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Descriptive set properties of collections of exceptional sets in Harmonic analysis

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Název práce: Deskriptivní vlastnosti systémů výjimečných množin v harmonické analýze

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Abstrakt: V této práci studujeme systémy malých množin, které se objevují v harmonické analýze. Zvláštní důraz je kladen na množiny jednoznačnosti $\mathcal{U}$ a přidružené systémy $H^{(N)}$, $N \in \mathbb{N}$, U a U_0. Zejména se zaměřujeme na porovnání velikostí těchto systémů, což provádíme pomocí tzv. polár - množin měr, které měří nulou všechny množiny z příslušného systému.

Lyons ukázal, že v tomto smyslu je systém $\bigcup_{N\in\mathbb{N}} H^{(N)}$ menší než U_0. Hlavním cílem této práce je studium otázky, zdali totéž platí, nahradíme-li U_0 podstatně menším systémem U. Za tímto účelem definujeme systém $H^{(\infty)}$ a systémy množin typu N pro $N \in \mathbb{N} \cup \{\infty\}$, a dokazujeme některé jejich vlastnosti, které by mohly přispět k vyřešení dané otázky.

Klíčová slova: množiny jednoznačnosti, deskriptivní teorie množin, harmonická analýza

Title: Descriptive set properties of collections of exceptional sets in Harmonic analysis

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Abstract: We study families of small sets which appear in Harmonic analysis. We focus on the systems $H^{(N)}$, $N \in \mathbb{N}$, U and U_0. In particular we compare their sizes via comparing the polars of these classes, i.e. the families of measures annihilating all sets from given class.

Lyons showed that in this sense, the family $\bigcup_{N\in\mathbb{N}} H^{(N)}$ is smaller than U_0. The main goal of this thesis is the study of the question whether this also holds when the system U_0 is replaced by the much smaller system U. To this end we define a new system $H^{(\infty)}$ and systems of sets of type N where $N \in \mathbb{N} \cup \{\infty\}$. We then prove some of their properties, which might be useful in solving the studied question.

Keywords: sets of uniqueness, descriptive set theory, harmonic analysis

Contents

1 Introduction

In this section we firstly present a brief overview of the theory of sets of uniqueness. We do not attempt to present all of the main results in the theory, which the interested reader can find for example in [KL], but we rather list the notions and problems which are required in order to describe the goal of this thesis. We then explain how the contents of this thesis are organized.

1.1 Brief historical overview

Trigonometric series and the problem of uniqueness:

A trigonometric series on $[0, 2\pi]$ is the formal expression $\sum_{k\in\mathbb{Z}} c_k \exp(kx)$, where $x \in [0, 2\pi]$ and $c_k \in \mathbb{C}$. Such series are often used in harmonic analysis, when we assign to a 2π -periodic (complex and integrable) function f its Fourier series. It is natural to ask whether the coefficients of a given trigonometric series $\sum_{k\in\mathbb{Z}} c_k \exp(kx)$ are unique, or if it is possible to find such $c'_k \in \mathbb{C}$ that we have

$$\forall x \in [0, 2\pi] : \sum_{k\in\mathbb{Z}} c_k \exp(kx) = \sum_{k\in\mathbb{Z}} c'_k \exp(kx),$$

but $(c_k)_{k\in\mathbb{Z}} \neq (c'_k)_{k\in\mathbb{Z}}$. Cantor showed in [Can] that the coefficients of any trigonometric series are indeed uniquely determined by the sum of this series on the whole interval $[0, 2\pi]$, when he proved the following statement: For every trigonometric series, we have

$$\sum_{k\in\mathbb{Z}} c_k \exp(kx) = 0 \text{ for every } x \in [0, 2\pi] \implies \forall k \in \mathbb{Z} : c_k = 0.$$

We can then ask whether we can replace the set $[0, 2\pi]$ in the previous statement by a smaller set E, such that the implication still holds. Note that this question is non-trivial, since for example when $\sum_{k\in\mathbb{Z}} c_k \exp(kx) = 0$ for some $x \in [0, 2\pi]$, this does not necessarily mean that all of the coefficients c_k are equal to zero. It is also not hard to prove (see Proposition 5.3) that whenever $E \subset [0, 2\pi]$ is a set with measure strictly less than 2π[1], then there exists coefficients $c_k \in \mathbb{C}$, $k \in \mathbb{Z}$, not all of them equal to zero, such that $\sum_{k\in\mathbb{Z}} c_k \exp(kx) = 0$ for every $x \in E$.

Definition of the system $\mathcal{U}$:

We say that $E \subset [0, 2\pi]$ is a *set of uniqueness*, denoting $E \in \mathcal{U}$, when for any

[1] Unless stated otherwise, we will assume that "a set of measure m" means "a set of Lebesgue measure m".

trigonometric series we have

$$\sum_{k\in\mathbb{Z}} c_k \exp(kx) = 0 \text{ for every } x \in [0,2\pi]\setminus E \implies \forall k \in \mathbb{Z} : \ c_k = 0.$$

When $E \notin \mathcal{U}$, we say that E is a *set of multiplicity*, writing $E \in \mathcal{M}$. In this notation, Cantor proved that the empty set belongs to $\mathcal{U}$. We note that in this situation it would be more intuitive to say that the set $[0,2\pi]\setminus E$ is of uniqueness, rather than E, but this notation is used from historical reasons...

Notes on the characterization problem and the union problem:

Since the introduction of the concept of $\mathcal{U}$-sets, this topic has received a lot of attention. However the problem of deciding whether a given set E is of uniqueness or of multiplicity turned out to be hard to solve. Finding some "nice" properties of $\mathcal{U}$ also proved to be difficult. For example by theorem of N.K. Bary ([Bar1]) when E and F are closed sets of uniqueness, we have $E \cup F \in \mathcal{U}$, but this does not hold for general E, $F \in \mathcal{U}$ (see Remark 5.6). We still do not know whether this holds for two G_δ sets, nor do we know whether there exist two measurable sets of uniqueness whose union is of multiplicity.

Approximating U by other systems:

We will now restrict ourselves to the system U of *closed sets of uniqueness*, where most of the theory lies. Failing to find a useful characterization of U-sets or at least enough "nice" properties which this system possesses, we can still turn to a different approach. Instead of working directly with the system U, we can "approximate" this collection by different systems $\mathcal{A} \subset U \subset \mathcal{B}$ of closed sets, which are easier to characterize and have better properties. This will partially solve the characterization problem and the problem of finding the properties of U. On the other hand, by working with approximations of U, we have to worry about a different question: How tight are the approximations $\mathcal{A} \subset U$ and $U \subset \mathcal{B}$? In this work we will discuss three ways of measuring the "tightness" of inclusion between two systems $\mathcal{S} \subset \mathcal{T}$, each of them stronger than the previous one. The first one is simply finding out whether *the inclusion $\mathcal{S} \subset \mathcal{T}$ is strict* or not. Then we can also check whether *there exist sets in $\mathcal{T}$ which cannot be covered by countably many sets from $\mathcal{S}$*, i.e. (for hereditary[2] $\mathcal{S}, \mathcal{T}$) whether we have $\mathcal{T} \setminus \mathcal{S}_\sigma \neq \emptyset$. Finally we can check whether *$\mathcal{T}$ is bigger than $\mathcal{S}$ in the sense of polars*, a concept which we define in Section 9.

Examples:

[2]Throughout the work, when working with a family of closed sets, the term "hereditary" will mean "hereditary with respect to *closed* subsets". Similarly when we say a family $\mathcal{F}$ of closed sets is a (σ-) ideal, it will mean that *closed* (countable) unions of $\mathcal{F}$-sets are again in $\mathcal{F}$.

Some examples of collections used for approximating U are the systems

$$H^{(1)} \subset H^{(2)} \subset \ldots \subset \bigcup_{N\in\mathbb{N}} H^{(N)} \subset U' \subset U \subset U_0$$

(which we introduce in more detail later on).

The system U_0:

When a closed set E supports a probability measure μ, such that its Fourier coefficients $\hat{\mu}(k) = \int \exp(-ikx)\,\mathrm{d}\mu(x)$ converge to 0 as $|k| \to \infty$, then clearly $E \in \mathcal{M}$. In this case we say that E is *of strict multiplicity*. When E is not of strict multiplicity, we say that E is a set of *extended uniqueness*. We denote the family of *closed sets of uniqueness* by symbol U_0. Clearly we have $U \subset U_0$ and Piatetski-Shapiro ([PS1]) proved that this inclusion is strict. The fact that U_0 is a σ-ideal is a simple consequence of its alternative definition (which we give in Section 6) and by [Bar1] the system U is a σ-ideal as well. This immediately implies that we have $U_0 \setminus U_\sigma \neq \emptyset$. Finally Kaufman ([Kau2]) proved that the inclusion $U \subset U_0$ is strict also in the sense of polars.

The systems U' and $H^{(N)}$:

Later in this work, we define the systems $H^{(N)}$, $N \in \mathbb{N}$ (see Section 8) and the family U' of *U-sets of rank 1* (Section 6). By a theorem of Piatetski-Shapiro ([PS1]), we have $\bigcup_{N\in\mathbb{N}} H^{(N)} \subset U'$, and as a corollary to the longstanding Borel basis problem ([DSR], or see Theorem 9.4 of this thesis), we have $U \supsetneq U'_\sigma \supset \left(\bigcup_N H^{(N)}\right)_\sigma$.

1.2 The goal and contents of this thesis

The goal of this thesis:

To the best of our knowledge, the question whether the inclusions $\bigcup_{N\in\mathbb{N}} H^{(N)} \subset U' \subset U$ are strict in the sense of polars still remains open. Vlasák recently proved in [Vla] that in the sense of polars, each of the inclusions $H^{(N)} \subset H^{(N+1)}$, $N \in \mathbb{N}$ is strict. The goal of this thesis was to prove his conjecture, which states that we can generalize the concept of $H^{(N)}$-sets and define the so-called $H^{(\infty)}$-sets, which have the following properties:

1. $\bigcup_N H^{(N)} \subset H^{(\infty)}$
2. Many of the $H^{(\infty)}$-sets can be used for witnessing that the inclusion $\bigcup_N H^{(N)} \subset H^{(\infty)}$ is strict in the sense of polars.
3. There exist $H^{(\infty)}$-sets satisfying 2. which belong to U' (and this can be proven by modifying the proof of the inclusion $\bigcup_N H^{(N)} \subset U$).

This would then witness that the inclusion $\bigcup_{N\in\mathbb{N}} H^{(N)} \subset U'$, and actually even the inclusion $\bigcup_{N\in\mathbb{N}} H^{(N)} \subset U'$, is strict in the sense of polars, thus solving the open problem.

Unfortunately we were unable to fully prove this conjecture. To be more specific, we successfully showed that the inclusion $\bigcup_{N\in\mathbb{N}} H^{(N)} \subset H^{(\infty)}$ is strict in the sense of polars and proved that this fact can be witnessed by any $H^{(\infty)}$-set which satisfies certain technical conditions, which are however not too limiting. We then attempted to prove the existence of $H^{(\infty)}$-sets from U', but it turned out that the original proof of inclusion $\bigcup_{N\in\mathbb{N}} H^{(N)} \subset U'$ cannot be modified to get this result, at least not in a direct way. Whether the existence of such sets can be proven in another way remains an open question.

Outline of the thesis:

We assume that the reader is familiar with basics of the descriptive set theory and knows the basic properties of the Fourier transform $\hat{}: L^1([0,2\pi]) \to c_0(\mathbb{Z})$. The facts from these two areas which we will use in this thesis can be found in the appendix Sections A.1 and A.2. We also refer to the appendix for some information on Hausdorff dimension, Cantor-Bendixson rank and Bernstein sets.

The main text is then organized as follows: We begin the Part I by introducing in Section 2 the notation which we are going to use. In Section 3 we present some examples of families of small sets which naturally appear in mathematical analysis. We then observe some of the common properties these systems have, which allows us to better understand what kind of results we can expect from the families $\mathcal{U}$ and U. In the last Section 4 of this part we then list the key known results related to the systems $\mathcal{U}$ and U and highlight some of the problems which are still open.

In Part II we discuss the sets $\mathcal{U}$ and U in more detail and define more of the related notions. We also include proofs for those theorems which are either relevant to our goal (i.e. the question whether the inclusion $\bigcup_{N\in\mathbb{N}} H^{(N)} \subset U'$ is strict in the sense of polars), or whose proofs are interesting from some other reasons. In Section 5 we prove the basic properties of the families $\mathcal{U}$ and U and in the next Section 6 we apply some of the tools from functional analysis to the theory of $\mathcal{U}$-sets, which allows us to define the collection U'. In the following two Sections 7 and 8 we introduce a few types of the so-called symmetric sets and define the $H^{(N)}$-sets. We also explore the properties of symmetric sets and $H^{(N)}$-sets and discuss the relation of these two families. In the last Section 9 of this part we define what it means for an inclusion between two families to be strict in the sense of polars. We then summarize the known results related to the "approximation problem" for U and highlight some open questions related to this topic.

In Part III we present our results, all of which are novel. We note that they are mostly inspired by the techniques used in Vlasák's proof of the fact that the inclusion $H^{(N)} \subset H^{(N+1)}$ is strict in the sense of polars ([Vla]). In the Section 10 we define the family $H^{(\infty)}$ and prove some of its properties. We then observe the similarity between the subfamily of "regular" $H^{(\infty)}$-sets and a certain family of symmetric sets. Lastly we give a few notes which explain the difficulties we had with attempts at finding $H^{(\infty)}$-sets of uniqueness.

In Section 11 we define the families of "sets of type N" for $N \in \mathbb{N} \cup \{\infty\}$ which generalize the families $H^{(N)}$ and $H^{(\infty)}$. We also define the system of L-sets of type N and regular sets of type N. Using the technique from [Vla] we prove the main theorem of this thesis, which states that every regular set of type $N \in \mathbb{N} \cup \{\infty\}$ supports a measure which measures every L-set of type $< N$ by zero. As a corollary of this theorem we get the fact that the inclusion $\bigcup_N H^{(N)} \subset H^{(\infty)}$ is strict in the sense of polars.

Part I

$\mathcal{U}$ as a family of thin sets

2 Notation

The unit sphere $\mathbb{T}$:

- By $\mathbb{T}$ we will denote the unit sphere $\{z \in \mathbb{C}|\ |z| = 1\}$ endowed with the topology inherited from $\mathbb{C}$. Note that we can identify $\mathbb{T}$ with the sphere in $\mathbb{R}^2$ via the mapping $a + ib \mapsto (a, b)$, or with the interval $[0, 2\pi)$ via the mapping $x \in [0, 2\pi) \mapsto e^{ix} \in \mathbb{T}$. We can also imagine $\mathbb{T}$ as the interval $[0, 2\pi]$ with points 0 and 2π identified. Using the mapping $x \in [0, 2\pi] \mapsto x/2\pi \in [0, 1]$ we can also identify $\mathbb{T}$ with the intervals $[0, 1]$ or $[0, 1)$. In all of the cases we will work with the topology received from the identification of $\mathbb{T}$ with a subspace of $\mathbb{C}$.

- Let $x, y \in [0, 2\pi)$. By $+_{\mathbb{T}}$ (or simply $+$) we will denote the *additive operation* on $\mathbb{T}$ defined as
$$x +_{\mathbb{T}} y := (x +_{\mathbb{R}} y) \mod 2\pi.$$
For $x \in [0, 2\pi)$ and $c \in \mathbb{R}$ we will define the *multiplication* on $\mathbb{T}$ by the formula
$$c \cdot_{\mathbb{T}} x := c \cdot x := cx := (c \cdot_{\mathbb{R}} x) \mod 2\pi.$$

Sequences:

- For sequences indexed by integers we will use the notation $x = (x_n)_{n=1}^{\infty} = (x_1, x_2, ...)$ resp. $x = (x_i)_{i=m}^{n} = (x_m, ..., x_n)$. For general sequences we write $x = (x_i)_{i \in I}$, where I is the index set. We will understand sequences as functions from the index set, which allows us to use the *restriction* operator $\restriction$. Sometimes when it is clear from the context over which set is the sequence indexed or which variable is used for indexing, we will omit these, writing simply $(x_i)_i$, $(x_i)_I$ or (x_i) instead of $(x_i)_{i \in I}$.

- When $x = (x_n)_{n=1}^{n_0}$ and $y = (y_n)_{n=1}^{n_1}$ are two sequences, where $n_0 \in \mathbb{N}$, $n_1 \in \mathbb{N} \cup \{\infty\}$, we will denote by $x\hat{\ }y$ the *concatenation of x and y* defined as
$$x\hat{\ }y = (x_1, ..., x_{n_0}, y_1, y_2, ...).$$

Binary operations:

- Let X, Y, Z be sets, $x \in X$, $S \subset Y$, $T \subset X$ and let $R : X \times Y \to Z$ be a binary operation. By xRS we will denote the set $\{xRs |\, s \in S\}$. We also set $TRS := \bigcup \{tRS |\, t \in T\}$. When there is no risk of confusion (e.g. R is the multiplication on $\mathbb{R}$ or $\mathbb{T}$) we will omit the symbol R and write simply xS instead of xRS.

- Assume that there is some canonical operation $+$ defined on X and that we have defined multiplication $\cdot$ of elements of X by real numbers. By a *shift* or *translation* of a set T we will then mean the set $x + T$ for some $x \in X$ and by a *dilatation* (resp. *contraction*) we will mean a set $r \cdot T$ for some $r > 1$ (resp. $r \in (0, 1)$).

- When $\vec{v} = (v_1, ..., v_n) \in Y^n$ is a vector, $x \in X$ and R is as above, we denote $xR\vec{v} := (xRv_1, ..., xRv_n)$. When we have $x, y \in \mathbb{R}^d$ we will denote by $x \cdot y$ or also xy the standard *scalar product* $x \cdot y = x_1y_1 + ... + x_ny_n$.

Miscellaneous:

- When $d \in \mathbb{N}$ and a set $S \subset \mathbb{R}^d$ is measurable, we will denote by $|S|$ the *d-dimensional Lebesgue measure* of S.

- Let X be a set. By $\mathcal{P}(X)$ we denote the *power set* of all subsets of X. When $\mathcal{S} \subset \mathcal{P}(X)$ and $S \in \mathcal{S}$, we say that the set S is an $\mathcal{S}$*-set*. By $\mathcal{S}_\sigma$ we denote the σ*-closure of* $\mathcal{S}$, defined as

$$\mathcal{S}_\sigma := \left\{ S \in \mathcal{P}(X) \,|\, \exists (S_n) \subset \mathcal{S} : \, S = \bigcup_{n=1}^{\infty} S_n \right\}.$$

  When $\mathcal{S}$ is finite, we denote by $\#\mathcal{S}$ the cardinality of $\mathcal{S}$. By a *countable* set we will understand a set which is at most countable, i.e. "countably infinite or finite".

- When $f : X \to \mathbb{R}$ is a function and $r \in \mathbb{R}$, we denote $\{f = r\} := \{x \in X |\, f(x) = r\}$ and define $\{f < r\}$, $\{f \leq r\}$ etc. analogically.

3 General families of thin sets

3.1 Examples of families of thin sets

First, we give some examples of families of small sets which naturally appear in various areas of mathematics.[3] One of them will be the sets of uniqueness, in

[3] Another important example of small sets is the class of Haar-null sets. However, we avoid discussing it in this work, as we mostly work in $\mathbb{T}$ or in $\mathbb{R}^n$ where the standard Lebesgue measure

which we will be interested in the remaining part of this thesis.

- $[X]^{<\omega}$ - the system of all finite subsets of a set X.
- $[X]^{\leq\omega}$ - the system of all at most countable subsets of a set X.
- $\mathcal{L}(X)$ - the negligible sets or (Lebesgue-) null sets, i.e. the subsets of $X \subset \mathbb{R}^n$ which are of Lebesgue measure zero. More generally, we can consider μ-null sets for general Radon measure on X.
- $NWD(X)$, $MGR(X)$ - nowhere dense and meager subsets of a topological space X.
- $\{F \leq \epsilon\}$, $\{F < \epsilon\}$, $\{F = \epsilon\}$ for $F : \mathcal{P}(X) \to [0, \infty)$ or $F : \mathcal{P}(X) \to On$ - the sets S for which $F(S)$ is small. For example the sets of small diameter, measure, cardinality, Hausdorff dimension[4] or Cantor-Bendixson rank[5].
- $\mathcal{U}$ - the sets of uniqueness on the unit circle $\mathbb{T}$. By definition, a set $S \subset \mathbb{T}$ is a set of uniqueness if it has the following property: whenever a trigonometric series $\sum_{k\in\mathbb{Z}} c_n e^{ikx}$ converges to 0 for all $x \in \mathbb{T} \setminus S$, then $c_k = 0$ for all $k \in \mathbb{Z}$.
- U - the family of all sets of uniqueness which are closed.
- U_0 - the closed sets of extended uniqueness, which we define later.

3.2 Basic properties

We observe that most of these families $\mathcal{F} \subset \mathcal{P}(X)$ have some, or even all, of the following properties:

1. $\emptyset \in \mathcal{F}$, $X \notin \mathcal{F}$ - non-triviality,
2. $S \in \mathcal{F}, T \subset S \implies T \in \mathcal{F}$ - being hereditary with respect to inclusion,
3. $S, T \in \mathcal{F} \implies S \cup T \in \mathcal{F}$ - closure under finite unions,
4. $S_n \in \mathcal{F}$ for $n \in \mathbb{N} \implies \bigcup_n S_n \in \mathcal{F}$ - closure under countable unions.

Definition 3.1. As in [BKR], we say that $\mathcal{F} \subset \mathcal{P}(X)$ is a *family of thin sets*, if it satisfies the first two conditions. If it also satisfies the condition 3, it is said to be an *ideal.* If all of the conditions are satisfied, it is said to be a *σ-ideal.*

is available. We also choose not to discuss the families of porous and σ-porous sets.

[4]See Section A.3 for definition and some details on d-dimensional Hausdorff measures $\mathcal{H}_d$ and Hausdorff dimension $\dim_{\mathcal{H}}$.

[5]For definition see Section A.4.

3.3 Motivation - which questions to ask?

The questions: When one encounters a family of thin sets, it is natural to ask the following questions:

- Is $\mathcal{F}$ an ideal? Is it a σ-ideal?
- Does $\mathcal{F}$ contain all singletons?
- What is the relation of $\mathcal{F}$ to other important families of thin sets? For example, which of the σ-ideals $\mathcal{L}$, MGR, and $[X]^{\leq\omega}$ are contained in $\mathcal{F}$ and vice versa.
- For $S \in \mathcal{F}$, does there always exist a "nice" (e.g. closed) set T with $S \subset T \in \mathcal{F}$?
- Is $\mathcal{F}$ closed under some other interesting operations, such as shifts $S \mapsto S + x$, dilatations $S \mapsto \alpha S$ or more generally, images under isometries or homeomorphisms.
- Is an "easy way" to tell whether a given set belongs to $\mathcal{F}$? Naturally, we already have some definition of $\mathcal{F}$, so we are looking for something simpler than this definition.

Example: For example, the negligible sets $\mathcal{L}$ form a σ-ideal, they contain all singletons and thus also countable sets. On the other hand there exist discontinua in $[0, 1]$ of positive Lebesgue measure (we discuss this later in Section 7) and such sets are meager. Consequently $\mathcal{L}$ does not contain MGR. Whenever S is a negligible set, by outer regularity of Lebesgue measure, we can find G_δ set $G \supset S$ which is also of measure zero. Also, $\mathcal{L}$ is closed under isometries. but not under homeomorphisms. Lastly, given a set S, we can use the regularity of Lebesgue measure to either find $\epsilon > 0$ and compact $K \subset S$ of measure at least ϵ witnessing that $S \notin \mathcal{L}$, or we find for each $\epsilon > 0$ an open set $G \supset S$ of measure at most ϵ, thus proving that $S \in \mathcal{L}$. The last question is rather vague, but the characterization of $\mathcal{L}$-sets we just described seems to be an example of the kind of "easier to work with" condition we were looking for.

In case of negative answer: Finally, whenever the answer to one of the above questions is negative, we usually ask under which conditions would the answer be positive. For example, if $\mathcal{F}$ is not a σ-ideal, what are the properties of the smallest σ-ideal $\mathcal{F}'$ containing $\mathcal{F}$? Is there a "nice" σ-ideal $\mathcal{F}'' \subset \mathcal{F}$ not "much smaller" than $\mathcal{F}$? $\mathcal{F}$ might not be closed under isometries, but what about $\mathcal{F}'$ and $\mathcal{F}''$? A good example of how this approach can be useful are the families NWD and MGR. A different direction to take would be to relax the conditions asked in our questions. In our example of negligible sets and closure under images under homeomorphisms, we know the answer would become positive, if we restricted ourselves to Lipschitz mappings. Another example would be the family $\mathcal{F} = \{\mu \leq \epsilon\}$ of sets of small measure for some measure μ. $\mathcal{F}$ is generally not hereditary with respect to inclusion, since there exist non-measurable sets. But if we only ask the μ-measurable subsets of $\mathcal{F}$-sets to be in $\mathcal{F}$, we will avoid such problems.

4 $\mathcal{U}$ as a family of thin sets

4.1 $\mathcal{U}$-sets and some negative results

Definition 4.1. Trigonometric series on $\mathbb{T}$ with coefficients $c_k \in \mathbb{C}$, $k \in \mathbb{Z}$ is a sum $\sum_{k\in\mathbb{Z}} c_k e^{ikx}$, $x \in \mathbb{T}$. We say that $E \subset \mathbb{T}$ is a set of *uniqueness*, writing $E \in \mathcal{U}$, if it has the following property: Whenever a trigonometric series $\sum c_k e^{ikx}$ converges to 0 for every $x \notin E$, then necessarily $c_k = 0$ for each $k \in \mathbb{Z}$. If $M \subset \mathbb{T}$ is not a set of uniqueness, we say that it is a set of *multiplicity*, writing $M \in \mathcal{M}$.

In Subsection 3.3 we noted a number of questions relevant to $\mathcal{U}$-sets. The simple answers to these questions are summarized in the following remark. We include it now from motivational reasons - most of the individual points of the remark will be stated and proved later on.

Remark 4.2 (Properties of $\mathcal{U}$ - simple answer). The family $\mathcal{U}$ has the following properties:

1. $\mathcal{U}$ is a family of thin sets ([Can]).
2. $\mathcal{U}$ is not an ideal (Remark 5.6).
3. Shifts of sets from $\mathcal{U}$ are again of uniqueness (straightforward). The system $\mathcal{U}$ is not closed under dilatations ([BKR, p. 481], see also Remark 5.6).
4. $\mathcal{U}$ contains every countable set ([You]). No inclusion holds between $\mathcal{U}$ and MGR or $\mathcal{L}$-sets (D. E. Menshov, see also Lemma 7.6 and Example 7.4 combined with Theorem 8.12).
5. For every $d \in (0, 1]$ there exist both $\mathcal{U}$-sets and $\mathcal{M}$-sets of Hausdorff dimension d (Example 10.3).

Remark. As mentioned earlier, the ”characterization” problem is rather vaguely stated, but as of now, no ”nice” characterization of $\mathcal{U}$-sets has been found.

4.2 U-sets and some positive results

For general sets of uniqueness, we only have a few positive results. In order to better characterize the sets in $\mathcal{U}$ for which some interesting results hold, a fair number of auxiliary families of sets were defined and relations between these families were studied. We focus on them in the following sections of this work. Now we define the family of closed sets of uniqueness and formulate its properties. When combined with the Remark 4.2, these properties give a somewhat more complete answer to the questions presented in Subsection 3.3.

Definition 4.3. Denote by $\mathcal{K}(\mathbb{T})$ the hyperspace of all compact subsets of $\mathbb{T}$ endowed with the Vietoris topology (see Appendix A.1 for definition). We define the system U of *closed sets of uniqueness* as $U = \mathcal{U} \cap \mathcal{K}(\mathbb{T})$ and the system M of *closed sets of multiplicity* as $M = \mathcal{K}(\mathbb{T}) \setminus \mathcal{U}$.

Remark 4.4 (Properties of U). The family U has the following properties:

- U is a σ-ideal of closed sets ([Bar1]).
- U is closed under shifts and dilatations ([KL, p. 180].
- Every U-set is both of measure zero and meager (in fact this holds for every measurable $\mathcal{U}$-set, resp. for every $\mathcal{U}$-set with the Baire property) (see e.g. Proposition 5.3 for the first proposition (which is straightforward) and [DSR] for the second).
- There exist closed null sets sets (and thus also closed meager sets) which are not in U (same as in Remark 4.2).
- U, as a subspace of $\mathcal{K}(\mathbb{T})$, is Π^1_1-complete (R. M. Solovay and independently [Kau1]).
- For every $d \in [0,1]$ (resp. $(0,1]$) there exists a set in U (resp. M) of Hausdorff dimension d (same as in Remark 4.2).

As the proposition shows, there are certain advantages to this approach - for one, the class of closed sets of uniqueness has much better properties than $\mathcal{U}$, avoiding pathologies such as non-measurable sets, leading to some positive results. Secondly, we now consider U as a subset of a Polish space $\mathcal{K}(\mathbb{T})$, which allows us to compute its complexity, showing that U is Π^1_1-complete. This explains why no "simple" description of U has been found - a "simple enough" description of U would imply that it is in fact Borel. In some sense, this result gives a negative answer to the problem of characterizing which sets are of uniqueness. This is something we could not have done with the whole class $\mathcal{U} \subset \mathcal{P}(\mathbb{T})$, because $\mathcal{U}$ is too large to be embedded in any Polish space and thus no notion of complexity is defined.

4.3 Open questions

The Propositions 4.2 and 4.4 partially answered the questions stated in Subsection 3.3. The two propositions however still leave some gaps to be filled. Specifically, the following questions are still open, at least to the best of our knowledge:

1. (Union problem) For which $E, F \in \mathcal{U}$ is $E \cup F \in \mathcal{U}$? This question is open even when both E and F are G_δ (or measurable).

2. (Interior problem) For given $E \in \mathcal{M}$, can we always find a closed set $F \subset E$ of multiplicity? Again, this is open even for G_δ-sets.

3. (Characterization problem) Find a "nice" necessary and sufficient condition, telling us whether a given perfect set E is of uniqueness or of multiplicity.

4. Are there "nice" families $\mathcal{A}$, $\mathcal{B}$ which approximate $\mathcal{U}$ (resp. U) well, in the sense that $\mathcal{A} \subset \mathcal{U} \subset \mathcal{B}$ and these inclusions are "not too strict"?

Part II

Properties of $\mathcal{U}$-sets and related systems

In this chapter, we will establish some of the classical notions related to the sets of uniqueness, while also explaining in more detail the properties of families $\mathcal{U}$ and U. However, the range of results in the theory of sets of uniqueness is very extensive, so we will focus on defining the following notions and proving the following properties, mostly in this order:

1. $\mathcal{U}$, U and the property of being ideal or σ-ideal,
2. Lebesgue measure and $\mathcal{U}$ sets,
3. countable sets and $\mathcal{U}$ sets,
4. Rajchman measures and U_0, the closed sets of extended uniqueness,
5. application of functional analysis to U, introduction of family U',
6. symmetric sets, $H^{(N)}$-sets and their relation to U,
7. the inclusions between the families $H^{(N)}$, U', U and U_0,
8. bases of σ-ideals and the relative sizes of the families $H^{(N)}$, U', U and U_0,
9. polars, p-bases of σ-ideals and the families $H^{(N)}$, U', U and U_0.

In particular, we avoid the discussion of the question whether $\mathcal{U}$ and U are closed under shifts or dilatations. We also focus mostly on the closed sets, leaving out notions such as $\mathcal{U}_0$ (the general version of U_0-sets), or further discussion of the union problem.

5 Basic properties of $\mathcal{U}$ and U

5.1 Basic examples of $\mathcal{U}$-sets

Theorem 5.1 ([Can]). *$\mathcal{U}$ is a family of thin sets.*

Proof. It is clear from the definition of $\mathcal{U}$ that it is hereditary with respect to inclusion. To observe that the whole set $[0, 2\pi]$ is not in $\mathcal{U}$, one can simply consider the constant function 1 and note that its Fourier transform is not a zero sequence. The remaining non-trivial fact that the empty set is a set of uniqueness is due to Cantor. □

Proposition 5.2 ([Can]). *$\mathcal{U}$ contains every singleton.*

Remark. Cantor actually proved a stronger result that every countable closed set of finite Cantor-Bendixson rank is a set of uniqueness.

Proposition 5.3. *Measurable $\mathcal{U}$-sets are of measure zero.*

Proof. Let $E \subset [0, 2\pi)$ be a set of positive measure. We will show that $E \in \mathcal{M}$. Let $K \subset E$ be a compact set of positive measure and consider the function $f = \chi_E$. By the standard Rieman's localization principle, we know that f vanishes on a neighborhood of every point $x \notin K$, therefore $S(f)$ converges to 0 at such points. In particular, $S(f)$ converges to 0 outside E. On the other hand, we have $\hat{f}(0) = |K| > 0$ and thus $\hat{f} \neq 0$. Consequently, f witnesses that E is a set of multiplicity. □

Proposition 5.4 (W.H.Young). *If $E \subset \mathbb{T}$ contains no perfect set, then E is a set of uniqueness.*

Proof. Let E be a set of multiplicity. Then there exists a nonzero trigonometric series $\sum c_k e^{ikx}$ with $\sum c_k e^{ikx} = 0$ on $\mathbb{T} \setminus E$. We denote

$$B := \mathbb{T} \setminus \left\{ x \in \mathbb{T} | \sum c_k e^{ikx} = 0 \right\} \subset E.$$

The series $\sum c_k e^{ikx}$ witnesses that $B \notin \mathcal{U}$ and thus by Corollary 5.9 B is uncountable. Clearly B is a borel set, and by Perfect set theorem, every uncountable Borel set contains a perfect set. Since E was arbitrary, we have shown that each set of multiplicity contains a perfect set, which proves the proposition. □

Example 5.5. Bernstein[6] sets are $\mathcal{U}$-sets.

Proof. This follows directly from the previous proposition and the fact that Bernstein sets contain no perfect subsets. □

5.2 Being ideal and σ-ideal

Remark 5.6. (1) $\mathcal{U}$ is not an ideal.

(2) There exists $x \in \mathbb{R}$ and $E \in \mathcal{U}$ such that $xE \notin \mathcal{U}$. In other words $\mathcal{U}$ is not closed under dilatations.

[6]For the definition of a Bernstein set, see Section A.5.

Proof. (1) Let E be a Bernstein set. By Example 5.5, both E and E^C are in $\mathcal{U}$. However, since $\mathcal{U}$ is a family of thin sets, we have $E \cup E^C = [0, 2\pi] \notin \mathcal{U}$, witnessing that $\mathcal{U}$ is not an ideal.

(2) This can be witnessed by the set $F = \left(\frac{1}{2}E\right) \cup \left(\frac{\pi}{2} + \frac{1}{2}E^C\right)$, which satisfies $2F = [0, 2\pi] \in \mathcal{M}$ (where E is as above). For details, see [BKR, p. 481]. □

Theorem 5.7 ([Bar1])**.** *Countable union of closed* $\mathcal{U}$*-sets is in* $\mathcal{U}$*.*

Corollary 5.8. *U is a σ-ideal of closed sets.*

Corollary 5.9. *$\mathcal{U}$ contains every countable set.*

Proof. Use Theorem 5.7 and Proposition 5.2. □

6 Applications of functional analysis in the theory of $\mathcal{U}$-sets

6.1 Spaces A, PF and PM, ideal $J(E)$

Remark 6.1 (Identification of $l^1(\mathbb{Z})$ and A). By the properties of Fourier transform, we can identify the space $l^1 = l^1(\mathbb{Z})$ with the subspace $A = \left\{ f \in \mathcal{C}(\mathbb{T}) \,|\, \hat{f} \in l^1 \right\}$ of the space $\mathcal{C}(\mathbb{T})$ via the bijection $f \mapsto \hat{f}$ (See Section A.2 for details). On A we consider the norm induced by the identification with l^1. Recall as well that $\widehat{fg} = \hat{f} * \hat{g}$ and that the space l^1 with convolution is a Banach algebra.

For any $f \in \left\{ f \in \mathcal{C}(\mathbb{T}) \,|\, \hat{f} \in l^1 \right\}$ the mapping $(c_k) \mapsto \sum_{k\in\mathbb{Z}} \hat{f}(k)\, c_k$ is clearly a continuous linear functional on $c_0 = c_0(\mathbb{Z})$, and for any $(b_k) \in l^\infty = l^\infty(\mathbb{Z})$ the mapping $f \mapsto \sum_{k\in\mathbb{Z}} b_k \hat{f}(k)$ is a continuous linear functional on the space $(\{g \in \mathcal{C}(\mathbb{T}) \,|\, \hat{g} \in l^1\}, \|\|)$ with the norm $\|g\| := \|\hat{g}\|_{l^1(\mathbb{Z})}$.

This leads to the following definition:

Definition 6.2. We denote by A (=$A(\mathbb{T})$) the Banach algebra of all *continuous functions* (on $\mathbb{T}$) *with absolutely convergent Fourier series.* On A, we consider the norm $\|f\|_A = \left\|\hat{f}\right\|_{l^1}$ and the standard pointwise multiplication of functions.

By PF we denote space of trigonometric series which have coefficients in c_0. Identifying PF with $(c_0, \|\|_\infty)$, we see that it is a predual of A (with duality $\langle S, f\rangle_{(PF,A)} := \left\langle S, \hat{f}\right\rangle_{(c_0,l^1)}$, $S \in PF$, $f \in A$). Similarly we denote by PM those trigonometric series which have l^∞ coefficients and identify this space with $l^\infty = l^\infty(\mathbb{Z})$ with the standard norm. PM is then the dual space to A, using the duality $\langle f, S\rangle_{(A,PM)} := \left\langle \hat{f}, S\right\rangle_{(l^1,l^\infty)}$, $f \in A$, $S \in PM$.

Proposition 6.3. $\mathcal{C}^1(\mathbb{T}) \subset A$.

Proof. This is an immediate consequence of [KL, Proposition II.1.1], which states that for absolutely continuous $f \in \mathcal{C}(\mathbb{T})$, we have $f' \in L^2(\mathbb{T}) \implies f \in A$. $\square$

Definition 6.4. For $E \subset \mathbb{T}$ we define the ideal $J(E)$ of functions from A which vanish on some open neighborhood of E:

$$J(E) = \{f \in A |\, f = 0 \text{ on } V \text{ for some } V \supset E \text{ open}\}.$$

Remark. Clearly $J(E)$ is a linear subspace of A. Recall that A is a Banach algebra with the standard pointwise multiplication of functions. Consequently $J(E)$ is closed under multiplication by functions from A, which justifies the word "ideal" in the previous definition.

6.2 The sets of extended uniqueness

Definition 6.5. By $\mathcal{R}$ we denote the set of *Rajchman measures*

$$\mathcal{R} = \left\{\mu \in \mathcal{M}(\mathbb{T}) \,|\, \hat{\mu}(n) \overset{|n|\to\infty}{\longrightarrow} 0\right\}.$$

We then define the *closed sets of extended uniqueness* as

$$U_0 = \{E \in \mathcal{K}(\mathbb{T}) \,|\, \mu(E) = 0 \textit{ for every } \mu \in \mathcal{R}\}$$

and *closed sets of restricted multiplicity* $M_0 = \mathcal{K}(\mathbb{T}) \setminus U_0$.

Remark 6.6. The family U_0 has the following properties:

1. $U_0 \supset U$,
2. U_0 is a σ-ideal,
3. every U_0-set is of measure zero.

Moreover, the family $\mathcal{R}$ satisfies $A \subset \mathcal{R} = PF \cap \mathcal{M}(\mathbb{T}) \subset PF$.

Proof. **1.** By [KL, Proposition II.6.5] the new definition of U_0 is equivalent with the one given in the introduction (page 3). This immediately implies that $U_0 \supset U$ (alternative proof using the later definition can be found in [KL, Proposition II.6.3]).

2. From Definition 6.5 it is clear that it is a σ-ideal of closed sets.

3. By Riemann-Lebesgue lemma the Lebesgue measure is a Rajchman measure, which gives the result.

Note as well that Rajchaman measures are those measures μ on $\mathbb{T}$, for which $\hat{\mu}$ is a pseudofunction, i.e. $\mathcal{R} = PF \cap \mathcal{M}(\mathbb{T})$. In particular we have $\mathcal{R} \supset A$, since $A \subset PF$ and $A \subset \mathcal{C}(\mathbb{T}) \subset \mathcal{M}(\mathbb{T})$. $\square$

6.3 Characterization of U and the definition of U'

Theorem 6.7 ([PS2]). *Let $E \subset \mathbb{T}$ be a closed set. Then E is in U if and only if the ideal $J(E)$ is w^*-dense in A.*

Recall here the w^*-topology on A is not metrizable and thus a closure of a set $S \subset A$ is equal to the set of all limits of nets of points from S. This however, is in general not the same as taking just all the limits of countable sequences of points from S.

Definition 6.8. We define the family U' of *closed sets of uniqueness of rank less or equal to 1* as

$$U' = \{E \in U |\, J(E) \text{ is } w^*\text{-sequentionally dense in } A\}.$$

Remark. We will not use the notion of rank in this thesis, so we refer the interested reader to, for example [KL, Chapter V]. We just note here that the only set of rank strictly less than 1 is the empty set, so it is correct to say that a nonempty set E is of rank 1 whenever $E \in U'$.

Remark 6.9. When working with the family U', it is useful to keep in mind the following simple observation

$$\begin{aligned} E \in U' \quad &\iff \quad 1 \in w^*\text{-sequential closure of } A \\ &\iff \quad \exists f_n \in A \text{ with } \operatorname{supp}(f_n) \subset \mathbb{T} \setminus E \text{ and } \sup \|f_n\|_A < \infty \\ &\qquad\quad \text{satisfying } \hat{f}_n(0) \to 1 \text{ and } \hat{f}_n(k) \to 0 \text{ for } k \neq 0. \end{aligned}$$

The first equivalence follows immediately from the fact that $J(E)$ is an ideal and multiplication on A is continuous, while the second equivalence is simply the description of w^*-convergence of sequences in l^1.

7 Symmetric sets

7.1 General construction

Remark 7.1. In mathematics, we often come across the following general construction of a set in $\mathbb{R}^d$. We have a bounded $B \subset \mathbb{R}^d$ and a sequence of sets $E_n \subset B$, $n \in \mathbb{N}$. We then construct a new set E as $E = \bigcap_n E_n$. If the sets E_n are closed in $\mathbb{R}^d$ and they form a centered system, E is a nonempty compact set. Later in Section 11, we will study a more general case, but for now, we assume that $d = 1$, B is either $[0, 1]$ or $[0, 2\pi]$ and the sets E_n are finite unions of closed intervals with disjoint interiors.

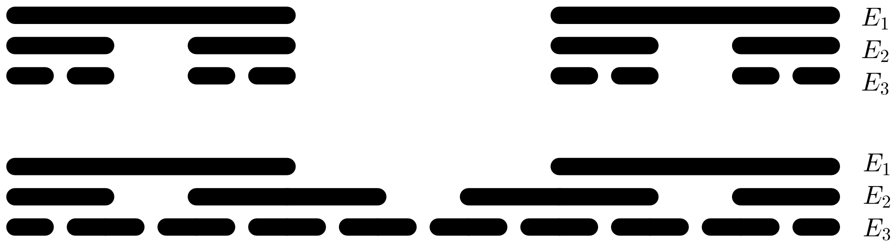

Figure 1: Cantor set $C = \bigcap E_n$ constructed in the classical way (up) and its $H^{(1)}$-representation (down).

In Sections 7 and 8, we focus on symmetric sets and $H^{(N)}$-sets, $N \in \mathbb{N}$, which are both of this type[7]. As the name suggests, the sets E_n of a symmetric set E will be somehow symmetric or "regular". Typical example of a symmetric set is the Cantor set (Figure 1). This set is also an example of a $H^{(1)}$-set. For higher N, the $H^{(N)}$-sets no longer have to be so symmetric, but they enjoy other properties instead. In the following sections, we will define these families, show that some symmetric sets and all of the $H^{(N)}$-sets are sets of uniqueness and finally state the Salem-Zygmund theorem, which in particular implies that certain symmetric sets are also in the family $H^{(N)}$.

7.2 The definition

Definition 7.2 (Symmetric set of constant dissection ratio (taken from [BKR])). For real numbers $a < b$ and $\xi \in \left(0, \frac{1}{2}\right)$, performing a *dissection of type* ξ *on* $[a, b]$ means replacing $[a, b]$ by the union of two closed intervals $[a, a_1]$ and $[b_1, b]$ of lengths $\xi\,(b - a)$.

Let $(\xi_n)_n$ be a sequence with $\xi_n \in \left(0, \frac{1}{2}\right)$ for $n \in \mathbb{N}$. A *symmetric perfect set with dissection ratios* ξ_n, $n \in \mathbb{N}$ is a set $E_{\xi_n} = \cap E_n$, where $E_0 = [0, 2\pi]$ and whenever E_{n-1} is a disjoint union of closed intervals I_k, we receive E_n by performing a dissection of type ξ_n on every I_k. If the sequence$(\xi, \xi, \ldots)$ is constant, we write simply E_ξ and we call such set a *symmetric perfect set of constant ratio of dissection* ξ.

Example 7.3. $E_{\frac{1}{3}}$ is the classical Cantor set in $[0, 2\pi]$.

Remark. When working with dissections of intervals, there are three important variables - the dissection ratio ξ, i.e. the (relative) measure of the remaining intervals, the number 2ξ - i.e. the (relative) measure of remaining set and $1 - 2\xi$, i.e. the (relative) measure of the set we removed. To make the matters worse, some authors index the symmetric perfect sets by the first of the mentioned

[7]To be more precise, each $H^{(N)}$-set is contained in a closed $H^{(N)}$-set which is of this type.

variables, as we do here, while others index them by the last variable and both notations coincide for (the) Cantor set.

Example 7.4. E_{ξ_n} is of measure zero if and only if the sum $\sum_{n=1}^{\infty}(1-2\xi_n)$ diverges. In particular there exist symmetric perfect sets of positive measure (and these are consequently in neither of the classes U and U_0).

Proof. Clearly $E_1 \supset E_2 \supset ... \supset E_{\xi_n}$. By induction we get that (the normalized) measure of E_n is $\prod_{k=1}^{n} 2\xi_k$, which is equal to $\prod_{k=1}^{n}(1-\epsilon_k)$, where $\epsilon_k = 1-2\xi_k$. Since $\epsilon_k \in (0,1)$, we get that $|E_{\xi_n}| = \prod_{n=1}^{\infty}(1-\epsilon_n)$ is equal to zero if and only if the sum $\sum_{n=1}^{\infty}\epsilon_n$ is infinite. □

Definition 7.5 (Homogeneous perfect set)**.** Let ξ,$\vec{\eta} = (\eta_0, ..., \eta_k)$ be numbers satisfying $0 = \eta_0 < \eta_1 < ... < \eta_k = 1$, $k > 1$, $\xi = 1-\eta_{k-1}$ and $\eta_i > \eta_{i-1} + \xi$ for each $i < k$. By performing a *dissection of type* $(\xi, \vec{\eta})$ *on* $[a, b]$, we mean replacing $[a, b]$ by the disjoint union of closed intervals $[a_i, b_i]$, $i = 0, .., k-1$ of lengths $\xi\,(b-a)$, where $a_i = (1-\eta_i)\,a + \eta_i b$.

Let $(\xi_n, \vec{\eta}_n)_n$ be a sequence of numbers $\xi_n \in \left(0, \frac{1}{2}\right)$ and vectors $\vec{\eta}_n = (\eta_{n,0}, ..., \eta_{n,k_n})$, $n \in \mathbb{N}$, each pair satisfying the above conditions. We generalize the Definition 7.2 and define a *symmetric perfect set* $E_{\xi_n,\vec{\eta}_n}$ *with dissection ratios* $(\xi_n, \vec{\eta}_n)$ in the obvious way, replacing "dissection of type ξ_n" with "dissection of type $(\xi_n, \vec{\eta}_n)$".

If the sequence $(\xi_n, \vec{\eta}_n)_n = (\xi, \vec{\eta})_n$ is constant, we call the resulting set $E_{\xi,\vec{\eta}}$ a *homogeneous perfect set* $E_{\xi,\vec{\eta}}$ *associated to* $(\xi, \vec{\eta})$.

Remark. Let E_{ξ_n} be a symmetric perfect set with dissection ratios ξ_n. If we take $\vec{\eta}_n = (0,\, 1-\xi_n,\, 1)$, we have $E_{\xi_n} = E_{\xi_n,\vec{\eta}_n}$. Therefore Definition 7.5 truly generalizes the previous Definition 7.2.

7.3 Some properties

Remark. Let $(\xi_n, \vec{\eta}_n)_n$ and $(k_n)_n$ be as in the above definition. Similarly to Example 7.4, we have

$$|E_{\xi_n,\vec{\eta}_n}| = 0 \iff \sum_{n=1}^{\infty}(1-k_n\xi_n) = \infty.$$

Lemma 7.6. *The symmetric sets* $E_{\xi_n,\vec{\eta}_n}$ *defined above are nowhere dense (and so, in particular, meager).*

Proof. By Remark 7.1, all of these sets are closed. Let $E = \cap E_n$ be such a set.

In Definition 7.5, we only allow non-trivial dissections - in each step, each interval is split into at least two disjoint intervals of the same length. Consequently, the E_n consists of at least 2^n disjoint intervals of length at most 2^{-n}. This implies that $\overline{E} = E = \cap E_n$ cannot contain an open set, and thus it is nowhere dense. □

Definition 7.7. A symmetric perfect set E_{ξ_n} is said to be *ultra-thin*, if the dissection ratios satisfy $\sum \xi_n^2 < \infty$.

Theorem 7.8 ([Mey, Chapter VIII, Theorem I])**.** *All ultra-thin symmetric sets are of uniqueness.*

Remark. When ξ_n decreases quickly enough, the set E_{ξ_n} is of uniqueness by the previous theorem. On the other hand, when ξ_n is high enough, the resulting set E_{ξ_n} will have positive measure, and consequently it will not be in U. However for general sequence $(\xi_n)_n$, there is no known characterization explaining when is the set E_{ξ_n} of uniqueness and when is it of multiplicity.

8 $H^{(N)}$-sets

8.1 The definition and basic properties

Notation. Recall that for $x, y \in \mathbb{R}^N$ we denote by xy the standard scalar product $\sum_{i=1}^N x_i y_i$ in $\mathbb{R}^N$.

In the following section, we will use the notation

$$\mathbf{x} = (x_n)_n = \left(x_n^1, ..., x_n^N\right)_n \in \left(\mathbb{R}^N\right)^{\mathbb{N}}.$$

Definition 8.1. Let $N \in \mathbb{N}$ and let $\mathbf{x} \in \left(\mathbb{R}^N\right)^{\mathbb{N}}$ be a sequence of vectors. We say that $\mathbf{x}$ is *quasi-independent*, if for every $0 \neq \alpha \in \mathbb{Z}^N$ we have $\lim_n |x_n \alpha| = \infty$. By Q_N we denote the set of all quasi-independent sequences in $\left(\mathbb{N}^N\right)^{\mathbb{N}}$ and by Q_{N*} the set of all quasi-independent sequences in $\left((\mathbb{R} \setminus \{0\})^N\right)^{\mathbb{N}}$.

Remark 8.2. It is easy to see that if $\mathbf{x}$ is quasi-independent, then necessarily $\lim_n \left|x_n^k\right| = \infty$ for each $k \leq N$. An example of a sequence which is not quasi-independent would be $x_n = n\alpha$ for $\alpha \in \mathbb{Z}^N$ (where $N > 1$). For $N = 1$, clearly any $(x_n)_n$ with $|x_n| \to \infty$ is quasi-independent.

For higher N, a sufficient condition for $\mathbf{x}$ to be quasi-independent is when for each $1 \leq k, l \leq N$, $k \neq l$ we have either $x_n^k / x_n^l \to \infty$ or $x_n^l / x_n^k \to \infty$ as $n \to \infty$. To see this, fix nonzero $\alpha \in \mathbb{Z}^N$. There exists an index i_0 satisfying $\alpha_{i_0} \neq 0$ and $x_n^{i_0} / x_n^k \to \infty$ as $n \to \infty$ for each $1 < k \leq N$ with $\alpha_k \neq 0$. We then have

$$\lim_n |x_n \alpha| = \lim_n x_n^{i_0} \left| \alpha_{i_0} + \sum_{k \neq i_0} \alpha_k \frac{x_n^k}{x_n^{i_0}} \right| = \infty \cdot \left(|\alpha_{i_0}| + \sum_{k \neq i_0} \alpha_k \cdot 0 \right) = \infty,$$

which implies that $\mathbf{x}$ is quasi-independent.

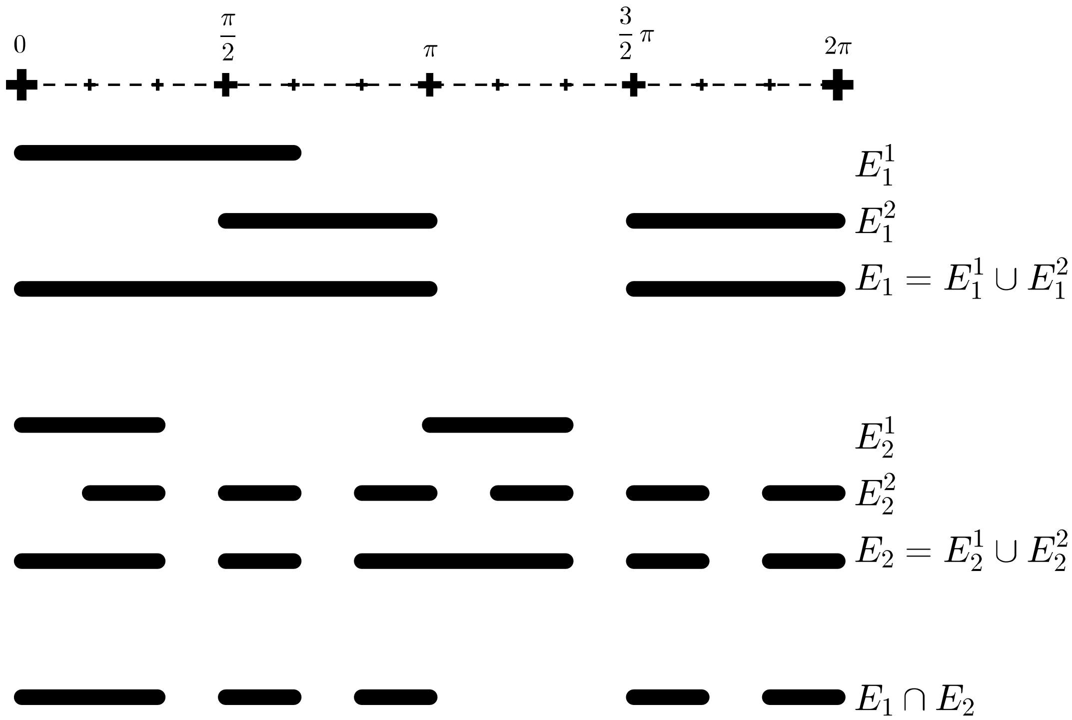

Figure 2: First two steps of the construction of a $H^{(2)}$-set E with $I_1 = \left(\frac{1}{3}\cdot 2\pi, 2\pi\right)$, $I_2 = \left(0, \frac{1}{2}\cdot 2\pi\right)$ and $x_1^1 = 1$, $x_1^2 = x_2^1 = 2$, $x_2^2 = 6$.

Definition 8.3. A set $I \subset \mathbb{T}^N$ is *an open interval,* if it is of the form $I = I_1 \times ... \times I_N$ where each I_k, $k = 1, ..., N$ is an open interval.

A set $E \subset \mathbb{T}$ is in $H^{(N)}$, if there exists $\mathbf{x} \in \mathcal{Q}_N$ and an open interval $I \subset \mathbb{T}^N$ such that for every $x \in E$ and $n \in \mathbb{N}$ the vector $x \cdot x_n \in \mathbb{T}^N$ is not in I (where $x \cdot x_n := \left(x \cdot_{\mathbb{T}} x_n^1, ..., x \cdot_{\mathbb{T}} x_n^N\right)$). Similarly E is in $H^{(N)*}$, if there exists $\mathbf{x} \in \mathcal{Q}_{N*}$ and an open interval $I \subset \mathbb{T}^N$ with the same property.

Remark 8.4. Let F be a $H^{(N)}$-set and suppose that this fact is witnessed by the sequence $\mathbf{x} \in \mathcal{Q}_N$ and open interval I. Clearly F is contained in a "true" $H^{(N)}$-set $E = \bigcap E_n = \bigcap \left(E_n^1 \cup ... \cup E_n^N\right)$, where the sets E_n^k are defined as

$$E_n^k := \left\{x \in \mathbb{T} | \, x \cdot x_n^k \in \mathbb{T} \setminus I_k\right\},$$

and this set E is closed. Thus we can focus our attention mostly on those closed $H^{(N)}$-sets E which are of the form

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{N} E_n^k =: H\left(N, I, \mathbf{x}\right)$$

for some quasi-independent sequence $\mathbf{x}$ and open interval I. In particular, from now on unless stated otherwise, all $H^{(N)}$-sets will be closed.

Remark. In the definition used by [Bar2], the property of being $H^{(N)*}$-set can actually be witnessed by quasi-independent sequence $\mathbf{x} \in \left(\mathbb{R}^N\right)^{\mathbb{N}}$ instead of $\mathbf{x} \in \left((\mathbb{R}\setminus\{0\})^N\right)^{\mathbb{N}}$. However the families $H^{(N)*}$ in either definition are hereditary. Also when the fact that $E \in H^{(N)*}$ is witnessed by sequence of vectors (x_n), it is also witnessed by any subsequence of (x_n). Combined with the fact that $\lim_n \left|x_n^k\right| = \infty$ holds for each $k \leq N$, we see that both of the possible definitions give the same object $H^{(N)*}$ - and the one we adopted avoids division by zero.

Proposition 8.5 (Properties of $H^{(N)}$-sets)**.** *Let $N \in \mathbb{N}$. Then we have the following:*

1. $H^{(N)} \subset H^{(N)*} \subset \left\{\bigcup_{k=1}^n E_k \middle| \, n \in \mathbb{N} \,\&\, \forall k = 1, ..., N: \, E_k \in H^{(N)}\right\}$.
2. $H^{(N)} \subset H^{(N+1)}$.
3. *Let $M \in \mathbb{N}$ and $E \in H^{(M)}$, $F \in H^{(N)}$. Then $E \cup F \in H^{(M+N)}$.*

The points 2. and 3. of the previous proposition also hold for the collections $H^{(N)*}$, with identical proofs. The first assertion immediately implies the following corollary. For definition of symbol $(\cdot)^{\perp}$ see Definition 27.

Corollary 8.6. *For $N \in \mathbb{N}$ we have $\left(H^{(N)}\right)_\sigma = \left(H^{(N)*}\right)_\sigma$ (and thus also $\left(H^{(N)}\right)^{\perp} = \left(H^{(N)*}\right)^{\perp}$).*

Proof. The first inclusion in 1. is trivial, while the second can be found in [Bar2].

In order to prove 2., let $E \subset H(N, I, \mathbf{x}) \in H^{(N)}$. Define $x_n^{N+1} := n \cdot \max\left\{x_n^k \middle| \, k \leq N\right\}$, $x_n' := \left(x_n^1, ..., x_n^{N+1}\right)$, $\mathbf{x}' = \left(x_n'\right)$ and $I' = I \times \mathbb{T}$. By Remark 8.2 we have $\mathbf{x}' \in Q_{N+1}$. Since clearly $x \cdot x_n' \notin I' \iff x \cdot x_n \notin I$ for any $x \in \mathbb{T}$, the fact that $E \in H^{(N+1)}$ is witnessed by $\mathbf{x}'$ and I'.

3. Let $E \subset H(M, I, \mathbf{x}) \in H^{(M)}$ and $F \subset H(N, J, \mathbf{y}) \in H^{(N)}$. Since for each $\beta \in \mathbb{Z}^N$ we have $|\beta y_n| \to \infty$, we can for each $k \in \mathbb{N}$ find such $n_k \in \mathbb{N}$ that $|\beta y_{n_k}| \geq 2\,|\alpha x_k|$ holds for each $(\alpha, \beta) \in \{-k, ..., k\}^{M+N}$. Denote $\mathbf{z} = (z_k)$, $z_k = (x_k, y_{n_k})$.

We check that $\mathbf{z} \in \mathcal{Q}_{M+N}$. Fix nonzero $\gamma = (\alpha, \beta) \in \mathbb{Z}^M \times \mathbb{Z}^N$. If either α or β is a zero vector, we have $|\gamma z_k| = |\beta y_{n_k}| \to \infty$ (or $|\gamma z_k| = |\alpha x_k| \to \infty$), since $\mathbf{x}$ and $\mathbf{z}$ are quasi-independent. On the other hand when $\alpha, \beta \neq 0$, for $k \geq \|\alpha\|_\infty \|\beta\|_\infty$ we have

$$|\gamma z_k| = |\alpha x_k + \beta y_{n_k}| \geq |\beta y_{n_k}| - |\alpha x_k| \geq |\alpha x_k| \to \infty.$$

Furthermore for any $k \in \mathbb{N}$, we have $x \cdot x_k \notin I \implies x \cdot z_k \notin I \times J$ and $x \cdot y_{n_k} \notin J \implies x \cdot z_k \notin I \times J$. Clearly we have $F \subset H(N, J, \mathbf{y}) \subset H(N, J, (y_{n_k}))$ and therefore

$$E \cup F \subset H(M+N, I \times J, \mathbf{z}) \in H^{(M+N)}.$$

□

The system $H^{(N)*}$ admits the following characterization of $H^{(N)*}$ sets, which we will need later:

Definition 8.7. We denote

$$\begin{aligned} H_L^{(N)*} = &\{E \in H^{(N)*} |\, \exists \mathbf{x} \in \mathcal{Q}_N\, \exists \text{ open interval } I:\ E \subset H(N, I, \mathbf{x}) \\ &\& \,\forall k < N\, \forall n \in \mathbb{N}: \left| \frac{x_n^{k+1} |I_k|}{x_n^k} \right| \geq L \}. \end{aligned}$$

Theorem 8.8 ([Vla])**.** *For any $N \in \mathbb{N}$, $L > 0$ we have $H^{(N)*} = H_L^{(N)*}$.*

8.2 The theorem of Piatetski-Shapiro

We now state one of the main theorems relevant to this thesis. We also include its proof, as we will refer to it later.

Theorem 8.9 ([PS1])**.** *For every $N \in \mathbb{N}$, a $H^{(N)}$-set E is in U'. Consequently we have $\bigcup_N H^{(N)} \subset U' \subset U$.*

Proof. **Step 1.** Let $E \subset H(N, I, \mathbf{x}) \in H^{(N)}$. As noted in Remark 6.9 it suffices to find a sequence f_n of functions from A with $\operatorname{supp}(f_n) \subset \mathbb{T} \setminus E$, such that $f_n \xrightarrow{w^*} 1$. We will take f_n as $f_n = f_n^1 \cdot ... \cdot f_n^N$ with $\operatorname{supp}(f_n^i) \subset \mathbb{T} \setminus E_n^i$, where $E_n^i = \left\{x \in \mathbb{T} |\, x \cdot x_n^k \in \mathbb{T} \setminus I_k\right\}$. Furthermore, denote by φ_i a fixed function from A with $\operatorname{supp}(\varphi_i) \subset I_i$ and $\hat{\varphi}_i(0) = 1$ (such a function exists, since by 6.3 any $f \in \mathcal{C}^1(\mathbb{T})$ is in A). We claim that $f_n^i(x) := \varphi_i(x_n^i x)$ are the functions we were looking for.

Step 2. We will denote $f_n^i = \varphi_i(x_n^i \cdot)$. Firstly we observe that

$$\|f_n\|_A \leq \prod_{i=1}^N \left\|f_n^i\right\|_A = \prod_{i=1}^N \|\varphi_i\|_A,$$

which then implies that $\sup\{\|f_n\| \,|\, n \in \mathbb{N}\} < \infty$. The first inequality is immediate from the fact that A is a Banach algebra. The equality then follows from the fact that for $x \in \mathbb{T}$:

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \widehat{f_n^i}(k) \exp(ikx) = f_n^i(x) = \varphi_i\left(x_n^i x\right) = \\ = \sum_{l\in\mathbb{Z}} \hat{\varphi}_i(l) \exp\left(il\left(x_n^i x\right)\right) = \sum_{l\in\mathbb{Z}} \hat{\varphi}_i(l) \exp\left(i\left(l x_n^i\right) x\right). \end{aligned}$$

This implies that the numbers $\left(\widehat{\varphi_i(x_n^i \cdot)}(k)\right)_k$, defined as $\widehat{\varphi_i(x_n^i \cdot)}(k) = \hat{\varphi}_i(l)$ when $k = l \cdot x_n^i$ for some $l \in \mathbb{Z}$ and $\widehat{\varphi_i(x_n^i \cdot)}(k) = 0$ otherwise are Fourier coefficients for the function f_n^i. Since these coefficients are uniquely determined, we have $\hat{f}_n^i = \widehat{\varphi_i\left(x_n^i \cdot\right)}$ and thus $\|f_n^i\|_A = \|\varphi_i\left(x_n^i\cdot\right)\|_A = \|\varphi_i\|_A$.

Step 3. It remains to prove that $\hat{f}_n(0) \to 1$ and $\hat{f}_n(k) \to 0$ for $k \neq 0$. Let $k \in \mathbb{Z}$ and $\epsilon > 0$. Using first the standard properties of Fourier transform and then the observation made in step 2, we get

$$\hat{f}_n(k) = \left(\widehat{\prod_{i=1}^N \varphi_i\left(x_n^i\cdot\right)}\right)(k) = \left(\widehat{\varphi_1\left(x_n^1\cdot\right)} * \ldots * \widehat{\varphi_N\left(x_n^N\cdot\right)}\right)(k) = \sum_{\substack{p\in\mathbb{Z}^N \\ p_1+\ldots+p_N=k}} \prod_{i=1}^N \widehat{\varphi_i\left(x_n^i\cdot\right)}(p_i) =$$

$$= \sum_{\substack{\alpha\in\mathbb{Z}^N \\ \alpha x_n=k}} \prod_{i=1}^N \hat{\varphi}_i(\alpha_i) = \sum_{\substack{\alpha x_n=k \\ \forall i:\, |\alpha_i|\leq m}} \prod_{i=1}^N \hat{\varphi}_i(\alpha_i) + \sum_{\substack{\alpha x_n=k \\ \exists i:\, |\alpha_i|>m}} \prod_{i=1}^N \hat{\varphi}_i(\alpha_i) = S_1 + S_2.$$

In the equation above, fix m such that $\sum_{|l|>m} |\hat{\varphi}_i(l)| < \epsilon$ holds for each i. Since $(x_n) \in \mathcal{Q}_N$, we have $|\alpha x_n| \to \infty$ for each nonzero $\alpha \in \mathbb{Z}^N$. This means that we can find n_0 high enough so that no α with $|\alpha_i| \leq m$ for each i satisfies $\alpha x_n = k$ for every $n \geq n_0$ (with the possible exception of $\alpha = 0$ when $k = 0$). Therefore for such m and n we have either $S_1 = 0$ when $k \neq 0$ or $S_1 = \prod \hat{\varphi}_i(0) = 1$ when $k = 0$. We can bound S_2 in the following way:

$$S_2 = \sum_{\substack{\alpha x_n=k \\ \exists i:\, |\alpha_i|>m}} \prod_{i=1}^N \hat{\varphi}_i(\alpha_i) \leq \sum_{\substack{\alpha\in\mathbb{Z}^N \\ \exists i:\, |\alpha_i|>m}} \prod_{i=1}^N \hat{\varphi}_i(\alpha_i) \leq \sum_{i_0=1}^N \sum_{\substack{\alpha\in\mathbb{Z}^N \\ \left|\alpha_{i_0}\right|>m}} \prod_{i=1}^N |\hat{\varphi}_i(\alpha_i)|$$

$$= \sum_{i_0=1}^N \left(\sum_{|l|>m} |\hat{\varphi}_{i_0}(l)|\right) \prod_{i\neq i_0}^N \left(\sum_{l\in\mathbb{Z}} |\hat{\varphi}_i(l)|\right) < \epsilon \cdot \sum_{i_0=1}^N \prod_{i\neq i_0}^N \|\varphi_i\|_A.$$

Thus for each $k \in \mathbb{Z}$ and $\epsilon > 0$ we proven the existence of such n_0 that for every $n \geq n_0$ we have $\left|\hat{f}_n(k) - \hat{1}(k)\right| < \epsilon$, which completes the proof. □

8.3 Relation of $H^{(N)}$-sets and symmetric sets

Remark 8.10 ("regular" $H^{(N)}$-sets)*.* To simplify the notation, we identify for the moment the unit circle $\mathbb{T}$ with the unit interval $[0,1]$. Let $E = H(N, I, \mathbf{x}) \in H^{(N)*}$ (where as before $N \in \mathbb{N}$, I is an open interval and $\mathbf{x} \in \mathcal{Q}_\infty$) and denote $E_n^k = \left\{x \in \mathbb{T} \middle| x \cdot x_n^k \in I_k^C\right\}$ as before. We will show that under certain conditions, E is a symmetric perfect set (in a slightly generalized sense).

- $E = \bigcap E_n$, where $E_n = E_n^1 \cup \ldots \cup E_n^N$ is the representation of the set E in the sense of Remark 7.1. However, the set E for general open interval I

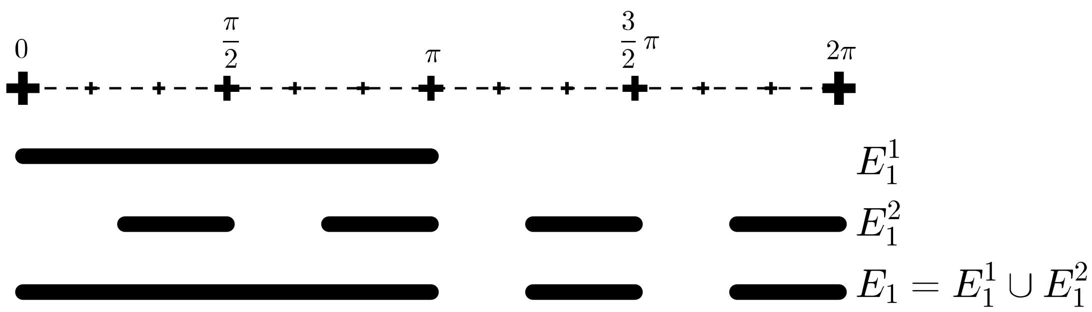

Figure 3: First step of the construction of a "regular" $H^{(2)}$-set E with $I_1 = \left(\frac{1}{2}\cdot 2\pi, 2\pi\right)$, $I_2 = \left(0, \frac{1}{2}\cdot 2\pi\right)$ and $x_1^1 = 1$, $x_1^2 = 2$.

and quasi-independent $\mathbf{x}$ can be rather complicated to draw or imagine.

- Assume that $I_k = (a_k, 1)$ is an open interval and $x_n^k \in \mathbb{N}$. The set E_n^k can then be written as

$$E_n^k = \left\{x \in \mathbb{T} \middle|\, x \cdot x_n^k \in I_k^C\right\} = \left(\frac{1}{x_n^k} \cdot_{\mathbb{R}} \left(\mathbb{Z} +_{\mathbb{R}} (I_k)^C\right)\right) \cap [0,1],$$

which shows that it is a disjoint union of x_n^k closed intervals of length $(1-a_k)/x_n^k$. Similar remark can be made for general $I_k = (a_k, b_k)$.

- Suppose that $I_k = \left(\frac{a_k}{q_k}, \frac{b_k}{q_k}\right)$, where $a_k, b_k \in \mathbb{N} \cup \{0\}$, $q_k \in \mathbb{N}$ and that $k < N$, x_n^k, $x_n^{k+1} \in \mathbb{N}$ and $x_n^k q_k | x_n^{k+1}$. The set E_n^{k+1} then "divides well" the set E_n^k in the sense that whenever J_1, $J_2 \subset E_n^k$ are intervals of the form $[j, j+j]/x_n^k q_k$, then the sets $E_n^{k+1} \cap J_i$, $i = 1,2$ are translations of each other (and the same thing also holds for open intervals $J_i \subset \left(E_n^k\right)^C$ of the form $(j, j+j)/x_n^k q_k$).

- When trying to "draw" or imagine $H^{(N)}$-sets, it is convenient to have the following order of elements of vectors of sequence $\mathbf{x}$ in mind: $x_1^1, x_1^2, ..., x_1^N, x_2^1, x_2^2, ...$

- Suppose that for each $k < \mathbb{N}$ the endpoints of $I_k = \left(\frac{a_k}{q_k}, \frac{b_k}{q_k}\right)$ satisfy $a_k, b_k \in \mathbb{N} \cup \{0\}$, $q_k \in \mathbb{N}$ and $x_n^k \in \mathbb{N}$. Moreover, assume that for all $n \in \mathbb{N}$, $k < N$ we have $x_n^k q_k | x_n^{k+1}$, $x_n^N q_N | x_{n+1}^1$. The set E_{n+1} (and thus also E_m for $m > n$ as well as E) then "looks the same" at each interval $J \subset E_n$ of the form $J = [j, j+j]/x_n^k q_k$ (in the above mentioned sense that if J_i, $i = 1,2$ are two such intervals, then $E \cap J_1$ is a translation of $E \cap J_2$). We say that such a set E is *regular*.

- For regular E, the sets E_n are unions of closed intervals of length $1/\left(x_n^N q_k\right)$, and these intervals have disjoint interiors and their endpoints are of the form $a/x_n^N q_k$ for some $a \in \left\{0, ..., x_n^N q_k\right\}$. We can relax the condition in the Definition 7.5 of dissection of type $(\xi, \vec{\eta})$ to only require $\eta_i \geq \eta_{i-1} + \xi$ instead of $\eta_i > \eta_{i-1} + \xi$. Then by the previous point, we get that each regular $H^{(N)}$-set is a symmetric perfect set with dissection ratios $(\xi_n, \vec{\eta}_n)$ for some ξ_n and $\vec{\eta}_n$.

- We could write the dissection ratios $(\xi_n, \vec{\eta}_n)$ for E explicitly - unfortunately, doing so does not seem to bring in any useful information, as this form of representation of E is far from being compact[8].

Regular $H^{(N)}$-sets then share some of the properties with "proper" symmetric perfect sets with dissection ratios $(\xi_n, \vec{\eta}_n)$. For one, we have $|\bigcap_{i=1}^{n} E_i| = |\bigcap_{i=1}^{n-1} E_i| \cdot |E_n|$ for E as above. This allows us the calculate the exact measure of E. Secondly, it is well known that we can represent the classical Cantor set C as a set of sums

$$C = \left\{ \sum_{n=1}^{\infty} \epsilon_n \left(\frac{1}{3}\right)^n | \, \epsilon_n \in \{0, 2\} \right\} = \left\{ \sum_{n=1}^{\infty} \epsilon_n \left(\frac{1}{3}\right)^{n-1} | \, \epsilon_n \in \left\{0, \frac{2}{3}\right\} \right\}$$

and that this way each $x \in C$ is represented uniquely. Recall that C is a symmetric perfect set with dissection ratios $(\xi, \vec{\eta})$, where $\xi = \frac{1}{3}$ and $\vec{\eta} = \left(0, \frac{2}{3}, 1\right)$, which gives meaning to the second equality in the above representation. It is easy to check that whenever E is a symmetric perfect set with dissection ratios$(\xi_n, \vec{\eta}_n)$, we can represent it in a similar way as

$$E = \left\{ \sum_{n=1}^{\infty} \epsilon_n \xi_1 ... \xi_{n-1} | \, \epsilon_n \in \{\eta_{n,0}, ..., \eta_{n,k_n-1}\} \right\}.$$

However if we had to relax the definition of symmetric perfect set with dissection ratios $(\xi_n, \vec{\eta}_n)$ to allow $\eta_{n,i} \geq \eta_{n,i-1} + \xi_n$, this representation will not be unique.

Next we present the Salem-Zygmund Theorem which relates the homogenous perfect sets and the sets of uniqueness, and $H^{(N)}$-sets in particular.

Definition 8.11. Real number $\theta > 1$ is called a *Pisot number* if θ is the root of some polynomial P which has coefficients from $\mathbb{Z}$, leading coefficient of P is 1 and all other roots of P have absolute value less than 1. *Order* of θ is the minimal degree of a polynomial P which witnesses that θ is a Pisot number.

Remark. Clearly every $n \in \mathbb{N}$ greater than 1 is trivially a Pisot number of degree 1 and $\mathbb{Q}(n) = \mathbb{Q}$ (recall that $\mathbb{F}(s)$ is the extension of the field $\mathbb{F}$ by the element s).

[8]The word "compact" is not used in the mathematical sense here.

Theorem 8.12 (Salem-Zygmund, 1955). *Let $\xi, \vec{\eta} = (\eta_0, ..., \eta_k)$ be numbers satisfying $0 = \eta_0 < \eta_1 < ... < \eta_k = 1$, $k > 1$, $\xi = 1 - \eta_{k-1}$ and $\eta_i > \eta_{i-1} + \xi$ for each $i < k$.*

(1) *Suppose that $\theta = \frac{1}{\xi}$ is a Pisot number of degree N and $\eta_1, ..., \eta_k \in \mathbb{Q}(\theta)$. Then $E_{\xi,\vec{\eta}}$ is a $H^{(N)}$-set. In particular $E_{\xi,\vec{\eta}}$ is a set of uniqueness.*

(2) *If either $\theta = \frac{1}{\xi}$ is not a Pisot number or one of the numbers $\eta_1, ..., \eta_k$ is not in $\mathbb{Q}(\theta)$, then $E_{\xi,\vec{\eta}}$ is a set of restricted multiplicity.*

Proof. [KL, Chapter III]. □

9 Other families as approximations of U

9.1 Bases and p-bases

Remark 9.1. In previous chapters, we listed several families of thin sets connected to U-sets. In particular we have the following inclusions

$$H^{(1)} \subset H^{(2)} \subset ... \subset \bigcup_{N=1}^{\infty} H^{(N)} \subset U' \subset U \subset U_0. \tag{1}$$

It is natural to ask whether these inclusions are strict and, in case of a positive answer, "how strict" they are.

Definition 9.2. Let $\mathcal{F}$ be a family of sets. By $\mathcal{F}_\sigma$ we denote its *σ-closure*, i.e. the family $\mathcal{F}_\sigma = \{\bigcup_{n=1}^{\infty} F_n | \, F_n \in \mathcal{F}\}$. Clearly if $\mathcal{F}$ is a family of thin sets, its σ-closure is a σ-ideal. If $\mathcal{I}$ is a σ-ideal, $\mathcal{F}$ a family of thin sets and $\mathcal{I} = \mathcal{F}_\sigma$, we say that $\mathcal{F}$ is a *basis of σ-ideal* $\mathcal{I}$. We also define the corresponding notions for families of closed sets in the obvious way.

Remark. For two families of thin sets $\mathcal{F} \subset \mathcal{G}$, clearly $\mathcal{F}_\sigma \not\supseteq \mathcal{G}$ implies $\mathcal{F}_\sigma \not\supseteq \mathcal{G}_\sigma$. On the other hand we also have $\mathcal{F}_\sigma \supseteq \mathcal{G} \implies \mathcal{F}_\sigma = \mathcal{G}_\sigma$. Consequently, it is the same to ask whether there exists a set $G \in \mathcal{G}$ which cannot be written as countable union of sets $F_n \in \mathcal{F}$, as it is to ask "Is the family $\mathcal{F}$ a basis for the σ-ideal $\mathcal{G}_\sigma$?".

When we compare two families of thin sets $\mathcal{F} \subset \mathcal{G}$, the first step is to check whether $\mathcal{F} \subsetneq \mathcal{G}$. If this is the case, we then usually proceed to ask whether $\mathcal{F}$ is a basis for $\mathcal{G}_\sigma$. Should the difference of two families turn out to be big in this sense as well, as it will for families in (1), it will be useful to have even stronger notion of difference between two families of thin sets, which we define now.

Definition 9.3. For $\mathcal{F} \subset \mathcal{K}(\mathbb{T})$ denote by

$$\mathcal{F}^{\perp} = \{\mu \in \mathcal{M}(\mathbb{T}) \, | \, \forall E \in F : \, \mu(E) = 0\}$$

the *polar of* $\mathcal{F}$. Analogically we define for $\mathcal{S} \subset \mathcal{M}(\mathbb{T})$ the set

$$\mathcal{S}^{\perp} = \{E \in \mathcal{K}(\mathbb{T}) \mid \forall \mu \in \mathcal{S} : \mu(E) = 0\}$$

. If $\mathcal{F} \subset \mathcal{G}$ has the property that $\mathcal{F}^{\perp} = \mathcal{G}^{\perp}$ and $\mathcal{G}$ is a σ-ideal, we say that $\mathcal{F}$ is a *polarity basis*, or also just *p-basis*, for $\mathcal{G}$.

Remark. By σ-aditivity of measures from $\mathcal{M}(\mathbb{T})$, if $\mathcal{F}$ is a basis for $\mathcal{G}_{\sigma}$, then trivially $\mathcal{F}$ is also a p-basis for $\mathcal{G}$. Thus "being smaller in the sense of polars" is a stronger notion than "being smaller in the sense of bases". Recall as well that by definition, $U_0 = \mathcal{R}^{\perp}$.

9.2 Results concerning the relations between $H^{(N)}$, U', U_0 and U

When Rajchman introduced the $H^{(1)}$-sets in 1923, he conjectured that $H_{\sigma}^{(1)} = U$. It was also expected that the inclusion $U \subset U_0$ might not be strict. However it later turned out that both of the inclusions $H_{\sigma}^{(1)} \subset U$ and $U \subset U_0$ are strict in a strong sense. We now list some of the results known about the inclusions in (1) in order to highlight the related open questions.

Lyons showed in [Lyo] that $U_0^{\perp} = \mathcal{R}$ and Kaufman [Kau2] complemented this by the (very difficult) result that $U^{\perp} \supsetneq \mathcal{R}$. Together, these two results imply that U is not a polarity basis for U_0. Debs and Saint-Raymond solved negatively the so-called Borel basis problem for U by proving the following theorem:

Theorem 9.4 ([DSR])**.** *Let $E \subset \mathbb{T}$ be of multiplicity. Then $U \cap \mathcal{K}(E)$ admits no Borel basis. In particular whenever $\mathcal{F} \subset \mathcal{K}(\mathbb{T})$ is Borel, then $\mathcal{F}$ cannot be a basis for U.*

The following lemma then implies that U' is not a basis for U.

Lemma 9.5. *The family U' is Σ_3^0 in $\mathcal{K}(\mathbb{T})$.*

Proof. By Remark 6.9 we have for $E \in \mathcal{K}(\mathbb{T})$

$$\begin{aligned} E \in U' \iff & \exists f_n \in A \text{ with } \operatorname{supp}(f_n) \subset \mathbb{T} \setminus E \text{ and } \sup \|f_n\|_A < \infty \\ & \text{satisfying } \hat{f}_n(0) \to 1 \text{ and } \hat{f}_n(k) \to 0 \text{ for } k \neq 0. \\ \iff & \exists n \in \mathbb{N}\, \forall m \in \mathbb{N}\, \exists f \in A : (E \subset \mathbb{T} \setminus \operatorname{supp}(f) \;\&\; \|f\|_A \leq n \\ & \& \forall k, |k| \leq m : \left| \hat{f}(k) - \hat{1}(k) \right| < \frac{1}{m}). \end{aligned}$$

Denoting

$$M_{m,n} = \left\{ f \in A \mid \|f\|_A \leq n, \forall |k| \leq m : \left| \hat{f}(k) - \hat{1}(k) \right| < \frac{1}{m} \right\},$$

we get that $U' = \bigcup_n \bigcap_m \bigcup_{f \in M_{m,n}} \{E \in \mathcal{K}(E) \,|\, E \subset \mathbb{T} \setminus \operatorname{supp}(f)\}$, which shows that U' is $G_{\delta\sigma}$. $\square$

In [Vla] we can then find the following lemma and its corollary:

Lemma 9.6. *Let $N \in \mathbb{N}$. Then there exists a set $E \in H^{(N+1)}$ and a probability measure $\mu \in \mathcal{M}(E)$, such that $\mu(F) = 0$ for every $F \in H_{10}^{(N)*}$.*

Corollary 9.7. $\left(H^{(N)}\right)^{\perp} \supsetneq \left(H^{(N+1)}\right)^{\perp}$ *holds for every $N \in \mathbb{N}$.*

Proof. By Lemma 9.6 we have $\left(H^{(N+1)}\right)^{\perp} \subsetneq \left(H_{10}^{(N)*}\right)^{\perp}$. Characterization of $H^{(N)*}$ sets from Theorem 8.8 then gives $\left(H_{10}^{(N)*}\right)^{\perp} = \left(H^{(N)*}\right)^{\perp}$. Finally by Corollary 8.6 we have $\left(H^{(N)*}\right)^{\perp} = \left(H^{(N)}\right)^{\perp}$, which implies the result. $\square$

We will not give the proof of Lemma 9.6 here, but we will later in Section 11 prove a more general result, from which this lemma can be derived.

9.3 Summary of relations between $H^{(N)}$, U', U_0 and U, open problems

Remark 9.8. We now summarize the known results about the inclusions in (1).

1. The following inclusions hold:

$$H^{(1)} \subset H^{(2)} \subset \ldots \subset \bigcup_{N=1}^{\infty} H^{(N)} \subset U' \subset U \subset U_0.$$

2. For each $N \in \mathbb{N}$, the family $H^{(N)}$ is much smaller than $H^{(N+1)}$ in the sense that $\left(H^{(N)}\right)^{\perp} \supsetneq \left(H^{(N+1)}\right)^{\perp}$.

3. $\bigcup_{N=1}^{\infty} H^{(N)}$ is smaller than U' in the sense that $\left(\bigcup_{N=1}^{\infty} H^{(N)}\right)_{\sigma} \subsetneq U'$.

4. U' is smaller than U in the sense that $U'_{\sigma} \subsetneq U$.

5. U is much smaller than U_0 in the sense that $U^{\perp} \supsetneq U_0^{\perp}$.

Proof. The only result not mentioned so far is 3. This follows from the fact that each $H^{(N)}$-set is σ-porous ([Zaj]), but there exists a U'-set which is not σ-porous ([ZP]). $\square$

This leaves the following problem, to the best of our knowledge, open:

Problem 9.9. Are the following inclusions strict?

1. $\left(\bigcup_{N=1}^{\infty} H^{(N)}\right)^{\perp} \supset (U')^{\perp}$

2. $(U')^{\perp} \supset U^{\perp}$

In particular, is it true that $\bigcup_{N=1}^{\infty} H^{(N)}$ is polarity basis for U?

Part III

$H^{(\infty)}$-sets and sets of type N

10 $H^{(\infty)}$-sets

Vlasák conjectured the following:

Conjecture 10.1 ($H^{(\infty)}$-sets)**.** *We can define the so-called $H^{(\infty)}$-sets, a family of sets which generalizes the concept of $H^{(N)}$-sets. These sets have, among others, the following properties:*

(i) *There exists a set $E \in H^{(\infty)}$ and a measure $\mu \in \mathcal{M}(E)$ which annihilates every set from $\bigcup_{N=1}^{\infty} H^{(N)}$.*

(ii) *In (i), the set E can be chosen such that $E \in U$ (or possibly even $E \in U'$).*

In particular, the conjecture would give an answer to Problem 9.9, as it implies that $\left(\bigcup H^{(N)}\right)^{\perp} \neq U^{\perp}$ (or possibly even $\left(\bigcup H^{(N)}\right)^{\perp} \neq (U')^{\perp}$). We were able to prove the part (i) of the conjecture and we present this result in Section 11. In this section we give the definition of $H^{(\infty)}$-sets and prove some of their basic properties. So far, we were unable to prove (ii) - our original intention was to do so using the Theorem 8.9 of Piatetski-Shapiro. Sadly this did not work, so we at least explain why the proof of the mentioned theorem cannot be modified in order to get the desired result (at least not in a straightforward way).

Remark 10.2. A natural question regarding the polars of H^N-sets is whether this topic could be somehow related to Hausdorff dimension and Hausdorff measures. For example one might wonder whether a d-dimensional Hausdorff measure $\mathcal{H}_d$ for some $d \in (0,1)$ restricted on some $E \in H^{(N+1)}$ could not witness the fact that $\left(H^{(N+1)}\right)^{\perp} \neq \left(H^{(N)}\right)^{\perp}$. Unfortunately this is not the case. Firstly the notion of polars only works with Radon measures, while the measure $\mathcal{H}_d$ for $d < 1$ is actually not a Radon measure as it is not finite on compact sets. A more significant obstruction is however caused by the following example which suggests that the notion of Hausdorff measure is not closely related to the theory of $H^{(N)}$-sets.

Example 10.3. (1) For any $d \in (0,1)$ there exists a $H^{(1)}$-set of Haudorff dimension at least d.

(2) There exists a $H^{(1)}_{\sigma}$-set of Haudorff dimension 1.

(3) For any $d \in (0,1)$ there exists a $\mathcal{M}$-set of Haudorff dimension less than d.

Proof. (2) follows from (1) using the fact that when $E_k \in \bigcup_{N\in\mathbb{N}} H^{(N)}$ for $k \in \mathbb{N}$ then $\bigcup E_k \in H^{(\infty)}$ (see Proposition 10.7 for the proof of this fact) and $1 \geq \dim_{\mathcal{H}} \bigcup E_k \geq \sup_k \dim_{\mathcal{H}} E_k = 1$.

(1) Let $d \in (0,1)$. There exists some $k \in \mathbb{N}$, such that $\log 2(k-1)/\log 2k \geq d$. Denote $E = H(N, I, \mathbf{x})$, where $I = (1/2 - 1/2k, 1/2 + 1/2k)$ and $\mathbf{x} = (x_n)$, $x_n = (2k)^{n-1}$. By the definition of Hausdorff dimension (see A.3) we have $\dim_{\mathcal{H}}(E) := \inf\left\{d \geq 0 | \mathcal{H}^d(E) = 0\right\}$, where

$$\mathcal{H}^d(E) = \lim_{n\to\infty} \mathcal{H}^d_{1/(2x_n k)}(E) = \lim_{n\to\infty} \inf\left\{\sum_{i=1}^{\infty} (\mathrm{diam} U_i)^d \,|\, \bigcup U_i \supset E,\, \mathrm{diam} U_i \leq \frac{1}{2k}\cdot\frac{1}{x_n}\right\}.$$

Fix $n \in \mathbb{N}$. Clearly we are only interested in $d \in [0,1]$. For such d, it is clear that the infimum in the above equation will be attained for

$$\{U_i\} = \mathcal{S}_n := \{R \in \mathcal{R}_n | \, R \subset E_1 \cap ... \cap E_n\},$$

where $E_n = \{x \in \mathbb{T} | \, x x_n \notin I\}$ and

$$\mathcal{R}_n = \{[j, j+1]/2kx_n | \, j = 0, ..., 2kx_n - 1\}.$$

Consequently we have

$$\mathcal{H}^d(E) = \lim_n (\#\mathcal{S}_n)/(2kx_n)^d = \lim_n (\#\mathcal{S}_n)/((2k)^n)^d.$$

By a simple induction we can prove that $\#\mathcal{S}_n = 2^n(k-1)^n$, which then implies that

$$\mathcal{H}^d(E) = \lim_n 2^n(k-1)^n/((2k)^n)^d = \lim_n \left(2(k-1)/(2k)^d\right)^n.$$

Clearly then $\mathcal{H}^d(E) \in (0,\infty) \iff 2(k-1)/(2k)^d = 1 \iff d = \log_{2k} 2(k-1)$, which means that $\dim_{\mathcal{H}}(E) = \log 2(k-1)/\log 2k \geq d$.

(3) Using Salem-Zygmund theorem, we get $E_\xi \in \mathcal{M}$ when $1/\xi = \theta$ is not a Pisot number. Since there exists arbitrarily high numbers which are not Pisot, we have $\mathcal{M}$-sets E_ξ for arbitrarily low ξ. The proposition then follows from the well known fact that when $\xi \searrow 0$, we have $\dim_{\mathcal{H}}(E_\xi) \to 0$ (This can be proven in a similar manner to (1). Alternatively see e.g. [Kar].). □

10.1 Definition and basic properties

Notation 10.4. In this chapter we will always use the following notation: $\mathcal{N} = (N_n)$ is a nondecreasing sequence of integers (with $\lim N_n \in \mathbb{N} \cup \{\infty\}$), $\mathbf{x}$ a sequence of vectors, where $\mathbf{x} = (x_n)$, $x_n = \left(x_n^1, ..., x_n^{N_n}\right) \in \mathbb{N}^{N_n}$ and $\mathcal{I} = (I_n)_{n=1}^{\lim N_n}$

a sequence of open sets in $\mathbb{T}$. We also denote

$$H(\mathcal{N},\mathcal{I},\mathbf{x}) := \bigcap_{n=1}^{\infty} \{x \in \mathbb{T} |\, x \cdot x_n \notin I_1 \times ... \times I_{N_n}\}.$$

Definition 10.5. Let $(\mathcal{N},\mathcal{I},\mathbf{x})$ be as above and suppose that $\lim N_n = \infty$. We say that $\mathbf{x}$ is *quasi-independent* if $\left(\left(x_n^1, ..., x_n^N\right)\right)_{n=1}^{\infty} \in \mathcal{Q}_N$ holds for each $N \in \mathbb{N}$ (where for $k > N_n$ we set $x_n^k = 0$). If this is the case, we denote $\mathbf{x} \in \mathcal{Q}_\infty$.

We say that a set $E \subset \mathbb{T}$ is a $H^{(\infty)}$*-set*, if there exists a tuple $(\mathcal{N},\mathcal{I},\mathbf{x})$ with $\mathbf{x} \in \mathcal{Q}_\infty$ such that

$$E \subset \bigcap_{n=1}^{\infty} \{x \in \mathbb{T} |\, x \cdot x_n \notin I_1 \times ... \times I_{N_n}\}.$$

We then say that $(\mathcal{N},\mathcal{I},\mathbf{x})$ *witnesses that* $E \in H^{(\infty)}$.

Let $L > 0$. If there exists a witnessing tuple $(\mathcal{N},\mathcal{I},\mathbf{x})$ for E, satisfying

$$\forall n \in \mathbb{N}\, \forall k < N_n : \left| \frac{x_n^{k+1} |I_k|}{x_n^k} \right| \geq L,$$

we write $E \in H_L^{(\infty)}$.

Remark 10.6. (1) Similarly to the case of $H^{(N)}$-sets, the closed $H^{(\infty)}$-sets and $H^{(\infty)}$-sets of the form $E = H(\mathcal{N},\mathcal{I},\mathbf{x})$ are of particular interest.

(2) Suppose that $(\mathcal{N},\mathcal{I},\mathbf{x})$ witnesses the fact that $E \in H^{(\infty)}$. Clearly for any increasing sequence (n_k) we have

$$H(\mathcal{N},\mathcal{I},\mathbf{x}) \subset H\left((N_{n_k})_k, \mathcal{I}, (x_{n_k})_k\right),$$

therefore the tuple $\left((N_{n_k})_k, \mathcal{I}, (x_{n_k})_k\right)$ also witnesses that $E \in H^{(\infty)}$.

(3) Let $(\mathcal{N},\mathcal{I},\mathbf{x})$ be as in Notation 10.4 and suppose that $\mathbf{x}$ is quasi-independent. Let $\mathcal{M} = (M_k)$ be a nondecreasing sequence of integers with $\lim M_k = M \in \mathbb{N} \cup \{\infty\}$, $\mathbf{y} \in \mathcal{Q}_M$ a quasi-independent sequence of vectors, where $\mathbf{y} = (y_k)$, $y_k = \left(y_k^1, ..., y_k^{M_k}\right) \in \mathbb{N}^{M_k}$ and $\mathcal{J} = (J_n)_{n=1}^{M}$ a sequence of open sets in $\mathbb{T}$.

Suppose $(\mathcal{M},\mathcal{J},\mathbf{y})$ is "contained" in $(\mathcal{N},\mathcal{I},\mathbf{x})$ in the following sense: For every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ and indices $i(1), ..., i\left(N_{n(k)}\right) \leq M_k$ such that $\forall j \leq N_{n(k)} : I_j = J_{i(j)} \,\&\, x_{n(k)}^j = y_k^{i(j)}$. Then

$$H(\mathcal{N},\mathcal{I},\mathbf{x}) \subset H(\mathcal{M},\mathcal{J},\mathbf{y}) \in H^{(M)}.$$

Proof. (2) is trivial. (3): Let $x \in H(\mathcal{N}, \mathcal{I}, \mathbf{x})$. We need to prove that

$$x \in \bigcap_{k} \{x' \in \mathbb{T} | \, x' \cdot y_k \notin J_1 \times ... \times J_{M_k}\}.$$

Let $k \in \mathbb{N}$. Since $x \in H(\mathcal{N}, \mathcal{I}, \mathbf{x})$, we have $x \cdot x_{n(k)} \notin I_1 \times ... \times I_{N_{n(k)}}$. Since $(\mathcal{M}, \mathcal{J}, \mathbf{y})$ is "contained" in $(\mathcal{N}, \mathcal{I}, \mathbf{x})$, this is equivalent to

$$x \cdot \left(y_k^{i(1)}, ..., y_k^{i\left(N_{n(k)}\right)} \right) \notin J_{i(1)} \times ... \times J_{i\left(N_{n(k)}\right)},$$

which implies $x \cdot y_k \notin J_1 \times ... \times J_{M_k}$. □

Proposition 10.7. (1) *For each* $N \in \mathbb{N}$ *we have* $H^{(N)} \subset H^{(\infty)}$.

(2) *The family* $H^{(\infty)}$ *is an ideal.*

Proof. (1) Let $E \subset H(N, I, \mathbf{x}) \in H^{(N)}$, where $I = I_1 \times ... \times I_N$ is an open interval and $\mathbf{x} \in \mathcal{Q}_N$. We set $N_n = \max\{N, n\}$, $\mathcal{N} = (N_n)$, $I_n = \mathbb{T}$ for $n > N$ and $\mathcal{I} = (I_n)$. Furthermore we set $y_n^k = x_n^k$ whenever $k \leq N$ and find y_n^k for $n \in \mathbb{N}$ and $N < k \leq N_n$ such that $y_n^k \geq n \cdot \max\left\{\left|y_n^l\right| | \, l < k\right\}$. We set $y_n = \left(y_n^1, ..., y_n^{N_n}\right)$. By Remark 8.2 $\mathbf{y} = (y_n)$ is quasi-independent, which implies that $H(\mathcal{N}, \mathcal{I}, \mathbf{y}) \in H^{(\infty)}$. Clearly $(\mathcal{N}, \mathcal{I}, \mathbf{y})$ is contained in $\left((N)_{n=1}^{\infty}, (I_n)_{n=1}^{N}, \mathbf{x}\right)$ in the sense of (3) from Remark 10.6 , which completes the proof, as we then have

$$E \subset H(N, I, \mathbf{x}) = H\left((N)_{n=1}^{N}, (I_n)_{n=1}^{N}, \mathbf{x}\right) \subset H(\mathcal{N}, \mathcal{I}, \mathbf{y}) \in H^{(\infty)}.$$

(2) Clearly $H^{(\infty)}$ is hereditary. It remains to show that the union of two $H^{(\infty)}$-sets is again in $H^{(\infty)}$. Let $E_i \in H^{(\infty)}$, $i = 1, 2$. By definition we have $E_i \subset H((\mathcal{N}^i, \mathcal{I}^i, \mathbf{x}^i)) \in H^{(\infty)}$ for some non-decreasing $\mathcal{N}^i = (N_n^i)$, $\mathcal{I}^i = (I_n^i)$ and $\mathbf{x}^i = (x_{i,n}) \in \mathcal{Q}_\infty$, $x_{i,n} = \left(x_{i,n}^1, ..., x_{i,n}^{N_n^i}\right)$.

By induction we will construct $\mathcal{N} = (N_k)$, $\mathcal{I} = (I_k)$ and $\mathbf{x} = (x_k)$ such that $\mathbf{x}$ is quasi-independent and for $i = 1, 2$, $(\mathcal{N}^i, \mathcal{I}^i, \mathbf{x}^i)$ is contained in $(\mathcal{N}, \mathcal{I}, \mathbf{x})$ in the sense of Remark 10.6. Granting these properties of $(\mathcal{N}, \mathcal{I}, \mathbf{x})$, we can then finish the proof by observing that for each $i \in \{1, 2\}$ we have

$$E_i \subset H\left(\mathcal{N}^i, \mathcal{I}^i, \mathbf{x}^i\right) \overset{10.6}{\subset} H(\mathcal{N}, \mathcal{I}, \mathbf{x}) \in H^{(\infty)}.$$

Construction: $k = 1$: Since $\mathbf{x}^2$ is quasi-independent, we find such $n_1 \in \mathbb{N}$ that

$$\left|\beta \cdot x_{2,n_1} \restriction \left\{1, ..., N_1^1\right\}\right| \geq 2 \left|\alpha \cdot x_{1,1}\right|$$

holds for each $\alpha, \beta \in \{-1, 0, 1\}^{N_1^1}$, $\beta \neq 0$. We then set $N_1 := N_1^1 + N_{n_1}^2$, $(I_n)_{n=1}^{N_1} = (I_n^1)_{n=1}^{N_1^1} \,\hat{}\, (I_n^2)_{n=1}^{N_{n_1}^2}$ and similarly $x_1 = x_{1,1} \,\hat{}\, x_{2,n_1}$.

$k-1 \mapsto k$: Assume that we have already constructed N_n and x_n for $n<k$ and I_i for $i \leq N_{k-1}$. As in the step $k=1$ we find such $n_k \in \mathbb{N}$ that

$$\left|\beta \cdot x_{2,n_k} \upharpoonright \left\{1,...,N_k^1\right\}\right| \geq 2\left|\alpha \cdot x_{1,k}\right| \tag{2}$$

for each $\alpha, \beta \in \{-k,...,k\}^{N_k^1}$, $\beta \neq 0$. Again we set $N_k := N_k^1 + N_{n_k}^2$ and

$$(I_n)_{n=1}^{N_k} := (I_n)_{n=1}^{N_{k-1}} {}^\frown \left(I_n^1\right)_{n=N_{k-1}^1+1}^{N_k^1} {}^\frown \left(I_n^2\right)_{n=N_{n_{k-1}}^2+1}^{N_{n_k}^2}.$$

We can assume that $n_k \geq n_{k-1}$, so that the sequence (N_k) is non-decreasing. Similarly the vector x_k will consist of elements of vectors $x_{1,k}$ and x_{2,n_k}. In order to be able to use Remark 10.6, it remains to make sure the numbers $x_{i,m}^l$ are ordered in the same as the open sets I_l^i. Therefore we take the sequences $x_{1,k} = \left(x_{1,k}^1, ..., x_{1,k}^{N_k^1}\right)$, $x_{2,n_k} = \left(x_{2,n_k}^1, ..., x_{2,n_k}^{N_{n_k}^2}\right)$ and arrange their elements into sequence $x_k = \left(x_k^1, ..., x_k^{N_k^1+N_{n_k}^2}\right)$ as follows:

$$\begin{aligned} x_k := & \left(x_{1,k}^1, ..., x_{1,k}^{N_1^1}\right) {}^\frown \left(x_{2,n_k}^1, ..., x_{2,n_k}^{N_{n_1}^2}\right) {}^\frown \left(x_{1,k}^{N_1+1}, ..., x_{1,k}^{N_2^1}\right) {}^\frown \left(x_{2,n_k}^{N_{n_1}^2+1}, ..., x_{2,n_k}^{N_{n_2}^2}\right) {}^\frown ... \\ & ... {}^\frown \left(x_{1,k}^{N_{k-1}^1+1}, ..., x_{1,k}^{N_k^1}\right) {}^\frown \left(x_{2,n_k}^{N_{n_{k-1}}^2+1}, ..., x_{2,n_k}^{N_{n_k}^2}\right). \end{aligned}$$

$(\mathcal{N}, \mathcal{I}, \mathbf{x})$ **has the desired properties:** From the construction of x_k we immediately get that $(\mathcal{N}, \mathcal{I}, \mathbf{x})$ is contained in $(\mathcal{N}^i, \mathcal{I}^i, \mathbf{x}^i)$ in the sense of Remark 10.6 for both $i=1$ and $i=2$. It remains to prove that $\mathbf{x} \in \mathcal{Q}_\infty$.

Let $\gamma \in \mathbb{Z}^N$ be a non-zero vector for some $N \in \mathbb{N}$. Since $N_n^1 \to \infty$, we can find $n_0 \in \mathbb{N}$ such that $2N_{n_0} \geq N$. Furthermore we can assume that $\gamma \in \{-n_0, ..., n_0\}^N$. Then there exist some $\alpha \in \{-n_0, ..., n_0\}^a$ and $\beta \in \{-n_0, ..., n_0\}^b$, such that for any $n \geq n_0$ we have

$$\begin{aligned} \gamma \cdot x_n \upharpoonright \{1, ...N\} &= \sum_{l=1}^{N} \gamma_l x_n^l = \sum_{l=1}^{a} \alpha_l x_{1,n}^l + \sum_{l=1}^{b} \beta_l x_{2,n_n}^l = \\ &= \alpha \cdot x_{1,n} \upharpoonright \{1, ..., a\} + \beta \cdot x_{2,n_n} \upharpoonright \{1, ..., b\}. \end{aligned} \tag{3}$$

If $\beta = 0$ we immediately get

$$|\gamma \cdot x_n \upharpoonright \{1, ...N\}| = |\alpha \cdot x_{1,n} \upharpoonright \{1, ..., a\}| \to \infty$$

since $\mathbf{x}^1 \in \mathcal{Q}_\infty$ and α and β cannot be both zero vectors. In the same way we can deal with the case $\alpha = 0$.

Assume now that $\alpha, \beta \neq 0$, fix $n \geq n_0$ and set $\tilde{\alpha} = \alpha {}^\frown (0, ..., 0)$, $\tilde{\beta} = \beta {}^\frown (0, ..., 0)$ such that $\tilde{\alpha}, \tilde{\beta} \in \{-n, ..., n\}^{N_n^1}$ (which we can do since $n \geq n_0$ and $a, b \leq N_{n_0}^1 \leq$

N_n^1). Using equations (2) and (3) we get

$$\begin{aligned}
|\gamma \cdot x_n \upharpoonright \{1, ...N\}| &\overset{(3)}{=} \left|\tilde{\alpha} \cdot x_{1,n} + \tilde{\beta} \cdot x_{2,n_n} \upharpoonright \left\{1, ..., N_n^1\right\}\right| \\
&\geq \left|\tilde{\beta} \cdot x_{2,n_n} \upharpoonright \left\{1, ..., N_n^1\right\}\right| - |\tilde{\alpha} \cdot x_{1,n}| \overset{(2)}{\geq} |\tilde{\alpha} \cdot x_{1,n}| \\
&= |\alpha \cdot x_{1,n} \upharpoonright \{1, ..., a\}| \overset{n \to \infty}{\longrightarrow} \infty,
\end{aligned}$$

where the last term tends to infinity because $\mathbf{x}^1 \in \mathcal{Q}_\infty$. □

10.2 Regular $H^{(\infty)}$-sets

In Remark .8.10 we made some observations concerning "regular" $H^{(N)}$-sets. For $H^{(\infty)}$-sets we will discuss these sets in more detail. One of the reasons for doing so is that these observations will illustrate the notions needed in Section 11. The regular $H^{(\infty)}$-sets will also serve as specific examples of $H^{(\infty)}$-sets. In particular we will be able to calculate the measure of such sets, showing that there exist $H^{(\infty)}$-sets of measure zero - a necessary condition for existence of $H^{(\infty)}$-sets of uniqueness.

Notation 10.8. As in Remark 8.10 we denote $E_n^i = \{x \in \mathbb{T} | \, x \cdot x_n^i \notin I_i\}$, $E_n = \bigcup_{i=1}^{N_n} E_n^i$ and to simplify the notation we identify $\mathbb{T}$ with the unit interval $[0, 1]$. We then set

$$E_{n,k} = (E_1 \cap ... \cap E_{n-1}) \cap \left(E_n^1 \cup ... \cup E_n^k\right).$$

For now we will deal with the sets of the form $E = H(\mathcal{N}, \mathcal{I}, \mathbf{x})$. For any sequence of integers $\mathcal{N}$, we can consider the following ordering of the set $\{(n, i) \,|\, n \in \mathbb{N}, \, i = 1, 2, ..., N_n\}$:

$$(1, 1) < (1, 2) < ... < (1, N_1) < (2, 1) < (2, 2) < ....$$

When referring to "the previous" set E_n^i, "the next" x_n^i etc. it will always be with respect to this ordering.

As noted earlier, we can imagine E as a "limit" of sets $E_1, E_2, ...$, where again each E_n is iteratively created in steps $E_{n,1}, E_{n,2}, ..., E_{n,N_n}$. Informally, we can imagine a *regular* $H^{(\infty)}$ set E as a set for which each step $E_{n,i}$ from the construction above "refines" the previous step and each step $E_{n,i}$ "looks the same" at each of the intervals $\left[j/x_m^k, (j+1)/x_m^k\right]$, $j = 0, ..., x_n^i - 1$ which intersect the interior of $E_{m,k}$, for every $(m, k) < (n, i)$. Example of such a set follows below:

Example 10.9 (Existence of regular sets)**.** For each $k \in \mathbb{N}$ let $I_k = \bigcup_{i=1}^{K_k} \left(\frac{a_i}{q_k}, \frac{b_i}{q_k}\right)$ where $0 < a_i < b_i < q_k$ and K_k are integers.. Furthermore for each $n \in \mathbb{N}$, $k \leq N_n$ let $x_n^k \in \mathbb{N}$ be such that $q_k x_n^k | x_n^{k+1}$ (resp. $q_k x_n^{N_n} | x_{n+1}^1$ in case that $k = N_n$). By

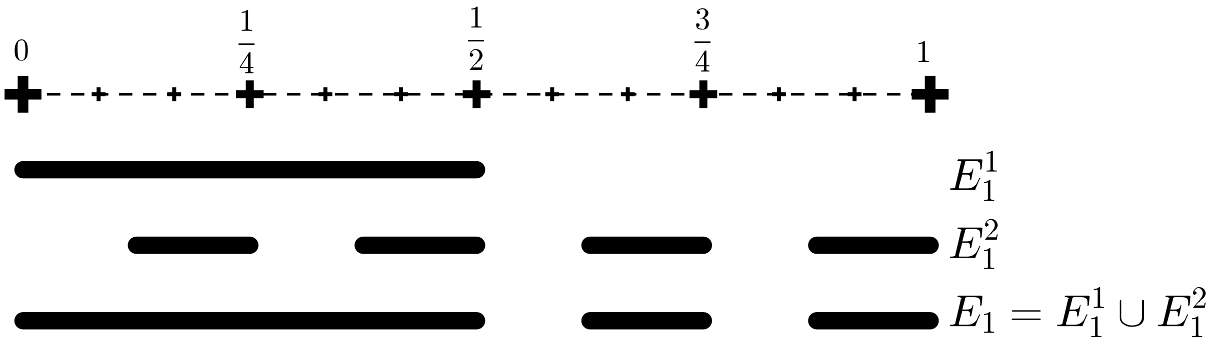

Figure 4: First step of a construction of a regular $H^{(\infty)}$-set E with $N_1 = 2$, $I_1 = \left(\frac{1}{2}, 1\right)$, $I_2 = \left(0, \frac{1}{2}\right)$, $x_1^1 = 1$, $x_1^2 = 4$. In this case we have $q_1 = q_2 = 2$, $\mathcal{G}_1^1 = \{[0,1]\}$, $\mathcal{R}_1^1 = \left\{\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right]\right\}$, $\mathcal{G}_1^2 = \{[j, j+1]/4 |\, j = 0, ..., 3\}$ and $\mathcal{R}_1^2 = \{[j, j+1]/8 |\, j = 0, ..., 7\}$. The system $\mathcal{E}_1^1$ (resp. $\mathcal{E}_1^2$, $\mathcal{E}_{1,2}$) consists of those sets from $\mathcal{R}_1^1$ (resp. $\mathcal{R}_1^2$) which lie in the set E_1^1 (resp. E_1^2, $E_{1,2}$).

Remark 8.2 we can choose x_n^k such that $\mathbf{x}$ is quasi-independent and therefore $E \in H^{(\infty)}$ (or $E \in H^{(N)}$ depending on whether $\limsup_n N_n$ is equal to ∞ or not). We call such E a *regular* $H^{(\infty)}$-set and denote for $n \in \mathbb{N}$, $k \leq N_n$

$$\begin{aligned}
\mathcal{G}_n^k &= \left\{\left[j/x_n^k, (j+1)/x_n^k\right] |\, j = 0, ..., x_n^k - 1\right\} \\
\mathcal{R}_n^k &= \left\{\left[j/qx_n^k, (j+1)/qx_n^k\right] |\, j = 0, ..., q_k x_n^k - 1\right\} \\
\mathcal{E}_n^k &= \left\{R \in \mathcal{R}_n^k |\, R \subset E_n^k\right\} \\
\mathcal{E}_{n,k} &= \left\{R \in \mathcal{R}_n^k |\, R \subset E_{n,k}\right\}
\end{aligned}$$

(the letter "$\mathcal{G}$" stands for "grid" and "$\mathcal{R}$" stands for "refinement" (of the grid)).

Remark 10.10. When $E \in H^{(\infty)}$ is a regular set, we can apply the assertions from Remark 8.10. In particular, we then have the following:

1. There exists a sequence of real numbers ξ_n and sequence of vectors $\vec{\eta_n}$, such that $E = E_{(\xi_n, \vec{\eta_n})}$, i.e. the set E is a symmetric perfect set with dissection ratios $(\xi_n, \vec{\eta_n})$ (in the generalized sense mentioned in the remark).

2. E can be represented as a set of sums

$$\left\{\sum_{n=1}^{\infty} \epsilon_n \xi_1 ... \xi_{n-1} |\, \epsilon_n \in \{\eta_{n,0}, ..., \eta_{n,k_n - 1}\}\right\}.$$

3. Let $(n,i) < (m,j)$ and $C_1, C_2 \in \mathcal{G}_{n,i}$. Suppose that either $C_1, C_2 \subset E_{n,i}$ or $\operatorname{Int}(C_1) \cap E_{n,i} = \operatorname{Int}(C_2) \cap E_{n,i} = \emptyset$. Then the set $C_1 \cap E_{m,j}$ is a translation of the set $C_2 \cap E_{m,j}$. Consequently $C_1 \cap E$ is a translation of $C_2 \cap E$.

Corollary 10.11. *Let $E = H(\mathcal{N}, \mathcal{I}, \mathbf{x})$ be a regular $H^{(\infty)}$-set. The measure of E is then equal to*

$$|E| = \prod_{n=1}^{\infty} (1 - |I_1| \cdot ... \cdot |I_{N_n}|).$$

In particular, there exist $H^{(\infty)}$-sets of measure zero.

Proof. Let $E = H(\mathcal{N}, \mathcal{I}, \mathbf{x})$ be a regular set. We easily observe that for any $n \in \mathbb{N}$, $i \leq N_n$ we have

$$|E_1 \cap ... \cap E_n| = |E_1| \cdot ... \cdot |E_n|, \tag{4}$$

$$\left| \left(E_n^1\right)^C \cap ... \cap \left(E_n^{N_n}\right)^C \right| = \left| \left(E_n^1\right)^C \right| \cdot ... \cdot \left| \left(E_n^{N_n}\right)^C \right| \tag{5}$$

((4) follows from Remark 10.10 3. and the fact that $q_i x_n^i | x_n^{i+1}$, for (5) see Figure 10.9). This allows us to compute $|E_n|$:

$$\begin{aligned} |E_n| &= \left| E_n^1 \cup ... \cup E_n^{N_n} \right| = 1 - \left| \left(E_n^1\right)^C \cap ... \cap \left(E_n^{N_n}\right)^C \right| \\ &\overset{(5)}{=} 1 - \left| \left(E_n^1\right)^C \right| \cdot ... \cdot \left| \left(E_n^{N_n}\right)^C \right| = 1 - |I_1| \cdot ... \cdot |I_{N_n}|, \end{aligned}$$

which, when combined with (4), immediately gives the formula

$$|E| = \prod_{n=1}^{\infty} (1 - |I_1| \cdot ... \cdot |I_{N_n}|).$$

To prove the "in particular" part of the corollary, we need to find such a tuple $(\mathcal{N}, \mathcal{I}, \mathbf{x})$ that the resulting set $E = H(\mathcal{N}, \mathcal{I}, \mathbf{x})$ is of measure zero. To this end we set $N_n := n$, $\mathcal{N} = (N_n)$ and fix a divergent series $\sum_{n=1}^{\infty} \tilde{l}_n$ with $\tilde{l}_n \in (0,1) \cap \mathbb{Q}$. For $n \in \mathbb{N}$ let I_n be an open interval in $\mathbb{T}$ with rational endpoints and length $l_n := \tilde{l}_n / \tilde{l}_{n-1}$ (where $\tilde{l}_0 := 1$) and $\mathcal{I} = (I_n)$. Finally let $\mathbf{x}$ be a quasi-independent sequence such that the set $E := H(\mathcal{N}, \mathcal{I}, \mathbf{x})$ is regular $H^{(\infty)}$-set (such quasi-independent $\mathbf{x}$ exists by Example 10.9). We then have $|E| = 0$, since

$$\prod_{n=1}^{\infty} (1 - |I_1| \cdot ... \cdot |I_n|) = 0 \iff \sum_{n=1}^{\infty} (|I_1| \cdot ... \cdot |I_n|) = \infty$$

and

$$\sum_{n=1}^{\infty} (|I_1| \cdot ... \cdot |I_n|) = \sum_{n=1}^{\infty} \left(\tilde{l}_1 / \tilde{l}_0 \cdot ... \cdot \tilde{l}_n / \tilde{l}_{n-1} \right) = \sum_{n=1}^{\infty} \tilde{l}_n = \infty.$$

□

10.3 $H^{(\infty)}$-sets and sets of uniqueness

In this section we briefly recall some of the results which might be relevant to the problem of deciding whether there exist $H^{(\infty)}$-sets of uniqueness. Since we did not succeed at using these results, we at least make a few notes which explain where our approach failed.

Remark 10.12 ($H^{(N)}$-sets are of uniqueness). . We recall here the Theorem 8.9 of Piatetski-Shapiro which states that each $H^{(N)}$-set E is in U'. The idea of the proof was the following: Suppose that $(\mathcal{I}, \mathbf{x})$ witnesses that $E \in H^{(N)}$. We find functions $0 \leq \varphi_i \in A$ with $\operatorname{supp}(\varphi_i) \subset I_i$ and $\hat{\varphi}_i(0) = 1$. We then set $f_n^i(x) := \varphi_i(x_n^i x)$, $f_n = f_n^1 \cdot ... \cdot f_n^N$ and claim that the sequence (f_n) witnesses that $E \in U'$ - i.e. that

(*i*) f_n have support disjoint from E,

(*ii*) $\sup_n \|f_n\|_A < \infty$ and

(*iii*) $\hat{f}_n(k) \to \hat{1}(k)$ as $n \to \infty$ for each $k \in \mathbb{Z}$.

Condition (*i*) then follows from the definition of f_n and φ_i, condition (*ii*) uses the inequality

$$\|f_n\|_A = \left\|f_n^1 \cdot ... \cdot f_n^N\right\|_A \leq \left\|f_n^1\right\|_A \cdot ... \cdot \left\|f_n^N\right\|_A = \|\varphi_1\|_A \cdot ... \cdot \|\varphi_N\|_A \tag{6}$$

and the last condition follows from the quasi-independence of $\mathbf{x}$.

It was our intent to modify this proof to work for $E \subset H(\mathcal{N}, \mathcal{I}, \mathbf{x}) \in H^{(\infty)}$ as well. Clearly we can define the functions f_n in the same way as above with the only change being $f_n := f_n^1 \cdot ... \cdot f_n^{N_n}$. Such a definition again guarantees that (*i*) is satisfied and under some additional conditions we were able to prove that (*iii*) holds as well (this later part was non-trivial, but possible). There is however a problem with the condition (*ii*) - the original functions φ_i used in the proof of Theorem 8.9 satisfied $\|\varphi_i\| \geq 2$. Therefore we were unable to use the inequality (6) as $\|\varphi_1\|_A \cdot ... \cdot \|\varphi_{N_n}\|_A \to \infty$. Our hope was that we could find such functions φ_n that the inequality $\|\prod f_n^i\|_A \leq \prod \|f_n^i\|_A$ would not be tight. However we were unable to find any such functions. The following example gives a partial explanation of this.

Example 10.13. Let $\mathcal{N}, \mathcal{I}, \mathbf{x}, \varphi_i, f_n^i$ and f_n be as above and suppose that $x_n^{i+1}/x_n^i \to \infty$ as $n \to \infty$ holds for all $i \in \mathbb{N}$. Later we will prove that for any $N \in \mathbb{N}$

$$\sup_n \left\|f_n^1 \cdot ... \cdot f_n^N\right\|_A \geq \|\varphi_1\|_\infty \cdot ... \|\varphi_N\|_\infty . \tag{7}$$

Observe that any non-negative continuous function $\varphi : \mathbb{T} \to \mathbb{R}$ with $\varphi(0) = 0$ and $\hat{\varphi}(0) = \int \varphi = 1$ satisfies $\|\varphi\|_\infty > 1$.

1. Assume that the observed inequality holds uniformly for all φ_i, i.e. there exists $c > 1$ such that $\forall i : \|\varphi_i\|_\infty \geq c > 1$. Equation (7) then implies that the sequence $f_n = \left(f_n^1 \cdot ... \cdot f_n^N\right) \cdot \left(f_n^{N+1} \cdot ... \cdot f_n^{N_n}\right)$ satisfies

$$\lim_N \sup_n \left\|f_n^1 \cdot ... \cdot f_n^N\right\|_A \geq \prod_{i=1}^{\infty} \|\varphi_i\|_\infty = \infty.$$

2. *Conjecture:* Assume that the functions φ_i satisfy $\liminf_i \|\varphi_i\|_\infty = 1$ (as opposite to 1.). We conjecture that in this case $\limsup_i \|\varphi_i\|_A = \infty$ holds and thus, in particular, we have

$$\lim_{n\to\infty} \max_{i \leq N_n} \left\|f_n^i\right\|_A = \infty.$$

Since for $f, g \in A$ we can have $\|fg\|_A < \|f\|_A \|g\|_A$, neither of the cases actually proves that the sequence f_n is unbounded. However, the above presented observations suggest that (f_n) is very unlikely to be bounded.

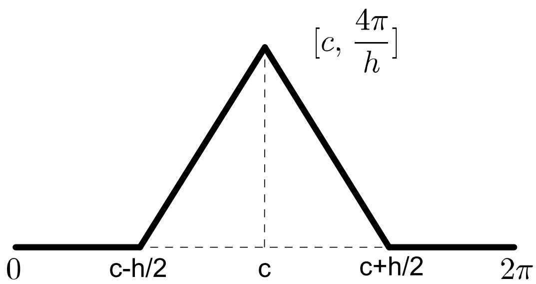

Figure 5: Triangle function φ with $\operatorname{supp}\varphi = [c - h/2, c + h/2]$

Remark. For example when φ_i are the triangle functions (Figure 5) used in the original proof, then $c = 2$. The assumptions we make about $\mathbf{x}$ are not necessarily met for every quasi-independent $\mathbf{x}$, but the sequences we will later use to witness that $\left(H^{(\infty)}\right)^\perp \neq \left(\cup H^{(N)}\right)^\perp$ always do have this property.

Proof of **(7)** . For any non-negative $\varphi \in A$ and $x \in \mathbb{T}$ we have

$$\varphi(x) = \sum \hat{\varphi}(k) e^{ikx} \leq \sum |\hat{\varphi}(k)| \left|e^{ikx}\right| = \sum |\hat{\varphi}(k)| = \|\varphi\|_A, \tag{8}$$

which immediately gives the inequality $\|\varphi\|_A \geq \|\varphi\|_\infty$. Therefore it is enough to show that

$$\left\|f_n^1 \cdot ... \cdot f_n^N\right\|_\infty \to \|\varphi_1\|_\infty \cdot ... \|\varphi_N\|_\infty. \tag{9}$$

We start with the following simple observation: Let $\varphi, \psi \in A$, $\epsilon > 0$ and denote $\psi_n(x) := \psi(nx)$. Fix $x_0 \in \mathbb{T}$ with $|\varphi(x_0)| = \|\varphi\|_\infty$ and $\delta > 0$ satisfying

$$x \in (x_0 - \delta, x_0 + \delta) \implies |\varphi(x)| \geq \|\varphi\|_\infty - \epsilon.$$

Clearly any $\psi_n$is $1/n$ periodic and for $n \geq 1/2\delta$, whole period of ψ_n fits into the interval $(x_0 - \delta, x_0 + \delta)$. Consequently there exists $x_1 \in (x_0 - \delta, x_0 + \delta)$ with $|\psi_n(x_1)| = \|\psi_n\|_\infty = \|\psi\|_\infty$ and for this x_1 we have $|\varphi(x_1)\psi_n(x_1)| \geq (\|\varphi\|_\infty - \epsilon)\|\psi\|_\infty$. This implies that $\|\varphi\psi_n\|_\infty \to \|\varphi\|_\infty \|\psi\|_\infty$.

We now prove 9 by modifying the above observation. Let $N \in \mathbb{N}$ and $\epsilon > 0$. Let $n_0 \in \mathbb{N}$ be high enough (depending on φ_i, $i \leq N$ and $\mathbf{x}$) and fix $n \geq n_0$. By a period of function f_n^i we mean any closed interval $P = [j/x_n^i, (j+1)/x_n^i]$ for some $j = 0, ..., x_n^i - 1$. Fix a period P_1 of f_n^1. Since $f_n^i(x) = \varphi_i(x_n^i x)$, we can find open interval $U_1 \subset P_1$ such that $f_n^1 \geq \|\varphi_1\|_\infty - \epsilon$ and the ratio $|U_1| / |P_1|$ does not depend on n. We know that $x_{n_0}^2 / x_{n_0}^1 \to \infty$, therefore for n_0 high enough, we can find a period P_2 of f_n^2, such that $P_2 \subset U_1$. We then proceed inductively to find periods P_i of f_n^i and open intervals U_i, $i = 1, ..., N$, such that

$$P_1 \supset U_1 \supset P_2 \supset U_2 \supset ... \supset U_N$$

and for any $i \leq N$ we have $f_n^i \geq \|\varphi_i\|_\infty - \epsilon$ on U_i. Then for any $x \in U_N$ we have

$$\left\|f_n^1 \cdot ... \cdot f_n^N\right\|_A \overset{(8)}{\geq} \left(f_n^1 \cdot ... \cdot f_n^N\right)(x) \geq \prod_{i=1}^{N} \left(\|\varphi_i\|_\infty - \epsilon\right),$$

which completes the proof. □

Remark 10.14 (Ultra-thin symmetric sets are of uniqueness). There is another result which might be relevant to the problem of deciding whether there exists $H^{(\infty)}$-sets of uniqueness. By Theorem 7.8, every ultra-thin symmetric set E_{ξ_n} is of uniqueness. Consequently one might wonder whether the proof of this theorem could be modified to work for regular $H^{(\infty)}$-sets as well, i.e. for the "generalized" sets of the form $E = E_{\xi_n, \vec{\eta}_n}$.

The first issue with this approach is that the original proof uses only the symmetric sets $E = E_{\xi_n}$. Since the main reason for this was the ability to represent E as a set of sums, this problem does not seem to be fundamental. However the proof also heavily relies on the fact that the inequalities $\eta_{n,i} \geq \eta_{n,i-1} + \xi_n$ are always strict for E, which means that the direct modification of the proof for regular $H^{(\infty)}$-sets is impossible. Of course this still does not rule out the possibility that an analogous result to Theorem 7.8 might hold for $H^{(\infty)}$-sets as well. An example of such a proposition might be the following: "Let

$E = H(\mathcal{N}, \mathcal{I}, \mathbf{x})$ be a regular $H^{(\infty)}$-set. If $|I_n| \to 1$ holds as $n \to \infty$ and this convergence is "fast enough", then $E \in U$."

11 Sets of type N

The aim of this section is to prove that $\left(H^{(\infty)}\right)^{\perp} \neq \left(\bigcup_N H^{(N)}\right)^{\perp}$. However, this problem has mostly geometrical flavor, rather than number theoretic or analytic. In particular the quasi-independent sequences, which play a central role in the theory of $H^{(N)}$-sets, are of small importance here. Therefore we will work in a slightly different setting which is also a bit more general.

Throughout this section we fix dimension $d \in \mathbb{N}$. For $N \in \mathbb{N} \cup \{\infty\}$ we will define sets of type N in $\mathbb{R}^d$ which somehow correspond to $H^{(N)}$-sets, L-sets of type N which correspond to $H_L^{(N)}$ sets and regular sets of type N which correspond to regular $H^{(N)}$-sets. In particular we will do this in such a way that for $d = 1$, every $H^{(N)}$-set is a set of type N, there exists $L_0 \in \mathbb{R}$ such that every $H_{L_0}^{(N)}$-set is an L-set of type N and every regular $H^{(N)}$-set is also a regular set of type N (see Example 11.9). Our goal will then be to prove the following theorem (which we state properly at the end of this section).

Theorem. (1) *Let $N \in \mathbb{N}$ and $N < M \in \mathbb{N} \cup \{\infty\}$.*

(a) *For any regular set E of type M which satisfies a certain technical condition, there exists $\mu \in \mathcal{M}^1(E)$ such that $\mu(F) = 0$ for every L-set F of type N.*

(b) *$H^{(M)}$-sets always satisfy the required technical condition.*

(2) *There exist regular sets of type ∞ which are of measure zero.*

In Remark 11.10 we observe that in case of $M = \infty$ and $d = 1$, the measure μ also annihilates every $H_{L_0}^{(N)*}$-set. Recall here Vlasák's characterization of $H^{(N)*}$-sets (Theorem 8.8), which implies that for any $N \in \mathbb{N}$ and $L_0 > 0$ we have $\left(H_{L_0}^{(N)*}\right)^{\perp} = \left(H^{(N)*}\right)^{\perp} = \left(H^{(N)}\right)^{\perp}$. Granting the properties of sets of type N listed above, this characterization immediately gives the following corollary.

Corollary. (1) *For any $N \in \mathbb{N}$, $\left(H^{(N+1)}\right)^{\perp} \subsetneq \left(H^{(N)}\right)^{\perp}$.*

(2) $\left(H^{(\infty)} \cap \mathcal{L}\right)^{\perp} \subsetneq \left(\bigcup_N H^{(N)}\right)^{\perp}$ *(where $\mathcal{L} = \{E \subset \mathbb{T} \mid |E| = 0\}$).*

11.1 Definition of a set of type N

Definition 11.1. When S, T are two sets in a topological space such that $\operatorname{Int}(S) \cap \operatorname{Int}(T) = \emptyset$, we say that *$S$ and T do not overlap.*

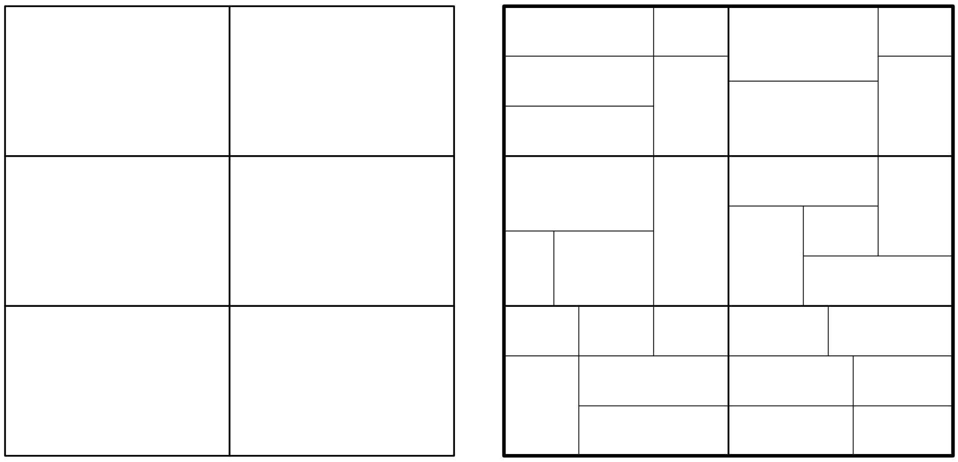

Figure 6: Grid and its refinement.

A finite system $\mathcal{G}$ of subsets of $[0,1]^d$ is a *grid*, if it satisfies the following conditions:

1. Each set $G \in \mathcal{G}$ is a product of closed intervals and $\mathcal{G}$ is a system of non-overlapping sets.

2. $\bigcup \mathcal{G} = [0,1]^d$.

3. For each two sets $G_1, G_2 \in \mathcal{G}$ the set G_1 is a translation of a set G_2.

A finite system $\mathcal{R}$ is a *refinement of a grid* $\mathcal{G}$ if it satisfies the conditions 1. and 2. from above and if its elements satisfy

4. $R \in \mathcal{R} \implies \exists! G \in \mathcal{G} : R \subset G.$

For any system $\mathcal{P}$ of subsets of $[0,1]^d$ and any $S \subset [0,1]^d$ we denote

$$\mathcal{P}^S := \{P \in \mathcal{P} | \, P \subset S\}.$$

We also set

$$\|\mathcal{P}\| := \inf \{\mathrm{diam} P | \, P \in \mathcal{P}\}.$$

Definition 11.2. Let $N \in \mathbb{N} \cup \{\infty\}$. A *scheme of type* N is a tuple $(\mathcal{N}, \mathcal{G}, \mathcal{R}, \mathcal{S})$ satisfying

1. $\mathcal{N} = (N(n))_{n=1}^{\infty}$ is a sequence of integers with $\liminf_n N(n) = N$.
2. $\mathcal{G} = (\mathcal{G}_n^i)$, $\mathcal{R} = (\mathcal{R}_n^i)$, $\mathcal{S} = (\mathcal{S}_n^i)$ are systems indexed by $n \in \mathbb{N}$, $i = 1, ..., N(n)$. For each n, i we have

(a) $\mathcal{G}_n^i$ is a grid,

(b) $\mathcal{R}_n^i$ is a refinement of the grid $\mathcal{G}_n^i$.

(c) $\mathcal{S}_n^i \subset \mathcal{R}_n^i$ satisfies $G_1, G_2 \in \mathcal{G}_n^i \implies 0 < \left|\bigcup \left(\mathcal{S}_n^i\right)^{G_1}\right| = \left|\bigcup \left(\mathcal{S}_n^i\right)^{G_2}\right| < |G_1| = |G_2|$.

Notation 11.3. In this situation we denote by $c_n^i \in (0,1)$ the *coverage ratio* of $\mathcal{S}_n^i$, a number satisfying $G \in \mathcal{G}_n^i \implies \left|\bigcup \left(\mathcal{S}_n^i\right)^{G}\right| = c_n^i |G|$. We also denote $l_n^i = 1 - c_n^i$ and call this number the *loss ratio* of $\mathcal{S}_n^i$. Furthermore we denote

$$\begin{aligned}
S_n^i &:= \bigcup \mathcal{S}_n^i \\
S_n &:= \bigcap_{k=1}^{n} \bigcup_{i=1}^{N(k)} S_k^i \\
S_{n,i} &:= S_{n-1} \cap \left(S_n^1 \cup ... \cup S_n^i\right) \text{ (where } S_0{:=}[0,1]^d \text{)} \\
\mathcal{S}_{n,i} &:= \left\{R \in \mathcal{R}_n^i \middle| \, R \subset S_{n,i}\right\} \\
T\left(\mathcal{N}, \mathcal{G}, \mathcal{R}, \mathcal{S}\right) &:= \bigcap_{n=1}^{\infty} S_n = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{N(n)} S_n^i = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{N(n)} S_{n,i}.
\end{aligned}$$

Definition 11.4. We say that $S \subset [0,1]^d$ is a *set of type* N when there exists a scheme $(\mathcal{N}, \mathcal{G}, \mathcal{R}, \mathcal{S})$ of type N such that $S \subset T\left(\mathcal{N}, \mathcal{G}, \mathcal{R}, \mathcal{S}\right)$.

Notation. As in the case of $H^{(N)}$-sets, we will mostly be interested in the sets S which satisfy $S = T\left(\mathcal{N}, \mathcal{G}, \mathcal{R}, \mathcal{S}\right)$. When a scheme of type N uses a different letters, i.e. $(\mathcal{N}, \mathcal{C}, \mathcal{B}, \mathcal{A})$, we will obviously not denote the respective sets by $\mathcal{G}_n^i$, S_n^i, S etc. but rather by $\mathcal{C}_n^i$, A_n^i, A etc. In order to avoid confusion, we will sometimes add the name of the set as an index to the related variables, i.e. $l_n^i = l_{n,A}^i$ for the loss ratio of $\mathcal{C}_n^i$.

11.2 Restriction to manageable sets

We will need to control the properties of sets of type N in some way. To this end we restrict ourselves to sets of type N which are created from non-flat sets:

Definition 11.5. From now on, $N_{\mathrm{f}} \geq 1$ will be a fixed constant. We say that a set M in a metric space is *non-flat*, if there exist $\eta > 0$, such that $B\left(x, \eta/N_{\mathrm{f}}\right) \subset M \subset B\left(x, \eta N_{\mathrm{f}}\right)$ holds for some $x \in M$.

The key reason for using the notion of being non-flat is the following technical property.

Lemma 11.6 (Key observation: non-flat sets and grids)**.** *There exist constants* $C_{\mathrm{dr}}, c_{\mathrm{mr}} > 0$ *(diameter and measure ratios), such that the following holds: Let*

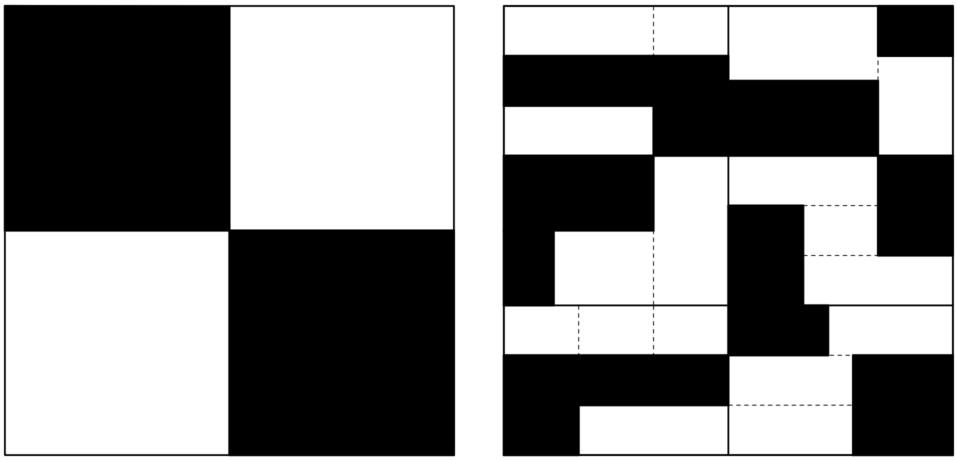

Figure 7: First two steps of a construction of a type 1 set S: S_1^1 for $\mathcal{G}_1^1 = \left\{[0,1]^2\right\}$, $\mathcal{R}_1^1 = \left\{\left[0+i, \frac{1}{2}+i\right] \times \left[0+j, \frac{1}{2}+j\right] \mid i,j \in \left\{0, \frac{1}{2}\right\}\right\}$ and $c_1^1 = \frac{1}{2}$ (left). S_2^1 with $c_2^1 = \frac{4}{9}$ using the grid and refinement from Figure 6 (right). For an example of a "nicer" scheme see Figure 8.

$\mathcal{G}$ be a grid and let $G \in \mathcal{G}$ be it's element. If a system $\mathcal{M}$ of measurable non-flat subsets of $\mathbb{R}^d$ satisfies $\forall M \in \mathcal{M}$: $\operatorname{diam} M \geq C_{\mathrm{dr}} \|\mathcal{G}\|$, then we have $\forall M \in \mathcal{M}$: $\#\mathcal{G}^M \geq c_{\mathrm{mr}} |M| / |G|$.

Proof. Let $G \in \mathcal{G}$, $M \in \mathcal{M}$ be sets and $x \in M$ the point from the definition of a non-flat set. By symbol $B_\eta = B(x, \eta)$ we will denote the closed balls centered at this point. We have $B_r \subset M \subset B_R$ for some $r, R > 0$ satisfying $\frac{r}{R} \geq N_{\mathrm{f}}^{-2}$.

Clearly we have

$$G \cap B_{r - \operatorname{diam} G} \neq \emptyset \implies G \subset M. \tag{10}$$

Thus we can bound the number $\#\mathcal{G}^M$:

$$\begin{aligned}
\#\mathcal{G}^M &= \frac{\left|\bigcup \mathcal{G}^M\right|}{|G|} \geq \frac{\left|\bigcup \mathcal{G}^{B_r}\right|}{|G|} \overset{(10)}{\geq} \frac{\left|B_{r-\|\mathcal{G}\|}\right|}{|G|} = \frac{|M|}{|G|} \frac{\left|B_{r-\|\mathcal{G}\|}\right|}{|B_r|} \frac{|B_r|}{|M|} \\
&\geq \frac{|M|}{|G|} \frac{\left|B_{r-\|\mathcal{G}\|}\right|}{|B_r|} \frac{|B_r|}{|B_R|} = \frac{|M|}{|G|} \left(\frac{r - \|\mathcal{G}\|}{r}\right)^d \left(\frac{r}{R}\right)^d.
\end{aligned}$$

We have $\frac{r - \|\mathcal{G}\|}{r} = 1 - \frac{\|\mathcal{G}\|}{\operatorname{diam} M} \frac{\operatorname{diam} M}{r} \geq 1 - \frac{\|\mathcal{G}\|}{\operatorname{diam} M} \frac{2R}{r} = 1 - \frac{\|\mathcal{G}\|}{\operatorname{diam} M} 2N_{\mathrm{f}}^2$, and so for $\frac{\operatorname{diam} M}{\|\mathcal{G}\|} \geq= 4N_{\mathrm{f}}^2$ we get

$$\#\mathcal{G}^M \geq \frac{|M|}{|G|} \left(\frac{1}{2}\right)^d \left(N_{\mathrm{f}}^{-2}\right)^d.$$

Therefore the numbers $C_{\mathrm{dr}} = 4N_{\mathrm{f}}^2$ and $c_{\mathrm{mr}} = \left(\frac{1}{2}\right)^d \left(N_{\mathrm{f}}^{-2}\right)^d$ are the desired constants. $\square$

Notation 11.7. If two systems $\mathcal{M}$ and $\mathcal{P}$ of subsets of $\mathbb{R}^d$ (not necessarily grids) satisfy the inequality $\forall M \in \mathcal{M} : \operatorname{diam} M \geq C_{\mathrm{dr}} \|\mathcal{P}\|$ from the previous lemma, we write $\mathcal{M} \gg \mathcal{P}$. If they satisfy $\forall M \in \mathcal{M} \operatorname{diam} M > \|\mathcal{P}\|$ (resp. $\geq$), we write $\mathcal{M} > \mathcal{P}$ (resp. $\geq$). Clearly for any scheme $(\mathcal{N}, \mathcal{G}, \mathcal{R}, \mathcal{S})$ of type N and any $n \in \mathbb{N}$, $i \leq N(n)$ we have $\mathcal{G}_n^i > \mathcal{R}_n^i \geq \mathcal{S}_n^i$.

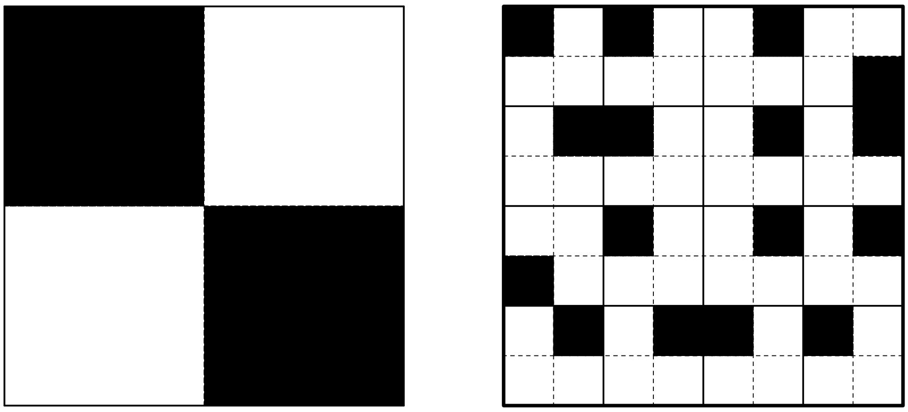

Figure 8: First two steps of the construction of a regular set of type 1. S_1^1 (left) is the same as on Figure 7. S_2^1 (right) satisfies $\mathcal{G}_2^1 = \left\{\left[0+i, \frac{1}{4}+i\right] \times \left[0+j, \frac{1}{4}+j\right] \mid i, j \in \left\{\frac{0}{4}, ..., \frac{3}{4}\right\}\right\}$, $\mathcal{R}_2^1 = \left\{\left[0+i, \frac{1}{8}+i\right] \times \left[0+j, \frac{1}{8}+j\right] \mid i, j \in \left\{\frac{0}{8}, ..., \frac{7}{8}\right\}\right\}$ and $c_2^1 = \frac{1}{4}$. Notice that $\mathcal{G}_2^1$ refines $\mathcal{R}_1^1$ and $\mathcal{R}_1^1$ actually a grid, not just a refinement of $\mathcal{G}_1^1$.

Definition 11.8. Let $S \subset T(\mathcal{N}, \mathcal{G}, \mathcal{R}, \mathcal{S})$ be a set of type $N \in \mathbb{N} \cup \{\infty\}$. We say that S is an *L-set* if it has the following key property

- for each $n \in \mathbb{N}$, $i \leq N(n)$ the systems $\mathcal{G}_n^i$ and $\mathcal{R}_n^i$ consist of non-flat sets and we have
$$i < N(n) \implies \mathcal{R}_n^i \setminus \mathcal{S}_n^i \gg \mathcal{G}_n^{i+1}$$

and if the following technical conditions holds

1. *Monotonicity of $\mathcal{N}$:* If $N \in \mathbb{N}$, then $\mathcal{N}$ is constant. If $N = \infty$, then $\mathcal{N}$ is non-decreasing.

2. *Measure loss control:* For each $i \in \mathbb{N}$ there exists $l_S^i > 0$ such that $l_n^i \geq l_S^i$ holds for each $n \in \mathbb{N}$ with $N(n) \geq i$.

3. *The refinements are not too fine:* For each $i \in \mathbb{N}$ there exists $d_S^i > 0$ such that $\operatorname{diam} R \geq d_S^i \|\mathcal{G}_n^i\|$ holds for each $n \in \mathbb{N}$ with $N(n) \geq i$ and each $R \in \mathcal{R}_n^i$.

We say that S is a *regular* set if it has the following key property

- for each $n \in \mathbb{N}$, $i \leq N(n)$, $\mathcal{R}_n^i$ is a grid and for each $n, m \in \mathbb{N}$, $i \leq N(n)$, $j \leq N(m)$ we have[9]

$$(n,i) < (m,j) \implies \mathcal{G}_m^j \text{ is a refinement of } \mathcal{R}_n^i$$

and if it satisfies the following technical conditions:

1. *S is an L-set.*

2. $\|\mathcal{R}_n^i\|$ *decreases quickly:* For any $(n,i) > (1,1)$ we denote by δ_n^i the number satisfying $\|\mathcal{R}_n^{i-1}\| = \delta_n^i \|\mathcal{R}_n^i\|$ (resp. $\left\|\mathcal{R}_{n-1}^{N(n)}\right\| = \delta_n^1 \|\mathcal{R}_n^1\|$ for $i = 1$). Then the following two conditions hold:

   (a) $\delta_n^i \overset{n\to\infty}{\longrightarrow} \infty$ holds for each i.

   (b) For each $n, m \in \mathbb{N}$, $i \leq N(n)$, $j \leq N(m)$ we have

   $$(n,i) < (m,j) \implies \mathcal{R}_n^i \gg \mathcal{G}_m^j.$$

3. *S is a "true" set of type N: $S = T(\mathcal{N}, \mathcal{G}, \mathcal{R}, \mathcal{S})$.*

Example 11.9. Let $N \in \mathbb{N} \cup \{\infty\}$ and $E \in H^{(N)}$. By Proposition 10.7 (1) we know that there exists some tuple $(\mathcal{N}, \mathcal{I}, \mathbf{x})$ as in Notation 10.4, such that $E \subset H(\mathcal{N}, \mathcal{I}, \mathbf{x})$, $\lim N_n = N$ and $\mathbf{x}$ is quasi-independent. We can assume without loss of generality that $I_i = (a_i, b_i)$, where $0 < a_i < b_i < 1$.

1. Every $H^{(N)}$-set is a set of type N:

We denote $\mathcal{N} = (N)_{n=1}^{\infty}$ and $\mathcal{G} = (\mathcal{G}_n^i)$, $\mathcal{R} = (R_n^i)$, $\mathcal{S} = (\mathcal{S}_n^i)$, where

$$\mathcal{G}_n^i = \left\{ \left[j/x_n^i, (j+1)/x_n^i \right] \mid 0 \leq j \leq x_n^i - 1 \right\},$$

$$\begin{aligned} \mathcal{S}_n^i &= \left\{ \left[j/x_n^i, (j+a_i)/x_n^i \right] \mid j = 0, ..., x_n^i - 1 \right\} \cup \\ &\cup \left\{ \left[(j+b_i)/x_n^i, (j+1)/x_n^i \right] \mid 0 \leq j \leq x_n^i - 1 \right\}, \end{aligned}$$

$$\mathcal{R}_n^i = S_n^i \cup \left\{ \left[(j+a_i)/x_n^i, (j+b_i)/x_n^i \right] \mid 0 \leq j \leq x_n^i - 1 \right\}.$$

These systems then witness that $E \subset T(\mathcal{N}, \mathcal{G}, \mathcal{R}, \mathcal{S})$ is a set of type N.

2. $H_{L_0}^{(N)}$-sets are L-sets for $L_0 \geq C_{\mathbf{dr}}$:

Suppose that the tuple $(\mathcal{N}, \mathcal{I}, \mathbf{x})$ also witnesses that $H_{L_0}^{(N)}$ for some $L_0 \geq C_{\mathrm{dr}}$. We will show that E is an L-set of type N. Clearly all of the systems from $(\mathcal{N}, \mathcal{G}, \mathcal{R}, \mathcal{S})$ consist of 1-dimensional intervals, i.e. of non-flat sets (in this case we could actually use $N_{\mathrm{f}} = 1$). By definition of $H_{L_0}^{(N)}$-set we have $x_n^{i+1} |I_i| / x_n^i \geq L_0$

[9] Recall here the definition of ordering from Notation 10.8.

for $i < N(n)$. Since $\|\mathcal{G}_n^{i+1}\| = 1/x_n^{i+1}$ and $\|\mathcal{R}_n^i \setminus \mathcal{S}_n^i\| = (b_i - a_i)/x_n^i = |I_i|/x_n^i$, this means that for $L_0 \geq C_{\text{dr}}$ we have

$$\left\|\mathcal{R}_n^i \setminus \mathcal{S}_n^i\right\| / \left\|\mathcal{G}_n^{i+1}\right\| = x_n^{i+1} |I_i| / x_n^i \geq L_0 \geq C_{\text{dr}},$$

which implies that E satisfies the key property $\mathcal{R}_n^i \setminus \mathcal{S}_n^i \gg \mathcal{G}_n^{i+1}$. We also have $l_n^i = l_E^i = b_i - a_i$ and $d_A^i = \min\{a_{,i}\ b_i - a_i,\ 1 - b_i\}$, therefore E also satisfies the technical conditions necessary for being an L-set.

Existence of $H^{(N)}$-sets which are also regular sets of type N: Suppose now that $E = H(\mathcal{N}, \mathcal{I}, \mathbf{x})$, $E \in H_{L_0}^{(N)}$ and for each $i \leq N$ we have $I_i = \left(\frac{a_i}{q_i}, \frac{b_i}{q_i}\right)$, where $0 < a_j < b_j < q_i$ are integers, and assume that for each $n \in \mathbb{N}$, $i \leq N$ we have $x_n^i \in \mathbb{N}$ and $q_i x_n^i | x_n^{i+1}$ (resp. $q_N x_n^N | x_{n+1}^1$ in case that $i = N$).

By the previous point, E is an L-set of type N as witnessed by the scheme $(\mathcal{N}, \mathcal{G}, \mathcal{R}, \mathcal{S})$ from the first point. We will either show that E is also a regular set or modify E slightly so that it becomes a regular set (but remains a $H^{(N)}$-set).

Generally, the regularity of E cannot be witnessed by $(\mathcal{N}, \mathcal{G}, \mathcal{R}, \mathcal{S})$, since the systems $\mathcal{R}_n^i$ are not necessarily grids. To remedy this we define $\tilde{\mathcal{R}} := \left(\tilde{\mathcal{R}}_n^i\right)$ and $\tilde{\mathcal{S}} = \left(\tilde{\mathcal{S}}_n^i\right)$, where

$$\tilde{\mathcal{R}}_n^i := \left\{[j, j+1]/q_i x_n^i |\, j = 0, ..., q_i x_n^i - 1\right\},$$

$$\tilde{\mathcal{S}}_n^i := \left\{R \in \tilde{\mathcal{R}}_n^i |\, R \subset \bigcup \mathcal{S}_n^i\right\}.$$

The key property of being a regular set is then satisfied for these systems, since $q_i x_n^i | x_m^j$ holds for $(n, i) < (m, j)$. It remains to prove that E satisfies the condition 2.

We have $\|\mathcal{G}_n^i\| = 1/x_n^i$ and $\left\|\tilde{\mathcal{R}}_n^i\right\| = \left\|\tilde{\mathcal{S}}_n^i\right\| = 1/q_n^i x_n^i$ and we know that $x_n^i \to \infty$. We can assume that for every $i \leq N$ we have $x_n^{i+1}/x_n^i \to \infty$ as $n \to \infty$ (resp. x_{n+1}^1/x_n^N in case that $i = N$) - if this was not true, we could take a different set $\tilde{E} = H(N, I, \tilde{\mathbf{x}})$, where $\tilde{x}_n^{i+1} := x_n^{i+1} x_n^i$, $\tilde{x}_{n+1}^1 := x_{n+1}^1 x_n^{N(n)}$. This implies the technical condition 2.(a) which in turn implies that 2.(b) is satisfied for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Consequently the set $E' = H(\mathcal{N}', \mathcal{I}, \mathbf{x}')$ where $\mathcal{N}' = (N_{n+n_0})_n$, $\mathbf{x}' = (x_{n+n_0})_n$is a $H^{(N)}$-set which is also a regular set of type N.

Note that by Corollary 10.11 we have

$$|E| = 0 \iff \left|\tilde{E}\right| = 0 \iff |E'| = 0.$$

Remark 11.10 ($H^{(N)*}$-sets). When $E = H(N, I, \mathbf{x}) \in H^{(N)*} \setminus H^{(N)}$, we have $x_n^i \in \mathbb{R} \setminus \mathbb{N}$ for some n and i. For $E \in H_{L_0}^{(N)*}$, we can assume without loss of generality that $x_n^i > 2$. Let $\mathcal{N}$, $\mathcal{G}_n^i$, $\mathcal{S}_n^i$ and $\mathcal{R}_n^i$ be as in the representation of

$H^{(N)}$-sets. The tuple $(\mathcal{N}, \mathcal{G}, \mathcal{R}, \mathcal{S})$ is not a scheme of type N, because the systems $\mathcal{G}_n^i$, $\mathcal{R}_n^i$ only cover the interval $[0, \lfloor x_n^i \rfloor / x_n^i]$ rather than the whole interval $[0,1]$ and we have $E \subsetneq \bigcap S_n$. We set

$$\tilde{\mathcal{G}}_n^i = \mathcal{G}_n^i \cup \left\{ \left[\lfloor x_n^i \rfloor / x_n^i, 1 \right] \right\},$$

$$\tilde{\mathcal{R}}_n^i = \mathcal{R}_n^i \cup \left\{ \left[\lfloor x_n^i \rfloor / x_n^i, 1 \right] \right\},$$

$$\tilde{\mathcal{S}}_n^i = \mathcal{S}_n^i \cup \left\{ \left[\lfloor x_n^i \rfloor / x_n^i, 1 \right] \right\}.$$

This new tuple $\left(\mathcal{N}, \tilde{\mathcal{G}}, \tilde{\mathcal{R}}, \tilde{\mathcal{S}}\right)$ is still not a scheme of type N since the elements of $\mathcal{G}_n^i$ necessarily do not have the same diameter, but we at least have $E \cup \{1\} = T\left(\mathcal{N}, \tilde{\mathcal{G}}, \tilde{\mathcal{R}}, \tilde{\mathcal{S}}\right)$.

1. In order to avoid complicating the notation even further, we will not attempt to generalize the definition of sets of type N in such a way that $H_{L_0}^{(N)*}$-sets become L-sets of type N. However by using the representation of $H^{(N)*}$-sets introduced by this remark, we can still prove the main result (i.e. the existence of measure μ, supported on $H^{(\infty)}$-set, which annihilates every $H_{C_{\mathrm{dr}}}^{(N)*}$-set) even for $H_{C_{\mathrm{dr}}}^{(N)*}$-sets. We claim that this will require only minor modifications to the proofs, namely the only affected proof is that of Proposition 11.19, where the addition of the interval $[\lfloor x_n^i \rfloor / x_n^i, 1]$ to the systems $\tilde{\mathcal{S}}_n^i$ might slightly change the value of α. However since $x_n^i > 2$, the new $\tilde{\alpha}$ will not be lower than $\alpha/2^N$. Consequently all of the later propositions will remain valid with the exact same proofs.

2. We only need the $H^{(N)*}$-sets in order to be able to use the results from [Vla] - but in fact, its author only uses rational quasi-independent sequences to prove his results. Therefore we can also observe that if a quasi-independent sequence $\mathbf{x}$ consists of rational numbers, it is possible to represent the resulting $H^{(N)*}$-set as a set of type N. This gives us an alternative to the modification of our proves suggested in 1.

11.3 Technical interlude

Notation 11.11. In the remainder of this chapter we will fix an L-set A of type N, $N \in \mathbb{N}$ and a regular set P of type $M > N$, $M \in \mathbb{N} \cup \{\infty\}$ and we will try to find a measure μ on P which annihilates A. For the set A we will use the notation

$$A = T\left((N), \mathcal{C}, \mathcal{B}, \mathcal{A}\right),$$

in all the following propositions we will have $k \in \mathbb{N}$, $1 \leq i, j \leq N$ and use these numbers for indexing the relevant systems and variables (i.e. $\mathcal{C} = (\mathcal{C}_k^i)$, $l_k^i = 1 - c_k^i$,

...). For the set P we will use the notation

$$P = T\left(\mathcal{N}, \mathcal{S}, \mathcal{R}, \mathcal{P}\right),$$

the integers m, n, p, q will always satisfy $m, n \in \mathbb{N}$, $1 \leq p \leq N(m)$, $1 \leq q \leq N(n)$ and they will be used for indexing the systems and variables related to P. If $M < \infty$, then we assume that $\mathcal{N} = (M)_{n=1}^{\infty}$.

The following simple, but very technically involved, facts will be repeatedly needed during the proofs of main results, so we formulate them in a separate lemma. Note that while the statements of this lemma might look rather complicated, most of them have an intuitive meaning as well as explanation.

Lemma 11.12 (Technical lemma: properties of regular sets)**.** *The following is true for P:*

(i) *$\mathcal{P}_{n,q}$ is a subset of a grid: $\forall n, q\, \forall D, D' \in \mathcal{P}_{n,q} : |D| = |D'|$ & $\operatorname{diam} D = \operatorname{diam} D' = \|\mathcal{P}_{n,q}\|$.*

(ii) *Each two sets from $\mathcal{P}_{m,N(m)}$ contain the same number of elements: For each $D, D' \in \mathcal{P}_{m,N(m)}$, $D \neq D'$ we have*

$$\forall m < n\, \forall q : \#\mathcal{P}_{n,q}^{D} = \#\mathcal{P}_{n,q}^{D'} \,\&\, \mathcal{P}_{n,q}^{D} \cap \mathcal{P}_{n,q}^{D'} = \emptyset.$$

(iii) *Number of elements can be compared via measure ratio: The identity*

$$\#\mathcal{P}_{n,q}^{X} / \#\mathcal{P}_{n,q}^{Y} = \left|\bigcup \mathcal{P}_{n,q}^{X}\right| / \left|\bigcup \mathcal{P}_{n,q}^{Y}\right|,$$

holds for each $X, Y \subset [0,1]^d$, n and q, provided we avoid division by zero. In particular for any $D \in \mathcal{P}_{n,q}$, $X \subset [0,1]^d$:

$$\left|\bigcup \mathcal{P}_{n,q}^{X}\right| = \#\mathcal{P}_{n,q}^{X} \cdot |D|.$$

(iv) *Measure of $P_{n,q}$: If $n = m + 1$ then*

$$\left|\bigcup \mathcal{P}_{n,q}^{D}\right| = |P_{n,q} \cap D| = |D| \left(1 - l_n^1 \cdot \ldots \cdot l_n^q\right)$$

holds for every q, p and $D \in \mathcal{P}_{m,p}$.

(v) *For every $m_0 < m < n$, $p_0 \leq N(m_0)$, $E \in \mathcal{P}_{m_0,p_0}$ and q we have*

$$\#\mathcal{P}_{n,q}^{E} = \sum_{D \in \mathcal{P}_{m,N(m)}^{E}} \#\mathcal{P}_{n,q}^{D} \overset{(ii)}{=} \#\mathcal{P}_{m,N(m)}^{E} \#\mathcal{P}_{n,q}^{\tilde{D}},$$

(where $\tilde{D} \in \mathcal{P}^E_{m,N(m)}$ is arbitrary).

(vi) *Generalization of (v): Under the assumptions of (v) we further have for $p \leq N(m)$ that*

$$\#\mathcal{P}^E_{n,q} = \frac{\left|P_{m,N(m)} \cap E\right|}{\left|P_{m,p} \cap E\right|} \sum_{D \in \mathcal{P}^E_{m,p}} \#\mathcal{P}^D_{n,q} \overset{(ii)}{=} \frac{\left|P_{m,N(m)}\right|}{\left|P_{m,p}\right|} \#\mathcal{P}^E_{m,p} \#\mathcal{P}^{\tilde{D}}_{n,q}$$

(where $\tilde{D} \in \mathcal{P}^E_{m,p}$ is arbitrary).

(vii) *Analogy of Lemma 11.6 for $\mathcal{P}$: Let $n = m+1$ and $H \in \mathcal{B}^j_k$ for some k, j. If we have $\mathcal{B}^j_k \gg \mathcal{S}^q_n$, then*

$$\left|\bigcup \mathcal{P}^H_{n,q}\right| \geq c_{\mathrm{mr}} \left(1 - l^q_n\right) |H| .$$

Moreover if $q \leq \tilde{q} \leq N(n)$, then

$$\left|\bigcup \mathcal{P}^H_{n,\tilde{q}}\right| \geq c_{\mathrm{mr}} \left(1 - l^q_n \cdot l^{q+1}_n \cdot \ldots \cdot l^{\tilde{q}}_n\right) |H| .$$

Using (iv) inductively, we get the following corollary:

Corollary 11.13. *For the regular set P, the measure of $P_{n,q}$ is equal to*

$$\left|P_{n,q}\right| = \left|P_{1,N(1)}\right| \prod_{m=2}^{n-1} \left(1 - l^1_m \cdot \ldots \cdot l^{N(p)}_m\right) \cdot \left(1 - l^1_n \cdot \ldots \cdot l^q_n\right) .$$

In particular P is of measure zero if and only if $\sum_{n=1}^{\infty} l^1_n \cdot \ldots \cdot l^{N(n)}_n = \infty$.

Proof of Lemma 11.12. (i) follows immediately from the fact that each system $\mathcal{P}_{n,q}$ is a subset of grid $\mathcal{R}_{n,q}$ (which consists of isometric sets). To get (ii), we combine the fact that $\mathcal{P}_{n,q} \subset \mathcal{R}_{n,q}$ with the key refinement property of L-sets with (ii). (iii) is again immediate from (i) and the fact that $\mathcal{P}_{n,q}$ is a subset of a grid.

(iv): Note that the first identity is trivial. We prove the statement for $p = N(m)$. The general version for $p \leq N(m)$ follows from the fact that each $D \in \mathcal{P}_{m,p}$ is a disjoint union of elements of $\mathcal{P}_{m,N(m)}$ (combined with (i)).

Tt is enough to show that $\left|D \setminus P_{n,q}\right| = |D|\, l^1_n \cdot \ldots \cdot l^q_n$. Let $q > 1$. The set $D \setminus P_{n,q-1}$ is a union of sets $R_f \in \mathcal{R}^{q-1}_n$, $f \in F$ for some finite index set F. Every R_f is refined by the system $\mathcal{R}^q_n$ and we have

$$\begin{aligned} D \setminus P_{n,q} &= \bigcup_{f \in F} \bigcup \left\{ G \in \left(\mathcal{R}^q_n\right)^{R_f} \mid G \notin \mathcal{P}_{n,q} \right\} \\ &= \bigcup_{f \in F} \bigcup_{S \in \left(\mathcal{S}^q_n\right)^{R_f}} \bigcup \left\{ G \in \left(\mathcal{R}^q_n\right)^{S} \mid G \notin \mathcal{P}_{n,q} \right\} . \end{aligned}$$

Since by definition of l_n^q (Notation 11.3) we have for any $S \in \mathcal{S}_n^q$

$$\left|\bigcup\left\{G \in (\mathcal{R}_n^q)^S \mid G \notin \mathcal{P}_{n,q}\right\}\right| \;=\; l_n^q \left|S\right|$$

we get

$$|D \setminus P_{n,q}| = \sum_F \sum_{S \in \left(\mathcal{S}_n^q\right)^{R_f}} l_n^q \left|S\right| = \sum_F l_n^q \left|R_f\right| = l_n^q \left|D \setminus P_{n,q-1}\right|.$$

By the exact same reasoning we get that $|D \setminus P_{n,1}| = l_n^1 \left|D\right|$. This implies that $|D \setminus P_{n,q}| = |D|\, l_n^1 \cdot \ldots \cdot l_n^q$ holds for each q, which finishes the proof of (iv).

(v) : By the refinement property, $\mathcal{R}_n^q$ refines $\mathcal{R}_m^{N(m)}$ and by the definition (Notation 11.3) we have $\bigcup \mathcal{P}_{n,q} \subset \bigcup \mathcal{P}_{m,N(m)}$. So for $E \in \mathcal{P}_{m_0,N(m_0)}$ we get $F \in \mathcal{P}_{n,q}^E \iff \exists D \in \mathcal{P}_{m,N(m)}^E : F \subset D \;\&\; F \in \mathcal{P}_{n,q}$. This implies $\mathcal{P}_{n,q}^E = \bigcup_{D \in \mathcal{P}_{m,N(m)}^E} \mathcal{P}_{n,q}^D$, and since this union is disjoint, we get the desired result.

(vi): We start with rewriting the sum on the right hand of the desired equality. We use the refinement property (r.p.) and then the fact that each of the elements of $\mathcal{P}_{m,N(m)}$ contains the same number of elements from $\mathcal{P}_{n,q}$ ((ii)):

$$\sum_{D \in \mathcal{P}_{m,p}^E} \#\mathcal{P}_{n,q}^D \overset{\text{r.p.}}{=} \#\mathcal{P}_{n,q}^{\bigcup \mathcal{P}_{m,p}^E} \overset{\text{r.p.}}{=} \#\mathcal{P}_{n,q}^{P_{m,p} \cap E} \overset{(ii)}{=} \#\mathcal{P}_{m,N(m)}^{P_{m,p} \cap E} \cdot \#\mathcal{P}_{n,q}^{\tilde{F}},$$

(where $\tilde{F} \in \mathcal{P}_{m,N(m)}$ is arbitrary).

By the exact same reasoning we see that

$$\#\mathcal{P}_{n,q}^E \overset{(v)}{=} \sum_{D \in \mathcal{P}_{m,N(m)}^E} \#\mathcal{P}_{n,q}^D = \#\mathcal{P}_{m,N(m)}^{P_{m,N(m)} \cap E} \cdot \#\mathcal{P}_{n,q}^{\tilde{F}}.$$

We also note that

$$\frac{\#\mathcal{P}_{m,N(m)}^{P_{m,N(m)} \cap E}}{\#\mathcal{P}_{m,N(m)}^{P_{m,p} \cap E}} \overset{(iii)}{=} \frac{\left|\bigcup \mathcal{P}_{m,N(m)}^{P_{m,N(m)} \cap E}\right|}{\left|\bigcup \mathcal{P}_{m,N(m)}^{P_{m,p} \cap E}\right|} \overset{(iv)}{=} \frac{\left|P_{m,N(m)} \cap E\right|}{\left|P_{m,p} \cap E\right|}.$$

Combining these three equations together finishes the proof.

(vii): Assume first that $\tilde{q} = q$. We have the inclusion $\mathcal{P}_{n,q}^H \supset \bigcup \left\{\mathcal{P}_{n,q}^S | S \in \left(\mathcal{S}_n^q\right)^H\right\}$. This implies $\left|\bigcup \mathcal{P}_{n,q}^H\right| \geq \sum_{S \in \left(\mathcal{S}_n^q\right)^H} \left|\bigcup \mathcal{P}_{n,q}^S\right|$. Since each such $S \in \mathcal{S}_n^q$ is a subset of $P_{m,N(m)}$, we get $\left|\bigcup \mathcal{P}_{n,q}^S\right| \geq |P_n^q \cap S| = c_n^q \left|S\right|$. Thus we can complete the proof as follows:

$$\left|\bigcup \mathcal{P}_{n,q}^H\right| \geq \sum_{S \in \left(\mathcal{S}_n^q\right)^H} \left|\bigcup \mathcal{P}_{n,q}^S\right| \geq \sum_{S \in \left(\mathcal{S}_n^q\right)^H} c_n^q \left|S\right| = \#\left(\mathcal{S}_n^q\right)^H c_n^q \left|\tilde{S}\right| \geq c_{\text{mr}} \frac{|H|}{\left|\tilde{S}\right|} c_n^q \left|\tilde{S}\right|,$$

where $\tilde{S} \in \mathcal{S}_n^q$ is arbitrary and the last inequality comes from Lemma 11.6 (using the definition of the symbol $\ll$ from Notation 11.7).

For general $\tilde{q}$, we observe that for $S \in \mathcal{S}_n^q$ we have

$$\left|\bigcup \mathcal{P}_{n,\tilde{q}}^S\right| = |P_{n,\tilde{q}} \cap S| \geq \left|\left(P_n^q \cup ... \cup P_n^{\tilde{q}}\right) \cap S\right| = \left(1 - l_n^q \cdot ... \cdot l_n^{\tilde{q}}\right)|S|$$

(the proof of this fact is exactly the same as the proof of (ii)) and use this estimate instead. $\square$

11.4 Canonical measure and its properties

Definition 11.14 (Canonical measure on a regular set)**.** We define the *canonical measure μ on $P = T(\mathcal{N}, \mathcal{S}, \mathcal{R}, \mathcal{P})$* by the formula

$$D \in \mathcal{R}_n^{N(n)} \Longrightarrow \mu(D) = \begin{cases} \frac{1}{\#\mathcal{P}_{n,N(n)}} & D \in \mathcal{P}_{n,N(n)} \\ 0 & D \notin \mathcal{P}_{n,N(n)} \end{cases}.$$

Proposition 11.15 (Properties of the canonical measure)**.** *The formula from the previous definition correctly defines a measure on P. The measure μ is a Radon continuous probability measure with* $\operatorname{supp}\mu = P$.

Proof. The standard mass distribution principle (e.g. Proposition 1.7 from [Fal]) implies that the set function μ as defined above can be extended to a measure which has the desired properties. $\square$

Remark 11.16 (Motivation for definition of Lr). Let $\mathcal{G}$ be a grid and let $M, D \subset \mathbb{R}^d$ be bounded measurable sets, such that $|M \setminus D| = 0$ and D is a union of a Lebesgue null set and finite number of elements of $\mathcal{G}$. Then $|D| = \left|\bigcup \mathcal{G}^D\right|$, $|D| = \left|\tilde{E}\right| \#\mathcal{G}^D$ (where $\tilde{E} \in \mathcal{G}$ is arbitrary) and we clearly have

$$\begin{aligned} |M| &\leq \#\left\{E \in \mathcal{G}^D \middle| E \cap M \neq \emptyset\right\} \sup\left\{|M \cap E| \,\middle|\, E \in \mathcal{G}^D\right\} \\ &= \frac{\#\left\{E \in \mathcal{G}^D \middle| E \cap M \neq \emptyset\right\}}{\#\mathcal{G}^D} \sup\left\{|M \cap E| \,\middle|\, E \in \mathcal{G}^D\right\} \#\mathcal{G}^D \\ &\leq \frac{\#\left\{E \in \mathcal{G}^D \middle| E \cap M \neq \emptyset\right\}}{\#\mathcal{G}^D} |D|. \end{aligned}$$

If we wanted a more precise bound on $|M|$, we could start from the second line of the computation above, find a finer grid $\mathcal{G}'$ and use exactly the same steps for each of the sets $M \cap E \subset E$.

In fact, this observation still holds if we replace Lebesgue measure and grid $\mathcal{G}$ by a different measure and arbitrary system of sets which all have the same measure and are disjoint up to a null set. In particular it also holds for the canonical measure μ and any of the systems $\mathcal{P}_{n,N(n)}$, $n \in \mathbb{N}$.

Notation 11.17 (Loss ratio Lr). As we will make extensive use of the previous observation, we denote some of the variables used there by special symbols. By $Cr_{\mathcal{P}}^{D}(M)$ we denote the *coverage ratio*

$$Cr_{\mathcal{P}}^{D}(M) = \frac{\#\left\{E \in \mathcal{P}^{D} \mid E \cap M \neq \emptyset\right\}}{\#\mathcal{P}^{D}}$$

(assuming that $\mathcal{P}^{D}$ is nonempty). We set $Cr_{n,q}^{D} = Cr_{\mathcal{P}_{n,q}}^{D}$ and $Cr_{n,q}^{m,p}(M) = \sup\left\{Cr_{n,q}^{D}(M) \mid D \in \mathcal{P}_{m,p}\right\}$. We also denote the *loss ratio*

$$Lr_{\mathcal{P}}^{D}(M) := 1 - Cr_{\mathcal{P}}^{D}(M) = \frac{\#\left\{E \in \mathcal{P}^{D} \mid E \cap M = \emptyset\right\}}{\#\mathcal{P}^{D}}$$

(and define the corresponding versions with more indices in the obvious way).

Lemma 11.18 (General properties of Cr and Lr)**.** *For any μ-measurable sets $X, Y \subset [0,1]^{d}$ we have the following:*

(1) *Monotonicity: $X \subset Y \implies Cr_{n,q}^{D}(X) \leq Cr_{n,q}^{D}(Y)$ holds for any n, q and any measurable $D \subset [0,1]^{d}$ with $\mathcal{P}^{D} \neq \emptyset$.*

(2) *Relation to μ: If $n_l \nearrow \infty$ is an increasing sequence of integers, then we have*

$$\mu(X) \leq \prod_{l=1}^{\infty} Cr_{n_{l+1},N(n_{l+1})}^{n_l,N(n_l)}(X).$$

(3) *Lr and unions: Suppose that $Lr_{n_1,q_1}^{m_1,p_1}(X) \geq \alpha$ and $Lr_{n_2,q_2}^{m_2,p_2}(Y) \geq \beta$ holds for some integers $m_1 < n_1 \leq m_2 < n_2 = m_2 + 1$ and $p_1 < q_1 \leq p_2 < q_2$. Then*

$$Lr_{n_2,q_2}^{m_1,p_1}(X \cup Y) \geq c\alpha\beta \text{ holds with } c = \frac{\left|P_{n_1,q_1}\right|}{\left|P_{n_1,N(n_1)}\right|}\frac{\left|P_{m_2,p_2}\right|}{\left|P_{m_2,N(m_2)}\right|}.$$

(4) *$Lr^{m,N(m)}$ vs $Lr^{m,1}$: If $m < m_1 < n$, then we have*

$$Lr_{n,N(n)}^{m,N(m)}(X) \geq c Lr_{n,N(n)}^{m_1,1}(X) \text{ for } c = \frac{\left|P_{m_1,1}\right|}{\left|P_{m_1,N(m)}\right|}.$$

Proof. (1) is trivial. For (2), we will use induction over $L \in \{0\} \cup \mathbb{N}$ to prove the inequality

$$\mu(X) \leq \left(\prod_{l=1}^{L} Cr_{n_{l+1},N(n_{l+1})}^{n_l,N(n_l)}(X)\right) \sup\left\{\mu(X \cap E) \mid E \in \mathcal{P}_{n_{L+1},N(n_{L+1})}\right\} \#\mathcal{P}_{n_{L+1},N(n_{L+1})}.$$

The result is then immediate, since for $E \in \mathcal{P}_{n_{L+1},N(n_{L+1})}$ we have $\mu(X \cap E) \leq \mu(E) = 1/\#\mathcal{P}_{n_{L+1},N(n_{L+1})}$.

$L = 0$: We clearly have

$$\begin{aligned}
\mu(X) &= \mu(X \cap P) \leq \mu\left(X \cap P_{n_1,N(n_1)}\right) \\
&= \sum_{E \in \mathcal{P}_{n_1,N(n_1)}} \mu(X \cap E) \leq \sup\left\{\mu(X \cap E) \,|\, E \in \mathcal{P}_{n_1,N(n_1)}\right\} \#\mathcal{P}_{n_1,N(n_1)}.
\end{aligned}$$

$L - 1 \mapsto L$: Let $L \in \mathbb{N}$ and suppose the inequality holds for $L - 1$. Using Remark 11.16 on $X \cap E$ for $E \in \mathcal{P}_{n_L,N(n_L)}$, we get

$$\begin{aligned}
\mu(X \cap E) &\leq \#\left\{F \in \mathcal{P}_{n_{L+1},N(n_{L+1})} | \, F \cap X \cap E \neq \emptyset\right\} \cdot \\
&\quad \cdot \sup\left\{\mu(F \cap X \cap E) \,|\, F \in \mathcal{P}_{n_{L+1},N(n_{L+1})}\right\} \\
&= \frac{\#\left\{F \in \mathcal{P}^E_{n_{L+1},N(n_{L+1})} | \, F \cap X \neq \emptyset\right\}}{\#\mathcal{P}^E_{n_{L+1},N(n_{L+1})}} \cdot \\
&\quad \cdot \sup\left\{\mu(X \cap F) \,|F \in \mathcal{P}^E_{n_{L+1},N(n_{L+1})}\right\} \#\mathcal{P}^E_{n_{L+1},N(n_{L+1})}.
\end{aligned}$$

Taking supremum over $E \in \mathcal{P}_{n_L,N(n_L)}$ we get

$$\sup\left\{\mu(X \cap E) \,|\, E \in \mathcal{P}_{n_L,N(n_L)}\right\} \leq$$

$$\leq Cr^{n_L,N(n_L)}_{n_{L+1},N(n_{L+1})} \cdot \sup\left\{\mu(X \cap F) \,|F \in \mathcal{P}_{n_{L+1},N(n_{L+1})}\right\} \#\mathcal{P}^{\tilde{E}}_{n_{L+1},N(n_{L+1})},$$

where $\tilde{E} \in \mathcal{P}_{n_L,N(n_L)}$ can be arbitrary, since the quantity $\#\mathcal{P}^E_{n_{L+1},N(n_{L+1})}$ is the same for every $E \in \mathcal{P}_{n_L,N(n_L)}$ (Properties of regular sets, Lemma 11.12 (ii)). Combining this inequality with the induction hypothesis, we get

$$\begin{aligned}
\mu(X) &\leq \left(\prod_{l=1}^{L+1} Cr^{n_l,N(n_l)}_{n_{l+1},N(n_{l+1})}(X)\right) \sup\left\{\mu(X \cap F) \,|F \in \mathcal{P}_{n_{L+1},N(n_{L+1})}\right\} \cdot \\
&\quad \cdot \#\mathcal{P}_{n_L,N(n_L)} \#\mathcal{P}^E_{n_{L+1},N(n_{L+1})}.
\end{aligned}$$

Using Lemma 11.12 (v) we get $\#\mathcal{P}_{n_L,N(n_L)} \#\mathcal{P}^E_{n_{L+1},N(n_{L+1})} = \#\mathcal{P}_{n_{L+1},N(n_{L+1})}$, which finishes the proof of (2).

(3) : Let $E \in \mathcal{P}_{m_1,p_1}$. We want to find sets in $\mathcal{P}^E_{n_2,q_2}$ which avoid both X and Y and show that there is a sufficient number of them. To this end we observe that if a set $D \in \mathcal{P}_{n_1,q_1}$ avoids X, then so does any $F \in \mathcal{P}^D_{m_2,p_2}$. This allows us to

bound the loss ratio $Lr^E_{m_2,p_2}(X)$:

$$Lr^E_{m_2,p_2}(X) \stackrel{\text{def}}{=} \frac{\#\left\{F \in \mathcal{P}^E_{m_2,p_2} \middle| F \cap X = \emptyset\right\}}{\#\mathcal{P}^E_{m_2,p_2}} \geq \frac{\sum_{D\in\mathcal{P}^E_{n_1,q_1}, D\cap X=\emptyset} \#\mathcal{P}^D_{m_2,p_2}}{\#\mathcal{P}^E_{m_2,p_2}} = (\star)$$

We continue by using (vi) from Lemma 11.12 ($\tilde{D}$ is again an arbitrary element of $\mathcal{P}^E_{n_1 q_1}$).

$$(\star) = \frac{\#\left\{D\in\mathcal{P}^E_{n_1,q_1}, D\cap X=\emptyset\right\}\#\mathcal{P}^{\tilde{D}}_{m_2,p_2}}{\frac{\left|P_{n_1,N(n_1)}\right|}{\left|P_{n_1,q_1}\right|}\#\mathcal{P}^E_{n_1,q_1}\#\mathcal{P}^{\tilde{D}}_{m_2,p_2}} = Lr^E_{n_1,q_1}(X)\frac{\left|P_{n_1,q_1}\right|}{\left|P_{n_1,N(n_1)}\right|}. \tag{11}$$

Using the same arguments, we now compute the desired loss ratio $Lr^E_{n_2,q_2}(X\cup Y)$:

$$\begin{aligned}
Lr^E_{n_2,q_2}(X\cup Y) \quad & \stackrel{\text{def}}{=} \quad \frac{\#\left\{F\in\mathcal{P}^E_{n_2,q_2}\middle| F\cap X=\emptyset \,\&\, F\cap Y=\emptyset\right\}}{\#\mathcal{P}^E_{n_2,q_2}} \\
& \geq \quad \frac{\sum_{D\in\mathcal{P}^E_{m_2,p_2}, D\cap X=\emptyset}\#\left\{F\in\mathcal{P}^D_{n_2,q_2}\middle| F\cap Y=\emptyset\right\}}{\mathcal{P}^E_{n_2,q_2}} \\
& \stackrel{\text{L11.12}(vi)}{=} \quad \frac{\sum_{D\in\mathcal{P}^E_{m_2,p_2}, D\cap X=\emptyset}\#\left\{F\in\mathcal{P}^D_{n_2,q_2}\middle| F\cap Y=\emptyset\right\}}{\#\mathcal{P}^E_{m_2,p_2}\#\mathcal{P}^{\tilde{D}}_{n_2,q_2}}\frac{\left|P_{m_2,p_2}\right|}{\left|P_{m_2,N(m_2)}\right|} \\
& \geq \quad \frac{\#\left\{D\in\mathcal{P}^E_{m_2,p_2}, D\cap X=\emptyset\right\}}{\#\mathcal{P}^E_{m_2,p_2}}\cdot \\
& \quad\quad \cdot \inf_{D\in\mathcal{P}_{m_2,q_2}}\frac{\#\left\{F\in\mathcal{P}^D_{n_2,q_2}\middle| F\cap Y=\emptyset\right\}}{\#\mathcal{P}^D_{n_2,q_2}}\cdot\frac{\left|P_{m_2,p_2}\right|}{\left|P_{m_2,N(m_2)}\right|} \\
& = \quad Lr^E_{m_2,p_2}(X)\,Lr^{m_2,p_2}_{n_2,q_2}(Y)\frac{\left|P_{m_2,p_2}\right|}{\left|P_{m_2,N(m_2)}\right|} \\
& \stackrel{(11)}{\geq} \quad Lr^{m_1,p_1}_{n_1,q_1}(X)\frac{\left|P_{n_1,q_1}\right|}{\left|P_{n_1,N(n_1)}\right|}Lr^{m_2,q_2}_{n_2,q_2}(Y)\frac{\left|P_{m_2,p_2}\right|}{\left|P_{m_2,N(m_2)}\right|}
\end{aligned}$$

(where $\tilde{D}\in\mathcal{P}_{m_2,q_2}$ is arbitrary).

(4): Fix $E\in\mathcal{P}_{m,N(m)}$. Since $P_{m_1,1}\subset P_{m_1,N(m_1)}$ we have

$$\begin{aligned}
\#\left\{F\in\mathcal{P}^E_{n,N(n)}\middle| F\cap X=\emptyset\right\} \quad & = \quad \sum_{D\in\mathcal{P}^E_{m_1,N(m_1)}}\#\left\{F\in\mathcal{P}^D_{n,N(n)}\middle| F\cap X=\emptyset\right\} \qquad (12)\\
& \geq \quad \sum_{D\in\mathcal{P}^E_{m_1,1}}\#\left\{F\in\mathcal{P}^D_{n,N(n)}\middle| F\cap X=\emptyset\right\}.
\end{aligned}$$

By (vi) in Lemma 11.12 we have

$$\#\mathcal{P}^E_{n,N(n)} = \frac{\left|P_{m_1,N(m_1)}\right|}{\left|P_{m_1,1}\right|}\sum_{D\in\mathcal{P}^E_{m_1,1}}\#\mathcal{P}^D_{n,N(n)} = \#\mathcal{P}^E_{m_1,1}\cdot\#\mathcal{P}^{\tilde{D}}_{n,N(n)}/c, \tag{13}$$

(where $\tilde{D} \in \mathcal{P}^E_{m_1,1}$ is arbitrary and $c = |P_{m_1,1}| / |P_{m_1,N(m_1)}|$). Therefore dividing (12) by $\#\mathcal{P}^E_{n,N(n)}$ yields

$$
\begin{aligned}
Lr^E_{n,N(n)}(X) &\overset{\text{def}}{=} \frac{\#\left\{F \in \mathcal{P}^E_{n,N(n)} \mid F \cap X = \emptyset\right\}}{\#\mathcal{P}^E_{n,N(n)}} \\
&\overset{(12),(13)}{\geq} \frac{\sum_{D \in \mathcal{P}^E_{m_1,1}} \#\left\{F \in \mathcal{P}^D_{n,N(n)} \mid F \cap X = \emptyset\right\}}{\#\mathcal{P}^E_{m_1,1} \#\mathcal{P}^{\tilde{D}}_{n,N(n)}/c} \\
&\geq c\frac{\#\mathcal{P}^E_{m_1,1} \cdot \inf\left\{\#\left\{F \in \mathcal{P}^D_{n,N(n)} \mid F \cap X = \emptyset\right\} \mid D \in \mathcal{P}^E_{m_1,1}\right\}}{\#\mathcal{P}^E_{m_1,1} \#\mathcal{P}^{\tilde{D}}_{n,N(n)}} \\
&\geq c \cdot \inf\left\{\frac{\#\left\{F \in \mathcal{P}^D_{n,N(n)} \mid F \cap X = \emptyset\right\}}{\#\mathcal{P}^D_{n,N(n)}} \mid D \in \mathcal{P}_{m_1,1}\right\} \\
&\overset{\text{def}}{=} cLr^{m_1,1}_{n,N(n)}(X).
\end{aligned}
$$

Since $Lr^{m,N(m)}_{n,N(n)}(X) = \inf\left\{Lr^E_{n,N(n)}(X) \mid E \in \mathcal{P}_{m,N(m)}\right\}$ and E was an arbitrary element of $\mathcal{P}_{m,N(m)}$, the inequality above implies the desired lower bound. $\square$

Remark. Recall here the Definition 11.2 and Notation 11.11: We have $N < M \in \mathbb{N} \cup \{\infty\}$ and the sets $A = T((N), \mathcal{C}, \mathcal{B}, \mathcal{A})$ (L-set of type N) and $P = T(\mathcal{N}, \mathcal{S}, \mathcal{R}, \mathcal{P})$ (regular set of type M), using the notation T (sequence, grid, refinement, subset

Proposition 11.19 (Loss ratio of L-sets)**.** (1) *Suppose that* $\mathcal{P}_{m,p} \gg \mathcal{C}^i_k > ... > \mathcal{C}^j_k \geq \mathcal{B}^j_k \gg \mathcal{S}^q_n$*, where* $n = m+1$*,* $i \geq j$*. Then for* $\tilde{q} \geq q$ *we have*

$$Lr^{m,p}_{n,\tilde{q}}\left(A^i_k \cup ... \cup A^j_k\right) \geq \alpha > 0, \text{ where } \alpha = c^{1+j-i+1}_{mr} \frac{1 - l^q_{n,P}}{1 - l^1_{n,P} \cdot ... \cdot l^q_{n,P}} l^i_{k,A} \cdot ... \cdot l^j_{k,A}.$$

(2) *There exists* $m_0 \in \mathbb{N}$ *(depending only on* A *and* P*), such that for every* $m \geq m_0$ *and* $n = m+1$

$$\|\mathcal{P}_{m,p+1}\| \sqrt{\delta^{p+1}_m} > \left\|\mathcal{C}^i_k\right\| > ... > \left\|\mathcal{C}^j_k\right\| \geq \|\mathcal{P}_{n,q}\| \sqrt{\delta^q_n}$$

is a sufficient condition for (1) *to hold.*

Proof. (1) : Let $E \in \mathcal{P}_{m,p}$.

Step 1: $\mathcal{H}^j(E)$

Firstly we define the system of holes in the set $A^i_k \cup ... \cup A^j_k$. For $M \subset \mathbb{R}^d$ and $l \in \{i, ..., j\}$ we denote by induction

$$\mathcal{H}^l(M) := \left\{H \in \left(\mathcal{B}^l_k \setminus \mathcal{A}^l_k\right)^M \mid l > i \implies H \subset \bigcup \mathcal{H}^{l-1}(M)\right\}.$$

We then get the following properties of $\mathcal{H}^L$:

($\mathcal{H}1$) $H \in \mathcal{H}^j(M) \Longrightarrow H \cap \left(A_k^i \cup \ldots \cup A_k^j\right) = \emptyset$. Thus also

$$\bigcup \mathcal{P}_{n,q}^H \cap \left(A_k^i \cup \ldots \cup A_k^j\right) = \emptyset.$$

($\mathcal{H}2$) We can rewrite $\left|\bigcup \mathcal{H}^i(E)\right|$ in the following way:

$$\begin{aligned}
\left|\bigcup \mathcal{H}^i(E)\right| &\geq \left|\bigcup\left\{\bigcup \mathcal{H}^i(C) \mid C \in \left(\mathcal{C}_k^i\right)^E\right\}\right| \\
&= \left|\bigcup\left\{H \in \left(\mathcal{B}_k^i \setminus \mathcal{A}_k^i\right)^C \mid C \in \left(\mathcal{C}_k^i\right)^E\right\}\right| \\
&= \sum_{C \in \left(\mathcal{C}_k^i\right)^E} \left|\bigcup \left(\mathcal{B}_k^i \setminus \mathcal{A}_k^i\right)^C\right| \\
&= \sum_{C \in \left(\mathcal{C}_k^i\right)^E} l_k^i \left|C\right| = \#\left(\mathcal{C}_k^i\right)^E l_k^i \left|\tilde{C}\right|
\end{aligned}$$

(where $\tilde{C} \in \mathcal{C}_k^i$ is arbitrary). Since $\mathcal{P}_{m,p} \gg \mathcal{C}_k^i$, Lemma 11.6 gives $\#\left(\mathcal{C}_k^i\right)^E \geq c_{\mathrm{mr}} \frac{|E|}{|\tilde{C}|}$. Consequently, we can rewrite the inequality above as

$$\left|\bigcup \mathcal{H}^i(E)\right| \geq \#\left(\mathcal{C}_k^i\right)^E l_{k,A}^i \left|\tilde{C}\right| \geq c_{\mathrm{mr}} \frac{|E|}{\left|\tilde{C}\right|} l_{k,A}^i \left|\tilde{C}\right| = c_{\mathrm{mr}} l_{k,A}^i \left|E\right|.$$

($\mathcal{H}3$) Let $l \in \{i, \ldots, j-1\}$. The systems $\mathcal{C}_k^l, \mathcal{B}_k^l$ correspond to an L-set, so by the definition we have $\mathcal{B}_k^l \setminus \mathcal{A}_k^l \gg \mathcal{C}_k^{l+1}$. Therefore, when $H \in \mathcal{H}^l(M) \subset \mathcal{B}_k^l \setminus \mathcal{A}_k^l$, we can use exactly the same reasoning as above to get $\left|\bigcup \mathcal{H}^{l+1}(H)\right| \geq c_{\mathrm{mr}} l_{k,A}^{l+1} \left|H\right|$. Also, it is clear from the definition of $\mathcal{H}^l(M)$ that

$$\mathcal{H}^{l+1}(M) \supset \bigcup \left\{B \in \mathcal{H}^{l+1}(H) \mid H \in \mathcal{H}^l(M)\right\}.$$

Consequently we get

$$\left|\bigcup \mathcal{H}^{l+1}(E)\right| \geq \sum_{H \in \mathcal{H}^l(E)} \left|\bigcup \mathcal{H}^{l+1}(H)\right| \geq \sum_{H \in \mathcal{H}^l(E)} c_{\mathrm{mr}} l_k^{l+1} \left|H\right| = c_{\mathrm{mr}} l_{k,A}^{l+1} \left|\bigcup \mathcal{H}^l(E)\right|.$$

Combining this inductively with ($\mathcal{H}2$) gives

$$\left|\bigcup \mathcal{H}^j(E)\right| \geq c_{\mathrm{mr}}^{j-i+1} l_{k,A}^i \cdot \ldots \cdot l_{k,A}^j \left|E\right|.$$

Step 2: Computation of Lr

In the following, symbols (i),...,(vii) refer to Lemma 11.12.

$$
\begin{aligned}
Lr^{E}_{n,q}\left(A^i_k\cup\ldots\cup A^j_k\right) &\overset{\text{def}}{=} \frac{\#\left\{D\in\mathcal{P}^E_{n,q}\,|\,D\cap\left(A^i_k\cup\ldots\cup A^j_k\right)=\emptyset\right\}}{\#\mathcal{P}^E_{n,q}}\\
&\overset{(\mathcal{H}1)}{\geq} \sum_{H\in\mathcal{H}^j(E)}\frac{\#\mathcal{P}^H_{n,q}}{\#\mathcal{P}^E_{n,q}} \overset{(iii)}{=} \sum_{H\in\mathcal{H}^j(E)}\frac{\left|\bigcup\mathcal{P}^H_{n,q}\right|}{\left|\bigcup\mathcal{P}^E_{n,q}\right|}\\
&\overset{(vii)}{=} \sum_{H\in\mathcal{H}^j(E)}\frac{\left|\bigcup\mathcal{P}^H_{n,q}\right|}{\left|P_{n,q}\cap E\right|}\\
&\overset{\mathcal{B}^j_k\gg\mathcal{S}^q_n}{\geq} \sum_{H\in\mathcal{H}^j(E)}\frac{c_{\mathrm{mr}}c^q_{n,P}\left|H\right|}{\left|P_{n,q}\cap E\right|}\\
&\overset{(iv)}{=} c_{\mathrm{mr}}\frac{1-l^q_{n,P}}{1-l^1_{n,P}\cdot\ldots\cdot l^q_{n,P}}\frac{\left|\bigcup\mathcal{H}^j\left(E\right)\right|}{|E|}\\
&\overset{(\mathcal{H}3)}{\geq} c_{\mathrm{mr}}\frac{1-l^q_{n,P}}{1-l^1_{n,P}\cdot\ldots\cdot l^q_{n,P}}c^{j-i+1}_{\mathrm{mr}}l^i_{k,A}\cdot\ldots\cdot l^j_{k,A}.
\end{aligned}
$$

For general $\tilde{q}\geq q$, the only difference is that we use the second estimate from (vii):

$$\left|\bigcup\mathcal{P}^H_{n,\tilde{q}}\right|\geq c_{\mathrm{mr}}\left(1-l^q_{n,P}\cdot\ldots\cdot l^{\tilde{q}}_{n,P}\right)|H|\geq c_{\mathrm{mr}}\left(1-l^q_{n,P}\right)|H|\,.$$

(2) : It remains to prove that $\mathcal{P}_{m,p}\gg\mathcal{C}^i_k$ and $\mathcal{B}^j_k\gg\mathcal{S}^q_n$.

$\mathcal{P}_{m,p}\gg\mathcal{C}^i_k$: Using the definition of δ^{p+1}_m, we have

$$\left\|\mathcal{P}_{m,p}\right\|=\sqrt{\delta^{p+1}_m}\left(\sqrt{\delta^{p+1}_m}\left\|\mathcal{P}_{m,p}+1\right\|\right)>\sqrt{\delta^{p+1}_m}\left\|\mathcal{C}^i_k\right\|.$$

To ensure that $\mathcal{P}_{m,p}\gg\mathcal{C}^i_k$ holds, we need to show that $\left\|\mathcal{P}_{m,p}\right\|\geq C_{\mathrm{dr}}\left\|\mathcal{C}^i_k\right\|$. Since $\delta^p_m\overset{m\to\infty}{\longrightarrow}\infty$, this will indeed be true for every m high enough.

$\mathcal{B}^j_k\gg\mathcal{S}^q_n$: Since both A and P are L-sets, we have the following:

$$\left\|\mathcal{B}^j_k\right\|\geq d^j_A\left\|\mathcal{C}^j_k\right\|\ \&\ \left\|\mathcal{C}^j_k\right\|\geq\sqrt{\delta^q_n}\left\|\mathcal{P}_{n,q}\right\|\ \&\ \left\|\mathcal{P}_{n,q}\right\|\geq d^q_P\left\|\mathcal{S}^q_n\right\|$$

$$\left\|\mathcal{B}^j_k\right\|\geq d^j_A\sqrt{\delta^q_n}d^q_P\left\|\mathcal{S}^q_n\right\|.$$

And once again, as δ^q_n goes to infinity, the inequality $\left\|\mathcal{B}^j_k\right\|\geq C_{\mathrm{dr}}\left\|S^q_n\right\|$ will be satisfied for all m high enough. $\square$

Lemma 11.20 (Key lemma: bounding $Lr\left(A^1_k\cup\ldots\cup A^N_k\right)$)**.** (1) *For every m, there exists $n>m$, k and $\epsilon_m>0$ such that*

$$Lr^{m,N(m)}_{n,N(n)}\left(A^1_k\cup\ldots\cup A^N_k\right)\geq\epsilon_m.$$

(2) *If the numbers $l^p_{n,P}$ are constant with respect to n (i.e. if $\forall m,n\,\forall p:\ l^p_{m,P}=l^p_{n,P}$), then $\epsilon_m=\epsilon$ does not depend on m (it only depends on the sets A and P).*

Proof. Let $m \in \mathbb{N}$. Firstly, we set n_0 to be the maximum of $m+1$, the least number $\tilde{m}$ satisfying $N(\tilde{m}) \geq N+1$ and m_0 from (2) in Proposition 11.19. Since $\|\mathcal{C}_k^1\| \searrow 0$, we can set k to be the minimal number satisfying $\sqrt{\delta_{n_0}^2}\mathcal{P}_{n_0,2} > \mathcal{C}_k^1$.

We divide the proof into two sections. In the first section, our goal will be to divide the numbers $\{1, ..., N\}$ into blocks $\{i_1, ..., j_1\}, ..., \{i_s, ..., j_s\}$, and find for each block $i_i, ..., j_i$ numbers m_i, n_i, p_i, q_i, such that we can apply Proposition 11.19 (with the already defined k) to get a bound on $Lr\left(A_k^{i_i} \cup ... \cup A_k^{j_i}\right)$. However, we need to do this in such a way that we can also apply (3) from Lemma 11.18 to bound $Lr\left(A_k^1 \cup ... \cup A_k^N\right)$ and be able to get (1) later. In the second section, we will apply the mentioned propositions in order to actually get (1), then we compute the value of ϵ_m and, grating the constancy of l_n^p-s, bound it from below to get ϵ and thus also (2).

step 1: We will find numbers $s, m_i, n_i, p_i, q_i, i_i, j_i$ for $i = 1, ..., s$ satisfying:

1. Conditions from Proposition 11.19:
   (a) $\|\mathcal{P}_{m_i,p_i+1}\| \sqrt{\delta_{m_i}^{p_i+1}} > \left\|\mathcal{C}_k^{i_i}\right\| > ... > \left\|\mathcal{C}_k^{j_i}\right\| \geq \|\mathcal{P}_{n_i,q_i}\| \sqrt{\delta_{n_i}^{q_i}}$
   (b) $m_i \geq n_0$, $n_i = m_i + 1$ and $i_i \leq j_i$
2. Conditions from Lemma 11.12 (3):
   (a) $p_1 = 1$, $q_s = N+1$ and $i_1 = 1$, $j_s = N$
   (b) $m_{i+1} \geq n_i$, $p_i < q_i = p_{i+1}$ and $i_{i+1} = j_i + 1$

For $i = 0, ..., s$ we denote by (IH_i) the following statement:

$$q_i = j_i + 1 \ \& \ \sqrt{\delta_{n_i}^{q_i+1}}\mathcal{P}_{n_i,q_i+1} > \mathcal{C}_k^{j_i+1}. \qquad (IH_i)$$

$i = 0$: We set $j_0 = 0$, $q_0 = 1$. Clearly we then have $q_0 = j_0 + 1$ and as stated above, we have $\sqrt{\delta_{n_0}^{q_0+1}}\|\mathcal{P}_{n_0,q_0+1}\| > \left\|\mathcal{C}_k^{j_0+1}\right\|$. In other words (IH_0) holds.

$i - 1 \mapsto i$: Let $i \in \{1, ..., s\}$. Assume that we have already constructed the numbers $m_{\tilde{i}}, n_{\tilde{i}}, p_{\tilde{i}}, q_{\tilde{i}}, i_{\tilde{i}}, j_{\tilde{i}}$ for all $\tilde{i} = 1, ..., i-1$, these numbers satisfy the conditions $1.(a)$, $1.(b)$ and $2.(b)$ and that (IH_{i-1}) holds. Our goal will be to define $m_i, n_i, p_i, q_i, i_i, j_i$ in such a way that these numbers also satisfy $1.(a)$, $1.(b)$ and $2.(b)$ and show that (IH_i) holds.

We set $i_i = j_{i-1} + 1$ and $p_i = q_{i-1}$. Since, by (IH_{i-1}), the number $\tilde{m} = n_{i-1}$ satisfies $\|\mathcal{P}_{\tilde{m},p_i+1}\| \sqrt{\delta_{\tilde{m}}^{p_i+1}} > \left\|\mathcal{C}_k^{i_i}\right\|$, we can define m_i as the maximum number with this property and set $n_i = m_i + 1$. The number $p_i = i_i = \tilde{j}$ then trivially satisfies $\tilde{j} \geq i_i$ and, by maximality of m_i, $\left\|\mathcal{C}_k^{\tilde{j}}\right\| \geq \|\mathcal{P}_{n_i,\tilde{j}+1}\| \sqrt{\delta_{n_i}^{\tilde{j}+1}}$. We define j_i as the highest number (not greater than N) with this property and set $q_i = j_i + 1$. Clearly the conditions $1.(a)$, $1.(b)$ and $2.(b)$ are satisfied.

If $j_i = N$, we set $s := i$, observing that 2. (a) holds, and denote $n := n_s$. Combination of Proposition 11.19, and Lemma 11.12 (3) then finishes the proof of (1). If $j_i < N$, we know that, by the maximality of j_i, (IH_i) holds and we continue with the next induction step.

step 2: By Proposition 11.19, we have for $i < s$ and $i = s$ (using the version for $\tilde{q} = N(n_s) \geq q$) the following bounds:

$$Lr^{m_i,p_i}_{n_i,q_i}\left(A^{i_i}_k \cup ... \cup A^{j_i}_k\right) \geq \alpha_i = c^{1+j_i-i_i+1}_{\text{mr}} \frac{1-l^{q_i}_{n_i,P}}{1-l^1_{n_i,P}\cdot ... \cdot l^{q_i}_{n_i,P}} l^{i_i}_{k,A}\cdot ... \cdot l^{j_i}_{k,A} > 0,$$

$$Lr^{m_s,p_s}_{n,N(n)}\left(A^{i_s}_k \cup ... \cup A^{N}_k\right) \geq \alpha_s = c^{1+N-i_s+1}_{\text{mr}} \frac{1-l^{q_s}_{n,P}}{1-l^1_{n,P}\cdot ... \cdot l^{q_s}_{n,P}} l^{i_s}_{k,A}\cdot ... \cdot l^{N}_{k,A} > 0.$$

Iterative use of (3) from Lemma 11.18 (on $X = A^{i_1}_k \cup ... \cup A^{j_1}_k$ and $Y = A^{i_2}_k \cup ... \cup A^{j_2}_k$, then on $X = A^{i_1}_k \cup ... \cup A^{j_2}_k$ and $Y = A^{i_3}_k \cup ... \cup A^{j_3}_k$ and so on) produces the following lower bound.

$$Lr^{m_1,i_1}_{n_s,N(n_s)}\left(A^{i_1}_k \cup ... \cup A^{j_s}_k\right) = Lr^{m_1,1}_{n,N(n)}\left(A^{1}_k \cup ... \cup A^{N}_k\right) \geq \alpha > 0, \text{ where}$$

$$\begin{aligned}
\alpha &= \prod_{i=1}^{s}\alpha_i \prod_{i=1}^{s-1}\frac{|P_{n_i,q_i}|}{\left|P_{n_i,N(n_i)}\right|}\prod_{i=2}^{s}\frac{|P_{m_i,p_i}|}{\left|P_{m_i,N(m_i)}\right|} \\
&= \prod_{i=1}^{s}\alpha_i \prod_{i=1}^{s-1}\frac{1-l^1_{n_i,P}\cdot ... \cdot l^{q_i}_{n_i,P}}{1-l^1_{n_i,P}\cdot ... \cdot l^{N(n_i)}_{n_i,P}}\prod_{i=2}^{s}\frac{1-l^1_{m_i,P}\cdot ... \cdot l^{p_i}_{m_i,P}}{1-l^1_{m_i,P}\cdot ... \cdot l^{N(m_i)}_{m_i,P}}.
\end{aligned}$$

Combining this with Lemma 11.18,(4), we get

$$\begin{aligned}
Lr^{m,N(m)}_{n,N(n)}\left(A^{1}_k \cup ... \cup A^{N}_k\right) &\geq Lr^{m_1,1}_{n,N(n)}\left(A^{1}_k \cup ... \cup A^{N}_k\right)\frac{|P_{m_1,1}|}{\left|P_{m_1,N(m)}\right|} \\
&= \alpha\frac{1-l^1_{m_1,P}}{1-l^1_{m_1,P}\cdot ... \cdot l^{N(m_1)}_{m_1,P}} = \epsilon_m.
\end{aligned}$$

Rearranging the product of α_i-s, bounding the denominators from above by 1 and adding the terms for $i = 1$ and $i = s$ in the products defining α, we can estimate the value of ϵ_m in the following way

$$\begin{aligned}
\epsilon_m \geq\ & (c_{\text{mr}})^{3N} l^1_{k,A}\cdot ... \cdot l^N_{k,A}\left(1-l^1_{m_1,A}\right)\cdot \\
& \cdot\prod_{i=1}^{s}\left[\left(1-l^1_{n_{i,},P}\cdot ... \cdot l^{q_i}_{n_i,P}\right)\left(1-l^1_{m_i,P}\cdot ... \cdot l^{p_i}_{m_i,P}\right)\left(1-l^{q_i}_{n_i,P}\right)\right].
\end{aligned}$$

By the definition of L-sets, we have $\forall k,i : l^i_{k,A} \geq l^i_A \geq l_A > 0$, where $l_A := \min\left\{l^1_A, ..., l^N_A\right\}$. We now use the assumption that the value of $l^p_{n,P}$ is independent of n. By the non-triviality of l-s from definition of an L-set we know that $0 < l_{n,P}^{p} < 1$. Therefore there exists a constant

$$l = l_{P,N} = \sup \left\{ l_{n,P}^{p} \middle| \, n \in \mathbb{N}, \, p \leq N+1 \right\} < 1,$$

dependent on N and P only, such that all of the l-s which appear in the estimate on ϵ_m can be bounded from above by this l. We can now specify the bound on ϵ_m:

$$\begin{aligned} \epsilon_m = \epsilon \;\; & \geq \;\; (c_{\mathrm{mr}})^{3N} \, l_A^N \, (1-l)^{N+1} \prod_{i=1}^{N} (1-l)^2 \\ & \geq \;\; c^N l_A^N \left(c_{P,N}^4 \right)^N , \end{aligned}$$

where we denoted $c = (c_{\mathrm{mr}})^3$ and $c_{P,N} = 1 - l_{P,N}$. □

Definition 11.21. If a set of type N has the property that $\forall m, n \, \forall p : \; l_{m,P}^{p} = l_{n,P}^{p}$, we call it a set of type N with constant loss ratios.

11.5 Main result and its application to $H^{(N)}$-sets

Theorem 11.22. (1) *Fix $M \in \mathbb{N} \cup \{\infty\}$, let P be any regular set of type M with constant loss ratios and denote by μ the canonical measure on P. Then $\mu(A) = 0$ for every L-set A of type strictly lower than M.*

(2) *There exist sets P of measure zero satisfying part* (1) *of this theorem.*

Proof. (1) : Fix an L-set A of type N, $N < M$. Set $n_0 = m_1 = 1$. By Lemma 11.20 we can find $n_1 > m_1$ and k_1 such that

$$Lr_{n,N(n)}^{m_1,N(m_1)} \left(A_{k_1}^1 \cup ... \cup A_{k_1}^N \right) \geq \epsilon$$

for some $\epsilon > 0$. We set $m_2 = n_1$. This way we will inductively find sequences m_l,n_l,k_l satisfying $m_l < n_l$, $m_{l+1} = n_l$ and

$$Lr_{n_l,N(n_l)}^{m_l,N(m_l)} \left(A_{k_l}^1 \cup ... \cup A_{k_l}^N \right) \geq \epsilon.$$

By the properties of Lr (Lemma 11.18), we have

$$\begin{aligned}
\mu(A) &\leq \prod_{l=0}^{\infty} Cr_{n_{l+1},N(n_{l+1})}^{n_l,N(n_l)}(A) \leq \prod_{l=0}^{\infty} Cr_{n_{l+1},N(n_{l+1})}^{n_l,N(n_l)}\left(A_{k_l}^1 \cup \ldots \cup A_{k_l}^N\right) \\
&= \prod_{l=1}^{\infty}\left(1 - Lr_{n_l,N(n_l)}^{m_l,N(m_l)}\left(A_{k_l}^1 \cup \ldots \cup A_{k_l}^N\right)\right) \\
&\leq \prod_{l=1}^{\infty}(1-\epsilon) = 0.
\end{aligned}$$

(2) : By Corollary 11.13 we have

$$\sum_{n=1}^{\infty} l_{n,P}^1 \cdot \ldots \cdot l_{n,P}^{N(n)} = \infty \implies P \text{ is of measure zero.}$$

Therefore if type of P is a finite number, P is automatically of measure zero. If $N = \infty$, set for example $N(n) = n$, $l_{n,P}^p = 1 - 2^{-p}$. Since $\sum_{p=1}^{\infty} 2^{-p} = 1 < \infty$ we get

$$\prod_{p=1}^{\infty} l_{n,P}^p = \prod_{p=1}^{\infty}\left(1 - 2^{-p}\right) = s > 0.$$

Consequently $\sum_{n=1}^{\infty} l_{n,P}^1 \cdot \ldots \cdot l_{n,P}^{N(n)} \geq \sum_{n=1}^{\infty} s = \infty$, which means that $|P| = 0$. $\square$

Example 11.23 (Counterexample: constant loss ratios are needed)**.** Let P be a regular set of type 2 satisfying $c_n^1 = 1 - l_n^1 = \frac{1}{2}$ and $\liminf c_n^2 = 0$.

Denote by n_k some increasing sequence of natural numbers satisfying $c_{n_k}^2 \leq 2^{-k}$. We set

$$\left(\mathcal{C}_k^1, \mathcal{B}_k^1, \mathcal{A}_k^1\right) = \left(\mathcal{S}_{n_k}^1, \mathcal{R}_{n_k}^1, \mathcal{P}_{n_k}^1\right)$$

and denote by A the set of type 1 defined by this sequence. Clearly A is an L-set (even regular). However, we will show that $\mu(A) > 0$.

On each level A_k^1 of the set A, the measure of A decreases exactly by the factor of $\frac{c_{n_k}^1}{c_{n_k}^1 + l_{n_k}^1 c_{n_k}^2}$; formally for $A_k = \bigcap_{l=1}^{k} A_k^1$, $P_n = \bigcap_{m=1}^{n}\left(P_m^1 \cup P_m^2\right)$, $k > 1$ we have the following:

$$\mu(A_k) = \mu(A_{k-1}) \frac{\left|P_{n_k}^1\right|}{\left|P_{n_k}^1 \cup P_{n_k}^2\right|} = \mu(A_{k-1}) \frac{|P_{k-1}|\, c_{n_k}^1}{|P_{k-1}|\left(c_{n_k}^1 + l_{n_k}^1 c_{n_k}^2\right)}.$$

Thus we can compute the measure of A as follows (assuming that $\mu(P) = 1$):

$$\begin{aligned}
\mu(A) &= \mu(P) \prod_{k=1}^{\infty} \frac{c_{n_k}^1}{c_{n_k}^1 + l_{n_k}^1 c_{n_k}^2} = \prod_{k=1}^{\infty} \frac{c_{n_k}^1}{c_{n_k}^1 + l_{n_k}^1 c_{n_k}^2} = \prod_{k=1}^{\infty} \frac{1}{1 + c_{n_k}^2} \\
&\geq \prod_{k=1}^{\infty} \frac{1}{1+2^{-k}} = \prod_{k=1}^{\infty} \left(1 - \frac{1+2^{-k}}{1+2^{-k}} + \frac{1}{1+2^{-k}}\right) \geq \prod_{k=1}^{\infty} \left(1 - \frac{2^{-k}}{1}\right) \\
&> 0, \text{ since } \sum_{k=1}^{\infty} 2^{-k} < \infty.
\end{aligned}$$

This implies that some kind of extra assumptions on the loss ratios of regular set P is needed.

Remark. Note that the same idea as we just described will work for any regular P, provided that there exists a subsequence n_k and indices $p_1, ..., p_s$, such that the loss ratios $l_{n_k}^{p_1}, ..., l_{n_k}^{p_s}$ are uniformly strictly less than 1 and the products of the remaining indices tend to 1 as $k \to \infty$.

However, this does not mean it is impossible to construct a different regular set P, such that μ annihilates all the L-sets of lesser type, but l_n^p-s tend to 1 as $n \to \infty$ (for example when P satisfies $\forall n: \ l_n^1 = l_n^2 = ... = l_n^{N(n)}$).

Corollary 11.24. (1) *For $N \in \mathbb{N}$ we have $\left(H^{(N+1)}\right)^{\perp} \subsetneq \left(H^{(N)}\right)^{\perp}$.*

(2) *Denote $\mathcal{L} = \{E \subset \mathbb{T} \,|\, |E| = 0\}$. Then we have $\left(H^{(\infty)} \cap \mathcal{L}\right)^{\perp} \subsetneq \left(\bigcup_{N\in\mathbb{N}} H^{(N)}\right)^{\perp}$.*

Proof. (1) Fix $N \in \mathbb{N}$. By Example 11.9 we know that for $L_0 \geq C_{\mathrm{dr}}$, every $H_{L_0}^{(N)}$-set is an L-set of type N and we also know that there exist $H^{(N+1)}$-sets which are also regular sets of type N. By the definition of $H^{(N)}$-sets, clearly any $E \in H^{(N)}$ is a set with constant loss ratios. Using Theorem 11.22 we then have $\left(H^{(N+1)}\right)^{\perp} \subsetneq \left(H_{L_0}^{(N)}\right)^{\perp}$ and by Remark 11.10 this also holds for $H^{(N)*}$-sets, i.e. $\left(H^{(N+1)}\right)^{\perp} \subsetneq \left(H_{L_0}^{(N)*}\right)^{\perp}$. We finish the proof by using Vlasák's characterization of $H^{(N)*}$-sets (Theorem 8.8) and the fact that every $H^{(N)*}$-set is a finite union of $H^{(N)}$-sets:

$$\left(H^{(N+1)}\right)^{\perp} \subsetneq \left(H_{L_0}^{(N)*}\right)^{\perp} = \left(H^{(N)*}\right)^{\perp} = \left(H^{(N)}\right)^{\perp}.$$

(2) The proof of the second part of the statement is identical to the proof of (1) - it only remains to prove that there exists a regular $H^{(\infty)}$-set of measure zero, which follows from Corollary 10.11. $\square$

Appendix

A Preliminaries

In the following section we list some of the basics of descriptive set theory. We do not give a complete introduction, instead we rather only present the notions which will be used in this work - i.e. the space $\mathcal{K}(\mathbb{T})$, the family of Borel sets, the collections Σ_3^0 and Π_2^1 and the notion of Π_2^1-completeness. For details on descriptive set theory we refer the interested reader to, for example, [Kec].

A.1 Descriptive set theory

Polish spaces: A *Polish space* is a topological space X which is separable and completely metrizable. Typical examples of such spaces are $\mathbb{R}, \mathbb{R}^d, \mathbb{C}, \mathbb{T}, \mathbb{Z}$ with their classical topologies, the Cantor set 2^ω or the Baire space $\mathcal{N} = \omega^\omega$ with the standard product topologies (where the sets $2 = \{0, 1\}$ and $\omega = \{0, 1, 2, ...\}$ are endowed with discrete topology). Also any metrizable compact space K is clearly a Polish.

Hyperspace $\mathcal{K}(X)$: Another example of a Polish space is given by the following proposition. We will be particularly interested in the space $\mathcal{K}(\mathbb{T})$.

Proposition A.1. *Let X be a metrizable compact space and denote by $\mathcal{K}(X)$ the hyperspace of all compact subsets of X*

$$\mathcal{K}(X) = \{F \subset X \mid F \text{ is closed}\} = \{F \subset X \mid F \text{ is compact}\}.$$

Furthermore we denote by $\mathcal{V}$ the so-called Vietoris topology, i.e. a topology generated by the collection of all sets of the form $\{F \subset X \mid F \subset U\}$ and $\{F \subset X \mid F \cap U \neq \emptyset\}$, where $U \subset X$ is open.

Endowed with the topology $\mathcal{V}$, the space $\mathcal{K}(X)$ is Polish.

Proof. This can be proven by showing that the topology $\mathcal{V}$ coincides with the topology generated by the so-called Hausdorff metric ϱ_H. For the complete proof see for example [Kec]. □

Borel hierarchy, Projective hierarchy: Let X be a fixed Polish space. *Borel sets* in X are the elements of the smallest σ-algebra containing all open sets. By Π_1^0 we denote the system of closed subsets of X and by Σ_3^0 we denote the system of all $G_{\delta\sigma}$ subsets of X (i.e. all sets S of the form $S = \bigcup_{n=1}^{\infty} H_n$ where the sets H_n are G_δ). We say that a subset A of X is *analytic in X*, if there exists a continuous mapping $\varphi : \omega^\omega \to X$, such that $\varphi(\omega^\omega) = A$. We say that a set

$C \subset X$ is *coanalytic in* X if the set $X \setminus C$ is analytic in X. We denote by $\Sigma_1^1(X)$ (resp. $\Pi_1^1(X)$) the system of all analytic (resp. coanalytic) subsets of X. Similarly we denote

$$\Sigma_2^1(X) := \left\{ S \subset X \mid \exists \varphi : \omega^\omega \to X \text{ continuous s.t. } S = \varphi(C) \text{ for some } C \in \Pi_1^1(\omega^\omega) \right\}$$

and $\Pi_2^1(X) := \{T \subset X \mid X \setminus T \in \Sigma_2^1(X)\}$.

Π_2^1-complete set: A topological space Y is said to be *0-dimensional* when its topology admits a clopen basis. A set S in some Polish space X is said to be Π_2^1-complete when $S \in \Pi_2^1(X)$ and for every 0-dimensional space Y and every $T \in \Pi_2^1(Y)$ there exists a continuous mapping $\varphi : Y \to X$ such that $\varphi^{-1}(X) = Y$.

A.2 Fourier transform on $\mathbb{T}$

Spaces c_0, l^1 and l^∞: We use the symbols $c_0 = c_0(\mathbb{Z})$, $l^1 = l^1(\mathbb{Z})$ and $l^\infty = l^\infty(\mathbb{Z})$ to denote the usual Banach spaces with their standard norms. Recall that we have $c_0^* = l^1$ and $(l^1)^* = l^\infty$ where the dualities are given by the mapping $\langle c, d \rangle \mapsto \sum_{k \in \mathbb{Z}} c(k)\, d(k)$ (where either $c \in c_0$ and $d \in l^1$ or $c \in l^1$ and $d \in l^\infty$). On l^1 we can also consider the w^*-topology. By a w^*-sequential closure of a set $M \subset l^1$ we will denote the set $\left\{ x \in l^1 \mid \exists x_n \in M,\, n \in \mathbb{N} :\, x_n \xrightarrow{w^*} x \right\}$. Note that this topology is not metrizable, therefore the w^*-sequential closure is usually not the same as w^*-closure. Furthermore we have the following equivalence for $x_n, x \in l^1$:

$$x_n \xrightarrow{w^*} x \iff \sup_n \|x_n\|_{l^1} < \infty \;\&\; \forall k \in \mathbb{Z} :\, x_n(k) \to x(k)\,.$$

We also note that l^1 with convolution $*$ defined as

$$c * d = e, \text{ where } e = (e(k))_{k \in \mathbb{Z}} \text{ is given by } e(k) := \sum_{m \in \mathbb{Z}} c(m)\, d(k-m)$$

is a Banach algebra.

Spaces $\mathcal{C}, L^1$ and $\mathcal{M}$: By λ we will denote the normalized Lebesgue measure on $\mathbb{T}$ given by the identification of $\mathbb{T}$ with $[0, 2\pi]$. By $L^1(\mathbb{T})$ we will denote the space of complex functions on $\mathbb{T}$ which are λ-integrable, endowed with the usual norm. By $\mathcal{C}(\mathbb{T})$ we denote the space of all continuous complex functions on $\mathbb{T}$ with the supremum norm. By $\mathcal{M}(\mathbb{T})$ we denote the space of all (complex) Radon measures on $\mathbb{T}$ with the norm $\|\mu\|_{\mathcal{M}(\mathbb{T})} = |\mu|(\mathbb{T})$, where $|\mu|$ is the total variation of the measure μ. For $E \subset \mathbb{T}$ the symbol $\mathcal{M}(E)$ will stand for all the measures from $\mathcal{M}(\mathbb{T})$ which are supported by E. We note that every function f from $\mathcal{C}(\mathbb{T})$ or $L^1(\mathbb{T})$ can be identified with a measure μ from $\mathcal{M}(\mathbb{T})$ by the formula

$\int g\mathrm{d}\mu = \int fg\mathrm{d}\lambda$.

Fourier transform on $\mathbb{T}$: A *trigonometric series* S is a formal expression $S \sim \sum_{k\in\mathbb{Z}} c_k e^{ikx}$ where $x \in \mathbb{T}$ and the coefficients c_k belong to $\mathbb{C}$. We say that S converges at x when the symmetric partial sums $\sum_{k=-N}^{N} c_k e^{ikx}$ converge to some $f(x) \in \mathbb{C}$. We say that S converges when it converges at every $x \in \mathbb{T}$.

Fourier series of a function $f \in L^1(\mathbb{T})$ is the sequence is the trigonometric series $S(f) \sim \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{ikx}$, where $\hat{f}(k)$ is the *k-th Fourier coefficient of f*, given by the formula

$$\hat{f}(k) = \int f(x) e^{-ikx} \mathrm{d}\lambda(x).$$

The mapping $\hat{}: f \mapsto \left(\hat{f}(k)\right)_{k\in\mathbb{Z}}$ is called the *Fourier transform.* More generally we can consider also the *Fourier transform of a measure* $\mu \in \mathcal{M}(\mathbb{T})$ given by the formula

$$\mu \mapsto (\hat{\mu}(k))_{k\in\mathbb{Z}}, \ \hat{\mu}(k) := \int e^{-ikx} \mathrm{d}\mu.$$

Fourier transform and spaces L^1 and $\mathcal{M}$: For $f \in L^1(\mathbb{T})$ we have by Riemann-Lebesgue lemma that $\hat{f}(k) \to 0$ as $|k| \to \infty$, which gives us

$$f \in L^1(\mathbb{T}) \implies \hat{f} \in c_0.$$

By the Uniqueness Theorem for Fourier series, this map is one-to-one. It is immediate from the definition of $\hat{\mu}(k)$ that for $\mu \in \mathcal{M}(\mathbb{T})$ we have $|\hat{\mu}(k)| \leq \|\mu\|_{\mathcal{M}(\mathbb{T})}$ for any $k \in \mathbb{Z}$. This implies that

$$\mu \in \mathcal{M} \implies \hat{\mu} \in l^\infty.$$

Whenever $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{T})$ are distinct, we have $\int f\mathrm{d}\mu_1 \neq \int f\mathrm{d}\mu_2$ for some $f \in \mathcal{C}(\mathbb{T})$. Since trigonometric polynomials are dense in $\mathcal{C}(\mathbb{T})$, $f = \sum c_k e^{ikx}$ can be a trigonometric polynomial. Finally because for such f we have $\int f\mathrm{d}\mu_i = \sum c_k \hat{\mu}_i(k)$, Fourier transform is necessarily one-to-one on $\mathcal{M}(\mathbb{T})$ as well. In this sense, every function $f \in L^1(\mathbb{T})$ can be identified with a sequence from c_0 and every measure $\mu \in \mathcal{M}(\mathbb{T})$ can be viewed as a bounded sequence. Note however that the inverse mappings

$$\check{}: (c(k))_{k\in\mathbb{Z}} \mapsto \left(x \in \mathbb{T} \mapsto \sum_{k\in\mathbb{Z}} c(k) e^{ikx} \in \mathbb{C}\right)$$

are *not* always correctly defined on either of the spaces c_0 and l^∞ and thus we cannot identify c_0 with $L^1(\mathbb{T})$ nor l^∞ with $\mathcal{M}(\mathbb{T})$. At best, every $c \in l^\infty$ can be identified with a linear operator on the set $A := A(\mathbb{T}) := \left\{f \in L^1(\mathbb{T}) \,|\, \hat{f} \in l^1\right\}$

by the formula $\langle f, c\rangle := \sum_{k\in\mathbb{Z}} \hat{f}(k)\, c(k)$.

Fourier transform and l^1: Clearly when $c \in l^1$, the fact that $\left\|e^{ikx}\right\|_\infty = 1$ gives the following:

$$f_N(x) = \sum_{|k|\leq N} c(k)\, e^{ikx} \rightrightarrows \sum_{k\in\mathbb{Z}} c(k)\, e^{ikx} =: f(x) \text{ on } \mathbb{T} \text{ as } N \to \infty.$$

Since the functions f_N are continuous we get that $f \in \mathcal{C}(\mathbb{T})$ holds as well. The fact that $\hat{}: \mathcal{C}(\mathbb{T}) \to c_0$ is continuous and $\widehat{\exp(ikx)}(l) = \delta_{kl}$ then implies that $\hat{f} = c$. Consequently we can identify l^1 with a subspace of $\mathcal{C}(\mathbb{T})$ - in fact this means we identify l^1 with the set A from the previous paragraph, which is actually of the form

$$A = \left\{ f \in \mathcal{C}(\mathbb{T}) \mid \hat{f} \in l^1 \right\}.$$

It is also well known that for $f, g \in L^1(\mathbb{T})$ with $\hat{f}, \hat{g} \in l^1$ we have $\widehat{fg} = \hat{f} * \hat{g}$. Thus the identification of A with l^1 by the Fourier transform also respects the natural multiplication operations defined on these spaces.

A.3 Hausdorff dimension

Hausdorff measure: Let X be a metric space, $S \subset X$ and $d \in [0, \infty)$. For $\delta > 0$ we define

$$\mathcal{H}^d_\delta(S) := \inf\left\{ \sum_{i=1}^{\infty} (\mathrm{diam} U_i)^d \mid \bigcup U_i \supset S,\ \mathrm{diam} U_i < \delta \right\}.$$

and set $\mathcal{H}^d(S) := \sup_{\delta>0} \mathcal{H}^d_\delta(S)$. The number $\mathcal{H}^d(S)$ is called the *outer d-dimensional Hausdorff measure of the set S*. Since $\mathcal{H}^d_\delta(S)$ is clearly monotone with respect to δ, we have $\mathcal{H}^d(S) = \lim_{\delta\to 0+} \mathcal{H}^d_\delta(S)$. When $d \in \mathbb{N}$ and $X = \mathbb{R}^d$, the measure $\mathcal{H}^d$ coincides with the d-dimensional Lebesgue measure, up to a multiplicative constant (on, for example, Borel sets).

Hausdorff dimension: *Hausdorff dimension* $\dim_{\mathcal{H}}(S)$ *of a set S* can be defined in one of the following equivalent ways:

$$\dim_{\mathcal{H}}(S) := \inf\left\{ d \geq 0 \mid \mathcal{H}^d(S) = 0 \right\} = \sup\left\{ d \geq 0 \mid H^d(S) = \infty \right\}$$

(where we set $\inf \emptyset = \infty$, $\sup \emptyset = 0$).

A.4 Cantor-Bendixson rank

Perfect sets: For $x \in \mathbb{R}^d$ we denote by $\mathcal{U}(x)$ the collection of all open neighborhoods of x. A point x of a set $C \subset \mathbb{R}^d$ is said to be *isolated (in C)* when there exists such $U \in \mathcal{U}(x)$ that $U \cap C = \{x\}$. The set C is said to be *perfect* if it contains no isolated points (in C).

Definition of r_{CB}: For $C \subset \mathbb{R}^d$ we define the *Cantor-Bendixson derivative C' of the set C as*

$$C' := C \setminus \{x \in C |\, x \text{ is isolated in } C\}.$$

We denote $C^{(0)} := C$ and for an ordinal $\alpha > 0$ we set

$$C^{(\alpha)} := \bigcap_{\beta<\alpha} \left(C^{(\beta)}\right)'.$$

Finally the *Cantor-Bendixson rank* $r_{CB}(C)$ *of a set* C is the least ordinal α such that $C^{(\alpha+1)} = C^{(\alpha)}$. Clearly everytime $C^{(\alpha+1)} \subsetneq C^{(\alpha)}$, there exists a basic set $B \subset C^{(\alpha)} \setminus C^{(\alpha+1)}$ which we have removed by taking the derivative. Since we can take a countable basis for $\mathbb{R}^d$, the rank of each $C \subset \mathbb{R}^d$ is necessarily at most countable and therefore r_{CB} is well defined. It is also immediate that $r_{\mathrm{CB}}(C) = 0 \iff C$ is perfect.

A.5 Bernstein sets

Definition A.2. A set $B \subset \mathbb{T}$ is said to be a *Bernstein set* if B intersects every closed perfect subset of $\mathbb{T}$ but contains none of them.

Existence: Assuming that the axiom of choice holds, we can find enumeration P_α, $\alpha < 2^\omega$ of all the closed perfect subsets of $\mathbb{T}$ (clearly 2^ω indices will suffice, as there is at most 2^ω closed subsets of $\mathbb{T}$). We then inductively find $x_\alpha, y_\alpha \in P_\alpha$ for $\alpha < 2^\omega$ such that $x_\alpha \neq y_\alpha$ and $x_\alpha, y_\alpha \in P_\alpha \setminus \bigcup_{\beta<\alpha} \{x_\beta, y_\beta\}$ (clearly this is possible since for every $\alpha < 2^\omega$ we have by Perfect set theorem

$$\mathrm{Card}\,(P_\alpha) = 2^\omega > \mathrm{Card}\,(\alpha) = \mathrm{Card}\left(\bigcup_{\beta<\alpha} \{x_\beta, y_\beta\}\right)).$$

The set $B := \{x_\alpha |\, \alpha < 2^\omega\}$ is then clearly a Bernstein set.

Non-measurability: We also note that whenever B is a Bernstein set, then B is Lebesgue-non-measurable: Recall that by Perfect Set theorem, every uncountable closed set contains a closed perfect subset. Consequently no Bernstein set can contain an uncountable compact subset and neither can $\mathbb{T}\setminus B$. This means that if B was measurable, we would get by inner regularity of Lebesgue measure that

$$|B| = \sup\{|K|\,|\, K \subset B \text{ is compact}\} = \sup\{|K|\,|\, K \subset B \text{ is compact and countable}\} = 0.$$

In the same way we would get that $|\mathbb{T} \setminus B| = 0$, which contradicts the assumption that B is measurable, since $|\mathbb{T}| > 0$ and $\mathbb{T} = B \cup \{\mathbb{T} \setminus B\}$.

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