INSTITUT DE MATHEMATIQUES
DE JUSSIEU
UMR 7586
Université Pierre et Marie Curie
4, Place Jussieu
75252 PARIS CEDEX 05
Tél. 01 44 27 75 68
Fax 01 44 27 73 21

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Report on the Doctoral Thesis : "Analysis in Banach spaces", presented by E. Pernecka.

The Memoir presented by Eva Pernecka, under the supervision of Professor Petr Hajek, consists of three chapters. The first chapter is a joint work with Gilles Lancien, published in 2013 in Journal of Functional Analysis. The second chapter is a joint work with Petr Hajek, published in Journal of Mathematical Analysis and Applications (2014). These two chapters are somewhat related. The third chapter concerns a different topic, and clearly contains enough material for providing a third publication. This Memoir is very carefully written, and pleasant to read. Let me now describe its contents.

Let M be a metric space, equipped for convenience with a distinguished point denoted 0. The space $Lip_0(M)$ of Lipschitz functions on M which vanish at 0 is a Banach space for the Lipschitz norm, and the closed linear span of the Dirac measures in its dual is the canonical predual of $Lip_0(M)$. This predual denoted $\mathscr{F}(M)$ is called the Lipschitz-free space (in short, free space) over M in the Studia 2003 article by Nigel Kalton and the reviewer. Its importance relies on the fact that its properties somehow reflect those of the metric space M. Despites the simplicity of the definition, this class of Banach spaces is far from being well understood, and its study became an active field of research in the last ten years.

The first chapter investigates the bounded approximation property (BAP) for free spaces $\mathscr{F}(M)$. A motivation for this investigation is that BAP for $\mathscr{F}(M)$ is tightly related with the existence of linear extension operators for Lipchitz functions defined on (e.g.) finite subsets of M to all of M. It is known that this property can fail, even for some compact spaces M. However, the very interesting corollary 1.2.2 states that BAP holds for all doubling metric spaces. Hence a "finite dimensionality" condition on the space M ensures BAP. The proof is also very nice since it uses the elaborate Lee-Naor K-gentle partitions of unity. In particular, if H is a closed subset of $\mathbb{R}^N$, then by Proposition 1.2.3 $\mathscr{F}(H)$ has BAP, with a constant controlled by $\sqrt{N}$. It is apparently not known if such an estimate is optimal. The second part of this chapter investigates in detail the free spaces over the finite-dimensional spaces l_1^N and the Banach space l_1. A tiling method allows to show (Theorem 1.3.1) that these spaces have a monotone finite-dimensional Schauder decomposition. The central geometric Lemma 1.3.2 crucially uses the special properties of the l_1 norm for showing that a function defined of the vertices of a straight cube can be extended to the whole cube with the same Lipschitz constant. Let us mention that it is an important open question to know if the space $\mathscr{F}(l_1)$ is complemented in its bidual. It would follow from a positive answer that a space which is Lipschitz-isomorphic to l_1 is actually linearly isomorphic to that space.

The second chapter provides an improvement of the second part of Chapter 1. Namely, the main result (Theorem 2.3.1) asserts in particular that the spaces $\mathscr{F}(l_1^N)$ and $\mathscr{F}(l_1)$ have a Schauder basis. The proofs rely again on the consideration of nested tilings but much more care must be applied in adding vertices "one at a time" to control the sequence of dimensions of the increasing sequence of spaces. The quite delicate technical Lemma 2.2.2 contains most of what is needed in the argument. A natural open question concludes this section : if M is an arbitrary subset of $\mathbb{R}^n$, does $\mathscr{F}(M)$ have a Schauder basis ? Along these lines, the recent work of Pedro Kaufmann (Products of Lipschitz-free spaces and applications, posted on Arxiv, March 2014) could be helpful.

Finally, the third chapter investigates an independent direction of research, which can be described as follows. It is known since Pelczynski's work (1962) that if there exists a non-compact linear operator T from c_0 into a Banach space Y then Y contains an isomorphic copy of c_0. It is natural to wonder if this result remains true when the assumption of linearity of T is relaxed and replaced by a smoothness condition. It turns out that the natural assumption is in this case uniform differentiability, a condition which means that the differential map is uniformly continuous on bounded subsets. It is clear that dropping such a powerful condition as linearity will lead to considerable difficulties. However, an argument of (non linear) extension to the bidual spaces joined to a linear result of Rosenthal's allows to show (see Hajek-Johanis's authoritative work [10]) that if there is a uniformly differentiable map from the unit ball of c_0 to a Banach space X with non-compact range then X^{**} contains a linear copy of l_∞. The main result of Chapter 3 (Theorem 3.2.1) shows that this result extends to maps which are not necessarily obtained by the extension procedure. More precisely, if a uniformly differentiable map from the unit ball of l_∞ to a Banach space Y is such that the image of the canonical basis of c_0 is not relatively compact in Y, then Y contains a copy of l_∞. This deep result implies in particular that there is no uniformly differentiable mapping from l_∞ into c_0 which fixes the basis. Note that a classical result of Lindenstrauss asserts the existence of a Lipschitz projection from l_∞ onto c_0, while a theorem of Kalton (Fundamenta Math. 2011) implies that such a Lipschitz projection cannot be quasi-additive. A neat compactness argument (naturally using ultraproducts) yields from Theorem 3.2.1 to a finite-dimensional statement (Theorem 3.2.3) which somewhat express the rigidity of cubes under uniformly differentiable mappings. An alternative application (Corollary 3.2.4) is a "concentration" statement for uniformly differentiable maps from a large finite-dimensional cube into a space of finite cotype.

This Memoir is writen with great care, and concluded with a proper list of references. It constitutes a very significant amount of original research, in difficult and competitive domains. Moreover its presentation clearly shows that Eva Pernecka is a well-trained young researcher of international level, who is fully aware of what goes on around the frontier where she works. In conclusion, this Thesis clearly proves Eva Pernecka's ability for creative scientific work. It constitutes a very good PhD Thesis, and I strongly recommend that the defense take place as soon as possible.

Gilles Godefroy
Directeur de Recherches au C.N.R.S.