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DOCTORAL THESIS

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Analysis in Banach spaces

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague 28th May 2014 Eva Pernecká

Název práce: Analýza v Banachových prostorech

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Abstrakt: Práce sestává ze dvou článků a jednoho preprintu. Oba články se věnují aproximačním vlastnostem Lipschitz-free prostorů. V prvním článku dokážeme, že Lipschitz-free prostor na metrickém prostoru, který je doubling, má omezenou aproximační vlastnost. Speciálně, Lipschitz-free prostor na uzavřené podmnožině $\mathbb{R}^n$ má omezenou aproximační vlastnost. Také ukážeme, že Lipschitz-free prostory na ℓ_1 a ℓ_1^n mají monotonní konečně dimenzionální Schauderovu dekompozici. Ve druhém článku posouváme tuto práci dále a dostáváme dokonce Schauderovu bázi v Lipschitz-free prostorech na ℓ_1 a ℓ_1^n. Tématem preprintu je rigidita ℓ_∞ a ℓ_∞^n vzhledem k uniformně diferencovatelným zobrazením. Náš hlavní výsledek je nelineární analogií klasického výsledku od Rosenthala o rigiditě ℓ_∞ vzhledem k lineárním operátorům, které nejsou slabě kompaktní, a zobecňuje větu o nekomplementovanosti c_0 v ℓ_∞ od Phillipse.

Klíčová slova: Lipschitz-free prostory, aproximační vlastnost, Schauderova dekompozice, Schauderova báze, uniformně diferencovatelné zobrazení

Title: Analysis in Banach spaces

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Abstract: The thesis consists of two papers and one preprint. The two papers are devoted to the approximation properties of Lipschitz-free spaces. In the first paper we prove that the Lipschitz-free space over a doubling metric space has the bounded approximation property. In particular, the Lipschitz-free space over a closed subset of $\mathbb{R}^n$ has the bounded approximation property. We also show that the Lipschitz-free spaces over ℓ_1 and over ℓ_1^n admit a monotone finite-dimensional Schauder decomposition. In the second paper we improve this work and obtain even a Schauder basis in the Lipschitz-free spaces over ℓ_1 and ℓ_1^n. The topic of the preprint is rigidity of ℓ_∞ and ℓ_∞^n with respect to uniformly differentiable mappings. Our main result is a non-linear analogy of the classical result on rigidity of ℓ_∞ with respect to non-weakly compact linear operators by Rosenthal, and it generalises the theorem on non-complementability of c_0 in ℓ_∞ due to Phillips.

Keywords: Lipschitz-free spaces, approximation property, Schauder decomposition, Schauder basis, uniformly differentiable mappings

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Introduction

This thesis is based on the following original research work:

- (with G. Lancien) *Approximation properties and Schauder decompositions in Lipschitz-free spaces*, J. Funct. Anal. 264 (10) (2013) 2323–2334 ([14]).
- (with P. Hájek) *On Schauder bases in Lipschitz-free spaces*, J. Math. Anal. Appl. 416 (2) (2014) 629–646 ([11]).
- *On uniformly differentiable mappings*, preprint ([21]).

The first paper is presented in Chapter 1, the second paper forms Chapter 2 and the last study constitutes Chapter 3.

Let us now outline the content of the chapters.

In Chapter 1 and Chapter 2, we deal with Lipschitz-free spaces in terms of their approximation properties. Consider the space $\mathrm{Lip}_0(M)$ of all real-valued Lipschitz functions on a metric space M which vanish at a fixed point of M, equipped with the norm given by the Lipschitz constant of a function. Then the *Lipschitz-free space* $\mathcal{F}(M)$ over the metric space M is the canonical predual of $\mathrm{Lip}_0(M)$.

Lipschitz-free spaces provide a way to abstractly linearize Lipschitz mappings in the following sense. If we use the Dirac map to identify metric spaces M and N with subsets of the corresponding Lipschitz-free spaces $\mathcal{F}(M)$ and $\mathcal{F}(N)$, respectively, then any Lipschitz map from the metric space M into the metric space N can be extended to a continuous linear map from $\mathcal{F}(M)$ into $\mathcal{F}(N)$ with the same Lipschitz constant (see [23] or Lemma 2.2 in [7]).

Although Lipschitz-free spaces are relatively simply defined, their linear structure is more difficult to analyse and is not thoroughly understood yet. Hence this subject provides a rich field to investigate and has become an active area of study. One can easily see that $\mathcal{F}(\mathbb{R})$ is isometric to L_1. On the other hand, $\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to any subspace of L_1 as Naor and Schechtman showed in [19] by adapting a Theorem of Kislyakov [13]. Then in [6] Godard characterized the metric spaces whose Lipschitz-free space is isometric to a subspace of L_1.

Chapter 1, consisting of paper [14], focuses on metric spaces M for which the Lipschitz-free space $\mathcal{F}(M)$ has the *bounded approximation property (BAP)* or a *finite-dimensional Schauder decomposition (FDD)*. Research in this area was initiated in the seminal publication of Godefroy and Kalton [7]. One of their main results states that a Banach space X has the λ-BAP if and only if $\mathcal{F}(X)$ has the λ-BAP. In particular, every Lipschitz-free space $\mathcal{F}(E)$, where E is a finite-dimensional Banach space, has the *metric approximation property (MAP)*. In [7] Godefroy and Kalton also prove the so-called isometric lifting property for any separable Banach space. A modification of this proof along with Enflo's fundamental result on the existence of a separable Banach space not admitting the *approximation property (AP)* was later used by Godefroy and Ozawa in [8] to establish the existence of a compact metric space M such that $\mathcal{F}(M)$ does not have AP. The presence of both positive and negative examples motivates the search for a description of the metric spaces over which the Lipschitz-free spaces have the BAP. Several related questions were raised in [8].

Section 1.2 is devoted to this problem. The key ingredient is the existence of linear extension operators of Lipschitz functions. Our general result concerning BAP for Lipschitz-free spaces, Theorem 1.2.1, uses the notion of *K-gentle partition of unity* introduced by Lee and Naor in [15]. For a $K > 0$ and a metric space Y having a K-gentle partition of unity with respect to some separable closed subset X of Y (the partition being a function from the product of Y and a measure space into $[0, \infty)$ of certain properties), they construct a linear extension operator from $\mathrm{Lip}_0(X)$ into $\mathrm{Lip}_0(Y)$ with norm not greater than $3K$. In Theorem 1.2.1, by composing such extension operators with restriction operators, we show that if M is a separable metric space such that there exists a constant $K > 0$ satisfying that for any closed subset X of M, M admits a K-gentle partition of unity with respect to X, then $\mathcal{F}(M)$ has the $3K$-BAP. By [15], a *doubling metric space M* with a doubling constant $D(M)$ has a $K(1 + \log(D(M)))$-gentle partition of unity with respect to any closed subset X, where K is a universal constant. Therefore the Lipschitz-free space $\mathcal{F}(M)$ has the $C(1 + \log(D(M)))$-BAP with a universal constant C. This is the statement of Corollary 1.2.2. For a natural number n, the space $\mathbb{R}^n$ with the Euclidean norm is a doubling metric space with the doubling constant bounded above by K^n, where K is a universal constant. Since this property is inherited by metric subspaces, Corollary 1.2.2 yields that for any closed subset F of the Euclidean space $\mathbb{R}^n$, the Lipschitz-free space $\mathcal{F}(F)$ has the Cn-BAP for some universal constant C. But in fact, the constant can be improved as is presented in Proposition 1.2.3. Indeed, applying results from [15], we obtain that if F is a closed subset of $\mathbb{R}^n$ equipped with the Euclidean norm, then the Lipschitz-free space $\mathcal{F}(F)$ is isometric to a $C\sqrt{n}$-complemented subspace of the Lipschitz-free space $\mathcal{F}(\mathbb{R}^n)$ for a universal constant C. This combined with the fact that $\mathcal{F}(\mathbb{R}^n)$ has MAP, proved in [7], gives that $\mathcal{F}(F)$ has the $C\sqrt{n}$-BAP, where C is a universal constant.

This result now gives rise to further interesting questions. As suggested by Gilles Godefroy, one may ask whether the constant $C\sqrt{n}$ is optimal, or, whether there exists $\lambda > 0$ such that for every $n \in \mathbb{N}$ and every closed subset M of $\mathbb{R}^n$, the Lipschitz-free space $\mathcal{F}(M)$ has the $\lambda-$BAP.

Another very recent positive result, which says that the Lipschitz-free space over a countable compact metric space has the MAP, is due to Dalet [3].

In Section 1.3 we proceed by looking for Banach spaces such that the corresponding Lipschitz-free spaces have stronger approximation properties. The study in this direction began with the work of Borel-Mathurin [1], where the existence of an FDD for the Lipschitz-free space $\mathcal{F}(\mathbb{R}^n)$ is proved. The decomposition constant in this result depends on the dimension n. In Theorem 1.3.1 we show that the Lipschitz-free spaces $\mathcal{F}(\ell_1^n)$ and $\mathcal{F}(\ell_1)$ admit a *monotone finite-dimensional Schauder decomposition*, i.e. an FDD with the decomposition constant equal to 1. Let us just mention here that later, in Chapter 2, refining the technique of the proof we improve this result and obtain even a Schauder basis for the spaces $\mathcal{F}(\ell_1)$ and $\mathcal{F}(\ell_1^n)$. The proof of Theorem 1.3.1 is based on finding a sequence $(T_k)_{k=1}^{\infty}$ of projections on the considered Lipschitz-free space $\mathcal{F}(X)$ (here X standing for ℓ_1^n or ℓ_1) which yields FDD. To this end, we construct a sequence $(P_k)_{k=1}^{\infty}$ of projections on $\mathcal{F}(X)^* = \mathrm{Lip}_0(X)$ so that they are adjoint and that the sequence $(T_k)_{k=1}^{\infty}$ of projections on $\mathcal{F}(X)$ given via the relation $T_k^* = P_k$ has desired properties. The crucial tool for building $(P_k)_{k=1}^{\infty}$ is a particular method for interpolating Lipschitz functions defined on the vertices of a hypercube in ℓ_1^n to the whole hypercube preserving the Lipschitz constant of the function. Let us now sketch the construction. We consider an increasing sequence of finite-dimensional hypercubes in X, growing both in the dimension and the size, which are decomposed into smaller hypercubes (we call this decomposition tiling) that are, on the other hand, shrinking. Then we define the corresponding projections on $\mathrm{Lip}_0(X)$ by fixing the values of the original function at all the vertices of all hypercubes forming the tiling, applying the interpolation method inside these hypercubes and finally by using a $1-$Lipschitz retraction outside the big hypercube.

The paper [11], forming Chapter 2, is a continuation of the work [14]. In Theorem 2.3.1 we show that if X is a product of countably many closed (possibly unbounded or degenerate) intervals with endpoints in $\mathbb{Z} \cup \{-\infty, \infty\}$, considered as a metric subspace of ℓ_1 equipped with the inherited metric, then the Lipschitz-free space $\mathcal{F}(X)$ has a *Schauder basis*. This implies in particular that the Lipschitz-free spaces $\mathcal{F}(\ell_1)$ and $\mathcal{F}(\ell_1^n)$ have a Schauder basis, strengthening thus the result on the existence of FDD in these spaces formulated in 1.3.1. In view of this positive example, it might be interesting to ask if an analogue of the aforementioned result on equivalence of BAP for a Banach space and BAP for its Lipschitz-free space [7] can be obtained for Schauder basis. That is, in general, is it true that a Banach space X has a Schauder basis if and only if the Lipschitz-free space $\mathcal{F}(X)$ has a Schauder basis?

The main idea of the proof of Theorem 2.3.1 is similar to that followed in 1.3.2 (see above). However, there, when we pass from a projection P_k to P_{k+1}, we increase the dimension of the big hypercube by one, double the length of its edge, refine the tiling by splitting each of its hypercubes, and define $P_{k+1}(f)$ so that it coincides with f at all the vertices in the tiling. This way the growth of the ranks of the differences of two consecutive projections is not controlled and we only arrive at FDD. In order to achieve a basis, one must obviously proceed slower. We do so in [11]. Here, when we go from step k to step $k+1$, we always include only at most one more vertex given by the tiling to the set of points at which $P_{k+1}(f)$ agrees with the original function f. Then we need to define $P_{k+1}(f)$ at the remaining vertices in the tiling in a way which does not ruin the Lipschitz constant. For that purpose we either use a 1-Lipschitz retraction to assign a value to $P_{k+1}(f)$ at a treated vertex, or we inductively compute the value as an average of the values at the neighbouring vertices in a careful order.

The proof of Theorem 2.3.1 is rather technical and special to the metric of ℓ_1, and does not seem to generalise in its present form to any infinite dimensional Banach space non-isomorphic to ℓ_1.

We conclude the chapter by raising an open problem.

Preprint [21], in the thesis introduced as Chapter 3, is concerned with the rigidity of the spaces ℓ_∞ and ℓ_∞^n with respect to *uniformly differentiable mappings*.

Recently, it has been shown that if Y is a Banach space and if f is a non-compact uniformly differentiable mapping from the unit ball of c_0 into Y, then Y^{**} contains a copy of ℓ_∞ and, when Y is a dual space, then even Y contains a copy of c_0 ([4], [10, Theorem 6.45]). This work addresses, in special case, the question of generalization of a classical Pełczyński theorem for linear operators to uniformly differentiable mappings. The linear result due to Pełczyński (see [20], [5, Theorem 4.51]) says that if T is a non-compact linear operator from

c_0 into a Banach space Y, then c_0 contains a linear subspace X isomorphic to c_0 such that $T|_X$ is an isomorphism. Hence, Y contains a copy of c_0. The considered problem is whether the existence of a non-compact uniformly differentiable mapping from the unit ball of c_0 into a Banach space Y implies that Y contains a copy of c_0. The general case remains open.

In our study we are interested in an analogous question for the space ℓ_∞ and show that it can be answered in positive. In the result by Deville and Hájek [4], roughly speaking, the isomorphism taking ℓ_∞ into Y^{**} is the derivative of the bidual extension of f at some point ot the unit ball of the bidual c_0^{**}. We generalise this to uniformly differentiable mappings which are not necessarily bidual extensions of uniformly differentiable mappings. Namely, Theorem 3.2.1 implies that if Y is a Banach space and f is a uniformly differentiable mapping from the unit ball of ℓ_∞ into Y such that $\{f(e_k), k \in \mathbb{N}\}$ is not relatively compact in Y, then there exists a linear subspace Z of ℓ_∞ isometric to ℓ_∞ and a point $x \in Z$ such that the operator $Df(x)|_Z$ is an isomorphism. In particular, Y contains a copy of ℓ_∞. This, in some way, transfers a well-known linear result by Rosenthal (see [16, Proposition 2.f.4]) to non-linear setting. He proved that if Y is a Banach space and T is a non-weakly compact linear operator from ℓ_∞ into Y, then ℓ_∞ contains a linear subspace X isomorphic to ℓ_∞ such that $T|_X$ is an isomorphism, thus, Y contains a copy of ℓ_∞. Moreover, Theorem 3.2.1 generalises the classical result on non-complementability of c_0 in ℓ_∞ due to Phillips (see [5, Theorem 5.6]) since it implies that there does not exist any uniformly differentiable mapping from ℓ_∞ into c_0 which fixes the basis.

In Section 3.1 we also discuss why the same method cannot lead to a solution of the problem in the case of c_0.

Finally, we give attention to ℓ_∞^n spaces. Using ultrapowers, we obtain a finite-dimensional result, Theorem 3.2.3, as a corollary of Theorem 3.2.1.

1. Approximation properties and Schauder decompositions in Lipschitz-free spaces

joint work with G. Lancien

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1.1 Introduction

For (M_1, d_1) and (M_2, d_2) metric spaces and $f : M_1 \to M_2$, we denote by $\mathrm{Lip}(f)$ the Lipschitz constant of f given by

$$\mathrm{Lip}(f) = \sup\left\{\frac{d_2(f(x), f(y))}{d_1(x, y)},\ x, y \in M_1,\ x \neq y\right\}.$$

Consider (M, d) a *pointed* metric space, i.e. a metric space equipped with a distinguished element (origin) denoted 0. Then, the space $\mathrm{Lip}_0(M)$ of all real-valued Lipschitz functions f on M which satisfy $f(0) = 0$, endowed with the norm

$$\|f\|_{\mathrm{Lip}_0(M)} = \mathrm{Lip}(f)$$

is a Banach space. The Dirac map $\delta : M \to \mathrm{Lip}_0(M)^*$ defined by $\langle g, \delta(p)\rangle = g(p)$ for $g \in \mathrm{Lip}_0(M)$ and $p \in M$ is an isometric embedding from M into $\mathrm{Lip}_0(M)^*$. The closed linear span of $\{\delta(p),\ p \in M\}$ is denoted $\mathcal{F}(M)$ and called the *Lipschitz-free space over* M (or free space in short). It follows from the compactness of the unit ball of $\mathrm{Lip}_0(M)$ with respect to the topology of pointwise convergence, that $\mathcal{F}(M)$ can be seen as the canonical predual of $\mathrm{Lip}_0(M)$. Then the weak*-topology induced by $\mathcal{F}(M)$ on $\mathrm{Lip}_0(M)$ coincides with the topology of pointwise convergence on the bounded subsets of $\mathrm{Lip}_0(M)$. Lipschitz-free spaces are a very useful tool for abstractly linearizing Lipschitz maps. Indeed, if we identify through the Dirac map a metric space M with a subset of $\mathcal{F}(M)$, then any Lipschitz map from the metric space M to a metric space N extends to a continuous linear map from $\mathcal{F}(M)$ to $\mathcal{F}(N)$ with the same Lipschitz constant (see [23] or Lemma 2.2 in [7]). A comprehensive reference for the basic theory of the spaces of Lipschitz functions and their preduals, which are called Arens-Eells spaces there, is the book [23] by Weaver.

Despite the simplicity of their definition, very little is known about the linear structure of Lipschitz-free spaces over separable metric spaces. It is easy to see that $\mathcal{F}(\mathbb{R})$ is isometric to L_1. However, adapting a theorem of Kislyakov [13], Naor and Schechtman proved in [19] that $\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to any subspace of L_1. Then the metric spaces whose Lipschitz-free space is isometric to a subspace of L_1 have been characterized by Godard in [6].

The aim of this paper is to study metric spaces M such that $\mathcal{F}(M)$ has the bounded approximation property (BAP) or admits a finite-dimensional Schauder decomposition (FDD). This kind of study was initiated in the fundamental paper by Godefroy and Kalton [7], where they proved that a Banach space X has the λ-BAP if and only if $\mathcal{F}(X)$ has the λ-BAP. In particular, for any finite-dimensional Banach space E, $\mathcal{F}(E)$ has the metric approximation property (MAP). Another major result from [7] is that any separable Banach space has the so-called isometric lifting property. Refining the techniques used in the proof of this result, Godefroy and Ozawa have proved in their recent work [8] that any separable Banach space failing the BAP contains a compact subset whose Lipschitz-free space also fails the BAP. It is then natural, as it is suggested in [8], to try to describe the metric spaces whose Lipschitz-free space has BAP. We address this question in Section 1.2. Our main result of this section (Corollary 1.2.2) is that for any doubling metric space M, the Lipschitz-free space $\mathcal{F}(M)$ has the BAP.

Then we try to find Banach spaces such that the corresponding Lipschitz-free spaces have stronger approximation properties. The first result in this direction is due to Borel-Mathurin, who proved in [1] that $\mathcal{F}(\mathbb{R}^N)$ admits a finite-dimensional Schauder decomposition. The decomposition constant obtained in [1] depends on the dimension N. In Section 1.3 we show that $\mathcal{F}(\ell_1^N)$ and $\mathcal{F}(\ell_1)$ admit a monotone finite-dimensional Schauder decomposition. For that purpose, we use a particular technique for interpolating Lipschitz functions on hypercubes of $\mathbb{R}^N$.

1.2 BAP for Lipschitz-free spaces and K-gentle partitions of unity

We first recall the definition of the bounded approximation property.

Let $1 \leq \lambda < \infty$. A Banach space X has the *λ-bounded approximation property* (λ-BAP) if, for every $\varepsilon > 0$ and every compact set $K \subset X$, there is a bounded finite-rank linear operator $T : X \to X$ with $\|T\| \leq \lambda$ and such that $\|T(x) - x\| \leq \varepsilon$ whenever $x \in K$. We say that X has the BAP if it has the $\lambda-$BAP for some $1 \leq \lambda < \infty$.

Obviously, if there is a bounded sequence of finite-rank linear operators on a Banach space X converging in the strong operator topology to the identity on X, then X has the BAP. For further information on the approximation properties of Banach spaces we refer the reader to [16] or [5].

We now detail a construction due to Lee and Naor [15] that we shall use. Let (Y, d) be a metric space, X a closed subset of Y, (Ω, Σ, μ) a measure space and $K > 0$. Following [15] we say that a function $\psi : \Omega \times Y \to [0, \infty)$ is a *K-gentle partition of unity of Y with respect to X* if it satisfies the following:
(i) For all $x \in Y \setminus X$, the function $\psi_x : \omega \mapsto \psi(\omega, x)$ is in $L_1(\mu)$ and $\|\psi_x\|_{L_1(\mu)} = 1$.
(ii) For all $\omega \in \Omega$ and all x in X, $\psi(\omega, x) = 0$.
(iii) There exists a Borel measurable function $\gamma : \Omega \to X$ such that for all $x, y \in Y$

$$\int_\Omega |\psi(\omega, x) - \psi(\omega, y)| d(\gamma(\omega), x)\, d\mu(\omega) \leq K d(x, y).$$

Then, for Y having a K-gentle partition of unity with respect to a separable subset X and for f Lipschitz on X, Lee and Naor define $E(f)$ by $E(f)(x) = f(x)$ if $x \in X$ and

$$E(f)(x) = \int_\Omega f(\gamma(\omega))\psi(\omega, x)\, d\mu(\omega) \quad \text{if } x \in Y \setminus X$$

and show that $\mathrm{Lip}(E(f)) \leq 3K\,\mathrm{Lip}(f)$ ([15] Lemma 2.1).

The proof of this lemma is quite elementary. However, let us emphasize that building a K-gentle partition of unity is highly non-trivial. The approach of Lee and Naor in [15] is to first construct random partitions of unity. Then, the key idea, as we understand it, is that a single smooth or "gentle" partition of unity can emerge by averaging a "good" random distribution of partitions of unity.

Our general result is then the following.

Theorem 1.2.1. *Let (M,d) be a pointed separable metric space such that there exists a constant $K>0$ so that for any closed subset X of M, M admits a K-gentle partition of unity with respect to X. Then $\mathcal{F}(M)$ has the $3K$-BAP.*

Proof. Our objective is to find a sequence of finite-rank linear operators on $\mathcal{F}(M)$ with norms bounded by $3K$ and converging to the identity on $\mathcal{F}(M)$ in the strong operator topology. To this end, we first construct a sequence of operators of appropriate qualities on the dual space $\mathrm{Lip}_0(M)$ so that they are adjoint operators and then pass to $\mathcal{F}(M)$. To be more precise, we build a sequence $(S_n)_{n=1}^{\infty}$ of $3K$-bounded finite-rank linear operators on $\mathrm{Lip}_0(M)$ that are pointwise continuous, and therefore weak* to weak*-continuous, on bounded subsets of $\mathrm{Lip}_0(M)$ and such that for all $f \in \mathrm{Lip}_0(M)$, $(S_n(f))_{n=1}^{\infty}$ converges pointwise to f. This will imply that $S_n = T_n^*$, where T_n is a finite-rank operator on $\mathcal{F}(M)$ (see [5], Corollary 3.94 for instance) and such that $(T_n)_{n=1}^{\infty}$ is converging to the identity for the weak operator topology on $\mathcal{F}(M)$. Recall now that M is assumed to be a separable metric space. So using the fact that the Dirac map δ is an isometry from M into $\mathcal{F}(M)$ whose image has a dense linear span, we see that $\mathcal{F}(M)$ is also separable. Then Mazur's Lemma and a standard diagonal argument will yield the existence of a bounded sequence of finite-rank operators converging to the identity for the strong operator topology on $\mathcal{F}(M)$. Note that the operators obtained in this last step are made of convex combinations of the T_n's. This preserves our control on their norms.

So, let $(x_n)_{n=1}^{\infty}$ be a dense sequence in M and 0 be the origin of M. Put $X_n = \{0, x_1, .., x_n\}$. For $f \in \mathrm{Lip}_0(M)$ we denote $R_n(f)$ the restriction of f to X_n. The operator R_n, defined from $\mathrm{Lip}_0(M)$ to $\mathrm{Lip}_0(X_n)$, is clearly of finite rank, pointwise continuous and such that $\|R_n\| \leq 1$.

Thanks to our assumption that M admits a K-gentle partition of unity with respect to X_n, we can apply Lee and Naor's construction to obtain an extension operator E_n from $\mathrm{Lip}_0(X_n)$ to $\mathrm{Lip}_0(M)$. Note that it follows immediately from the definition of E_n and Lebesgue's dominated convergence theorem that E_n is pointwise continuous on bounded subsets of $\mathrm{Lip}_0(X_n)$.

Finally, we set $S_n = E_n R_n$. The sequence $(S_n)_{n=1}^{\infty}$ is indeed a sequence of finite-rank linear operators from $\mathrm{Lip}_0(M)$ to $\mathrm{Lip}_0(M)$ that are pointwise continuous on bounded subsets of $\mathrm{Lip}_0(M)$ and so that $\|S_n\| \leq 3K$ for all $n \in \mathbb{N}$. To finish the proof, we only need to show that for any $f \in \mathrm{Lip}_0(M)$, the sequence $(S_n(f))_{n=1}^{\infty}$ converges pointwise to f. So let us fix $x \in M$, $f \in \mathrm{Lip}_0(M)$ and $\varepsilon > 0$. Let $n_0 \in \mathbb{N}$ such that $d(x, x_{n_0}) \leq \varepsilon$. Then, for any $n \geq n_0$,

$$|f(x) - f(x_{n_0})| \leq \varepsilon \|f\|_{\mathrm{Lip}_0(M)} \quad \text{and} \quad |S_n(f)(x) - f(x_{n_0})| \leq 3K\varepsilon \|f\|_{\mathrm{Lip}_0(M)}.$$

Therefore $|S_n(f)(x) - f(x)| \leq (1+3K)\varepsilon \|f\|_{\mathrm{Lip}_0(M)}$. This concludes our proof. □

We recall that a metric space (M,d) is called *doubling* if there exists a constant $D(M) > 0$ such that any open ball $B(p,R)$ in M can be covered with at most $D(M)$ open balls of radius $\frac{R}{2}$. We can now state the main application of Theorem 1.2.1.

Corollary 1.2.2. *Let (M,d) be a pointed doubling metric space. Then the Lipschitz-free space $\mathcal{F}(M)$ has the $C(1+\log(D(M)))$-BAP, where C is a universal constant.*

Proof. We combine some of the important results from [15]. Namely, it follows from Lemma 3.8, Corollary 3.12 and Theorem 4.1 in [15] that if M is a doubling metric space and X is a closed subset of M, then M admits a $K(1+\log(D(M)))$-gentle partition of unity with respect to X (where K is a universal constant). The conclusion is now a direct application of Theorem 1.2.1. □

Remarks. 1) Let us mention that an extension operator with these properties could also be obtained from a later construction due to A. Brudnyi and Y. Brudnyi in [2], where they use the notion of metric space of homogeneous type. A Borel measure μ on a metric space (M,d) is called *doubling* if the measure of every open ball is strictly positive and finite and if there is a constant $\delta(\mu) > 0$ such that $\mu(B(p,2R)) \leq \delta(\mu)\mu(B(p,R))$ for all $p \in M$ and $R > 0$. A metric space endowed with a doubling measure is said to be of *homogeneous type*. Clearly, a space of homogeneous type is doubling. Conversely, Luukkainen and Saksman proved in [17] that every complete doubling metric space (M,d) carries a doubling measure μ such that $\delta(\mu) \leq c(D(M))$, where $c(D(M))$ is a constant depending only on $D(M)$. More on doubling metric spaces and spaces of homogeneous type can be found in [22] and [12].

2) We refer the reader to Lee and Naor's paper [15] for other examples of metric spaces admitting K-gentle partitions of unity such as negatively curved manifolds, special graphs or surfaces of bounded genus.

Let us conclude this section with a few words on the Lipschitz-free spaces over subsets of $\mathbb{R}^N$. It is easily checked that for $N \in \mathbb{N}$, the space $\mathbb{R}^N$ with the Euclidean norm is a doubling metric space with doubling constant bounded above by K^N, where K is a universal constant. This property is inherited by metric subspaces. So, it follows from Corollary 1.2.2 that for any closed subset F of the Euclidean space $\mathbb{R}^N$, $\mathcal{F}(F)$ has the CN-BAP for some universal constant C. It turns out that a better result can be derived from [15] and [7].

Proposition 1.2.3. *Let $N \in \mathbb{N}$ and F be a closed subset of $\mathbb{R}^N$ equipped with the Euclidean norm. Then the Lipschitz-free space $\mathcal{F}(F)$ is isometric to a $C\sqrt{N}$-complemented subspace of the Lipschitz-free space $\mathcal{F}(\mathbb{R}^N)$. In particular, $\mathcal{F}(F)$ has the $C\sqrt{N}$-BAP.*

Proof. We may assume, after translating F, that $0 \in F$. The restriction to F defined from $\mathrm{Lip}_0(\mathbb{R}^N)$ to $\mathrm{Lip}_0(F)$ is the adjoint operator of an isometry J from $\mathcal{F}(F)$ into $\mathcal{F}(\mathbb{R}^N)$. We can now apply a more precise result on extensions of Lipschitz functions coming from [15]. Indeed, it follows from Lemma 3.16 and Theorem 4.1 in [15] that $\mathbb{R}^N$ equipped with the Euclidean norm admits a $K\sqrt{N}$-gentle partition of unity with respect to F, where K is a universal constant. So, there exists a linear operator $E : \mathrm{Lip}_0(F) \to \mathrm{Lip}_0(\mathbb{R}^N)$ which is weak* to weak*-continuous on bounded subsets of $\mathrm{Lip}_0(F)$ and such that $E(f)|_F = f$ for every $f \in \mathrm{Lip}_0(F)$ and $\|E\| \leq 3K\sqrt{N}$. Due to the weak*-continuity of E, there exists a bounded linear operator $P : \mathcal{F}(\mathbb{R}^N) \to \mathcal{F}(F)$ satisfying $P^* = E$. Moreover, thanks to the fact that E is an extension operator and by the Hahn-Banach theorem, $JP(\mu) = \mu$ for every $\mu \in J(\mathcal{F}(F))$. Hence JP is a linear projection from $\mathcal{F}(\mathbb{R}^N)$ onto $J(\mathcal{F}(F))$ such that $\|JP\| \leq 3K\sqrt{N}$. This shows that $\mathcal{F}(F)$ is isometric to a $C\sqrt{N}$-complemented subspace of $\mathcal{F}(\mathbb{R}^N)$, where C is a universal constant. On the other hand, it is proved in [7] that $\mathcal{F}(\mathbb{R}^N)$ has the MAP. Therefore $\mathcal{F}(F)$ has the $C\sqrt{N}$-BAP. □

1.3 FDD of the Lipschitz-free space $\mathcal{F}(\ell_1)$

We recall the notion of finite-dimensional Schauder decomposition following the monograph of Lindenstrauss and Tzafriri [16].

Let X be a Banach space. A sequence $(X_n)_{n=1}^\infty$ of finite-dimensional subspaces of X is called a *finite-dimensional Schauder decomposition* of X (FDD) if every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^\infty x_n$ with $x_n \in X_n$ for every $n \in \mathbb{N}$.

If $(S_n)_{n=0}^\infty$, where $S_0 \equiv 0$, is a sequence of projections on X satisfying $S_n S_m = S_{\min\{m,n\}}$ such that $0 < \dim(S_n - S_{n-1})(X) < \infty$ and converging in the strong operator topology to the identity on X, then $\big((S_n - S_{n-1})(X)\big)_{n=1}^\infty$ is an FDD of X, for which the S_n's are the partial sum projections. Then the sequence $(S_n)_{n=1}^\infty$ is bounded and $K = \sup_{n\in\mathbb{N}} \|S_n\|$ is called the *decomposition constant*. If $K = 1$, then the decomposition is called *monotone*.

For $N \in \mathbb{N}$, the space $\mathbb{R}^N$ equipped with the norm $\|x\|_1 = \sum_{i=1}^N |x_i|$ is denoted ℓ_1^N. The space $\{x = (x_i)_{i=1}^\infty \in \mathbb{R}^\mathbb{N},\ \sum_{i=1}^\infty |x_i| < \infty\}$ equipped with the norm $\|x\|_1 = \sum_{i=1}^\infty |x_i|$ is denoted ℓ_1. We write $\mathbf{0^N}$ for the origin in ℓ_1^N and $\mathbf{0}$ for the origin in ℓ_1. Our result is the following.

Theorem 1.3.1. *The Lipschitz-free spaces $\mathcal{F}(\ell_1)$ and $\mathcal{F}(\ell_1^N)$ admit monotone finite-dimensional Schauder decompositions.*

Let X be ℓ_1 or ℓ_1^N. It follows from the classical theory that we only need to build a sequence of contractive finite-rank linear projections $(S_n)_{n=1}^\infty$ on $\mathcal{F}(X)$ such that $S_n S_m = S_{\min\{m,n\}}$ for all $m, n \in \mathbb{N}$ and that $\overline{\bigcup_{n=1}^\infty S_n(\mathcal{F}(X))} = \mathcal{F}(X)$. As in the proof of Theorem 1.2.1 we shall work on the dual space and construct a sequence of contractive finite-rank linear projections on $\mathrm{Lip}_0(X)$, that are pointwise continuous on bounded subsets of $\mathrm{Lip}_0(X)$, possess the commuting property and converge to the identity on $\mathrm{Lip}_0(X)$ in the weak*-operator topology. The general idea will be to take an increasing sequence of closed bounded subsets of X and associate with each of these sets a finite-rank linear operator on $\mathrm{Lip}_0(X)$ so that the image of a function under this operator has values close to the values of the original function at the points of the considered closed set. However, unlike the situation in our previous section, we have the linear structure of the metric space X at our disposal. This will enable us to work accurately enough to obtain a monotone FDD for $\mathcal{F}(X)$.

1.3.1 Notation and interpolation lemma

Put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and fix $N \in \mathbb{N}$. We denote by $C(y, R)$, where $y \in \mathbb{R}^N$ and $0 < R < \infty$, the hypercube

$$C(y,R) = \left\{ x \in \mathbb{R}^N,\ \sup_{1\leq i\leq N} |x_i - y_i| \leq \frac{R}{2} \right\}.$$

For $y \in \mathbb{R}^N$, $0 < R < \infty$ and $\delta \in \{-1,1\}^N$, the symbol $A_\delta(y,R)$ stands for the vertex $y + \frac{R}{2}\delta$ of the hypercube $C(y,R)$.

The following interpolation on $C(y,R)$ of a function defined on its vertices will be the crucial tool for our proof. Let $y \in \mathbb{R}^N$, $0 < R < \infty$, $x \in C(y,R)$ and let $f : \mathrm{dom}(f) \to \mathbb{R}$ satisfy $\left\{A_\delta(y,R),\ \delta \in \{-1,1\}^N\right\} \subset \mathrm{dom}(f) \subset \mathbb{R}^N$. We define inductively:

$$\begin{aligned}\Lambda_\gamma(f,\, C(y,R))(x) =& \frac{x_1 - y_1 + \frac{R}{2}}{R} f\left(A_{(1,\gamma_1,\dots,\gamma_{N-1})}(y,R)\right) \\ &+ \left(1 - \frac{x_1 - y_1 + \frac{R}{2}}{R}\right) f\left(A_{(-1,\gamma_1,\dots,\gamma_{N-1})}(y,R)\right)\end{aligned}$$

for each $\gamma = (\gamma_1, \dots, \gamma_{N-1}) \in \{-1,1\}^{N-1}$,

$$\begin{aligned}\Lambda_\gamma(f,\, C(y,R))(x) =& \frac{x_j - y_j + \frac{R}{2}}{R} \Lambda_{(1,\gamma_1,\dots,\gamma_{N-j})}(f,\, C(y,R))(x) \\ &+ \left(1 - \frac{x_j - y_j + \frac{R}{2}}{R}\right) \Lambda_{(-1,\gamma_1,\dots,\gamma_{N-j})}(f,\, C(y,R))(x)\end{aligned}$$

for each $j \in \{2, \dots, N-1\}$ and $\gamma = (\gamma_1, \dots, \gamma_{N-j}) \in \{-1,1\}^{N-j}$, and

$$\begin{aligned}\Lambda(f,\, C(y,R))(x) =& \frac{x_N - y_N + \frac{R}{2}}{R} \Lambda_{(1)}(f,\, C(y,R))(x) \\ &+ \left(1 - \frac{x_N - y_N + \frac{R}{2}}{R}\right) \Lambda_{(-1)}(f,\, C(y,R))(x). \qquad (1.1)\end{aligned}$$

Let us use the following convention: $\{-1,1\}^0 := \{\emptyset\}$ and $\Lambda_\emptyset = \Lambda$.

Let $I_1, \dots, I_N$ be closed intervals in $\mathbb{R}$. We shall say that a function Φ from $I_1 \times \cdots \times I_N$ into $\mathbb{R}$ has the property (AF) on $I_1 \times \cdots \times I_N \subset \mathbb{R}^N$ if its restriction to any segment lying in $I_1 \times \cdots \times I_N$ and parallel to one of the coordinate axes is affine. A function having the property (AF) on $I_1 \times \cdots \times I_N$ is uniquely determined by its values at the vertices of $I_1 \times \cdots \times I_N$. Observe that $\Lambda(f,\, C(y,R))$ has the property (AF) on $C(y,R)$ and coincides with the function f at the vertices of $C(y,R)$.

We now state and prove our basic interpolation lemma.

Lemma 1.3.2. *Let $y \in \mathbb{R}^N$, $0 < R < \infty$ and let $f : \mathrm{dom}(f) \to \mathbb{R}$ be a function satisfying $\left\{A_\delta(y,R),\ \delta \in \{-1,1\}^N\right\} \subset \mathrm{dom}(f) \subset \mathbb{R}^N$. Consider $\mathbb{R}^N$ equipped with the ℓ_1-norm. Then*

$$\mathrm{Lip}\left(\Lambda(f,\, C(y,R))\right) = \mathrm{Lip}\left(f|_{\{A_\delta(y,R),\ \delta\in\{-1,1\}^N\}}\right).$$

Proof. It follows clearly from its definition that $\Lambda\left(f, C\left(y,R\right)\right)$ is differentiable in the interior of $C\left(y,R\right)$. We shall prove that for any $1 \leq i \leq N$ and any x in the interior of $C\left(y,R\right)$

$$\left|\frac{\partial \Lambda\left(f,\, C\left(y,R\right)\right)}{\partial x_i}(x)\right| \leq \text{Lip}\left(f|_{\{A_\delta(y,R),\ \delta\in\{-1,1\}^N\}}\right).$$

Since $\mathbb{R}^N$ is equipped with $\|\ \|_1$, the conclusion of our lemma will then follow directly from the mean value theorem.

So let x be an interior point of $C\left(y,R\right)$, that is x so that $y_i - \frac{R}{2} < x_i < y_i + \frac{R}{2}$ for all $1 \leq i \leq N$. Consider first $\gamma, \tilde{\gamma} \in \{-1,1\}^{N-1}$ such that there exists a unique $k \in \{1, \ldots, N-1\}$ satisfying $\gamma_k \neq \tilde{\gamma}_k$. Then

$$\begin{aligned}
&\left|\Lambda_\gamma\left(f,\, C\left(y,R\right)\right)(x) - \Lambda_{\tilde{\gamma}}\left(f,\, C\left(y,R\right)\right)(x)\right| \\
&= \left|\left(1 - \frac{x_1 - y_1 + \frac{R}{2}}{R}\right)\left(f\left(A_{(-1,\gamma_1,\ldots,\gamma_{N-1})}\left(y,R\right)\right) - f\left(A_{(-1,\tilde{\gamma}_1,\ldots,\tilde{\gamma}_{N-1})}\left(y,R\right)\right)\right)\right. \\
&\qquad \left. + \frac{x_1 - y_1 + \frac{R}{2}}{R}\left(f\left(A_{(1,\gamma_1,\ldots,\gamma_{N-1})}\left(y,R\right)\right) - f\left(A_{(1,\tilde{\gamma}_1,\ldots,\tilde{\gamma}_{N-1})}\left(y,R\right)\right)\right)\right| \\
&\leq R \text{ Lip}\left(f|_{\{A_\delta(y,R),\ \delta\in\{-1,1\}^N\}}\right).
\end{aligned}$$

Further, one shows by induction on $j \in \{1, \ldots, N-1\}$ that for every couple $\gamma, \tilde{\gamma} \in \{-1,1\}^{N-j}$ such that there exists a unique $k \in \{1, \ldots, N-j\}$ satisfying $\gamma_k \neq \tilde{\gamma}_k$ we have

$$\begin{aligned}
&\left|\Lambda_\gamma\left(f,\, C\left(y,R\right)\right)(x) - \Lambda_{\tilde{\gamma}}\left(f,\, C\left(y,R\right)\right)(x)\right| \\
&= \left|\left(1 - \frac{x_j - y_j + \frac{R}{2}}{R}\right)\left(\Lambda_{(-1,\gamma_1,\ldots,\gamma_{N-j})}\left(f,\, C\left(y,R\right)\right)(x)\right.\right. \\
&\qquad\qquad \left. - \Lambda_{(-1,\tilde{\gamma}_1,\ldots,\tilde{\gamma}_{N-j})}\left(f,\, C\left(y,R\right)\right)(x)\right) \\
&\quad + \frac{x_j - y_j + \frac{R}{2}}{R}\left(\Lambda_{(1,\gamma_1,\ldots,\gamma_{N-j})}\left(f,\, C\left(y,R\right)\right)(x)\right. \\
&\qquad\qquad \left.\left. - \Lambda_{(1,\tilde{\gamma}_1,\ldots,\tilde{\gamma}_{N-j})}\left(f,\, C\left(y,R\right)\right)(x)\right)\right| \\
&\leq R \text{ Lip}\left(f|_{\{A_\delta(y,R),\ \delta\in\{-1,1\}^N\}}\right).
\end{aligned} \tag{1.2}$$

Now, for $\gamma \in \{-1,1\}^{N-1}$ and $i \in \{1, \ldots, N\}$,

$$\left|\frac{\partial \Lambda_\gamma\left(f,\, C\left(y,R\right)\right)}{\partial x_i}(x)\right| = \begin{cases} \left|\frac{f\left(A_{(1,\gamma_1,\ldots,\gamma_{N-1})}(y,R)\right)}{R} - \frac{f\left(A_{(-1,\gamma_1,\ldots,\gamma_{N-1})}(y,R)\right)}{R}\right| & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$$

Therefore

$$\left|\frac{\partial \Lambda_\gamma\left(f,\, C\left(y,R\right)\right)}{\partial x_i}(x)\right| \leq \text{Lip}\left(f|_{\{A_\delta(y,R),\ \delta\in\{-1,1\}^N\}}\right).$$

Further, for $j \in \{2, \dots, N\}$, $\gamma \in \{-1, 1\}^{N-j}$ and $i \in \{1, \dots, N\}$,

$$\left|\frac{\partial \Lambda_\gamma\left(f, C\left(y, R\right)\right)}{\partial x_i}(x)\right| = \begin{cases} \left|\frac{R-\left(x_j-y_j+\frac{R}{2}\right)}{R}\frac{\partial \Lambda_{(-1,\gamma_1,\dots,\gamma_{N-j})}(f, C(y,R))}{\partial x_i}(x)\right. \\ \quad \left. + \frac{x_j-y_j+\frac{R}{2}}{R}\frac{\partial \Lambda_{(1,\gamma_1,\dots,\gamma_{N-j})}(f, C(y,R))}{\partial x_i}(x)\right| & \text{if } i < j, \\ \left|\frac{\Lambda_{(1,\gamma_1,\dots,\gamma_{N-j})}(f, C(y,R))(x)}{R}\right. \\ \quad \left. - \frac{\Lambda_{(-1,\gamma_1,\dots,\gamma_{N-j})}(f, C(y,R))(x)}{R}\right| & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

Consequently, using (1.2) and an induction on j, one gets that for all $j \in \{1, \dots, N\}$, $i \in \{1, \dots, N\}$ and $\gamma \in \{-1, 1\}^{N-j}$,

$$\left|\frac{\partial \Lambda_\gamma\left(f, C\left(y, R\right)\right)}{\partial x_i}(x)\right| \leq \operatorname{Lip}\left(f|_{\{A_\delta(y,R),\ \delta \in \{-1,1\}^N\}}\right).$$

This concludes the proof. □

We now finish setting our notation. Provided that $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N) \in \{-1, 1\}^N$, $y \in \mathbb{R}^N$, $h = (h_1, \dots, h_N) \in \mathbb{N}_0^N$ and $k \in \mathbb{N}_0$, we denote

$$x_{h,k}^{\varepsilon,y} = y + 2^{-k-1}\varepsilon + 2^{-k}(\varepsilon_1 h_1, \dots, \varepsilon_N h_N).$$

Next, if $0 < R < \infty$ and $t \in \mathbb{R}$, we define $\pi_R(t)$ to be the nearest point to t in $[-\frac{R}{2}, \frac{R}{2}]$. Then we define $\Pi_R^N(x) = (\pi_R(x_1), \dots, \pi_R(x_N))$ for all $x \in \mathbb{R}^N$. It is easily checked that Π_R^N is a retraction from ℓ_1^N onto $C\left(\mathbf{0^N}, R\right)$ and that $\operatorname{Lip}(\Pi_R^N) = 1$. In fact, Π_R^N is the nearest point mapping to $C\left(\mathbf{0^N}, R\right)$ and is 1-Lipschitz in both $\|\ \|_1$ and $\|\ \|_2$ on $\mathbb{R}^N$.

Finally, we define ρ_N to be the canonical projection from ℓ_1 onto ℓ_1^N and τ_N to be the canonical injection from ℓ_1^N into ℓ_1. Namely $\rho_N(x) = (x_1, \dots, x_N)$ for any $x = (x_i)_{i=1}^\infty \in \ell_1$ and $\tau_N(x) = (x_1, \dots, x_N, 0, \dots)$ for every $x = (x_1, \dots, x_N) \in \ell_1^N$.

1.3.2 Proof of Theorem 1.3.1

We detail the argument for $\mathcal{F}(\ell_1)$. As we have announced in the note below the formulation of Theorem 1.3.1, we perform first a construction of projections having the desired qualities on $\operatorname{Lip}_0(\ell_1)$.

So, for $f \in \operatorname{Lip}_0(\ell_1)$, $n \in \mathbb{N}$ and $x \in \ell_1$ we define

$$Q_n(f)(x) = P_n(f \circ \tau_n)(\rho_n(x)),$$

with

$$P_n(g)(u) = \Lambda\left(g, C\left(x_{h,n-1}^{\varepsilon,\mathbf{0^n}}, 2^{1-n}\right)\right)(\Pi_{2^n}^n(u)), \quad \text{for } g \in \operatorname{Lip}_0(\ell_1^n) \text{ and } u \in \mathbb{R}^n,$$

where $\varepsilon \in \{-1, 1\}^n$ and $h \in \{0, \dots, 2^{2n-2} - 1\}^n$ are chosen so that

$$\Pi_{2^n}^n(u) \in C\left(x_{h,n-1}^{\varepsilon,\mathbf{0^n}}, 2^{1-n}\right).$$

Note that the symbols $x^{\varepsilon,\mathbf{0^n}}_{h,n-1}$, $C\left(x^{\varepsilon,\mathbf{0^n}}_{h,n-1}, 2^{1-n}\right)$ and $\Lambda\left(g, C\left(x^{\varepsilon,\mathbf{0^n}}_{h,n-1}, 2^{1-n}\right)\right)$ in the above construction are meant in $\mathbb{R}^n$, or acting on $\mathbb{R}^n$. In the sequel, the information on the dimension considered for hypercubes or for points $x^{\varepsilon,y}_{h,k}$ shall be carried by the centre of a hypercube or by y respectively, which most of the time will be $\mathbf{0^n}$. Finally, we denote V_n the set of all vertices of all cubes $C\left(x^{\varepsilon,\mathbf{0^n}}_{h,n-1}, 2^{1-n}\right)$ for $\varepsilon \in \{-1,1\}^n$ and $h \in \{0, \ldots, 2^{2n-2}-1\}^n$.

Before we proceed with the proof, let us describe the operator Q_n. The hypercube $C(\mathbf{0^n}, 2^n)$ of $\mathbb{R}^n$ is split into small hypercubes of edge length equal to 2^{1-n}. If x belongs to one of the small hypercubes, then $Q_n(f)(x)$ is the value obtained by performing the interpolation Λ for the restriction of f to the vertices of this small hypercube. If x does not belong to $C(\mathbf{0^n}, 2^n)$, then $Q_n(f)(x)$ is defined to be $Q_n(f)(r_n(x))$, where $r_n = \Pi^n_{2^n} \circ \rho_n$ is the natural retraction from ℓ_1 onto $C(\mathbf{0^n}, 2^n)$. In rough words, let us say that as we go from step n to step $n+1$, we perform the three following operations: we add one dimension to our hypercubes, we double the edge length of the large hypercube and divide by two the edge length of the small hypercubes.

We now make a simple observation.

Lemma 1.3.3. *Let $m > n$ in $\mathbb{N}$. Assume that $g \in \mathrm{Lip}_0(\ell_1^n)$. Then the function $P_n(g)$ has the property (AF) on each hypercube $C\left(x^{\varepsilon,\mathbf{0^n}}_{h,m-1}, 2^{1-m}\right)$ where $\varepsilon \in \{-1,1\}^n$ and $h \in \mathbb{N}_0^n$ (note here that these hypercubes are considered in $\mathbb{R}^n$).*

Proof. The assertion is clear if the hypercube $C\left(x^{\varepsilon,\mathbf{0^n}}_{h,m-1}, 2^{1-m}\right)$ lies inside the hypercube $C\left(\mathbf{0^n}, 2^n\right)$. Assume now that it is not the case. First, it is easily checked that $\Pi^n_{2^n}$ has the property (AF) on $C\left(x^{\varepsilon,\mathbf{0^n}}_{h,m-1}, 2^{1-m}\right)$. Besides, the image by $\Pi^n_{2^n}$ of a segment in $C\left(x^{\varepsilon,\mathbf{0^n}}_{h,m-1}, 2^{1-m}\right)$ that is parallel to a coordinate axis is either a point or a segment parallel to a coordinate axis. Finally, $\Pi^n_{2^n}\left(C\left(x^{\varepsilon,\mathbf{0^n}}_{h,m-1}, 2^{1-m}\right)\right)$ is included in a face of one of the hypercubes in the tilling of $C\left(\mathbf{0^n}, 2^n\right)$. On this face $P_n(g)$ has the property (AF). The conclusion follows. □

Let us proceed with the proof of Theorem 1.3.1. Fix $n \in \mathbb{N}$. First, it is clear that

$$Q_n(f)(\mathbf{0}) = f(\mathbf{0}) = 0.$$

Then, using Lemma 1.3.2 and the fact that $1 = \mathrm{Lip}(\tau_n) = \mathrm{Lip}(\rho_n) = \mathrm{Lip}(\Pi^n_{2^n})$, we get that for all $x, y \in \ell_1$

$$\begin{aligned}|Q_n(f)(x) - Q_n(f)(y)| &\leq \|f \circ \tau_n\|_{\mathrm{Lip}_0(\ell_1)} \|\Pi^n_{2^n}(\rho_n(x)) - \Pi^n_{2^n}(\rho_n(y))\|_1 \\ &\leq \|f\|_{\mathrm{Lip}_0(\ell_1)} \|x-y\|_1 .\end{aligned}$$

The map $f \mapsto \Lambda\left(f, C\left(y, R\right)\right)$ is clearly linear. Then, the linearity of Q_n follows easily. Moreover, $Q_n(f)$ is uniquely determined by the values of f at the elements of the finite set V_n. Hence $Q_n : \mathrm{Lip}_0(\ell_1) \to \mathrm{Lip}_0(\ell_1)$ is a well defined finite-rank linear operator with $\|Q_n\| \leq 1$.

Consider now $m, n \in \mathbb{N}$ so that $m \leq n$. Then $Q_n(f) \circ \tau_m = f \circ \tau_m$ on V_m. Indeed, for $A = (A_1, \ldots, A_m) \in V_m$, we have that $\rho_n(\tau_m(A)) \in V_n$. So

$$Q_n(f)(\tau_m(A)) = f(\tau_n(A_1, \ldots, A_m, \underbrace{0, \ldots, 0}_{n-m})) = f(\tau_m(A)).$$

Thus $Q_m(Q_n(f)) = Q_m(f)$ on ℓ_1 by definition.

Suppose now $m > n$ and assume that $j \in \{1, \ldots, m\}$, $\lambda \in [0,1]$ and that $x, \tilde{x} \in C\left(x^{\varepsilon,\mathbf{0^m}}_{h,m-1}, 2^{1-m}\right)$, where $\varepsilon \in \{-1,1\}^m$ and $h \in \{0, \ldots, 2^{2m-2}-1\}^m$, are such that $x_i = \tilde{x}_i$ for $i \neq j$. Then

$$\begin{aligned} Q_n(f)(\tau_m(\lambda x + (1-\lambda)\tilde{x})) &= P_n(f \circ \tau_n)(\rho_n(\tau_m(\lambda x + (1-\lambda)\tilde{x}))) \\ &= P_n(f \circ \tau_n)(\lambda \rho_n(\tau_m(x)) + (1-\lambda)\rho_n(\tau_m(\tilde{x}))) \\ &= \lambda P_n(f \circ \tau_n)(\rho_n(\tau_m(x))) \\ &\quad + (1-\lambda) P_n(f \circ \tau_n)(\rho_n(\tau_m(\tilde{x}))) \\ &= \lambda Q_n(f)(\tau_m(x)) + (1-\lambda) Q_n(f)(\tau_m(\tilde{x})). \end{aligned}$$

In the above we have used that ρ_n and τ_m are affine, that

$$\rho_n\left(\tau_m\left(C\left(x^{\varepsilon,\mathbf{0^m}}_{h,m-1}, 2^{1-m}\right)\right)\right) = C\left(x^{(\varepsilon_1,\ldots,\varepsilon_n),\mathbf{0^n}}_{(h_1,\ldots,h_n),m-1}, 2^{1-m}\right)$$

and the fact that $P_n(f \circ \tau_n)$ has the property (AF) on $C\left(x^{(\varepsilon_1,\ldots,\varepsilon_n),\mathbf{0^n}}_{(h_1,\ldots,h_n),m-1}, 2^{1-m}\right)$ (see Lemma 1.3.3). So, $Q_n(f) \circ \tau_m$ has the property (AF) on each hypercube $C\left(x^{\varepsilon,\mathbf{0^m}}_{h,m-1}, 2^{1-m}\right)$, where $\varepsilon \in \{-1,1\}^m$ and $h \in \{0, \ldots, 2^{2m-2}-1\}^m$. It follows by the uniqueness of the function admitting property (AF) on a hypercube $C\left(x^{\varepsilon,\mathbf{0^m}}_{h,m-1}, 2^{1-m}\right)$ and coinciding with $Q_n(f) \circ \tau_m$ at the vertices of this hypercube that for all $f \in \mathrm{Lip}_0(\ell_1)$ and $x \in \ell_1$

$$P_m(Q_n(f) \circ \tau_m)(\rho_m(x)) = Q_n(f)\left(\tau_m\left(\Pi^m_{2^m}\left(\rho_m(x)\right)\right)\right).$$

Therefore, we obtain that for all $x \in \ell_1$ and $f \in \mathrm{Lip}_0(\ell_1)$

$$\begin{aligned} Q_m(Q_n(f))(x) &= P_m(Q_n(f) \circ \tau_m)(\rho_m(x)) \\ &= Q_n(f)\left(\tau_m\left(\Pi^m_{2^m}\left(\rho_m(x)\right)\right)\right) \\ &= P_n(f \circ \tau_n)\left(\rho_n\left(\tau_m\left(\Pi^m_{2^m}\left(\rho_m(x)\right)\right)\right)\right) \\ &= P_n(f \circ \tau_n)\left(\pi_{2^m}(x_1), \ldots, \pi_{2^m}(x_n)\right) \\ &= P_n(f \circ \tau_n)\left(\Pi^n_{2^n}\left(\pi_{2^m}(x_1), \ldots, \pi_{2^m}(x_n)\right)\right). \end{aligned}$$

We now use the fact that $\pi_{2^n}\pi_{2^m} = \pi_{2^n}$ to get

$$\begin{aligned} Q_m(Q_n(f))(x) &= P_n(f \circ \tau_n)\left(\pi_{2^n}(x_1), \ldots, \pi_{2^n}(x_n)\right) \\ &= P_n(f \circ \tau_n)\left(\Pi^n_{2^n}\left(\rho_n(x)\right)\right) = Q_n(f)(x). \end{aligned}$$

Hence the formula $Q_m Q_n = Q_{\min\{m,n\}}$ is also satisfied for $m > n$.

By construction, for each n in $\mathbb{N}$, Q_n is pointwise continuous and therefore weak* to weak*-continuous on bounded subsets of $\mathrm{Lip}_0(\ell_1)$.

Furthermore, $\left(Q_n(f)\right)_{n=1}^{\infty}$ converges pointwise to f for every $f \in \mathrm{Lip}_0(\ell_1)$. Indeed, for given $f \in \mathrm{Lip}_0(\ell_1)$, $x \in \ell_1$ and $\eta > 0$, we can find $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\|f\|_{\mathrm{Lip}_0(\ell_1)} \sum_{i=n+1}^{\infty} |x_i| < \frac{\eta}{4}, \quad \rho_n(x) \in C(\mathbf{0^n}, 2^n) \quad \text{and} \quad n2^{1-n}\|f\|_{\mathrm{Lip}_0(\ell_1)} < \frac{\eta}{4}.$$

Thus, for any $n \geq n_0$, we get

$$\begin{aligned}|Q_n(f)(x) - f(x)| \leq &|Q_n(f)(x) - Q_n(f)(\tau_n(\rho_n(x)))| \\ &+ |Q_n(f)(\tau_n(\rho_n(x))) - f(\tau_n(A))| \\ &+ |f(\tau_n(A)) - f(\tau_n(\rho_n(x)))| + |f(\tau_n(\rho_n(x))) - f(x)|,\end{aligned}$$

where $A \in \mathbb{R}^n$ is a vertex of a hypercube $C\left(x_{h,n-1}^{\varepsilon,\mathbf{0^n}}, 2^{1-n}\right)$, with $\varepsilon \in \{-1,1\}^n$ and $h \in \{0, \ldots, 2^{2n-2}-1\}^n$, containing $\rho_n(x)$.

Since f and $Q_n(f)$ are $\|f\|_{\mathrm{Lip}_0(\ell_1)}$-Lipschitz and $\|\tau_n(A) - \tau_n(\rho_n(x))\|_1 \leq n2^{1-n}$, we deduce that

$$\begin{aligned}|Q_n(f)(x) - f(x)| &\leq 2\|f\|_{\mathrm{Lip}_0(\ell_1)} \left(\|\tau_n(\rho_n(x)) - x\|_1 + \|\tau_n(A) - \tau_n(\rho_n(x))\|_1\right) \\ &\leq 2\|f\|_{\mathrm{Lip}_0(\ell_1)} \left(\sum_{i=n+1}^{\infty} |x_i| + n2^{1-n}\right) < \eta.\end{aligned}$$

Now, it follows from the weak*-continuity of Q_n on bounded subsets of $\mathrm{Lip}_0(\ell_1)$ that $Q_n = S_n^*$, where $(S_n)_{n=1}^{\infty}$ is a sequence of finite-rank bounded linear projections on $\mathcal{F}(\ell_1)$. The sequence $(S_n)_{n=1}^{\infty}$ satisfies that $\|S_n\| \leq 1$ for each $n \in \mathbb{N}$ and that $S_m S_n = S_{\min\{m,n\}}$ for every $m, n \in \mathbb{N}$.

The convergence of $(Q_n)_{n=1}^{\infty}$ to the identity with respect to the weak*-operator topology then implies that $(S_n(\mu))_{n=1}^{\infty}$ converges weakly to μ for every $\mu \in \mathcal{F}(\ell_1)$. Therefore $\overline{\bigcup_{n=1}^{\infty} S_n(\mathcal{F}(\ell_1))} = \mathcal{F}(\ell_1)$. In view of these properties, the sequence $(S_n)_{n=1}^{\infty}$ determines a monotone FDD of $\mathcal{F}(\ell_1)$. The proof of Theorem 1.3.1 is now complete.

Remark. The proof for ℓ_1^N is clearly simpler and the sequence $(Q_n)_{n=1}^{\infty}$ can be directly given by

$$Q_n(f)(x) = \Lambda\left(f, C\left(x_{h,n-1}^{\varepsilon,\mathbf{0^N}}, 2^{1-n}\right)\right)\left(\Pi_{2^n}^N(x)\right),$$

where $\varepsilon \in \{-1,1\}^N$ and $h \in \{0, \ldots, 2^{2n-2}-1\}^N$ are such that

$$\Pi_{2^n}^N(x) \in C\left(x_{h,n-1}^{\varepsilon,\mathbf{0^N}}, 2^{1-n}\right).$$

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2. On Schauder bases in Lipschitz-free spaces

joint work with P. Hájek

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2.1 Introduction

Let (M_1, d_1) and (M_2, d_2) be metric spaces and $f : M_1 \to M_2$ be a Lipschitz mapping. By $\mathrm{Lip}(f)$ we denote the Lipschitz constant of f defined as

$$\mathrm{Lip}(f) = \sup\left\{\frac{d_2(f(x), f(y))}{d_1(x, y)},\ x, y \in M_1,\ x \neq y\right\}.$$

For a given metric space (M, d) and a fixed point in M, for convenience denoted by 0, we will consider the space $\mathrm{Lip}_0(M)$ of all real-valued Lipschitz functions f on M which satisfy $f(0) = 0$, endowed with the norm

$$\|f\|_{\mathrm{Lip}_0(M)} = \mathrm{Lip}(f).$$

This is easily seen to be a Banach space. The Dirac map $\delta : M \to \mathrm{Lip}_0(M)^*$ defined by $\langle g, \delta(p)\rangle = g(p)$ for $g \in \mathrm{Lip}_0(M)$ and $p \in M$ is an isometric embedding from M into $\mathrm{Lip}_0(M)^*$. The closed linear span of $\{\delta(p),\ p \in M\}$ is denoted $\mathcal{F}(M)$ and called the *Lipschitz-free space over* M (or free space in short). It follows from the compactness of the unit ball of $\mathrm{Lip}_0(M)$ with respect to the topology of pointwise convergence, that $\mathcal{F}(M)$ can be seen as the canonical predual of $\mathrm{Lip}_0(M)$. The weak*-topology induced by $\mathcal{F}(M)$ on $\mathrm{Lip}_0(M)$ coincides with the topology of pointwise convergence on the bounded subsets of $\mathrm{Lip}_0(M)$. Lipschitz-free spaces serve for linearizing Lipschitz maps. Indeed, identifying the metric space M with a subset of $\mathcal{F}(M)$ (through the Dirac map), any Lipschitz map from the metric space M to a metric space N extends to a continuous linear map from $\mathcal{F}(M)$ to $\mathcal{F}(N)$ with the same Lipschitz constant (see [23] or Lemma 2.2 in [7]). We refer the reader to the book [23] by Weaver for the basic theory of the spaces of Lipschitz functions and their preduals, which are called Arens-Eells spaces there.

The linear structure of Lipschitz-free spaces over metric spaces has recently become a rather active field of study. It is easy to see that $\mathcal{F}(\mathbb{R})$ is isometric to L_1. However, adapting a theorem of Kislyakov [13], Naor and Schechtman proved a surprising result in [19] that $\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to any subspace of L_1. Then the metric spaces whose Lipschitz-free space is isometric to a subspace of L_1 have been characterized by Godard in [6].

Recently, a number of interesting results were obtained in the direction of the (bounded) approximation property of $\mathcal{F}(M)$ for various metric and in particular Banach spaces. In the seminal paper of Godefroy and Kalton [7] it is proved that a Banach space X has the λ-BAP if and only if $\mathcal{F}(X)$ has the same property. It is known thanks to the fundamental work of Enflo that there are separable Banach spaces which fail to have the AP. Combining this fact with some analytic work, Godefroy and Ozawa [8] construct metric compact spaces M such that $\mathcal{F}(M)$ has no AP. On the positive side, one of the results in [1] shows that $\mathcal{F}(\mathbb{R}^n)$ have an FDD, strengthening the results in [7]. Finally, in [14] the BAP was shown for all doubling metric spaces M, so in particular for all subsets of $\mathbb{R}^n$.

Our present paper focuses on finding Schauder bases for $\mathcal{F}(M)$. Our main result, Theorem 2.3.1, implies in particular that the Lipschitz-free spaces $\mathcal{F}(\ell_1)$ and $\mathcal{F}(\mathbb{R}^n)$, have a Schauder basis.

Our proof is rather technical and special to the metric of ℓ_1, and does not seem to generalize in its present form to any infinite dimensional Banach space non-isomorphic to ℓ_1.

2.2 Preliminaries

Notation 2.2.1. In the sequel, ℓ_1 denotes the vector space

$$\left\{ x = (x_i)_{i=1}^{\infty} \in \mathbb{R}^{\mathbb{N}},\ \sum_{i=1}^{\infty} |x_i| < \infty \right\}$$

equipped with the norm $\|(x_i)_{i=1}^{\infty}\| = \sum_{i=1}^{\infty} |x_i|$, and ℓ_1^d, where $d \in \mathbb{N}$, denotes the vector space $\mathbb{R}^d$ endowed with the norm $\|(x_i)_{i=1}^{d}\| = \sum_{i=1}^{d} |x_i|$. The symbols 0, $\|\cdot\|$ and e_i stand for the origin, the norm and the unit vector $(0, \dots, 0, \underbrace{1}_{i}, 0, \dots)$, respectively, in the space ℓ_1 or ℓ_1^d for some $d \in \mathbb{N}$.

By a degenerate interval $[a, a] \subset \mathbb{R}$ we mean the singleton $\{a\} \subset \mathbb{R}$.

A hypercube H in ℓ_1^d, where $d \in \mathbb{N}$, is a product of d bounded closed intervals in $\mathbb{R}$ such that there exists $t \in (0, \infty)$ so that each interval is either degenerate or has length t. The set of all vertices of H in ℓ_1^d is denoted U_H.

If $d \in \mathbb{N}$ and if $F \subset \ell_1^d$ is a product of d closed bounded intervals in $\mathbb{R}$, i.e. $F = \Pi_{i=1}^{d}[p_i, q_i]$, where $p_i, q_i \in \mathbb{R}$ for all $i \in \{1, \dots, d\}$, then we define π_F to be the nearest point mapping from ℓ_1^d to F. Note that π_F is single-valued and that $\pi_F(x) = \left(\pi_{[p_i,q_i]}(x_i)\right)_{i=1}^{d} \in F$ for $x = (x_i)_{i=1}^{d} \in \ell_1^d$. Moreover, $\mathrm{Lip}(\pi_F) = 1$.

Consider $d \in \mathbb{N}$ and the lexicographic ordering $\prec$ on ℓ_1^d, which means that for vectors $x = (x_i)_{i=1}^{d} \in \ell_1^d$ and $y = (y_i)_{i=1}^{d} \in \ell_1^d$, $x \prec y$ whenever either $x = y$ or $x_{\min\{i \in \{1,\dots,d\},\, x_i \neq y_i\}} < y_{\min\{i \in \{1,\dots,d\},\, x_i \neq y_i\}}$. Then $\prec$ is a linear ordering on ℓ_1^d and we denote by $\prec_{\max}(Z)$ (resp. by $\prec_{\min}(Z)$) the maximal (resp. the minimal) element of a finite set $Z \subset \ell_1^d$ with respect to $\prec$.

We now recall the key ingredient of the proof of Theorem 3.1 in [14], an interpolation formula for functions defined on the vertices of a hypercube. Let $d \in \mathbb{N}$ and let H be a hypercube in ℓ_1^d. A function $g : H \to \mathbb{R}$ is said to have the property (AF) on H if its restriction to any segment lying in H and parallel to one of the coordinate axes is affine. For a function $f : \mathrm{dom}(f) \to \mathbb{R}$ such that $U_H \subset \mathrm{dom}(f) \subset \ell_1^d$, we define $\Lambda(f, H) : H \to \mathbb{R}$ to be the unique function which has the property (AF) on H and coincides with the function f on U_H. Note that the uniqueness of $\Lambda(f, H)$ is obvious from the definition. For the existence we refer to the explicit formula given in Section 3.1 of [14].

An important feature of the function $\Lambda(f, H)$ is observed in Lemma 3.2 in [14], which reads

$$\mathrm{Lip}(\Lambda(f, H)) = \mathrm{Lip}(f|_{U_H}).$$

Lemma 2.2.2 is rather technical but will be crucial in the proof of Theorem 2.3.1. It shows that with our construction of projections on the space of Lipschitz functions we can keep control over the Lipschitz constants of the images of functions. We sketch the situation by a few words before a precise formulation. Consider a hypercube in ℓ_1^d for some $d \in \mathbb{N}$ split into hypercubes with half edge length and a function defined on a subset V of the set M of all vertices of the smaller hypercubes, where V satisfies that it contains all vertices of the big hypercube and that if a point from M lying on a j dimensional face of the big hypercube for some $j \in \{1, \dots, d\}$ belongs to V, then also all points from M lying on the faces with dimension less than j belong to V. We extend the function to whole M inductively by taking the convex combinations of the values on the neighbouring predecessors, by which we mean the neighbouring points from M lying on a face of the big hypercube with one less dimension. And the lemma says that the extension preserves the Lipschitz constant up to multiplication by a universal numerical constant.

Lemma 2.2.2. *Let $d \in \mathbb{N}$, $z \in \ell_1^d$ and $t \in (0, \infty)$. Denote*

$$\begin{aligned} G^0 &= \left\{z + t\delta, \delta \in \{-1,1\}^d\right\} \\ &= \left\{z + t\delta, \delta \in \{-1,0,1\}^d, \operatorname{card}\left(\{i \in \{1,\dots,d\}, \delta_i = 0\}\right) = 0\right\}, \\ G^1 &= \left\{z + t\delta, \delta \in \{-1,0,1\}^d, \operatorname{card}\left(\{i \in \{1,\dots,d\}, \delta_i = 0\}\right) = 1\right\}, \\ &\vdots \\ G^d &= \{z\} = \left\{z + t\delta, \delta \in \{-1,0,1\}^d, \operatorname{card}\left(\{i \in \{1,\dots,d\}, \delta_i = 0\}\right) = d\right\} \end{aligned}$$

and, for $x \in G^j$ for some $j \in \{1, \dots, d\}$, put

$$A_x = \left\{x' \in G^{j-1}, \|x' - x\| = t\right\}.$$

Further, let $V \subset \bigcup_{i=0}^d G^i$ satisfy that $G^0 \subset V$ and that if $V \cap G^j \neq \emptyset$ for any $j \in \{1, \dots, d\}$, then $\bigcup_{i=0}^{j-1} G^i \subset V$. If f is a real-valued function on V and if $\Phi(f)$ is the real-valued function on $\bigcup_{i=0}^d G^i$ given by

$$\Phi(f)(x) = \begin{cases} f(x) & \text{if } x \in V, \\ \sum_{x' \in A_x} \frac{\Phi(f)(x')}{\operatorname{card}(A_x)} & \text{if } x \in \bigcup_{i=0}^d G^i \setminus V, \end{cases}$$

then

$$\operatorname{Lip}\left(\Phi(f)\right) \leq 3 \operatorname{Lip}(f).$$

Note that $\Phi(f)$ is well-defined since it is constructed on G^i's by induction on i and since $G^0 \subset V$.

Proof. For simplicity, we will perform the proof for a particular case when $z = 0$ and $t = 1$. In a general case, the proof can be carried out along the same lines.

Let us begin with estimating $\operatorname{Lip}(\Phi(f)|_{G^k})$ for all $k \in \{0, \dots, d\}$. We proceed by induction on the index k. As $G^0 \subset V$, we have that $\operatorname{Lip}(\Phi(f)|_{G^0}) \leq \operatorname{Lip}(f)$. For the inductive step take $k \in \{1, \dots, d\}$ and suppose that $\operatorname{Lip}(\Phi(f)|_{G^{k-1}}) \leq \frac{3}{2} \operatorname{Lip}(f)$. The choice of the constant $\frac{3}{2}$ on the right-hand side will become clear immediately. Certainly, the estimate holds for G^0. Let $x, y \in G^k$ and $x \neq y$. Observe that then $\|x - y\| \geq 2$. We distinguish three cases.

The first one is when both $x \in V$ and $y \in V$. Then

$$|\Phi(f)(x) - \Phi(f)(y)| = |f(x) - f(y)| \leq \mathrm{Lip}(f)\|x - y\|.$$

The second situation is when $x \in V$ and $y \notin V$. Then $G^{k-1} \subset V$ according to the assumption on V. Therefore

$$\begin{aligned}
|\Phi(f)(x) - \Phi(f)(y)| &= \left| f(x) - \sum_{y' \in A_y} \frac{f(y')}{\mathrm{card}\,(A_y)} \right| = \left| \sum_{y' \in A_y} \frac{f(x) - f(y')}{\mathrm{card}\,(A_y)} \right| \\
&\leq \sum_{y' \in A_y} \frac{|f(x) - f(y')|}{\mathrm{card}\,(A_y)} \leq \sum_{y' \in A_y} \frac{\mathrm{Lip}(f)\|x - y'\|}{\mathrm{card}\,(A_y)} \\
&\leq \sum_{y' \in A_y} \frac{\mathrm{Lip}(f)\,(\|x - y\| + \|y - y'\|)}{\mathrm{card}\,(A_y)} = \mathrm{Lip}(f)\,(\|x - y\| + 1) \\
&\leq \frac{3}{2}\,\mathrm{Lip}(f)\|x - y\|.
\end{aligned}$$

The last case is when $x, y \notin V$. If $I_x = \{i \in \{1, \ldots, d\}, x_i = 0\}$ and $I_y = \{i \in \{1, \ldots, d\}, y_i = 0\}$, then $\mathrm{card}\,(I_x) = k = \mathrm{card}\,(I_y)$ and by definition,

$$\Phi(f)(x) = \sum_{x' \in A_x} \frac{\Phi(f)(x')}{\mathrm{card}\,(A_x)} = \sum_{i \in I_x} \frac{\Phi(f)(x + e_i) + \Phi(f)(x - e_i)}{2k}$$

and

$$\Phi(f)(y) = \sum_{y' \in A_y} \frac{\Phi(f)(y')}{\mathrm{card}\,(A_y)} = \sum_{i \in I_y} \frac{\Phi(f)(y + e_i) + \Phi(f)(y - e_i)}{2k}.$$

Let $m = \mathrm{card}\,(I_x \cap I_y)$ and let $(i_j^x)_{j=1}^{k-m}$ and $(i_j^y)_{j=1}^{k-m}$ be the increasing sequences formed by the elements of the sets $I_x \setminus I_y$ and $I_y \setminus I_x$ respectively. Denote by $J_{x,y}$ the set consisting of those indices $j \in \{1, \ldots, k - m\}$ for which $y_{i_j^x} = -x_{i_j^y}$ (note that $|y_{i_j^x}| = 1 = |x_{i_j^y}|$ because $i_j^x \notin I_y$ and $i_j^y \notin I_x$). A straightforward computation yields

$$\|x + e_{i_j^x} - y - e_{i_j^y}\| = \|x - y\| = \|x - e_{i_j^x} - y + e_{i_j^y}\|$$

for $j \in J_{x,y}$, and

$$\|x + e_{i_j^x} - y + e_{i_j^y}\| = \|x - y\| = \|x - e_{i_j^x} - y - e_{i_j^y}\|$$

for $j \in \{1, \ldots, k-m\} \setminus J_{x,y}$. So, we obtain

$$\begin{aligned}
|\Phi(f)(x) - \Phi(f)(y)| = \Bigg| & \sum_{i \in I_x \cap I_y} \frac{\Phi(f)(x+e_i) - \Phi(f)(y+e_i)}{2k} \\
& + \sum_{i \in I_x \cap I_y} \frac{\Phi(f)(x-e_i) - \Phi(f)(y-e_i)}{2k} \\
& + \sum_{j \in J_{x,y}} \frac{\Phi(f)\left(x + e_{i_j^x}\right) - \Phi(f)\left(y + e_{i_j^y}\right)}{2k} \\
& + \sum_{j \in J_{x,y}} \frac{\Phi(f)\left(x - e_{i_j^x}\right) - \Phi(f)\left(y - e_{i_j^y}\right)}{2k} \\
& + \sum_{j \in \{1,\ldots,k-m\} \setminus J_{x,y}} \frac{\Phi(f)\left(x + e_{i_j^x}\right) - \Phi(f)\left(y - e_{i_j^y}\right)}{2k} \\
& + \sum_{j \in \{1,\ldots,k-m\} \setminus J_{x,y}} \frac{\Phi(f)\left(x - e_{i_j^x}\right) - \Phi(f)\left(y + e_{i_j^y}\right)}{2k} \Bigg| \\
\leq & \sum_{i \in I_x \cap I_y} \frac{\operatorname{Lip}\left(\Phi(f)|_{G_{k-1}}\right) \|(x+e_i) - (y+e_i)\|}{2k} \\
& + \sum_{i \in I_x \cap I_y} \frac{\operatorname{Lip}\left(\Phi(f)|_{G_{k-1}}\right) \|(x-e_i) - (y-e_i)\|}{2k} \\
& + \sum_{j \in J_{x,y}} \frac{\operatorname{Lip}\left(\Phi(f)|_{G_{k-1}}\right) \left\|\left(x + e_{i_j^x}\right) - \left(y + e_{i_j^y}\right)\right\|}{2k} \\
& + \sum_{j \in J_{x,y}} \frac{\operatorname{Lip}\left(\Phi(f)|_{G_{k-1}}\right) \left\|\left(x - e_{i_j^x}\right) - \left(y - e_{i_j^y}\right)\right\|}{2k} \\
& + \sum_{j \in \{1,\ldots,k-m\} \setminus J_{x,y}} \frac{\operatorname{Lip}\left(\Phi(f)|_{G_{k-1}}\right) \left\|\left(x + e_{i_j^x}\right) - \left(y - e_{i_j^y}\right)\right\|}{2k} \\
& + \sum_{j \in \{1,\ldots,k-m\} \setminus J_{x,y}} \frac{\operatorname{Lip}\left(\Phi(f)|_{G_{k-1}}\right) \left\|\left(x - e_{i_j^x}\right) - \left(y + e_{i_j^y}\right)\right\|}{2k} \\
\leq & \operatorname{Lip}\left(\Phi(f)|_{G_{k-1}}\right) \|x-y\| \leq \frac{3}{2} \operatorname{Lip}(f) \|x-y\|.
\end{aligned}$$

Hence

$$\operatorname{Lip}\left(\Phi(f)|_{G^k}\right) \leq \frac{3}{2} \operatorname{Lip}(f)$$

for every $k \in \{0, \ldots, d\}$.

We now examine the behaviour of $\Phi(f)$ on two sets with consecutive indices G^{k-1} and G^k and afterwards we will derive an upper estimate for the Lipschitz constant of $\Phi(f)$ on the whole union $\bigcup_{i=0}^{d} G^i$.

So, let $k \in \{1, \dots, d\}$, $x \in G^{k-1}$ and $y \in G^k$. Such x and y satisfy that $\|x - y\| \geq 1$. If $y \in V$, then, thanks to the properties of V, also $x \in V$ and

$$|\Phi(f)(x) - \Phi(f)(y)| = |f(x) - f(y)| \leq \mathrm{Lip}(f)\|x - y\|.$$

If $y \notin V$, then

$$\Phi(f)(y) = \sum_{y' \in A_y} \frac{\Phi(f)(y')}{\mathrm{card}(A_y)}$$

and

$$\begin{aligned}
|\Phi(f)(x) - \Phi(f)(y)| &= \left| \Phi(f)(x) - \sum_{y' \in A_y} \frac{\Phi(f)(y')}{\mathrm{card}(A_y)} \right| \\
&= \left| \sum_{y' \in A_y} \frac{\Phi(f)(x) - \Phi(f)(y')}{\mathrm{card}(A_y)} \right| \\
&\leq \sum_{y' \in A_y} \frac{|\Phi(f)(x) - \Phi(f)(y')|}{\mathrm{card}(A_y)} \\
&\leq \sum_{y' \in A_y} \frac{\mathrm{Lip}\left(\Phi(f)|_{G^{k-1}}\right) \|x - y'\|}{\mathrm{card}(A_y)} \\
&\leq \sum_{y' \in A_y} \frac{\mathrm{Lip}\left(\Phi(f)|_{G^{k-1}}\right) (\|x - y\| + \|y - y'\|)}{\mathrm{card}(A_y)} \\
&= \mathrm{Lip}\left(\Phi(f)|_{G^{k-1}}\right) (\|x - y\| + 1) \\
&\leq 2\,\mathrm{Lip}\left(\Phi(f)|_{G^{k-1}}\right) \|x - y\| \\
&\leq 3\,\mathrm{Lip}(f)\|x - y\|.
\end{aligned}$$

We show next that the foregoing estimate implies that for $k, l \in \{0, \dots, d\}$ such that $k < l$ and for $x \in G^k$, $y \in G^l$, we have that

$$|\Phi(f)(x) - \Phi(f)(y)| \leq 3\,\mathrm{Lip}(f)\|x - y\|.$$

Indeed, for each $j \in \{k, \dots, l-1\}$ choose $u^j \in G^j$ so that

$$\left\|x - u^k\right\| + \sum_{j=k}^{l-2} \left\|u^j - u^{j+1}\right\| + \left\|u^{l-1} - y\right\| = \|x - y\|.$$

Such a criterion is met for instance by $\left(u^j = \left(u_i^j\right)_{i=1}^d\right)_{j=k}^{l-1}$ given as

$$u_i^{l-1} = \begin{cases} x_i & \text{if } \ i = \min\left\{h \in \{1, \dots, d\}, y_h = 0 \neq x_h\right\}, \\ y_i & \text{otherwise} \end{cases}$$

and

$$u_i^j = \begin{cases} x_i & \text{if } \ i = \min\left\{h \in \{1, \dots, d\}, u_h^{j+1} = 0 \neq x_h\right\}, \\ u_i^{j+1} & \text{otherwise} \end{cases}$$

for $j \in \{k, \ldots, l-2\}$. Then

$$\begin{aligned}|\Phi(f)(x) - \Phi(f)(y)| &\leq \left|\Phi(f)(x) - \Phi(f)\left(u^k\right)\right| + \sum_{j=k}^{l-2} \left|\Phi(f)\left(u^j\right) - \Phi(f)\left(u^{j+1}\right)\right| \\ &\quad + \left|\Phi(f)\left(u^{l-1}\right) - \Phi(f)(y)\right| \\ &\leq 3\,\mathrm{Lip}(f)\left(\left\|x - u^k\right\| + \sum_{j=k}^{l-2} \left\|u^j - u^{j+1}\right\| + \left\|u^{l-1} - y\right\|\right) \\ &= 3\,\mathrm{Lip}(f)\|x - y\|.\end{aligned}$$

Hence,

$$\mathrm{Lip}\left(\Phi(f)\right) \leq 3\,\mathrm{Lip}(f).$$

Hereby we finished the proof of the lemma. □

2.3 Schauder basis of the Lipschitz-free space over a product of closed intervals in ℓ_1.

In this section we state and prove a theorem on the existence of a Schauder basis in Lipschitz-free spaces over products of closed intervals in $\mathbb{R}$ understood as metric subspaces of ℓ_1. In order to do so, we first recall the definition of a Schauder basis.

Let X be an infinite-dimensional Banach space. A sequence $(x_n)_{n=1}^{\infty} \subset X$ is called a *Schauder basis* of X if for every $x \in X$ there is a unique sequence of scalars $(a_n)_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} a_n x_n$.

From the classical theory it follows that if there is a sequence of uniformly bounded linear projections $(T_n)_{n=1}^{\infty}$ on a Banach space X which satisfies that $T_m T_n = T_{\min\{m,n\}}$ for all $m, n \in \mathbb{N}$, $\overline{\bigcup_{n=1}^{\infty} T_n(X)} = X$ and that $\dim(T_1(X)) < \infty$ and $\dim((T_{n+1} - T_n)(X)) = 1$ for all $n \in \mathbb{N}$, then X has a Schauder basis.

A comprehensive reference for the basic theory of Schauder bases and other related notions are the monographs [16] and [5].

Our main result is the following.

Theorem 2.3.1. *Let X be a product of countably many closed (possibly unbounded or degenerate) intervals in $\mathbb{R}$ with endpoints in $\mathbb{Z} \cup \{-\infty, \infty\}$, considered as a metric subspace of ℓ_1 equipped with the inherited metric. Then the Lipschitz-free space $\mathcal{F}(X)$ has a Schauder basis.*

In particular, the Lipschitz-free spaces $\mathcal{F}(\ell_1)$ and $\mathcal{F}\left(\ell_1^d\right)$, where $d \in \mathbb{N}$, have a Schauder basis.

Notation 2.3.2. By the assumption,

$$X = \Pi_{i=1}^{\infty} X_i \subset \ell_1,$$

where for all $i \in \mathbb{N}$, $X_i = [a_i, b_i]$ with $a_i, b_i \in \mathbb{Z}$ (possibly $a_i = b_i$), or $X_i = (-\infty, b_i]$ with $b_i \in \mathbb{Z}$, or $X_i = [a_i, \infty)$ with $a_i \in \mathbb{Z}$. For $d \in \mathbb{N}$ we consider $\Pi_{i=1}^{d} X_i$ and its subsets as metric subspaces of ℓ_1^d. Denote $\mathcal{O} = (\mathcal{O}_i)_{i=1}^{\infty}$ the unique nearest point to the origin 0 of ℓ_1 in X and let it be assigned the role of the origin of X. Observe that for every $i \in \mathbb{N}$ we have that $\mathcal{O}_i \in \{0, a_i, b_i\}$, in particular $\mathcal{O} \in \mathbb{Z}^{\mathbb{N}}$.

For $d, D \in \mathbb{N} \cup \{\infty\}$, $d < D$, let $\rho_{D,d}$ be the canonical projection from ℓ_1^D onto ℓ_1^d given by $\rho_{D,d}(x) = (x_i)_{i=1}^d$ for any $x = (x_i)_{i=1}^D \in \ell_1^D$, and let $\tau_{d,D}$ be an injection from ℓ_1^d into ℓ_1^D defined by $\tau_{d,D}(x) = (y_i)_{i=1}^D$, where

$$y_i = \begin{cases} x_i & \text{if } 1 \leq i \leq d, \\ \mathcal{O}_i & \text{otherwise,} \end{cases}$$

for every $x = (x_i)_{i=1}^d \in \ell_1^d$. Suppose, furthermore, that d and D are such that $\operatorname{card}(\{i \in \{1, \dots, D\}, X_i \text{ is non-degenerate}\}) = d$. Denote $(j_k)_{k=1}^d$ the increasing sequence formed by the elements of the set $\{i \in \{1, \dots, D\}, X_i \text{ is non-degenerate}\}$. We define $\psi_{D,d}$ to be the projection from ℓ_1^D onto ℓ_1^d given by $\psi_{D,d}(x) = (x_{j_k})_{k=1}^d$ for any $x = (x_i)_{i=1}^D \in \ell_1^D$, and $\sigma_{d,D}$ to be the injection from ℓ_1^d into ℓ_1^D defined by $\sigma_{d,D}(x) = (y_i)_{i=1}^D$, where

$$y_i = \begin{cases} x_k & \text{if } i = j_k \text{ for some } 1 \leq k \leq d, \\ \mathcal{O}_i & \text{otherwise,} \end{cases}$$

for every $x = (x_i)_{i=1}^d \in \ell_1^d$.

Proof of Theorem 2.3.1. In view of the comment below the definition of a Schauder basis, it is enough to prove the existence of a bounded sequence of finite-rank linear projections on $\mathcal{F}(X)$ with so-called commuting property such that the union of their ranges is dense in $\mathcal{F}(X)$ and, moreover, that the ranges of the differences of two consecutive projections are one-dimensional. We will do so by finding an appropriate sequence of adjoint operators $(P_n)_{n=1}^\infty$ on the dual space $\mathrm{Lip}_0(X)$. The sought projections on the predual $\mathcal{F}(X)$ will then be the corresponding operators to which P_n's are adjoint. Precisely, we are looking for a sequence $(P_n)_{n=1}^\infty$ of uniformly bounded finite-rank linear projections on $\mathrm{Lip}_0(X)$ that are weak* to weak*-continuous on bounded subsets of $\mathrm{Lip}_0(X)$, hence adjoint, and that converge to the identity on $\mathrm{Lip}_0(X)$ in the weak*-operator topology and such that $\dim\left((P_{n+1} - P_n)(\mathrm{Lip}_0(X))\right) = 1$ for all $n \in \mathbb{N}$ and that $P_m P_n = P_{\min\{m,n\}}$ for all $m, n \in \mathbb{N}$. Such a sequence then gives rise to a sequence $(T_n)_{n=1}^\infty$ of operators on $\mathcal{F}(X)$ via the relation $T_n^* = P_n$. Furthermore, $(T_n)_{n=1}^\infty$ shares some properties with $(P_n)_{n=1}^\infty$, namely, it is a bounded sequence (with the same bound) of finite-rank linear projections admitting the commuting property and satisfying that $\dim\left((T_{n+1} - T_n)(\mathcal{F}(X))\right) = 1$ for all $n \in \mathbb{N}$. In addition, the convergence of $(P_n)_{n=1}^\infty$ to the identity operator with respect to the weak*-operator topology implies the convergence of $(T_n)_{n=1}^\infty$ to the identity operator with respect to the weak-operator topology. Therefore, $\overline{\bigcup_{n=1}^\infty T_n(\mathcal{F}(X))}^w = \mathcal{F}(X)$. But thanks to the commuting property of T_n's, we have that

$$T_1(\mathcal{F}(X)) \subset T_2(\mathcal{F}(X)) \subset \cdots,$$

thus $\overline{\bigcup_{n=1}^\infty T_n(\mathcal{F}(X))} = \mathcal{F}(X)$ as wanted.

So, we devote the rest of the proof to building a suitable sequence $(P_n)_{n=1}^\infty$. We divide this work into two parts - construction of a sequence and verification of its properties.

Construction of projections on $\mathrm{Lip}_0(X)$. Before rigorous definitions, we will outline the way we proceed. We would like to improve the method used for proving the monotone FDD for $\mathcal{F}(\ell_1)$ in [14] so that we obtain even a Schauder basis. That means that we want the sequence $(P_n)_{n=1}^{\infty}$ to satisfy one more condition in this case, which is that $\dim((P_{n+1} - P_n)(\mathrm{Lip}_0(X))) = 1$ for all $n \in \mathbb{N}$. The main idea is the same, namely, to exhaust X with an increasing sequence of products of intervals, growing both in the length and number of intervals, decomposed into hypercubes that are, on the other hand, shrinking (we call this decomposition tiling) and to define the corresponding projections on $\mathrm{Lip}_0(X)$ by asking that the image of a function has the property (AF) on the hypercubes of the tiling, coincides with the original function on growing subset of their vertices and does not ruin the Lipschitz constant at the remaining vertices, and by using the retraction π outside the product of intervals. Then we can apply Lemma 3.2 from [14] and the fact that π is 1-Lipschitz. In [14], one step consisted of adding one dimension, doubling the edge length of the big hypercube and refining its tiling by bisecting the sides of all hypercubes in the tiling, and the control on the Lipschitz constant of the image of a function under a projection at the vertices of the hypercubes of the tiling was guaranteed by simply assigning directly the values of the original function. This natural definition however causes ranks of the differences of two consecutive projections to tend to infinity. Therefore, here, in one step we always include only at most one more vertex given by the tiling to the set of vertices at which the image of a function agrees with the original function and to the remaining vertices we apply either retraction π (when enlarging the set) or the function Φ from Lemma 2.2.2 (when refining the tiling). So, in the construction, we alternate the processes of adding one dimension, gradual enlarging the product of the intervals until the length of the intervals for which it is still possible doubles and gradual raising the accuracy of the projections by assigning the original values to the vertices of the tiling of the product of intervals which is half as fine as the one at the end of enlarging. Each enlarging and refining process consists of several steps, whose number increases with every iteration. Note that in our construction below we obtain in fact $\dim((P_{n+1} - P_n)(\mathrm{Lip}_0(X))) \leq 1$. Hence, by passing to a suitable infinite subsequence, we may assume without loss of generality that $\dim((P_{n+1} - P_n)(\mathrm{Lip}_0(X))) = 1$, for all $n \in \mathbb{N}$.

Now, we start building the sequence $(P_n)_{n=1}^{\infty}$. We proceed inductively, beginning with the projection P_1 by setting the following objects: $d_1 = 1$, $M_1 = \mathbb{Z} \cap X_1$,

$$p_1^1 = \begin{cases} \mathcal{O}_1 - 1 & \text{if } \mathcal{O}_1 - 1 \in X_1, \\ \mathcal{O}_1 & \text{otherwise,} \end{cases}$$

$$q_1^1 = \begin{cases} \mathcal{O}_1 + 1 & \text{if } \mathcal{O}_1 + 1 \in X_1, \\ \mathcal{O}_1 & \text{otherwise,} \end{cases}$$

$F_1 = [p_1^1, q_1^1]$ and $V_1 = M_1 \cap F_1$. Then, for $f \in \mathrm{Lip}_0(X)$ and $x = (x_i)_{i=1}^{\infty} \in X$, we define

$$P_1(f)(x) = \begin{cases} \Lambda\left(f \circ \tau_{d_1,\infty}, [p_1, \mathcal{O}_1]\right)(\pi_{F_1}(x_1)) & \text{if } \ x_1 \leq \mathcal{O}_1, \\ \Lambda\left(f \circ \tau_{d_1,\infty}, [\mathcal{O}_1, q_1]\right)(\pi_{F_1}(x_1)) & \text{otherwise.} \end{cases}$$

Let us describe the roles of the appearing objects in a general step l. The natural number d_l stands for the dimension in which we work in the $l-$th step. The set F_l is a subset of $\Pi_{i=1}^{d_l} X_i$ given as a product of closed bounded intervals

$\Pi_{i=1}^{d_l}[p_l^i, q_l^i]$. The set M_l is a mesh in $\Pi_{i=1}^{d_l} X_i$ determining how fine the tiling of F_l is. So, from now on, by the tiling of F_l we will always mean the family of hypercubes delimited by the uniformly discrete set $M_l \cap F_l$, that is the family of hypercubes H in $\Pi_{i=1}^{d_l} X_i$ such that $U_H = M_l \cap F_l \cap H$. The set $V_l \subset M_l \cap F_l$ consists then of those vertices of hypercubes in the tiling of F_l at which a function mapped by the projection P_l coincides with its image.

Further, having a projection P_{n-1} and all objects necessary for its definition that are listed above, we want to build a projection P_n. As we already mentioned in the brief description of the construction, it is based on repeating three main actions, namely, increasing the dimension, enlarging the set and refining its tiling. Therefore the definition of P_n differs with respect to the fact into which of these three stages of the process the step n falls. We discuss the three variants of the definition of P_n separately.

Increasing the dimension. Assume that we have just finished one iteration of increasing the dimension, enlarging the set and consecutive refining the projections and we are about to begin the next one. This can be more precisely expressed by saying that the identity

$$V_{l-1} = \left\{\rho_{\infty,d_{l-1}}(\mathcal{O}) + 2^{1-d_{l-1}}\xi, \xi \in \mathbb{Z}^{d_{l-1}}\right\} \cap F_{l-1} \tag{2.1}$$

holds for $l = n$. This is the case for instance when $n = 2$. Set

$$d_n = d_{n-1} + 1$$

and

$$M_n = \left\{\rho_{\infty,d_n}(\mathcal{O}) + 2^{2-d_n}\xi, \xi \in \mathbb{Z}^{d_n}\right\} \cap \Pi_{i=1}^{d_n} X_i.$$

The set $F_n \subset \Pi_{i=1}^{d_n} X_i$ will be the product of intervals with the end points $p_n^i = p_{n-1}^i$, $q_n^i = q_{n-1}^i$ for all $i \in \{1, \ldots, d_{n-1}\}$, and $p_n^{d_n} = q_n^{d_n} = \mathcal{O}_{d_n}$. That is

$$F_n = \Pi_{i=1}^{d_n}\left[p_n^i, q_n^i\right] = F_{n-1} \times \{\mathcal{O}_{d_n}\}.$$

Further, put $r_n^i = p_n^i$ and $s_n^i = q_n^i$ for all $i \in \{1, \ldots, d_n\}$ and

$$E_n = \Pi_{i=1}^{d_n}\left[r_n^i, s_n^i\right] \subset \Pi_{i=1}^{d_n} X_i.$$

So, $E_n = F_n$ in this case. Now, let V_n contain all the vertices of the hypercubes in the tiling of F_n, i.e. $V_n = M_n \cap F_n = V_{n-1} \times \{\mathcal{O}_{d_n}\}$, and let $W_n = (M_n \cap F_n) \setminus V_n = \emptyset$. Finally, for $f \in \mathrm{Lip}_0(X)$ and $x \in X$, let

$$P_n(f)(x) = \Lambda\left(f \circ \tau_{d_n,\infty}, H\right)\left(\pi_{F_n}\left(\rho_{\infty,d_n}(x)\right)\right),$$

where H is a hypercube in the tiling of F_n such that $\pi_{F_n}(\rho_{\infty,d_n}(x)) \in H$.

Enlarging the set. We gradually enlarge the set F_{n-1} first in the direction $-e_{d_{n-1}}$ until we reach $\max\{a_{d_{n-1}}, \mathcal{O}_{d_{n-1}} - 2^{-1+d_{n-1}}\}$, then in the direction $e_{d_{n-1}}$ until we reach $\min\{b_{d_{n-1}}, \mathcal{O}_{d_{n-1}} + 2^{-1+d_{n-1}}\}$ and so on for $-e_{-1+d_{n-1}}, \ldots, e_1$. We do it by adding points from the mesh M_{n-1} to the set V_{n-1} one at a time in individual steps.

So, suppose now that we are in the enlarging phase of an iteration, that is, for $l = n$,

$$V_{l-1} \subsetneq \left(\left\{\rho_{\infty,d_{l-1}}(\mathcal{O}) + 2^{2-d_{l-1}}\xi, \xi \in \mathbb{Z}^{d_{l-1}}\right\} \cap Y_{l-1}\right), \tag{2.2}$$

where

$$Y_{l-1} = \Pi_{i=1}^{d_{l-1}} \left[\mathcal{O}_i - 2^{-1+d_{l-1}}, \mathcal{O}_i + 2^{-1+d_{l-1}}\right] \cap \Pi_{i=1}^{d_{l-1}} X_i.$$

Then we put

$$d_n = d_{n-1}$$

and

$$M_n = M_{n-1}.$$

The rest of the definition is divided into two cases according to if we have already exhausted the set W_{n-1} in the previous step or not.

If $W_{n-1} = \emptyset$, set $k_n = \max\{k \in \{1, \ldots, d_n\}, q_{n-1}^k < \min\{b_k, \mathcal{O}_k + 2^{-1+d_n}\}\}$. Note that by virtue of the order in which we enlarge the set F_{n-1}, described in brief above, along with the assumptions, namely that (2.2) is true for $l = n$ and that $W_{n-1} = \emptyset$, the set on the right hand side is nonempty. Put $p_n^i = p_{n-1}^i$ and $q_n^i = q_{n-1}^i$ for all $i \in \{1, \ldots, d_n\} \setminus \{k_n\}$. Besides, let $r_n^i = p_{n-1}^i$ and $s_n^i = q_{n-1}^i$ for all $i \in \{1, \ldots, d_n\}$. Provided that $p_{n-1}^{k_n} > \max\{a_{k_n}, \mathcal{O}_{k_n} - 2^{-1+d_n}\}$, define $p_n^{k_n} = p_{n-1}^{k_n} - 2^{2-d_n}$ and $q_n^{k_n} = q_{n-1}^{k_n}$. Otherwise, i.e. on condition that $p_{n-1}^{k_n} = \max\{a_{k_n}, \mathcal{O}_{k_n} - 2^{-1+d_n}\}$, put $p_n^{k_n} = p_{n-1}^{k_n}$ and $q_n^{k_n} = q_{n-1}^{k_n} + 2^{2-d_n}$. Having the endpoints of the intervals, we continue defining

$$F_n = \Pi_{i=1}^{d_n} \left[p_n^i, q_n^i\right]$$

and

$$E_n = \Pi_{i=1}^{d_n} \left[r_n^i, s_n^i\right].$$

To conclude setting the objects important for establishing P_n, let

$$V_n = V_{n-1} \cup \{\prec_{\max} (M_n \cap (F_n \setminus E_n))\},$$

and

$$W_n = (M_n \cap F_n) \setminus V_n.$$

Next, for $f \in \mathrm{Lip}_0(X)$, define $\Phi_n(f) : M_n \cap F_n = V_n \cup W_n \to \mathbb{R}$ by

$$\Phi_n(f)(x) = \begin{cases} f(\tau_{d_n,\infty}(x)) & \text{if } x \in V_n, \\ f\left(\tau_{d_n,\infty}\left(\pi_{E_n}(x)\right)\right) & \text{otherwise.} \end{cases} \tag{2.3}$$

Finally, P_n is for $f \in \mathrm{Lip}_0(X)$ and $x \in X$ given by

$$P_n(f)(x) = \Lambda\left(\Phi_n(f), H\right)\left(\pi_{F_n}(\rho_{\infty,d_n}(x))\right),$$

where H is a hypercube in the tiling of F_n such that $\pi_{F_n}(\rho_{\infty,d_n}(x)) \in H$.

If $W_{n-1} \neq \emptyset$, then for all $i \in \{1, \ldots, d_n\}$ let $p_n^i = p_{n-1}^i$, $q_n^i = q_{n-1}^i$, $r_n^i = r_{n-1}^i$ and $s_n^i = s_{n-1}^i$. Thus F_n and E_n will agree with F_{n-1} and E_{n-1} respectively. Further, set

$$V_n = V_{n-1} \cup \{\prec_{\max} (W_{n-1})\}$$

and

$$W_n = W_{n-1} \setminus V_n = ((M_{n-1} \cap F_{n-1}) \setminus V_{n-1}) \setminus V_n = (M_n \cap F_n) \setminus V_n.$$

For $f \in \mathrm{Lip}_0(X)$, define $\Phi_n(f) : M_n \cap F_n = V_n \cup W_n \to \mathbb{R}$ again by

$$\Phi_n(f)(x) = \begin{cases} f(\tau_{d_n,\infty}(x)) & \text{if } x \in V_n, \\ f\left(\tau_{d_n,\infty}\left(\pi_{E_n}(x)\right)\right) & \text{otherwise.} \end{cases} \tag{2.4}$$

Then, the mapping P_n is for $f \in \mathrm{Lip}_0(X)$ and $x \in X$ given by the same formula as before, namely

$$P_n(f)(x) = \Lambda\left(\Phi_n(f), H\right)\left(\pi_{F_n}(\rho_{\infty,d_n}(x))\right),$$

where H is a hypercube in the tiling of F_n such that $\pi_{F_n}(\rho_{\infty,d_n}(x)) \in H$.

Refining the projections. We add vertices from the tiling of F_{n-1} to V_{n-1} one by one in separate steps in a way which allows us to apply Lemma 2.2.2. So, assume that it is time to begin the refining process of an iteration or that we are already in the middle of it. To be precise, suppose that for $l = n$,

$$\left(\left\{\rho_{\infty,d_{l-1}}(\mathcal{O}) + 2^{2-d_{l-1}}\xi, \xi \in \mathbb{Z}^{d_{l-1}}\right\} \cap Y_{l-1}\right) \subset V_{l-1} \tag{2.5}$$

and

$$V_{l-1} \subsetneq \left(\left\{\rho_{\infty,d_{l-1}}(\mathcal{O}) + 2^{1-d_{l-1}}\xi, \xi \in \mathbb{Z}^{d_{l-1}}\right\} \cap Y_{l-1}\right), \tag{2.6}$$

where

$$Y_{l-1} = \Pi_{i=1}^{d_{l-1}}\left[\mathcal{O}_i - 2^{-1+d_{l-1}}, \mathcal{O}_i + 2^{-1+d_{l-1}}\right] \cap \Pi_{i=1}^{d_{l-1}} X_i.$$

Then we define

$$d_n = d_{n-1},$$
$$M_n = \left\{\rho_{\infty,d_n}(\mathcal{O}) + 2^{1-d_n}\xi, \xi \in \mathbb{Z}^{d_n}\right\} \cap \Pi_{i=1}^{d_n} X_i \tag{2.7}$$

and $p_n^i = p_{n-1}^i$ and $q_n^i = q_{n-1}^i$ for all $i \in \{1, \ldots, d_n\}$. Hence

$$F_n = \Pi_{i=1}^{d_n}\left[p_n^i, q_n^i\right] = F_{n-1} = \Pi_{i=1}^{d_{n-1}}\left[\mathcal{O}_i - 2^{-1+d_{n-1}}, \mathcal{O}_i + 2^{-1+d_{n-1}}\right] \cap \Pi_{i=1}^{d_{n-1}} X_i.$$

Denote G_n^0 the set of those elements of $M_n \cap F_n$, whose distance from the origin $\mathcal{O}$ in every direction parallel to some coordinate axis is an even multiple of 2^{1-d_n}. Next, the set G_n^1 consists of those elements of $M_n \cap F_n$, whose every but one coordinate is of the stated form, and so on for increasing index j of the sets G_n^j. In other words, consider the decomposition of F_n given by $\left\{\rho_{\infty,d_n}(\mathcal{O}) + 2^{2-d_n}\xi, \xi \in \mathbb{Z}^{d_n}\right\} \cap F_n$. The set G_n^j contains exactly the centres of all $j-$dimensional faces of all hypercubes in the decomposition. In detail,

$$\begin{aligned}
G_n^0 &= \left\{x \in M_n \cap F_n, \operatorname{card}\left(\left\{i \in \{1, \ldots, d_n\}, 2^{-1+d_n}(\mathcal{O}_i - x_i) \in 2\mathbb{Z}\right\}\right) = d_n\right\} \\
G_n^1 &= \left\{x \in M_n \cap F_n, \operatorname{card}\left(\left\{i \in \{1, \ldots, d_n\}, 2^{-1+d_n}(\mathcal{O}_i - x_i) \in 2\mathbb{Z}\right\}\right) = d_n - 1\right\} \\
&\vdots \\
G_n^{d_n} &= \left\{x \in M_n \cap F_n, \operatorname{card}\left(\left\{i \in \{1, \ldots, d_n\}, 2^{-1+d_n}(\mathcal{O}_i - x_i) \in 2\mathbb{Z}\right\}\right) = 0\right\}.
\end{aligned}$$

So, we have that $\bigcup_{i=0}^{d_n} G_n^i = M_n \cap F_n$. However, note that there might exist $l \in \{1, \ldots, d_n\}$ such that $\bigcup_{i=l}^{d_n} G_n^i = \emptyset$. Given $x \in G_n^j$ for some $j \in \{1, \ldots, d_n\}$, we put

$$A_x = \left\{x' \in G_n^{j-1}, \|x' - x\| = 2^{1-d_n}\right\}.$$

By the assumption that (2.6) is satisfied by $l = n$, we have that

$$\left\{i \in \{0, \dots, d_n\}, G_n^i \not\subset V_{n-1}\right\} \neq \emptyset.$$

Let k_n be its minimal element and let

$$V_n = V_{n-1} \cup \left\{\prec_{\min} \left(G_n^{k_n} \setminus V_{n-1}\right)\right\}.$$

For $f \in \mathrm{Lip}_0(X)$ define $\Phi_n(f) : M_n \cap F_n = \bigcup_{i=0}^{d_n} G_n^i \to \mathbb{R}$ by

$$\Phi_n(f)(x) = \begin{cases} f\left(\tau_{d_n,\infty}(x)\right) & \text{if } x \in V_n, \\ \sum_{x' \in A_x} \frac{\Phi_n(f)(x')}{\mathrm{card}(A_x)} & \text{otherwise.} \end{cases} \tag{2.8}$$

Function $\Phi(f)$ is well-defined because it is constructed on G_n^i's by induction on i and $G_n^0 \subset V_n$. Similarly to the previous cases, for $f \in \mathrm{Lip}_0(X)$ and $x \in X$, define

$$P_n(f)(x) = \Lambda\left(\Phi_n(f), H\right)\left(\pi_{F_n}\left(\rho_{\infty,d_n}(x)\right)\right),$$

where H is a hypercube in the tiling of F_n such that $\pi_{F_n}(\rho_{\infty,d_n}(x)) \in H$.

Properties of projections on $\mathrm{Lip}_0(X)$. Now, we shall verify that the sequence $(P_n)_{n=1}^{\infty}$ meets the requirements stated at the beginning of the proof. In order to make this part more clear, we organize it into several claims.

Claim 2.3.3. The sequence $(P_n)_{n=1}^{\infty}$ is a bounded sequence of finite-rank linear operators on $\mathrm{Lip}_0(X)$.

Proof. To begin with, we shall show that for each $n \in \mathbb{N}$, P_n is a well-defined bounded linear operator from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(X)$. Fix $n \in \mathbb{N}$ and $f \in \mathrm{Lip}_0(X)$.

First, observe that $P_n(f)(\mathcal{O}) = 0$ because $\rho_{\infty,d_n}(\mathcal{O}) \in V_n$ and $f(\mathcal{O}) = 0$. Next, we shall prove that for a hypercube H in the tiling of F_n,

$$\mathrm{Lip}\left(P_n(f) \circ \tau_{d_n,\infty}|_{U_H}\right) \leq 3\|f\|_{\mathrm{Lip}_0(X)}.$$

Then, by virtue of the definition of P_n, Lemma 3.2 in [14] and the fact that $\mathrm{Lip}(\pi_{F_n}) = 1 = \mathrm{Lip}(\rho_{\infty,d_n})$, also

$$\mathrm{Lip}(P_n(f)) \leq 3\|f\|_{\mathrm{Lip}_0(\mathbb{R}^N)}.$$

So, assume that $n = 1$ or that $n > 1$ and (2.1) is true for $l = n$. Then $U_H \subset V_n$, therefore $P_n(f) \circ \tau_{d_n,\infty}|_{U_H} = f \circ \tau_{d_n,\infty}|_{U_H}$, hence

$$\mathrm{Lip}\left(P_n(f) \circ \tau_{d_n,\infty}|_{U_H}\right) \leq \|f\|_{\mathrm{Lip}_0(X)}.$$

Providing that $l = n$ satisfies (2.2), let $\Phi_n(f)$ be the auxiliary function on $M_n \cap F_n = V_n \cup W_n$ defined by (2.4) and let $x, y \in U_H$ be two distinct vertices of the hypercube H, whose edge length in this case is 2^{2-d_n}. Then, by the definition of Φ_n, we have that

$$\Phi_n(f)(x) = f(\tau_{d_n,\infty}(\tilde{x})) \text{ and } \Phi_n(f)(y) = f(\tau_{d_n,\infty}(\tilde{y})),$$

where

$$\tilde{x} = \begin{cases} x & \text{if } x \in V_n, \\ \pi_{E_n}(x) & \text{if } x \in W_n \end{cases}$$

and similarly

$$\tilde{y} = \begin{cases} y & \text{if } y \in V_n, \\ \pi_{E_n}(y) & \text{if } y \in W_n. \end{cases}$$

Note that $\|\tilde{y} - y\|, \|\tilde{x} - x\| \in \{0, 2^{2-d_n}\}$. Therefore

$$\begin{aligned} |\Phi_n(f)(x) - \Phi_n(f)(y)| &\leq \|f\|_{\mathrm{Lip}_0(\mathbb{R}^N)} \|\tau_{d_n,\infty}(\tilde{x}) - \tau_{d_n,\infty}(\tilde{y})\| \\ &= \|f\|_{\mathrm{Lip}_0(\mathbb{R}^N)} \|\tilde{x} - \tilde{y}\| \\ &\leq \|f\|_{\mathrm{Lip}_0(\mathbb{R}^N)} \left(\|\tilde{x} - x\| + \|x - y\| + \|y - \tilde{y}\|\right) \\ &\leq \|f\|_{\mathrm{Lip}_0(\mathbb{R}^N)} \left(2^{2-d_n} + \|x - y\| + 2^{2-d_n}\right) \\ &\leq 3\|f\|_{\mathrm{Lip}_0(\mathbb{R}^N)} \|x - y\|. \end{aligned}$$

Hence,

$$\mathrm{Lip}\,(P_n(f)|_{U_H}) = \mathrm{Lip}\,(\Phi_n(f)|_{U_H}) \leq 3\|f\|_{\mathrm{Lip}_0(X)}.$$

If $l = n$ satisfies conditions (2.5) and (2.6), we can use Lemma 2.2.2. For that purpose, let C stand for the unique hypercube lying in $\Pi_{i=1}^{d_n} X_i$ such that $U_C = \{\rho_{\infty,d_n}(\mathcal{O}) + 2^{2-d_n}\xi, \xi \in \mathbb{Z}^{d_n}\} \cap F_n \cap C$ and that $H \subset C$. We apply Lemma 2.2.2 to card$\{i \in \{1, \ldots, d_n\}, X_i$ is non-degenerate$\}$ assigned to the parameter d, the image of the centre of the hypercube C under the mapping $\psi_{d_n,d}$ as the parameter z, the edge length of H, equal 2^{1-d_n}, as t, the set $\psi_{d_n,d}(V_n \cap C)$ as V and to the function $f \circ \tau_{d_n,\infty} \circ \sigma_{d,d_n}|_V$ (see Notation 2.3.2 and Lemma 2.2.2) and we obtain again that

$$\mathrm{Lip}\,(P_n(f)|_{U_H}) \leq 3\|f\|_{\mathrm{Lip}_0(X)}.$$

Thus, for all $n \in \mathbb{N}$, P_n is a well-defined bounded mapping from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(X)$. The linearity of P_n is straightforward.

Since for every $n \in \mathbb{N}$ and any $f \in \mathrm{Lip}_0(X)$ the function $P_n(f)$ is uniquely determined by the values of f on the finite set $\tau_{d_n,\infty}(V_n)$, the operator P_n is of finite rank.

Moreover, the sequence $(P_n)_{n=1}^\infty$ is bounded because $\|P_n\| \leq 3$ for every $n \in \mathbb{N}$ by the above.

△

We continue with proving the commuting property of the sequence $(P_n)_{n=1}^\infty$.

Claim 2.3.4. For every $m, n \in \mathbb{N}$, we have that $P_m P_n = P_{\min\{m,n\}}$.

Proof. Take $m, n \in \mathbb{N}$ so that $m \leq n$ and $f \in \mathrm{Lip}_0(X)$. Then $P_m(P_n(f)) = P_m(f)$ on X because the image of a function under the operator P_m is uniquely determined by the values of this function at the elements of the set $\tau_{d_m,\infty}(V_m)$ and $P_n(f)$ coincides with f on $\tau_{d_n,\infty}(V_n) \supset \tau_{d_m,\infty}(V_m)$ by definition.

Assume now that $m, n \in \mathbb{N}$ satisfy the inequality $m > n$. Let $f \in \mathrm{Lip}_0(X)$. We want to prove that then $P_m(P_n(f)) = P_n(f)$ on X. To this end, it suffices to show that $P_n(f) \circ \tau_{d_m,\infty}$ has the property (AF) on every hypercube in the tiling of F_m and that $P_m(P_n(f)) \circ \tau_{d_m,\infty} = P_n(f) \circ \tau_{d_m,\infty}$ at the vertices of these hypercubes. Indeed, we obtain thus that $P_m(P_n(f)) = P_n(f)$ on $\tau_{d_m,\infty}(F_m)$ as the function $P_m(P_n(f)) \circ \tau_{d_m,\infty}$ has the property (AF) on each hypercube in the tiling of F_m by definition and such a function is uniquely determined by its values at the vertices of the hypercubes. Then, using this identity (for the second equality in the following equation) and due to formula $\pi_{[a,b]}\pi_{[c,d]} = \pi_{[a,b]} = \pi_{[c,d]}\pi_{[a,b]}$ holding for any real numbers $c \le a \le b \le d$ (applied to the argument in the fourth line), we obtain that

$$
\begin{aligned}
&P_m\left(P_n\left(f\right)\right)\left(x\right)\\
&\quad = P_m\left(P_n\left(f\right)\right)\left(\tau_{d_m,\infty}\left(\pi_{F_m}\left(\rho_{\infty,d_m}\left(x\right)\right)\right)\right)\\
&\quad = P_n\left(f\right)\left(\tau_{d_m,\infty}\left(\pi_{F_m}\left(\rho_{\infty,d_m}\left(x\right)\right)\right)\right)\\
&\quad = P_n\left(f\right)\left(\tau_{d_n,\infty}\left(\pi_{F_n}\left(\rho_{\infty,d_n}\left(\tau_{d_m,\infty}\left(\pi_{F_m}\left(\rho_{\infty,d_m}\left(x\right)\right)\right)\right)\right)\right)\right)\\
&\quad = P_n\left(f\right)\left(\tau_{d_n,\infty}\left(\pi_{[p_n^1,q_n^1]}\left(\pi_{[p_m^1,q_m^1]}\left(x_1\right)\right),\dots,\pi_{\left[p_n^{d_n},q_n^{d_n}\right]}\left(\pi_{\left[p_m^{d_n},q_m^{d_n}\right]}\left(x_{d_n}\right)\right)\right)\right)\\
&\quad = P_n\left(f\right)\left(\tau_{d_n,\infty}\left(\pi_{[p_n^1,q_n^1]}\left(x_1\right),\dots,\pi_{\left[p_n^{d_n},q_n^{d_n}\right]}\left(x_{d_n}\right)\right)\right)\\
&\quad = P_n\left(f\right)\left(\tau_{d_n,\infty}\left(\pi_{F_n}\left(\rho_{\infty,d_n}\left(x\right)\right)\right)\right)\\
&\quad = P_n\left(f\right)\left(x\right)
\end{aligned}
$$

for all $x \in X$ as desired.

So, we now study the function $P_n(f) \circ \tau_{d_m,\infty}$ on hypercubes in the tiling of F_m and at their vertices.

Since the $n-$th step precedes the $m-$th one, the edge length of the hypercubes in the tiling of F_n is greater than or equal to the edge length of the hypercubes in the tiling of F_m. Therefore the function $P_n(f) \circ \tau_{d_n}$ clearly has the property (AF) on every hypercube $H \subset \Pi_{i=1}^{d_n} X_i$ with $U_H = \rho_{d_m,d_n}(M_m) \cap F_n \cap H$. The retraction π_{F_n} has the property (AF) on every hypercube $H \subset \Pi_{i=1}^{d_n} X_i$ such that $U_H = \rho_{d_m,d_n}(M_m) \cap H$. In addition, if a subset L of such a hypercube H is a segment parallel to one of the coordinate axes, then $\pi_{F_n}(L)$ is a segment parallel to a coordinate axis or a point lying on the face of some hypercube $C \subset F_n$ satisfying that $U_C = \rho_{d_m,d_n}(M_m) \cap F_n \cap C$. Hence $P_n(f) \circ \tau_{d_n,\infty}$ has the property (AF) on every hypercube $H \subset \Pi_{i=1}^{d_n} X_i$, where $U_H = \rho_{d_m,d_n}(M_m) \cap H$ (cf. the proof of Lemma 3.3 in [14]). Let now $H \subset \Pi_{i=1}^{d_m} X_i$ be a hypercube in the tiling of F_m and let $L \subset H$ be a segment parallel to one of the coordinate axes. Then $\rho_{d_m,d_n}(L)$ is a point or a segment parallel to one of the coordinate axes in the hypercube $C \subset \Pi_{i=1}^{d_n} X_i$ satisfying that $C = \rho_{d_m,d_n}(H)$ and therefore that $U_C = \rho_{d_m,d_n}(M_m) \cap C$. Finally, the property (AF) possessed by the function $P_n(f) \circ \tau_{d_n,\infty}$ on the hypercube C along with the linearity of ρ_{d_m,d_n} yields that

$$P_n(f) \circ \tau_{d_m,\infty} = P_n(f) \circ \tau_{d_n,\infty} \circ \rho_{d_m,d_n}$$

has the property (AF) on H. So, the function $P_n(f) \circ \tau_{d_m,\infty}$ has the property (AF) on all hypercubes in the tiling of F_m.

Thus, to finish the proof of the commuting property for $(P_n)_{n=1}^{\infty}$, we are left with showing that $P_m(P_n(f)) \circ \tau_{d_m,\infty} = P_n(f) \circ \tau_{d_m,\infty}$ on the set $M_m \cap F_m$. The definition of P_m gives that $P_m(P_n(f))(\tau_{d_m,\infty}(x)) = P_n(f)(\tau_{d_m,\infty}(x))$ for all $x \in V_m$.

We are done if $l = m$ satisfies (2.1) because in such case $M_m \cap F_m = V_m$.

If $l = m$ satisfies condition (2.2) and if $x \in W_m$ (recall that the set W_m is defined as the complement of V_m in $M_m \cap F_m$), then

$$
\begin{aligned}
&P_m(P_n(f))\left(\tau_{d_m,\infty}(x)\right)\\
&\quad = P_n(f)\left(\tau_{d_m,\infty}\left(\pi_{E_m}(x)\right)\right)\\
&\quad = P_n(f)\left(\tau_{d_n,\infty}\left(\pi_{F_n}\left(\rho_{d_m,d_n}\left(\pi_{E_m}(x)\right)\right)\right)\right)\\
&\quad = P_n(f)\left(\tau_{d_n,\infty}\left(\pi_{[p_n^1,q_n^1]}\left(\pi_{[r_m^1,s_m^1]}(x_1)\right),\dots,\pi_{\left[p_n^{d_n},q_n^{d_n}\right]}\left(\pi_{\left[r_m^{d_n},s_m^{d_n}\right]}(x_{d_n})\right)\right)\right).
\end{aligned}
$$

The first equality is merely the definition of P_m, the second one the definition of P_n and the last one holds because the retractions π_{F_n} and π_{E_m} act coordinatewise. In order to obtain the desired expression on the right hand side, i.e. $P_n(f)\left(\tau_{d_m,\infty}(x)\right)$, one should observe that by construction necessarily either $[p_n^i, q_n^i] \subset [r_m^i, s_m^i]$ for all $i \in \{1, \dots, d_n\}$ or there exists an $i_0 \in \{1, \dots, d_n\}$ such that $[r_m^{i_0}, s_m^{i_0}] \subsetneq [p_n^{i_0}, q_n^{i_0}]$ and $[p_n^i, q_n^i] = [r_m^i, s_m^i]$ for all $i \in \{1, \dots, d_n\} \setminus \{i_0\}$. In the first case we get by the commuting property of the retractions π that

$$P_m(P_n(f))\left(\tau_{d_m,\infty}(x)\right) = P_n(f)\left(\tau_{d_n,\infty}\left(\pi_{F_n}\left(\rho_{d_m,d_n}(x)\right)\right)\right) = P_n(f)\left(\tau_{d_m,\infty}(x)\right).$$

The second situation implies that $d_n = d_m$, $F_n = F_m$, $W_m \subset W_n$ and $E_n = E_m$. Therefore, applying the commuting property of π's again,

$$P_m(P_n(f))\left(\tau_{d_m,\infty}(x)\right) = P_n(f)\left(\tau_{d_n,\infty}\left(\pi_{E_n}\left(\rho_{d_m,d_n}(x)\right)\right)\right) = P_n(f)\left(\tau_{d_m,\infty}(x)\right).$$

Thus

$$P_m(P_n(f)) \circ \tau_{d_m,\infty} = P_n(f) \circ \tau_{d_m,\infty}$$

on the set $M_m \cap F_m$ if $l = m$ satisfies (2.2).

Now, assume that (2.5) and (2.6) is true for $l = m$ and that H is a hypercube in $\Pi_{i=1}^{d_m} X_i$ such that

$$U_H = \left\{\rho_{\infty,d_m}(\mathcal{O}) + 2^{2-d_m}\xi, \xi \in \mathbb{Z}^{d_m}\right\} \cap F_m \cap H$$

(recall that the tiling of F_m is finer, see (2.7)). We proceed by induction on the index k of sets $G_m^k \cap H$ in order to show that $P_m(P_n(f)) \circ \tau_{d_m,\infty} = P_n(f) \circ \tau_{d_m,\infty}$ on the set $M_m \cap F_m \cap H = \bigcup_{i=0}^{d_m} G_m^i \cap H$. By definition,

$$P_m(P_n(f)) \circ \tau_{d_m,\infty} = \Phi_m(P_n(f))$$

holds on $M_m \cap F_m$, where Φ_m is given by formula (2.8). We know that

$$\Phi_m(P_n(f)) = P_n(f) \circ \tau_{d_m,\infty}$$

holds on G_m^0 because $G_m^0 \subset V_m$. Suppose that this equality is true on $G_m^{k-1} \cap H$ and that $x \in \left(G_m^k \cap H\right) \setminus V_m$ for a given $k \in \{1, \dots, d_m\}$. If the function $P_n(f) \circ \tau_{d_m,\infty}$ has the property (AF) on H and

$$I_x = \left\{i \in \{1, \dots, d_m\}, 2^{-1+d_m}(\mathcal{O}_i - x_i) \in \mathbb{Z} \setminus 2\mathbb{Z}\right\},$$

then

$$\begin{aligned}
&P_m(P_n(f))\left(\tau_{d_m,\infty}(x)\right) \\
&\quad = \Phi_m(P_n(f))(x) = \sum_{x' \in A_x} \frac{\Phi_m(P_n(f))(x')}{\mathrm{card}(A_x)} = \sum_{x' \in A_x} \frac{P_n(f)\left(\tau_{d_m,\infty}(x')\right)}{\mathrm{card}(A_x)} \\
&\quad = \frac{2}{\mathrm{card}(A_x)} \sum_{i \in I_x} \frac{1}{2} P_n(f)\left(\tau_{d_m,\infty}(x + 2^{1-d_m} e_i)\right) + \frac{1}{2} P_n(f)\left(\tau_{d_m,\infty}(x - 2^{1-d_m} e_i)\right) \\
&\quad = P_n(f)\left(\tau_{d_m,\infty}(x)\right)
\end{aligned}$$

as $A_x = \{x + \varepsilon 2^{1-d_m} e_i, \varepsilon \in \{-1, 1\}, i \in I_x\}$. If the function $P_n(f) \circ \tau_{d_m,\infty}$ does not have the property (AF) on the whole H (recall that it does have it on the hypercubes of half the size as shown above), then necessarily $d_n = d_m$, $M_n = M_m$, $F_n = F_m$ and the $n-$th step is also a refining step, i.e. $l = n$ satisfies (2.5) and (2.6). Thus, according to the induction assumption and the fact that $V_n \subset V_m$, we obtain

$$\begin{aligned}
P_m(P_n(f))\left(\tau_{d_m,\infty}(x)\right) = \Phi_m(P_n(f))(x) &= \sum_{x' \in A_x} \frac{\Phi_m(P_n(f))(x')}{\mathrm{card}(A_x)} \\
&= \sum_{x' \in A_x} \frac{P_n(f)\left(\tau_{d_m,\infty}(x')\right)}{\mathrm{card}(A_x)} = \sum_{x' \in A_x} \frac{\Phi_n(f)(x')}{\mathrm{card}(A_x)} \\
&= \Phi_n(f)(x) = P_n(f)\left(\tau_{d_m,\infty}(x)\right).
\end{aligned}$$

This concludes the proof of the identity

$$P_m(P_n(f)) \circ \tau_{d_m,\infty} = P_n(f) \circ \tau_{d_m,\infty}$$

on $M_m \cap F_m \cap H$ and, since H was chosen arbitrarily, also on the whole set $M_m \cap F_m$ provided that $l = m$ meets conditions (2.5) and (2.6).

The property (AF) of $P_n(f) \circ \tau_{d_m,\infty}$ on the hypercubes of the tiling of F_m along with the identity $P_m(P_n(f)) \circ \tau_{d_m,\infty} = P_n(f) \circ \tau_{d_m,\infty}$ on their vertices yield $P_m(P_n(f)) = P_n(f)$ on X for $m > n$ and $f \in \mathrm{Lip}_0(X)$.

Altogether, for any $m, n \in \mathbb{N}$ we have that $P_m P_n = P_{\min\{m,n\}}$. Thus also P_n is a projection for all $n \in \mathbb{N}$.

$\triangle$

Further, we show that the identity operator on $\mathrm{Lip}_0(X)$ is the limit of the sequence $(P_n)_{n=1}^{\infty}$ with respect to the weak*-operator topology.

Claim 2.3.5. For every $f \in \mathrm{Lip}_0(X)$, the sequence $(P_n(f))_{n=1}^{\infty}$ converges weak* to f.

Proof. The uniform boundedness of operators P_n combined with the inclusions

$$\tau_{d_1,\infty}(V_1) \subset \tau_{d_2,\infty}(V_2) \subset \tau_{d_3,\infty}(V_3) \subset \dots$$

and the identity $\overline{\bigcup_{n=1}^{\infty} \tau_{d_n,\infty}(V_n)} = X$ implies that $(P_n(f))_{n=1}^{\infty}$ converges pointwise to f for every $f \in \mathrm{Lip}_0(X)$. Since the topology of pointwise convergence agrees with the weak*-topology on bounded subsets of $\mathrm{Lip}_0(X)$, the sequence $(P_n)_{n=1}^{\infty}$ converges to the identity on $\mathrm{Lip}_0(X)$ in the weak*-operator topology.

$\triangle$

Now, we observe that P_n's are adjoint operators.

Claim 2.3.6. For every $n \in \mathbb{N}$, the operator P_n is weak* to weak*-continuous on bounded subsets of $\mathrm{Lip}_0(X)$.

Proof. One can see from its definition that the operator P_n is continuous on $\mathrm{Lip}_0(X)$ with the topology of pointwise convergence for every $n \in \mathbb{N}$. Therefore it is weak* to weak*-continuous on bounded subsets of $\mathrm{Lip}_0(X)$ as the weak*-topology coincides with the topology of pointwise convergence on bounded subsets of $\mathrm{Lip}_0(X)$.

$\triangle$

To conclude, we prove that the growth of the dimensions of the ranges of the operators P_n is controlled.

Claim 2.3.7. For every $n \in \mathbb{N}$, the operator $P_{n+1} - P_n$ is of rank 0 or 1.

Proof. Thanks to the commuting property of the sequence $(P_n)_{n=1}^{\infty}$ and the linearity of the operators P_n,

$$P_{n+1}(f) - P_n(f) = P_{n+1}(f - P_n(f))$$

on X for every $n \in \mathbb{N}$ and every $f \in \mathrm{Lip}_0(X)$. Recall that the function $P_{n+1}(f - P_n(f))$ is determined by the values of the function $f - P_n(f)$ on the set $\tau_{d_{n+1},\infty}(V_{n+1})$. But, since by definition $P_n(f)$ agrees with f on the set $\tau_{d_n,\infty}(V_n)$, these values are zero except possibly on $\tau_{d_{n+1},\infty}(V_{n+1}) \setminus \tau_{d_n,\infty}(V_n)$, which is either a singleton or an empty set. Therefore

$$\dim\left((P_{n+1} - P_n)(\mathrm{Lip}_0(X))\right) \leq 1$$

for all $n \in \mathbb{N}$.

△

By the discussion at the beginning of the proof, Theorem 2.3.1 follows.

□

2.4 Final remarks.

It is easy to see that $\mathcal{F}(M)$ has a Schauder basis whenever M is a bounded and convex subset of some $\mathbb{R}^n$. Indeed, suppose that M is a bounded and convex subset of $\mathbb{R}^n$. We may assume without loss of generality that M is closed, and it contains the origin as an interior point (by considering the smallest n for which there exists an imbedding of M into $\mathbb{R}^n$). Now the mapping taking the boundary points of the unit hypercube $[-1,1]^n$ in $\mathbb{R}^n$ onto the boundary points of M is bi-Lipschitz, and extends into a unique homothetic and bi-Lipschitz mapping between M and the hypercube $[-1,1]^n$. It is a general fact which follows from the definition that if two metric spaces are bi-Lipschitz equivalent then their respective Lipschitz-free spaces are linearly isomorphic. Hence the existence of a Schauder basis in one of them ensues the existence of a Schauder basis in the other.

It is not clear to us which subsets M of $\mathbb{R}^n$ share the above bi-Lipschitz condition, in particular we pose the following open problem.

Problem 2.4.1. Does the space $\mathcal{F}(M)$ have a Schauder basis for every $M \subset \mathbb{R}^n$?

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3. On uniformly differentiable mappings

3.1 Introduction

We begin by recalling some classical results in linear Banach space theory. According to Pełczyński, if Y is a Banach space and $T : c_0 \to Y$ is a non-compact linear operator, then c_0 contains a linear subspace X isomorphic to c_0 such that $T|_X$ is an isomorphism (see [20], [5, Theorem 4.51]). In particular, Y contains a copy of c_0. Similarly, Rosenthal showed that if Y is a Banach space and $T : \ell_\infty \to Y$ is a non-weakly compact linear operator, then ℓ_∞ contains a linear subspace X isomorphic to ℓ_∞ such that $T|_X$ is an isomorphism (see [16, Proposition 2.f.4]). In particular, Y contains a copy of ℓ_∞.

There has been a recent attempt ([4], [10, Theorem 6.45]) to generalise the first mentioned result into non-linear setting, namely, for *uniformly differentiable mappings* from the unit ball of c_0 in the sense of the following definitions. Our principal reference for the theory of smooth mappings on Banach spaces is the monograph [10].

Let X, Y be normed linear spaces and let $V \subset X$ be convex with non-empty interior $\operatorname{Int} V$. Then $\mathcal{C}^1(V;Y)$ denotes the locally convex space of continuous mappings $f : \operatorname{Int} V \to Y$ with continuous Fréchet derivative $Df : \operatorname{Int} V \to Y$ such that f and Df have a continuous extension to the whole V and are bounded on closed convex bounded subsets of V, endowed with the topology of uniform convergence of f and Df on closed convex bounded subsets of V. The subspace of $\mathcal{C}^1(V;Y)$ consisting of mappings f such that Df is uniformly continuous on closed convex bounded subsets of V is denoted by $\mathcal{C}^{1,+}(V;Y)$. In scalar case, we use shortened notation $\mathcal{C}^{1,+}(V) = \mathcal{C}^{1,+}(V;\mathbb{R})$.

If $f \in \mathcal{C}^{1,+}(B_X;Y)$ for some Banach spaces X and Y, then there exists a *bidual extension* f^{**} of the mapping f such that $f^{**} \in \mathcal{C}^{1,+}(B_{X^{**}};Y^{**})$. The construction uses the Converse Taylor theorem and the powerful ultrapower construction based on the principle of local reflexivity, and can be found in Section 6.2 of [10].

Theorem 6.45 in [10] implies that if Y is a Banach space and if $f \in \mathcal{C}^{1,+}(B_{c_0};Y)$ is a non-compact mapping, then there exists a point $x^{**} \in B_{c_0^{**}}$ such that $D\left(f^{**}\right)\left(x^{**}\right)$ is a non-weakly compact bounded linear operator from ℓ_∞ into Y^{**}. In particular, Y^{**} contains a copy of ℓ_∞.

In some special cases, e.g. when Y is a dual space, this result implies that Y contains a copy of c_0. The general case, however, remains an open question. That is, does for any Banach space Y the existence of a non-compact uniformly differentiable mapping from the unit ball of c_0 into Y imply that Y contains a copy of c_0? It should be noted that the problem cannot be solved by means of differentiation, in view of the next simple example. Indeed, choosing a surjective increasing C^∞-smooth function $\phi\colon \mathbb{R} \to \mathbb{R}$ such that $\phi(0) = 0$ and $D\phi(0) = 0$, one can show that the mapping $\Phi\colon c_0 \to c_0$ defined by $\Phi((x_k)_{k=1}^\infty) = (\phi(x_k))_{k=1}^\infty$ belongs to $\mathcal{C}^{1,+}(c_0;c_0)$, it is surjective, but $D\Phi(x)$ is a compact linear operator from c_0 into c_0 for every $x \in c_0$.

In the present note we will consider a variant of this problem when the initial space is ℓ_∞ and show that in this case the analogous question has positive answer, introducing thus an approach to Rosenthal's result in non-linear setting. In addition, by passing to infinite-dimensional case via ultrapowers, we will derive a finite-dimensional counterpart of the result.

To this end we will generalise Theorem 6.45 from [10] for uniformly differentiable mappings which are not necessarily bidual extensions of uniformly differentiable mappings.

In order to find the right assumptions, recall that ℓ_2 is a linear quotient of ℓ_∞. Therefore, by Theorem 6.68 in [10], there exists a surjective second degree polynomial from ℓ_∞ onto ℓ_1. So the non-compactness (or non-weak compactness) of the image of the unit ball B_{ℓ_∞} is not sufficient for concluding that Y contains a copy of ℓ_∞.

The proper generalization is presented in Section 3.2 as Theorem 3.2.1 and followed by a finite-dimensional version of the statement, Theorem 3.2.3.

3.2 Uniformly differentiable mappings from ℓ_∞ and ℓ_∞^n

Let us first fix some notation that will be used throughout this section. The linear space $\{x = (x_i)_{i=1}^\infty \in \mathbb{R}^\mathbb{N},\ \sup\{|x_i|,\ i \in \mathbb{N}\} < \infty\}$ equipped with the norm given as $\sup\{|x_i|,\ i \in \mathbb{N}\}$ is denoted by ℓ_∞. Similarly, if $n \in \mathbb{N}$, then ℓ_∞^n is the linear space $\{x = (x_i)_{i=1}^n \in \mathbb{R}^n,\ \sup\{|x_i|,\ i \in \{1,\dots,n\}\} < \infty\}$ with the norm defined as $\sup\{|x_i|,\ i \in \{1,\dots,n\}\}$. For $x \in \ell_\infty$ (resp. $x \in \ell_\infty^n$), we put $\operatorname{supp}(x) = \{i \in \mathbb{N}, x_i \neq 0\}$ (resp. $\operatorname{supp}(x) = \{i \in \{1,\dots,n\}, x_i \neq 0\}$). The symbol e_i stands for the unit vector $(0,\dots,0,\underbrace{1}_{i},0,\dots)$ in the space ℓ_∞ or ℓ_∞^n. We write $\|\cdot\|$ for the norm in any normed linear space. For a normed linear space X, we use standard notation B_X, U_X and S_X for the closed unit ball, open unit ball and the unit sphere of X, respectively. The set $B^+_{\ell_\infty}$ (resp. $B^+_{\ell_\infty^n}$) is then the subset of B_{ℓ_∞} (resp. $B_{\ell_\infty^n}$) consisting of vectors with non-negative coordinates. If X, Y are Banach spaces and $f\colon X \to Y$ has the Fréchet derivative Df at $x \in X$, we denote by $Df(x)[u] \in Y$ the evaluation of $Df(x)$ at $u \in X$.

The key ingredient of the proof of the main result, Theorem 3.2.1, will be Lemma 6.27 from [10] (see also [9]). Before stating its formulation, let us recall the notion of modulus of continuity of a uniformly continuous mapping.

Let (P,ρ) and (Q,σ) be metric spaces. The *minimal modulus of continuity* ω_f of a uniformly continuous mapping $f\colon P \to Q$ is for $\delta \in [0,+\infty)$ defined as

$$\omega_f(\delta) = \sup\{\sigma(f(x), f(y)), x, y \in P, \rho(x,y) \leq \delta\}.$$

Clearly, ω_f is continuous at 0. A non-decreasing function $\omega\colon [0,+\infty) \to [0,+\infty]$ continuous at 0 with $\omega(0) = 0$ is called a *modulus*. The set of all moduli is denoted by $\mathcal{M}$. We say that $\omega \in \mathcal{M}$ is a modulus of continuity of a uniformly continuous mapping $f\colon P \to Q$ if $\omega_f \leq \omega$.

Lemma 6.27 in [10] says that for each $\omega \in \mathcal{M}$ with $\omega(1) < \infty$, for every $L > 0$ and every $\varepsilon > 0$, there is an $N(\omega, L, \varepsilon) \in \mathbb{N}$ such that if $n \geq N(\omega, L, \varepsilon)$ and if $f \in \mathcal{C}^{1,+}(B^+_{\ell_\infty^n})$ is an $L-$Lipschitz function whose derivative Df has modulus of continuity ω, then there exists $j \in \{1,\dots,n\}$ for which $|f(e_j) - f(0)| < \varepsilon$.

Our main result is the following.

Theorem 3.2.1. *Let Y be a Banach space and let $f \in \mathcal{C}^{1,+}(B_{\ell_\infty};Y)$ be such that $\{f(e_k), k \in \mathbb{N}\}$ is not relatively compact in Y. Then there exists an infinite subset $\mathcal{K}$ of $\mathbb{N}$ and a closed interval $I \subset (0,1)$ such that for the subspace $Z = \{z \in \ell_\infty, \operatorname{supp}(z) \subset \mathcal{K}\}$ of ℓ_∞ and for every point $x \in Z$ satisfying that $x_k \in I$ for all $k \in \mathcal{K}$, the operator $Df(x)|_Z$ is an isomorphism. In particular, Y contains a copy of ℓ_∞.*

Proof. Applying a translation in Y, we may without loss of generality assume that $f(0) = 0$.

Let ω be the modulus of continuity of Df and let $L = \sup\{\|Df(x)\|, x \in B_{\ell_\infty}\}$. By the assumption, $L < \infty$.

After possible passing to a subsequence of the sequence $(e_k)_{k=1}^\infty$, we can find a bounded sequence $(\varphi_k)_{k=1}^\infty$ of functionals lying in Y^* satisfying that $\varphi_k(f(e_k)) = 1$. Denote C the real number $\sup\{\|\varphi_k\|, k \in \mathbb{N}\}$.

Now, we show that we may also assume that for every $k \in \mathbb{N}$, the value of $\varphi_k \circ f$ at any $x \in B^+_{\ell_\infty}$ is determined only by the $k-$th coordinate of x, up to a fixed error.

Claim 3.2.2. Let f, $(\varphi_k)_{k=1}^\infty$ and $(e_k)_{k=1}^\infty$ be as above. For any $\varepsilon > 0$ there exists an increasing sequence of natural numbers $(k_i)_{i=1}^\infty$ such that for every $l \in \mathbb{N}$ and every $x = (x_k)_{k=1}^\infty \in B^+_{\ell_\infty}$ with $\operatorname{supp}(x) \subset \{k_i, i \in \mathbb{N}\}$, the inequality

$$|\varphi_{k_l}(f(x)) - \varphi_{k_l}(f(x_{k_l} e_{k_l}))| < \varepsilon \tag{3.1}$$

holds.

Proof. Let $\varepsilon > 0$. When C and L are the constants defined earlier, take $q \in \mathbb{N}$ such that

$$\frac{1}{q} < \frac{\varepsilon}{4CL}. \tag{3.2}$$

Lemma 6.27 in [10] says that there exists $N\left(C\omega, CL, \frac{\varepsilon}{4}\right) \in \mathbb{N}$, where ω is the modulus of continuity of Df, satisfying that for every $n \geq N\left(C\omega, CL, \frac{\varepsilon}{4}\right)$ and every $g \in \mathcal{C}^{1,+}(B^+_{\ell^n_\infty})$ which is CL-Lipschitz and such that Dg has modulus of continuity $C\omega$, there is $j \in \{1, \dots, n\}$ for which $|g(e_j) - g(0)| < \frac{\varepsilon}{4}$. Put

$$n_1 = qN\left(C\omega, CL, \frac{\varepsilon}{4}\right) - q + 1$$

and

$$N_1 = qn_1N\left(C\omega, CL, \frac{\varepsilon}{4}\right) - qn_1 + 1.$$

Consider arbitrary N_1 infinite mutually disjoint subsets $A^1_1, \dots, A^1_{N_1}$ of $\mathbb{N}$ containing only elements greater than n_1.

There must be $M_1 \in \{1, \dots, N_1\}$ such that for every $u = (u_k)_{k=1}^\infty \in B^+_{\ell_\infty}$ with $\operatorname{supp}(u) \subset A^1_{M_1}$, the inequality

$$|\varphi_j(f(\beta_1 e_j + u)) - \varphi_j(f(\beta_1 e_j))| < \frac{\varepsilon}{4}$$

holds for each $j \in \{1, \dots, n_1\}$ and each $\beta_1 \in \left\{\frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}, 1\right\}$. Indeed, for a contradiction suppose that for every $i \in \{1, \dots, N_1\}$ there is $u(i) \in B^+_{\ell_\infty}$ with

$\mathrm{supp}(u(i)) \subset A_i^1$ and there is $j(i) \in \{1, \ldots, n_1\}$ and $\beta_1(i) \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}, 1\right\}$ such that

$$\left|\varphi_{j(i)}\left(f\left(\beta_1(i)e_{j(i)} + u(i)\right)\right) - \varphi_{j(i)}\left(f\left(\beta_1(i)e_{j(i)}\right)\right)\right| \geq \frac{\varepsilon}{4}.$$

Then there is an $h \in \{1, \ldots, n_1\}$, $\gamma_1 \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}, 1\right\}$ and an increasing sequence $(i_m)_{m=1}^{N\left(C\omega, CL, \frac{\varepsilon}{4}\right)}$ of elements of the set $\{1, \ldots, N_1\}$ for which $h = j(i_m)$ and $\gamma_1 = \beta_1(i_m)$ for all $m \in \left\{1, \ldots, N\left(C\omega, CL, \frac{\varepsilon}{4}\right)\right\}$. Thus we get a contradiction with Lemma 6.27 in [10] by considering the function $g : B^+_{\ell_\infty^{N\left(C\omega, CL, \frac{\varepsilon}{4}\right)}} \to \mathbb{R}$ defined by

$$g(\alpha) = \varphi_h\left(f\left(\gamma_1 e_h + \sum_{m=1}^{N\left(C\omega, CL, \frac{\varepsilon}{4}\right)} \alpha_m u(i_m)\right)\right)$$

for $\alpha = (\alpha_m)_{m=1}^{N\left(C\omega, CL, \frac{\varepsilon}{4}\right)} \in B^+_{\ell_\infty^{N\left(C\omega, CL, \frac{\varepsilon}{4}\right)}}$. So, hereby we proved the existence of the M_1.

Next, for every $i \in A^1_{M_1}$ there is a $j(i) \in \{1, \ldots, n_1\}$ such that

$$\left|\varphi_i\left(f\left(\beta_1 e_{j(i)} + \beta_0 e_i\right)\right) - \varphi_i\left(f\left(\beta_0 e_i\right)\right)\right| < \frac{\varepsilon}{4}$$

for all $\beta_1, \beta_0 \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}, 1\right\}$. Indeed, if there were $i \in A^1_{M_1}$ such that for every $j \in \{1, \ldots, n_1\}$ there exist $\beta_1(j), \beta_0(j) \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}, 1\right\}$ for which

$$\left|\varphi_i\left(f\left(\beta_1(j) e_j + \beta_0(j) e_i\right)\right) - \varphi_i\left(f\left(\beta_0(j) e_i\right)\right)\right| \geq \frac{\varepsilon}{4},$$

then by the value of n_1 there exists a $\gamma_0 \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}, 1\right\}$ and an increasing sequence $(j_m)_{m=1}^{N\left(C\omega, CL, \frac{\varepsilon}{4}\right)}$ of indices belonging to the set $\{1, \ldots, n_1\}$ such that $\gamma_0 = \beta_0(j_m)$ for all $m \in \left\{1, \ldots, N\left(C\omega, CL, \frac{\varepsilon}{4}\right)\right\}$. Hence, we arrive at a contradiction with Lemma 6.27 in [10] applied to the function $g : B^+_{\ell_\infty^{N\left(C\omega, CL, \frac{\varepsilon}{4}\right)}} \to \mathbb{R}$ defined by

$$g(\alpha) = \varphi_i\left(f\left(\sum_{m=1}^{N\left(C\omega, CL, \frac{\varepsilon}{4}\right)} \alpha_m \beta_1(j_m) e_{j_m} + \gamma_0 e_i\right)\right)$$

for $\alpha = (\alpha_m)_{m=1}^{N\left(C\omega, CL, \frac{\varepsilon}{4}\right)} \in B^+_{\ell_\infty^{N\left(C\omega, CL, \frac{\varepsilon}{4}\right)}}$. So, as $A^1_{M_1}$ is infinite, we can find an infinte subset A^1 of $A^1_{M_1}$ and a $k_1 \in \{1, \ldots, n_1\}$ satisfying that

$$\left|\varphi_i\left(f\left(\beta_1 e_{k_1} + \beta_0 e_i\right)\right) - \varphi_i\left(f\left(\beta_0 e_i\right)\right)\right| < \frac{\varepsilon}{4}$$

for every $\beta_0, \beta_1 \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}, 1\right\}$ and for every $i \in A^1$.

We construct the sought sequence of indices $(k_i)_{i=1}^\infty$ by induction. Let $l \in \mathbb{N}$. Suppose that we have an increasing sequence of natural numbers $(k_i)_{i=1}^l$ and an infinite subset A^l of $\mathbb{N} \setminus \{1, \ldots, k_l\}$. Then, using Lemma 6.27 from [10] now for parameters $C\omega$, CL and $\frac{\varepsilon}{2^{l+2}}$ we obtain a natural number $N\left(C\omega, CL, \frac{\varepsilon}{2^{l+2}}\right)$ such that for every $n \geq N\left(C\omega, CL, \frac{\varepsilon}{2^{l+2}}\right)$ and every $CL-$Lipschitz function $g \in \mathcal{C}^{1,+}(B^+_{\ell^n_\infty})$ whose derivative Dg has modulus of continuity $C\omega$, there exists $j \in \{1, \ldots, n\}$ such that $|g(e_j) - g(0)| < \frac{\varepsilon}{2^{l+2}}$. Set

$$n_{l+1} = q^{l+1} N\left(C\omega, CL, \frac{\varepsilon}{2^{l+2}}\right) - q^{l+1} + 1$$

and

$$N_{l+1} = q^{l+1} n_{l+1} N\left(C\omega, CL, \frac{\varepsilon}{2^{l+2}}\right) - q^{l+1} n_{l+1} + 1.$$

Denote $(a_j^{l+1})_{j=1}^{n_{l+1}}$ the increasing sequence of the first n_{l+1} elements of A^l and consider some N_{l+1} infinite mutually disjoint subsets $A_1^{l+1}, \ldots, A_{N_{l+1}}^{l+1}$ of A^l whose elements are greater than $a_{n_{l+1}}^{l+1}$.

Then, based on the same argument as above, there exists $M_{l+1} \in \{1, \ldots, N_{l+1}\}$ satisfying that for every $u \in B^+_{\ell_\infty}$ with $\mathrm{supp}(u) \subset A_{M_{l+1}}^{l+1}$, every $j \in \{1, \ldots, n_{l+1}\}$ and all coefficients $\beta_1, \ldots, \beta_{l+1} \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}, 1\right\}$, we have that

$$\begin{aligned}\Bigg|\varphi_{a_j^{l+1}}\Bigg(f\Bigg(\sum_{h=1}^{l} \beta_h e_{k_h} + \beta_{l+1} e_{a_j^{l+1}} + u\Bigg)\Bigg) \\ - \varphi_{a_j^{l+1}}\Bigg(f\Bigg(\sum_{h=1}^{l} \beta_h e_{k_h} + \beta_{l+1} e_{a_j^{l+1}}\Bigg)\Bigg)\Bigg| < \frac{\varepsilon}{2^{l+2}}.\end{aligned} \tag{3.3}$$

Besides, for every $i \in A_{M_{l+1}}^{l+1}$ there is a $j(i) \in \{1, \ldots, n_{l+1}\}$ satisfying that for all $\beta_0, \ldots, \beta_{l+1} \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}, 1\right\}$, we have that

$$\begin{aligned}\Bigg|\varphi_i\Bigg(f\Bigg(\sum_{h=1}^{l} \beta_h e_{k_h} + \beta_{l+1} e_{a_{j(i)}^{l+1}} + \beta_0 e_i\Bigg)\Bigg) \\ - \varphi_i\Bigg(f\Bigg(\sum_{h=1}^{l} \beta_h e_{k_h} + \beta_0 e_i\Bigg)\Bigg)\Bigg| < \frac{\varepsilon}{2^{l+2}}.\end{aligned} \tag{3.4}$$

This can be proved by a contradiction again. So, assume that it is not true. That is, there exists $i \in A_{M_{l+1}}^{l+1}$ such that for every $j \in \{1, \ldots, n_{l+1}\}$, there are coefficients $\beta_0(j), \ldots, \beta_{l+1}(j) \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}, 1\right\}$ for which

$$\begin{aligned}\Bigg|\varphi_i\Bigg(f\Bigg(\sum_{h=1}^{l} \beta_h(j) e_{k_h} + \beta_{l+1}(j) e_{a_j^{l+1}} + \beta_0(j) e_i\Bigg)\Bigg) \\ - \varphi_i\Bigg(f\Bigg(\sum_{h=1}^{l} \beta_h(j) e_{k_h} + \beta_0(j) e_i\Bigg)\Bigg)\Bigg| \geq \frac{\varepsilon}{2^{l+2}}.\end{aligned}$$

The definition of n_{l+1} implies that we can find $\gamma_0, \ldots, \gamma_l \in \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}, 1\right\}$ and a sequence $(j_m)_{m=1}^{N\left(C\omega, CL, \frac{\varepsilon}{2^{l+2}}\right)}$ of indices from the set $\{1, \ldots, n_{l+1}\}$ so that

$\gamma_h = \beta_h(j_m)$ for each $h \in \{0, \dots, l\}$ and each $m \in \left\{1, \dots, N\left(C\omega, CL, \frac{\varepsilon}{2^{l+2}}\right)\right\}$. Then the function $g : B^+_{\ell_\infty^{N\left(C\omega, CL, \frac{\varepsilon}{2^{l+2}}\right)}} \to \mathbb{R}$ given by

$$g(\alpha) = \varphi_i \left(f \left(\sum_{h=1}^{l} \gamma_h e_{k_h} + \sum_{m=1}^{N\left(C\omega, CL, \frac{\varepsilon}{2^{l+2}}\right)} \alpha_m \beta_{l+1}(j_m) e_{a^{l+1}_{j_m}} + \gamma_0 e_i \right) \right)$$

for $\alpha = (\alpha_m)_{m=1}^{N\left(C\omega, CL, \frac{\varepsilon}{2^{l+2}}\right)} \in B^+_{\ell_\infty^{N\left(C\omega, CL, \frac{\varepsilon}{2^{l+2}}\right)}}$ is $CL-$Lipschitz and its derivative Dg has modulus of continuity $C\omega$, but the conclusion of Lemma 6.27 in [10] does not hold for g. This is a contradiction. Thus we have established (3.4). Since $A^{l+1}_{M_{l+1}}$ is infinite, there is an infinite subset A^{l+1} of $A^{l+1}_{M_{l+1}}$ and an index $k_{l+1} \in \left\{a^{l+1}_1, \dots, a^{l+1}_{n_{l+1}}\right\}$ such that

$$\left| \varphi_i \left(f \left(\sum_{h=1}^{l+1} \beta_h e_{k_h} + \beta_0 e_i \right) \right) - \varphi_i \left(f \left(\sum_{h=1}^{l} \beta_h e_{k_h} + \beta_0 e_i \right) \right) \right| < \frac{\varepsilon}{2^{l+2}} \tag{3.5}$$

for all $i \in A^{l+1}$ and all $\beta_0, \dots, \beta_{l+1} \in \left\{\frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}, 1\right\}$.

We now combine the foregoing results to show (3.1) for the constructed sequence $(k_i)_{i=1}^\infty$. So, let $l \in \mathbb{N}$ and let $x = (x_i)_{i=1}^\infty \in B^+_{\ell_\infty}$ be such that $\operatorname{supp}(x) \subset \{k_i, i \in \mathbb{N}\}$. Put $u = x - \sum_{i=1}^l x_{k_i} e_{k_i}$. We choose $\beta_1, \dots, \beta_l \in \left\{\frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}, 1\right\}$ so that $|x_{k_i} - \beta_i| \le \frac{1}{q}$ for all $i \in \{1, \dots, l\}$ and write

$$\begin{aligned}
&|\varphi_{k_l}(f(x)) - \varphi_{k_l}(f(x_{k_l} e_{k_l}))| \\
&\quad \le \left| \varphi_{k_l}(f(x)) - \varphi_{k_l} \left(f \left(\sum_{i=1}^{l} \beta_i e_{k_i} + u \right) \right) \right| \\
&\qquad + \left| \varphi_{k_l} \left(f \left(\sum_{i=1}^{l} \beta_i e_{k_i} + u \right) \right) - \varphi_{k_l} \left(f \left(\sum_{i=1}^{l} \beta_i e_{k_i} \right) \right) \right| \\
&\qquad + \left| \varphi_{k_l} \left(f \left(\sum_{i=1}^{l} \beta_i e_{k_i} \right) \right) - \varphi_{k_l}(f(\beta_l e_{k_l})) \right| \\
&\qquad + |\varphi_{k_l}(f(\beta_l e_{k_l})) - \varphi_{k_l}(f(x_{k_l} e_{k_l}))| .
\end{aligned}$$

The fact that $\varphi_{k_l} \circ f$ is $CL-$Lipschitz along with the choice of q (see (3.2)) gives that

$$\left| \varphi_{k_l}(f(x)) - \varphi_{k_l} \left(f \left(\sum_{i=1}^{l} \beta_i e_{k_i} + u \right) \right) \right| < \frac{\varepsilon}{4}$$

and

$$|\varphi_{k_l}(f(\beta_l e_{k_l})) - \varphi_{k_l}(f(x_{k_l} e_{k_l}))| < \frac{\varepsilon}{4}.$$

Moreover, since $\operatorname{supp}(u) \subset A^l \subset A^l_{M_l}$ and as $k_l \in \{a^l_1, \dots, a^l_{n_l}\}$ (here we put $a^1_j = j$ for all $j \in \{1, \dots, n_1\}$), (3.3) yields that

$$\left| \varphi_{k_l} \left(f \left(\sum_{i=1}^{l} \beta_i e_{k_i} + u \right) \right) - \varphi_{k_l} \left(f \left(\sum_{i=1}^{l} \beta_i e_{k_i} \right) \right) \right| < \frac{\varepsilon}{2^{l+1}}.$$

Thus, if $l = 1$, we conclude that

$$|\varphi_{k_1}(f(x)) - \varphi_{k_1}(f(x_{k_1}e_{k_1}))| < \frac{3}{4}\varepsilon < \varepsilon.$$

If $l > 1$, then $k_l \in A^i$ for all $i \in \{1, \ldots, l-1\}$. Hence, due to (3.5),

$$\begin{aligned}
&\left|\varphi_{k_l}\left(f\left(\sum_{i=1}^{l}\beta_i e_{k_i}\right)\right) - \varphi_{k_l}(f(\beta_l e_{k_l}))\right| \\
&\quad\leq \sum_{j=1}^{l-1}\left|\varphi_{k_l}\left(f\left(\sum_{i=1}^{l-j}\beta_i e_{k_i} + \beta_l e_{k_l}\right)\right) - \varphi_{k_l}\left(f\left(\sum_{i=1}^{l-j-1}\beta_i e_{k_i} + \beta_l e_{k_l}\right)\right)\right| \\
&\quad\leq \sum_{j=1}^{l-1}\frac{\varepsilon}{2^{l-j+1}},
\end{aligned}$$

where we set $\sum_{i=1}^{0}\beta_i e_{k_i} = 0$. Finally,

$$|\varphi_{k_l}(f(x)) - \varphi_{k_l}(f(x_{k_l}e_{k_l}))| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2^{l+1}} + \left(\sum_{j=1}^{l-1}\frac{1}{2^{l-j}}\right)\frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon$$

as desired.

$\triangle$

Let us continue with the proof of Theorem 3.2.1. Find $\Delta \in (0, \frac{1}{2})$ so that $\omega(\Delta) < \frac{1}{16C}$. According to Claim 3.2.2, by passing to a subsequence of the sequence $(e_k)_{k=1}^{\infty}$ we may without loss of generality assume that for every $k \in \mathbb{N}$ and every $x = (x_i)_{i=1}^{\infty} \in B_{\ell_\infty}^{+}$,

$$|\varphi_k(f(x)) - \varphi_k(f(x_k e_k))| < \frac{\Delta}{32}. \tag{3.6}$$

For each $k \in \mathbb{N}$,

$$1 = \varphi_k(f(e_k)) = \int_0^1 \varphi_k(Df(te_k)[e_k])\,\mathrm{d}t.$$

Therefore there exists an $r \in [0,1]$ such that $\varphi_k(Df(re_k)[e_k]) \geq 1$. Then for every $t \in [r-\Delta, r+\Delta] \cap [0,1]$, $\|Df(te_k) - Df(re_k)\| \leq \frac{1}{16C}$. Hence, there is an interval $[a_k, b_k] \subset [0,1]$ of length Δ such that

$$\varphi_k(Df(te_k)[e_k]) \geq \frac{15}{16} \tag{3.7}$$

for each $t \in [a_k, b_k]$.

Passing to a subsequence of $(e_k)_{k=1}^{\infty}$, we may assume that there is an interval $[a,b] \subset [0,1]$ such that $b - a = \frac{\Delta}{2}$ and that (3.7) holds for every $t \in [a,b]$ and every $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$ and $x = (x_i)_{i=1}^\infty \in B_{\ell_\infty}^+$. Thanks to (3.6) and (3.7) we get the following.

$$\begin{aligned}
\int_a^b \varphi_k\Bigg(Df\Bigg(\sum_{i=1}^{k-1} x_i e_i + t e_k + x - \sum_{i=1}^{k} x_i e_i\Bigg)[e_k]\Bigg)\,\mathrm{d}t
&= \varphi_k\left(f\left(\sum_{i=1}^{k-1} x_i e_i + b e_k + x - \sum_{i=1}^{k} x_i e_i\right)\right) \\
&\quad - \varphi_k\left(f\left(\sum_{i=1}^{k-1} x_i e_i + a e_k + x - \sum_{i=1}^{k} x_i e_i\right)\right) \\
&> \varphi_k(f(be_k)) - \varphi_k(f(ae_k)) - \frac{\Delta}{16} \\
&= \int_a^b \varphi_k(Df(te_k)[e_k])\,\mathrm{d}t - \frac{\Delta}{16} \geq \frac{15}{32}\Delta - \frac{\Delta}{16} = \frac{13}{32}\Delta.
\end{aligned}$$

The foregoing lower estimate yields the existence of an $s \in [a, b]$ for which

$$\varphi_k\left(Df\left(\sum_{i=1}^{k-1} x_i e_i + s e_k + x - \sum_{i=1}^{k} x_i e_i\right)[e_k]\right) \geq \frac{13}{16}.$$

As $b - a = \frac{\Delta}{2}$, and Δ was chosen to satisfy that $\omega(\Delta) < \frac{1}{16C}$, we derive that for all $t \in [a, b]$,

$$\begin{aligned}
&\varphi_k\Bigg(Df\Bigg(\sum_{i=1}^{k-1} x_i e_i + t e_k + x - \sum_{i=1}^{k} x_i e_i\Bigg)[e_k]\Bigg) \\
&\qquad \geq \varphi_k\left(Df\left(\sum_{i=1}^{k-1} x_i e_i + s e_k + x - \sum_{i=1}^{k} x_i e_i\right)[e_k]\right) - \frac{1}{16} \geq \frac{3}{4}.
\end{aligned} \tag{3.8}$$

We show that $Df(x)$ is an isomorphism on ℓ_∞ for any $x = (x_i)_{i=1}^\infty \in \ell_\infty$ with $x_i \in [a + \lambda, b - \lambda]$ for all $i \in \mathbb{N}$, where $\lambda = \frac{\Delta}{5}$.

So, take any $x = (x_i)_{i=1}^\infty \in \ell_\infty$ with $x_i \in [a + \lambda, b - \lambda]$ for all $i \in \mathbb{N}$. Let $z = (z_i)_{i=1}^\infty \in \lambda S_{\ell_\infty}$ be such that $z_k \geq \frac{11}{12}\lambda$ for some $k \in \mathbb{N}$. By virtue of the assumption (3.6), we can write

$$\begin{aligned}
\int_0^1 \varphi_k\left(Df(x + tz)[z]\right)\mathrm{d}t &= \varphi_k(f(x + z)) - \varphi_k(f(x)) \\
&\geq \varphi_k(f(x + z)) - \varphi_k(f(x + z - z_k e_k) - \frac{\Delta}{16} \\
&= \int_0^1 \varphi_k(Df(x + z - z_k e_k + t z_k e_k)[z_k e_k])\,\mathrm{d}t - \frac{\Delta}{16}.
\end{aligned}$$

Denote $y(t) = x + z - z_k e_k + t z_k e_k$. Since $y(t)_i \in [a, b]$ for every $i \in \mathbb{N}$ and every $t \in [0, 1]$, (3.8) implies that

$$\varphi_k\left(Df(x + z - z_k e_k + t z_k e_k)[z_k e_k]\right) \geq \frac{3}{4} z_k.$$

Therefore there exists an $s \in [0,1]$ satisfying that

$$\varphi_k\left(Df(x+sz)[z]\right) \geq \frac{3}{4}z_k - \frac{\Delta}{16}.$$

Finally, as

$$\|Df(x) - Df(x+sz)\| \leq \omega\left(\frac{\Delta}{5}\right) \leq \frac{1}{16C},$$

we obtain that

$$\begin{aligned}\varphi_k(Df(x)[z]) &\geq \varphi_k\left(Df(x+sz)[z]\right) - \frac{\lambda}{16}\\ &\geq \frac{3}{4}z_k - \frac{\Delta}{16} - \frac{\lambda}{16}\\ &\geq \frac{11}{16}\lambda - \frac{\Delta}{16} - \frac{\lambda}{16} = \frac{\Delta}{16}.\end{aligned}$$

Now take any $z \in \ell_\infty$ and choose $k \in \mathbb{N}$ so that $|z_k| \geq \frac{11}{12}\|z\|$. Then

$$\|Df(x)[z]\| \geq \frac{1}{\|\varphi_k\|}\left|\varphi_k\left(Df(x)\left[\operatorname{sgn}(z_k)\lambda\frac{z}{\|z\|}\right]\right)\right|\frac{1}{\lambda}\|z\| \geq \frac{5}{16C}\|z\|.$$

So, $Df(x)$ is an isomorphism and the proof of the theorem is finished. □

As a corollary, it follows that there does not exist any uniformly differentiable mapping from ℓ_∞ into c_0 which fixes the basis. This generalises the classical theorem of Phillips which claims that c_0 is not complemented in ℓ_∞ (see [5, Theorem 5.6]).

In view of Corollary 3.2.4, Theorem 3.2.3 below can be seen as a vector version of Lemma 6.27 from [10], which played a crucial role in the previous proof. We obtain it from Theorem 3.2.1 by applying the ultrapower construction.

We now recall the notion of ultrapower following Section 4.1 of [10]. Let X be a Banach space and let $\ell_\infty(\mathbb{N}; X)$ be the Banach space

$$\{(x_n)_{n=1}^\infty, x_n \in X, \sup\{\|x_n\|, n \in \mathbb{N}\} < \infty\}$$

with the norm given by $\sup\{\|x_n\|, n \in \mathbb{N}\}$. If $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$, we define the *ultrapower of* X as the quotient space

$$(X)_\mathcal{U} = \ell_\infty(\mathbb{N}; X)\Big/\Big\{(x_n)_{n=1}^\infty \in \ell_\infty(\mathbb{N}; X), \lim_{\mathcal{U}}\|x_n\| = 0\Big\}$$

endowed with the canonical quotient norm. Here $\lim_\mathcal{U}\|x_n\| \in \mathbb{R}$ is the limit with respect to the ultrafilter $\mathcal{U}$. Then, $(X)_\mathcal{U}$ is a Banach space and $\|(x_n)_\mathcal{U}\| = \lim_\mathcal{U}\|x_n\|$ for every $(x_n)_\mathcal{U} \in (X)_\mathcal{U}$ represented by $(x_n)_{n=1}^\infty \in \ell_\infty(\mathbb{N}; X)$.

Here comes the finite-dimensional result.

Theorem 3.2.3. *For each $\omega \in \mathcal{M}$ with $\omega(1) < \infty$, $L > 0$, $m \in \mathbb{N}$ and $\varepsilon > 0$, there is $N(\omega, L, m, \varepsilon) \in \mathbb{N}$ such that if $n \geq N(\omega, L, m, \varepsilon)$, Y is a separable Banach space and $f \in \mathcal{C}^{1,+}(B_{\ell_\infty^n}; Y)$ is an L-Lipschitz mapping whose derivative Df has modulus of continuity ω and for which $\|f(e_i) - f(e_j)\| \geq \varepsilon$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$, then there exists $\mathcal{J} \subset \{1, \dots, n\}$ with $\operatorname{card}(\mathcal{J}) = m$ and $x \in \ell_\infty^n$ such that $Df(x)|_{\operatorname{span}\{e_j, j \in \mathcal{J}\}}$ is an isomorphism.*

Proof. Let $\omega \in \mathcal{M}$ be finite at 1, let $L > 0$, $m \in \mathbb{N}$ and $\varepsilon > 0$. Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of mappings such that for every $n \in \mathbb{N}$, $f_n \in \mathcal{C}^{1,+}(B_{\ell_\infty^n}; Y_n)$ for some separable Banach space Y_n, f_n is L-Lipschitz, $f_n(0) = 0$, Df_n has modulus of continuity ω, and $\|f_n(e_i) - f_n(e_j)\| \geq \varepsilon$ for each $i, j \in \{1, \dots, n\}$, $i \neq j$. We show that then there exists $n_0 \in \mathbb{N}$, a set $\mathcal{J} \subset \{1, \dots, n_0\}$ of cardinality m and a point $x \in \ell_\infty^{n_0}$ such that $Df_{n_0}(x)|_{\operatorname{span}\{e_j, j \in \mathcal{J}\}}$ is an isomorphism. Since $(f_n)_{n=1}^{\infty}$ is an arbitrary sequence with the listed properties, the statement of the theorem follows.

We may regard all f_n's as mappings from B_{ℓ_∞} into ℓ_∞ by composing with the projections $P_n \colon \ell_\infty \to \ell_\infty^n$ given by $P_n\left((x_i)_{i=1}^{\infty}\right) = (x_i)_{i=1}^{n}$ and by identifying Y_n with a subspace of ℓ_∞. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. We can define a mapping $f \colon B_{\ell_\infty} \to (\ell_\infty)_{\mathcal{U}}$ by

$$f(x) = (f_n(x))_{\mathcal{U}}$$

for $x \in B_{\ell_\infty}$. Since the mappings f_n are equi-Lipschitz, f is well-defined.

Moreover, $f \in \mathcal{C}^{1,+}(B_{\ell_\infty}; (\ell_\infty)_{\mathcal{U}})$. Indeed, Corollary 1.99 in [10] says in particular that

$$\|g(x+u) - g(x) - Dg(x)[u]\| \leq \sup_{t \in [0,1]} \|Dg(x+tu) - Dg(x)\| \|u\|$$

for every $g \in \mathcal{C}^1(U; Y)$, where U is an open convex subset of a Banach space X and Y is a Banach space, and for every $x \in U$ and every $u \in X$ such that $x + u \in U$. Applying it to the functions f_n, we obtain that

$$\left\|f_n(x+u) - f_n(x) - Df_n(x)[u]\right\| \leq \omega(\|u\|) \|u\| \tag{3.9}$$

for every $x \in U_{\ell_\infty}$ and $u \in \ell_\infty$ such that $x + u \in U_{\ell_\infty}$, and every $n \in \mathbb{N}$. Take $x \in U_{\ell_\infty}$. Define $S \colon \ell_\infty \to (\ell_\infty)_{\mathcal{U}}$ by $S(u) = \left(Df_n(x)[u]\right)_{\mathcal{U}}$ for $u \in \ell_\infty$. It is easy to see that S is a bounded linear operator from ℓ_∞ into $(\ell_\infty)_{\mathcal{U}}$. Let $u \in \ell_\infty$ be such that $x + u \in U_{\ell_\infty}$. Then by (3.9),

$$\begin{aligned}\|f(x+u) - f(x) - S(u)\| &= \left\|\left(f_n(x+u) - f_n(x) - Df_n(x)[u]\right)_{\mathcal{U}}\right\| \\ &\leq \omega(\|u\|) \|u\|.\end{aligned}$$

From Theorem 1.114 in [10] it follows that $f \in \mathcal{C}^{1,+}(B_{\ell_\infty}; (\ell_\infty)_{\mathcal{U}})$ and that the modulus of continuity of Df is $\tau\omega$ for some constant $\tau \geq 1$.

Besides, $\{f(e_k), k \in \mathbb{N}\}$ is not relatively compact in $(\ell_\infty)_{\mathcal{U}}$ as

$$\|f(e_k) - f(e_l)\| = \|(f_n(e_k) - f_n(e_l))_{\mathcal{U}}\| \geq \varepsilon$$

for all $k, l \in \mathbb{N}$, $k \neq l$.

So, we can apply Theorem 3.2.1 to the mapping f. We obtain an infinite set $\mathcal{K}$ of natural numbers and a point $x \in U_{\ell_\infty}$ such that $\operatorname{supp}(x) = \mathcal{K}$ and that $Df(x)|_Z$, where $Z = \{z \in \ell_\infty, \operatorname{supp}(z) \subset \mathcal{K}\}$, is an isomorphism. From the proof of Theorem 3.2.1 it follows that there exists $\xi \geq \frac{5}{32}\varepsilon$ such that $\|Df(x)[z]\| \geq \xi\|z\|$ for every $z \in Z$. Denote $\mathcal{J} \subset \mathcal{K}$ the set of the first m elements of $\mathcal{K}$.

If $\zeta > 0$ satisfies that $\omega(\zeta) < \frac{\xi}{4\tau}$ and that $x + \zeta B_{\ell_\infty} \subset U_{\ell_\infty}$ and if $u \in \zeta S_Z$, then by Corollary 1.99 in [10],

$$\|f(x+u) - f(x)\| \geq \|Df(x)[u]\| - \tau\omega(\|u\|)\|u\| \geq (\xi - \tau\omega(\zeta))\zeta > \frac{3}{4}\xi\zeta.$$

Choose $q \in \mathbb{N}$ so that $\frac{1}{q} < \frac{\xi}{4L}$. Denote

$$Q = \left\{ \sum_{j \in \mathcal{J}} \sigma_j \beta_j e_j, \sigma_j \in \{-1, 1\} \text{ and } \beta_j \in \left\{ 0, \frac{1}{q}\zeta, \ldots, \frac{q-1}{q}\zeta, \zeta \right\} \text{ for all } j \in \mathcal{J} \right\}.$$

For each $v \in Q \cap \zeta S_Z$, the set

$$N_v = \left\{ n \in \mathbb{N}, \|f_n(x+v) - f_n(x)\| > \frac{3}{4}\xi\zeta \right\}$$

belongs to the ultrafilter $\mathcal{U}$. Therefore the intersection $\bigcap_{v \in Q \cap \zeta S_Z} N_v$ is an infinite set. Take $n_0 \in \bigcap_{v \in Q \cap \zeta S_Z} N_v$ such that $n_0 \geq \max \mathcal{J}$. Then, given $u \in \zeta S_{\operatorname{span}\{e_j, j \in \mathcal{J}\}}$, we find $v \in Q \cap \zeta S_Z$ so that $\|u - v\| \leq \frac{1}{q}\zeta$ and obtain that

$$\begin{aligned} \|f_{n_0}(x+u) - f_{n_0}(x)\| &\geq \|f_{n_0}(x+v) - f_{n_0}(x)\| - \|f_{n_0}(x+u) - f_{n_0}(x+v)\| \\ &> \frac{3}{4}\xi\zeta - L\frac{1}{q}\zeta > \frac{1}{2}\xi\zeta. \end{aligned}$$

In view of the Corollary 1.99 in [10] again, for $u \in \zeta S_{\operatorname{span}\{e_j, j \in \mathcal{J}\}}$ we have that

$$\|Df_{n_0}(x)[u]\| \geq \|f_{n_0}(x+u) - f_{n_0}(x)\| - \omega(\zeta)\zeta > \frac{1}{2}\xi\zeta - \frac{1}{4\tau}\xi\zeta \geq \frac{1}{4}\xi\zeta \geq \frac{5}{128}\varepsilon\zeta.$$

Hence, $Df_{n_0}(x)|_{\operatorname{span}\{e_j, j \in \mathcal{J}\}}$ is an isomorphism. This finishes the proof. □

We conclude by deriving a corollary that witnesses the relation between just proved Theorem 3.2.3 and Lemma 6.27 in [10].

Corollary 3.2.4. *Let $\omega \in \mathcal{M}$ with $\omega(1) < \infty$, $L > 0$, $\varepsilon > 0$ and let Y be a Banach space with non-trivial cotype. Then there is an $N(\omega, L, \varepsilon, Y) \in \mathbb{N}$ such that if $n \geq N(\omega, L, \varepsilon, Y)$ and if $f \in \mathcal{C}^{1,+}(B_{\ell_\infty^n}; Y)$ is an $L-$Lipschitz mapping whose derivative Df has modulus of continuity ω, then there exist $i, j \in \{1, \ldots, n\}$, $i \neq j$, for which $\|f(e_i) - f(e_j)\| < \varepsilon$.*

Proof. Suppose that the statement does not hold. Then for every $m \in \mathbb{N}$ there is $n \geq N(\omega, L, m, \varepsilon)$, where $N(\omega, L, m, \varepsilon)$ is the constant obtained in Theorem 3.2.3, and there is $f \in \mathcal{C}^{1,+}(B_{\ell_\infty^n}; Y)$ which is $L-$Lipschitz and such that its derivative Df has modulus of continuity ω and that $\|f(e_i) - f(e_j)\| \geq \varepsilon$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$. So, according to Theorem 3.2.3, Y contains a subspace Y_m isomorphic to ℓ_∞^m. Moreover, from the proof of Theorem 3.2.3 it follows that $d(Y_m, \ell_\infty^m) \leq C(\omega, L, \varepsilon)$, where $d(Y_m, \ell_\infty^m)$ is the Banach-Mazur distance of Y_m and ℓ_∞^m and $C(\omega, L, \varepsilon)$ is a constant which depends on ω, L, ε but does not depend on m. This is by Maurey and Pisier's characterization [18] a contradiction with the non-trivial cotype of Y. □

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