corresponding to the limiting aggregate economy (1.43) and (1.2) of the row type is less than the weight of the agent used in calculating aggregate expectations, i.e.*

$$\left(\sum_i \psi_i\left|a^h_{0ij}\right| + \rho_f\sum_i \psi_i\left|a^h_{1ij}\right| + ... + \rho_f\sum_i \psi_i\left|a^h_{\tau ij}\right|\right)/\psi_j < \phi_h \forall j, \forall h, \forall l,$$

*where* $\rho_f = \sum_{r=1}^{k}|f_{lr}|$. *(for Model III* $a^h_{0ij} \equiv 0$*), then the economy described by (1.3) and (1.2),* Model I, *((1.10) and (1.2),* Model III*) under heterogeneous RLS learning is* $\delta$*-stable.*

**Proof.** See Appendix A.7.15. □

**Proposition 1.16** *In the general (non-diagonal) case, if for each row aggregation type ($f$ and $b$ -type) of economies from the corresponding set of the current value aggregate expectation models for the limiting aggregate economy for the associated "unlagged" economy of the economy described by (1.6) and (1.2),* Model II *((1.13) and (1.2),* Model IV*) under heterogeneous RLS learning, there exists at least one pair of vectors of weights $\psi$ for the aggregation of endogenous variables and weights $\phi$ for the aggregation of agents such that for each agent, every weighted endogenous variable's "own" expectations relative coefficient corresponding to the limiting aggregate associated "unlagged" economy (1.44) and (1.2), ((1.45) and (1.2)) is less than the weight of the agent used in calculating aggregate expectations, i.e.*

$$\exists(\psi,\phi) - weights \vdots \sum_i \psi_i \left( \sum_{r=1}^{k} \rho_f \left| a_{1ij}^h \right| + \left| a_{0ij}^h + \left( A_1^h \bar{b} \right)_{ij} \right| \right) / \psi_j < \phi_h \forall j, \forall h, \forall l,$$

*and*

$$\exists(\psi,\phi) - weights \vdots \sum_i \psi_i \left( \sum_{r=1}^{n} \rho_q \left| a_{1ij}^h \right| + \left| a_{0ij}^h + \left( A_1^h \bar{b} \right)_{ij} \right| \right) / \psi_j < \phi_h \forall j, \forall h, \forall q,$$

*where $\rho_f = \sum_{r=1}^{k} |f_{lr}|$, $\rho_q = \sum_{r=1}^{n} \left|\bar{b}_{qr}\right|$ (for Model IV $a_{0ij}^h \equiv 0$), then the economy described by (1.6) and (1.2),* Model II *((1.13) and (1.2),* Model IV*), in which all roots of $\bar{b}$ defined in (1.9) for Model II and in (1.16) for Model IV lie inside the unit circle, is $\delta$-stable under heterogeneous RLS learning.*

**Proof.** See Appendix A.7.16. □

### 1.5.3 Sufficient conditions for $\delta$-stability in terms of $E$-stability of maximal aggregate economies

However, the propositions above do not give a real rule of thumb (as they imply looking for systems of weights) that could be used to say if a particular economy is stable under heterogeneous learning. For this purpose, I go even further looking for upper boundaries by considering not only any possible signs of $a_{ij}$, but also the values of weights $\psi$ and $\phi$. These boundaries can be derived for four different subsets of limiting aggregate economies (Models I and III under mixed RLS/SG learning in the diagonal environment) and for economies from the corresponding sets of current value aggregate expectation models (Models I, II, III, and IV under heterogeneous RLS learning in the

| Subset | | $\beta_s^{AG\max} =$ |
|---|---|---|
| $s=1$ | $\psi$–any, $\phi$–any | $\max\limits_l S \sum\limits_j \max\limits_{h,i} \left\lvert a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right\rvert$ |
| $s=2$ | $\psi$–any, $\phi = \frac{1}{S}$ | $\max\limits_l \max\limits_i \sum\limits_h \sum\limits_j \left\lvert a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right\rvert$ |
| $s=3$ | $\psi = \frac{1}{n}$, $\phi$–any | $\max\limits_l S \sum\limits_i \max\limits_{h,j} \left\lvert a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right\rvert$ |
| $s=4$ | $\psi = \frac{1}{n}$, $\phi = \frac{1}{S}$ | $\max\limits_l \sum\limits_h \max\limits_j \sum\limits_i \left\lvert a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right\rvert$ |

Table 1.1: Maximal aggregate $\beta$–coefficients for maximal aggregate economies for the associated current value expectations models corresponding to models without lags (Model I and Model III) under mixed RLS/SG learning in the diagonal case

general (non-diagonal) case) depending on the weights $\psi$ and $\phi$: with arbitrary weights of agents and endogenous variables, and with either equal weights of agents, $\frac{1}{S}$, or equal weights of endogenous variables, $\frac{1}{n}$, or both.

So, I formulate the following definitions.

**Definition 1.11** *Given the weights of aggregation across endogenous variables $\psi_i > 0$, $\sum\limits_{i=1}^{n} \psi_i = 1$, and across agent types $\phi_h > 0$, $\sum\limits_{h=1}^{S} \phi_h = 1$, each aggregate economy from a particular subset of limiting aggregate economies for the set of associated current value expectations models corresponding to Model I (III) under mixed RLS/SG learning in the diagonal case is bounded above by the following* ***maximal aggregate economy***

*for Model I:*

$$y_t^{AG} = \sum_i \psi_i y_{it} \le y_t^{AG\,\mathrm{mod}} = \sum_i \psi_i \left\lvert y_{it} \right\rvert \le y_t^{AG\max} =$$
$$= \sum_i \psi_i \left\lvert \alpha_i \right\rvert + \beta_s^{AG\max} \hat{E}_{t-1}^{AG\max} \left( y_t^{AG\max} \right) + \left\lvert \left( \sum_i \psi_i B^i \right) w_t \right\rvert + \left\lvert \left( \sum_i \psi_i \varsigma^i \right) \varepsilon_t \right\rvert;$$

*for Model III:*

$$y_t^{AG} = \sum_i \psi_i y_{it} \le y_t^{AG\,\mathrm{mod}} = \sum_i \psi_i \left\lvert y_{it} \right\rvert \le y_t^{AG\max} =$$
$$= \sum_i \psi_i \left\lvert \alpha_i \right\rvert + \beta_s^{AG\max} \hat{E}_{t-1}^{AG\max} \left( y_t^{AG\max} \right) + \left\lvert \left( \sum_i \psi_i B^i \right) w_t \right\rvert + \left\lvert \left( \sum_i \psi_i \varsigma^i \right) \varepsilon_t \right\rvert,$$

*where $\beta_s^{AG\max}$ is defined in Table 1.1 (for Model III $a_{0ij}^h = 0$).*

**Note 1.1** *This set of maximal aggregate $\beta-$coefficients extends the set derived in Bogomolova and Kolyuzhnov [5] and Kolyuzhnov [40]. All of the above coefficients have the same structure (except for discounting) as in these papers. However, I have been able to find that it is possible to determine a lower boundary for the third set, thus obtaining a* *smaller aggregate coefficient than in these papers. This coeffcient is*

$$\max_l S \max_{h,j} \sum_i \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right|.$$

*It can be shown that if it is less than one, then the economy is* $\delta$*-stable (Take* $\psi_i = \frac{1}{n}, \varphi_h = \frac{1}{S}$ *in the corresponding proposition). However, this coefficient, valid for the diagonal (symmetric) structure, as is shown below, does not allow for the* $\delta$*-stability condition (in terms of the column type coeffcients) in the non-diagonal case, and instead, the coefficient generated in these two papers must be used.*

*Also though it is not mentioned in the papers cited above, it is clear that the maximal aggregate* $\beta$*-coefficient for the second set is always no greater than the maximal aggregate* $\beta$*-coeffcient for the first set and that the maximal aggregate* $\beta$*-coefficient for the fourth set is always no greater than the maximal aggregate* $\beta$*-coefficient for the third set. This means that I can use only the second and fourth* $\beta$*-coefficients in the sufficient conditions for* $\delta$*-stability, as they provide stronger conditions for* $\delta$*-stability and include a set of* $\delta$*-stable economies generated by the conditions on greater maximal aggregate* $\beta$*-coefficients. However, I prefer to mention all four maximal aggregate* $\beta$*-coefficients as the first and third aggregate* $\beta$*-coefficients (namely, their structure) turn out to be quite useful for sufficient conditions analyzed further in this paper in non-symmetric (non-diagonal) cases.*

*It can also be shown that all economies from all four sets are bounded above by the "universal" maximal aggregate* $\beta-$*coefficient* $nS\max_{i,j,h} \left| a_{ij}^h \right|$*. It can be shown that if it is less than one, then the economy is* $\delta$*-stable (Take* $\psi_i = \frac{1}{n}, \varphi_h = \frac{1}{S}$ *in the corresponding proposition). It provides the weakest sufficient condition and the narrowest set of* $\delta-$*stable economies among the ones considered.*

**Definition 1.12** *Given the weights of aggregation across endogenous variables* $\psi_i > 0$, $\sum_{i=1}^{n} \psi_i = 1$, *and across agent types* $\phi_h > 0$, $\sum_{h=1}^{S} \phi_h = 1$, *each economy from a particular subset of aggregate economies from the corresponding set of current value aggregate expectation models for the limiting aggregate economy for the economy described by (1.3) and (1.2),* Model I *((1.10) and (1.2),* Model III*) is bounded above by the following* ***maximal***

| Subset | Type | $\beta_{weighted\ s}^{AG\max}$ |
|---|---|---|
| $s=1$ | $f-$row | $\max\limits_{l} S \sum\limits_{j} \max\limits_{h,i} \left( \left\| a_{0ij}^{h} \right\| + \sum_{r=1}^{k} \left\| f_{lr} \right\| \left\| a_{1ij}^{h} \right\| + ... + \sum_{r=1}^{k} \left\| f_{lr}^{\tau} \right\| \left\| a_{\tau ij}^{h} \right\| \right)$ |
| $s=2$ | $f-$column | $\max\limits_{l} \max\limits_{i} \sum\limits_{h} \sum\limits_{j} \left( \left\| a_{0ij}^{h} \right\| + \sum_{r=1}^{k} \left\| f_{rl} \right\| \left\| a_{1ij}^{h} \right\| + ... + \sum_{r=1}^{k} \left\| f_{rl}^{\tau} \right\| \left\| a_{\tau ij}^{h} \right\| \right)$ |
| $s=3$ | $f-$column | $\max\limits_{l} S \sum\limits_{i} \max\limits_{h,j} \left( \left\| a_{0ij}^{h} \right\| + \sum_{r=1}^{k} \left\| f_{rl} \right\| \left\| a_{1ij}^{h} \right\| + ... + \sum_{r=1}^{k} \left\| f_{rl}^{\tau} \right\| \left\| a_{\tau ij}^{h} \right\| \right)$ |
| $s=4$ | $f-$row | $\max\limits_{l} \sum\limits_{h} \max\limits_{j} \sum\limits_{i} \left( \left\| a_{0ij}^{h} \right\| + \sum_{r=1}^{k} \left\| f_{lr} \right\| \left\| a_{1ij}^{h} \right\| + ... + \sum_{r=1}^{k} \left\| f_{lr}^{\tau} \right\| \left\| a_{\tau ij}^{h} \right\| \right)$ |

Table 1.2: Maximal aggregate $\beta$–coefficients for maximal aggregate economies for the current value aggregate expectation model for models without lags (Model I and Model III (with $a_{0ij}^{h} \equiv 0$))

***aggregate economy***

$$\begin{aligned} y_t^{AG} &= \sum_i \psi_i y_{it} \le y_t^{AG\,\mathrm{mod}} = \sum_i \psi_i \left| y_{it} \right| \le y_t^{AG\max} = \\ &= \sum_i \psi_i \left| \alpha_i \right| + \beta_{weighted\ s}^{AG\max} \hat{E}_{weighted\ l\ t-1}^{AG\max} \left( y_t^{AG\max} \right) + \\ &+ \left| \left( \sum_i \psi_i B^i \right) w_t \right| + \left| \left( \sum_i \psi_i \varsigma^i \right) \varepsilon_t \right|, \end{aligned}$$

*where $\beta_{weighted\ s}^{AG\max}$ is defined in Table 1.2.*

Similar definitions can be formulated for Models II and IV.

**Definition 1.13** *Given the weights of aggregation across endogenous variables $\psi_i > 0$, $\sum\limits_{i=1}^{n} \psi_i = 1$, and across agent types $\phi_h > 0$, $\sum\limits_{h=1}^{S} \phi_h = 1$, each aggregate economy from the corresponding set of current value aggregate expectation models for the limiting aggregate economy for the associated "unlagged" economy of an economy described by (1.6) and (1.2),* Model II *((1.13) and (1.2),* Model IV*) is bounded above by the following **maximal aggregate economy***

*for Model II:*

$$\begin{aligned} y_t^{AG} &= \sum_i \psi_i y_{it} \le y_t^{AG\,\mathrm{mod}} = \sum_i \psi_i \left| y_{it} \right| \le y_t^{AG\max} = \\ &= \sum_i \psi_i \left| \alpha_i \right| + \beta_{weighted\ s}^{AG\max} \hat{E}_{weighted\ l\ t-1}^{AG\max} \left( y_t^{AG\max} \right) + \\ &+ \left| \left( \sum_i \psi_i B^i \right) w_t \right| + \left| \left( \sum_i \psi_i \varsigma^i \right) \varepsilon_t \right|, \ (1.2); \end{aligned}$$

| Model | Type | Subset | $\beta^{AG\max}_{weighted\ s}$ |
| --- | --- | --- | --- |
| | $f-$row | $s=1$ | $\max_l S \sum_j \max_{h,i} \left( \left\| a^h_{0ij} + \left(A^h_1 \bar{b}\right)_{ij} \right\| + \sum_r \|f_{lr}\| \left\| a^h_{1ij} \right\| \right)$ |
| | $f-$column | $s=2$ | $\max_l \max_i \sum_h \sum_j \left( \left\| a^h_{0ij} + \left(A^h_1 \bar{b}\right)_{ij} \right\| + \sum_r \|f_{rl}\| \left\| a^h_{1ij} \right\| \right)$ |
| Model II | $f-$column | $s=3$ | $\max_l S \sum_i \max_{h,j} \left( \left\| a^h_{0ij} + \left(A^h_1 \bar{b}\right)_{ij} \right\| + \sum_r \|f_{rl}\| \left\| a^h_{1ij} \right\| \right)$ |
| and | $f-$row | $s=4$ | $\max_l \sum_h \max_j \sum_i \left( \left\| a^h_{0ij} + \left(A^h_1 \bar{b}\right)_{ij} \right\| + \sum_r \|f_{lr}\| \left\| a^h_{1ij} \right\| \right)$ |
| Model IV | $b-$row | $s=1$ | $\max_q S \sum_j \max_{h,i} \left( \left\| a^h_{0ij} + \left(A^h_1 \bar{b}\right)_{ij} \right\| + \sum_r \left\|\bar{b}_{qr}\right\| \left\| a^h_{1ij} \right\| \right)$ |
| (with $a^h_{0ij} \equiv 0$) | $b-$column | $s=2$ | $\max_q \max_i \sum_h \sum_j \left( \left\| a^h_{0ij} + \left(A^h_1 \bar{b}\right)_{ij} \right\| + \sum_r \left\|\bar{b}_{rq}\right\| \left\| a^h_{1ij} \right\| \right)$ |
| | $b-$column | $s=3$ | $\max_q S \sum_i \max_{h,j} \left( \left\| a^h_{0ij} + \left(A^h_1 \bar{b}\right)_{ij} \right\| + \sum_r \left\|\bar{b}_{rq}\right\| \left\| a^h_{1ij} \right\| \right)$ |
| | $b-$row | $s=4$ | $\max_q \sum_h \max_j \sum_i \left( \left\| a^h_{0ij} + \left(A^h_1 \bar{b}\right)_{ij} \right\| + \sum_r \left\|\bar{b}_{qr}\right\| \left\| a^h_{1ij} \right\| \right)$ |

Table 1.3: Maximal aggregate $\beta$–coefficients for maximal aggregate economies for the current value aggregate expectation model for the associated "unlagged" economy of models with lags (Model II, IV)

*for Model IV:*

$$\begin{aligned} y^{AG}_t &= \sum_i \psi_i y_{it} \leq y^{AG\,\mathrm{mod}}_t = \sum_i \psi_i \left| y_{it} \right| \leq y^{AG\max}_t = \\ &= \sum_i \psi_i \left|\alpha_i\right| + \beta^{AG\max}_{weighted\ s}\ \hat{E}^{AG\max}_{weighted\ l\ t-1}\left(y^{AG\,\mathrm{mod}}_t\right) + \\ &+ \left| \left( \sum_i \psi_i \left(BF^{-1}\right)^i \right) w_t \right| + \left| \left( \sum_i \psi_i \varsigma^i \right) \varepsilon_t \right|, \quad (1.2), \end{aligned}$$

*where $\beta^{AG\max}_{weighted\ s}$ is defined in Table 1.3 .*

These **maximal aggregate $\beta$-coefficients** are actually upper boundaries for the corresponding $\beta^{AG\,\mathrm{mod}}_l(\psi,\phi)$ and $\beta^{AG\,\mathrm{mod}}_{weighted\ l}(\psi,\phi)$ for different subsets of aggregate economies. Formally, the result can be written in the form of the following proposition.

**Proposition 1.17** *Maximal aggregate $\beta$-coefficients defined in Tables 1.1, 1.2, and 1.3 are upper boundaries for $\beta^{AG\,\mathrm{mod}}_l(\psi,\phi)$ and $\beta^{AG\,\mathrm{mod}}_{weighted\ l}(\psi,\phi)$ for the corresponding subsets of aggregate economies.*

**Proof.** See Appendix A.7.17. □

Thus, I have managed to aggregate the economy into one dimension and to find the maximal aggregate economies that bound all such aggregate economies within a particular subset. If one of these maximal aggregate economies for each type *of the current value aggregate expectation model* is $E$-stable (i.e. if at least one of the maximal aggregate $\beta$-coefficients is less than one), then all aggregate subeconomies from a particular subset of aggregate economies are $E$-stable. I have already mentioned the concept of a subeconomy, and now I introduce its formal definition as this concept is convenient to use in proofs and conditions for $\delta$-stability.

**Definition 1.14** ***A subeconomy*** $(h_1, ..., h_p)$ *of size* $p$ *for economy (1.1) and (1.2) is defined as consisting only of a part of the agents from the original economy:*

$$y_t = \alpha + \sum_{i=1}^{d} L_i y_{t-i} + \sum_{k=1}^{p} \sum_{b=0}^{m} \sum_{f=b}^{n} A_{bf}^{h_k} \hat{E}_{t-b}^{h_k} y_{t-b+f} + B w_t + \zeta \varepsilon_t, \;\; A_{00}^{h_k} \equiv 0, \quad (1.46)$$

*(1.2),*

*where* $(h_1, ..., h_p) \subseteq (1, ..., S)$ *is a set of numbers of agent types present in the subeconomy. A single economy is the particular case of a subeconomy with only one type of agent.*

Now I am ready to formulate the result in two propositions for the model without lags and with lags, respectively, which stresses the key role of $E$-stability of the aggregate economy in the stability of the original, structurally heterogeneous economy under heterogeneous learning with possibly different degrees of inertia (recall Proposition 2 and Proposition 3 in Honkapohja and Mitra [36]). The key results are as follows.

**Proposition 1.18** *If one of the maximal aggregate economies of the associated current value expectations models for models without lags (Model I and Model III) under mixed RLS/SG learning in the diagonal case is* $E$*-stable (i.e., one of the maximal aggregate* $\beta$*-coefficients is less than one), then the economy described by the original Model (I or III) under mixed RLS/SG learning in the diagonal case is* $\delta$*-stable. Notice that all subeconomies are also* $\delta$*-stable under this condition.*

**Proof.** See Appendix A.7.18. □

It is also possible to write down sufficient conditions for $\delta$-stability for Models without lags (I and III) under mixed RLS/SG learning in the diagonal case in terms of the $E$-stability of maximal aggregate economies of the original models (contrary to the $E$-stability of maximal aggregate economies of the associated models in Proposition 1.18). These conditions, though more restrictive (stronger), are presented in the following Corollary.

**Corollary 1.19** *If one of the maximal aggregate economies for the current value aggregate expectation model for models without lags (Model I and Model III) under mixed RLS/SG learning in the diagonal case is E-stable (i.e., one of the maximal aggregate $\beta$-coefficients is less than one), then the economy described by the original Model (I or III) under mixed RLS/SG learning in the diagonal case is $\delta$-stable. Notice that all subeconomies are also $\delta$-stable under this condition.*

**Proof.** It is easy to notice that the aggregated $\beta$-coefficients from Table 1.1 are less than or equal to the corresponding aggregated $\beta$-coefficients for the maximal aggregate economies of this Corollary.

$$\max_l S \sum_j \max_{h,i} \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \leq \max_l S \sum_j \max_{h,i} \left( \left| a_{0ij}^h \right| + |\rho_l| \left| a_{1ij}^h \right| + ... + |\rho_l^\tau| \left| a_{\tau ij}^h \right| \right)$$

$$\max_l \max_i \sum_h \sum_j \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \leq \max_l \max_i \sum_h \sum_j \left( \left| a_{0ij}^h \right| + |\rho_l| \left| a_{1ij}^h \right| + ... + |\rho_l^\tau| \left| a_{\tau ij}^h \right| \right)$$

$$\max_l S \sum_i \max_{h,j} \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \leq \max_l S \sum_i \max_{h,j} \left( \left| a_{0ij}^h \right| + |\rho_l| \left| a_{1ij}^h \right| + ... + |\rho_l^\tau| \left| a_{\tau ij}^h \right| \right)$$

$$\max_l \sum_h \max_j \sum_i \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \leq \max_l \sum_h \max_j \sum_i \left( \left| a_{0ij}^h \right| + |\rho_l| \left| a_{1ij}^h \right| + ... + |\rho_l^\tau| \left| a_{\tau ij}^h \right| \right)$$

Thus, if one of the latter coeffcients is less than one, this means that the corresponding former coefficient is less than one, which leads to $\delta$–stability of the original economy by Proposition 1.18. □

**Remark 1.2** *The E-stability condition for all $\rho_l$-types of maximal aggregate economies for the current value aggregate expectation model for models without lags (Model I and III) is equivalent to* the *E-stability condition for the maximal aggregate economies for the current value aggregate expectation model for models without lags (Model I and III) with $\rho_0 = 1$. Thus, this condition alone is sufficient for the $\delta$-stability of Models I and III under mixed RLS/SG learning in the diagonal case.*

**Proposition 1.20** *If one of the maximal aggregate economies for the current value aggregate expectation model for models without lags (Model I and Model III) under heterogeneous*

*RLS learning in the general (non-diagonal) case is E-stable (i.e., one of the maximal aggregate $\beta$-coefficients is less than one), then the economy described by the original Model (I, III) under heterogeneous RLS learning in the general (non-diagonal) case is $\delta$-stable. Notice that all subeconomies are also $\delta$-stable under this condition.*

**Proof.** See Appendix A.7.19. □

**Proposition 1.21** *If one of the maximal aggregate economies of each type (f-type and b-type) of current value aggregate expectation model for the associated "unlagged" economy of models with lags (Model II and Model IV) under heterogeneous RLS learning in the general (non-diagonal) case is E-stable (i.e., one of the maximal aggregate $\beta$-coefficients is less than one), then the economy described by the original Model (II or IV), in which all roots of $\bar{b}$ defined in (1.9) for Model II and in (1.16) for Model IV lie inside the unit circle, is $\delta$-stable under heterogeneous RLS learning in the general (non-diagonal) case.*

**Proof.** See Appendix A.7.20. □

It is possible to derive less restrictive (weaker) sufficient conditions for $\delta$-stability of models with lags under heterogeneous RLS learning in the diagonal environment case. They could be received as a Corollary to Propositions 1.18 and 1.21 above using the sufficient conditions for $\delta$-stability under heterogeneous RLS models with lags in Propositions 1.5 and 1.10. First, (due to Propositions 1.5 and 1.10) the stability properties of the MSV REE under heterogenous RLS of the associated "unlagged" economy of Models with lags (II, IV), similarly to the results for Models I and III, in the diagonal environment case are equivalent to the simultaneous stability of the MSV REE of the set of the associated current value expectations models (reflecting stability of $D_1\Omega_{\rho_l}$).

Model II and Model IV (with $A_0^h \equiv 0$)

$$y_t = \alpha + \sum_{h=1}^{S}(A_0^h + \left(A_1^h\bar{b}\right) + \rho_l A_1^h)\hat{E}_{t-1}^h y_t + Bw_t + \zeta\varepsilon_t, \qquad (1.47)$$

$$\textit{and } (1.2)$$

and of matrix $D_y\Omega_b$. The first part of the conditions gives the first part of the sufficient condition in terms of maximal aggregate economies of associated current value expectations models for models with lags as in Proposition 1.18. The second part of the conditions gives the second part of the sufficient conditions in terms of $b$-type aggregation as in Proposition 1.21. The result is reflected in the following Corollary.

| Model | Type | Subset | $\beta^{AG\max}_{weighted\ s}$ |
|---|---|---|---|
| Model II | $f-$row | $s=1$ | $\max_l S\sum_j \max_{h,i}\left\|a^h_{0ij}+\left(A^h_1\bar{b}\right)_{ij}+\rho_l a^h_{1ij}\right\|$ |
| and | $f-$column | $s=2$ | $\max_l \max_i \sum_h \sum_j \left\|a^h_{0ij}+\left(A^h_1\bar{b}\right)_{ij}+\rho_l a^h_{1ij}\right\|$ |
| Model IV | $f-$column | $s=3$ | $\max_l S\sum_i \max_{h,j}\left\|a^h_{0ij}+\left(A^h_1\bar{b}\right)_{ij}+\rho_l a^h_{1ij}\right\|$ |
| (with $a^h_{0ij}\equiv 0$) | $f-$row | $s=4$ | $\max_l \sum_h \max_j \sum_i \left\|a^h_{0ij}+\left(A^h_1\bar{b}\right)_{ij}+\rho_l a^h_{1ij}\right\|$ |

Table 1.4: Maximal aggregate $\beta$–coefficients for maximal aggregate economies of the associated current value expectations models under heterogeneous RLS learning in the diagonal case for the associated "unlagged" economy of models with lags (Model II and Model IV)

**Corollary 1.22** *If for the associated "unlagged" economy of models with lags (Model II and Model IV) under heterogeneous RLS learning in the diagonal case, at least one of the exonomies from the set that includes both the maximal aggregate economies of the associated current value expectations models and the maximal b-type aggregate economies of the current value aggregate expectation model is E-stable (i.e., one of the maximal aggregate $\beta$-coefficients defined in Table 1.3 (for b-type) and in Table 1.4 is less than one), then the economy described by the original Model (II or IV), in which all roots of $\bar{b}$ defined in (1.9) for Model II and in (1.16) for Model IV lie inside the unit circle, is $\delta$-stable under heterogeneous RLS learning in the diagonal case.*

## 1.6 Conclusion

In my paper I extend the results of Honkapohja and Mitra [36], Bogomolova and Kolyuzhnov [5], and Kolyuzhnov [40]. I provide the general criteria for stability under heterogeneous mixed RLS/SG learning for four classes of models considered: models without lags and with lags of the endogenous variable and with $t$- or $t-1$-dating of expectations. I also provide conditions for stability and $\delta$-stability in some simpler cases, where simplifications include the diagonal structure of the shock process behaviour, heterogeneous RLS learning, and equal degrees of inertia for each type of learning algorithm. The results on sufficient conditions for $\delta$-stability in terms of the $E$-stability of an aggregate economy derived in this paper are primarily based upon the negative diagonal dominance approach. The results based on the alternative definition of $D$-stability and the necessary conditions based on the characteristic equation approach in terms of the "same sign" conditions and the $E$-stability of a suitably defined average economy and its subeconomies are considered in a companion paper.

All the results of this paper can be summarized as follows.

I provide (in terms of stability of the corresponding Jacobian matrices):

- for the case of a general (**non-diagonal**) structure of the shock, **general criteria for stability** under heterogeneous **mixed RLS/SG learning** in terms of structural and learning heterogeneity *for all types of models* considered: models without lags and with lags of the endogenous variable and with $t$- or $t-1$-dating of expectations.
- for the case of a **diagonal** structure of the shock, **criteria for stability** under heterogeneous **mixed RLS/SG learning** in terms of structural and learning heterogeneity *for models without lags* of the endogenous variable of both types.
- **sufficient conditions for stability** for the case of a general (**non-diagonal**) structure of the shock under **heterogeneous RLS learning** in terms of structural and learning heterogeneity *for models with lags* of the endogenous variable of both types.

For the case of a general (**non-diagonal**) structure of the shock, I provide criteria for stability under heterogeneous **mixed RLS/SG learning with equal degrees of inertia of agents for each type of learning algorithm** in terms of structural and learning heterogeneity *for all types of models* considered in terms of the stability of a suitably defined, structurally heterogeneous, average economy under heterogeneous learning of two agents.

I provide **sufficient conditions for $\boldsymbol{\delta}$-stability** (that is, the stability that does not depend on such learning heterogeneity characteristics as different degrees of inertia and different starting values of learning algorithms) in terms of $E$-stability of suitably defined maximal aggregate economies:

- for the case of a **diagonal** structure of the shock *for models without lags* of endogenous variables under heterogeneous **mixed RLS/SG learning**,
- for the case of a general (**non-diagonal**) structure of the shock *for all types of models* considered under **heterogeneous RLS learning**,
- and (as a mixture of the above) for the case of a diagonal structure of the shock *for all types of models* considered under heterogeneous RLS learning.

Though for the ease of exposition of models with lags, I considered the models with one lag of the endogenous variables and one lead of expectations; the results derived are easily extendable for models with a larger amount of lags and leads. The unconsidered case of a forward-looking model with a lag when the information set includes current value of the endogenous variable to be used to predict the future value of this variable clearly falls under this paper's technical constructions with some modifications and is a matter for my future research.

The fundamental nature of the approach adopted in the paper allows one to apply its results to a vast majority of the existing and prospective linear and linearized economic models (including estimated DSGE models) with the adaptive learning of agents.

# Chapter 2

# Heterogeneous Learning: Beyond The Aggregate Economy Sufficient Conditions for Stability

# Heterogeneous Learning: Beyond The Aggregate Economy Sufficient Conditions for Stability*

Anna Bogomolova†

CERGE–EI‡
Politických vězňů 7, 111 21 Praha 1,
Czech Republic

**Abstract**

I provide criteria and sufficient and necessary conditions for the stability of a structurally heterogeneous economy under the heterogeneous learning of agents, thus extending the results of Honkapohja and Mitra [36], Bogomolova and Kolyuzhnov [5], Kolyuzhnov [40], and Bogomolova [4]. Using the alternative definition of the $D$-stability approach, I provide alternative (to criteria written in terms of the corresponding Jacobian matrices in Kolyuzhnov [40] and Bogomolova [4]) general criteria for stability under mixed RLS/SG learning for four classes of models: models without lags and with lags of the endogenous variable and with $t$- or $t-1$-dating of expectations, and provide alternative sufficient conditions for stability for some simpler cases. This approach also allows me to provide criteria for $\delta$-stability (that is, stability in terms of the structural heterogeneity independent of the heterogeneity in learning) for univariate models without lags of the endogenous variable under mixed RLS/SG learning in economically meaningful terms, such as "same sign" conditions and the $E$-stability of a suitably defined average economy and its subeconomies, and to provide quite weak sufficient conditions for $\delta$-stability for univariate models with a lag of the endogenous variable using the same economic terms. Using the characteristic equation approach, I provide quite strong, economically tractable, necessary conditions that can be used as an easy quick test for non-$\delta$-stability. The fundamental nature of the approach adopted in the paper allows one to apply its results to a vast majority of the existing and prospective linear and linearized economic models (including estimated DSGE models) with the adaptive learning of agents.

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†*Anna.Bogomolova@cerge-ei.cz.*

‡CERGE–EI is a joint workplace of the Center for Economic Research and Graduate Education, Charles University, and the Economics Institute of the Academy of Sciences of the Czech Republic.

## 2.1 Introduction

Adaptive learning is a form of bounded rationality that has arisen to question the rational expectations (RE) hypothesis usually used in macroeconomic models with expectations[1]. One of the widely used forms of adaptive learning is adaptive learning à la Evans and Honkapohja or the adaptive econometric learning where agents are considered as econometricians who update the estimated parameters of their forecast functions using statistical (econometric) approaches each time new information arrives. The essence of such an approach is well advocated by Sargent: If economists themselves do not know true models and have to estimate them econometrically, then we should not expect more from economic agents in general. Thus, it is suggested to consider them as behaving in a way that resembles the behaviour of econometricians (or statisticians). One of the roles of adaptive learning then, is to check the validity of the RE hypothesis, whether agents may learn to be rational; that is, whether a particular model under adaptive learning would converge to an RE equilibrium (REE).

Another hypothesis made in macroeconomic models that has to be questioned is related to adaptive learning itself. Usually it is assumed that agents in the model use the same learning procedure — the case of the so called homogeneous adaptive learning is considered. A question arises whether the stability properties generated by homogeneous learning based on the representative agent hypothesis are replicated in the case of heterogeneous adaptive learning when agents differ in the way they learn. This question is studied e.g. in Giannitsarou [31], who assumes that agents are homogeneous in all respects but in the way they learn; Honkapohja and Mitra [36], who consider a structurally heterogeneous economy meaning that, other than heterogeneity in learning, agents may also differ in structural parameters such as technologies, preferences, etc.; Bogomolova and Kolyuzhnov [5] and Kolyuzhnov [40], who consider conditions for stability independent of heterogeneity in the learning of a structurally heterogeneous forward-looking model with one lead in expectations and with the diagonal structure of shocks; and in a companion paper by Bogomolova [4].

Heterogeneity in learning in these papers comes in the form of different types of learning algorithms used by agents, different speeds of reacting to innovations, different initial perceptions and different shares of agents using a particualr type of learning

[1] including DSGE models

algorithm. The structure of a typical learning algorithm assumes that the updated *belief* parameter (in the simplest case, it is the regression coefficient, and in a recursive least squares (RLS) algorithm, it also includes elements of the second moments matrix) equals the previous value of the parameter plus the gain coefficient (usually presented by a decreasing-in-time sequence) multiplied by the error correction function that depends on the most recent forecast error. The different types of learning algorithms are presented by the RLS (derived as a recursive formulation of usual least squares) and the stochastic gradient (SG) algorithms, where the former differs from the latter only by the fact that it updates the second moments matrix, while the latter keeps it fixed, which allows us to model "less sophisticated" agents. The different speeds of reacting to innovation in the simplest case are modeled as relative weights before the gain sequence common for all agents. Different initial perceptions, in turn, are modeled as different starting values for learning algorithms for different agents. The type of heterogeneous learning that encompasses all types of learning heterogeneity is presented by learning when some agents use RLS and others use SG, and all of them have different degrees of inertia and different starting values for learning. This type of learning is called heterogeneous mixed RLS/SG learning with different degrees of inertia.

In my paper, I, following Bogomolova and Kolyuzhnov [5], Kolyuzhnov [40], and Bogomolova [4], solve the open question posed by Honkapohja and Mitra [36]: to find the conditions for stability of a structurally heterogeneous economy under mixed RLS/SG learning with (possibly) different degrees of inertia in terms of structural heterogeneity only, independent of heterogeneity in learning.

Though Honkapohja and Mitra [36] have formulated a general criterion for such a stability and have been able to solve for sufficient conditions for the case of a univariate model (a model with one endogenous variable), they did not derive the conditions (necessary and/or sufficient) in terms of the model's structure only, independent of the learning characteristics, for the general forward-looking (multivariate) case. Though Bogomolova and Kolyuzhnov [5] and Kolyuzhnov [40] consider conditions for stability irrespective of heterogeneity in learning, they consider only a forward-looking model with one lead and without lags of the endogenous variable and the diagonal environment case that implies a diagonal structure of the AR (1) coefficients matrix in the shock process. It leaves aside many economic models, such as DSGE models with a lag of the endogenous variable. In the companion paper, I substantially extend the results of Honkapohja and Mitra [36], Bogomolova and Kolyuzhnov [5], and Kolyuzhnov [40]. There I provide sufficient conditions for stability under heterogeneous mixed RLS/SG learning for four classes of models: models without lags and with lags of the endogenous variable and with $t$- or $t-1$-dating of expectations, and provide sufficient conditions for stability for some simpler cases, where simplifications include either the diagonal structure of the shock process or heterogeneous RLS learning. However, that paper provides only one part of the results for stability, independent of heterogeneity in learning, which follows mainly from the particular approach to studying stability, namely, the so-called negative diagonal dominance approach.

In the current paper, I also consider the same four types of classes considered in the companion paper by Bogomolova [4] and use the same concept of stability independent of the learning characteristics defined there (that slightly differs from the definition in Bogomolova and Kolyuzhnov [5] and Kolyuzhnov [40]: it does not include shares of agents using a particular type of learning algorithm); and I study the stability properties of the mimimal state variable rational expectations equilibrium solution (MSV REE) of these models under heterogeneous mixed RLS/SG learning.

The stability properties of the MSV REE of the models written in the form of SRAs (the theory on SRA representation of models and stability results for SRAs can be found e.g. in Evans and Honkapohja [24]) can be studied using the associated ordinary differential equations (ODEs). Studying the stability of the MSV REE under the associated ODEs is, in turn, transformed (using first-order approximations around the MSV REE) into studying the stability of the corresponding first-order derivatives matrices of the right-hand side of the ODEs evaluated at the REE, that is, the Jacobians.

The problem of finding the conditions for stability of the corresponding Jacobians results in finding conditions for stability of a matrix (matrices) of a $D\Omega$ type, where $D$ is a positive diagonal matrix. The problem of $D$-stability was studied, for example, in Johnson [37]. There are several approaches considered to tackle this problem, e.g., in Bogomolova and Kolyuzhnov [5] and Kolyuzhnov [40]. Among these approaches are the negative diagonal dominance approach[2], the alternative definition of the $D$-stability approach[3], the characteristic equation approach, the Routh-Hurwitz conditions[4] and an

[2]See Theorem A.4 in Appendix A.4.
[3]See Theorem A.6 in Appendix A.6.
[4]See Theorem A.5 in Appendix A.5.

approach based on the Lyapunov Theorem[5]. The description and discussion of these approaches can be found, for example, in Bogomolova [4][6]

While in that companion paper I essentially use the negative diagonal dominance approach, which allows me to derive a sufficient condition for the $D$-stability of matrices due to the MacKenzie Theorem[7], here I look at the problem from a different angle and try to find the conditions for stability that are not possible to derive using that approach.

I use the alternative definition of $D$-stability approach, which means that the problem of $D$-stability can be equivalently substituted with the problem of finding the stability of $\Omega$ and checking that $i$ is not an eigenvalue of some specially constructed matrix. Moreover, matrix $D$ in all the models considered is not only a positive diagonal matrix, but its diagonal consists of blocks of the same numbers of equal length. It has allowed Kolyuzhnov [40] to introduce the definition of blocked $D$-stability ($D_b$-stability). The alternative definition of $D_b$-stability allows me to write down the criteria for stability under heterogeneous mixed RLS/SG learning for all four classes of models considered in the general (non-diagonal) case. Using this approach, I also derive simplified alternative (to the criteria written in terms of the corresponding Jacobian matrices in Kolyuzhnov [40] and Bogomolova [4]) criteria for the stability of models without lags of both types with the diagonal structure of shocks. It also allows me to derive alternative (to the sufficient conditions written in terms of the corresponding Jacobian matrices in Bogomolova [4]) sufficient conditions (in terms of structural and learning heterogeneity) for stability of models with lags of the endogenous variable of both types with the general (non-diagonal) structure of shocks under heterogeneous RLS learning.

Thus, combined with the blocked structure of matrix $\Omega$, this $D_b$-stability allows me to obtain finer results for sufficient conditions, which in simple univariate cases turn into the weakest possible sufficient conditions, thus becoming necessary and resulting in a criterion. The first (companion) paper (Bogomolova [4]) does not provide a criterion for $\delta$-stability under general heterogeneous mixed RLS/SG learning. Kolyuzhnov [40] provides a criterion only for a univariate model of the forward-looking type with one expectation lead and no lags of the endogenous variable. Here, I provide criteria for $\delta$-stability for univariate models (with either $t$- or $t-1$-dating of expectations) without lags of the endogenous

[5] See Theorem A.2 in Appendix A.3.

[6] See Apppendix A of that paper for formal definitions.

[7] See Theorem A.4 in Appendix A.4

variable under mixed RLS/SG learning in economically meaningful terms. I also provide quite weak, economically tractable, sufficient conditions for $\delta$-stability in univariate models with a lag of the endogenous variable. The results based on the alternative definition of $D_b$-stability produce forms of the "same sign" conditions as considered by Honkapohja and Mitra [36]. In the simplest case, these conditions mean that the endogenous variable reacts to the same sign changes in expectations of different agents in the same direction.

In addition, for all types of models with general (non-diagonal) structure of shocks, using the characterisic equation approach, I provide (quite strong) necessary conditions (in terms of structural and learning heterogeneity) for stability and $\delta$-stability under heterogeneous mixed RLS/SG learning written in terms of stability of a suitably defined structurally heterogeneous average economy under heterogeneous learning of two agents. Using the same approach, for models without lags of endogenous variables with general (non-diagonal) structure of shocks, I provide necessary conditions for stability and $\delta$-stability under heterogeneous mixed RLS/SG learning written in terms of subeconomies for economies from a set of associated current value expectations models. Quite strong necessary conditions can be used as an easy quick test for non-$\delta$-stability.

The fundamental nature of the approach adopted in the paper allows one to apply its results to a vast majority of the existing and prospective linear and linearized economic models with the adaptive learning of agents. For example, those include (estimated) DSGE models with the introduced learning of agents. In this sense, the results derived could be very helpful in terms of checking the robustness of a particular DSGE model[8] to an expectation formation hypothesis, that is usually taken to be RE, and the validity of the representative agent assumption.

The rest of the paper is structured as follows. In the next section, I present the four classes of structurally heterogeneous models under the heterogeneous adaptive learning of agents. Section 3 provides the starting point for my derivations in the paper

[8] A typical DSGE model in structural form looks like

$$A_0 \begin{bmatrix} y_{t-1} \\ w_{t-1} \end{bmatrix} + A_1 \begin{bmatrix} y_t \\ w_t \end{bmatrix} + A_2 E_t y_{t+1} + B_0 \varepsilon_t = const.$$

After estimating (for example by DYNARE [38]), the solution of the model under rational expectations is given by

$$\begin{bmatrix} y_t \\ w_t \end{bmatrix} = \mu + T \begin{bmatrix} y_{t-1} \\ w_{t-1} \end{bmatrix} + R\varepsilon_t.$$

See, for example, Slobodyan and Wouters [53].

— the results of the companion paper (Bogomolova [4]) in the form of criteria and sufficient conditions for stability under heterogeneous learning for each class of models considered in general and simpler cases and formulates the concepts of heterogeneous expectational stability and of $\delta$-stability. In Section 4, using the alternative definition of $D$-stability approach, I provide the alternative, to those in Section 3, general criteria and sufficient conditions for stability for the general case and for some simpler cases discussed above for all classes of models considered. In the same section, I also provide the criteria for $\delta$-stability for univariate models without lags of the endogenous variable under mixed RLS/SG learning in economically meaningful terms, such as the "same sign" conditions and the $E$-stability of a suitably defined average economy and its subeconomies. There I also provide quite weak sufficient conditions for $\delta$-stability for univariate models with a lag of the endogenous variable using the same economic terms and provide the "same sign" sufficient conditions for $\delta$-stability for bivariate models without lags of the endogenous variables in the diagonal environment case. In Section 5, for all model classes, using the characteristic equation approach, I provide a set of quite strong, economically tractable, necessary conditions. Section 6 concludes with a summary of the results.

## 2.2 The setup of linear model classes under heterogeneous adaptive learning

### 2.2.1 Classes of structurally heterogeneous linear models with expectations

As earlier stated, I consider the same general setup of structurally heterogeneous linear models with expectations and four classes of models: models without lags and with lags of the endogenous variable with $t$- or $t-1$-dating of expectations, analyzed in Bogomolova [4][9]. The reduced form of the general class of structurally heterogeneous linear models with $S$ types of agents with different forecasts is given by

$$y_t = \alpha + \sum_{i=1}^{d} L_i y_{t-i} + \sum_{h=1}^{S} \sum_{b=0}^{m} \sum_{f=b}^{n} A_{bf}^{h} \hat{E}_{t-b}^{h} y_{t-b+f} + B w_t + \zeta \varepsilon_t, \ A_{00}^{h} \equiv 0, \quad (2.1)$$

$$w_t = F w_{t-1} + v_t, \quad (2.2)$$

[9] Here I provide a brief description of the setup in that paper. For a full description of models and discussion, please see Bogomolova [4].

where $y_t$ is an $n \times 1$ vector of endogenous variables; $w_t$ is a $k \times 1$ vector of exogenous variables; $v_t$ and $\varepsilon_t$ are vectors of (independent) white noise shocks; $\hat{E}_t^h y_{t+1}$ are (in general, non-rational) expectations of the vector of endogenous variables by agent $h$; $L_i$, $A_{bf}^h$, $B$, and $\zeta$ are conformable matrices. Further, it is assumed that $F$ (a $k \times k$ matrix) is such that $w_t$ follows a stationary VAR(1) process with $M_w = \lim_{t\to\infty} w_t w_t'$ being a positive definite matrix.

The structural heterogeneity of the model is expressed through matrices $A_{bf}^h = \zeta_h \cdot \tilde{A}_{bf}^h$, with $\zeta_h$ being the mass of each agent type, where $\tilde{A}_{bf}^h$'s (that in the general case are different for different types of agents) contain structural parameters characterizing a given economy, such as the basic characteristics of agents: preferences, technology, endowments, etc. When $\tilde{A}_{bf}^h = A_{bf}$, $\forall\ h$, and $\sum \zeta_h = 1$, the economy is structurally homogenous.

The four classes of models[10] are obtained with the following parameter values: $d = 0$, $m = 1, n$ -any, $A_{0f}^h \equiv 0$ (Model I); $d = 1$, $m = 1, n = 2$, $A_{0f}^h \equiv 0$ (Model II); $d = 0$, $m = 0, n$ -any (Model III); and $d = 1$, $m = 0, n = 1$ (Model IV).

The first group of classes of models considered are the two classes of structurally heterogeneous models with $t-1$-dating of expectations, where the first class (Model I) is presented by models without lags of the endogenous variable

$$\begin{aligned} y_t &= \alpha + \sum_{h=1}^{S} A_0^h \hat{E}_{t-1}^h y_t + \sum_{h=1}^{S} A_1^h \hat{E}_{t-1}^h y_{t+1} + \ldots + \sum_{h=1}^{S} A_\tau^h \hat{E}_{t-1}^h y_{t+\tau} + B w_t + \zeta \varepsilon_t, \quad (2.3) \\ & \textit{and } (2.2) \end{aligned}$$

and the second class (Model II) is presented by models with one lag of the endogenous variable and one forward-looking term in expectations

$$\begin{aligned} y_t &= \alpha + L y_{t-1} + \sum_{h=1}^{S} A_0^h \hat{E}_{t-1}^h y_t + \sum_{h=1}^{S} A_1^h \hat{E}_{t-1}^h y_{t+1} + B w_t + \zeta \varepsilon_t, \quad (2.4) \\ & \textit{and } (2.2), \end{aligned}$$

with the definitions of variables and matrices being the same as for the general class of structurally heterogeneous linear models with $S$ types of agents with different forecasts above.

The second group of classes of models considered are the two classes of structurally heterogeneous models with $t$-dating of expectations, where the first class (Model

[10] Though my discussion here and in Bogomolova [4] is easily extendable to the general model setup, in order to simplify the exposition of the results, I restrict my analysis to the examples used the most in the literature.

III) is presented by models without lags of the endogenous variable

$$y_t = \alpha + \sum_{h=1}^{S} A_1^h \hat{E}_t^h y_{t+1} + ... + \sum_{h=1}^{S} A_\tau^h \hat{E}_t^h y_{t+\tau} + Bw_t + \zeta\varepsilon_t, \quad \textit{and } (2.2), \tag{2.5}$$

and the second class (Model IV)[11] is presented by models with one lag of the endogenous variable, the $(1, y'_{t-1}, w'_t)$ information set, and one forward-looking term in expectations

$$y_t = \alpha + Ly_{t-1} + \sum_{h=1}^{S} A_1^h \hat{E}_t^h y_{t+1} + Bw_t + \zeta\varepsilon_t, \quad \textit{and } (2.2), \tag{2.6}$$

where the variables and the matrices are defined as above.

In all four classes of models, agents of each type $h$ are assumed to form their expectations $\hat{E}_{t-1}^h y_{t+r}$, $r = 0, 1, ..., \tau$ (in the case of $t-1$-dating of expectations) or $\hat{E}_t^h y_{t+r}$, $r = 1, ..., \tau$ (in the case of $t$-dating of expectations) about the endogenous variables believing that the economic system follows a model called the agents' **perceived law of motion (PLM)** that corresponds to the **minimal state variable (MSV) rational expectations equilibrium (REE)** solution:

$$y_t = a_{h,t-1}^h + b_{h,t-1} w_{t-1} \text{ (for Model I)},$$

$$y_t = a_{h,t-1}^h + b_{h,t-1} y_{t-1} + c_{h,t-1} w_{t-1} \text{ (for Model II)},$$

$$y_t = a_{h,t} + b_{h,t} w_t \text{ (for Model III)},$$

$$y_t = a_{h,t} + b_{h,t} y_{t-1} + c_{h,t} w_t \text{ (for Model IV)}.$$

After plugging the forecasts of each agent based on the corresponding PLM into the reduced form of the model and then equating the parameters of the corresponding mapping (called the $T$-map) from the parameters of the PLM to the parameters of the **actual law of motion (ALM)**, one may obtain the **MSV REE** in each class of models, with our main interest being in the matrix coefficient before the lagged endogenous

[11] In order to keep the presentation of results concise, in this paper, I do not consider the case of the $(1, y'_t, w'_t)$ information set analyzed, for example, in Evans and Honkapohja [22] for a structurally homogeneous economy under homogeneous learning. Instead, I consider a realistic situation where the value of the endogenous variable at time $t$ cannot be used to predict the future value of this variable since it is not yet known. Thus, simultaneity between $y_t$ and $\hat{E}_t^h y_{t+1}$ is avoided. The case of the $(1, y'_t, w'_t)$ information set clearly falls under this paper's technical constructions with some modifications and is a matter for my future research.

variables in Models II and IV

$$L + \sum_{h=1}^{S} A_0^h \bar{b} + \left( \sum_{h=1}^{S} A_1^h \right) \bar{b}^2 = \bar{b}, \tag{2.7}$$

where $A_0^h \equiv 0$ for Model IV.

### 2.2.2 Heterogeneous adaptive learning in various classes of linear models

In all classes of structurally heterogeneous linear models with expectations presented above, it is assumed that agents use heterogeneous **mixed RLS/SG learning** (discussed in the Introduction) when part of the agents, $h = \overline{1, S_0}$, are assumed to use the RLS learning algorithm, while others, $h = \overline{S_0 + 1, S}$, are assumed to use the SG learning algorithm.

For classes of models with $t-1$-dating of expectations (Model I and Model II)

RLS: for $h = \overline{1, S_0}$

$$\Phi_{h,t+1} = \Phi_{h,t} + \alpha_{h,t+1} R_{h,t+1}^{-1} z_t \left( y_{t+1} - \Phi'_{h,t} z_t \right)' \tag{2.8a}$$

$$R_{h,t+1} = R_{h,t} + \alpha_{h,t+1} \left( z_t z'_t - R_{h,t} \right) \tag{2.8b}$$

SG: for $h = \overline{S_0 + 1, S}$

$$\Phi_{h,t+1} = \Phi_{h,t} + \alpha_{h,t+1} z_t \left( y_{t+1} - \Phi'_{h,t} z_t \right)'. \tag{2.9}$$

For classes of models with $t$-dating of expectations (Model III and Model IV)

RLS: for $h = \overline{1, S_0}$

$$\Phi_{h,t+1} = \Phi_{h,t} + \alpha_{h,t+1} R_{h,t+1}^{-1} z_t \left( y_t - \Phi'_{h,t} z_t \right)' \tag{2.10a}$$

$$R_{h,t+1} = R_{h,t} + \alpha_{h,t+1} \left( z_t z'_t - R_{h,t} \right) \tag{2.10b}$$

SG: for $h = \overline{S_0 + 1, S}$

$$\Phi_{h,t+1} = \Phi_{h,t} + \alpha_{h,t+1} z_t \left( y_t - \Phi'_{h,t} z_t \right)', \tag{2.11}$$

where $z'_t = (1, w'_t)$ (Model I and III), $z'_t = (1, y'_t, w'_t)$ (Model II ), $z'_t = (1, y'_{t-1}, w'_t)$ (Model IV); $\Phi'_{h,t} = (a_{h,t}, b_{h,t})$ (Model I and III), $\Phi'_{h,t} = (a_{h,t}, b_{h,t}, c_{h,t})$ (Model II and IV); and where

$$\alpha_{h,t} = \delta_h \alpha_t,$$

where $\alpha_t$ is a deterministic, decreasing, and positive gain sequence that satisfies the usual conditions: $\sum_{t=1}^{\infty}\alpha_t = \infty$, $\sum_{t=1}^{\infty}\alpha_t^2 < \infty$, and $\lim\sup_{t\to\infty}\left[\left(\frac{1}{\alpha_{t+1}}\right)-\left(\frac{1}{\alpha_t}\right)\right] < \infty$, $\delta_h > 0$ are degrees of inertia given here in the formulation of Giannitsarou [31] as constant coefficients before the deterministic decreasing gain sequence in the learning algorithm, which is common for all agents[12].

In Bogomolova [4], I provide the criteria for stability under mixed RLS/SG learning of the MSV REE for all four classes of models in terms of the stability of the corresponding Jacobian matrices. I present these results (without proof) here for the reader's convenience as I use them as a starting pont for the derivation of new results in this paper.

## 2.3 Criteria, sufficient conditions, and the concepts of $HE$- and $\delta$-stability

**Criterion 2.1**[13] *In economy (2.3) and (2.2),* Model I *((2.5) and (2.2),* Model III*), mixed RLS/SG learning converges globally (almost surely) to the minimal state variable (MSV) solution if and only if the corresponding matrices $D_1\Omega$ and $D_w\Omega_F$ have eigenvalues with negative real parts, where*

$$D_1 = diag\left(\delta_1 I_n, ..., \delta_S I_n\right),$$

$$\Omega = \begin{pmatrix} A_0^1 + A_1^1 ... + A_\tau^1 - I_n & \cdots & A_0^S + A_1^S ... + A_\tau^S \\ \vdots & \ddots & \vdots \\ A_0^1 + A_1^1 ... + A_\tau^1 & \cdots & A_0^S + A_1^S ... + A_\tau^S - I_n \end{pmatrix}, \qquad (2.12)$$

$D_w{=}diag\left(D_{w1},...,D_{wS}\right)$*, with* $D_{wh}{=}\delta_h I_{nk}$ *for* $h{=}\overline{1,S_0}$ *and* $D_{wh}{=}\delta_h\left(M_w\otimes I_n\right)$ *for* $h{=}\overline{S_0+1,S}$,

$$\Omega_F{=}\begin{pmatrix} F'^{\tau}\otimes A_\tau^1{+}...{+}F'\otimes A_1^1{+}I_k\otimes A_0^1\text{-}I_{nk} & \cdots & F'^{\tau}\otimes A_\tau^S{+}...{+}F'\otimes A_1^S{+}I_k\otimes A_0^S \\ \vdots & \ddots & \vdots \\ F'^{\tau}\otimes A_\tau^1{+}...{+}F'\otimes A_1^1{+}I_k\otimes A_0^1 & \cdots & F'^{\tau}\otimes A_\tau^S{+}...{+}F'\otimes A_1^S{+}I_k\otimes A_0^S\text{-}I_{nk} \end{pmatrix},$$

*and for Model III,* $A_0^h \equiv 0$.

**Criterion 2.2**[14] *In economy (2.4) and (2.2),* Model II *((2.6) and (2.2),* Model IV*), in which all roots of $\bar{b}$ defined in (2.7) lie inside the unit circle, mixed RLS/SG learning*

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[12] Honkapohja and Mitra [36] use a more general form of degrees of inertia.
[13] that corresponds to Criteria 1.1 and 1.6 in Bogomolova [4]
[14] that corresponds to Criteria 1.2 and 1.7 in Bogomolova [4]

*converges (almost surely) to the minimal state variable (MSV) solution if and only if the corresponding matrix* $D_{1yw}\Omega_{1bF}$ *has eigenvalues with negative real parts, where*
$D_{1yw} = diag\left(D_{1yw1},...,D_{1ywS}\right),$ *with* $D_{1ywh} = \delta_h I_{n+n^2+nk}$ *for* $h = \overline{1,S_0}$ *and* $D_{1ywh} = \delta_h\left(M_{1yw}\otimes I_n\right)$ *for* $h = \overline{S_0+1,S},$

$$\Omega_{1bF} = \begin{bmatrix} R^1 - I_{n+n^2+nk} & R^1 & \cdots & R^1 \\ R^2 & R^2 - I_{n+n^2+nk} & \cdots & R^2 \\ \vdots & \vdots & \ddots & \vdots \\ R^S & R^S & \cdots & R^S - I_{n+n^2+nk} \end{bmatrix},$$

$$R^h = \begin{bmatrix} A_1^h + \left(A_0^h + A_1^h\bar{b}\right) & \bar{a}'\otimes A_1^h & 0 \\ 0 & \bar{b}'\otimes A_1^h + I_n\otimes\left(A_0^h + A_1^h\bar{b}\right) & 0 \\ \vdots & \bar{c}'\otimes A_1^h & F'\otimes A_1^h + I_k\otimes\left(A_0^h + A_1^h\bar{b}\right) \end{bmatrix},$$

*and for Model IV,* $A_0^h \equiv 0.$

These necessary and sufficient stability conditions for all classes of models can be conveniently presented as the requirement for stability of the matrix

$$\Omega_{KR} = \begin{bmatrix} \delta_1\left(R^1\text{-}I\right) & \cdots & \delta_1 R^{S_0} & \delta_1 R^{S_0+1} & \cdots & \delta_1 R^S \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \delta_{S_0}R^1 & \cdots & \delta_{S_0}\left(R^{S_0}\text{-}I\right) & \delta_{S_0}R^{S_0+1} & \cdots & \delta_{S_0}R^S \\ \delta_{S_0+1}KR^1 & \cdots & \delta_{S_0+1}KR^{S_0} & \delta_{S_0+1}\left(KR^{S_0+1}\text{-}K\right) & \cdots & \delta_{S_0+1}KR^S \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \delta_S KR^1 & \cdots & \delta_S KR^{S_0} & \delta_S KR^{S_0+1} & \cdots & \delta_S\left(KR^S\text{-}K\right) \end{bmatrix}, \tag{2.13}$$

where for Model I and Model III (with $A_0^h \equiv 0$)

$$K = \begin{bmatrix} 1 & 0 \\ 0 & M_w \end{bmatrix},\ R^h = \begin{bmatrix} 1 & 0 \\ 0 & F'^{\tau} \end{bmatrix}\otimes A_\tau^h + ... + \begin{bmatrix} 1 & 0 \\ 0 & F' \end{bmatrix}\otimes A_1^h + I_{k+1}\otimes A_0^h,$$

and for Model II and Model IV (with $A_0^h \equiv 0$)

$$K = \left(M_{1yw}\otimes I_n\right),\ M_{1yw} = \begin{bmatrix} M_{1y} & 0 \\ 0 & M_w \end{bmatrix},$$

$$R^h = \begin{bmatrix} A_1^h + \left(A_0^h + A_1^h\bar{b}\right) & \bar{a}'\otimes A_1^h & 0 \\ 0 & \bar{b}'\otimes A_1^h + I_n\otimes\left(A_0^h + A_1^h\bar{b}\right) & 0 \\ \vdots & \bar{c}'\otimes A_1^h & F'\otimes A_1^h + I_k\otimes\left(A_0^h + A_1^h\bar{b}\right) \end{bmatrix}.$$

These conditions are written in terms of a mixture of structural and learning heterogeneity. Similar to Bogomolova [4], here I refer to these conditions for the stability of an REE as **heterogeneous expectational ($HE$-) stability** (or stability in heterogeneous expectations) and refer to the stability of an REE written in terms of structural heterogeneity only as **$\delta$-stability**. Formally, the definition of ***$\delta$-stability*** can be given in the following way.

**Definition 2.1** ***$\delta$-stability*** *is the stability of an REE under heterogeneous (either RLS, SG, or mixed RLS/SG) learning for any positive values of degrees of inertia and any starting values of learning algorithms.*

Note that this definition does not include independence on shares of agents using a particular type of learning algorithm, and in this sense, this definition differs from the one introduced in Bogomolova and Kolyuzhnov [5] and Kolyuzhnov [40].

It is possible to simplify the conditions written above for the diagonal structure of the shocks process

$$F = diag(\rho_1, ..., \rho_k),\ M_w = \lim_{t\to\infty} w_t w_t' = diag\left(\frac{\sigma_1^2}{1-\rho_1^2}, ..., \frac{\sigma_k^2}{1-\rho_k^2}\right), \tag{2.14}$$

and/or for the heterogeneous RLS learning ($S = S_0$), or for the mixed RLS/SG learning with equal degrees of inertia for each type of learning algorithm. Later, it will allow us to derive sufficient conditions, necessary conditions (and criteria in some cases) that have some economic meaning in terms of structural heterogeneity only, that is, conditions for ***$\delta$-stability***. The results from Bogomolova [4] are as follows.

**Proposition 2.3**[15] *(The criterion for the stability of* Model I *(*Model III*) under mixed RLS/SG learning for the diagonal environment case under any (possibly different) degrees of inertia of agents, $\delta > 0$). In the structurally heterogeneous economy (2.3), (2.2), and (2.14), Model I ((2.3), (2.2), and (2.14), Model III), mixed RLS/SG learning converges globally (almost surely) to an MSV REE solution for any (possibly different) degrees of inertia of agents, $\delta > 0$, if and only if matrices $D_1\Omega_{\rho l}$ are stable for any $\delta > 0$, where*

$$D_1 = diag\left(\delta_1 I_n, ..., \delta_S I_n\right), \tag{2.15}$$

$$\Omega_{\rho_l} = \begin{pmatrix} A_0^1 + \rho_l A_1^1 + ... + \rho_l^\tau A_\tau^1 - I_n & \cdots & A_0^S + \rho_l A_1^S + ... + \rho_l^\tau A_\tau^S \\ \vdots & \ddots & \vdots \\ A_0^1 + \rho_l A_1^1 + ... + \rho_l^\tau A_\tau^1 & \cdots & A_0^S + \rho_l A_1^S + ... + \rho_l^\tau A_\tau^S - I_n \end{pmatrix}, \forall l = 0, ..., k, (\rho_0 = 1), \tag{2.16}$$

[15] that corresponds to Propositions 1.4 and 1.9 in Bogomolova [4]

*and for Model III,* $A_0^h \equiv 0$.

**Proposition 2.4**[16] *(Sufficient conditions for the stability of the MSV REE of Model II (IV) under heterogeneous RLS learning for the general (non-diagonal) environment case) In economy (2.4) and (2.2),* Model II *((2.6) and (2.2),* Model IV*), in which all roots of* $\bar{b}$ *defined in (2.7) lie inside the unit circle, heterogeneous RLS learning converges (almost surely) to the minimal state variable (MSV) solution if the corresponding matrices* $D_1\Omega$, $D_y\Omega_b$*, and* $D_w\Omega_F$ *have eigenvalues with negative real parts, where* $D_1$ is given in *(2.15),*

$$\Omega = \begin{pmatrix} A_0^1 + A_1^1 + A_1^1\bar{b} - I_n & \cdots & A_0^S + A_1^S + A_1^S\bar{b} \\ \vdots & \ddots & \vdots \\ A_0^1 + A_1^1 + A_1^1\bar{b} & \cdots & A_0^S + A_1^S + A_1^S\bar{b} - I_n \end{pmatrix}, \tag{2.17}$$

$$D_y = diag\left(\delta_1 I_{n^2}, ..., \delta_S I_{n^2}\right), \tag{2.18}$$

$$\Omega_b = \begin{pmatrix} \bar{b}' \otimes A_1^1 + I_n \otimes \left(A_0^1 + A_1^1\bar{b}\right) - I_{n^2} & \cdots & \bar{b}' \otimes A_1^S + I_n \otimes \left(A_0^S + A_1^S\bar{b}\right) \\ \vdots & \ddots & \vdots \\ \bar{b}' \otimes A_1^1 + I_n \otimes \left(A_0^1 + A_1^1\bar{b}\right) & \cdots & \bar{b}' \otimes A_1^S + I_n \otimes \left(A_0^S + A_1^S\bar{b}\right) - I_{n^2} \end{pmatrix}, \tag{2.19}$$

$$D_w = diag\left(\delta_1 I_{nk}, ..., \delta_S I_{nk}\right), \tag{2.20}$$

$$\Omega_F = \begin{pmatrix} F' \otimes A_1^1 + I_k \otimes \left(A_0^1 + A_1^1\bar{b}\right) - I_{nk} & \cdots & F' \otimes A_1^S + I_k \otimes \left(A_0^S + A_1^S\bar{b}\right) \\ \vdots & \ddots & \vdots \\ F' \otimes A_1^1 + I_k \otimes \left(A_0^1 + A_1^1\bar{b}\right) & \cdots & F' \otimes A_1^S + I_k \otimes \left(A_0^S + A_1^S\bar{b}\right) - I_{nk} \end{pmatrix}, \tag{2.21}$$

*and for Model IV,* $A_0^h \equiv 0$.

**Proposition 2.5**[17] *(Sufficient conditions for the stability of* Model II *(*Model IV*) under heterogeneous RLS learning for the diagonal environment case under any (possibly different) degrees of inertia of agents,* $\delta > 0$*) In the structurally heterogeneous economy (2.4), (2.2), and (2.14), Model II ((2.6), (2.2), and (2.14), Model IV), in which all roots of* $\bar{b}$ *defined in (2.7) lie inside the unit circle, heterogeneous RLS learning converges (almost surely) to an MSV REE solution for any (possibly different) degrees of inertia of agents,*

[16]that corresponds to Corollaries 1.3 and 1.8 in Bogomolova [4]
[17]that corresponds to Propositions 1.5 and 1.10 in Bogomolova [4]

$\delta > 0$, *if matrices* $D_y\Omega_b$ and $D_1\Omega_{\rho_l}$ *are stable for any* $\delta > 0$, *where* $D_1$ *is given in (2.15),*

$$\Omega_{\rho_l} = \begin{pmatrix} A_0^1 + \rho_l A_1^1 + A_1^1\bar{b} - I_n & \cdots & A_0^S + \rho_l A_1^S + A_1^S\bar{b} \\ \vdots & \ddots & \vdots \\ A_0^1 + \rho_l A_1^1 + A_1^1\bar{b} & \cdots & A_0^S + \rho_l A_1^S + A_1^S\bar{b} - I_n \end{pmatrix}, \forall l = 0, ..., k, (\rho_0 = 1), \tag{2.22}$$

*and for Model IV,* $A_0^h \equiv 0$.

**Proposition 2.6**[18] *(The criterion for stability under mixed RLS/SG learning with equal degrees of inertia of agents for each type of learning algorithm,* $\delta > 0$*) In the structurally heterogeneous economy (2.3) and (2.2), Model I ((2.5) and (2.2), Model III) or (2.4) and (2.2), Model II ((2.6) and (2.2), Model IV), in which all roots of* $\bar{b}$ *defined in (2.7) lie inside the unit circle, mixed RLS/SG learning with equal degrees of inertia of agents for each type of learning algorithm,* $\delta_i = \delta^{RLS}$, $\forall i = 1, ..., S_0$, $\delta_i = \delta^{SG}$, $\forall i = S_0 + 1, ..., S$, *converges globally (for Models I and III) or locally (for Models II and IV) (almost surely) to an MSV REE, if and only if the REE is a locally asymptotically stable fixed point of the corresponding average economy with two agents, defined for the general setup (2.1) and (2.2) as*

$$y_t = \alpha + \sum_{i=1}^{d} L_i y_{t-i} + \sum_{b=0}^{m}\sum_{f=b}^{n}\left(\sum_{h=1}^{S_0} A_{bf}^h\right)\hat{E}_{t-b}^{RLS} y_{t-b+f} + \tag{2.23}$$

$$+ \sum_{b=0}^{m}\sum_{f=b}^{n}\left(\sum_{h=S_0+1}^{S} A_{bf}^h\right)\hat{E}_{t-b}^{SG} y_{t-b+f} + Bw_t + \zeta\varepsilon_t,$$

*and (2.2),* $A_{00}^h \equiv 0$,

*where the agent with coefficients* $\left(\sum_{h=1}^{S_0} A_{bf}^h\right)$ *learns by RLS with the degree of inertia* $\delta^{RLS}$, *and the agent with coefficients* $\left(\sum_{h=S_0+1}^{S} A_{bf}^h\right)$ *learns by SG with the degree of inertia* $\delta^{SG}$.

As it is clear from the Propositions and Criteria above for the general and simpler cases, the problem of finding conditions for stability of the corresponding Jacobians results in finding conditions for stability of a matrix (matrices) of $D\Omega$ type, where $D$ is a positive diagonal matrix. The problem of $D$-stability was studied, for example, in Johnson [37]. There are several approaches considered to tackle this problem, e.g., in Bogomolova and Kolyuzhnov [5] and Kolyuzhnov [40]. Among these approaches are the negative diagonal

[18] that corresponds to Proposition 1.11 and Definition 1.2 in Bogomolova [4]

dominance approach[19], the alternative definition of $D$-stability approach[20], the characteristic equation approach, the Routh-Hurwitz conditions[21], and an approach based on the Lyapunov Theorem[22]. The description and dicussion of these approaches can be found, for example, in Bogomolova [4] (see Apppendix A.2 of that paper for formal definitions).

While in that companion paper I essentially use the negative diagonal dominance approach, which allows me to derive the sufficient condition for the $D$-stability of matrices due to the MacKenzie Theorem[23] and finally allows me to obtain sufficient conditions written in terms of $E$-stability of suitably defined maximal aggregate economies for all four classes of models for general and simpler cases, here I look at the problem from a different angle and try to find conditions for stability that are not possible to derive using that approach.

## 2.4 The alternative definition of $D$-stability approach and the "same sign" sufficient conditions for $\delta$-stability

### 2.4.1 Alternative criteria and sufficient conditions

I use the alternative definition of $D$-stability approach, which means that the problem of $D$-stability can be equivalently substituted with the problem of finding conditions for stability of $\Omega$ and checking that $i$ is not an eigenvalue of some specially constructed matrix. Moreover, matrix $D$ in all the models considered is not only a positive diagonal matrix, but its diagonal consists of blocks of the same numbers of equal length. It has allowed Kolyuzhnov [40] to introduce a definition of blocked $D$-stability ($D_b$-stability) and an alternative definition of it.

**Definition 2.2** *($D_b$-stability) Matrix $A$ of size $nS \times nS$ is $D_b$-stable if $D_bA$ is stable for any positive blocked-diagonal matrix $D_b = diag(\delta_1, ..., \delta_1, ..., \delta_S, ..., \delta_S)$.*

**Proposition 2.7** *(An alternative definition of $D_b$-stability) Consider $M_{nS}(C)$, the set of all complex $nS \times nS$ matrices, and $D_{bnS}$, the set of all $nS \times nS$ blocked–diagonal matrices with positive diagonal entries. Take $A \in M_{nS}(C)$ and suppose there is $F \in D_{bnS}$ such that $FA$ is stable. Then, $A$ is $D_b$-stable if and only if $A \pm iD_b$ is non–singular for all*

[19] See Theorem A.4 in Appendix A.4.
[20] See Theorem A.6 in Appendix A.6.
[21] See Theorem A.5 in Appendix A.5.
[22] See Theorem A.2 in Appendix A.3.
[23] See Theorem A.4 in Appendix A.4.

*$D_b \in D_{bnS}$. If $A \in M_{nS}(R)$ — the set of all $nS \times nS$ real matrices, then "$\pm$" in the above condition may be replaced with "+" since, for a real matrix, any complex eigenvalues come in conjugate pairs.*

Taking $F$ as an identity matrix and $D$ as $diag(\frac{1}{\delta_1}, ..., \frac{1}{\delta_1}, ..., \frac{1}{\delta_S}, ..., \frac{1}{\delta_S})$, $\delta_h > 0, h = \overline{1, S}$, in the alternative definition to $D_b$-stability above, I obtain the following necessary and sufficient conditions (alternative criteria) for stability under heterogeneous mixed RLS/SG learning for all four classes of models considered in the general (non-diagonal) case:

**Criterion 2.8** *In the structurally heterogeneous economy (2.3) and (2.2), Model I ((2.5) and (2.2), Model III) or (2.4) and (2.2), Model II ((2.6) and (2.2), Model IV), in which all roots of $\bar{b}$ defined in (2.7) lie inside the unit circle, mixed RLS/SG learning converges globally (almost surely) to an MSV REE if and only if the corresponding matrix $\Omega_{KR}$, defined in (2.13), is stable and*

$$\det\left[\sum_{h=1}^{S_0}\left(\frac{-R_h}{I-\frac{i}{\delta_h}I}\right)+\sum_{h=S_0+1}^{S}\left(\frac{-KR_h}{K-\frac{i}{\delta_h}I}\right)+I\right]=$$

$$=\det\left[\left(\sum_{h=1}^{S_0}\frac{1}{I+\frac{1}{\delta_h^2}I}(-R_h)+\sum_{h=S_0+1}^{S}\frac{K}{K^2+\frac{1}{\delta_h^2}I}(-R_h)+I\right)+\right.$$

$$\left.+i\left(\sum_{h=1}^{S_0}\frac{\frac{1}{\delta_h}}{I+\frac{1}{\delta_h^2}I}(-R_h)+\sum_{h=S_0+1}^{S}\frac{\frac{1}{\delta_h}K}{K^2+\frac{1}{\delta_h^2}I}(-R_h)\right)\right]\neq 0$$

$$\forall\delta_h>0, h=\overline{1,S},$$

*where to shorten notation $\frac{A}{B}$ means $B^{-1}A$.*

**Proof.** See Appendix B.1.1. □

The alternative definition to $D_b$-stability approach allows me to write down simplified alternative stability criteria (to criteria in terms of the corresponding Jacobian matrices [*given in Proposition 2.3 here*] in Kolyuzhnov [40] and Bogomolova [4]) for models without lags of both types (Models I and III) in the case of the diagonal structure of the shock behaviour.

**Proposition.2.9** *(An alternative criterion for the stability of models without lags (*Model I *and Model III) under mixed RLS/SG learning for the diagonal environment case under* *any (possibly different) degrees of inertia of agents, $\delta > 0$) In the structurally heterogeneous economy (1.3), (1.2), and (1.22), mixed RLS/SG learning (2.8), (1.20), and (1.4) converges globally (almost surely) to an MSV REE solution for any (possibly different) degrees of inertia of agents, $\delta > 0$, if and only if the corresponding matrix $\Omega_{\rho_l}$, defined in (1.24), is stable and*

$$\det\left[\sum_{h=1}^{S}\left(\frac{-(A_0^h+\rho_l A_1^h+...+\rho_l^\tau A_\tau^h)}{1+\frac{i}{\delta_h}}\right)+I\right]=$$

$$=\det\left[\left(\sum_{h=1}^{S}\frac{-1}{1+\frac{1}{\delta_h^2}}\left(A_0^h+\rho_l A_1^h+...+\rho_l^\tau A_\tau^h\right)+I\right)+\right.$$

$$\left.+i\left(\sum_{h=1}^{S}\frac{-\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}}\left(A_0^h+\rho_l A_1^h+...+\rho_l^\tau A_\tau^h\right)\right)\right]\neq 0,$$

$$\forall\delta_h>0, h=\overline{1,S}, \forall l=0,1,...,k, (\rho_0=1).$$

*For the univariate case ($n = 1$), this condition simplifies to $\Omega_{\rho_l}$ — stable and*

$$\left(\sum_{h=1}^{S}\frac{-1}{1+\frac{1}{\delta_h^2}}\left(A_0^h+\rho_l A_1^h+...+\rho_l^\tau A_\tau^h\right)+1\right)\neq 0 \text{ or } \sum_{h=1}^{S}\frac{-\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}}\left(A_0^h+\rho_l A_1^h+...+\rho_l^\tau A_\tau^h\right)\neq 0,$$

$$\text{or both}, \forall\delta_h > 0, h=\overline{1,S}, \forall l=0,1,...,k, (\rho_0=1),$$

*where $A_0^h \equiv 0$ for model III everywhere above.*

**Proof.** See Appendix B.1.2. □

This approach also allows me to write down alternative sufficient stability conditions (to the sufficient conditions in terms of the corresponding Jacobian matrices [*given in Proposition 2.4 here*] in Bogomolova [4]) for models with lags of the endogenous variable of both types (Models II and IV) in the case of the general (non-diagonal) structure of shocks under heterogeneous RLS learning — conditions written in terms of structural and learning heterogeneity.

**Proposition.2.10** *(Alternative sufficient conditions for stability of models with lags (*Model II, IV*) under heterogeneous RLS learning for the general (non-diagonal) environment case under any (possibly different) degrees of inertia of agents, $\delta > 0$) In the structurally heterogeneous economy (2.4) and (2.2), Model II ((2.6) and (2.2), Model IV), in which all roots of $\bar{b}$ defined in (2.7) lie inside the unit circle, heterogeneous RLS learning converges (almost surely) to an MSV REE solution for any (possibly different) degrees* *of inertia of agents, $\delta > 0$, if the corresponding matrices $\Omega$, $\Omega_b$, and $\Omega_F$ defined in (2.17), (2.19), and (2.21), respectively, are stable and*

$$\det\left[\sum_{h=1}^{S}\left(\frac{-(A_0^h+A_1^h+A_1^h\bar{b})}{1+\frac{i}{\delta_h}}\right)+I\right]=$$

$$=\det\left[\left(\sum_{h=1}^{S}\frac{-1}{1+\frac{1}{\delta_h^2}}\left(A_0^h+A_1^h+A_1^h\bar{b}\right)+I\right)+i\left(\sum_{h=1}^{S}\frac{-\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}}\left(A_0^h+A_1^h+A_1^h\bar{b}\right)\right)\right]\neq 0$$

$$\forall\delta_h>0, h=\overline{1,S};$$

*and*

$$\det\left[\sum_{h=1}^{S}\left(\frac{-(\bar{b}'\otimes A_1^h+I_n\otimes\left(A_0^h+A_1^h\bar{b}\right))}{1+\frac{i}{\delta_h}}\right)+I\right]=$$

$$=\det\left[\left(\sum_{h=1}^{S}\frac{-1}{1+\frac{1}{\delta_h^2}}\left(\bar{b}'\otimes A_1^h+I_n\otimes\left(A_0^h+A_1^h\bar{b}\right)\right)+I\right)+\right.$$

$$\left.+i\left(\sum_{h=1}^{S}\frac{-\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}}\left(\bar{b}'\otimes A_1^h+I_n\otimes\left(A_0^h+A_1^h\bar{b}\right)\right)\right)\right]\neq 0$$

$$\forall\delta_h>0, h=\overline{1,S};$$

*and*

$$\det\left[\sum_{h=1}^{S}\left(\frac{-(F'\otimes A_1^h+I_k\otimes\left(A_0^h+A_1^h\bar{b}\right))}{1+\frac{i}{\delta_h}}\right)+I\right]=$$

$$=\det\left[\left(\sum_{h=1}^{S}\frac{-1}{1+\frac{1}{\delta_h^2}}\left(F'\otimes A_1^h+I_k\otimes\left(A_0^h+A_1^h\bar{b}\right)\right)+I\right)+\right.$$

$$\left.+i\left(\sum_{h=1}^{S}\frac{-\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}}\left(F'\otimes A_1^h+I_k\otimes\left(A_0^h+A_1^h\bar{b}\right)\right)\right)\right]\neq 0$$

$$\forall\delta_h>0, h=\overline{1,S}.$$

*For the case $n=1$, $k=1$, this condition simplifies to $\Omega$, $\Omega_b$, and $\Omega_F$ — stable and*

$$\left(\sum_{h=1}^{S}\frac{-1}{1+\frac{1}{\delta_h^2}}\left(A_0^h+A_1^h+A_1^h\bar{b}\right)+1\right)\neq 0 \text{ or } \sum_{h=1}^{S}\frac{-\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}}\left(A_0^h+A_1^h+A_1^h\bar{b}\right)\neq 0,$$
$$\text{or both}, \forall\delta_h > 0, h=\overline{1,S};$$

*and*

$$\left(\sum_{h=1}^{S}\frac{-1}{1+\frac{1}{\delta_h^2}}\left(A_0^h+2A_1^h\bar{b}\right)+1\right)\neq 0 \text{ or } \sum_{h=1}^{S}\frac{-\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}}\left(A_0^h+2A_1^h\bar{b}\right)\neq 0,$$
$$\text{or both}, \forall\delta_h > 0, h=\overline{1,S};$$

*and*

$$\left(\sum_{h=1}^{S}\frac{-1}{1+\frac{1}{\delta_h^2}}\left(A_0^h+A_1^h\rho+A_1^h\bar{b}\right)+1\right) \neq 0 \text{ or } \sum_{h=1}^{S}\frac{-\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}}\left(A_0^h+A_1^h\rho+A_1^h\bar{b}\right)\neq 0,$$
$$\text{or both}, \forall\delta_h > 0, h=\overline{1,S}.$$

*For Model IV,* $A_0^h \equiv 0$ *everywhere above.*

**Proof.** See Appendix B.1.3. □

Sufficient conditions written above can be simplified further by assuming also the diagonal environment setting (2.14). This results in alternative sufficient stability conditions (to sufficient conditions in terms of the corresponding Jacobian matrices [*given in Proposition 2.5 here*] in Bogomolova [4]) for models with lags of the endogenous variable of both types (Models II and IV) in the case of the diagonal structure of shocks under heterogeneous RLS learning — conditions written in terms of structural and learning heterogeneity .

**Proposition.2.11** *(Alternative sufficient conditions for stability of models with lags (*Model II, IV*) under heterogeneous RLS learning for the diagonal environment case under any (possibly different) degrees of inertia of agents,* $\delta > 0$*) In the structurally heterogeneous economy (2.4), (2.2), and (2.14), Model II ((2.6), (2.2), and (2.14), Model IV), in which all roots of* $\bar{b}$ *defined in (2.7) lie inside the unit circle, heterogeneous RLS learning converges (almost surely) to an MSV REE solution for any (possibly different) degrees of inertia of agents,* $\delta > 0$*, if the corresponding matrices* $\Omega_{\rho_l}$ *and* $\Omega_b$*, defined in (2.22) and (2.19), are stable and*

$$\det\left[\sum_{h=1}^{S}\left(\frac{-(A_0^h+\rho_l A_1^h+A_1^h\bar{b})}{1+\frac{i}{\delta_h}}\right)+I\right]=$$

$$=\det\left[\left(\sum_{h=1}^{S}\frac{-1}{1+\frac{1}{\delta_h^2}}\left(A_0^h+\rho_l A_1^h+A_1^h\bar{b}\right)+I\right)+i\left(\sum_{h=1}^{S}\frac{-\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}}\left(A_0^h+\rho_l A_1^h+A_1^h\bar{b}\right)\right)\right]\neq 0$$
$$\forall\delta_h>0, h=\overline{1,S}, \forall l=0,1,...,k, (\rho_0=1),$$

*and*

$$\det\left[\sum_{h=1}^{S}\left(\frac{-(\bar{b}'\otimes A_1^h+I_n\otimes\left(A_0^h+A_1^h\bar{b}\right))}{1+\frac{i}{\delta_h}}\right)+I\right]=$$
$$=\det\left[\left(\sum_{h=1}^{S}\frac{-1}{1+\frac{1}{\delta_h^2}}\left(\bar{b}'\otimes A_1^h+I_n\otimes\left(A_0^h+A_1^h\bar{b}\right)\right)+I\right)+\right.$$

$$+i\left(\sum_{h=1}^{S}\frac{-\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}}\left(\bar{b}'\otimes A_1^h+I_n\otimes\left(A_0^h+A_1^h\bar{b}\right)\right)\right)\Bigg]\neq 0$$

$$\forall\delta_h>0, h=\overline{1,S}.$$

*For the univariate case ($n=1$), this condition simplifies to* $\Omega_{\rho_l}$ and $\Omega_b$ *— stable and*

$$\begin{aligned}\left(\sum_{h=1}^{S}\frac{-1}{1+\frac{1}{\delta_h^2}}\left(A_0^h+\rho_l A_1^h+A_1^h\bar{b}\right)+1\right) &\neq 0 \text{ or } \sum_{h=1}^{S}\frac{-\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}}\left(A_0^h+\rho_l A_1^h+A_1^h\bar{b}\right)\neq 0,\\ \text{or both}, \forall\delta_h &> 0, h=\overline{1,S}, \forall l=0,1,...,k, (\rho_0=1),\end{aligned}$$

*and*

$$\begin{aligned}\left(\sum_{h=1}^{S}\frac{-1}{1+\frac{1}{\delta_h^2}}\left(A_0^h+2A_1^h\bar{b}\right)+1\right) &\neq 0 \text{ or } \sum_{h=1}^{S}\frac{-\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}}\left(A_0^h+2A_1^h\bar{b}\right)\neq 0,\\ \text{or both}, \forall\delta_h &> 0, h=\overline{1,S}.\end{aligned}$$

*For Model IV, $A_0^h\equiv 0$ everywhere above.*

**Proof.** See Appendix B.1.4. □

### 2.4.2 The "same sign" sufficient conditions for $\delta$-stability

The alternative definition of $D_b$-stability approach allows me to derive "same sign" conditions, considered by Honkapohja and Mitra [36] for lower dimension cases $n=1,2$, and necessary and sufficient conditions for $\delta$-stability for $n=1$.

The first (companion) paper did not provide a criterion for $\delta$-stability in the general heterogeneous mixed RLS/SG learning case. Kolyuzhnov [40] provides a criterion only for the univariate model of the forward-looking type with one expectation lead and no lags of the endogenous variable. Here, I provide criteria for $\delta$-stability for univariate models (with either $t$- or $t-1$-dating of expectations) without lags of the endogenous variable (Models I and III) under mixed RLS/SG learning in economically meaningful terms such as subeconomies.

**Definition 2.3** ***A subeconomy*** $(h_1,...,h_p)$ *of size $p$ for economy (2.1) and (2.2) is defined as consisting only of a part of agents from the original economy:*

$$y_t = \alpha+\sum_{i=1}^{d}L_i y_{t-i}+\sum_{k=1}^{p}\sum_{b=0}^{m}\sum_{f=b}^{n}A_{bf}^{h_k}\hat{E}_{t-b}^{h_k}y_{t-b+f}+Bw_t+\zeta\varepsilon_t,\ A_{00}^{h_k}\equiv 0, \quad (2.24)$$

*(2.2),*

*where* $(h_1, ..., h_p) \subseteq (1, ..., S)$ *is a set of numbers of agent types present in the subeconomy. A single economy is a particular case of a subeconomy with only one type of agent.*

Before formulating the criteria for $\delta$-stability for univariate Models I and III under mixed RLS/SG learning in the diagonal environment case, first note that the stability properties of the MSV REE under mixed RLS/SG learning of **Model I** in the diagonal environment case is equivalent to the ones of the MSV REE of ***the set of associated current value expectations models***[24]

$$\begin{aligned} y_t &= \alpha + \sum_{h=1}^{S} (A_0^h + \rho_l A_1^h + ... + \rho_l^\tau A_\tau^h) \hat{E}_{t-1}^h y_t + B w_t + \zeta \varepsilon_t, \\ & \textit{and } (1.2). \end{aligned} \tag{2.25}$$

Similarly, the stability properties of the MSV REE under mixed RLS/SG learning of **Model III** in the diagonal environment case is equivalent to the ones of the MSV REE of ***the set of associated current value expectations models***

$$\begin{aligned} y_t &= \alpha + \sum_{h=1}^{S} (\rho_l A_1^h + ... + \rho_l^\tau A_\tau^h) \hat{E}_{t-1}^h y_t + B w_t + \zeta \varepsilon_t, \\ & \textit{and } (1.2). \end{aligned} \tag{2.26}$$

The next criterion is formulated in terms of ***this set of associated current value expectations models*** as well.

**Proposition 2.12** *(A criterion for $\delta$-stability in the univariate case of models without lags,* Model I *and Model III, under mixed RLS/SG learning for the diagonal environment case) In the case of* $n = 1$, *the structurally heterogeneous economy (2.3), (2.2), and (2.14), Model I ((2.5), (2.2), and (2.14), Model III) under mixed RLS/SG learning is $\delta$-stable if and only if each economy from the set of associated current value expectations models corresponding to Model I (III) is $E$-stable (that is, the corresponding matrix* $\Omega_{\rho_l}$, *defined in* (2.16), *is stable) and for each economy from the set of associated current value expectations models corresponding to Model I (III), at least one of the following holds true: the "same sign" condition (all* $\left(A_0^h + \rho_l A_1^h + ... + \rho_l^\tau A_\tau^h\right)$ *are greater than or equal to zero, and at least one is strictly greater than zero, or all* $\left(A_0^h + \rho_l A_1^h + ... + \rho_l^\tau A_\tau^h\right)$

[24] The same definition is given in Bogomolova [4]. It has turned out to be useful in deriving sufficient conditions in terms of $E$-stability of a suitably defined maximal aggregate economy.

*are less than or equal to zero, and at least one is strictly less than zero), or all average economies with* $(A_0 + \rho_l A_1 + ... + \rho_l^\tau A_\tau)_{(h_1,...,h_p)} = \sum_{(h_1,...,h_p)} \left(A_0^h + \rho_l A_1^h + ... + \rho_l^\tau A_\tau^h\right)$ *corresponding to subeconomies* $(h_1, ..., h_p)$ *of all sizes* $p$ *are not* $E$*-unstable, and for each* $l = 0, 1, ..., k$ $(\rho_0 = 1)$ *there exists at least one average economy corresponding to subeconomy* $(h_1^*(l), ..., h_p^*(l))$ *in each size* $p$ *for which the stability coefficient*

$$\sum_{(h_1^*(l),...,h_p^*(l))} \left(A_0^h + \rho_l A_1^h + ... + \rho_l^\tau A_\tau^h\right)$$

*is strictly less than one. For Model III,* $A_0^h \equiv 0$ *everywhere above.*

**Proof.** See Appendix B.1.5. □

I also provide quite weak, economically tractable, sufficient conditions for $\delta$-stability for univariate models with a lag of the endogenous variable (Models II and IV) under heterogeneous RLS learning. Before formulating these conditions for $\delta$-stability for univariate Models II and IV under heterogeneous RLS learning in the diagonal environment case, I have to formulate several definitions from the companion paper as it has turned out that "same sign" conditions can be applied to the same sets of aggregate economies considered there.

First note that for models with a lag of the endogenous variable (Models II and IV), one can construct an economy without a lag corresponding to the model considered that has the same asymptotic behaviour around the REE. In Bogomolova [4], I call this model (by analogy to the associated ODE) the associated "unlagged" economy. With respect to Models II and IV, ***the associated "unlagged" model corresponding to Model II and IV*** is given in the following proposition (for the proof see Bogomolova [4]).

**Proposition 2.13** ***The associated "unlagged" model corresponding to Model II (IV)*** *(that is, the model that has the same asymptotic behaviour as Model II (IV), where the component for the lag coefficient is fixed at the MSV REE value) is the model: for Model II:*

$$y_t = \alpha + \sum_{h=1}^{S} \left(A_0^h + A_1^h \bar{b}\right) \hat{E}_{t-1}^h y_t + \sum_{h=1}^{S} A_1^h \hat{E}_{t-1}^h y_{t+1} + B w_t + \zeta \varepsilon_t,$$

*and (2.2);*

*for Model IV:*

$$y_t = \alpha + \sum_{h=1}^{S} \left(A_1^h \bar{b}\right) \hat{E}_{t-1}^h y_t + \sum_{h=1}^{S} A_1^h \hat{E}_{t-1}^h y_{t+1} + BF^{-1} w_t + \zeta \varepsilon_t,$$

*and (2.2),*

*where $\bar{b}$ is defined in (2.7).*

**The set of associated current value expectations models for this associated "unlagged" economy** is then naturally defined as for models without lags (Models I and III) above. This set may also be considered as the set of aggregate models of the associated "unlagged" economy, where aggregation is done to reduce the number of forward-looking terms in expectations using $\rho_l$'s as discounting factors. However, such discounting can also be done using elements of $\bar{b}$ instead of $\rho_l$'s since the endogenous variable at the REE depends on its lag with matrix coefficient $\bar{b}$. In the univariate case, the choice of elements of $\bar{b}$ to be used for discounting is obvious. One has to replace $\rho_l$'s with $\bar{b}$ in the definition of the set of aggregate models above to obtain the **$b$-type current value aggregate expectation model for the associated "unlagged" economy of the model with a lag, Model II (Model IV)**. The following sufficient conditions are formulated in terms of the properties of these aggregate economies and their subeconomies.

**Proposition 2.14** *(Sufficient conditions for $\delta$-stability in the univariate case of Models with lags (*Model II, IV*) under heterogeneous RLS learning for the diagonal environment case) In the case of $n = 1$, the structurally heterogeneous economy (2.4), (2.2), and (2.14), Model II ((2.6), (2.2), and (2.14), Model IV), in which the value of $\bar{b}$ defined in (2.7) is less than one, under heterogeneous RLS learning is $\delta$-stable if each economy from the set of associated current value expectations models and the b-type current value aggregate expectation model for the associated "unlagged" economy of the model with a lag, Model II (Model IV), are E-stable (that is, the corresponding matrices $\Omega_{\rho_l}$ and $\Omega_b$, defined in (2.22) and (2.19), are stable)* ***and*** *for each economy from the set of associated current value expectations models and for the b-type current value aggregate expectation model for the associated "unlagged" economy of the model with a lag, Model II (Model IV), at least one of the following holds true: the "same sign" condition (all $\left(A_0^h + \rho_l A_1^h + A_1^h \bar{b}\right)$ (for b-type $\left(A_0^h + 2A_1^h \bar{b}\right)$ ) are greater than or equal to zero, and at least one is strictly greater than zero or all $\left(A_0^h + \rho_l A_1^h + A_1^h \bar{b}\right)$ are less than or equal to zero, and at least one is strictly less than zero), or all average economies with $\left(A_0^h + \rho_l A_1^h + A_1^h \bar{b}\right)_{(h_1,\ldots,h_p)} =$*

$\sum\limits_{(h_1,...,h_p)} \left(A_0^h + \rho_l A_1^h + A_1^h \bar{b}\right)$.*(For b-type* $\left(A_0^h + 2A_1^h \bar{b}\right)_{(h_1,...,h_p)} = \sum\limits_{(h_1,...,h_p)} \left(A_0^h + 2A_1^h \bar{b}\right)$ *) corresponding to subeconomies* $(h_1, ..., h_p)$ *of all sizes p are not E-unstable and for each aggregate economy there exists at least one average economy corresponding to subeconomy* $(h_1^*(l), ..., h_p^*(l))$ *in each size p for which the stability coefficient* $\sum\limits_{(h_1^*(l),...,h_p^*(l))} \left(A_0^h + \rho_l A_1^h + A_1^h \bar{b}\right)$ *(for b-*type $\sum\limits_{(h_1^*(l),...,h_p^*(l))} \left(A_0^h + 2A_1^h \bar{b}\right)$ *) is strictly less than one. For Model IV,* $A_0^h \equiv 0$ *everywhere above.*

**Proof.** See Appendix B.1.6. □

**Proposition 2.15** *In the case of* $n = 2$*, the structurally heterogeneous economy (1.3), (1.2), and (1.22), Model I (III), under mixed RLS/SG learning (2.8), (1.20), and (1.4) in the diagonal environment case is* $\delta$*-stable, if each economy from the set of associated current value expectations models corresponding to Model I (III) is E-stable (that is, the corresponding matrix* $\Omega_{\rho_l}$*, defined in (*1.24*), is stable) and the following "same sign" condition for each of these economies holds true:*

$$\{\det\left(-\rho_l\left(A_0^i + \rho_l A_1^i + ... + \rho_l^\tau A_\tau^i\right)\right) \geq 0,$$

$$\Big[\det mix\left(-\rho_l\left(A_0^i + \rho_l A_1^i + ... + \rho_l^\tau A_\tau^i\right), -\rho_l\left(A_0^j + \rho_l A_1^j + ... + \rho_l^\tau A_\tau^j\right)\right) +$$

$$+ \det mix\left(-\rho_l\left(A_0^j + \rho_l A_1^j + ... + \rho_l^\tau A_\tau^j\right), -\rho_l\left(A_0^i + \rho_l A_1^i + ... + \rho_l^\tau A_\tau^i\right)\right)\Big] \geq 0, i \neq j,$$

$$M_1(-\rho_l\left(A_0^i + \rho_l A_1^i + ... + \rho_l^\tau A_\tau^i\right)) \geq 0\}$$

*or*

$$\{\det\left(-\rho_l\left(A_0^i + \rho_l A_1^i + ... + \rho_l^\tau A_\tau^i\right)\right) \leq 0,$$

$$\Big[\det mix\left(-\rho_l\left(A_0^i + \rho_l A_1^i + ... + \rho_l^\tau A_\tau^i\right), -\rho_l\left(A_0^j + \rho_l A_1^j + ... + \rho_l^\tau A_\tau^j\right)\right) +$$

$$+ \det mix\left(-\rho_l\left(A_0^j + \rho_l A_1^j + ... + \rho_l^\tau A_\tau^j\right), -\rho_l\left(A_0^i + \rho_l A_1^i + ... + \rho_l^\tau A_\tau^i\right)\right)\Big] \leq 0, i \neq j,$$

$$M_1(-\rho_l\left(A_0^i + \rho_l A_1^i + ... + \rho_l^\tau A_\tau^i\right)) \leq 0\}$$

$$\forall l = 0, 1, ..., k, (\rho_0 = 1),$$

*where* $mix\left(-\rho_l A_i, -\rho_l A_j\right)$ *denotes a matrix of structural parameters of a pairwise–mixed economy and is composed by mixing columns of a pair of matrices* $\rho_l A_i$*,* $\rho_l A_j$*, for any i,* $j = \overline{1, S}$*. For Model III* $A_0^h \equiv 0$ everywhere above.

**Proof.** See Appendix B.1.7. □

## 2.5 Necessary Conditions for $HE$- and $\delta$-stability

In addition, using **the characterisic equation** approach, for the case of the general (non-diagonal) structure of shocks, I provide (quite strong) necessary conditions for stability and $\delta$-stability in terms of structural and learning heterogeneity under heterogeneous mixed RLS/SG learning for all types of models considered in terms of stability of a suitably defined structurally heterogeneous average economy under the heterogeneous learning of two agents.

**Proposition 2.16** *(A universal necessary condition for stability under mixed RLS/SG learning) For the structurally heterogeneous economy (2.3) and (2.2), Model I ((2.5) and (2.2), Model III) or (2.4) and (2.2), Model II ((2.6) and (2.2), Model IV), in which all roots of* $\bar{b}$ *defined in (2.7) lie inside the unit circle, under mixed RLS/SG learning in the general (non-diagonal) case, to be* $\delta$*-stable, it is necessary that the corresponding average economy (2.23) and (2.2) (with the same MSV REE) under mixed RLS/SG learning of two agents, where one learns by RLS with the degree of inertia* $\delta^{RLS}$ *and the other by SG with the degree of inertia* $\delta^{SG}$*, is stable for any positive values of* $\delta^{RLS}$ *and* $\delta^{SG}$*.*

**Proof.** Follows directly from Proposition 2.6. □

Using **the characterisic equation** approach, for the case of the general (non-diagonal) structure of shocks, I also provide necessary conditions for $HE$-stability and $\delta$-stability for models without lags of the endogenous variables (Model I and III) under heterogeneous mixed RLS/SG learning in terms of subeconomies for an economy from a set of associated current value expectations models.

**Proposition 2.17** *A necessary condition for* $HE$*-stability of models without lags,* Model I *and Model III, under mixed RLS/SG learning in the general (non-diagonal) environment case: For the structurally heterogeneous economy (2.3) and (2.2), Model I ((2.5) and (2.2), Model III) under mixed RLS/SG learning to be* $HE$*-stable, it is necessary that all sums of the same-size principal minors of* $D_{1r}\,(-\Omega_r)$ *are nonnegative for all subeconomies* $r = (h_1, ..., h_p)$ *for all* $p$ *of the economy from the set of associated current value expectations models corresponding to Model I (III) with* $\rho_0 = 1$ *for all positive block–diagonal matrices* $D_{1r}$*, where* $D_{1r}$ *and* $\Omega_r$ *defined similar to* $D_1$ *and* $\Omega$ *in (2.15) and (2.12) (where* $A_0^h \equiv 0$ *for Model III), respectively, correspond to a subeconomy* $r = (h_1, ..., h_p)$*.*

**Proof.** See Appendix B.1.8. □

The condition above can not be used as a test for non-$\delta$-stability, as it requires checking all subeconomies' sums of minors for all possible $D_{1r}$. That is why I present below a condition that has a direct testing application.

**Proposition 2.18** *A necessary condition for $\delta$-stability of Models without lags,* Model I *and Model III, under mixed RLS/SG learning in the general (non-diagonal) environment case: For the structurally heterogeneous economy (2.3) and (2.2), Model I ((2.5) and (2.2), Model III) under mixed RLS/SG learning to be $\delta$-stable, it is necessary that all sums of the same-size principal minors of minus matrices corresponding to subeconomies $(-\Omega_r)$ of the economy from the set of associated current value expectations models corresponding to Model I (III) with $\rho_0 = 1$ are non-negative for each corresponding subeconomy $r = (h_1, ..., h_p)$.*

**Proof.** See Appendix B.1.8. □

Stronger sufficient conditions can be derived for $HE$-stability and $\delta$-stability for models without lags of the endogenous variables (Model I and III) under heterogeneous mixed RLS/SG learning in the case of the diagonal structure of the shock behaviour in terms of subeconomies for economies from a set of associated current value expectations models.

**Proposition 2.19** *A necessary condition for $HE$–stability of Models without lags,* Model I *and Model III, under mixed RLS/SG learning in the diagonal environment case: For the structurally heterogeneous economy (2.3) and (2.2), Model I ((2.5) and (2.2), Model III) under mixed RLS/SG learning to be $HE$-stable, it is necessary that all sums of the same-size principal minors of $D_{1r}\left(-\Omega_{r\rho_l}\right)$ are nonnegative for all subeconomies $r = (h_1, ..., h_p)$ for all $p$ of all economies from the set of associated current value expectations models corresponding to Model I (III) for all positive block-diagonal matrices $D_{1r}$, where $D_{1r}$ and $\Omega_{r\rho_l}$ defined similar to $D_1$ and $\Omega_{\rho_l}$ in (2.15) and (2.16) (with $A_0^h \equiv 0$ for Model III), respectively, correspond to a subeconomy $r = (h_1, ..., h_p)$.*

**Proof.** See Appendix B.1.8. □

Again, the condition above cannot be used as a test for non-$\delta$-stability as it requires checking all subeconomies' sums of minors for all possible $D_{1r}$. That is why I present below a condition that has a direct testing application.

**Proposition 2.20** *A necessary condition for $\delta$-stability of Models without lags,* Model I *and Model III, under mixed RLS/SG learning in the diagonal environment case: For the structurally heterogeneous economy (2.3) and (2.2), Model I ((2.5) and (2.2), Model III) under mixed RLS/SG learning to be $\delta$-stable, it is necessary that all sums of the same-size principal minors of minus matrices corresponding to subeconomies $\left(-\Omega_{r\rho_l}\right)$ of all economies from the set of associated current value expectations models corresponding to Model I (III) are non-negative for each corresponding subeconomy $r = (h_1, ..., h_p)$.*

**Proof.** See Appendix □

Quite strong necessary conditions can be used as an easy quick test for non-$\delta$-stability. I think that this is quite a strong necessary condition, which implies that many models will not satisfy it, and will not be $\delta$-stable. Note that the stability of each single economy and subeconomies is a sufficient condition for the condition above to hold true.

## 2.6 Conclusion

In this paper, I extend the results of Honkapohja and Mitra [36], Bogomolova and Kolyuzhnov [5], Kolyuzhnov [40], and of the companion paper by Bogomolova [4]. I provide sufficient and necessary conditions for stability under heterogeneous mixed RLS/SG learning for four classes of models considered: models without lags and with lags of the endogenous variable and with $t$- or $t-1$-dating of expectations. While in Bogomolova [4] I essentially use the negative diagonal dominance approach that allows me to derive a sufficient condition for the $D$-stability of matrices due to the MacKenzie Theorem, here I look at the problem from a different angle and try to find conditions for stability that were not possible to derive using that approach. The alternative definition of $D_b$-stability approach allows me to derive for all four classes of models considered in the general (non-diagonal) case, alternative stability criteria (to the criteria in terms of the corresponding Jacobian matrices in Kolyuzhnov [40] and Bogomolova [4]) under heterogeneous mixed RLS/SG learning. It also allows me to obtain simplified alternative stability criteria for both types of models without lags for the case of the diagonal structure of shocks; and for the case of the general (non-diagonal) structure of shocks, alternative sufficient stability conditions (to the sufficient conditions in terms of the corresponding Jacobian matrices in Bogomolova [4]) for models with lags of the endogenous variable of both types under heterogeneous RLS learning — conditions written in terms of structural and learning heterogeneity.

The alternative definition to $D_b$-stability approach allows me to provide the criteria for $\delta$-stability for univariate models (with either $t$- or $t-1$-dating of expectations) without lags of the endogenous variable under mixed RLS/SG learning in economically meaningful terms, such as "same sign" conditions and $E$-stability of a suitably defined average economy and its subeconomies. I also provide quite weak sufficient conditions for $\delta$-stability for univariate models with a lag of the endogenous variable using the same economic terms. Using the characteristic equation approach, I also derive quite strong necessary conditions that can be used as an easy quick test for non-$\delta$-stability. Necessary conditions are derived for the general (non-diagonal) case of the shock process.

The results for sufficient conditions for $\delta$-stability based primarily on the negative diagonal dominance approach in terms of $E$-stability of maximal aggregate economies are considered in the companion paper. All the results of this paper can be summarized in the following way.

I provide (using the alternative definition of $D_b$-stability):

- for the case of the general (**non-diagonal**) structure of shocks, general alternative **stability criteria** (to the general criteria in terms of the corresponding Jacobian matrices in Honkapohja and Mitra [36] and Bogomolova [4]) *for all types of models* considered: a model without lags and with lags of the endogenous variable and with $t$- or $t-1$-dating of expectations, under heterogeneous **mixed RLS/SG learning** — criteria written in terms of structural and learning heterogeneity;

- for the case of the **diagonal** structure of shocks, alternative **stability criteria** (to the criteria in terms of the corresponding Jacobian matrices in Kolyuzhnov [40] and Bogomolova [4]) in terms of structural and learning heterogeneity *for both types of models without lags* of the endogenous variable, under heterogeneous **mixed RLS/SG learning** — criteria written in terms of structural and learning heterogeneity;

- for the case of the general (**non-diagonal**) structure of shocks, alternative **sufficient stability conditions** (to the sufficient conditions in terms of the corresponding Jacobian matrices in Bogomolova [4]) *for both types of models with lags* of the endogenous variable, under **heterogeneous RLS learning** — conditions written in terms of structural and learning heterogeneity;

For the case of the **diagonal** structure shocks and a **univariate** endogenous variable, I provide**:**

- **criteria for $\delta$-stability** (that is, stability that does not depend on such learning heterogeneity characteristics as different degrees of inertia and different starting values of learning algorithms) *for models without lags* of the endogenous variables under heterogeneous **mixed RLS/SG learning** in terms of the "same sign" conditions and $E$-stability of a suitably defined average economy and its subeconomies.

- (quite weak) **sufficient conditions for $\delta$-stability** *for models with lags* of the endogenous variables under **heterogeneous RLS learning** in terms of the "same sign" conditions and $E$-stability of a suitably defined average economy and its subeconomies.

For the case of the **diagonal** structure of shocks and a **bivariate** endogenous variable, I provide **sufficient conditions for $\delta$-stability** *for models without lags* of the endogenous variables under heterogeneous **mixed RLS/SG learning** in terms of the "same sign" conditions.

For the case of the general (**non-diagonal**) structure of shocks, I provide:

- (quite strong) **necessary conditions for stability** in terms of structural and learning heterogeneity under heterogeneous **mixed RLS/SG learning** *for all types of models* considered in terms of stability of a suitably defined structurally heterogeneous average economy under heterogeneous learning of two agents. These **conditions** are also **necessary** (for any fixed degrees of inertia) **for $\delta$-stability** under heterogeneous **mixed RLS/SG leaning** *for all types of models* considered.

- **necessary conditions for stability** *for models without lags* of the endogenous variables under heterogeneous **mixed RLS/SG learning** in terms of subeconomies for economies from a set of associated current value expectations models. These **conditions** are also **necessary** (for any fixed degrees of inertia) **for $\boldsymbol{\delta}$-stability** under heterogeneous **mixed RLS/SG learning** *for models without lags* of the endogenous variables.

The alternative criteria and sufficient conditions for stability under heterogeneous learning derived using the alternative definition of $D_b$-stability allow for further ellaboration for various cases and are a subject for future research. The fundamental nature of the approach adopted in the paper allows one to apply its results to a vast majority of existing and prospective linear and linearized economic models (including estimated DSGE models) with the adaptive learning of agents.

# Chapter 3

# Optimal Monetary Policy Rules: The Problem of Stability Under Heterogeneous Learning

# Optimal Monetary Policy Rules: The Problem of Stability Under Heterogeneous Learning*†

Anna Bogomolova and Dmitri Kolyuzhnov‡

CERGE–EI§
Politických vězňů 7, 111 21 Praha 1,
Czech Republic

**Abstract**

In this paper, we extend the analysis of optimal monetary policy rules in terms of the stability of an economy, started by Evans and Honkapohja [26], to the case of heterogeneous private agents learning. Following Giannitsarou [31], we pose the question about the applicability of the representative agent hypothesis to learning. This hypothesis was widely used in the learning literature in early stages to demonstrate the convergence of an economic system under adaptive learning of agents to one of the rational expectations equilibria in the economy. We test these monetary policy rules in the general setup of the New Keynesian model that is the work horse of monetary policy models today. It turns out that the results obtained by Evans and Honkapohja [26] for the homogeneous learning case are replicated for the case where the representative agent hypothesis is lifted.

JEL Classification: C62, D83, E31, E52
Keywords: monetary policy rules, New Keynesian model, adaptive learning, stability of equilibrium, heterogeneous agents

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‡*{Anna.Bogomolova, Dmitri.Kolyuzhnov }@cerge-ei.cz.*

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## 3.1 Introduction

The stabilization monetary policy design problem is very often studied in the New Keynesian model. Using the environment of this model, we may study different monetary policy rules to find out which is more efficient in smoothing business cycle fluctuations and also which monetary policy rule would not lead to an indeterminacy of equilibria in our model. For a comprehensive overview of various interest rate rules in the New Keynesian model, one can see Woodford [57]. Also, frequently cited papers on monetary policy design are Clarida, Gali, and Gertler [17, 18]. Svensson [54] gives a clear distinction between instrument and target rules and the implications of their use.

A number of recent studies also consider the New Keynesian model environment with the adaptive learning of agents. Examples are Evans and Honkapohja [25, 26], Bullard and Mitra [9], and Honkapohja and Mitra [35] on the stability of an economy under various policy rules. Evans and Honkapohja [25, 26] take up the issue of stability under learning for optimal monetary policies in economies with adaptive learning.

The concept of the adaptive learning of agents in economic models is introduced as a specific form of bounded rationality advocated by Sargent [50]. According to his argument, it is more natural to assume that agents face the same limitations economists face (in a sense that economists have to learn the model structure and its parameter values themselves) and view agents as econometricians when forecasting the future state of the economy.

Using adaptive learning in an economy makes it possible to test the validity of the rational expectations hypothesis by checking if a given dynamic model converges over time to the rational expectations equilibrium (REE) implied by the model. It can also be used as a selection device in models with multiple equilibria. Even if the model has a unique REE, it is still of interest to see if the rational expectations (RE) hypothesis holds under learning, which is done by checking if our model under learning converges to a given REE. In both cases (multiple or unique REE), one has to check certain stability conditions. After this analysis of stability conditions, the next step could be to study the policy rules for effectiveness and indeterminacy, assuming or making sure that the stability conditions on the model structure are satisfied.

Therefore, before we start analyzing particular monetary policies for efficiency (evaluating a particular type of policy: Taylor rule or an optimization-based rule with or without commitment), we should take a general type of a linear policy feedback rule, plug it into our structural form of the New Keynesian model, and obtain some general linear reduced form (RF) of this model. All things being equal (the same structural equations: Phillips and IS curves), we can obtain different RFs depending on the policy rule used by the policy maker. Hence, we obtain different REEs and different stability results. Then we should study a given reduced form for stability in order to see if a given REE is chosen. In this paper, we study the stability of a New Keynesian model under the following classification of policy rules introduced by Evans and Honkapohja [26].

Depending on the assumptions of the central bank about the expectations of private agents (firms, households), Evans and Honkapohja [26] divide all policy rules into fundamentals–based rules and expectations–based rules. The fundamentals–based rule is obtained if the policy maker assumes the RE of private agents, while the expectations–based rule takes into account possibly non–rational expectations of agents (assuming that these expectations are observable to the central bank).[1]

We consider the stability question under the assumption of heterogeneous learning of agents. As has been shown in Giannitsarou [31] and Honkapohja and Mitra [36], stability results may be different under homogeneous and heterogeneous learning. Honkapohja and Mitra [36] also demonstrate that stability may depend on the interaction of structural heterogeneity and learning heterogeneity, and Honkapohja and Mitra [35] examine how structural heterogeneity in the New Keynesian model may affect stability results under various types of policy rules.

Note that though Honkapohja and Mitra [35] consider heterogeneity in learning in the New Keynesian model, their definition of heterogeneity implies a situation where the central bank and private agents have (possibly) different learning algorithms with (possibly) different parameters of these algorithms. They essentially consider the situation when all private agents could be considered as one representative agent, and in this sense learning of private agents considered by Honkapohja and Mitra [35] is homogeneous. In some sense, the situation considered by Honkapohja and Mitra [35] could be called two-sided learning in a structurally heterogeneous bivariate economy.

[1]We should note here that in Taylor–type rules, the current value of the interest rate depends on the current values of inflation and the output gap. In this paper, we study stability under feedback rules that are derived from the policy maker minimization problem, in particular, we study their two categories according to Evans and Honkapohja [26]: fundamentals–based and expectations–based. Stability under Taylor–type rules, which do not fall under this classification, will be studied later in a separate work.

In this paper, we do not consider learning of the central bank and assume, following Evans and Honkapohja [26], that the policy maker takes the expectations of private agents as given or assumes and knows the exact structure of their rational expectations; at the same time, we fully exploit the case where private agents have heterogeneous learning. The case of internal central bank forecasting (that includes Taylor rules) in the situation of heterogeneous learning of private agents, which develops the model of Honkapohja and Mitra [35] since they only consider the situation of a representative private agent, is the topic of our further research.

It turns out that under the fundamentals–based linear feedback policy rule (optimization-based), learning in our model never converges to the REE of the model. Evans and Honkapohja [26] demonstrate this instability result for the homogeneous recursive least squares (RLS) and for the stochastic gradient (SG) learning,[2] while we obtain a similar instability result for the three types of heterogeneous learning considered by Giannitsarou [31].

The other category of policy rules — expectations–based rules — is supposed to react to agents' expectations. Under certain conditions, we can have stability under such rules. Evans and Honkapohja [26] obtain a stability result for homogeneous RLS or for SG learning. We obtain a stability result (with conditions on the model structure) for the case of the three types of heterogeneous learning considered by Giannitsarou [31].

Originally, when heterogeneous learning in a general setup of self-referential linear stochastic models was studied by Giannitsarou [31], the purpose of introducing the heterogeneous learning of agents was to see if the representative agents hypothesis influences stability results, i.e., if one may always apply this hypothesis. For some cases, it is demonstrated that it does make sense to consider the heterogeneous setup. Our paper is about stability under monetary policy rules, so, though we, in fact, prove that the representative agent hypothesis holds true for the New Keynesian model, the accent of our paper is shifted away from testing the importance (influence) of the representative agent

[2]We in this paper and Honkapohja and Mitra [36] consider two possible algorithms used to reflect bounded rationality of agents: RLS and SG learning algorithms (which are examples of econometric learning). Their description can be found, e.g., in Evans and Honkapohja [24], Honkapohja and Mitra [36], Giannitsarou [31], and Evans, Honkapohja and Williams [27]. Both are used by agents to update the estimates of the model parameters. Essentially, the difference is as follows. The RLS algorithm has two updating equations: one—for updating parameters entering the forecast functions, and the other—for updating the second moments matrix (of the model state variables). The SG algorithm assumes this matrix fixed.

hypothesis.

We, essentially, apply the stability analysis of the model under heterogeneous learning in the same manner the stability analysis of the model under homogeneous (when all agents can be substituted with a representative agent) learning is applied in Evans and Honkapohja [26].[3] In our paper, we link the study of stability conditions under a certain category of linear monetary policy rules of [26] with the study of stability under heterogeneous learning of Giannitsarou [31].

We first show that in the New Keynesian–type models, stability can be analyzed using structural parameters, whatever the type of heterogeneous learning, using the general criterion of Honkapohja and Mitra [36]. These results are the structural matrix eigenvalues sufficient and necessary conditions for the stability of a structurally homogeneous model derived in this paper, and the aggregate economy sufficient conditions derived in Kolyuzhnov [40], where the concept of stability under heterogeneous learning, termed as $\delta$–stability, is introduced. Then we apply these results to derive stability and instability results under heterogeneous learning for the two categories of feedback rules: fundamentals–based and expectations–based, in the model with an arbitrary number of agent types.

Summarizing all the above, our work now looks, on the one hand, like a link between the study of stability under monetary policy rules for homogeneous learning of Evans and Honkapohja [26] and the study of stability conditions under heterogeneous learning of Giannitsarou [31] — the link through the $\delta$–stability conditions that we derived for the general setup of Honkapohja and Mitra [36] and through the general stability criterion of Honkapohja and Mitra [36]. On the other hand, this study can serve as one more economic example demonstrating the application of $\delta$–stability sufficient and necessary conditions.

The structure of the paper is as follows. In the next section, we present the basic New Keynesian model. In Section 3, we discuss the general stability results under hetero-

[3]Evans and Honkapohja [26] study stability conditions under monetary policy rules for the case of homogeneous learning. Their major input is (both for the one–sided learning and the two–sided learning) to have shown that under fundamentals–based rules the REE of the model is always unstable, while under the expectations based rule there is always stability. In the two cases, the reduced form of the model is different, which has, as a consequence, the difference in the stability results. So, the policy implication of such a stability analysis is that, given the structure of the model (the two structural New Keynesian equations), the central bank can influence (determine) the outcome of its policy by selecting the appropriate optimal monetary policy: the one that guarantees convergence to a particular REE.

geneous learning and the concept of $\delta$–stability introduced in Kolyuzhnov [40]. In Section 4, we give necessary and sufficient conditions for $\delta$–stability for structurally homogeneous models. Section 5 describes the two types of optimal policy rules and the structure of the reduced forms under each type. In Section 6, we provide stability and instability results for the types of optimal monetary policies considered in application to the New Keynesian model. Section 7 concludes.

## 3.2 Model

The model that we consider is a general New Keynesian model with observable stationary AR(1) shocks. The structural form of the model looks as follows:

$$x_t = c_1 - \phi\left(i_t - \widehat{E}_t\pi_{t+1}\right) + \widehat{E}_t x_{t+1} + \chi_1' w_t \tag{3.1}$$

$$\pi_t = c_2 + \lambda x_t + \beta \widehat{E}_t \pi_{t+1} + \chi_2' w_t, \tag{3.2}$$

where the first equation is for the IS curve, and the second equation is for the Phillips curve. $w_t = \left[\begin{array}{ccc} w_{1t} & ... & w_{kt} \end{array}\right]'$ is a vector of observable AR(1) shocks[4],

$$w_{it} = \rho_i w_{it-1} + \nu_{it}, |\rho_i| < 1, \nu_{it} \sim iid\left(0, \sigma_i^2\right), i = \overline{1,k} \tag{3.3}$$

To introduce heterogeneity into the model, we assume that we have $S$ types of private agents characterized by their share $\zeta_h > 0$ in the economy, $\sum\limits_{h=1}^{S} \zeta_h = 1$. So, $\widehat{E}_t x_{t+1} =$

[4]Typically, New Keynesian models include only an observable component, which is assumed to follow an AR(1) process. However, there are specifications including both observable and unobservable shocks. For example, Evans and Honkapohja [27], who study stability rules under recursive least squares learning, include unobservable shocks to the New Keynesian model equations. In our case, a more general specification with unobservable shocks would contain the additional term $\Omega_1 \epsilon_t$ in the IS curve and $\Omega_2 \epsilon_t$ in the Phillips curve, where $\epsilon_t = \left[\begin{array}{ccc} \epsilon_{1_t} & ... & \epsilon_{m_t} \end{array}\right]'$ are unobservable shocks, $\epsilon_{i_t} \sim iid\left(0, \gamma_i^2\right)$, $i = 1, ..., m$, not correlated with observable shocks $g_t$.

Of course, these unobservables do not bring a difference into the stability results (that is why we omit them in the model analyzed), but introducing them into the setup has its own reasoning. For example, it makes sense to introduce unobservable shocks into structural equations when we consider central bank learning structural coefficients of the model. If we have only observable shocks (which play a role of just another regressor — some exogenous variable) as well as other observable regressors, we will evaluate the equations' coefficients exactly if we have a sufficient number of observations. In this case, learning is trivial: the convergence will be very quick if initially we did not have enough observations but gained them over a short period of time.

If we think of how these unobservable shocks can emerge at the micro foundations level, we may think of the following economic interpretation. For example, let us assume that preference and technology shocks consist of observable and unobservable components. As for preference shocks, we can imagine a qualitative change in our preferences, such that we know how the shock has changed our preferences qualitatively, but we cannot precisely measure this change quantitatively. A similar interpretation can be given to the technological shock. What we have measured enters as an observable component, while the measurement error (which always exists since we assume that our quantitative measurement of any change is imprecise) is treated as an unobservable component.

$\sum_{h=1}^{S} \zeta_h \widehat{E}_t^h x_{t+1}$, $\widehat{E}_t \pi_{t+1} = \sum_{h=1}^{S} \zeta_h \widehat{E}_t^h \pi_{t+1}$, where $\widehat{E}_t^h x_{t+1}$ and $\widehat{E}_t^h \pi_{t+1}$ are expectations (in general, non–rational) of private agent of type $h$ made at time $t$ about the next period output gap and inflation, respectively.

The model (3.1), (3.2), and (3.3) is a general formulation of models derived from microfoundations that are considered in macroeconomics and monetary economics literature[5]. The two basic equations of the New Keynesian model, which are the Phillips curve and the IS curve, are derived from the optimal problems of the representative household and the representative monopolistically competitive firm, with the assumption of the Calvo [10] pricing mechanism in the firms' price-setting decision. So the two New Keynesian curves are derived using the optimality conditions of the private agents (households and firms). The derivation of these two curves for the standard New Keynesian model setup can be found, e.g., in Walsh [55]. The description of the New Keynesian model can also be found in Woodford [56, 57] and in Christiano, Eichenbaum, and Evans [16].

In solving their optimization problems, private agents are assumed to take the interest rate (entering the IS curve equation) as given. The interest rate, in turn, is set by the policy maker — the central bank. In various studies of monetary policy issues (in the New Keynesian framework), it is normally assumed that the policy maker uses some linear feedback rule to set the interest rate. In general, a feedback rule that is derived from the loss function minimization problem determines how the interest rate reacts to the expected values of the model's endogenous variables (inflation and output gap in the New Keynesian model) and the model's exogenous variables (various shocks, e.g., technology, preference, and cost-push shocks). Instrument rules, like Taylor–type rules, are designed to respond to the target variables (e.g., inflation and the output gap). As is noted in the introduction, Taylor–type rules will be considered in a separate study.

Plugging the feedback rule into the IS curve equation, we obtain the model reduced form. Using the same New Keynesian equations (IS and Phillips curves), we can obtain different reduced forms for different policy rules, i.e. other things being equal, the reduced form structure depends on the policy rule. It depends not only on the type of it (Taylor or optimization–based) but, as is demonstrated by Evans and Honkapohja [26],

[5]Our NK model includes the standard NK model. One may have different shocks for the IS and Phillips curves by having appropriate zeroes in $\chi_1'$ and $\chi_2'$ vectors of coefficients. Learning and forecasting inflation and the output gap, of course, use all the shocks that appear in the rational expectations equilibrium processes of these variables.

on the assumption of the central bank about private agents expectations, resulting either in the fundamentals–based or in the expectations–based category of feedback rules.

After plugging in some monetary policy rule of the central bank $i_t$, assuming that the central bank knows the expectations of private agents or assumes and knows the form of rational expectations of agents (we will talk about the types of optimal monetary policy rules later), the model can be written in the reduced form that has a general representation of a bivariate system with a stationary AR(1) observable shocks process

$$y_t = \alpha + A\hat{E}_t y_{t+1} + Bw_t, \tag{3.4}$$

$$y_t = \begin{bmatrix} \pi_t & x_t \end{bmatrix}' \tag{3.5}$$

and (3.3).

In what follows, for the derivation of our stability results, we may allow for some generalization (as it is just a matter of notation compared to the bivariate model) and consider a multivariate (not just a bivariate) system (3.4) with a stationary AR(1) observable shocks process (3.3).

In our notation, the reduced form is written in such a way that it includes all factors that appear in the structural form. This means that the absence of some factor in the reduced form in our notation is expressed by the corresponding zero column of matrix $B$. Note that here we adopt such a notation in order to be able later to consider different specifications of learning algorithms that include factors from different sets.[6] So our notation is the most general possible.

In adaptive learning models of bounded rationality, it is assumed that agents do not know the rational expectations equilibrium and instead have their own understanding of the relation between variables in the model. The coefficients in this relation (that are called beliefs) are updated each period as new information on observed variables arrives (in this respect, agents are modeled as if they were statisticians, or econometricians). For the beginning, we assume that agents have the following perceived relation among the variables in the economy, which is called the perceived law of motion (PLM)

$$y_t = a^h + \Gamma^h w_t,$$

[6] An example when a model reduced form may not include all shocks that are present as factors in the model structural form can be found in Evans and Honkapohja [26], who used the New Keynesian model setup of Clarida, Gali, Gertler [17].

$$\text{with } a^h = \begin{bmatrix} a_1^h & a_2^h \end{bmatrix}', \Gamma^h = \begin{bmatrix} \gamma_{11}^h & \gamma_{12}^h & ... & \gamma_{1k}^h \\ \gamma_{21}^h & \gamma_{22}^h & ... & \gamma_{2k}^h \end{bmatrix} \text{ in the bivariate case,}$$

that includes all components of $w_t$. A bit later, we weaken this assumption. Though we assume that the parameters of the PLM may differ across agents, we assume that the structure of the PLMs is the same for all agents. We may also write the average (or aggregate) PLM using the weights of agents.

$$y_t = a + \Gamma w_t, \text{ where } a = \sum_{h=1}^{S} \zeta_h a^h, \Gamma = \sum_{h=1}^{S} \zeta_h \Gamma^h. \tag{3.6}$$

Thus, agents have the following forecast functions based on their PLMs:

$$\widehat{E}_t^h y_{t+1} = a^h + \Gamma^h diag(\rho_1, ..., \rho_k) w_t,$$

and consequently the average forecast function is given by

$$\widehat{E}_t y_{t+1} = \sum_{h=1}^{S} \zeta_h \left( a^h + \Gamma^h diag(\rho_1, ..., \rho_k) w_t \right) = a + \Gamma diag(\rho_1, ..., \rho_k) w_t. \tag{3.7}$$

After plugging the average forecast function (3.7) corresponding to the average PLM (3.6) into the reduced form (3.4), we derive the actual law of motion (ALM):

$$y_t = Aa + \alpha + (A\Gamma diag(\rho_1, ..., \rho_k) + Bw_t). \tag{3.8}$$

The rational expectations equilibrium (REE) defined as $E_t y_{t+1} = \widehat{E}_t y_{t+1} = \widehat{E}_t^i y_{t+1}$ (see, e.g., Sargent [50] or Evans and Honkapohja [24] for the meaning of the RE concept) can be calculated by equating the parameters of the average PLM (3.6) with the corresponding parameters of the ALM (3.8). If we define the $T$–map as a mapping of beliefs from the average PLM (3.6) to the ALM (3.8),

$$T(a, \Gamma) = (Aa + \alpha, A\Gamma diag(\rho_1, ..., \rho_k) + B), \tag{3.9}$$

we will be able to write the REE condition as $T(a, \Gamma) = (a, \Gamma)$.

Now, we will widen the set of PLMs considered. Let us start with the following definition.

**Definition 3.1** *The active factors set is a subset of a set of histories of* $w_{i_t}$ *up to time* $t$ *and a constant used by agents in their PLMs.*[7]

[7] Note that by the active factors set we mean not the variables that agents are actually aware of at time $t$, but essentially those that are used by agents in their PLMs (a subset that may be smaller than the subset of actually available variables).

Following the definition, we renumber the subscripts corresponding to regressors that are included into the agents' active factors set from 1 to $k'$, and denote the set of subscripts taken from $\{1, ..., k\}$ corresponding to the active factors set as $\widetilde{I}$. Assuming, as before, that all agents have the same structure of their individual PLMs, agents now are assumed to have the following average perceived law of motion (PLM):

$$y_t = a + \widetilde{\Gamma}\widetilde{w}_t$$

with $a = \begin{bmatrix} a_1 & a_2 \end{bmatrix}', \widetilde{\Gamma} = \begin{bmatrix} \widetilde{\gamma}_{11} & \widetilde{\gamma}_{12} & \cdots & \widetilde{\gamma}_{1k'} \\ \widetilde{\gamma}_{21} & \widetilde{\gamma}_{22} & \cdots & \widetilde{\gamma}_{2k'} \end{bmatrix}$ in the bivariate case,

where $\widetilde{w}_t$ consists of the components of $w_t$ included in the agents' active factors set. Consequently, $T$–map (3.9) can be rewritten as

$$\widetilde{T}(a, \widetilde{\Gamma}) = \left(Aa + \alpha, A\widetilde{\Gamma}diag(\rho_1, ..., \rho_k) + \widetilde{B}\right),$$

where $\widetilde{B}$ consists of columns of matrix $B$ that correspond to the active factors set.

Similarly, one may try to write the REE condition as $\widetilde{T}(a, \widetilde{\Gamma}) = (a, \widetilde{\Gamma})$. However, in this case, it is clear that for the existence of a REE, agents have to include into their active factors set those factors $w_{i_t}$ that correspond to non-zero columns of matrix $B$ in the reduced form. A PLM which consists only of such factors is a PLM that corresponds to the so–called minimal state variable (MSV) solution. Also, in the above PLMs, we have used the following assumption.

**Assumption 3.1** *Agents include in their PLM of each endogenous variable all factors from their active factors set.*[8]

Essentially, Assumption 3.1 postulates that we may write each agent's PLM equations in matrix form, without a priori setting coefficients at some factors to zero. In addition, we assume that all agents use the same set of factors (which in matrix form means that they use the same vector). We also note here that a similar assumption on

[8]So, we exclude situations when agents do not include into the PLM equation of one endogenous variable some factor having a zero coefficient in matrix $B$ of the reduced form, while including the same factor in the PLM equation of the other endogenous variable, with this factor having a non–zero coefficient in matrix $B$ of the reduced form. We assume that agents do not know the true structure of the reduced form and use all the available information to form their expectations. So, if one factor is present in one PLM equation, it is present in another PLM equation.

the matrix formulation of PLMs has been made by Giannitsarou [31] and Honkapohja and Mitra [36].[9]

The Propositions below state the necessary and sufficient conditions for the existence and uniqueness of a REE in a general multivariate model with stationary AR(1) observable shocks. These conditions are well known, but we prefer to state them here for the reader's convenience. To formulate the following propositions, we return to the initial numbering of shocks, denote the constant term in the active factors set of agents as $w_0$, and take $\rho_0 = 1$ and $B^0 = \alpha$. So now, $i$ takes integer values from 0 to $k$. We will denote this set as $I_0$ and the corresponding set of subscripts taken from $I_0 = \{0, 1, ..., k\}$ as $\widetilde{I}_0$. Note that the constant term is always included as a factor in any active factors set; therefore, 0 always belongs to $I_0$.

**Proposition 3.1** *(Necessary and sufficient conditions for the existence of a REE) Under Assumption 3.1, a REE solution exists if and only if the agents' active factors set includes, among others, all $w_i$ such that $B^i \neq 0$ in the reduced form and $rank(\rho_i A - I) = rank(\rho_i A - I, B^{\dot{i}})$ for $i$ such that* $\det(\rho_i A - I) = 0$ *and $B^i \neq 0$.*

**Proof.** See Appendix C.1. □

**Proposition 3.2** *(Necessary and sufficient conditions for the existence and uniqueness of a REE): Under Assumption 3.1, a REE solution exists and is unique if and only if the agents' active factors set includes, among others, all $w_i$ such that $B^i \neq 0$ in the reduced form and for all $w_i$ included,* $\det(\rho_i A - I) \neq 0$.

**Proof.** See Appendix C.1. □

So in what follows, we always assume that Assumption 3.1 and the necessary and sufficient conditions[10] for the existence of a REE hold true. Basically, we assume that

[9] Notice that here we also do not consider situations of the restricted perceptions equilibrium (RPE), the discussion of which may be found, for example in Evans and Honkapohja [24]. In our terminology, for the situation of the RPE, one has to assume that agents do not include into their active factors set some of the factors that are present in a unique REE, that is, factors that correspond to non-zero coefficients in matrix $B$. Here, we introduce the notion of the active factors set only to allow for a considering of the PLMs not only corresponding to the MSV, but also those that include more factors than enough to determine a unique REE. It is done to derive the "strong $\delta$–stability" or "strong $\delta$–instability result." (Compare to the notion of the "strong $E$–stability" in the homogeneous learning literature.)

[10] The propositions above have a similar meaning to Proposition 1 of Honkapohja and Mitra [36]: again, the condition requires matrices participating in the derivation of the RE values of beliefs to be invertible. So, the above propositions stress that we are aware of cases when an REE may not exist and of the conditions that are required for its existence (and uniquness).

in both equations of their PLM, agents use at least all the regressors that appear in the right–hand side of the reduced form (3.4), and that the REE solution (either unique or multiple) exists under this PLM. That is, in principle, we consider all possible PLMs that satisfy these conditions.

After specifying PLMs of agents and conditions for the existence and uniqueness of the REE, we are ready to introduce heterogeneous learning of agents into the economy considered and derive conditions for the stability of the REE under this learning. Then, we use these conditions to study stability under heterogenous learning in the general New Keynesian model when optimal monetary policy rules are applied.

## 3.3 Heterogeneous Learning and the Concept of $\delta$–stability

The model (3.4) and (3.3) that we consider belongs to the class of multivariate forward–looking economic models. Thus, we naturally employ the general framework and notation from Honkapohja and Mitra [36], who were the first to formulate the general criterion for the stability of a multivariate forward–looking economy under heterogeneous learning.

Honkapohja and Mitra [36] consider the class of linear structurally heterogeneous forward-looking models with $S$ types of agents with different forecasts presented by

$$
\begin{aligned}
y_t &= \alpha + \sum_{h=1}^{S} A_h \hat{E}_t^h y_{t+1} + B w_t, && (3.10) \\
w_t &= F w_{t-1} + v_t, && (3.11)
\end{aligned}
$$

where $y_t$ is an $n \times 1$ vector of endogenous variables, $w_t$ is a $k \times 1$ vector of exogenous variables, $v_t$ is white noise, $\hat{E}_t^h y_{t+1}$ are (in general, non-rational) expectations of the endogenous variable by agent type $h$, $M_w = \lim_{t\to\infty} w_t w_t'$ is positive definite, and $F$ is such that $w_t$ follows a stationary VAR process.

The PLM is presented by (3.6). A part of agent types, $h = \overline{1, S_0}$, is assumed to use the RLS learning algorithm, while the rest, $h = \overline{S_0 + 1, S}$, are assumed to use the SG learning algorithm.[11] Moreover, all of them are assumed to use possibly different degrees of responsiveness to the updating function that are presented by different degrees of inertia

[11] Essentially, the part of agents using RLS are assumed to be more sophisticated in their learning because from an econometric point of view, the RLS algorithm is more efficient since it uses information on the second moments.

$\delta_i > 0$, constant coefficients before the common for all agents decreasing gain sequence in the learning algorithm.[12]

It is worth noting that the model (3.4) and (3.3) we consider belongs to the subclass of models considered by Honkapohja and Mitra [36], namely, a class of structurally homogeneous forward–looking models. Structural heterogeneity in the setup of Honkapohja and Mitra [36] is expressed through matrices $A_h$, which are assumed to incorporate mass $\zeta_h$ of each agent type. That is, $A_h = \zeta_h \cdot \hat{A}_h$, where $\hat{A}_h$ is defined as describing how agents of type $h$ respond to their forecasts. So these are the structural parameters characterizing a given economy. Those may be basic characteristics of agents, like the ones describing their preferences, endowments, and technology. Structural heterogeneity means that all $\hat{A}_h$'s are different for different types of agents. When $\hat{A}_h = A$ and $\sum \zeta_h = 1$, the economy is structurally homogenous.

When we apply the conditions for a structurally homogeneous economy, $A_h = \zeta_h A$, where $\sum_{h=1}^{S} \zeta_h = 1$, and $1 > \zeta_h > 0$, to the model (3.10) and (3.11) considered by Honkapohja and Mitra [36], we get

$$y_t = \alpha + \sum_{h=1}^{S} A_h \hat{E}_t^h y_{t+1} + Bw_t = \alpha + \sum_{h=1}^{S} \zeta_h A \hat{E}_t^h y_{t+1} + Bw_t =$$

$$= \alpha + A \underbrace{\sum_{h=1}^{S} \zeta_h \hat{E}_t^i y_{t+1}}_{\hat{E}_t^{aver} y_{t+1}} + Bw_t,$$

which is exactly the formulation of the structurally homogeneous model considered by Giannitsarou [31].[13] Thus, the conditions for stability valid for the (more general) class of structurally heterogeneous forward–looking models remain valid for the class of structurally homogeneous models.

After denoting $z_t = (1, w_t)$ and $\Phi_{h,t} = (a_{h,t}, \Gamma_{h,t})$, the formal presentation of the learning algorithms in this model can be written as follows[14].

---

[12] Honkapohja and Mitra [36] use a more general formulation of the degrees of inertia.

[13] Heterogeneous learning in the structurally homogeneous case was considered by Giannitsarou [31] for a more general class of self–referential linear stochastic models, which includes in itself the case of forward–looking models. Since our setup does not assume lagged endogenous variables, we concentrate on the structurally homogeneous case of forward–looking models that are a subclass of models considered by Giannitsarou [31] and at the same time are a special case of the setup of Honkapohja and Mitra [36].

[14] We assume that the elements of matrix $F$ are known to agents. Adding the learning of the shocks process will basically not change anything in case agents do not misspecify the structure of the shocks process as learning (through expectations) does not influence the behaviour of the exogenous shocks process,

RLS: for $h = \overline{1, S_0}$

$$\begin{aligned} \Phi_{h,t+1} &= \Phi_{h,t} + \alpha_{h,t+1} R_{h,t}^{-1} z_t \left( y_t - \Phi_{h,t}' z_t \right)' \\ R_{h,t+1} &= R_{h,t} + \alpha_{h,t+1} \left( z_{t-1} z_{t-1}' - R_{h,t} \right) \end{aligned} \tag{3.12}$$

SG: for $h = \overline{S_0 + 1, S}$

$$\Phi_{h,t+1} = \Phi_{h,t} + \alpha_{h,t+1} z_t \left( y_t - \Phi_{h,t}' z_t \right)'. \tag{3.13}$$

Honkapohja and Mitra [36] show that the stability of the REE, $\Phi_t$, in this model is determined by the stability of the ODE[15]:

$$\begin{aligned} \frac{d\Phi_h}{d\tau} &= \delta_h \left( T(\Phi')' - \Phi_h \right), h = \overline{1, S_0} \\ \frac{d\Phi_h}{d\tau} &= \delta_h M_z \left( T(\Phi')' - \Phi_h \right), h = \overline{S_0 + 1, S}, \end{aligned}$$

where $M_z = \lim_{t \to \infty} E z_t z_t'$.

The conditions for the stability of this ODE give the general criterion for the stability result for this class of models presented in Proposition 5 in Honkapohja and Mitra [36]. In the economy (3.10) and (3.11), the mixed RLS/SG learning (3.12) and (3.13) converges globally (almost surely) to the minimal state variable (MSV) solution if and only if matrices $D_1 \Omega$ and $D_w \Omega_F$ have eigenvalues with negative real parts, where

$$D_1 = \begin{pmatrix} \delta_1 I_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_n \end{pmatrix}, \Omega = \begin{pmatrix} A_1 - I_n & \cdots & A_S \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_S - I_n \end{pmatrix} \tag{3.14}$$

$$D_w = \begin{pmatrix} D_{w1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{wS} \end{pmatrix}, \quad \begin{aligned} D_{wh} &= \delta_h I_{nk}, h = \overline{1, S_0} \\ D_{wh} &= \delta_h \left( M_w \otimes I_n \right), h = \overline{S_0 + 1, S} \end{aligned}$$

$$\Omega_F = \begin{pmatrix} F' \otimes A_1 - I_{nk} & \cdots & F' \otimes A_S \\ \vdots & \ddots & \vdots \\ F' \otimes A_1 & \cdots & F' \otimes A_S - I_{nk} \end{pmatrix},$$

and after some iterations the estimates of the elements of matrix $F$ will almost converge to their true values. Since we consider local convergence properties of the REE, one may say that we consider learning from the point in time when these values are already known.

[15] In the general case, to obtain the associated ODE, one has to take the math expectation of the RHS term (at the gain sequence) from the stochastic recursive algorithm (SRA) specification of a learning algorithm, with respect to the limiting distribution of the state vector. See Ch. 6.2 in Evans and Honkapohja [24] for assumptions on the learning rule and state dynamics that have to hold so that we are able to apply the theory on SRA and local convergence analysis and the general formula for ODE (6.5) on p. 126.

with $\otimes$ denoting the Kronecker product.

Note that agents in the setup of Honkapohja and Mitra [36] are assumed to use PLMs that correspond to the MSV solution, i.e., include all factors that appear in the right-hand side of the reduced form. However, Honkapohja and Mitra [36] in their proof of conditions for the stability of the system do not have restrictions on matrix $B$. This means that we may, in principle, consider additional factors in learning that enter the reduced form with zero coefficients in matrix $B$ for all agents. This means that we may consider the criterion conditions for all possible PLMs that include (among others) all factors that appear in the right-hand side of the reduced form, satisfying conditions for the existence specified in the previous chapter.

Kolyuzhnov [40] shows that in the "diagonal" environment, namely

$$F = diag(\rho_1, ..., \rho_k), M_w = diag\left(\frac{\sigma_1^2}{1-\rho_1^2}, ..., \frac{\sigma_k^2}{1-\rho_k^2}\right), \tag{3.15}$$

which we consider in this paper, the problem of finding stability conditions for both $D_1\Omega$ and $D_w\Omega_F$ is simplified to finding stability conditions for $D_1\Omega$ and $D_1\Omega_{\rho l}$,where $\Omega_{\rho_l}$ is obtained from $\Omega$ by substituting all $A_h$ with $\rho_l A_h$, where $|\rho_l| < 1$ as $w_t$ follows a stationary VAR(1) process.

$$\Omega_{\rho_l} = \begin{pmatrix} \rho_l A_1 - I_n & \cdots & \rho_l A_S \\ \vdots & \ddots & \vdots \\ \rho_l A_1 & \cdots & \rho_l A_S - I_n \end{pmatrix}, \forall l = 0, ..., k, (\rho_0 = 1). \tag{3.16}$$

Kolyuzhnov [40] uses a special blocked—diagonal structure of matrix $D_1$, which is the feature of the dynamic environment in this class of models. In a sense, these positive diagonal $D$–matrices may now be called positive blocked—diagonal $\delta$–matrices. This makes it possible to formulate the concept of $\delta$–stability by analogy to the terminology of the concept of $D$–stability about matrices that remain stable under multiplication by a diagonal matrix with positive elements, studied for example in Johnson [37].

**Definition 3.2** *Given $n$, the number of endogenous variables, and $S$, the number of agent types, $\delta$–stability is defined as the stability of the economy under structurally heterogeneous mixed RLS/SG learning for any (possibly different) degrees of inertia of agents, $\delta > 0$.*

$\delta$–stability, thus formulated, has the same meaning in models with heterogeneous learning described above as the $E-$stability condition in models with homogeneous RLS learning. The $E-$stability condition is a condition for the asymptotic stability of an REE under homogeneous RLS learning. The REE of the model is stable if it is locally asymptotically stable under the following ODE:

$$\frac{d\theta}{d\tau} = T(\theta) - \theta,$$

where $\theta$ are the estimated parameters from agents PLMs, $T(\theta)$ is a mapping of the PLM parameters into the parameters of the actual law of motion (ALM), which is obtained when we plug the forecast functions based on the agents' PLMs into the reduced form of the model, and $\tau$ is a "notional" ("artificial") time. The fixed point of this ODE is the REE of the model.[16]

Note that the $\delta$–stability concept comprises stability under the three types of heterogeneous learning considered by Giannitsarou [31]. It is worth noting that in the case of heterogeneous learning in a structurally homogeneous economy, which we employ in the current setup, the criterion of Honkapohja and Mitra [36] is simplified to conditions on the Jacobians considered by Giannitsarou [31]. First, to get the structurally homogeneous economy as discussed earlier, one has to replace $A_i$ in the setup of Honkapohja and Mitra [36] with $\zeta_i A$. After that, one has to make the following simplifications in the setup corresponding to a particular type of heterogeneous learning considered.

The first type of heterogeneous learning is characterized by different initial perceptions of agents and equal degrees of inertia. This type is termed transiently heterogeneous learning by Honkapohja and Mitra [36]. The condition for stability under this learning is easily derived from the criterion above by setting all $\delta$'s to be equal, and setting $S_0$ to $S$ or to 0 in order to get transiently heterogeneous RLS or SG learning, respectively.

The second type of heterogeneous learning considered by Giannitsarou [31] is such that agents use different degrees of inertia and the same type of learning algorithm (RLS or SG). This is what Honkapohja and Mitra [36] call persistently heterogeneous learning in a weak form. The Jacobians, for this case, are easily derived by setting $S_0$ to $S$ or to 0 in order to have heterogeneous RLS or SG learning, respectively, and by allowing

[16] Notice that $\delta$–stability conditions on the Jacobian in the general forward–looking model of Honkapohja and Mitra [36] do not depend on the particular equilibrium point (in the case of multiple equilibria) because the system of differential equations is linear in this setup, in which case the first derivatives of the RHS of the associated ODE do not depend on a particular value of a RE equilibrium. So if stability conditions are satisfied for a given Jacobian, then all equilibrium points are stable. Convergence to a particular point depends on the initial conditions. In this paper, we do not consider how equilibrium selection is made.

| Type of heterogeneity | Type of learning | Assumptions in the general H&M (2006) model | |
|---|---|---|---|
| | | structurally heterogeneous | structurally homogeneous |
| | | $A_h = \zeta_h \hat{A}_h$ | $A_h = \zeta_h A$ |
| I Different initial perceptions (transiently heterogeneous learning) | RLS<br>SG | $\delta_h = \delta$ for all $h$, $S_0 = S$<br>$\delta_h = \delta$ for all $h$, $S_0 = 0$ | |
| II Different degrees of inertia (persistently heterogeneous learning in a weak form) | RLS<br>SG | $S_0 = S$<br>$S_0 = 0$ | |
| III Different learning algorithms (persistently heterogeneous learning in a strong form) | RLS and SG | | |

Table 3.1: Types of heterogeneity in learning.

for possibly different $\delta$'s.

The third type of heterogeneous learning considered by Giannitsarou [31] is characterized by possibly different initial perceptions, possibly different degrees of inertia, and by different agents using different learning algorithms (RLS or SG). Such type of learning Honkapohja and Mitra [36] call persistently heterogeneous learning in a strong form. The stability Jacobians for this case are derived by writing the general criterion for stability for the structurally homogeneous case, i.e., by setting $A_i = \zeta_i A$.

The relation between the above–described formulations of the types of heterogeneous learning by Giannitsarou [31] and by Honkapohja and Mitra [36] can be conveniently summarized in the following table[17]:

Notice that in the "diagonal" case (3.15), $\delta$–stability does not depend on $S_0$. Thus, if the economy (3.10), (3.11), and (3.15) is $\delta$–stable, it is stable under all three types of heterogeneous learning and under RLS and SG homogeneous learning.

[17]Note that there is one type of heterogeneous learning that was not introduced by Giannitsarou [31] but is introduced here. It is heterogeneity in the degrees of inertia under which all types of agents use the SG learning algorithm. Although Honkapohja and Mitra [36] have the general criterion for stability in this case (as discussed above), their formulation includes only forward–looking models. In the general setup of self-referential structurally homogeneous models of Giannitsarou [31], the stability conditions under such a type of learning (in Giannitsarou [31] notation, naturally extended from her proofs) would depend on the stability of matrix $J_2^{SG}(\Phi_f) = diag(\delta_1, ..., \delta_S) \otimes I \otimes M(\Phi_f) J_1^{LS}(\Phi_f)$, where $\Phi_f$ is an REE, $M(\Phi_f)$ is defined similarly to $M_z$, and $J_1^{LS}(\Phi_f)$ is a Jacobian that defines stability in case of the first type of heterogeneity (different initial perceptions of agents) when all agents use RLS learning. For details, see Giannitsarou [31] . Again, it is clear that in the forward–looking case these conditions for stability fall under the general stability criterion of Honkapohja and Mitra [36] with $S_0 = 0$ (see the table above).

## 3.4 Conditions for $\delta$–stability of Structurally Homogeneous Models

After establishing the universal role of the concept of $\delta$–stability for stability under all three types of heterogeneous learning discussed above, we present necessary and sufficient conditions. First, we provide the reader with a set of sufficient conditions for $\delta$–stability applicable to our setup, that is, for a class of structurally homogeneous models. We present (without proofs) the so–called aggregate economy sufficient condition for the case of a structurally homogeneous model and the "same sign" sufficient condition for the case of a structurally heterogeneous bivariate economy that were derived in Kolyuzhnov [40]

**Proposition 3.3** *For the structurally homogeneous economy (3.4) and (3.3) to be $\delta$–stable, it is sufficient that at least one of the following maximal aggregated $\beta$–coefficients (which are the coefficients before the expectation term of a one–dimensional forward–looking aggregate economy model, for details see Kolyuzhnov [40]):* $\max_i \sum_j |a_{ij}|$ *and* $\max_j \sum_i |a_{ij}|$ *are less than one, where $a_{ij}$ denotes an element in the $i^{th}$ row and the $j^{th}$ column of $A$.*

**Proposition 3.4** *In case $n = 2$, the economy (3.10), (3.11), and (3.15) is $\delta$–stable if the corresponding matrix $\Omega$, defined in (3.14), is stable and the following "same sign" condition holds true:*

$$\det(-\rho_l A_i) \geq 0, [\det mix(-\rho_l A_i, -\rho_l A_j) + \det mix(-\rho_l A_j, -\rho_l A_i)] \geq 0, i \neq j, M_1(-\rho_l A_i) \geq 0$$

$$or$$

$$\det(-\rho_l A_i) \leq 0, [\det mix(-\rho_l A_i, -\rho_l A_j) + \det mix(-\rho_l A_j, -\rho_l A_i)] \leq 0, i \neq j, M_1(-\rho_l A_i) \leq 0,$$
$$\forall l = 0, 1, ..., k, (\rho_0 = 1),$$

*where $mix(-\rho_l A_i, -\rho_l A_j)$ denotes a matrix of structural parameters of a pairwise–mixed economy and is composed by mixing columns of a pair of matrices $\rho_l A_i$, $\rho_l A_j$, for any $i$, $j = \overline{1, S}$.*

It is also possible to derive some necessary conditions and sufficient conditions for $\delta$–stability in the structurally homogeneous case in terms of eigenvalues of the matrix of structural parameters of the reduced form, $A$. It is possible by a direct application of the characteristic equation approach, where one requires all roots of the polynomial (that are eigenvalues of the Jacobian matrix) to be less than zero for stability, the latter being equivalent to the well-known Routh–Hurwitz conditions.

**Proposition 3.5** *If all eigenvalues of $A$ are real and less than one, then the structurally homogeneous system (3.4) and (3.3) with two agents is $\delta$–stable, that is, stable under the three types of heterogeneous learning: agents with different initial perceptions with RLS or SG learning, agents with possibly different degrees of inertia with RLS or SG learning, and agents with different learning algorithms, RLS and SG. For the structurally homogeneous system (3.4) and (3.3) with any number of agents to be $\delta-$stable, it is necessary that all real roots of $A$ be less than one. This gives a test for non–$\delta$–stability.*

**Proof.** See Appendix C.1. □

In the proof of the proposition above, using the structure of the Jacobian matrices in our setup, we have derived a sufficient condition for stability under all three types of heterogeneous learning with two agent types. We did this using the criterion for stability of Honkapohja and Mitra [36]. For the case of real roots of $A$, we have shown that in this setup, the analysis of stability of a particular Jacobian turns into the analysis of stability of $A$, which gives us very simple eigenvalues conditions. Also, using the general criterion of Honkapohja and Mitra [36], we have proved here the necessary conditions for $\delta$–stability (the failure of which is sufficient for non–$\delta$–stability) for the case of an arbitrary number of agent types.

## 3.5 Optimal Policy Rules and the Structure of the Reduced Forms

After deriving and stating the conditions for stability under the three types of heterogeneous learning discussed in the previous section, we are ready to study the general New Keynesian model (3.1), (3.2), and (3.3) for stability under heterogenous learning when optimal monetary policy rules are applied. Here we describe the types of optimal policy rules that are analyzed in this study.

The policy maker is assumed to use the loss function minimization problem, which comes from the flexible inflation targeting approach (a policy regime adopted in several countries in the 1990s), described and defended by Svensson [54]. The central bank here has two options: adopt a discretionary policy, by solving the problem every period, or commit to a rule that is once and for all derived from the minimization of the infinite horizon loss function. Svensson [54] and Cecchetti [12] advocate the first option, which is essentially commitment to a certain behavior (minimizing the loss function) with a reconsidering of the optimal rule every period to take new information into account. They provide various arguments, like inefficiency (in general) of instrument rules designed to respond only to target variables or the way monetary policy decisions are made in practice.

The infinite horizon loss function of the policy maker for the flexible inflation targeting approach looks as follows:

$$\frac{1}{2}E_t\sum_{i=0}^{\infty}\beta^i\left[\alpha\left(x_{t+i}-\bar{x}\right)^2+\left(\pi_{t+i}-\bar{\pi}\right)^2\right].$$

According to the discussion above, we assume the discretionary policy of the policy maker and the problem of minimizing the loss function simplifies to solving each period

$$\min\frac{1}{2}\left[\alpha\left(x_t-\bar{x}\right)^2+\left(\pi_t-\bar{\pi}\right)^2\right]+R_t \tag{3.17}$$

subject to

$$\pi_t=c_2+\lambda x_t+F_t$$

(the central bank takes the remainder terms of the loss function $R_t$, and the constraint $F_t=\beta\widehat{E}_t\pi_{t+1}+\chi_2 w_t$ as given).

The classification of the loss–function–optimization–based rules into fundamentals–based and expectations–based rules provided below is due to Evans and Honkapohja [26]. The derivation of these rules and of the corresponding reduced forms is done by Evans and Honkapohja [26] for a slightly more narrow setup than is assumed here (we assume a general structure of autoregressive shocks); therefore in the derivations that follow, we basically repeat their steps extending them for our setup.

### 3.5.1 Expectations–based Optimal Policy Rules

The expectations–based policy rule implies the central bank's reaction to (possibly non-rational) expectations of private agents, assuming that these expectations are observable (or can be estimated). Its general form is $i_t = \delta_0 + \delta_\pi \hat{E}_t \pi_{t+1} + \delta_x \hat{E}_t x_{t+1} + \delta_w' w_t$. The coefficients of this rule are obtained by solving the equilibrium conditions: structural equations with non–rational expectations of private agents (3.1) and (3.2) and the first-order conditions (FOC) of the optimization problem of the central bank (3.17), $\lambda(\pi_t - \bar{\pi}) + \alpha(x_t - \bar{x}) = 0$. Thus, the expectations–based policy rule is as follows:

$$\begin{aligned} & i_t = \delta_0 + \delta_\pi \hat{E}_t \pi_{t+1} + \delta_x \hat{E}_t x_{t+1} + \delta_w' w_t, \text{ where} \\ & \delta_0 = -\left(\lambda^2 + \alpha\right)^{-1} \phi^{-1} \left(\lambda\bar{\pi} + \alpha\bar{x} - \lambda c_2 - \left(\alpha + \lambda^2\right) c_1\right), \\ & \delta_\pi = 1 + \left(\lambda^2 + \alpha\right)^{-1} \phi^{-1} \lambda\beta,\ \delta_x = \phi^{-1},\ \delta_w = \phi^{-1}\chi_1 + \left(\lambda^2 + \alpha\right)^{-1} \phi^{-1} \lambda \chi_2. \end{aligned} \tag{3.18}$$

After plugging this policy rule into the IS curve equation, we get the following reduced form.

$$\begin{aligned} y_t &= c^E + A^E \hat{E}_t y_{t+1} + \chi^E w_t, \\ w_t &= F w_{t-1} + \nu_t, \\ y_t &= \begin{bmatrix} \pi_t & x_t \end{bmatrix}', \text{ where } F = diag(\rho_i),\ |\rho_i| < 1, \nu_{i_t} \sim iid\left(0, \sigma_i^2\right), i = \overline{1,n}, \\ A^E &= \begin{pmatrix} \beta\alpha\left(\lambda^2 + \alpha\right)^{-1} & 0 \\ -\beta\lambda\left(\lambda^2 + \alpha\right)^{-1} & 0 \end{pmatrix}, \\ c^E &= \begin{pmatrix} c_2 + \lambda\left(c_1 - \phi\delta_0\right) \\ c_1 - \phi\delta_0 \end{pmatrix}, \chi^E = \begin{pmatrix} \chi_2'\left[1 - \frac{\lambda^2}{\lambda^2 + \alpha}\right] \\ -\frac{\lambda^2}{\lambda^2 + \alpha}\chi_2' \end{pmatrix}. \end{aligned} \tag{3.19}$$

Note that the REE solution is not needed either for deriving matrix $A^E$ or for deriving the coefficients of the optimal expectations–based policy rule. The REE solution will be needed for deriving the optimal fundamentals–based policy rule and, therefore, will be derived in the corresponding part of the text.

### 3.5.2 Fundamentals–based Optimal Policy Rules

In general, the fundamentals–based policy rule (not necessarily optimal) has the form

$$i_t = \psi_0 + \sum_{i=1}^{n} \psi_{w_i} w_{i_t} = \psi_0 + \psi_w' w_t. \tag{3.20}$$

Later, we show that there exists a unique fundamentals–based optimal policy rule in this setup and derive this rule.

Plugging this policy rule into structural form (3.1) and (3.2), we get the following reduced form:

$$
\begin{aligned}
y_t &= c^F + A^F \widehat{E}_t y_{t+1} + \chi^F w_t, \\
w_t &= F w_{t-1} + \nu_t, \\
y_t &= \left[ \begin{array}{cc} \pi_t & x_t \end{array} \right]', \text{ where } F = diag(\rho_i),\ \ |\rho_i| < 1, \nu_{i_t} \sim iid\left(0, \sigma_i^2\right), i = \overline{1, n}, \\
A^F &= \left( \begin{array}{cc} \beta + \lambda\phi & \lambda \\ \phi & 1 \end{array} \right), \\
c^F &= \left( \begin{array}{c} c_1 - \phi\psi_0 \\ c_2 + \lambda\left(c_1 - \phi\psi_0\right) \end{array} \right),\ \chi_F = \left( \begin{array}{c} \lambda\left(-\phi\psi_w' + \chi_1'\right) + \chi_2' \\ -\phi\psi_w' + \chi_1' \end{array} \right).
\end{aligned}
\tag{3.21}
$$

The optimal fundamentals–based rule, under the central banks' discretionary policy, is obtained from the loss function minimization, with the central bank assuming that private agents have RE. With the REE structure being $y_t = a + \Gamma w_t$, its general form is $i_t = \psi_0 + \psi_w' w_t$, where $w_t$ is a vector of exogenous variables. Using the equilibrium conditions (economy's structural equations (3.1) and (3.2), with the REE structure entering them and the FOC of the central bank's optimization problem), we obtain the coefficients of the REE and of the optimal fundamentals–based policy rule.

To get the REE, one has to write the ALM using the Phillips curve (3.2), the FOC of the central bank's optimization problem and the PLM in the general form, $y_t = a + \Gamma w_t$, and then according to the RE principle, equate coefficients of the resulting ALM ($T$–mapping) with the corresponding coefficients of the PLM. The resulting ALM looks like

$$
\begin{aligned}
\pi_t &= \frac{c_2 + \lambda\left[\lambda\bar{\pi} + \alpha\bar{x}\right]}{\lambda^2 + \alpha} + \frac{\alpha\beta}{\lambda^2 + \alpha}\left[a_1 + \gamma_{11}\rho_1 w_{1t} + ... + \gamma_{1n}\rho_n w_{nt}\right] + \frac{\alpha}{\lambda^2 + \alpha}\chi_2' w_t \\
x_t &= \frac{\lambda\bar{\pi} + \alpha\bar{x}}{\alpha} - \frac{\lambda}{\alpha}\pi_t,
\end{aligned}
$$

and the REE looks like

$$
\begin{aligned}
\pi_t &= a_1^* + \sum_{i=1}^{n} \gamma_{1i}^* w_{it} \\
x_t &= a_2^* + \sum_{i=1}^{n} \gamma_{2i}^* w_{it}, \text{ where} \\
a_1^* &= \frac{c_2 + \lambda\left[\lambda\bar{\pi} + \alpha\bar{x}\right]}{\lambda^2 + \alpha\left(1 - \beta\right)}, a_2^* = \frac{\lambda\bar{\pi} + \alpha\bar{x}}{\alpha} - \frac{\lambda}{\alpha}a_1^* = \frac{-\frac{\lambda}{\alpha}c_2 + \left(1 - \beta\right)\left[\lambda\bar{\pi} + \alpha\bar{x}\right]}{\lambda^2 + \alpha\left(1 - \beta\right)}, \\
\gamma_{1i}^* &= \frac{\alpha\chi_{2i}\rho_i}{\alpha\left(1 - \beta\rho_i\right) + \lambda^2}, \gamma_{2i}^* = -\frac{\lambda}{\alpha}\gamma_{1i}^* = -\frac{\lambda\chi_{2i}\rho_i}{\alpha\left(1 - \beta\rho_i\right) + \lambda^2}, i = \overline{1, n}.
\end{aligned}
\tag{3.22}
$$

To get the *optimal* fundamentals–based policy rule, one has to express $i_t$ using the IS curve (3.1), plugging into it the REE solution (3.22) derived above.

$$i_t = -\frac{1}{\phi}\left(a_2^* + \sum_{i=1}^{n} \gamma_{2i}^* w_{it}\right) + \left(a_1^* + \sum_{i=1}^{n} \gamma_{1i}^* \rho_i w_{it}\right) + \frac{1}{\phi}\left(a_2^* + \sum_{i=1}^{n} \gamma_{2i}^* \rho_i w_{it}\right) + \frac{1}{\phi}\chi_1' w_t.$$

As a result, the optimal fundamentals–based policy rule is

$$\begin{aligned} i_t &= \psi_0^* + \psi_w^{*\prime} w_t, \text{ where} \qquad (3.23)\\ \psi_0^* &= a_1^*,\ \psi_w^* = \frac{1}{\phi}\left[\begin{pmatrix} \gamma_{21}(\rho_1 - 1) & \ldots & \gamma_{2n}(\rho_n - 1) \end{pmatrix} + \chi_1\right] + \begin{pmatrix} \gamma_{11}\rho_1 & \ldots & \gamma_{1n}\rho_n \end{pmatrix}. \end{aligned}$$

In both cases of optimal monetary policy rules, we plug the corresponding policy rule into the structural equations and obtain the corresponding reduced form of the model. These reduced forms were studied for stability under homogeneous RLS learning in the Clarida, Gali, and Gertler [17, 18] formulation of the New Keynesian model by Evans and Honkapohja [26] , who derived the stability results for the expectations–based rule and the instability results for the fundamentals–based rule. We study stability and instability for the two categories of rules under the heterogeneous learning of private agents in the general setup of the New Keynesian model (3.1), (3.2), and (3.3).

## 3.6 Stability Problem in the New Keynesian Model

After deriving the reduced forms corresponding to the optimal monetary policy rules, we are ready to check them for $\delta$–stability. To do this, we have to test the resulting matrix $A$ of the reduced form (3.19) or (3.21) for the applicability of the sufficient and necessary conditions for $\delta$–stability. For the optimal expectations–based policy rule, we have the following result.

**Proposition 3.6** *The general New Keynesian model with a stationary AR(1) observable shocks process (3.1), (3.2), and (3.3) is $\delta$–stable when the optimal expectations–based policy rule (3.18) is applied.*[18]

**Proof.** We know that the corresponding $A$ matrix in the optimal expectations–based policy rule case is $A^E = \begin{pmatrix} \beta\alpha\left(\lambda^2+\alpha\right)^{-1} & 0 \\ -\beta\lambda\left(\lambda^2+\alpha\right)^{-1} & 0 \end{pmatrix}$. Using the sufficient condition in Proposition 3.4, we have that $\Omega$ is stable since its eigenvalues are determined from the following

[18] This result is not very surprising as Evans, Honkapohja, and Williams [27] have a convergence result under the optimal expectations–based policy rule when all agents use SG learning.

characteristic equation $\det\left(A^E - I_2(1+\mu)\right)(1+\mu)^{2(S-1)} = 0$ and, therefore, are equal to $-1$ and $\beta\alpha\left(\lambda^2+\alpha\right)^{-1} - 1$, i. e., are negative, and we have that $\det(-\rho_l A_i) = 0$, $[\det mix\,(\text{-}\rho_l A_i, \text{-}\rho_l A_j) + \det mix\,(\text{-}\rho_l A_j, \text{-}\rho_l A_i)]{=}0$, $i \neq j$, $M_1(\text{-}\rho_l A_i){=}\text{-}\rho_l\zeta_h\beta\alpha\left(\lambda^2{+}\alpha\right)^{-1} \geq (\leq)\,0$, for all $l = 0, 1, ..., k$ $(\rho_0 = 1)$, so the "same sign" condition holds true. Notice that using the "aggregate economy" sufficient condition from Proposition 3.3, we can write two aggregate $\beta$–coefficients in the expectations–based policy rule case. These are $\beta_1^{\max} = \max_i \sum_j |a_{ij}| = \max\left\{\beta\alpha\left(\lambda^2+\alpha\right)^{-1}, \beta\lambda\left(\lambda^2+\alpha\right)^{-1}\right\}$ and $\beta_2^{\max} = \max_j \sum_i |a_{ij}| = \beta(\alpha+\lambda)\left(\lambda^2+\alpha\right)^{-1}$. It is clear that both coefficients are less than one if $\lambda \geq 1$. So, the "aggregate economy" sufficient condition for $\delta$–stability is a more restrictive condition compared to the "same sign" condition since it requires additional assumptions on the structure of the economy. However, it can be with success applied in more than two dimensional economies, where similar "same sign" conditions are not sufficient for $\delta$–stability (see Kolyuzhnov [40]). □

Note that Evans and Honkapohja [26] have a similar result for homogeneous learning. The proposition below presents the instability result for the fundamentals–based monetary policy rule.

**Proposition 3.7** *The general New Keynesian model with a stationary AR(1) observable shocks process (3.1), (3.2), and (3.3) is non–$\delta$–stable when the fundamentals–based policy rule (3.20), as well as the optimal fundamentals–based policy rule (3.23), is applied.*

**Proof.** We know that the corresponding matrix $A$ in the fundamentals–based policy rule case is $A^F = \begin{pmatrix} \beta+\lambda\phi & \lambda \\ \phi & 1 \end{pmatrix}$. Using the "eigenvalues" necessary condition from Proposition 3.5,[19] we get the eigenvalues of this matrix:.$\mu_{1,2} = 1 + \frac{\beta+\lambda\phi-1}{2} \pm \sqrt{\left(\frac{\beta+\lambda\phi-1}{2}\right)^2 + \lambda\phi}$. Both of these eigenvalues are real, and eigenvalue $\mu_1 = 1 + \frac{\beta+\lambda\phi-1}{2} + \sqrt{\left(\frac{\beta+\lambda\phi-1}{2}\right)^2 + \lambda\phi}$ is greater than one. So, the sufficient condition for non–$\delta$–stability is satisfied. □

Again, Evans and Honkapohja [26] have a similar result for homogeneous learning.

[19] In principle, we could also use our necessary conditions for $\delta$–stability (derived in Kolyuzhnov [40]) to show the instability of the fundamentals–based rule. However, these may be more difficult to check than the necessary conditions on eigenvalues derived in this paper. Besides, the necessary conditions on eigenvalues work for the case of an arbitrary number of agent types.

Proposition 3.6 means that the REE in this model, resulting after implementing the optimal expectations–based policy rule, is stable under the recursive least squares and the stochastic gradient homogeneous learning and the three types of heterogeneous learning: agents with different initial perceptions with the RLS or SG learning, agents with different degrees of inertia with RLS or SG learning, and agents with different learning algorithms, RLS and SG. Proposition 3.7 claims that the REE of this model with the fundamentals–based policy rule is always unstable under any type of heterogeneous and homogeneous learning of agents.

## 3.7 Conclusion

We have used the environment of the New Keynesian model to explore the question of stability of two categories of optimal monetary policy rules under the assumption of heterogeneous learning of private agents.

These two categories were introduced by Evans and Honkapohja [26], and this division is based on the assumption about the central bank's perception of private agents' expectations: RE or possibly non-rational. Under the central bank assuming private agents have RE, the fundamentals–based rule is obtained, while the case of the central bank assuming possibly non-rational expectations of private agents results in the expectations–based rule.

The purpose of this research was, on the one hand, to explore whether, given structural homogeneity of the model, heterogeneity in the learning of agents influences the stability results implied by the application of either of the two categories of policy rules.

Using the general criterion for stability of Honkapohja and Mitra [36] and the sufficient $\delta$–stability conditions derived in Kolyuzhnov [40] for the case of heterogeneous learning, we obtain results similar to those obtained by Evans and Honkapohja [26] for the case of homogeneous learning. In particular, under the fundamentals–based policy rule, the model economy is always unstable, so there is no convergence to the associated REE of the model, while there is stability under the optimal expectations–based rule, and the economy converges to the REE corresponding to the optimal monetary policy without commitment.

The above–described results have been obtained using only the structure of the model, so there is no dependence on heterogeneity of any type considered. This implies that in the New Keynesian model, the stability results are independent of heterogeneity in learning, so the representative agent hypothesis is applicable in this setup.

The method of analysis presented in this paper allows us to check the applicability of this hypothesis in the case of the heterogeneous learning of private agents in the New Keynesian economy under Taylor–type rules (the case of internal central bank forecasting), which do not fall under the classification of Evans and Honkapohja [26]. This issue will be considered in a separate study.

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# Appendix A

# Appendix to Chapter 1

## A.1 Assumptions on the SRA from the stochastic approximation literature (Benveniste, Métivier and Priouret [3]) as they are given in Evans and Honkapohja [24, pp. 124-125]

(A.1) Assumption A in the paper

(A.2) For any compact subset $Q \subset D$, there exist $C_1$,$C_2$, $q_1$, and $q_2$ such that $\forall \theta \in Q$ and $\forall t$ :

(i) $|H(\theta, x)| \leq C_1 \left(1 + |x|^{q_1}\right),$

(ii) $|\rho_t(\theta, x)| \leq C_2 \left(1 + |x|^{q_2}\right).$

(A.3) For any compact subset $Q \subset D$, the function $H(\theta, x)$ satisfies $\forall \theta$, $\theta' \in Q$ and $x_1$, $x_2 \in \mathbb{R}^k$ :

(i) $\left|\frac{\partial H(\theta,x_1)}{\partial x} - \frac{\partial H(\theta,x_2)}{\partial x})\right| \leq L_1 \left|x_1 - x_2\right|,$

(ii) $\left|H(\theta, 0) - H(\theta', 0)\right| \leq L_2 \left|\theta - \theta'\right|,$

(iii) $\left|\frac{\partial H(\theta,x)}{\partial x} - \frac{\partial H(\theta',x)}{\partial x}\right| \leq L_2 \left|\theta - \theta'\right|,$

for some constants $L_1$, $L_2$.

(B.1) $W_t$ is iid with finite absolute moments.

(B.2) For any compact subset $Q \subset D$:

$$\sup_{\theta \in Q} |B(\theta)| \leq M \text{ and } \sup_{\theta \in Q} |A(\theta)| \leq \rho < 1,$$

for some matrix norm $|\cdot|$, and $A(\theta)$ and $B(\theta)$ satisfy Lipschitz conditions on $Q$.

Here, I provide the reader with definitions and theorems adapted from the mathematics literature that I used for deriving conditions for $\delta$-stability. These results are structured according to the approach that is used for deriving stability conditions.

## A.2 The general definition of stability and $D$-stability of a matrix

**Definition A.1** *Matrix $A$ is stable if all the solutions of the system of ordinary differential equations $\dot{x}(t) = Ax(t)$ converge toward zero as $t$ converges to infinity.*

**Theorem A.1** *Matrix $A$ is stable if and only if all its eigenvalues have negative real parts.*

**Definition A.2** *($D-$stability) Matrix $A$ is $D$–stable if $DA$ is stable for any positive diagonal matrix $D$.*

## A.3 The Lyapunov theorem approach

**Theorem A.2** *(Lyapunov) A real $n \times n$ matrix $A$ is a stable matrix if and only if there exists a positive definite matrix $H$ such that $A'H + HA$ is negative definite.*

**Theorem A.3** *(Arrow-McManus, 1958) Matrix $A$ is $D$-stable if there exists a positive diagonal matrix $C$ such that $A'C + CA$ is negative definite.*

## A.4 The negative diagonal dominance approach

**Definition A.3** *(introduced by McKenzie) A real $n \times n$ matrix $A$ is dominant diagonal if there exist $n$ real numbers $d_j > 0, j = 1, ..., n$, such that $d_j|a_{jj}| > \sum d_i|a_{ij}| : i \neq j)$, $j = 1, \ldots, n$. This is called the "column" diagonal dominance. The "row" diagonal dominance is defined as the existence of $d_i > 0$ such that $d_i|a_{ii}| > \sum d_j|a_{ij}| : j \neq i), i = 1, \ldots, n$.*

**Theorem A.4** *(a sufficient condition for stability, McKenzie, 1960): If an $n \times n$ matrix $A$ is dominant diagonal and its diagonal is composed of negative elements ($a_{ii} < 0$, all $i = 1, \ldots, n$), then the real parts of all its eigenvalues are negative, i.e., $A$ is stable.*

**Corollary A.1** *If $A$ has negative diagonal dominance, then it is $D$–stable.*

## A.5 The characteristic equation approach

**Theorem A.5** *(Routh-Hurwitz necessary and sufficient conditions for the negativity of eigenvalues of a matrix) Consider the following characteristic equation:*

$$|\lambda I - A| = \lambda^n + b_1\lambda^{n-1} + ... + b_{n-1}\lambda + b_n = 0$$

*determining $n$ eigenvalues $\lambda$ of a real $n \times n$ matrix $A$, where $I$ is the identity matrix. Then eigenvalues $\lambda$ all have negative real parts if and only if $\Delta_1 > 0, \Delta_2 > 0, ..., \Delta_n > 0$, where*

$$\Delta_k = \begin{vmatrix} b_1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ b_3 & b_2 & b_1 & 1 & 0 & \cdots & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{2k-1} & b_{2k-2} & b_{2k-3} & b_{2k-4} & b_{2k-5} & \cdots & b_k \end{vmatrix}.$$

## A.6 The alternative definition of $D-$stability approach

**Theorem A.6** *(From Observation (iv) in Johnson [37]). Consider $M_n(C)$, the set of all complex $n \times n$ matrices, and $D_n$, the set of all $n \times n$ diagonal matrices with positive diagonal entries. Take $A \in M_n(C)$ and suppose that there is an $F \in D_n$ such that $FA$ is stable. Then $A$ is $D$–stable if and only if $A \pm iD$ is non–singular for all $D \in D_n$. If $A \in M_n(R)$, the set of all $n \times n$ real matrices, then "$\pm$" in the above condition may be replaced with "$+$" since, for a real matrix, any complex eigenvalues come in conjugate pairs.*

## A.7 Proofs of propositions in Chapter 1

### A.7.1 Proof for the form of the associated ODE for models with $t-1$-dating of expectations and information available up to $t-1$ (Models I and II)

I have that Models I and II under mixed RLS/SG learning are presented in the standard form of SRA

$$\theta_t = \theta_{t-1} + \alpha_t H\left(\theta_{t-1}, X_t\right) + \alpha_t^2 \rho_t\left(\theta_{t-1}, X_t\right),$$

$$X_t = A(\theta_{t-1})X_{t-1} + B(\theta_{t-1})W_t,$$

where for the law of motion of beliefs for the RLS part for agent type $h$, I have

RLS: for $h = \overline{1, S_0}$

$$H_{\Phi_h}() = \delta_h \left[S_{h,t-1}^{-1} z_{t-1} z_{t-1}' \left[T(\Phi_{t-1}) - \Phi_{t-1}\right] + S_{h,t-1}^{-1} z_{t-1} \left(B\nu_t + \zeta\varepsilon_t\right)'\right], \rho_{\Phi_h}() = 0,$$

$$H_{S_h}() = (\delta_h)\left(z_t z_t' - S_{h,t-1}\right), \rho_{S_h}() = \left(\frac{\alpha_{t+1} - \alpha_t}{\alpha_t^2}\right)(\delta_h)\left(z_t z_t' - S_{h,t-1}\right),$$

where $\rho_{S_h}()$ is bounded since $(z_t z_t' - S_{h,t-1})$ has a limit, and $\frac{\alpha_{t+1} - \alpha_t}{\alpha_t^2} \leq \frac{1}{\alpha_{t+1}} - \frac{1}{\alpha_t}$[1] is bounded due to the additional technical assumption [2] in Assumption A

$$\limsup_{t\to\infty} \left[\left(\frac{1}{\alpha_{t+1}}\right) - \left(\frac{1}{\alpha_t}\right)\right] < \infty;$$

thus, Assumtion A2 (ii) is satisfied, and other required assumptions for the derivation of the associated ODE of SRAs, see Apendix A.1 (conditions A1-A3, B1-B2 for Models I and II are also satisfied).

So, the right-hand side of the associated ODE $\frac{d\theta}{d\tau} = h(\theta)$, where $h(\theta) = \lim_{t\to\infty} EH\left(\theta, X_t(\theta)\right),$ for the RLS part looks as

RLS: for $h = \overline{1, S_0}$

$$\begin{aligned} \lim_{t\to\infty} H_{\Phi_h}(\theta, X_t) &= \delta_h S_h^{-1} M_z \left[T(\Phi) - \Phi\right] \\ \lim_{t\to\infty} H_{S_h}(\theta, X_t) &= \delta_h \left[(M_z - S_h)\right]. \end{aligned}$$

Similarly, for the SG case

SG: for $h = \overline{S_0 + 1, S}$

$$H_{\Phi_h}() = \delta_h \left[z_{t-1} z_{t-1}' \left[T(\Phi_{t-1}) - \Phi_{t-1}\right] + z_{t-1}\left(B\nu_t + \zeta\varepsilon_t\right)'\right], \rho_{\Phi_h}() = 0$$

$$\lim_{t\to\infty} EH_{\Phi_h}(\theta, X_t) = \delta_h M_z \left(T(\Phi) - \Phi\right).$$

So, the associated ODE looks as

$$\begin{aligned} \frac{d\Phi_h}{d\tau} &= \delta_h S_h^{-1} M_z \left(T(\Phi')' - \Phi_h\right), h = \overline{1, S_0} \\ \frac{dS_h}{d\tau} &= \delta_h \left[(M_z - S_h)\right] \\ \frac{d\Phi_h}{d\tau} &= \delta_h M_z \left(T(\Phi')' - \Phi_h\right), h = \overline{S_0 + 1, S}. \end{aligned}$$

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[1] See footnote 24 on p.42 in Evans and Honkapohja [22].

[2] imposed as well by Evans and Honakapohja [22, p.32]

As $S_h \longrightarrow M_z$, this ODE asymptotically behaves as (can be proved along the lines of Evans and Honkapohja [22, p. 42] by using Marcet and Sargent [43, Prop. 3]):

$$\begin{aligned}\frac{d\Phi_h}{d\tau} &= \delta_h \left(T(\Phi) - \Phi_h\right), h = \overline{1, S_0} \\ \frac{d\Phi_h}{d\tau} &= \delta_h M_z \left(T(\Phi) - \Phi_h\right), h = \overline{S_0 + 1, S},\end{aligned}$$

*Q.E.D.*

### A.7.2 Proof for the form of the associated ODE for models with $t$-dating of expectations (Models III and IV)

The proof is essentially similar to the proof for models with $t-1$ dating of expectations. Models III and IV under mixed RLS/SG learning are presented in the standard form of SRA

$$\theta_t = \theta_{t-1} + \alpha_t H\left(\theta_{t-1}, X_t\right) + \alpha_t^2 \rho_t \left(\theta_{t-1}, X_t\right),$$

$$X_t = A(\theta_{t-1}) X_{t-1} + B(\theta_{t-1}) W_t,$$

where for the law of motion of beliefs for the RLS part for agent type $h$, I have

RLS: for $h = \overline{1, S_0}$

$$H_{\Phi_h}() = \delta_h \left[S_{h,t-1}^{-1} z_{t-1} z_{t-1}' \left[T(\Phi_{t-1}) - \Phi_{t-1}\right] + S_{h,t-1}^{-1} z_{t-1} \left(\zeta \varepsilon_t\right)'\right], \rho_{\Phi_h}() = 0$$

$$H_{S_h}() = (\delta_h)\left(z_t z_t' - S_{h,t-1}\right), \rho_{S_h}() = \left(\frac{\alpha_{t+1} - \alpha_t}{\alpha_t^2}\right)(\delta_h)\left(z_t z_t' - S_{h,t-1}\right),$$

where $\rho_{S_h}()$ is bounded since $(z_t z_t' - S_{h,t-1})$ has a limit, and $\frac{\alpha_{t+1} - \alpha_t}{\alpha_t^2} \leq \frac{1}{\alpha_{t+1}} - \frac{1}{\alpha_t}$ [3] is bounded due to the additional technical assumption [4] in Assumption A

$$\limsup_{t \to \infty} \left[\left(\frac{1}{\alpha_{t+1}}\right) - \left(\frac{1}{\alpha_t}\right)\right] < \infty;$$

thus, Assumtion A2 (ii) is satisfied, and other required assumptions for the derivation of the associated ODE of SRAs, see Apendix A.1 (conditions A1-A3, B1-B2 for Models III and IV are also satisfied).

So, the right-hand side of the associated ODE $\frac{d\theta}{d\tau} = h(\theta)$, where $h(\theta) = \lim_{t \to \infty} EH\left(\theta, X_t(\theta)\right)$ for the RLS part looks as

[3] See footnote 24 on p.42 in Evans and Honkapohja [22].

[4] imposed as well by Evans and Honkapohja [22, p.32]

RLS: for $h=\overline{1,S_0}$

$$\begin{aligned}\lim_{t\to\infty} H_{\Phi_h}(\theta,X_t) &= \delta_h S_h^{-1} M_z\left[T(\Phi)-\Phi\right]\\ \lim_{t\to\infty} H_{S_h}(\theta,X_t) &= \delta_h\left[(M_z-S_h)\right].\end{aligned}$$

Similarly, for the SG case

SG: for $h=\overline{S_0+1,S}$

$$H_{\Phi_h}()=\delta_h\left[z_{t-1}z_{t-1}'\left[T(\Phi_{t-1})-\Phi_{t-1}\right]+z_{t-1}\left(\zeta\varepsilon_t\right)'\right],\rho_{\Phi_h}()=0$$

$$\lim_{t\to\infty} EH_{\Phi_h}(\theta,X_t)=\delta_h M_z\left(T(\Phi)-\Phi\right).$$

So, the associated ODE looks as

$$\begin{aligned}\frac{d\Phi_h}{d\tau} &= \delta_h S_h^{-1} M_z\left(T(\Phi')'-\Phi_h\right), h=\overline{1,S_0}\\ \frac{dS_h}{d\tau} &= \delta_h\left[(M_z-S_h)\right]\\ \frac{d\Phi_h}{d\tau} &= \delta_h M_z\left(T(\Phi')'-\Phi_h\right), h=\overline{S_0+1,S}.\end{aligned}$$

As $S_h\longrightarrow M_z$, this ODE asymptotically behaves as (can be proved along the lines of Evans and Honkapohja [22, p. 42] by using Marcet and Sargent [43, Prop. 3]):

$$\begin{aligned}\frac{d\Phi_h}{d\tau} &= \delta_h\left(T(\Phi)-\Phi_h\right), h=\overline{1,S_0}\\ \frac{d\Phi_h}{d\tau} &= \delta_h M_z\left(T(\Phi)-\Phi_h\right), h=\overline{S_0+1,S},\end{aligned}$$

*Q.E.D.*

### A.7.3 Proof of Criterion 1.1

**The associated ODE** for the SRA of Model I looks like

$$\begin{aligned}\frac{d\Phi_h}{d\tau} &= \delta_h\left(T(\underbrace{(\Phi_1',...,\Phi_s')'}_{\Phi})-\Phi_h\right), h=\overline{1,S_0}\\ \frac{d\Phi_h}{d\tau} &= \delta_h M_z\left(T(\underbrace{(\Phi_1',...,\Phi_s')'}_{\Phi})-\Phi_h\right), h=\overline{S_0+1,S},\end{aligned}$$

where the $T$-map is given by

$$T\begin{bmatrix} a_{1,t} \\ b_{1,t} \\ \vdots \\ a_{S,t} \\ b_{S,t} \end{bmatrix} = \begin{bmatrix} \left[\alpha + \sum_{h=1}^{S}\left[A_0^h a_{h,t} + A_1^h a_{h,t} + ... + A_\tau^h a_{h,t}\right]\right]' \\ \left[\left(\sum_{h=1}^{S} A_0^h b_{h,t}\right) + \left(\sum_{h=1}^{S} A_1^h b_{h,t}\right) F + ... + \left(\sum_{h=1}^{S} A_\tau^h b_{h,t}\right) F^\tau + BF.\right]' \end{bmatrix} = \begin{bmatrix} T_a(\Phi_t) \\ T_b(\Phi_t) \end{bmatrix}.$$

First, transpose the ODE

$$\begin{aligned} \frac{d\Phi_h'}{d\tau} &= \delta_h\left(T'(\Phi) - \Phi_h'\right), h = \overline{1, S_0} \\ \frac{d\Phi_h'}{d\tau} &= \delta_h\left(T'(\Phi) - \Phi_h'\right) M_z', h = \overline{S_0 + 1, S}, \end{aligned}$$

then vectorize with the $vec$ operator using the rules $vec(AB) = B' \otimes I_n vec(A)$, where $A$ has dimension $n \times l$ and $B$ - $l \times m$, take $(T'(\Phi) - \Phi_h')$ as $A$ , $M_z'$ - as $B$, $d\,(vec(X)) = vec(dX)$

$$\begin{aligned} \frac{dvec\,(\Phi_h')}{d\tau} &= \delta_h vec\left(T'(\Phi) - \Phi_h'\right), h = \overline{1, S_0} \\ \frac{dvec(\Phi_h')}{d\tau} &= \delta_h M_z \otimes I_n vec\left(T'(\Phi) - \Phi_h'\right), h = \overline{S_0 + 1, S}. \end{aligned}$$

After substituting for $T'(\Phi)$, I obtain:

for $h = \overline{1, S_0}$

$$\frac{dvec\,(\Phi_h')}{d\tau} = \delta_h vec\left(\left[\alpha + \sum_{h=1}^{S}\sum_{r=0}^{\tau} A_r^h a_{h,t}, \sum_{r=0}^{\tau}\left(\sum_{h=1}^{S} A_r^h b_{h,t}\right) F^r + BF\right] - [a_{h,t}, b_{h,t}]\right)$$

for $h = \overline{S_0 + 1, S}$

$$\frac{dvec(\Phi_h')}{d\tau} = \delta_h M_z \otimes I_n vec\left(\left[\alpha + \sum_{h=1}^{S}\sum_{r=0}^{\tau} A_r^h a_{h,t}, \sum_{r=0}^{\tau}\left(\sum_{h=1}^{S} A_r^h b_{h,t}\right) F^r + BF\right] - [a_{h,t}, b_{h,t}]\right).$$

Using $vec(ABC) = (C' \otimes A)vec(B)$:

for $h = \overline{1, S_0}$

$$\frac{dvec\,(\Phi_h')}{d\tau} = \delta_h\left(\left[\alpha + \sum_{h=1}^{S}\sum_{r=0}^{\tau} A_r^h a_{h,t}, \sum_{r=0}^{\tau} F^{r\prime} \otimes \sum_{h=1}^{S} A_r^h vec b_{h,t} + vecBF\right] - \begin{bmatrix} a_{h,t} \\ b_{h,t}^1 \\ \vdots \\ b_{h,t}^k \end{bmatrix}\right)$$

for $h = \overline{S_0 + 1, S}$

$$\frac{dvec\left(\Phi_h'\right)}{d\tau} = \delta_h M_z \otimes I_n \left( \left[ \alpha + \sum_{h=1}^{S} \sum_{r=0}^{\tau} A_r^h a_{h,t}, \sum_{r=0}^{\tau} F^{r\prime} \otimes \sum_{h=1}^{S} A_r^h vecb_{h,t} + vecBF \right] - \begin{bmatrix} a_{h,t} \\ b_{h,t}^1 \\ \vdots \\ b_{h,t}^k \end{bmatrix} \right).$$

Now, I have to compute the Jacobian of the right hand side of this ODE, that is, to take the derivative with respect to $dvec(\Phi')=\left( (a_{1,t})', (b_{1,t}^1)', ..., (b_{1,t}^k)', ..., (a_{S,t})', \left(b_{S,t}^1\right)', ..., \left(b_{S,t}^k\right)' \right)'$.

Use $M_z = \begin{bmatrix} 1 & 0 \\ 0 & M_w \end{bmatrix}$.

It will have the following structure.

The first $S_0$ row blocks look like

$$\begin{bmatrix} \delta_1 \left[ \sum\limits_{r=0}^{\tau} A_r^1 \text{-} I_n \right] & 0 & \cdots & \delta_1 \sum\limits_{r=0}^{\tau} A_\tau^{S_0} & 0 & \cdots \\ 0 & \delta_1 \left[ \sum\limits_{r=0}^{\tau} F^{r\prime} \otimes A_r^1 \text{-} I_{nk} \right] & \cdots & 0 & \delta_1 \sum\limits_{r=0}^{\tau} F^{r\prime} \otimes A_\tau^{S_0} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \delta_{S_0} \sum\limits_{r=0}^{\tau} A_\tau^1 & 0 & \cdots & \delta_{S_0} \left[ \sum\limits_{r=0}^{\tau} A_\tau^{S_0} \text{-} I_n \right] & 0 & \cdots \\ 0 & \delta_{S_0} \sum\limits_{r=0}^{\tau} F^{r\prime} \otimes A_\tau^1 & \cdots & 0 & \delta_{S_0} \left[ \sum\limits_{r=0}^{\tau} F^{r\prime} \otimes A_\tau^{S_0} \text{-} I_{nk} \right] & \cdots \end{bmatrix}$$

The last $S - S_0$ row blocks look like

$$\begin{bmatrix} \delta_{S_0+1} [\sum\limits_{r=0}^{\tau} A_r^{S_0+1} \text{-} I] & 0 & \cdots & \delta_{S_0+1} \sum\limits_{r=0}^{\tau} A_\tau^S & 0 \\ 0 & \delta_{S_0+1} K [\sum\limits_{r=0}^{\tau} F^{r\prime} \otimes A_r^{S_0+1} \text{-} I] & \cdots & 0 & \delta_{S_0+1} K \sum\limits_{r=0}^{\tau} F^{r\prime} \otimes A_\tau^S \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_S \sum\limits_{r=0}^{\tau} A_\tau^{S_0+1} & 0 & \cdots & \delta_S [\sum\limits_{r=0}^{\tau} A_\tau^S \text{-} I] & 0 \\ 0 & \delta_S K \sum\limits_{r=0}^{\tau} F^{r\prime} \otimes A_\tau^{S_0+1} & \cdots & 0 & \delta_S K [\sum\limits_{r=0}^{\tau} F^{r\prime} \otimes A_\tau^S \text{-} I] \end{bmatrix}$$

$K = M_w \otimes I_n$.

From the blocked structure, it is clear that the stability of this Jacobian can be studied using two unrelated matrix blocks — for $a$'s and for $b$' s — the ones stated in Criterion 1.1, *Q.E.D.*

### A.7.4 Proof of Criterion 1.2

**The associated ODE** for the SRA of Model II looks like

$$\begin{aligned}\frac{d\Phi_h}{d\tau} &= \delta_h\left(T(\underbrace{(\Phi_1',...,\Phi_s')'}_{\Phi})-\Phi_h\right), h=\overline{1,S_0}\\ \frac{d\Phi_h}{d\tau} &= \delta_h M_{1yw}\left(T(\underbrace{(\Phi_1',...,\Phi_s')'}_{\Phi})-\Phi_h\right), h=\overline{S_0+1,S},\end{aligned}$$

where the $T$-map is given by

$$T\begin{bmatrix} a_{1,t}\\ b_{1,t}\\ c_{1,t}\\ \vdots\\ a_{S,t}\\ b_{S,t}\\ c_{S,t}\end{bmatrix}=\begin{bmatrix}\left[\alpha+\sum\limits_{h=1}^{S}\left[A_0^h a_{h,t}+A_1^h a_{h,t}+A_1^h b_{h,t}a_{h,t}\right]\right]'\\ \left[L+\left(\sum\limits_{h=1}^{S}A_0^h b_{h,t}\right)+\left(\sum\limits_{h=1}^{S}A_1^h b_{h,t}^2\right)\right]'\\ \left[\left(\sum\limits_{h=1}^{S}A_0^h c_{h,t}\right)+\left(\sum\limits_{h=1}^{S}A_1^h b_{h,t}c_{h,t}\right)+\left(\sum\limits_{h=1}^{S}A_1^h c_{h,t}F\right)+BF\right]'\end{bmatrix}\equiv\begin{bmatrix}T_a(\Phi_t)\\ T_b(\Phi_t)\\ T_c(\Phi_t)\end{bmatrix}.$$

First transpose the ODE

$$\begin{aligned}\frac{d\Phi_h'}{d\tau} &= \delta_h\left(T'(\Phi)-\Phi_h'\right), h=\overline{1,S_0}\\ \frac{d\Phi_h'}{d\tau} &= \delta_h\left(T'(\Phi)-\Phi_h'\right)M_{1yw}', h=\overline{S_0+1,S},\end{aligned}$$

then vectorize with the $vec$ operator using the rules $vec(AB)=B'\otimes I_n vec(A)$, where $A$ has dimension $n\times l$ and $B$ - $l\times m$, take $(T'(\Phi)-\Phi_h')$ as $A$ , $M_z'$ as $B$, $d\left(vec(X)\right)=vec(dX)$

$$\begin{aligned}\frac{dvec\left(\Phi_h'\right)}{d\tau} &= \delta_h vec\left(T'(\Phi)-\Phi_h'\right), h=\overline{1,S_0}\\ \frac{dvec(\Phi_h')}{d\tau} &= \delta_h M_{1yw}\otimes I_n vec\left(T'(\Phi)-\Phi_h'\right), h=\overline{S_0+1,S}.\end{aligned}$$

After substituting for $T'(\Phi)$, I obtain:

for $h=\overline{1,S_0}$

$$\frac{dvec\left(\Phi_h'\right)}{d\tau}=\delta_h vec\left(\begin{bmatrix}\left[\alpha+\sum_{h=1}^{S}\left[A_0^h a_{h,t}+A_1^h a_{h,t}+A_1^h b_{h,t}a_{h,t}\right]\right]' \\ \left[L+\left(\sum_{h=1}^{S}A_0^h b_{h,t}\right)+\left(\sum_{h=1}^{S}A_1^h b_{h,t}^2\right)\right]' \\ \left[\left(\sum_{h=1}^{S}A_0^h c_{h,t}\right)+\left(\sum_{h=1}^{S}A_1^h b_{h,t}c_{h,t}\right)+\left(\sum_{h=1}^{S}A_1^h c_{h,t}F\right)+BF\right]'\end{bmatrix}' - \begin{bmatrix}a_{h,t}' \\ b_{h,t}' \\ c_{h,t}'\end{bmatrix}'\right);$$

for $h=\overline{S_0+1,S}$

$$\frac{dvec(\Phi_h')}{d\tau}=\delta_h M_{1yw}\otimes I_n vec\left(\begin{bmatrix}\left[\alpha+\sum_{h=1}^{S}\left[A_0^h a_{h,t}+A_1^h a_{h,t}+A_1^h b_{h,t}a_{h,t}\right]\right]' \\ \left[L+\left(\sum_{h=1}^{S}A_0^h b_{h,t}\right)+\left(\sum_{h=1}^{S}A_1^h b_{h,t}^2\right)\right]' \\ \left[\sum_{h=1}^{S}(A_0^h+A_1^h b_{h,t})c_{h,t}+\left(\sum_{h=1}^{S}A_1^h c_{h,t}F\right)+BF\right]'\end{bmatrix}' - \begin{bmatrix}a_{h,t}' \\ b_{h,t}' \\ c_{h,t}'\end{bmatrix}'\right).$$

Using $vec(ABC) = (C' \otimes A)vec(B)$:

for $h=\overline{1,S_0}$

$$\frac{dvec\left(\Phi_h'\right)}{d\tau}=\delta_h\left(\begin{bmatrix}\left[\alpha+\sum_{h=1}^{S}\left[A_0^h a_{h,t}+A_1^h a_{h,t}+A_1^h b_{h,t}a_{h,t}\right]\right] \\ vec\left[L+\left(\sum_{h=1}^{S}A_0^h b_{h,t}\right)+\left(\sum_{h=1}^{S}A_1^h b_{h,t}^2\right)\right] \\ \left[I_k\otimes\sum_{h=1}^{S}\left(A_0^h+A_1^h b_{h,t}\right)+F'\otimes\sum_{h=1}^{S}A_1^h\right]vecc_{h,t}+vecBF\end{bmatrix}-\begin{bmatrix}a_{h,t} \\ b_{h,t}^1 \\ \vdots \\ b_{h,t}^n \\ c_{h,t}^1 \\ \vdots \\ c_{h,t}^k\end{bmatrix}\right);$$

for $h=\overline{S_0+1,S}$

$$\frac{dvec(\Phi_h')}{d\tau}=\delta_h M_{1yw}\otimes I_n\left(\begin{bmatrix}\left[\alpha+\sum_{h=1}^{S}\left[A_0^h a_{h,t}+A_1^h a_{h,t}+A_1^h b_{h,t}a_{h,t}\right]\right] \\ vec\left[L+\left(\sum_{h=1}^{S}A_0^h b_{h,t}\right)+\left(\sum_{h=1}^{S}A_1^h b_{h,t}^2\right)\right] \\ \left[I_k\otimes\sum_{h=1}^{S}\left(A_0^h+A_1^h b_{h,t}\right)+F'\otimes\sum_{h=1}^{S}A_1^h\right]vecc_{h,t}+vecBF\end{bmatrix}-\begin{bmatrix}a_{h,t} \\ b_{h,t}^1 \\ \vdots \\ b_{h,t}^n \\ c_{h,t}^1 \\ \vdots \\ c_{h,t}^k\end{bmatrix}\right).$$

Now, I have to compute the Jacobian of the right-hand side of this ODE, that is, to take the derivative with respect to $dvec(\Phi')=\left((a_{1,t})',(b_{1,t}^1)',...,(b_{1,t}^k)',...,(a_{S,t})',\left(b_{S,t}^1\right)',...,\left(b_{S,t}^k\right)'\right)'$.

Using (as $d(AX^2)=A\left[(dX)X+XdX\right]$, $vec(AB)=(B'\otimes I_n)\,vec(A)=(I_m\otimes A)\,vecB$, where $A$ has dimension $n\times l$ and $B$ - $l\times m$) $d\,(vec(X))=vec(dX)$, $vec(ABC)=(C'\otimes A)vec(B)$)

$$dvec\left[L+\left(\sum_{h=1}^{S}A_0^h b_{h,t}\right)+\left(\sum_{h=1}^{S}A_1^h b_{h,t}^2\right)\right]/dvec\,(b_{h,t})=$$

$$=vecd\left[L+\left(\sum_{h=1}^{S}A_0^h b_{h,t}\right)+\left(\sum_{h=1}^{S}A_1^h b_{h,t}^2\right)\right]/dvec\,(b_{h,t})=$$

$$=\left(vec\sum_{h=1}^{S}A_0^h db_{h,t}+vec\left[\sum_{h=1}^{S}A_1^h\,(db_{h,t})\,b_{h,t}+A_1^h b_{h,t}\,(db_{h,t})\right]\right)/dvec\,(b_{h,t})=$$

$$=\left(\sum_{h=1}^{S}I_n\otimes\left(A_0^h+A_1^h b_{h,t}\right)dvecb_{h,t}+\sum_{h=1}^{S}\left(b'_{h,t}\otimes A_1^h\right)dvecb_{h,t}\right)/dvec\,(b_{h,t})=$$

$$=I_n\otimes\left(A_0^h+A_1^h b_{h,t}\right)+b'_{h,t}\otimes A_1^h.$$

I arrive at the following structure of the Jacobian

$$J=\begin{bmatrix}\delta_1\left[R^1-I\right] & \cdots & \delta_1 R^{S_0} & \delta_1 R^{S_0+1} & \cdots & \delta_1 R^S\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \delta_{S_0}R^1 & \cdots & \delta\left[_{S_0}R^{S_0}-I\right] & \delta_{S_0}R^{S_0+1} & \cdots & \delta_{S_0}R^S\\ \delta_{S_0+1}KR^1 & \cdots & \delta_{S_0+1}KR^{S_0} & \delta_{S_0+1}K\left[R^{S_0+1}-I\right] & \cdots & \delta_{S_0+1}KR^S\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \delta_S KR^1 & \cdots & \delta_S KR^{S_0} & \delta_S KR^{S_0+1} & \cdots & \delta_S K\left[R^S-I\right]\end{bmatrix},$$

where $I=I_{n+n^2+nk}, K=M_{1yw}\otimes I_n$,

$$R^h=\begin{bmatrix}A_1^h+\left(A_0^h+A_1^h\bar{b}\right) & \bar{a}'\otimes A_1^h & 0\\ 0 & \bar{b}'\otimes A_1^h+I_n\otimes\left(A_0^h+A_1^h\bar{b}\right) & 0\\ \vdots & \bar{c}'\otimes A_1^h & F'\otimes A_1^h+I_k\otimes\left(A_0^h+A_1^h\bar{b}\right)\end{bmatrix}.$$

This Jacobian is clearly the same as the one stated in Criterion 1.2, $Q.E.D.$

### A.7.5 Proof of Corollary 1.3

From the proof of Criterion 1.2, it follows that under heterogeneous RLS learning, the stability of the MSV REE of Model II is governed by the stability of the Jacobian

$$J = \begin{bmatrix} \delta_1 \left[R^1 - I_{n+n^2+nk}\right] & \cdots & \delta_1 R^S \\ \vdots & \ddots & \vdots \\ \delta_S R^1 & \cdots & \delta_S \left[R^S - I_{n+n^2+nk}\right] \end{bmatrix},$$

where

$$R^h = \begin{bmatrix} A_1^h + \left(A_0^h + A_1^h \bar{b}\right) & \bar{a}' \otimes A_1^h & 0 \\ 0 & \bar{b}' \otimes A_1^h + I_n \otimes \left(A_0^h + A_1^h \bar{b}\right) & 0 \\ \vdots & \bar{c}' \otimes A_1^h & F' \otimes A_1^h + I_k \otimes \left(A_0^h + A_1^h \bar{b}\right) \end{bmatrix}.$$

It is clear that since the stability part for $b$'s does not depend on $a$'s and $c$'s, one may require this part to be stable.

$$D_1\Omega_b = \begin{pmatrix} \delta_1 \left[\bar{b}' \otimes A_1^1 + I_n \otimes \left(A_0^1 + A_1^1 \bar{b}\right) - I_{n^2}\right] & \cdots & \delta_1 \left[\bar{b}' \otimes A_1^S + I_n \otimes \left(A_0^S + A_1^S \bar{b}\right)\right] \\ \vdots & \ddots & \vdots \\ \delta_S \left[\bar{b}' \otimes A_1^1 + I_n \otimes \left(A_0^1 + A_1^1 \bar{b}\right)\right] & \cdots & \delta_S \left[\bar{b}' \otimes A_1^S + I_n \otimes \left(A_0^S + A_1^S \bar{b}\right) - I_{n^2}\right] \end{pmatrix}.$$

This would mean that $b$ converges to $\bar{b}$. And given this convergence of $b$'s, the convergence of $a$'s and $c$'s is provided by the remaining "own" Jacobian subsystems for $a$'s and $c$'s, respectively

$$D_1\Omega = \begin{pmatrix} \delta_1 \left[A_0^1 + A_1^1 + A_1^1 \bar{b} - I_n\right] & \cdots & \delta_1 \left[A_0^S + A_1^S + A_1^S \bar{b}\right] \\ \vdots & \ddots & \vdots \\ \delta_S \left[A_0^1 + A_1^1 + A_1^1 \bar{b}\right] & \cdots & \delta_S \left[A_0^S + A_1^S + A_1^S \bar{b} - I_n\right] \end{pmatrix},$$

$$D_w\Omega_F = \begin{pmatrix} \delta_1 \left[F' \otimes A_1^1 + I_k \otimes \left(A_0^1 + A_1^1 \bar{b}\right) - I\right] & \cdots & \delta_1 \left[F' \otimes A_1^S + I_k \otimes \left(A_0^S + A_1^S \bar{b}\right)\right] \\ \vdots & \ddots & \vdots \\ \delta_S \left[F' \otimes A_1^1 + I_k \otimes \left(A_0^1 + A_1^1 \bar{b}\right)\right] & \cdots & \delta_S \left[F' \otimes A_1^S + I_k \otimes \left(A_0^S + A_1^S \bar{b}\right) - I\right] \end{pmatrix},$$

$I = I_{nk}$.

These are the sufficient conditions of Corollary 1.3, *Q.E.D.*

### A.7.6 Proof of Proposition 1.4

I have to consider conditions for stability for any positive $(\delta_1, ..., \delta_S)$ of the following matrices:

$$D_1\Omega = \begin{pmatrix} \delta_1 I_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_n \end{pmatrix} \begin{pmatrix} A_0^1 + ... + A_\tau^1 - I_n & \cdots & A_0^S + ... + A_\tau^S \\ \vdots & \ddots & \vdots \\ A_0^1 + ... + +A_\tau^1 & \cdots & A_0^S + ... + A_\tau^S - I_n \end{pmatrix}$$

and

$$D_w\Omega_F = \begin{pmatrix} D_{w1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{wS} \end{pmatrix} \times$$

$$\times \begin{pmatrix} F'^\tau \otimes A_\tau^1 + ... + F' \otimes A_1^1 + I_k \otimes A_0^1 - I_{nk} & \cdots & F'^\tau \otimes A_\tau^S + ... + F' \otimes A_1^S + I_k \otimes A_0^S \\ \vdots & \ddots & \vdots \\ F'^\tau \otimes A_\tau^1 + ... + F' \otimes A_1^1 + I_k \otimes A_0^1 & \cdots & F'^\tau \otimes A_\tau^S + ... + F' \otimes A_1^S + I_k \otimes A_0^S - I_{nk} \end{pmatrix},$$

where $\begin{matrix} D_{wh} = \delta_h I_{nk}, h = \overline{1, S_0} \\ D_{wh} = \delta_h (M_w \otimes I_n), h = \overline{S_0 + 1, S} \end{matrix}$, $F = diag(\rho_1, ..., \rho_k)$, $M_w = diag\left(\frac{\sigma_1^2}{1-\rho_1^2}, ..., \frac{\sigma_k^2}{1-\rho_k^2}\right)$.

The expression for $D_w\Omega_F$ in the diagonal case looks as follows:

$$D_w\Omega_F = diag(\underbrace{\delta_1, \ldots, \delta_1}_{nk}, \ldots, \underbrace{\delta_{S_0}, \ldots, \delta_{S_0}}_{nk}, \underbrace{\frac{\delta_{S_0+1}\sigma_1^2}{1-\rho_1^2}, \ldots, \frac{\delta_{S_0+1}\sigma_1^2}{1-\rho_1^2}}_{n}, \ldots, \underbrace{\frac{\delta_{S_0+1}\sigma_k^2}{1-\rho_k^2}, \ldots, \frac{\delta_{S_0+1}\sigma_k^2}{1-\rho_k^2}}_{n}, \ldots$$
$$\ldots, \underbrace{\frac{\delta_S\sigma_1^2}{1-\rho_1^2}, \ldots, \frac{\delta_S\sigma_1^2}{1-\rho_1^2}}_{n}, \ldots, \underbrace{\frac{\delta_S\sigma_k^2}{1-\rho_k^2}, \ldots, \frac{\delta_S\sigma_k^2}{1-\rho_k^2}}_{n}) \times$$

$$\times \begin{pmatrix} \sum_{i=0}^{\tau} \rho_1^i A_i^1 - I_n & \cdots & 0 & \cdots & \sum_{i=0}^{\tau} \rho_1^i A_i^S & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{i=0}^{\tau} \rho_k^i A_i^1 - I_n & \cdots & 0 & \cdots & \sum_{i=0}^{\tau} \rho_k^i A_i^S \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sum_{i=0}^{\tau} \rho_1^i A_i^1 & \cdots & 0 & \cdots & \sum_{i=0}^{\tau} \rho_1^i A_i^S - I_n & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{i=0}^{\tau} \rho_k^i A_i^1 & \cdots & 0 & \cdots & \sum_{i=0}^{\tau} \rho_k^i A_i^S - I_n \end{pmatrix}.$$

After some permutations of rows and columns that do not change the absolute value of the determinant of $D_w\Omega_F - \mu I$, I obtain that the following characteristic equation for eigenvalues $\mu$ of $D_w\Omega_F$

$$\det\left[D_w\Omega_F - \mu I\right] = 0$$

is equivalent to

$$\begin{aligned} 0 \;=\; & \det[diag((\underbrace{\delta_1,\ldots,\delta_1}_{n},\ldots,\underbrace{\delta_{S_0},\ldots,\delta_{S_0}}_{n},\underbrace{\tfrac{\delta_{S_0+1}\sigma_1^2}{1-\rho_1^2},\ldots,\tfrac{\delta_{S_0+1}\sigma_1^2}{1-\rho_1^2}}_{n},\ldots,\underbrace{\tfrac{\delta_S\sigma_1^2}{1-\rho_1^2},\ldots,\tfrac{\delta_S\sigma_1^2}{1-\rho_1^2}}_{n}),\ldots \\ & \ldots,(\underbrace{\delta_1,\ldots,\delta_1}_{n},\ldots,\underbrace{\delta_{S_0},\ldots,\delta_{S_0}}_{n},\underbrace{\tfrac{\delta_{S_0+1}\sigma_k^2}{1-\rho_k^2},\ldots,\tfrac{\delta_{S_0+1}\sigma_k^2}{1-\rho_k^2}}_{n},\ldots,\underbrace{\tfrac{\delta_S\sigma_k^2}{1-\rho_k^2},...,\tfrac{\delta_S\sigma_k^2}{1-\rho_k^2}}_{n}))\times \\ & \times diag(\begin{bmatrix} \sum\limits_{i=0}^{\tau}\rho_1^iA_i^1 - I_n - \frac{\mu I_n}{\delta_1} & \cdots & \sum\limits_{i=0}^{\tau}\rho_1^iA_i^S \\ \vdots & \ddots & \vdots \\ \sum\limits_{i=0}^{\tau}\rho_1^iA_i^1 & \cdots & \sum\limits_{i=0}^{\tau}\rho_1^iA_i^S - I_n - \frac{(1-\rho_1^2)\mu I_n}{\delta_S\sigma_1^2} \end{bmatrix},\ldots \\ & \ldots,\begin{bmatrix} \sum\limits_{i=0}^{\tau}\rho_k^iA_i^1 - I_n - I_n - \frac{\mu I_n}{\delta_1} & \cdots & \sum\limits_{i=0}^{\tau}\rho_k^iA_i^S \\ \vdots & \ddots & \vdots \\ \sum\limits_{i=0}^{\tau}\rho_k^iA_i^1 - I_n & \cdots & \sum\limits_{i=0}^{\tau}\rho_k^iA_i^S - I_n - \frac{(1-\rho_k^2)\mu I_n}{\delta_S\sigma_k^2} \end{bmatrix})], \end{aligned}$$

or, in matrix form:

$$0 = \det\begin{bmatrix} \tilde{D}_1\Omega_{\rho_1} - \mu I_{nS} & \\ & \\ & \tilde{D}_k\Omega_{\rho k} - \mu I_{nS} \end{bmatrix} = \prod_{l=1}^{k}\det\left[\tilde{D}_l\Omega_{\rho_l} - \mu I_{nS}\right],$$

where

$$\begin{aligned} \tilde{D}_l &= \begin{pmatrix} \delta_1 I_n & & \cdots & & 0 \\ & \ddots & & & \\ & & \delta_{S_0} I_n & & \\ \vdots & & & \frac{\delta_{S_0+1}\sigma_l^2}{1-\rho_l^2}I_n & \vdots \\ & & & \ddots & \\ 0 & & \cdots & & \frac{\delta_S\sigma_l^2}{1-\rho_l^2}I_n \end{pmatrix}, \\ \Omega_{\rho_l} &= \begin{pmatrix} A_0^1 + \rho_l A_1^1 + ... + \rho_l^\tau A_\tau^1 - I_n & \cdots & A_0^S + \rho_l A_1^S + ... + \rho_l^\tau A_\tau^S \\ \vdots & \ddots & \vdots \\ A_0^1 + \rho_l A_1^1 + ... + \rho_l^\tau A_\tau^1 & \cdots & A_0^S + \rho_l A_1^S + ... + \rho_l^\tau A_\tau^S - I_n \end{pmatrix}, \\ l &= \overline{1,k}. \end{aligned}$$

Thus, the analysis of the stability of $D_w\Omega_F$ is equivalent to the analysis of the stability of $\tilde{D}_l\Omega_{\rho_l}$, $\forall l = \overline{1,k}$.

So, the analysis of the stability of $D_w\Omega_F$ can be split into the analysis of the stability of the unrelated matrix blocks. Changing notation $\delta_h := \frac{\delta_h\sigma_l^2}{1-\rho_l^2} > 0$ for $h = \overline{S_0+1,S}$ for each case $l = \overline{1,k}$, I obtain that the analysis of stability of $D_w\Omega_F$ for any $\delta > 0$ is equivalent to the analysis of stability of $k$ matrices $D_1\Omega_{\rho_l}$. Introducing notation $\rho_0 = 1$, I can write the general criterion for stability of a structurally heterogeneous economy under mixed RLS/SG learning for the diagonal environment case under any (possibly different) degrees of inertia of agents, $\delta > 0$, as follows: $D_1\Omega_{\rho_l}$ is stable for all $l = 0, 1, ..., k$. *Q.E.D.*

### A.7.7 Proof of Proposition 1.5

I have to consider the conditions for stability for any positive $(\delta_1, ..., \delta_S)$ of the following matrices $D_1\Omega$ , $D_y\Omega_b$ and $D_w\Omega_F$, where $F = diag(\rho_1, ..., \rho_k)$.

The analysis of stability of $D_w\Omega_F$ and $D_1\Omega$ is equivalent to the analysis of stability of $D_1\Omega_{\rho_l}$, $\forall l = \overline{0,k}$ $(\rho_0 = 1)$ — the result follows from the proof of Proposition 1.4 above, where one has to replace $A_0^h$ with $A_0^h + A_1^h\bar{b}$, set $A_i^h, i > 1$ to zero, and set $S = S_0$. *Q.E.D.*

### A.7.8 Proof of Criterion 1.6

The proof directly follows from the proof of Criterion 1.1, setting $A_0^h \equiv 0$. *Q.E.D.*

### A.7.9 Proof of Criterion 1.7

The proof directly follows from the proof of Criterion 1.2, setting $A_0^h \equiv 0$. *Q.E.D.*

### A.7.10 Proof of Corollary 1.8

The proof directly follows from the proof of Corollary 1.3, setting $A_0^h \equiv 0$. *Q.E.D.*

### A.7.11 Proof of Proposition 1.9

Follows directly from the proof of Proposition 1.4. Set $A_0^h$ to zero.*Q.E.D.*

### A.7.12 Proof of Proposition 1.10

Follows directly from the proof of Proposition 1.5. Set $A_0^h$ to zero. *Q.E.D.*

### A.7.13 Proof of Proposition 1.11

The characteristic equation for eigenvalues of $\Omega_{KR}$ is given by

$$\begin{vmatrix} R^1 - I - \frac{\mu}{\delta_1} I & \cdots & R^{S_0} & R^{S_0+1} & \cdots & R^S \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ R^1 & \cdots & R^{S_0} - I - \frac{\mu}{\delta_{S_0}} I & R^{S_0+1} & \cdots & R^S \\ KR^1 & \cdots & KR^{S_0} & KR^{S_0+1} - K - \frac{\mu}{\delta_{S_{0+1}}} I & \cdots & KR^S \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ KR^1 & \cdots & KR^{S_0} & KR^{S_0+1} & \cdots & KR^S - K - \frac{\mu}{\delta_S} I \end{vmatrix} = 0.$$

For equal degrees of inertia of agents for each type of learning algorithm, $\delta_i = \delta^1$, $\forall i = 1, ..., S_0$, $\delta_i = \delta^1$, $\forall i = S_0 + 1, ..., S$, it simplifies to

$$\begin{vmatrix} R^1 - I - \frac{\mu}{\delta^1} I & \cdots & R^{S_0} & R^{S_0+1} & \cdots & R^S \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ R^1 & \cdots & R^{S_0} - I - \frac{\mu}{\delta^1} I & R^{S_0+1} & \cdots & R^S \\ KR^1 & \cdots & KR^{S_0} & KR^{S_0+1} - K - \frac{\mu}{\delta^2} I & \cdots & KR^S \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ KR^1 & \cdots & KR^{S_0} & KR^{S_0+1} & \cdots & KR^S - K - \frac{\mu}{\delta^2} I \end{vmatrix} = 0.$$

Then, it is possible to obtain the following equivalent algebraic representations for this characteristic equation

$$\begin{vmatrix} -I - \frac{\mu}{\delta^1} I & \cdots & 0 & I + \frac{\mu}{\delta^1} I & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -I - \frac{\mu}{\delta^1} I & I + \frac{\mu}{\delta^1} I & 0 & \cdots & 0 & 0 \\ R^1 & \cdots & R^{S_0-1} & R^{S_0} - I - \frac{\mu}{\delta^1} I & R^{S_0+1} & \cdots & R^{S-1} & R^S \\ 0 & \cdots & 0 & 0 & -K - \frac{\mu}{\delta^2} I & \cdots & 0 & K + \mu I \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & -K - \frac{\mu}{\delta^2} I & K + \mu I \\ KR^1 & \cdots & KR^{S_0-1} & KR^{S_0} & KR^{S_0+1} & \cdots & KR^{S-1} & KR^S - K - \frac{\mu}{\delta^2} I \end{vmatrix} = 0,$$

$$\begin{vmatrix} -I-\frac{\mu}{\delta^1}I & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -I-\frac{\mu}{\delta^1}I & 0 & 0 & \cdots & 0 & 0 \\ R^1 & \cdots & R^{S_0-1} & \sum\limits_1^{S_0} R^h-I-\frac{\mu}{\delta^1}I & R^{S_0+1} & \cdots & R^{S-1} & \sum\limits_{S_0+1}^{S} R^h \\ 0 & \cdots & 0 & 0 & -K-\frac{\mu}{\delta^2}I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & -K-\frac{\mu}{\delta^2}I & 0 \\ KR^1 & \cdots & KR^{S_0-1} & K\sum\limits_1^{S_0} R^h & KR^{S_0+1} & \cdots & KR^{S-1} & K\sum\limits_{S_0+1}^{S} R^h-K-\frac{\mu}{\delta^2}I \end{vmatrix} = 0,$$

$$\left(-1-\frac{\mu}{\delta^1}\right)^{q(S_0-1)} \begin{vmatrix} \sum\limits_{h=1}^{S_0} R^h - I - \frac{\mu}{\delta^1}I & R^{S_0+1} & \cdots & R^{S-1} & \sum\limits_{h=S_0+1}^{S} R^h \\ 0 & -K-\frac{\mu}{\delta^1}I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -K-\frac{\mu}{\delta^2}I & 0 \\ K\sum\limits_{h=1}^{S_0} R^h & KR^{S_0+1} & \cdots & KR^{S-1} & K\sum\limits_{h=S_0+1}^{S} R^h - K - \frac{\mu}{\delta^2}I \end{vmatrix} = 0,$$

$$\left(-1-\frac{\mu}{\delta^1}\right)^{q(S_0-1)} \left[\det\left(-K-\frac{\mu}{\delta^2}I\right)\right]^{S-S_0-1} \begin{vmatrix} \sum\limits_{h=1}^{S_0} R^h - I - \frac{\mu}{\delta^1}I & \sum\limits_{h=S_0+1}^{S} R^h \\ K\sum\limits_{h=1}^{S_0} R^h & K\sum\limits_{h=S_0+1}^{S} R^h - K - \frac{\mu}{\delta^2}I \end{vmatrix} = 0.$$

As $\sum R^h = \begin{bmatrix} 1 & 0 \\ 0 & F'^{\tau} \end{bmatrix} \otimes \sum A_\tau^h + ... + \begin{bmatrix} 1 & 0 \\ 0 & F' \end{bmatrix} \otimes \sum A_1^h + I_{k+1} \otimes \sum A_0^h$, and $K$ is positive definite, then the condition for stability of Model I under mixed RLS/SG learning with equal degrees of inertia of agents for each type of learning algorithm, $\delta_i = \delta^{RLS}$, $\forall i = 1, ..., S_0$, $\delta_i = \delta^{SG}$, $\forall i = S_0+1, ..., S$, is exactly defined by the conditions of stability of the corresponding average economy under mixed RLS/SG learning of two agents with equal degrees of inertia for each type of learning algorithm, $\delta^{RLS}$ for RLS and $\delta^{SG}$ for SG:

$$\begin{vmatrix} \sum\limits_{h=1}^{S_0} R^h - I - \frac{\mu}{\delta^{RLS}}I & \sum\limits_{h=S_0+1}^{S} R^h \\ K\sum\limits_{h=1}^{S_0} R^h & K\sum\limits_{h=S_0+1}^{S} R^h - K - \frac{\mu}{\delta^{SG}}I \end{vmatrix} = 0.$$

Since

$$R^h = \begin{bmatrix} \sum A_1^h + \sum\left(A_0^h + A_1^h\bar{b}\right) & \bar{a}' \otimes \sum A_1^h & 0 \\ 0 & \bar{b}' \otimes \sum A_1^h + I_n \otimes \sum\left(A_0^h + A_1^h\bar{b}\right) & 0 \\ \vdots & \bar{c}' \otimes \sum A_1^h & F' \otimes \sum A_1^h + I_k \otimes \sum\left(A_0^h + A_1^h\bar{b}\right) \end{bmatrix},$$

$\bar{b}$ corresponds to the average economy and is defined by (1.9) for Model II and in (1.16) for Model IV, which may be rewritten as

$$L + \left[ \left( \sum_{h=1}^{S_0} A_0^h \right) + \left( \sum_{h=S_0+1}^{S} A_0^h \right) \right] \bar{b} + \left[ \left( \sum_{h=1}^{S_0} A_1^h \right) + \left( \sum_{h=S_0+1}^{S} A_1^h \right) \right] \bar{b}^2 = \bar{b}$$

(with $A_0^h \equiv 0$ for Model IV), and $K$ is positive definite, the condition for stability of Model II under mixed RLS/SG learning with equal degrees of inertia is exactly defined by the conditions of stability of the corresponding average economy under mixed RLS/SG learning of two agents with equal degrees of inertia.

The proof for Model III is completely similar to the proof for Model I above. Set $A_0^h \equiv 0$.

The proof for Model IV is completely similar to the proof for Model II above. Set $A_0^h \equiv 0$. *Q.E.D.*

### A.7.14 Proof of Proposition 1.12 (for Model I and Model III without lags, the diagonal case, mixed RLS/SG learning)

For $\Omega_{\rho_l}$, repeat basically the same steps as in Kolyuzhnov [40]. Use the "columns" negative diagonal dominance of $\Omega_{\rho_l}$, which is sufficient for the real parts of the eigenvalues of $D_1\Omega_{\rho_l}$ to be negative; look for a condition that would be sufficient for negative diagonal dominance in this setup. As weights for rows use $(\phi_1(\psi_1, ..., \psi_n), ..., \phi_s(\psi_1, ..., \psi_n))$, $\phi_i > 0$, $\psi_h > 0$, $\sum\limits_i \psi_i = 1, \sum\limits_h \phi_h = 1$.

For any $l$ take any block $h$ and any column $j$

$$\left\{ \begin{array}{l} a_{0jj}^h + \rho_l a_{1jj}^h + ... + \rho_l^\tau a_{1jj}^h - 1 < 0 \text{ - negative diagonal} \\ \phi_h \psi_j \left| a_{0jj}^h + \rho_l a_{1jj}^h + ... + \rho_l^\tau a_{1jj}^h - 1 \right| > \\ > (\phi_1 + ... + \phi_s) \sum\limits_i \psi_i \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{1ij}^h \right| - \\ -\phi_h \psi_j \left| a_{0jj}^h + \rho_l a_{1jj}^h + ... + \rho_l^\tau a_{1jj}^h \right| \text{ - dominance} \end{array} \right. \quad \forall j, \forall h, \forall l$$

$$\Updownarrow$$

$$\left\{ \begin{array}{l} a_{0jj}^h + \rho_l a_{1jj}^h + ... + \rho_l^\tau a_{1jj}^h - 1 < 0 \\ -\phi_h \psi_j \left( a_{0jj}^h + \rho_l a_{1jj}^h + ... + \rho_l^\tau a_{1jj}^h \right) + \phi_h \psi_j > \\ > (\phi_1 + ... + \phi_s) \sum\limits_i \psi_i \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{1ij}^h \right| - \\ -\phi_h \psi_j \left| a_{0jj}^h + \rho_l a_{1jj}^h + ... + \rho_l^\tau a_{1jj}^h \right| \end{array} \right. \quad \forall j, \forall h, \forall l$$

$$\Updownarrow$$

$$\text{Case 1} \left\{ \begin{array}{l} 0 \le a_{0jj}^h + \rho_l a_{1jj}^h + ... + \rho_l^\tau a_{1jj}^h < 1 \\ \sum\limits_i \psi_i \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{1ij}^h \right| < \frac{\phi_h \psi_j}{\underbrace{\phi_1 + ... + \phi_S}_{=1}} \end{array} \right. \quad \forall j, \forall h, \forall l$$

$$\cup$$

$$\text{Case 2}\begin{cases} a_{0jj}^h+\rho_l a_{1jj}^h+...+\rho_l^\tau a_{1jj}^h<0 \\ \sum_i \psi_i \left| a_{0ij}^h+\rho_l a_{1ij}^h+...+\rho_l^\tau a_{1ij}^h \right| < \frac{\phi_h\psi_j}{\underbrace{\phi_1+...+\phi_S}_{=1}} - \\ -\frac{2\varphi_h\psi_j}{\underbrace{\phi_1+...+\phi_S}_{=1}}\left(a_{0jj}^h+\rho_l a_{1jj}^h+...+\rho_l^\tau a_{1jj}^h\right) \end{cases} \forall j,\forall h,\forall l.$$

Since in the second case $\rho_l\left(a_{0jj}^h+\rho_l a_{1jj}^h+...+\rho_l^\tau a_{1jj}^h\right)<0$, one may formulate the following sufficient condition $\sum_i \psi_i \left| a_{0ij}^h+\rho_l a_{1ij}^h+...+\rho_l^\tau a_{1ij}^h \right| < \phi_h\psi_j\ \forall j,\forall h,\forall l$. The condition $1>a_{0jj}^h+\rho_l a_{1jj}^h+...+\rho_l^\tau a_{1jj}^h$ is implied by this relation, and the condition of case 2 is also satisfied. To prove that $1>a_{0jj}^h+\rho_l a_{1jj}^h+...+\rho_l^\tau a_{1jj}^h$, notice that

$$\sum_i \psi_i \left| a_{0ij}^h+\rho_l a_{1ij}^h+...+\rho_l^\tau a_{1ij}^h \right| < \phi_h\psi_j \Longrightarrow$$

$$\Longrightarrow \underbrace{\frac{\sum_{i\neq j}\psi_i\left|a_{0ij}^h+\rho_l a_{1ij}^h+...+\rho_l^\tau a_{1ij}^h\right|}{\psi_j}}_{>0}+\underbrace{\left|a_{0jj}^h+\rho_l a_{1jj}^h+...+\rho_l^\tau a_{1jj}^h\right|}_{>0}<\phi_h<1\Longrightarrow$$

$$\Longrightarrow \left|a_{0jj}^h+\rho_l a_{1jj}^h+...+\rho_l^\tau a_{1jj}^h\right|<1\Longrightarrow a_{0jj}^h+\rho_l a_{1jj}^h+...+\rho_l^\tau a_{1jj}^h<1.$$

So this condition alone is sufficient. The derived sufficient condition follows from

$$\sum_i \psi_i\left(\left|a_{0ij}^h\right|+|\rho_l|\left|a_{1ij}^h\right|+...+|\rho_l^\tau|\left|a_{1ij}^h\right|\right)<\phi_h\psi_j\forall j,\forall h,\forall l.$$

This is the condition of Proposition 1.13.(For Model III, set $a_{0ij}^h=0$ everywhere) $Q.E.D.$

### A.7.15 Proof of Proposition 1.15 (for Model I, III without a lag, the general non-diagonal case, heterogeneous RLS learning)

Prove that $\Omega$ and $\Omega_F$ are $D$-stable. May prove just for $\Omega_F$ as a more general case, with the part for $\Omega$ derived then by setting $F=I$.

For $\Omega_F$ use the "columns" negative diagonal dominance of $\Omega_F$, which is sufficient for the real parts of the eigenvalues of $D_w\Omega_F$ to be negative; look for a condition that would be sufficient for negative diagonal dominance in this setup. As weights for rows use $(\underbrace{\phi_1(\psi_1,...,\psi_n),...,\phi_1(\psi_1,...,\psi_n)}_{k},...,\underbrace{\phi_S(\psi_1,...,\psi_n),...,\phi_S(\psi_1,...,\psi_n)}_{k})$, $\phi_i>0$, $\psi_h>0$, $\sum_i \psi_i=1,\sum_h \phi_h=1.$

Take any block $h$, any row $l$ of $F$, and any column $j$

$$\begin{cases} f_{ll}^{\tau}a_{\tau jj}^{h}+...+f_{ll}a_{1jj}^{h}+a_{0jj}^{h}-1<0 \text{ - negative diagonal} \\ \phi_h\psi_j\left|f_{ll}^{\tau}a_{\tau jj}^{h}+...+f_{ll}a_{1jj}^{h}+a_{0jj}^{h}-1\right|>(\phi_1+...+\phi_s)\sum\limits_i\psi_i(\sum_{r=1}^{k}\left|f_{lr}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+... \\ +\sum_{r=1}^{k}\left|f_{lr}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|)-\phi_h\psi_j\left|f_{ll}^{\tau}a_{\tau jj}^{h}+...+f_{ll}a_{1jj}^{h}+a_{0jj}^{h}\right| \text{ - dominance} \end{cases} \forall j,\forall h,\forall l$$

$$\Updownarrow$$

$$\begin{cases} f_{ll}^{\tau}a_{\tau jj}^{h}+...+f_{ll}a_{1jj}^{h}+a_{0jj}^{h}-1<0 \\ -\phi_h\psi_j\left(f_{ll}^{\tau}a_{\tau jj}^{h}+...+f_{ll}a_{1jj}^{h}+a_{0jj}^{h}\right)+\phi_h\psi_j>(\phi_1+...+\phi_s)\sum\limits_i\psi_i(\sum_{r=1}^{k}\left|f_{lr}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+... \\ +\sum_{r=1}^{k}\left|f_{lr}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|)-\phi_h\psi_j\left|f_{ll}^{\tau}a_{\tau jj}^{h}+...+f_{ll}a_{1jj}^{h}+a_{0jj}^{h}\right| \end{cases} \forall j,\forall h,\forall l$$

$$\Updownarrow$$

$$\text{Case 1}\begin{cases} 0\le f_{ll}^{\tau}a_{\tau jj}^{h}+...+f_{ll}a_{1jj}^{h}+a_{0jj}^{h}<1 \\ \sum\limits_i\psi_i\left(\sum_{r=1}^{k}\left|f_{lr}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{lr}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|\right)<\frac{\phi_h\psi_j}{\underbrace{\phi_1+...+\phi_S}_{=1}} \end{cases} \forall j,\forall h,\forall l$$

$$\cup$$

$$\text{Case 2}\begin{cases} f_{ll}^{\tau}a_{\tau jj}^{h}+...+f_{ll}a_{1jj}^{h}+a_{0jj}^{h}<0 \\ \sum\limits_i\psi_i\left(\sum_{r=1}^{k}\left|f_{lr}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{lr}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|\right)< \\ <\frac{\phi_h\psi_j}{\underbrace{\phi_1+...+\phi_S}_{=1}}-\frac{2\varphi_h\psi_j}{\underbrace{\phi_1+...+\phi_S}_{=1}}\left(f_{ll}^{\tau}a_{\tau jj}^{h}+...+f_{ll}a_{1jj}^{h}+a_{0jj}^{h}\right) \end{cases} \forall j,\forall h,\forall l.$$

Since in the second case $f_{ll}^{\tau}a_{\tau jj}^{h}+...+f_{ll}a_{1jj}^{h}+a_{0jj}^{h}<0$, formulate the following sufficient condition

$$\sum_i\psi_i\left(\sum_{r=1}^{k}\left|f_{lr}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{lr}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|\right)<\phi_h\psi_j\forall j,\forall h,\forall l.$$

The condition $1>f_{ll}^{\tau}a_{\tau jj}^{h}+...+f_{ll}a_{1jj}^{h}+a_{0jj}^{h}$ is implied by this relation, and the condition of case 2 is also satisfied. To prove that $1>f_{ll}^{\tau}a_{\tau jj}^{h}+...+f_{ll}a_{1jj}^{h}+a_{0jj}^{h}$, notice that

$$\sum_i\psi_i\left(\sum_{r=1}^{k}\left|f_{lr}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{lr}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|\right)<\phi_h\psi_j\Longrightarrow$$

$$\Longrightarrow\underbrace{\frac{\sum\limits_{i\neq j}\psi_i(\sum_{r=1}^{k}|f_{lr}^{\tau}||a_{\tau ij}^{h}|+...+\sum_{r=1}^{k}|f_{lr}||a_{1ij}^{h}|+|a_{0ij}^{h}|)}{\psi_j}}+$$

$$+\underbrace{\left(\sum_{r=1}^{k}\left|f_{lr}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{lr}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|\right)}_{>0}<\phi_h<1\Longrightarrow$$

$$\Longrightarrow\sum_{r=1}^{k}\left|f_{lr}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{lr}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|<1\Longrightarrow$$

$$\Longrightarrow |f_{ll}^{\tau}| \left|a_{\tau ij}^{h}\right| + ... + |f_{ll}| \left|a_{1ij}^{h}\right| + \left|a_{0ij}^{h}\right| < 1 \Longrightarrow \left|f_{ll}^{\tau} a_{\tau ij}^{h} + ... + f_{ll} a_{1ij}^{h} + a_{0ij}^{h}\right| < 1 \Longrightarrow$$

$$\Longrightarrow f_{ll}^{\tau} a_{\tau jj}^{h} + ... + f_{ll} a_{1jj}^{h} + a_{0jj}^{h} < 1.$$

So this condition alone is sufficient for $\delta$-stability. For Model III, set $a_{0ij}^{h}$ to zero everywhere above. Thus, this is the condition of Proposition 1.15. *Q.E.D.*

### A.7.16 Proof of Proposition 1.16

Prove that $\Omega$, $\Omega_F$, and $\Omega_b$ corresponding to the models are $D$-stable. For $\Omega$ and $\Omega_F$, the proof is similar to the proof of Proposition 1.15 above. Replace $a_{0ij}^{h}$ with $a_{0ij}^{h} + \left(A_1^h \bar{b}\right)_{ij}$ and set $\sum_{r=1}^{k} |f_{lr}^{\tau}|$ to zero for $\tau > 1$ for Model II, and replace $a_{0ij}^{h}$ with $\left(A_1^h \bar{b}\right)_{ij}$ and set $\sum_{r=1}^{k} |f_{lr}^{\tau}|$ to zero for $\tau > 1$ for Model IV.

Proof for $\Omega_b$ is similar to the proof for $\Omega_F$. As weights for rows use

$$(\underbrace{\phi_1(\psi_1, ..., \psi_n), ..., \phi_1 (\psi_1, ..., \psi_n)}_{n}, ..., \underbrace{\phi_S(\psi_1, ..., \psi_n), ..., \phi_S (\psi_1, ..., \psi_n)}_{n}), \phi_i > 0, \psi_h > 0,$$

$\sum\limits_i \psi_i = 1, \sum\limits_h \phi_h = 1$, where the number of repetitions $k$ from the previous case is replaced for $n$ to reflect the dimension of $\bar{b}$. Then everywhere in the proof, replace $F$ with $\bar{b}$ and use summations of elements in rows of $\bar{b}$ up to $n$ and not up to $k$. *Q.E.D.*

### A.7.17 Proof of Proposition 1.17

**For** $\beta_l^{AG\,\mathrm{mod}}(\psi, \phi)$:

**For Model I**

1. $$\beta_l^{AG\,\mathrm{mod}}(\psi, \phi)\Big|_{\substack{\phi-any \\ \psi-any}} = S \sum_h \phi_h \sum_i \psi_i \sum_j \left|a_{0ij}^{h} + \rho_l a_{1ij}^{h} + ... + \rho_l^{\tau} a_{\tau ij}^{h}\right| \leq$$

$$\leq S \sum_h \phi_h \sum_j \sum_i \psi_i \max_{h,i} \left|a_{0ij}^{h} + \rho_l a_{1ij}^{h} + ... + \rho_l^{\tau} a_{\tau ij}^{h}\right| =$$

$$= S \sum_j \underbrace{\left(\sum_h \sum_i \phi_h \psi_i\right)}_{=1} \max_{h,i} \left|a_{0ij}^{h} + \rho_l a_{1ij}^{h} + ... + \rho_l^{\tau} a_{\tau ij}^{h}\right| \leq$$

$$= \max_l S \sum_j \max_{h,i} \left|a_{0ij}^{h} + \rho_l a_{1ij}^{h} + ... + \rho_l^{\tau} a_{\tau ij}^{h}\right| = \beta_1^{AG\,\max}.$$

2. $$\beta^{AG\,\mathrm{mod}}(\psi, \phi)\Big|_{\substack{\phi=\frac{1}{S} \\ \psi-any}} = S \sum_h \underbrace{\frac{1}{S}}_{\phi_h} \sum_i \psi_i \sum_j \left|a_{0ij}^{h} + \rho_l a_{1ij}^{h} + ... + \rho_l^{\tau} a_{\tau ij}^{h}\right| =$$

$$= \sum_h \sum_i \psi_i \sum_j \left|a_{0ij}^{h} + \rho_l a_{1ij}^{h} + ... + \rho_l^{\tau} a_{\tau ij}^{h}\right| \leq$$

$$\leq \underbrace{\left(\sum_i \psi_i\right)}_{=1} \max_i \sum_h \sum_j \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \leq$$

$$\leq \max_l \max_i \sum_h \sum_j \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| = \beta_2^{AG\max}.$$

3. $\beta^{AG\bmod}(\psi, \phi)\big|_{\substack{\phi-any \\ \psi=\frac{1}{n}}} = S \sum_h \phi_h \sum_i \underbrace{\frac{1}{n}}_{\psi_i} \sum_j \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \leq$

$$\leq S \sum_i \frac{1}{n} \sum_h \sum_j \phi_h \max_{h,j} \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| =$$

$$= S \sum_i \frac{1}{n} \max_{h,j} \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \underbrace{\left( \sum_h \sum_j \phi_h \right)}_{=n} =$$

$$= S \sum_i \max_{h,j} \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \leq$$

$$\leq \max_l S \sum_i \max_{h,j} \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| = \beta_3^{AG\max}.$$

4. $\beta^{AG\bmod}(\psi, \phi)\big|_{\substack{\phi=\frac{1}{S} \\ \psi=\frac{1}{n}}} = S \sum_h \underbrace{\frac{1}{S}}_{\phi_h} \sum_i \underbrace{\frac{1}{n}}_{\psi_i} \sum_j \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| =$

$$= \sum_h \sum_i \frac{1}{n} \sum_j \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \leq$$

$$\leq \sum_h \frac{1}{n} \sum_j \max_j \sum_i \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| =$$

$$= \sum_h \max_j \sum_i \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \underbrace{\frac{1}{n} \sum_j 1}_{=1} \leq$$

$$\leq \max_l \sum_h \max_j \sum_i \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| = \beta_4^{AG\max}.$$

To prove the proposition for $\beta^{AG\bmod}(\psi, \phi)$ for **Model III**, set $a^h_{0ij} = 0$ in the proof above.

**For** $\beta^{AG\bmod}_{weighted\ l}(\psi, \phi)$:

To prove **for Model I,** replace $a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij}$ with $\left|a^h_{0ij}\right| + \sum_{r=1}^k |f_{lr}| \left|a^h_{1ij}\right| + ... + \sum_{r=1}^k |f^\tau_{lr}| \left|a^h_{\tau ij}\right|$ in the proof above for steps 1 and 4 and with $\left|a^h_{0ij}\right| + \sum_{r=1}^k |f_{rl}| \left|a^h_{1ij}\right| + ... + \sum_{r=1}^k |f^\tau_{rl}| \left|a^h_{\tau ij}\right|$ for steps 2 and 3.

To prove the proposition for $\beta^{AG\bmod}_{weighted\ l}(\psi, \phi)$ for **Model III,** set $a^h_{0ij} = 0$ in the proof for Model I.

For $\beta^{AG\bmod}_{weighted\ p}(\psi, \phi)$:

**for f-type aggregation**

To prove the proposition for **Model II,** replace $a^h_{0ij}$ with $a^h_{0ij} + \left(A^h_1\bar{b}\right)_{ij}$ and set $\sum_{r=1}^k |f^\tau_{lr}|$ and $\sum_{r=1}^k |f^\tau_{lr}|$ to zero for $\tau > 1$ in the proof for Model I.

To prove the proposition for **Model IV,** replace $a^h_{0ij}$ with $\left(A^h_1\bar{b}\right)_{ij}$ and set $\sum_{r=1}^k |f^\tau_{lr}|$ and $\sum_{r=1}^k |f^\tau_{rl}|$ to zero for $\tau > 1$ in the proof for Model I.

**for b-type aggregation**

The proof is analogous to the proof for **f-type aggregation**. One only uses $b$ instead of $f$, changes the index of rows and columns from $l$ to $q$, and uses summation up to $n$ and not up to $k.Q.E.D.$

### A.7.18 Proof of Proposition 1.18

**For Model I**

1. for $\beta_1^{AG\max} = \max\limits_l S \sum\limits_j \max\limits_{h,i} \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right|$:

$\beta_1^{AG\max} = \max\limits_l S \sum\limits_j \max\limits_{h,i} \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| < 1$. Let's prove that there exist weights $\psi$ and $\phi$ such that

$$\sum_i \psi_i \left( a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right) / \psi_j < \phi_h \forall j, \forall h, \forall l.$$

Let us take $\phi_h = \frac{1}{S}$ $\forall h$, and

$$\psi_j = S \max_{h,i} \left( \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \right) + \frac{\overbrace{1 - S \sum_j \max_{h,i} \left( \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \right)}^{>0}}{n} \forall j, \forall l.$$

These can be considered as weights since

$$\sum_{h=1}^S \phi_h = 1, 0 < \phi_h < 1 \text{ and } \sum_{j=1}^n \psi_j = 1, 0 < \psi_j < 1.$$

Notice that

$$\frac{\psi_j}{S} > \max_{h,i} \left( \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \right) = \sum_i \psi_i \max_{h,i} \left( \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \right) \geqq$$

$$\geqq \sum_i \psi_i \left( \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \right), \forall j, \forall h, \forall l,$$

or after rewriting: $\sum\limits_i \psi_i \left( \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \right) < \psi_j \underbrace{\phi_h}_{=\frac{1}{S}}, \forall j, \forall h, \forall l.$

4. for $\beta_4^{AG\max} = \max\limits_l \sum\limits_h \max\limits_j \sum\limits_i \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right|$

$\beta_4^{AG\max} = \max\limits_l \sum\limits_h \max\limits_j \sum\limits_i \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| < 1$. Let's prove that there exist weights $\psi$ and $\phi$ such that

$$\sum_i \psi_i \left( \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \right) / \psi_j < \phi_h \forall j, \forall h, \forall l.$$

Let us take $\psi_j = \frac{1}{n}\ \forall j$,

$$\phi_h = \max_j \sum_i \left( \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \right) + \frac{\overbrace{1 - \sum\limits_h \max\limits_j \sum\limits_i \left( \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \right)}^{>0}}{S} \forall h, \forall l.$$

These are weights as $\sum\limits_{h=1}^{S} \phi_h = 1, 0 < \phi_h < 1$ and $\sum\limits_{j=1}^{S} \psi_j = 1, 0 < \psi_j < 1$.

Notice that

$$\phi_h > \max_j \sum_i \left( \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \right) \geqq \sum_i \left( \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \right), \forall j, \forall h, \forall l$$

or after rewriting: $\dfrac{\sum\limits_i \overbrace{\psi_i}^{\frac{1}{n}} \left( \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \right)}{\underbrace{\psi_j}_{\frac{1}{n}}} < \phi_h, \forall j, \forall h, \forall l.$

To prove the proposition for $\beta_2^{AG\max} = \max\limits_l \max\limits_i \sum\limits_h \sum\limits_j \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right|$ and $\beta_3^{AG\max} = \max\limits_l S \sum\limits_i \max\limits_{h,j} \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right|$, I first derive a sufficient condition for $\delta -$stability that follows from the "rows" diagonal dominance condition, which is also sufficient for stability of matrices $D_1 \Omega_{\rho l}$. Therefore, my derivation of this condition resembles the steps in the proof of Proposition 1.12. Use $(d_1, ..., d_n, ..., d_1, ..., d_n)$, $d_i > 0$, $\sum\limits_i d_i = 1$ as weights for columns.

For any $l$, take any block $h$ and any row $i$.

$$\begin{cases} a_{0ii}^h + \rho_l a_{1ii}^h + ... + \rho_l^\tau a_{\tau ii}^h - 1 < 0 \text{ - negative diagonal} \\ d_i \left| a_{0ii}^h + \rho_l a_{1ii}^h + ... + \rho_l^\tau a_{\tau ii}^h - 1 \right| > \sum\limits_h \sum\limits_j d_j \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| - \quad \forall i, \forall h, \forall l \\ -d_i \left| a_{0ii}^h + \rho_l a_{1ii}^h + ... + \rho_l^\tau a_{\tau ii}^h \right| \text{ - dominance} \end{cases}$$

$\Updownarrow$

$$\begin{cases} a_{0ii}^h + \rho_l a_{1ii}^h + ... + \rho_l^\tau a_{\tau ii}^h - 1 < 0 \\ -d_i \left( a_{0ii}^h + \rho_l a_{1ii}^h + ... + \rho_l^\tau a_{\tau ii}^h \right) + d_i > \sum\limits_h \sum\limits_j d_j \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| - \quad \forall i, \forall h, \forall l \\ -d_i \left| a_{0ii}^h + \rho_l a_{1ii}^h + ... + \rho_l^\tau a_{\tau ii}^h \right| \end{cases}$$

$\Updownarrow$

$$\text{Case 1}\begin{cases} 0 \leq a^h_{0ii} + \rho_l a^h_{1ii} + ... + \rho^\tau_l a^h_{\tau ii} < 1 \\ \sum\limits_h \sum\limits_j d_j \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| < d_i \end{cases} \forall i, \forall h, \forall l$$

$$\cup$$

$$\text{Case 2}\begin{cases} a^h_{0ii} + \rho_l a^h_{1ii} + ... + \rho^\tau_l a^h_{\tau ii} < 0 \\ \sum\limits_h \sum\limits_j d_j \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| < d_i\text{-}2d_i \left( a^h_{0ii} + \rho_l a^h_{1ii} + ... + \rho^\tau_l a^h_{\tau ii} \right) \end{cases} \forall i, \forall h, \forall l.$$

Since in the second case $a^h_{0ii} + \rho_l a^h_{1ii} + ... + \rho^\tau_l a^h_{\tau ii} < 0$, formulate the following sufficient condition: $\sum\limits_h \sum\limits_j d_j \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| < d_i \ \forall i, \forall h, \forall l$. The condition $1 > a^h_{0ii} + \rho_l a^h_{1ii} + ... + \rho^\tau_l a^h_{\tau ii}$ is implied by this relation, and the condition of case 2 is also satisfied. To prove that $1 > a^h_{0ii} + \rho_l a^h_{1ii} + ... + \rho^\tau_l a^h_{\tau ii}$, notice that $\sum\limits_h \sum\limits_j d_j \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| < d_i \Longrightarrow$

$$\Longrightarrow \underbrace{\sum_h \sum_{j \neq i} d_j \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right|}_{>0} +$$

$$\underbrace{+ \sum_h d_i \left| a^h_{0ii} + \rho_l a^h_{1ii} + ... + \rho^\tau_l a^h_{\tau ii} \right|}_{>0} < d_i < 1 \Longrightarrow$$

$$\Longrightarrow \left| a^h_{0ii} + \rho_l a^h_{1ii} + ... + \rho^\tau_l a^h_{\tau ii} \right| < 1 \Longrightarrow a^h_{0ii} + \rho_l a^h_{1ii} + ... + \rho^\tau_l a^h_{\tau ii} < 1.$$

So this condition alone is sufficient for $\delta$–stability.

Next, I use the derived sufficient condition to prove Proposition 4.18 for $\beta_2^{AG\max} = \max\limits_l \max\limits_i \sum\limits_h \sum\limits_j \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right|$ and $\beta_3^{AG\max} = \max\limits_l S \sum\limits_i \max\limits_{h,j} \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right|$.

2. for $\beta_2^{AG\max} = \max\limits_l \max\limits_i \sum\limits_h \sum\limits_j \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right|$:

$\beta_2^{AG\max} = \max\limits_l \max\limits_i \sum\limits_h \sum\limits_j \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| < 1$. Let's prove that there exist weights $d = (d_1, ..., d_n, ..., d_1, ..., d_n)$, $d_i > 0$, $\sum\limits_i d_i = 1$, such that

$$\sum_h \sum_j d_j \left( \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \right) < d_i \forall i, \forall h, \forall l.$$

Let us take $d_j = \frac{1}{n} \ \forall j$.

Notice that

$$\sum_h \sum_j \left( \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \right) \leqq \max_i \sum_h \sum_j \left( \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \right) < 1, \forall i, \forall h, \forall l,$$

or after rewriting: $\sum\limits_h \sum\limits_j \underbrace{\frac{1}{n}}_{d_j} \left( \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right| \right) < \underbrace{\frac{1}{n}}_{d_i}, \forall i, \forall h, \forall l.$

3. for $\beta_3^{AG\max} = \max\limits_l S \sum\limits_i \max\limits_{h,j} \left| a^h_{0ij} + \rho_l a^h_{1ij} + ... + \rho^\tau_l a^h_{\tau ij} \right|$:

$\beta_3^{AG\max} = \max_l S \sum_i \max_{h,j} \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| < 1$. Let's prove that there exist weights $d = (d_1, ..., d_n, ..., d_1, ..., d_n)$, $d_i > 0$, $\sum_i d_i = 1$, such that

$$\sum_h \sum_j d_j \left( \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \right) < d_i \forall i, \forall h, \forall l.$$

Let us take

$$d_i = S \max_{h,j} \left( \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \right) + \frac{\overbrace{1 - S \sum_i \max_{h,j} \left( \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \right)}^{>0}}{n} \forall i, \forall l.$$

These can be taken as weights since $\sum_{i=1}^n d_i = 1, 0 < d_i < 1$.

Notice that

$$d_i > S \max_{h,j} \left( \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \right) = \underbrace{\sum_{j=1}^n d_j}_{=1} \underbrace{\sum_{h=1}^S}_{=S} \max_{h,j} \left( \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \right) \geqq$$

$$\geqq \sum_h \sum_j d_j \left( \left| a_{0ij}^h + \rho_l a_{1ij}^h + ... + \rho_l^\tau a_{\tau ij}^h \right| \right), \forall i, \forall h, \forall l.$$

To prove the proposition for **Model III**, just set $a_{0ij}^h = 0$ in the proof above.

### A.7.19 Proof of Proposition 1.20

**For Model I**

1. for $\beta_1^{AG\max} = \max_l S \sum_j \max_{h,i} \left( \left| a_{0ij}^h \right| + \sum_{r=1}^k |f_{lr}| \left| a_{1ij}^h \right| + ... + \sum_{r=1}^k |f_{lr}^\tau| \left| a_{\tau ij}^h \right| \right)$:

$$\beta_1^{AG\max} = \max_l S \sum_j \max_{h,i} \left( \left| a_{0ij}^h \right| + \sum_{r=1}^k |f_{lr}| \left| a_{1ij}^h \right| + ... + \sum_{r=1}^k |f_{lr}^\tau| \left| a_{\tau ij}^h \right| \right) < 1.$$

Let's prove that there exist weights $\psi$ and $\phi$ such that

$$\sum_i \psi_i \left( \left| a_{0ij}^h \right| + \sum_{r=1}^k |f_{lr}| \left| a_{1ij}^h \right| + ... + \sum_{r=1}^k |f_{lr}^\tau| \left| a_{\tau ij}^h \right| \right) / \psi_j < \phi_h \forall j, \forall h, \forall l.$$

Let us take $\phi_h = \frac{1}{S}$ $\forall h$, and

$$\psi_j = S \max_{h,i} \left( \left| a_{0ij}^h \right| + \sum_{r=1}^k |f_{lr}| \left| a_{1ij}^h \right| + ... + \sum_{r=1}^k |f_{lr}^\tau| \left| a_{\tau ij}^h \right| \right) +$$

$$+ \frac{\overbrace{1 - S \sum_j \max_{h,i} \left( \left| a_{0ij}^h \right| + \sum_{r=1}^k |f_{lr}| \left| a_{1ij}^h \right| + ... + \sum_{r=1}^k |f_{lr}^\tau| \left| a_{\tau ij}^h \right| \right)}^{>0}}{n} \forall j, \forall l.$$

These can be considered as weights since

$$\sum_{h=1}^{S}\phi_h = 1, 0<\phi_h<1 and \sum_{j=1}^{n}\psi_j = 1, 0<\psi_j<1.$$

Notice that

$$\frac{\psi_j}{S} > \max_{h,i}\left(\left|a_{0ij}^h\right| + \sum_{r=1}^{k}|f_{lr}|\left|a_{1ij}^h\right| + ... + \sum_{r=1}^{k}|f_{lr}^{\tau}|\left|a_{\tau ij}^h\right|\right) =$$

$$= \sum_i \psi_i \max_{h,i}\left(\left|a_{0ij}^h\right| + \sum_{r=1}^{k}|f_{lr}|\left|a_{1ij}^h\right| + ... + \sum_{r=1}^{k}|f_{lr}^{\tau}|\left|a_{\tau ij}^h\right|\right) \geqq$$

$$\geqq \sum_i \psi_i \left(\left|a_{0ij}^h\right| + \sum_{r=1}^{k}|f_{lr}|\left|a_{1ij}^h\right| + ... + \sum_{r=1}^{k}|f_{lr}^{\tau}|\left|a_{\tau ij}^h\right|\right), \forall j, \forall h, \forall l,$$

or after rewriting:

$$\sum_i \psi_i \left(\left|a_{0ij}^h\right| + \sum_{r=1}^{k}|f_{lr}|\left|a_{1ij}^h\right| + ... + \sum_{r=1}^{k}|f_{lr}^{\tau}|\left|a_{\tau ij}^h\right|\right) < \psi_j \underbrace{\phi_h}_{=\frac{1}{S}}, \forall j, \forall h, \forall l.$$

4. for $\beta_4^{AG\max} = \max\limits_l \sum\limits_h \max\limits_j \sum\limits_i \left(\left|a_{0ij}^h\right| + \sum_{r=1}^{k}|f_{lr}|\left|a_{1ij}^h\right| + ... + \sum_{r=1}^{k}|f_{lr}^{\tau}|\left|a_{\tau ij}^h\right|\right)$:

$$\beta_4^{AG\max} = \max_l \sum_h \max_j \sum_i \left(\left|a_{0ij}^h\right| + \sum_{r=1}^{k}|f_{lr}|\left|a_{1ij}^h\right| + ... + \sum_{r=1}^{k}|f_{lr}^{\tau}|\left|a_{\tau ij}^h\right|\right) < 1.$$

Let's prove that there exist weights $\psi$ and $\phi$ such that

$$\sum_i \psi_i \left(\left|a_{0ij}^h\right| + \sum_{r=1}^{k}|f_{lr}|\left|a_{1ij}^h\right| + ... + \sum_{r=1}^{k}|f_{lr}^{\tau}|\left|a_{\tau ij}^h\right|\right)/\psi_j < \phi_h \forall j, \forall h, \forall l.$$

Let us take $\psi_j = \frac{1}{n}\ \forall h$,

$$\phi_h = \max_j \sum_i \left(\left|a_{0ij}^h\right| + \sum_{r=1}^{k}|f_{lr}|\left|a_{1ij}^h\right| + ... + \sum_{r=1}^{k}|f_{lr}^{\tau}|\left|a_{\tau ij}^h\right|\right) +$$

$$+\frac{\overbrace{1 - \sum_h \max_j \sum_i \left(\left|a_{0ij}^h\right| + \sum_{r=1}^{k}|f_{lr}|\left|a_{1ij}^h\right| + ... + \sum_{r=1}^{k}|f_{lr}^{\tau}|\left|a_{\tau ij}^h\right|\right)}^{>0}}{S} \forall j, \forall l.$$

These are weights as $\sum\limits_{h=1}^{S}\phi_h = 1, 0<\phi_h<1$ and $\sum\limits_{j=1}^{S}\psi_j = 1, 0<\psi_j<1$.

Notice that

$$\phi_h > \max_j \sum_i \left(\left|a_{0ij}^h\right| + \sum_{r=1}^{k}|f_{lr}|\left|a_{1ij}^h\right| + ... + \sum_{r=1}^{k}|f_{lr}^{\tau}|\left|a_{\tau ij}^h\right|\right) \geqq$$

$$\geqq \sum_i \left( \left| a_{0ij}^h \right| + \sum_{r=1}^k |f_{lr}| \left| a_{1ij}^h \right| + ... + \sum_{r=1}^k |f_{lr}^\tau| \left| a_{\tau ij}^h \right| \right), \forall j, \forall h, \forall l,$$

or after rewriting:

$$\frac{\sum_i \overbrace{\psi_i}^{\frac{1}{n}} \left( \left| a_{0ij}^h \right| + \sum_{r=1}^k |f_{lr}| \left| a_{1ij}^h \right| + ... + \sum_{r=1}^k |f_{lr}^\tau| \left| a_{\tau ij}^h \right| \right)}{\underbrace{\psi_j}_{\frac{1}{n}}} < \phi_h, \forall j, \forall h, \forall l.$$

To prove the proposition for

$$\beta_2^{AG\max} = \max_l \max_i \sum_h \sum_j \left( \left| a_{0ij}^h \right| + \sum_{r=1}^k |f_{rl}| \left| a_{1ij}^h \right| + ... + \sum_{r=1}^k |f_{rl}^\tau| \left| a_{\tau ij}^h \right| \right)$$

and $\beta_3^{AG\max} = \max_l S \sum_i \max_{h,j} \left( \left| a_{0ij}^h \right| + \sum_{r=1}^k |f_{rl}| \left| a_{1ij}^h \right| + ... + \sum_{r=1}^k |f_{rl}^\tau| \left| a_{\tau ij}^h \right| \right)$, I first derive a sufficient condition for $\delta-$stability that follows from the "rows" diagonal dominance condition, which is also sufficient for $\delta$-stability of matrices $\Omega$ and $\Omega_F$, prove for $\Omega_F$ as a more general case, with the part for $\Omega$ then derived by setting $F = I$. Therefore, my derivation of this condition resembles the steps in the proof of Proposition 1.15. Use $(d_1, ..., d_n, ..., d_1, ..., d_n)$, $d_i > 0$, $\sum_i d_i = 1$ as weights for columns.

Take any block $h$, any column of $F$ $l$, and any row $i$

$$\left\{ \begin{array}{l} f_{ll}^\tau a_{\tau ii}^h + ... + f_{ll} a_{1ii}^h + a_{0ii}^h - 1 < 0 \text{ - negative diagonal} \\ d_i \left| f_{ll}^\tau a_{\tau ii}^h + ... + f_{ll} a_{1ii}^h + a_{0ii}^h - 1 \right| > \sum_h \sum_j d_j (\sum_{r=1}^k |f_{rl}^\tau| \left| a_{\tau ij}^h \right| + ... \\ + \sum_{r=1}^k |f_{rl}| \left| a_{1ij}^h \right| + \left| a_{0ij}^h \right|) - \\ -d_i \left| f_{ll}^\tau a_{\tau ii}^h + ... + f_{ll} a_{1ii}^h + a_{0ii}^h \right| \text{ - dominance} \end{array} \right. \quad \forall i, \forall h, \forall l$$

$\Updownarrow$

$$\left\{ \begin{array}{l} f_{ll}^\tau a_{\tau ii}^h + ... + f_{ll} a_{1ii}^h + a_{0ii}^h - 1 < 0 \\ -d_i \left( f_{ll}^\tau a_{\tau ii}^h + ... + f_{ll} a_{1ii}^h + a_{0ii}^h \right) + d_i > \sum_h \sum_j d_j (\sum_{r=1}^k |f_{rl}^\tau| \left| a_{\tau ij}^h \right| + ... \\ + \sum_{r=1}^k |f_{rl}| \left| a_{1ij}^h \right| + \left| a_{0ij}^h \right|) - \\ -d_i \left| f_{ll}^\tau a_{\tau ii}^h + ... + f_{ll} a_{1ii}^h + a_{0ii}^h \right| \end{array} \right. \quad \forall i, \forall h, \forall l$$

$\Updownarrow$

$$\text{Case 1} \left\{ \begin{array}{l} 0 \leq f_{ll}^\tau a_{\tau ii}^h + ... + f_{ll} a_{1ii}^h + a_{0ii}^h < 1 \\ \sum_h \sum_j d_j (\sum_{r=1}^k |f_{rl}^\tau| \left| a_{\tau ij}^h \right| + ... + \sum_{r=1}^k |f_{rl}| \left| a_{1ij}^h \right| + \left| a_{0ij}^h \right|) < d_i \end{array} \right. \quad \forall i, \forall h, \forall l$$

$\cup$

$$\text{Case 2}\begin{cases} f_{ll}^{\tau}a_{\tau ii}^{h}+...+f_{ll}a_{1ii}^{h}+a_{0ii}^{h}<0 \\ \sum\limits_{h}\sum\limits_{j}d_j(\sum_{r=1}^{k}\left|f_{rl}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+...+ \\ +\sum_{r=1}^{k}\left|f_{rl}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|)<d_i-2d\left({}_{i}f_{ll}^{\tau}a_{\tau ii}^{h}+...+f_{ll}a_{1ii}^{h}+a_{0ii}^{h}\right)\end{cases}\forall i,\forall h,\forall l.$$

Since in the second case $f_{ll}^{\tau}a_{\tau ii}^{h}+...+f_{ll}a_{1ii}^{h}+a_{0ii}^{h}<0$, one may formulate the following sufficient condition:

$$\sum_{h}\sum_{j}d_j(\sum_{r=1}^{k}\left|f_{rl}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{rl}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|)<d_i\forall i,\forall h,\forall l.$$

The condition $1>f_{ll}^{\tau}a_{\tau ii}^{h}+...+f_{ll}a_{1ii}^{h}+a_{0ii}^{h}$ is implied by this relation, and the condition of case 2 is also satisfied. To prove that $1>f_{ll}^{\tau}a_{\tau ii}^{h}+...+f_{ll}a_{1ii}^{h}+a_{0ii}^{h}$, notice that

$$\sum_{h}\sum_{j}d_j(\sum_{r=1}^{k}\left|f_{rl}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{rl}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|)<d_i\Longrightarrow$$

$$\underbrace{\sum_{h}\sum_{j\neq i}d_j(\sum_{r=1}^{k}\left|f_{rl}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{rl}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|)+}_{>0}$$

$$\underbrace{+\sum_{h}d_i(\sum_{r=1}^{k}\left|f_{rl}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{rl}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|)}_{>0}<d_i\Longrightarrow$$

$$\Longrightarrow\sum_{r=1}^{k}\left|f_{rl}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{rl}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|<1\Longrightarrow$$

$$\Longrightarrow\left|f_{ll}^{\tau}\right|\left|a_{\tau ij}^{h}\right|+...+\left|f_{ll}\right|\left|a_{1ij}^{h}\right|+\left|a_{0ij}^{h}\right|<1\Longrightarrow f_{ll}^{\tau}a_{\tau ii}^{h}+...+f_{ll}a_{1ii}^{h}+a_{0ii}^{h}<1$$

So this condition alone is sufficient for $\delta$–stability.

Next, I use the derived sufficient condition to prove Proposition 1.20 for

$$\beta_2^{AG\max}=\max_{l}\max_{i}\sum_{h}\sum_{j}\left(\left|a_{0ij}^{h}\right|+\sum_{r=1}^{k}\left|f_{rl}\right|\left|a_{1ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{rl}^{\tau}\right|\left|a_{\tau ij}^{h}\right|\right)$$

and

$$\beta_3^{AG\max}=\max_{l}S\sum_{i}\max_{h,j}\left(\left|a_{0ij}^{h}\right|+\sum_{r=1}^{k}\left|f_{rl}\right|\left|a_{1ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{rl}^{\tau}\right|\left|a_{\tau ij}^{h}\right|\right).$$

2. for $\beta_2^{AG\max}=\max_{l}\max_{i}\sum_{h}\sum_{j}\left(\left|a_{0ij}^{h}\right|+\sum_{r=1}^{k}\left|f_{rl}\right|\left|a_{1ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{rl}^{\tau}\right|\left|a_{\tau ij}^{h}\right|\right)$:

$$\beta_2^{AG\max}=\max_{l}\max_{i}\sum_{h}\sum_{j}\left(\left|a_{0ij}^{h}\right|+\sum_{r=1}^{k}\left|f_{rl}\right|\left|a_{1ij}^{h}\right|+...+\sum_{r=1}^{k}\left|f_{rl}^{\tau}\right|\left|a_{\tau ij}^{h}\right|\right)<1.$$

Let's prove that there exist weights $d=(d_1,...,d_n,...,d_1,...,d_n)$, $d_i>0$, $\sum\limits_i d_i=1$, such that $\sum\limits_h\sum\limits_j d_j(\sum_{r=1}^k |f_{rl}^\tau|\left|a_{\tau ij}^h\right|+...+\sum_{r=1}^k |f_{rl}|\left|a_{1ij}^h\right|+\left|a_{0ij}^h\right|)<d_i\ \forall i, \forall h,\ \forall l.$

Let us take $d_j=\frac{1}{n}\ \forall j$.

Notice that

$$\sum_h\sum_j\left(\sum_{r=1}^k |f_{rl}^\tau|\left|a_{\tau ij}^h\right|+...+\sum_{r=1}^k |f_{rl}|\left|a_{1ij}^h\right|+\left|a_{0ij}^h\right|\right)\leqq$$

$$\leqq\max_i\sum_h\sum_j\left(\sum_{r=1}^k |f_{rl}^\tau|\left|a_{\tau ij}^h\right|+...+\sum_{r=1}^k |f_{rl}|\left|a_{1ij}^h\right|+\left|a_{0ij}^h\right|\right)<1, \forall i, \forall h, \forall l,$$

or after rewriting:

$$\sum_h\sum_j\underbrace{\frac{1}{n}}_{d_j}\left(\sum_{r=1}^k |f_{rl}^\tau|\left|a_{\tau ij}^h\right|+...+\sum_{r=1}^k |f_{rl}|\left|a_{1ij}^h\right|+\left|a_{0ij}^h\right|\right)<\underbrace{\frac{1}{n}}_{d_i}, \forall i, \forall h, \forall l.$$

3. for $\beta_3^{AG\max}=\max\limits_l S\sum\limits_i\max\limits_{h,j}\left(\left|a_{0ij}^h\right|+\sum_{r=1}^k |f_{rl}|\left|a_{1ij}^h\right|+...+\sum_{r=1}^k |f_{rl}^\tau|\left|a_{\tau ij}^h\right|\right)$:

$\beta_3^{AG\max}=\max\limits_l S\sum\limits_i\max\limits_{h,j}\left(\left|a_{0ij}^h\right|+\sum_{r=1}^k |f_{rl}|\left|a_{1ij}^h\right|+...+\sum_{r=1}^k |f_{rl}^\tau|\left|a_{\tau ij}^h\right|\right)<1.$

Let's prove that there exist weights $d=(d_1,...,d_n,...,d_1,...,d_n)$, $d_i>0$, $\sum\limits_i d_i=1$, such that $\sum\limits_h\sum\limits_j d_j(\sum_{r=1}^k |f_{rl}^\tau|\left|a_{\tau ij}^h\right|+...+\sum_{r=1}^k |f_{rl}|\left|a_{1ij}^h\right|+\left|a_{0ij}^h\right|)<d_i\ \forall i, \forall h,\ \forall l.$

Let us take $d_i=S\max\limits_{h,j}\left(\sum_{r=1}^k |f_{rl}^\tau|\left|a_{\tau ij}^h\right|+...+\sum_{r=1}^k |f_{rl}|\left|a_{1ij}^h\right|+\left|a_{0ij}^h\right|\right)+$

$$+\frac{\overbrace{1-S\sum\limits_i\max\limits_{h,j}\left(\sum\limits_{r=1}^k |f_{rl}^\tau|\left|a_{\tau ij}^h\right|+...+\sum\limits_{r=1}^k |f_{rl}|\left|a_{1ij}^h\right|+\left|a_{0ij}^h\right|\right)}^{>0}}{n}$$

$\forall i,\ \forall l$. These can be taken as weights since $\sum\limits_{i=1}^n d_i=1, 0<d_i<1$.

Notice that $d_i>S\max\limits_{h,j}\left(\sum_{r=1}^k |f_{rl}^\tau|\left|a_{\tau ij}^h\right|+...+\sum_{r=1}^k |f_{rl}|\left|a_{1ij}^h\right|+\left|a_{0ij}^h\right|\right)=$

$$=\underbrace{\sum_{j=1}^n d_j}_{=1}\underbrace{\sum_{h=1}^S}_{=S}\max_{h,j}\left(\sum_{r=1}^k |f_{rl}^\tau|\left|a_{\tau ij}^h\right|+...+\sum_{r=1}^k |f_{rl}|\left|a_{1ij}^h\right|+\left|a_{0ij}^h\right|\right)\geqq$$

$$\geqq\sum_h\sum_j d_j(\sum_{r=1}^k |f_{rl}^\tau|\left|a_{\tau ij}^h\right|+...+\sum_{r=1}^k |f_{rl}|\left|a_{1ij}^h\right|+\left|a_{0ij}^h\right|), \forall i, \forall h,\ \forall l.$$

To prove the proposition for **Model III**, set $a_{0ij}^h=0$ in the proof above.

### A.7.20 Proof of Proposition 1.21

For the maximal aggregate $\beta$-coefficients of $f$-type, the proof is a direct repetition of proof 1.20, where one has to replace $a_{0ij}^h$ with $a_{0ij}^h+\left(A_1^h\bar{b}\right)_{ij}$ and to set $\sum_{r=1}^k |f_{lr}^\tau|$ and

$\sum_{r=1}^{k} |f_{rl}^{\tau}|$ to zero for $\tau > 1$ for Model II, and to replace $a_{0ij}^{h}$ with $\left(A_1^h \bar{b}\right)_{ij}$ and to set $\sum_{r=1}^{k} |f_{lr}^{\tau}|$ and $\sum_{r=1}^{k} |f_{rl}^{\tau}|$ to zero for $\tau > 1$ for Model IV.

For the maximal aggregate $\beta$-coefficients of $b$-type, the proof is similar to the proof for the maximal aggregate $\beta-$coefficients of $f-$type. Use $b$ instead of $f$, change the index of rows and columns from $l$ to $q$, and use summation up to $n$ and not up to $k$. $Q.E.D.$

# Appendix B

# Appendix to Chapter 2

## B.1 Proofs of propositions in Chapter 2

### B.1.1 Proof of Criterion 2.8

Model I (III) (Model II (IV), in which all roots of $\bar{b}$ lie inside the unit circle) is $\delta$-stable if and only if the corresponding matrix $\Omega_{KR}$ is $D_b$-stable. Take $F$ as an identity matrix, and $D$ as $diag(\frac{1}{\delta_1}, ..., \frac{1}{\delta_1}, ..., \frac{1}{\delta_S}, ..., \frac{1}{\delta_S})$, $\delta_h > 0, h = \overline{1, S}$ in the alternative definition of $D_b$-stability of $\Omega_{KR}$ and write down the characteristic equation for eigenvalues with $-i$ substituted for eigenvalue. No eigenvalue must equal $-i$, so the following determinant must not equal zero.

$$\begin{vmatrix} R^1\text{-}(I\text{-}\frac{i}{\delta_1}I) & \cdots & R^{S_0} & R^{S_0+1} & \cdots & R^S \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ R^1 & \cdots & R^{S_0}\text{-}(I\text{-}\frac{i}{\delta_{S_0}}I) & R^{S_0+1} & \cdots & R^S \\ KR^1 & \cdots & KR^{S_0} & KR^{S_0+1}\text{-}(K\text{-}\frac{i}{\delta_{S_0+1}}I) & \cdots & KR^S \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ KR^1 & \cdots & KR^{S_0} & KR^{S_0+1} & \cdots & KR^S\text{-}(K\text{-}\frac{i}{\delta_S}I) \end{vmatrix} \neq 0.$$

By subtracting the $S_0$ row block from the row blocks from 1 to $S_0 - 1$ and subtracting the $S$ row block from the row blocks from $S_0 + 1$ to $S - 1$, I obtain an equivalent condition

$$\begin{vmatrix} \text{-}(I\text{-}\frac{i}{\delta_1}I) & \cdots & 0 & \text{-}(I\text{-}\frac{i}{\delta_{S_0}}I) & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \text{-}(I\text{-}\frac{i}{\delta_{S_0-1}}I) & \text{-}(I\text{-}\frac{i}{\delta_{S_0}}I) & 0 & \cdots & 0 \\ R^1 & \cdots & R^{S_0-1} & R^{S_0}\text{-}(I\text{-}\frac{i}{\delta_{S_0}}I) & R^{S_0+1} & \cdots & R^S \\ 0 & \cdots & 0 & 0 & \text{-}(K\text{-}\frac{i}{\delta_{S_0+1}}I) & \cdots & \text{-}(K\text{-}\frac{i}{\delta_S}I) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ KR^1 & \cdots & KR^{S_0-1} & KR^{S_0} & KR^{S_0+1} & \cdots & KR^S\text{-}(K\text{-}\frac{i}{\delta_S}I) \end{vmatrix} \neq 0$$

By multiplying each column block from the right by $(I - \frac{i}{\delta_h}I)^{-1}$ for $h = \overline{1, S_0}$ and by $(K - \frac{i}{\delta_h}I)^{-1}$ for $h = \overline{S_0 + 1, S}$. (it is possible to do since $\frac{i}{\delta_h}$ is not an eigenvalue of $I$ or $K$ as $I$ and $K$ are positive definite), and adding all column blocks from 1 to $S_0 - 1$ to the $S_0^{\text{th}}$ column block and adding all column blocks from $S_0 + 1$ to $S - 1$ to the $S^{\text{th}}$

column block, I obtain the following equivalent condition

$$\left|\begin{array}{ccccccc} -(I-\frac{i}{\delta_1}I) & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -(I-\frac{i}{\delta_{S_0-1}}I) & 0 & 0 & \cdots & 0 \\ R^1 & \cdots & R^{S_0-1} & \sum_{h=1}^{S_0}\left(\frac{R_h}{I-\frac{i}{\delta_h}I}\right)\text{-}I & R^{S_0+1} & \cdots & \sum_{h=S_0+1}^{S}\left(\frac{R_h}{K-\frac{i}{\delta_h}I}\right) \\ 0 & \cdots & 0 & 0 & -(K-\frac{i}{\delta_{S_0+1}}I) & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ KR^1 & \cdots & KR^{S_0-1} & \sum_{h=1}^{S_0}\left(\frac{KR_h}{I-\frac{i}{\delta_h}I}\right) & KR^{S_0+1} & \cdots & \sum_{h=S_0+1}^{S}\left(\frac{KR_h}{K-\frac{i}{\delta_h}I}\right)\text{-}I \end{array}\right| \neq 0,$$

where, to simplify notation, $\frac{A}{B}$ means $B^{-1}A$

Due to the blocked structure of the matrix under the determinant and due to the already mentioned fact that $\frac{i}{\delta_h}$ is not an eigenvalue of $I$ or $K$, the condition  is equivalent to

$$\left|\begin{array}{cc} \sum_{h=1}^{S_0}\left(\left(I-\frac{i}{\delta_h}I\right)^{-1}R_h\right)-I & \sum_{h=S_0+1}^{S}\left(\left(K-\frac{i}{\delta_h}I\right)^{-1}R_h\right) \\ \sum_{h=1}^{S_0}\left(\left(I-\frac{i}{\delta_h}I\right)^{-1}KR_h\right) & \sum_{h=S_0+1}^{S}\left(\left(K-\frac{i}{\delta_h}I\right)^{-1}KR_h\right)-I \end{array}\right| \neq 0.$$

Since $K$, $K-\frac{i}{\delta_h}I$ ,and $I-\frac{i}{\delta_h}I$ are symmetric the last condition is equivalen to

$$\left|\begin{array}{cc} \sum_{h=1}^{S_0}\left(\left(I-\frac{i}{\delta_h}I\right)^{-1}R_h\right)-I & \sum_{h=S_0+1}^{S}\left(\left(K-\frac{i}{\delta_h}I\right)^{-1}R_h\right) \\ K\sum_{h=1}^{S_0}\left(\left(I-\frac{i}{\delta_h}I\right)^{-1}R_h\right) & K\sum_{h=S_0+1}^{S}\left(\left(K-\frac{i}{\delta_h}I\right)^{-1}R_h\right)-I \end{array}\right| \neq 0.$$

Multiply the second row block from the right by $K^{-1}$ and then subtract the result in the row block from the first row block (it is possible to do as $K$ is positive definite) to obtain

$$\left|\begin{array}{cc} -I & K^{-1} \\ \sum_{h=1}^{S_0}\left(\left(I-\frac{i}{\delta_h}I\right)^{-1}R_h\right) & \sum_{h=S_0+1}^{S}\left(\left(K-\frac{i}{\delta_h}I\right)^{-1}R_h\right)-K^{-1} \end{array}\right| \neq 0.$$

Multiply the second column block from the right by $K$ and then add the first column block to the second one to obtain the following equivalent condition

$$\left|\begin{array}{cc} -I & 0 \\ \sum_{h=1}^{S_0}\left(\left(I-\frac{i}{\delta_h}I\right)^{-1}R_h\right) & \sum_{h=1}^{S_0}\left(\left(I-\frac{i}{\delta_h}I\right)^{-1}R_h\right)+K\sum_{h=S_0+1}^{S}\left(\left(K-\frac{i}{\delta_h}I\right)^{-1}R_h\right)-I \end{array}\right| \neq 0.$$

This last condition is equivalent to

$$\det\left[\sum_{h=1}^{S_0}\left(\frac{-R_h}{I-\frac{i}{\delta_h}I}\right)+\sum_{h=S_0+1}^{S}\left(\frac{-KR_h}{K-\frac{i}{\delta_h}I}\right)+I\right] \neq 0.$$

Further derivations are striaightforward, $Q.E.D.$

### B.1.2 Proof of Proposition 2.9

The proof follows from the proof of Criterion 2.8. Take $R_h = A_0^h + \rho_l A_1^h + ... + \rho_l^\tau A_\tau^h$ and $K = I$. $Q.E.D.$

### B.1.3 Proof of Proposition 2.10

The proof follows from the proof of Criterion 2.8. Take, subsequently, $R_h = A_0^h + A_1^h + A_1^h \bar{b}$ and $K = I$, $R_h = \bar{b}' \otimes A_1^h + I_n \otimes \left(A_0^h + A_1^h \bar{b}\right)$ and $K = I$, and $R_h = F' \otimes A_1^h + I_k \otimes \left(A_0^h + A_1^h \bar{b}\right)$ and $K = I$, setting $A_0^h \equiv 0$ for Model IV, $Q.E.D.$

### B.1.4 Proof of Proposition 2.11

The proof follows from the proof of Criterion 2.8. Take subsequently $R_h = A_0^h + \rho_l A_1^h + A_1^h \bar{b}$ and $K = I$, $R_h = \bar{b}' \otimes A_1^h + I_n \otimes \left(A_0^h + A_1^h \bar{b}\right)$ and $K = I$, Model IV setting $A_0^h \equiv 0$ for Model IV. $Q.E.D.$

### B.1.5 Proof of Proposition 2.12

For the case of $n = 1$, the condition of the alternative criterion for stability of Models without lags (Model I and Model III) under mixed RLS/SG learning for the diagonal environment case under any (possibly different) degrees of inertia of agents, $\delta > 0$, according to Proposition 2.9, simplifies the requirement for $\Omega_{\rho_l}$ to be stable and for at least one of the following to hold true

$$\sum_{h=1}^{S} \frac{\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}} \left(-A_0^h - \rho_l A_1^h - ... - \rho_l^\tau A_\tau^h\right) \neq 0$$

$$\left(\sum_{h=1}^{S} \frac{1}{1+\frac{1}{\delta_h^2}} \left(-A_0^h - \rho_l A_1^h - ... - \rho_l^\tau A_\tau^h\right) + 1\right) \neq 0, \textit{ for all } l = 0, 1, ..., k(\rho_0 = 1),$$

*with* $A_0^h \equiv 0$ *for Model III everywhere above.*

The first "same sign" condition follows directly from the first inequality above as $\delta$'s take any positive values. The second condition that follows from the second inequality is proved below.

**Necessity:** Follows directly from the proof of Proposition 2.19. Note that in the univariate economy setup, any sum of minors $M_k$ consists of elements

$$\delta_{h_1}\delta_{h_2}...\delta_{h_k}(\left(-A_0^{h_1}-\rho_l A_1^{h_1}-...-\rho_l^{\tau}A_{\tau}^{h_1}\right)+...+\left(-A_0^{h_k}-\rho_l A_1^{h_k}-...-\rho_l^{\tau}A_{\tau}^{h_k}\right)+1),$$

where $(h_1,...,h_p)$ are indeces of agents from a subeconomy, and that if the sum of nonnegative elements $(\left(-A_0^{h_1}-\rho_l A_1^{h_1}-...-\rho_l^{\tau}A_{\tau}^{h_1}\right)+...+\left(-A_0^{h_k}-\rho_l A_1^{h_k}-...-\rho_l^{\tau}A_{\tau}^{h_k}\right)+1$ is greater or equal to zero as $\delta$'s take any positive values) is strictly greater than zero, then at least one of them has to be strictly positive.

**Sufficiency:** I have

$$\left(-A_0^{h_1}-\rho_l A_1^{h_1}-...-\rho_l^{\tau}A_{\tau}^{h_1}\right)+...+\left(-A_0^{h_k}-\rho_l A_1^{h_k}-...-\rho_l^{\tau}A_{\tau}^{h_k}\right)+1\geqq 0$$

for any subeconomy $(h_1,...,h_p)$ and for each group of subeconomies of size $p$, $\exists\left(h_1^*(l),...,h_p^*(l)\right)\vdots$

$$\left(-A_0^{h_1^*}-\rho_l A_1^{h_1^*}-...-\rho_l^{\tau}A_{\tau}^{h_1^*}\right)+...+\left(-A_0^{h_p^*}-\rho_l A_1^{h_p^*}-...-\rho_l^{\tau}A_{\tau}^{h_p^*}\right)+1>0,$$

and have to prove that $\left(\sum_{h=1}^{S}\frac{1}{1+\frac{1}{\delta_h^2}}\left(-A_0^h-\rho_l A_1^h-...-\rho_l^{\tau}A_{\tau}^h\right)+1\right)\neq 0$ .

I group separately the terms corresponding to the non-positive $\left(A_0^h+\rho_l A_1^h+...+\rho_l^{\tau}A_{\tau}^h\right)$'s and the terms corresponding to the strictly positive $\left(A_0^h+\rho_l A_1^h+...+\rho_l^{\tau}A_{\tau}^h\right)$'s.

Schematically, I will have

$$\underbrace{\left[\frac{1}{1+\frac{1}{\delta_1^2}}\left(A_0^1+\rho_l A_1^1+...+\rho_l^{\tau}A_{\tau}^1\right)^-+...+\frac{1}{1+\frac{1}{\delta_k^2}}\left(A_0^k+\rho_l A_1^k+...+\rho_l^{\tau}A_{\tau}^k\right)^-\right]}_{\leq 0}+$$

$$+\underbrace{\left[\frac{1}{1+\frac{1}{\delta_1^2}}\left(A_0^1+\rho_l A_1^1+...+\rho_l^{\tau}A_{\tau}^1\right)^++...+\frac{1}{1+\frac{1}{\delta_m^2}}\left(A_0^m+\rho_l A_1^m+...+\rho_l^{\tau}A_{\tau}^m\right)^+\right]}_{\leq 1}-1.$$

If the first sum is strictly less than zero, then the whole expression is less than zero since the sum of something less than zero with something that at maximum equals one is less than one. If the first sum is equal to zero, then the second sum (if there are any positive $\left(A_0^h+\rho_l A_1^h+...+\rho_l^{\tau}A_{\tau}^h\right)$'s at all) has to be less than 1. The last result follows from the fact that for the whole economy, I have to have that $\left(-A_0^1-\rho_l A_1^1-...-\rho_l^{\tau}A_{\tau}^1\right)+...+\left(-A_0^S-\rho_l A_1^S-...-\rho_l^{\tau}A_{\tau}^S\right)+1>0$, a necessary condition that follows directly from the stability of $\Omega_{\rho_l}$ (see Proposition 2.19); that is, excluding zero terms, I have to have $-\rho_l A_1^+-...-\rho_l A_m^++1>0$, which proves the claim as $0<\frac{1}{1+\frac{1}{\delta_1^2}}<1$. This proves the sufficiency part of the second condition in Proposition 2.12. $Q.E.D.$

### B.1.6 Proof of Proposition 2.14

For the case of $n = 1$, alternative sufficient conditions for $\delta$-stability in the univariate case of Models with lags (*Model II, IV*), where $\bar{b} < 1$, under heterogeneous RLS learning for the diagonal environment case under any (possibly different) degrees of inertia of agents, $\delta > 0$, according to Proposition 2.10, are simplified to the requirements:

$\Omega_{\rho_l}$ is stable and at least one of the following holds true

$$\sum_{h=1}^{S} \frac{\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}} \left(-A_0^h - \rho_l A_1^h - A_1^h \bar{b}\right) \neq 0$$

$$\left(\sum_{h=1}^{S} \frac{1}{1+\frac{1}{\delta_h^2}} \left(-A_0^h - \rho_l A_1^h - A_1^h \bar{b}\right) + 1\right) \neq 0 \text{ , for all } l = 0, 1, ..., k \ (\rho_0 = 1)$$

**and**

$\Omega_b$ is stable and at least one of the following holds true

$$\sum_{h=1}^{S} \frac{\frac{1}{\delta_h}}{1+\frac{1}{\delta_h^2}} \left(-A_0^h - 2A_1^h \bar{b}\right) \neq 0$$

$$\left(\sum_{h=1}^{S} \frac{1}{1+\frac{1}{\delta_h^2}} \left(-A_0^h - 2A_1^h \bar{b}\right) + 1\right) \neq 0,$$

where $A_0^h \equiv 0$ for Model IV everywhere above.

The proof for the $\Omega_{\rho_l}$ part of the statement follows directly from the proof of Proposition 2.12 in the sufficiency part. Replace $A_0^h$ with $A_0^h + A_1^h \bar{b}$ and set $A_r^h \equiv 0, r > 1$.

The proof for the $\Omega_b$ part of the statement also follows from the proof of Proposition 2.12. Replace $A_0^h$ with $A_0^h + A_1^h \bar{b}$, set $A_r^h \equiv 0, r > 1$ and use $\bar{b}$ instead of $\rho_l$. In the corresponding results of Proposition 2.19, use $\bar{b}$ instead of $\rho_l$ that leaves the result of this proposition intact since we consider a univariate model, and $\bar{b}$ has dimension one.

### B.1.7 Proof of Proposition 2.15

For the case of $n = 2$, the condition of the alternative criterion for stability of models without lags (Model I and Model III) under mixed RLS/SG learning for the diagonal environment case under any (possibly different) degrees of inertia of agents, $\delta > 0$, according to Proposition 2.9, contains the requirement for $\Omega_{\rho_l}$ to be stable and for the following condition to hold true:

$$\det\left[\sum_{h=1}^{S}\left(\frac{-(A_0^h+\rho_l A_1^h+...+\rho_l^\tau A_\tau^h)}{1+\frac{i}{\delta_h}}\right)+I\right] = 1+\det\frac{-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1)}{1+\frac{i}{\delta_1}}+...+\det\frac{-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S)}{1+\frac{i}{\delta_S}}+$$

$$+\frac{M_1(-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1))}{1+\frac{i}{\delta_1}} + ... + \frac{M_1(-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S))}{1+\frac{i}{\delta_S}}+$$

$$+\det mix\left(\frac{-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1)}{1+\frac{i}{\delta_1}},\frac{-(A_0^2+\rho_l A_1^2+...+\rho_l^\tau A_\tau^2)}{1+\frac{i}{\delta_2}}\right)+...+$$

$$+\det mix\left(\frac{-(A_0^{S-1}+\rho_l A_1^{S-1}+...+\rho_l^\tau A_\tau^{S-1})}{1+\frac{i}{\delta_{S-1}}},\frac{-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S)}{1+\frac{i}{\delta_S}}\right)=$$

$$=1+\left(\frac{1-\frac{i}{\delta_1}}{1+\frac{1}{\delta_1^2}}\right)^2\det\left[-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1)\right]+...+\left(\frac{1-\frac{i}{\delta_S}}{1+\frac{1}{\delta_S^2}}\right)^2\det\left[-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S)\right]+$$

$$+\left(\frac{1-\frac{i}{\delta_1}}{1+\frac{1}{\delta_1^2}}\right)M_1\left(-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1)\right)+...+\left(\frac{1-\frac{i}{\delta_S}}{1+\frac{1}{\delta_S^2}}\right)M_1\left(-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S)\right)+...+$$

$$+\left(\frac{1-\frac{i}{\delta_1}}{1+\frac{1}{\delta_1^2}}\right)\left(\frac{1-\frac{i}{\delta_2}}{1+\frac{1}{\delta_2^2}}\right)[\det mix\left(-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1),-(A_0^2+\rho_l A_1^2+...+\rho_l^\tau A_\tau^2)\right)+$$

$$+\det mix\left(-(A_0^2+\rho_l A_1^2+...+\rho_l^\tau A_\tau^2),-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1)\right)]+...+$$

$$+\left(\frac{1-\frac{i}{\delta_{S-1}}}{1+\frac{1}{\delta_{S-1}^2}}\right)\left(\frac{1-\frac{i}{\delta_S}}{1+\frac{1}{\delta_S^2}}\right)[\det mix\left(-(A_0^{S-1}+\rho_l A_1^{S-1}+...+\rho_l^\tau A_\tau^{S-1}),-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S)\right)+$$

$$+\det mix\left(-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S),-(A_0^{S-1}+\rho_l A_1^{S-1}+...+\rho_l^\tau A_\tau^{S-1})\right)]\neq 0$$

for all $l=0,1,...,k$, $(\rho_0=1)$.

Now, take real and imaginary parts to obtain

$$\operatorname{Re}\det\left[\sum_{h=1}^{S}\left(\frac{-(A_0^h+\rho_l A_1^h+...+\rho_l^\tau A_\tau^h)}{1+\frac{i}{\delta_h}}\right)+I\right]=1+\frac{1-\frac{1}{\delta_1^2}}{\left(1+\frac{1}{\delta_1^2}\right)^2}\det\left[-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1)\right]+...+$$

$$+\frac{1-\frac{1}{\delta_S^2}}{\left(1+\frac{1}{\delta_S^2}\right)^2}\det\left[-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S)\right]+\frac{1}{1+\frac{1}{\delta_1^2}}M_1(-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1))+...+$$

$$+\frac{1}{1+\frac{1}{\delta_S^2}}M_1(-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S))+...+$$

$$+\frac{1-\frac{1}{\delta_1\delta_2}}{\left(1+\frac{1}{\delta_1^2}\right)\left(1+\frac{1}{\delta_2^2}\right)}\det[mix\left(-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1),-(A_0^2+\rho_l A_1^2+...+\rho_l^\tau A_\tau^2)\right)+$$

$$+\det mix\left(-(A_0^2+\rho_l A_1^2+...+\rho_l^\tau A_\tau^2),-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1)\right)]+...+$$

$$+\frac{1-\frac{1}{\delta_{S-1}\delta_S}}{\left(1+\frac{1}{\delta_{S-1}^2}\right)\left(1+\frac{1}{\delta_S^2}\right)}[\det mix\left(-(A_0^{S-1}+\rho_l A_1^{S-1}+...+\rho_l^\tau A_\tau^{S-1}),-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S)\right)+$$

$$+\det mix\left(-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S),-(A_0^{S-1}+\rho_l A_1^{S-1}+...+\rho_l^\tau A_\tau^{S-1})\right)]$$

$$\operatorname{Im}\det\left[\sum_{h=1}^{S}\left(\frac{-(A_0^h+\rho_l A_1^h+...+\rho_l^\tau A_\tau^h)}{1+\frac{i}{\delta_h}}\right)+I\right]=\frac{-\frac{2i}{\delta_1}}{\left(1+\frac{1}{\delta_1^2}\right)^2}\det\left[-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1)\right]+...+$$

$$+\frac{-\frac{2i}{\delta_S}}{\left(1+\frac{1}{\delta_S^2}\right)^2}\det\left[-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S)\right]+\frac{-\frac{i}{\delta_1}}{1+\frac{1}{\delta_1^2}}M_1(-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1))+...+$$

$$+\frac{-\frac{i}{\delta_S}}{1+\frac{1}{\delta_S^2}}M_1(-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S))+...+$$

$$+\frac{-i\left(\frac{1}{\delta_1}+\frac{1}{\delta_2}\right)}{\left(1+\frac{1}{\delta_1^2}\right)\left(1+\frac{1}{\delta_2^2}\right)}[\det mix\left(-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1),-(A_0^2+\rho_l A_1^2+...+\rho_l^\tau A_\tau^2)\right)+$$

$$+\det mix\left(-(A_0^2+\rho_l A_1^2+...+\rho_l^\tau A_\tau^2),-(A_0^1+\rho_l A_1^1+...+\rho_l^\tau A_\tau^1)\right)]+...+$$

$$+\frac{-i\left(\frac{1}{\delta_{S-1}}+\frac{1}{\delta_S}\right)}{\left(1+\frac{1}{\delta_{S-1}^2}\right)\left(1+\frac{1}{\delta_S^2}\right)}[\det mix\left(-(A_0^{S-1}+\rho_l A_1^{S-1}+...+\rho_l^\tau A_\tau^{S-1}),-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S)\right)+$$

$$+\det mix\left(-(A_0^S+\rho_l A_1^S+...+\rho_l^\tau A_\tau^S),-(A_0^{S-1}+\rho_l A_1^{S-1}+...+\rho_l^\tau A_\tau^{S-1})\right)]$$

for all $l=0,1,...,k$ $(\rho_0=1)$ for all $l=0,1,...,k$ $(\rho_0=1)$.

The "same sign" sufficient condition for this case can be seen from the Im part. They are sufficient for the Im part to be either nonnegative or nonpositive.

For all $l=0,1,...,k$ $(\rho_0=1)$.

$$\det\left[-(A_0^h+\rho_l A_1^h+...+\rho_l^\tau A_\tau^h)\right]\geq 0,$$

$$[\det mix\left(-(A_0^i+\rho_l A_1^i+...+\rho_l^\tau A_\tau^i),-(A_0^j+\rho_l A_1^j+...+\rho_l^\tau A_\tau^j)\right)+$$

$$+\det mix\left(-(A_0^j+\rho_l A_1^j+...+\rho_l^\tau A_\tau^j),-(A_0^i+\rho_l A_1^i+...+\rho_l^\tau A_\tau^i)\right)]\geq 0,\forall i\neq j,$$

$$M_1(-(A_0^h+\rho_l A_1^h+...+\rho_l^\tau A_\tau^h))\geq 0,$$

or

$$\det\left[-(A_0^h+\rho_l A_1^h+...+\rho_l^\tau A_\tau^h)\right]\leq 0,$$

$$[\det mix\left(-(A_0^i+\rho_l A_1^i+...+\rho_l^\tau A_\tau^i),-(A_0^j+\rho_l A_1^j+...+\rho_l^\tau A_\tau^j)\right)+$$

$$+\det mix\left(-(A_0^j+\rho_l A_1^j+...+\rho_l^\tau A_\tau^j),-(A_0^i+\rho_l A_1^i+...+\rho_l^\tau A_\tau^i)\right)]\leq 0,\forall i\neq j,$$

$$M_1(-(A_0^h+\rho_l A_1^h+...+\rho_l^\tau A_\tau^h))\leq 0.$$

If all inequalities above hold with equality (that would mean a zero Im part), then the Re part equals 1 from the expression for Re above. This proves that these conditions are sufficient for $\delta$–stability in this setting. For Model III, set $A_0^h\equiv 0$ everywhere above.

### B.1.8 Proof of Propositions 2.17, 2.18, 2.19, and 2.20

First, prove Proposition 2.19. Proposition 2.17 is a special case of Proposition 2.19 for $l = 0$ ($\rho_0 = 1$). According to Proposition 2.3, the necessary condition for stability in this case is the stability of matrix $D_1\Omega_{\rho_l}$, defined in (2.15) and (2.16) (where $A_0^h \equiv 0$ for Model III), respectively. The rest of the proof essentially follows the lines of the proof of Proposition 12 in Kolyuzhnov [40]. I have to consider matrix $\Gamma = D(-\Omega)$. A necessary and sufficient condition for stability (in a mathematics definition that is opposite to the definition used throughout the text of this paper) of this matrix is that the real parts of the eigenvalues of $D(-\Omega)$ must be greater than zero. For the condition on eigenvalues to hold true, it is necessary that all sums of the principal minors of $D\left(-\Omega\right)$ grouped by the same size are greater than zero.

It follows from the fact that, on the one hand, the characteristic equation for eigenvalues of $\Gamma$ has the form

$\det\left(\Gamma + I\mu\right) = \det\Gamma + \mu M_{n-1} + \mu^2 M_{n-2} + ... + \mu^{n-1}M_1 + \mu^n = 0$, where $\lambda = -\mu$ is the eigenvalue of $\Gamma$, and $M_k$ is the sum of all principal minors of $\Gamma$ of size $k$,

while, on the other hand, the same characteristic equation can be written in terms of the product decomposition of the polynomial:

$$(\mu+\lambda_1)\cdots(\mu+\lambda_n) = \underbrace{\lambda_1...\lambda_n}_{>0} + ... + \mu^{n-2}\underbrace{(\lambda_1\lambda_2 + ... + \lambda_{n-1}\lambda_n)}_{>0} + \mu^{n-1}\underbrace{(\lambda_1 + ... + \lambda_n)}_{>0} + \mu^n = 0.$$

Thus, all $M_k > 0$.

By writing this condition in terms of $D(-\Omega)$, one obtains that in each size group, the sum of minors is subdivided into groups of sums of minors that contain the same number of columns of each block of $(-\Omega)$, i.e. $-(A_0^h + \rho_l A_1^h + ... + \rho_l^\tau A_\tau^h - I_n)$. The coefficient before such particular sum has the form $(\delta_{h_1})^{j_1}(\delta_{h_2})^{j_2}...\left(\delta_{h_p}\right)^{j_p}$. This coefficient uniquely specifies the sum of minors by the size, the number of columns from each block, and from which subeconomy it is formed, $(h_1, ..., h_p)$. The size of minors in such a group is equal to the total power of the coefficients, $j_1 + ... + j_p$, and the subscripts of $\delta$'s denote from which block of $(-\Omega)$ the columns are taken, while the power of each $\delta$ indicates how many columns are taken from this particular block.

Let us fix one subeconomy (say, formed by blocks 1, 2, and 3) and consider the limit of inequalities for the sum of minors, with $\delta$'s for other blocks going to zero. Doing the same operation for all subeconomies, I will derive the condition in the statement of

Proposition 2.19. Therefore, Proposition 2.17 also holds true, and Propositions 2.18 and 2.20 are derived from Propositions 2.17 and 2.19, respectively, by setting all $\delta$'s for all subeconomies equal to 1.

# Appendix C

# Appendix to Chapter 3

## C.1 Proofs of propositions in Chapter 3

### C.1.1 Proof of Propositions 3.1 and 3.2

The PLM in general form is $y_t = a + \Gamma w_t$. If $w_i$ is not included in the PLM, it is reflected in the corresponding zero column of $\Gamma$. The REE conditions can be written as $(\rho_i A - I_n) \begin{bmatrix} \gamma_{1i} & \dots & \gamma_{ni} \end{bmatrix}' + B^i = 0$, $i \in I_0$.

It is clear that in case $i$ is not included into the active factors set, that is $\begin{bmatrix} \gamma_{1i} & \dots & \gamma_{ni} \end{bmatrix}' = 0$, then in order to have a REE solution, $B^i$ has to be equal to 0, so that one can omit only those factors in the PLM that have a zero column in $B$ in the reduced form. Equivalently, it is clear that if $B_i \neq 0$, then, in order to have a REE solution, one should not have $\begin{bmatrix} \gamma_{1i} & \dots & \gamma_{ni} \end{bmatrix}' = 0$, that is, one has to include $w_i$ into the active factors set.

In case $i$ is included in the active factors set, that is $\begin{bmatrix} \gamma_{1i} & \dots & \gamma_{ni} \end{bmatrix}' \neq 0$, the REE solution exists if and only if the following conditions hold true.

$B^i = 0$, or ($B^i \neq 0$ and $\det(\rho_i A - I) \neq 0$), or ($B^i \neq 0$ and $\det(\rho_i A - I) = 0$ and $rank(\rho_i A - I) = rank(\rho_i A - I, B^i)$).

Combining the two cases we get the statement in Proposition 3.1.

For Proposition 3.2, one has only to transform the last conditions to guarantee the uniqueness of the solution.

In case $i$ is included in the active factors set, that is $\begin{bmatrix} \gamma_{1i} & \dots & \gamma_{ni} \end{bmatrix}' \neq 0$, the REE solution exists and is unique if and only if the following condition holds true.

$\det(\rho_i A - I) \neq 0$.

### C.1.2 Proof of Proposition 3.5 (Necessary conditions and sufficient conditions in terms of eigenvalues for the structurally homogeneous case)

We have to study matrix $D_1 \Omega_{\rho_l}$ for stability under any $\delta_h > 0$, where $D_1$ and $\Omega_{\rho_l}$ are defined in (3.14) and (3.16), respectively. Thus, we consider

$$\begin{aligned} \det\left(\Omega_{\rho_l} - D_1^{-1}\mu I\right) &= \det \begin{bmatrix} \rho_l A_1 - \left(1 + \frac{\mu}{\delta_1}\right) I & \cdots & \rho_l A_S \\ \vdots & \ddots & \vdots \\ \rho_l A_1 & \cdots & \rho_l A_S - \left(1 + \frac{\mu}{\delta_S}\right) I \end{bmatrix} = 0, \\ \forall l &= 0, ..., k, (\rho_0 = 1), \end{aligned}$$

where $A_h = \zeta_h A$, $\sum \zeta_h = 1$.

It is clear from the structure of the matrix above that $\mu = -\delta_{i_0}$ is a root if and only if at least one of the following holds true: $\bar{A}$ is singular or there exists at least one other $\delta_j$ that equals $\delta_{i_0}$. (If $A$ is singular, then $\mu_h = -\delta_h$, $h = \overline{1,S}$ are the roots. That is, if none of $-\delta'$s is the root, then $A$ is non-singular.)

Assume that $A$ is non-singular and all $\delta_h$'s are different, that is, assume that none of $-\delta'$s is the root. If there are roots other than $-\delta_h'$s (the case of eigenvalues $\mu_h = -\delta_h < 0$ is obvious), then they satisfy the characteristic equation for obtaining the eigenvalues of $D_1\Omega_{\rho_l}$ that are not equal to $-\delta_h$:

$$\det\left(\Omega_{\rho_l} - D_1^{-1}\mu I\right) = \det\begin{bmatrix} \rho_l A_1 - \left(1+\frac{\mu}{\delta_1}\right) I & \cdots & \rho_l A_S \\ \vdots & \ddots & \vdots \\ \rho_l A_1 & \cdots & \rho_l A_S - \left(1+\frac{\mu}{\delta_S}\right) I \end{bmatrix} =$$

(subtracting the last row from other rows)

$$= \det\begin{bmatrix} -\left(1+\frac{\mu}{\delta_1}\right) I & 0 & \cdots & 0 & \left(1+\frac{\mu}{\delta_S}\right) I \\ 0 & -\left(1+\frac{\mu}{\delta_2}\right) I & \cdots & 0 & \left(1+\frac{\mu}{\delta_S}\right) I \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\left(1+\frac{\mu}{\delta_{S-1}}\right) I & \left(1+\frac{\mu}{\delta_S}\right) I \\ \rho_l A_1 & \rho_l A_2 & \cdots & \rho_l A_{S-1} & \rho_l A_S - \left(1+\frac{\mu}{\delta_S}\right) I \end{bmatrix} =$$

(for $\mu \neq \delta_h \ \forall h$)

$$= \left(1+\frac{\mu}{\delta_1}\right) \times \ldots \times \left(1+\frac{\mu}{\delta_S}\right) \det\begin{bmatrix} -I & \cdots & 0 & I \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -I & I \\ \frac{\rho_l A_1}{\left(1+\frac{\mu}{\delta_1}\right)} & \cdots & \frac{\rho_l A_{S-1}}{\left(1+\frac{\mu}{\delta_{S-1}}\right)} & \frac{\rho_l A_S}{\left(1+\frac{\mu}{\delta_S}\right)} - I \end{bmatrix} =$$

(adding all columns to the last one)

$$= \left(1+\frac{\mu}{\delta_1}\right) \times \ldots \times \left(1+\frac{\mu}{\delta_S}\right) \det\begin{bmatrix} -I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -I & 0 \\ \frac{\rho_l A_1}{\left(1+\frac{\mu}{\delta_1}\right)} & \cdots & \frac{\rho_l A_{S-1}}{\left(1+\frac{\mu}{\delta_{S-1}}\right)} & \left[\frac{\rho_l A_1}{1+\frac{\mu}{\delta_1}} + \ldots + \frac{\rho_l A_S}{1+\frac{\mu}{\delta_S}} - I\right] \end{bmatrix} =$$

$$= \left(1+\frac{\mu}{\delta_1}\right) \times ... \times \left(1+\frac{\mu}{\delta_S}\right)(-1)^{n(S-1)} \det\left[\frac{\rho_l A_1}{1+\frac{\mu}{\delta_1}} + ... + \frac{\rho_l A_S}{1+\frac{\mu}{\delta_S}} - I\right] = 0.$$

As we consider $\mu \neq -\delta_h$, the last equation is equivalent to

$$\det\left[\frac{-\rho_l A_1}{1+\frac{\mu}{\delta_1}} + ... + \frac{-\rho_l A_S}{1+\frac{\mu}{\delta_S}} + I\right] = 0, \text{ where } A_h = \zeta_h A, \sum \zeta_h = 1.$$

After some calculations, we obtain

$$\det\left[\rho_l A \left(\frac{-\zeta_1}{1+\frac{\mu}{\delta_1}} + ... + \frac{-\zeta_S}{1+\frac{\mu}{\delta_S}}\right) + I\right] = 0,$$

and finally

$$\rho_l \lambda_k \left(\frac{\zeta_1}{1+\frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1+\frac{\mu}{\delta_S}}\right) = 1$$

for those $\lambda_k$, eigenvalues of $A$, that are not equal to zero. If all $\lambda_k = 0$, then $A$ is a zero matrix, and the only eigenvalues of $D\Omega$ are $-\delta_h$'s.

As complex eigenvalues of a real matrix $A$ come in conjugate pairs, the system above is equivalent to

$$\begin{cases} \rho_l \operatorname{Re}(\lambda_k) \operatorname{Re}\left(\frac{\zeta_1}{1+\frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1+\frac{\mu}{\delta_S}}\right) - \rho_l \operatorname{Im}(\lambda_k) \operatorname{Im}\left(\frac{\zeta_1}{1+\frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1+\frac{\mu}{\delta_S}}\right) = 1 \\ \rho_l \operatorname{Im}(\lambda_k) \operatorname{Re}\left(\frac{\zeta_1}{1+\frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1+\frac{\mu}{\delta_S}}\right) + \rho_l \operatorname{Re}(\lambda_k) \operatorname{Im}\left(\frac{\zeta_1}{1+\frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1+\frac{\mu}{\delta_S}}\right) = 0 \end{cases}$$

for each pair of conjugate eigenvalues. In case of a real eigenvalue, $\operatorname{Im}(\lambda_k) = 0$, the corresponding system simplifies to

$$\rho_l \operatorname{Re}(\lambda_k)\left(\frac{\zeta_1}{1+\frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1+\frac{\mu}{\delta_S}}\right) = \rho_l \lambda_k \left(\frac{\zeta_1}{1+\frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1+\frac{\mu}{\delta_S}}\right) = 1.$$

For any $S$ we have that for eigenvalues $\mu$ to be negative, it is necessary that $\frac{\frac{1}{\rho_l \lambda_k} - 1}{\frac{1}{\rho_l \lambda_k \delta_1 ... \delta_S}} > 0$ and therefore that $\rho_l \lambda_k < 1, \forall l = 0, ..., k, (\rho_0 = 1)$. As $|\rho_l| < 1, \forall l = \overline{1,k}$, the latter condition is equivalent to $\lambda_k < 1$.

For $S = 2$, the system corresponding to a real eigenvalue looks as follows:

$$\begin{cases} \rho_l \lambda_k \left(\frac{\zeta_1}{1+\frac{\mu}{\delta_1}} + \frac{\zeta_2}{1+\frac{\mu}{\delta_2}}\right) = 1 \\ \mu^2 + \mu \frac{\frac{1}{\rho_l \lambda_k}\left(\frac{1}{\delta_1} + \frac{1}{\delta_2}\right) - \left(\frac{\zeta_1}{\delta_2} + \frac{\zeta_2}{\delta_1}\right)}{\frac{1}{\rho_l \lambda_k \delta_1 \delta_2}} + \frac{\frac{1}{\rho_l \lambda_k} - 1}{\frac{1}{\rho_l \lambda_k \delta_1 \delta_2}} = 0. \end{cases}$$

The Routh–Hurwitz conditions for the negativity of real parts of $\mu$ are necessary and sufficient and look as follows:

$$\begin{cases} \frac{\frac{1}{\rho_l \lambda_k} - 1}{\frac{1}{\rho_l \lambda_k \delta_1 \delta_2}} > 0 \\ \frac{\frac{1}{\rho_l \lambda_k}\left(\frac{1}{\delta_1} + \frac{1}{\delta_2}\right) - \left(\frac{\zeta_1}{\delta_2} + \frac{\zeta_2}{\delta_1}\right)}{\frac{1}{\rho_l \lambda_k \delta_1 \delta_2}} > 0 \end{cases}.$$

The system of inequalities above is equivalent to

$$\begin{cases} \rho_l \lambda_k < 1 \\ \rho_l \lambda_k < \frac{\frac{1}{\delta_1}+\frac{1}{\delta_2}}{\frac{\zeta_1}{\delta_2}+\frac{\zeta_2}{\delta_1}} \end{cases}.$$

Since $\frac{\frac{1}{\delta_1}+\frac{1}{\delta_2}}{\frac{\zeta_1}{\delta_2}+\frac{\zeta_2}{\delta_1}} > 1$, as $\frac{1-\zeta_1}{\delta_1} + \frac{1-\zeta_2}{\delta_2} > 0$, the last system of inequalities is equivalent to $\rho_l \lambda_k < 1, \forall l = 0, ..., k, (\rho_0 = 1)$. As $|\rho_l| < 1, \forall l = \overline{1,k}$, the latter condition is equivalent to $\lambda_k < 1$.

Thus, the sufficient condition for stability for the case of $S = 2$ is that all eigenvalues of $A$ are real and less than 1; and the necessary condition for stability for any $S$ is that all real eigenvalues of $A$ have to be less than 1. $Q.E.D.$