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MASTER THESIS

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Recurrent properties of products and skew-products of finitely -valued random processes

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Rekurentní vlastnosti součinů a skosných součinů konečně stavových náhodných procesů

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Abstrakt: V této práci se zabýváme dobami vstupu a dobami návratu v pravděpodobnostních dynamických systémech. Uvažujeme speciální typ skosného součinu dvou Bernoulliho posunů jako model pro náhodný pohyb po náhodné abecedě. Pro tyto systémy, za předpokladu modelu generovaného procesem nezávislých stejně rozdělených náhodných veličin s konečným rozptylem a nenulovou střední hodnotou, nebo s nulovou střední hodnotou a konečnou abecedou, je limitní rozdělení normalizovaných dob vstupů do cylindrů rostoucích délek exponenciální. Nakonec se zabýváme mixujícími vlastnostmi skosného součinu, abychom porovnali výsledky této práce s již známými výsledky pro přeškálované doby vstupu v silně mixujících systémech.
Klíčová slova: Pravděpodobnostní dynamické systémy, Bernoulliho posun, skosný součin, doba prvního vstupu.

Title: Recurrent properties of products and skew-products of finitely -valued random processes

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Abstract: In this work, we study return and hitting times in measure-preserving dynamical systems. We consider a special type of skew-products of two Bernoulli schemes, called a random walk in random scenery. For these systems, the limit distribution of normalized hitting times for cylinders of increasing length is proved to be exponential under the assumption of finite variance of the first order distribution of the Bernoulli scheme representing the walk, and provided the drift is non-zero or the scenery alphabet is finite. Mixing properties of the skew-products are discussed in order to relate our work with some known results on rescaled hitting times for strongly-mixing systems.
Keywords: Measure-preserving dynamical system, Bernoulli scheme, skew-product, hitting time.

Contents

1 Introduction

In this master's project we study the properties of time until the first occurrence of a cylinder of positive measure in measure-preserving dynamical systems.

In a measure-preserving dynamical system for any set of positive measure holds that almost every element of this set returns in this set infinitely many times by Poincare recurrence theorem. This is only a qualitative result. There is interest in studying their statistical properties to model physical phenomena like metastability or intermittency. Other applications are in biology(gene occurrence in DNA), linguistics(the rhythm of language) or computer science(data compression algorithms).

We focus on the limiting distribution of normalized hitting time $\mu(A_n)\tau_{A_n}$ to a cylinder A_n of length n. To this subject was presented many papers (some references can be found in [2]). In [10] was proved for Markov chains that $\mu(A_n)\tau_{A_n}$ converges in distribution to a random variable with exponential distribution, more precisely with the distribution function $E(t) = \max\{0, 1-\exp(-t)\}$. In recent papers was proved that the limit distribution of the rescaled hitting time $\lambda(A_n)\mu(A_n)\tau_{A_n}$ is exponential for different strong-mixing systems (ψ-mixing summable in [5], ϕ-mixing or α-mixing summable in [1], α-mixing in [2]).

The master's thesis focused on special case of skew-products, so-called random walks in random scenery. We show that the limit distribution of normalized hitting time $\mu(A_n)\tau_{A_n}$ is exponential in the skew-product of two Bernoulli schemes. However the title of the thesis suggest only finite-state processes, the result is proved in slightly more general form where countable-state processes are taken into account. Since the work on this result had appeared to be quite extensive, we have resigned of treating another types of skew-products. In particular, despite the title, standard products of random processes are not considered in the thesis.

We consider the skew-product of two Bernoulli schemes as a model for random movement of a reading device along a sequence of countable- or finite- valued random variables indexed by integers. This work builds on my bachelor's project where was shown limit distribution of $\mu(A_n)\tau_{A_n}$ in the model of random movement along a sequence of random variables which does not allow backward movement. Unlike bachelor's thesis we consider a more nature definition of cylinders (with respect to other papers) and we admit the more general movement. We show that the limit distribution of $\mu(A_n)\tau_{A_n}$ for suitable cylinders A_n is exponential for a model with movement generated by process of independent identically distributed random variables with a finite variance and a positive expectation along countable- or finite- valued random variables and for a model with movement generated by process of independent identically distributed random variables with a finite variance and with a zero expectation along finite-valued random variables (Main theorem 1).

Structure of the master's thesis is following. In Section 2 we formulate some general properties of probability space. In Section 3 we define the skew-product resp. the skew-product of two Bernoulli schemes and its basic characteristic. At the end of this section we formulate Main theorem 1. The proof takes Sections 4, 5 and 6. The proof is based on relations of the distribution function of normalized hitting time and the distribution function of normalized return time (Definition 3). These relations are formulated in Section 4. For verifying of assumptions of these lemmas we distinguished the model with movement with positive expectation (Section 5) and the model with movement with zero expectation (Section 6).

In the proof we do not use the mixing properties, but we use properties of product measure. In the last Section 7 we provide some mixing properties for specific skew-products. We clarify that some of skew-products in this paper satisfying assumptions of [2, Theorem 1]. We show that the skew-product is not ϕ-mixing in general. It is not easy to verify that the skew-product presented in this paper is at least α-mixing in general and it is not clear if it fulfills the assumptions of [2, Theorem 1].

2 Extrema and Sums of i.i.d. variables

Lemma 1. *Let $Z_1, Z_2, \ldots$ be independent identically distributed random variables on probability space $(\Omega,\, \mathcal{A}, P)$ such that $EZ_1^2 < \infty$. Then*

$$\frac{1}{n}\sum_{k=1}^{n}(Z_k - EZ_k) \to 0 \quad a.s.$$

Proof. [8, Theorem 5.16] □

Corollary 1. *Especially, if $Z_1, Z_2, \ldots$ are independent identically distributed random variables such that $EZ_1 > 0$ and $EZ_1^2 < \infty$, then*

$$P(\sum_{k=1}^{n} Z_k \geq \frac{nEZ_1}{2}) \to 1.$$

Proof. Take $0 < \epsilon < \frac{1}{2}EZ_1$, then the convergence almost surely assures

$$P(|\overline{Z}_n - EZ_1| > \epsilon) \to 0, \tag{1}$$

resp.

$$P(\overline{Z}_n \in [EZ_1 - \epsilon, EZ_1 + \epsilon]) \to 1,$$

where we denote by $\overline{Z}_n$ the sample mean i.e.,

$$\overline{Z}_n = \frac{\sum\limits_{k=1}^{n} Z_k}{n}.$$

Since (1) and $\epsilon < \frac{1}{2}EZ_1$, we get

$$\begin{aligned} P(\sum_{k=1}^{n} Z_k \geq \frac{nEZ_1}{2}) &\geq P(\sum_{k=1}^{n} Z_k \geq n(EZ_1 - \epsilon)) \\ &\geq P(\overline{Z}_n \in [EZ_1 - \epsilon, EZ_1 + \epsilon]) \to 1. \end{aligned}$$

□

Lemma 2. *Let $Z_1, Z_2, \ldots$ be a martingale or non-negative submartingale on probabilistic space $(\Omega,\, \mathcal{F}, P)$. Then for every $p \geq 1$,*

$$P(\max_{k=1,\ldots,n} |Z_k| \geq a) \leq a^{-p}E|Z_n|^p.$$

Proof. [8, Theorem 11.2] □

Corollary 2. *Let $Z_1, Z_2 \dots$ be independent identically distributed random variables such that $E|Z_1| < \infty$. Define $S_n = \sum_{j=1}^{n} (Z_j - EZ_1)$ for $n \in \mathbb{N}$. Then $\{S_n\}$ is a martingale.*

Corollary 3. *Let $Z_1, Z_2 \dots$ be independent identically distributed random variables such that $E|Z_1| < \infty$. Define $S_n = \sum_{j=1}^{n} Z_j - nEZ_1$ for $n \in \mathbb{N}$ and $S_0 = 0$. Then for each $p \geq 1$,*

$$P(\max_{0 \leq j \leq n} \{S_j\} - \min_{0 \leq j \leq n} \{S_j\} \geq a) \leq 2^p \frac{E|S_n|^p}{a^p}.$$

Proof. Since $\max_{0 \leq j \leq n} \{S_j\} \geq 0$ and $\min_{0 \leq j \leq n} \{S_j\} \leq 0$,

$$\max_{0 \leq j \leq n} \{S_j\} - \min_{0 \leq j \leq n} \{S_j\} \geq a$$

implies

$$2 \max_{1 \leq j \leq n} \{|S_j|\} \geq a.$$

Since $\{S_j\}_{j=1}^{\infty}$ is martingale, $\{|S_j|\}_{j=1}^{\infty}$ is submartingale and Lemma 2 concludes the proof. ☐

Lemma 3. *For non-trivial random walk $S_n = \sum_{j=1}^{n} Z_j$ such that $E|Z_1| < \infty$,*

a) $EZ_1 > 0$ implies $S_n \to \infty$ a.s.

b) $EZ_1 < 0$ implies $S_n \to -\infty$ a.s.

c) $EZ_1 = 0$ implies $-\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = \infty$ a.s.

Proof. [7, Proposition 8.14] ☐

Lemma 4. *Let $Z_1, Z_2 \dots$ be independent identically distributed random variables on probability space $(\Omega, \mathcal{A}, P)$ with $EZ_1 = \mu$ and $var(Z_1) = \sigma^2 \in (0, \infty)$ and such that $P(Z_n \in \mathbb{Z}) = 1$. Define $S_n = \sum_{i=1}^{n} (Z_i - \mu)$, then*

$$\sqrt{n} \sup_{x \in \mathbb{Z}} |P(S_n = x) - \frac{1}{\sigma\sqrt{n}} \varphi(\frac{x - n\mu}{\sigma\sqrt{n}})| \to 0,$$

where φ is a density of standard normal distribution. Especially, if $EZ_1 = 0$, then

$$\sup_{x \in \mathbb{Z}} P(S_n = x) \leq \frac{c}{\sqrt{n}}. \tag{2}$$

Proof. The first part is version of the local limit theorem and can be found in [9, Chapter VII, Theorem 1].

Especially, for $EZ_1 = 0$, we know from previous that there exists n_0 such that for every $n > n_0$ and for every $x \in \mathbb{Z}$,

$$P(S_n = x) - \frac{1}{\sigma\sqrt{n}}\varphi(\frac{x}{\sigma\sqrt{n}}) \in (-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}).$$

It follows that

$$P(S_n = x) \leq \frac{1 + \frac{1}{\sigma}\varphi(\frac{x}{\sigma\sqrt{n}})}{\sqrt{n}}$$

and since φ is bounded, we have

$$P(S_n = x) \leq \frac{c}{\sqrt{n}}$$

for each $x \in \mathbb{Z}$. □

3 Basic definitions and main result

3.1 Measure-preserving dynamical system

Definition 1. *Let* $(\Omega, \mathcal{A}, \mu, T)$ *be such that*

1. Ω *is a set,*
2. $\mathcal{A}$ *is a* σ*-algebra of subsets of* Ω*,*
3. $\mu\colon \mathcal{A} \to [0,1]$ *is a probability measure,*
4. $T\colon \Omega \to \Omega$ *is a measurable mapping such that each* $A \in \mathcal{A}$ *satisfies* $\mu(T^{-1}(A)) = \mu(A)$*,*

then $(\Omega, \mathcal{A}, \mu, T)$ *is a* ***measure-preserving dynamical system****.*

Definition 2. *Let* $(\Omega, \mathcal{A}, \mu, T)$ *be a measure-preserving dynamical system and* $A \in \mathcal{A}$*. Then the* ***hitting time to*** A *is defined by*

$$\tau_A(\omega) = \inf\{k \geq 1 \mid T^k(\omega) \in A\},$$

where $\omega \in \Omega$*.*

Definition 3. *Let* $(\Omega, \mathcal{A}, \mu, T)$ *be a measure-preserving dynamical system. For* $A \in \mathcal{A}$ *let* τ_A *be the hitting time to* A*. The* ***distribution function of normalized hitting time to*** A *is the function*

$$F_A(t) = \mu(\omega \in \Omega \mid \mu(A)\tau_A(\omega) \leq t)$$

for $t \in \mathbb{R}$*. For* $A \in \mathcal{A}$*,* $\mu(A) > 0$*, we define the* ***distribution function of return time to*** A *as the function*

$$\widetilde{F}_A(t) = \frac{1}{\mu(A)}\mu(\omega \in A \mid \mu(A)\tau_A(\omega) \leq t)$$

for $t \in \mathbb{R}$*.*

3.2 Bernoulli scheme

Definition 4. *Consider a measure-preserving dynamical system* $(E^{\mathbb{N}_0}, \mathcal{F}, \eta, T)$ *such that* E *is an arbitrary countable or finite set and* $\mathcal{F}$ *is the product* σ*-algebra* *on $E^{\mathbb{N}_0}$. Further, assume η is the measure on $\mathcal{F}$ and T is the* ***shift*** *i.e., $T : E^{\mathbb{N}_0} \to E^{\mathbb{N}_0}$ is a surjective mapping such that*

$$(T(x))_i = x_{i+1} \ \ for \ \ x \in E^{\mathbb{N}_0} \ \ and \ \ i \in \mathbb{N}_0.$$

The measure-preserving dynamical system $(E^{\mathbb{N}_0}, \mathcal{F}, \eta, T)$ is called the ***discrete-time random process****. If η is the product measure and*

$$0 < \eta(x \in E^{\mathbb{N}_0} \mid x_0 = e) < 1 \ for \ every \ e \in E,$$

then the measure-preserving dynamical system $(E^{\mathbb{N}_0}, \mathcal{F}, \eta, T)$ is called the ***one-sided Bernoulli scheme****.*

Definition 5. *Consider a measure-preserving dynamical system $(E^{\mathbb{Z}}, \mathcal{F}, \eta, T)$ such that E is an arbitrary countable or finite set and $\mathcal{F}$ is the product σ-algebra on $E^{\mathbb{Z}}$. Further, assume η is the product measure on $\mathcal{F}$ and T is the* ***shift*** *i.e., $T : E^{\mathbb{Z}} \to E^{\mathbb{Z}}$ is a bijective mapping such that*

$$(T(x))_i = x_{i+1} \ \ for \ \ x \in E^{\mathbb{Z}} \ \ and \ \ i \in \mathbb{Z}.$$

If

$$0 < \eta(x \in E^{\mathbb{Z}} \mid x_0 = e) < 1 \ for \ every \ e \in E,$$

then the measure-preserving dynamical system $(E^{\mathbb{Z}}, \mathcal{F}, \eta, T)$ is called the ***two-sided Bernoulli scheme****.*

3.3 Skew-product

Definition 6. *Let $(\Omega_1, \mathcal{A}_1, \mu_1, T_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2, T_2)$ be two measure-preserving dynamical systems. Suppose a measurable mapping*

$$\Gamma : (\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2) \to (\Omega_1, \mathcal{A}_1)$$

such that for every $\omega_2 \in \Omega_2$, the mapping

$$\Gamma_{\omega_2} : \Omega_1 \to \Omega_1,$$

$$\Gamma_{\omega_2}(\omega_1) = \Gamma(\omega_1, \omega_2)$$

satisfies

$$\Gamma_{\omega_2} \circ T_1 = T_1 \circ \Gamma_{\omega_2}$$

and for every $A \in \mathcal{A}_1$

$$\mu_1(\Gamma_{\omega_2}^{-1}(A)) = \mu_1(A).$$

Define

$$U : \Omega_1 \times \Omega_2 \to \Omega_1 \times \Omega_2$$

$$U(\omega_1, \omega_2) = (\Gamma_{\omega_2}(\omega_1), T_2(\omega_2)).$$

Then we say that $(\Omega_1 \times \Omega_2,\ \mathcal{A}_1 \times\ \mathcal{A}_2, \mu_1 \times \mu_2, U)$ *is the* ***skew-product*** *with the base* $(\Omega_2,\ \mathcal{A}_2, \mu_2, T_2)$*.*

Definition 7. *Let* $M \subseteq \mathbb{Z}$ *be a non-empty subset of integers,* $(E^{\mathbb{Z}},\ \mathcal{A}, \eta, T)$ *be a two-sided Bernoulli scheme and* $(M^{\mathbb{N}_0},\ \mathcal{G}, \nu, T)$ *be a one-sided Bernoulli scheme. Let us take a skew-product* $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0},\ \mathcal{F} =\ \mathcal{A} \times\ \mathcal{G}, \mu = \eta \times \nu, U)$*, where a mapping*

$$U : E^{\mathbb{Z}} \times M^{\mathbb{N}_0} \to E^{\mathbb{Z}} \times M^{\mathbb{N}_0}$$

is defined as

$$U((\ldots, x_{-1}, x_0, x_1, \ldots), (y_0, \ldots)) \to (T^{y_0}(\ldots, x_{-1}, x_0, x_1, \ldots), (y_1, \ldots)).$$

Then $(E^{\mathbb{Z}}\times M^{\mathbb{N}_0},\ \mathcal{F}, \mu, U)$ *is called the* ***skew-product of two Bernoulli schemes****.*

Remark 1. *Denote*

$$\Gamma(x, y) = T^{y_0}(x),$$

then for every $A \in\ \mathcal{A}$*,*

$$\begin{aligned}\Gamma^{-1}(A) &= \{(x, y) \in E^{\mathbb{Z}} \times M^{\mathbb{N}_0} \mid T^{y_0}(x) \in A\} \\ &= \bigcup_{z\in\mathbb{Z}} (T^{-z}(A) \times \{y \in M^{\mathbb{N}_0} \mid y_0 = z\}) \in\ \mathcal{F}\end{aligned}$$

and

$$\begin{aligned}\Gamma(Tx, y) &= T^{y_0}(T(x)) \\ &= T(T^{y_0}(x)) = T(\Gamma(x, y)).\end{aligned}$$

Therefore the skew-product of two Bernoulli schemes is special case of skew-product.

Remark 2. *The skew-product of two Bernoulli schemes* $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0},\ \mathcal{F}, \mu, U)$ *is measure-preserving dynamical system.*

Proof. We have

$$\begin{aligned}\mu(U^{-1}(A\times B)) &= \mu\{(x,y)\in E^{\mathbb{Z}}\times M^{\mathbb{N}_0} \mid T^{y_0}(x)\in A, T(y)\in B\}\\ &= \mu(\bigcup_{z\in M}\{(x,y)\in E^{\mathbb{Z}}\times M^{\mathbb{N}_0} \mid T^{z}(x)\in A, T(y)\in B, y_0=z\})\\ &= \mu(\bigcup_{z\in M}(T^{-z}(A)\times(T^{-1}(B)\cap\{y\in M^{\mathbb{N}_0}\mid y_0=z\})))\\ &= \sum_{z\in M}\mu(T^{-z}(A)\times(T^{-1}(B)\cap\{y\in M^{\mathbb{N}_0}\mid y_0=z\}))\\ &= \eta(A)\sum_{z\in M}\mu(T^{-1}(B)\cap\{y\in M^{\mathbb{N}_0}\mid y_0=z\})\\ &= \eta(A)\nu(B)=\mu(A\times B).\end{aligned}$$

We have shown that

$$U^{-1}(A\times B)=\bigcup_{z\in M}(T^{-z}(A)\times(T^{-1}(B)\cap\{y\in M^{\mathbb{N}_0}\mid y_0=z\}))\in\mathcal{F}$$

and

$$\mu(U^{-1}(A\times B))=\mu(A\times B).$$

Since $\mathcal{S}=\{A\times B\mid A\in\mathcal{A}, B\in\mathcal{G}\}$ generates $\mathcal{F}$ and since $\mathcal{S}$ is closed on finite intersections, $(E^{\mathbb{Z}}\times M^{\mathbb{N}_0}, \mathcal{F},\mu,U)$ is measure-preserving dynamical system. ☐

Remark 3. *Let* $(E^{\mathbb{Z}}\times M^{\mathbb{N}_0}, \mathcal{F},\mu,U)$ *be a skew-product of two Bernoulli schemes. Define* $V_0(x,y)=(x_0,y_0)$ *and for* $k\in\mathbb{N}$ *define* $V_k(x,y)=V_0(U^k(x,y))$*. Then the random sequence* $\{V_k, k\geq 0\}$ *is strictly stationary.*

Proof. Let $k\in\mathbb{N}$, $t_1,\dots,t_k,h\in\mathbb{N}_0$, $y_1,\dots,y_k\in M$ and $e_1,\dots,e_k\in E$ be arbitrary. Then

$$\begin{aligned}&\mu(V_{t_1+h}=(e_1,y_1),V_{t_2+h}=(e_2,y_2),\dots V_{t_k+h}=(e_k,y_k))\\ &\quad=\mu(U^{-h-t_1}V_0^{-1}(e_1,y_1)\cap\dots\cap U^{-h-t_k}V_0^{-1}(e_k,y_k))\\ &\quad=\mu(U^{-h}(U^{-t_1}V_0^{-1}(e_1,y_1)\cap\dots\cap U^{-t_k}V_0^{-1}(e_k,y_k))\\ &\quad=\mu(U^{-t_1}V_0^{-1}(e_1,y_1)\cap\dots\cap U^{-t_k}V_0^{-1}(e_k,y_k))\\ &\quad=\mu(V_{t_1}=(e_1,y_1),V_{t_2}=(e_2,y_2),\dots V_{t_k}=(e_k,y_k))\end{aligned}$$

and random sequence is strictly stationary. ☐

Remark 4. *Because we do not use other skew-products except of skew-product of two Bernoulli schemes we will write skew-product instead of skew-product of two Bernoulli schemes. The model which is described by the skew-product of two Bernoulli schemes is also known as a random walk in a random scenery (see [4]). In our case,* $x \in E^{\mathbb{Z}}$ *represents scenery and for* $y \in M^{\mathbb{N}_0}$ *a sequence* $\{T^{y_0}(x), (T^{y_1} \circ T^{y_0})(x), \ldots\}$ *represents a walk along the scenery* x.

Remark 5. *Let* $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ *be a skew-product. For* $n \in \mathbb{Z}$ *we define mappings*

$$X_n : E^{\mathbb{Z}} \times M^{\mathbb{N}_0} \to E,$$

$$X_n(x, y) = x_n$$

and

$$\widetilde{X}_n : E^{\mathbb{Z}} \to E,$$

$$\widetilde{X}_n(x) = x_n.$$

Further, for $m \in \mathbb{N}_0$*, we define mappings*

$$Y_m : E^{\mathbb{Z}} \times M^{\mathbb{N}_0} \to M,$$

$$Y_m(x, y) = y_m$$

and

$$\widetilde{Y}_m : M^{\mathbb{N}_0} \to M,$$

$$\widetilde{Y}_m(y) = y_m.$$

By definition, we have

$$X_n(x, y) = \widetilde{X}_n(x),$$

$$Y_n(x, y) = \widetilde{Y}_n(y).$$

$X = \{X_n, n \in \mathbb{Z}\}$ *resp.* $\widetilde{X} = \{\widetilde{X}_n, n \in \mathbb{Z}\}$ *are processes of independent identically distributed random variables on probability space* $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu)$ *resp.* $(E^{\mathbb{Z}}, \mathcal{A}, \eta)$ *and* $Y = \{Y_n, n \in \mathbb{N}_0\}$ *resp.* $\widetilde{Y} = \{\widetilde{Y}_n, n \in \mathbb{N}_0\}$ *are processes of independent identically distributed random variables on probability space* $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu)$ *resp.* $(M^{\mathbb{N}_0}, \mathcal{G}, \nu)$*. Finally, denote*

$$q = \max_{m \in M} \{\mu(Y_1 = m)\}$$

and if E is finite, denote

$$p = \min_{e \in E} \{\mu(X_1 = e)\}.$$

Remark 6. *For $n \in \mathbb{Z} \cup \{-\infty\}$, $m \in \mathbb{Z} \cup \{\infty\}$, $m \geq n$, we denote*

$$\mathcal{A}_n^m = \sigma(\{X_j\}_{j=n}^m) \subseteq \mathcal{F}$$

and for $n \in \mathbb{N}_0$, $m \in \mathbb{N}_0 \cup \{\infty\}$, $m \geq n$

$$\mathcal{G}_n^m = \sigma(\{Y_j\}_{j=n}^m) \subseteq \mathcal{F}.$$

Remark 7. *Let us take $y \in M^{\mathbb{N}_0}$ resp. process Y. For $m \geq n \geq 0$, denote*

$$S_n(y) = \sum_{j=0}^{n-1} y_j, \;\; S_n^m(y) = \sum_{j=n}^{m-1} y_j$$

resp.

$$S_n = S_n(Y) = \sum_{j=0}^{n-1} Y_j, \;\; S_n^m = S_n^m(Y) = \sum_{j=n}^{m-1} Y_j,$$

especially denote

$$S_0(y) = 0 \;\; \textit{resp.} \;\; S_n^n(y) = 0.$$

For $m \geq n$, the notation $S_n(y)$ resp. $S_n^m(y)$ make sense also for vectors $y = (y_0, y_1, \ldots, y_{n-1}) \in \mathbb{Z}^n$ resp. $y = (y_n, \ldots, y_{m-1}) \in \mathbb{Z}^{m-n}$. For $(x, y) \in E^{\mathbb{Z}} \times M^{\mathbb{N}_0}$ the n-th composition of the mapping U can be rewritten as

$$U^n(x, y) = (T^{S_n(y)}x, T^n y).$$

Definition 8. *Let $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ be a skew-product. Denote a partition $\mathcal{V} = \{V_{e,m}\}_{e \in E, m \in M}$, where*

$$V_{e,m} = \{X_0 = e, Y_0 = m\} = \{(\widetilde{X}_0 = e) \times (\widetilde{Y}_0 = m)\}.$$

A ***cylinder of length*** **n** *is an event of the form*

$$D_n = V_{x_0,y_0} \cap U^{-1}(V_{x_1,y_1}) \cap \ldots \cap U^{-(n-1)}(V_{x_{n-1},y_{n-1}}),$$

where for each $i \in \{0, \ldots n-1\}$ is $x_i \in E$ and $y_i \in M$. We denote by C_n the set of all cylinders of length n of nonzero probability.

Example 1. *Consider* $x_0, x_1 \in E, x_0 \neq x_1, y_0 = 0$ *and arbitrary* $y_1 \in M$*, then the event*

$$\begin{aligned} D_2 &= V_{x_0,y_0} \cap U^{-1}(V_{x_1,y_1}) \\ &= \{X_0 = x_0, X_{Y_0} = x_1, Y_0 = 0, Y_1 = y_1\} \\ &= \{X_0 = x_0, X_0 = x_1, Y_0 = 0, Y_1 = y_1\} = \emptyset \end{aligned}$$

is the zero probability event and $D_2 \notin C_2$*.*

Remark 8. *Let*

$$D_n = \{V_{x_0,y_0} \cap U^{-1}(V_{x_1,y_1}) \cap \ldots \cap U^{-(n-1)}(V_{x_{n-1},y_{n-1}})\}$$

be a cylinder of length n*. Denote* $y = (y_0, \ldots, y_{n-1})$*, clearly*

$$D_n = \{Y_k = y_k, X_{S_k(y)} = x_k; 0 \leq k \leq n-1\}.$$

Define

$$E_{min}(y) = \min_{0 \leq j \leq n-1}\{S_j(y)\} \quad and \quad E_{max}(y) = \max_{0 \leq j \leq n-1}\{S_j(y)\}$$

Assume that $\mu(D_n) > 0$*, then there exist* $0 < m \leq n-1$*, a sequence of indexes* $i_0 = E_{min}(y) < i_1 < \ldots < i_m = E_{max}(y)$*, where* $i_j \in \{S_k(y)\}_{k=0}^{n-1}$ *and a vector* $(x_{i_0}, x_{i_1}, \ldots, x_{i_m}) \in E^m$ *such that the cylinder* D_n *can be written as*

$$D_n = A'_n \times B'_n,$$

where

$$A'_n = \{\widetilde{X}_{i_0} = x_{i_0}, \ldots, \widetilde{X}_{i_m} = x_m\} \in \mathcal{A}$$

$$B'_n = \{\widetilde{Y}_0 = y_0, \ldots, \widetilde{Y}_{n-1} = y_{n-1}\} \in \mathcal{G}.$$

We will usually use the representation

$$D_n = A_n \cap B_n,$$

where

$$A_n = \{X_{i_0} = x_0, \ldots, X_{i_m} = x_m\} \in \mathcal{F},$$

$$B_n = \{Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}\} \in \mathcal{F}.$$

Further, we consider a shift T on the skew-product, which is defined as

$$T : E^{\mathbb{Z}} \times M^{\mathbb{Z}} \to E^{\mathbb{Z}} \times M^{\mathbb{Z}}$$
$$T(x, y) = (Tx, Ty),$$

and we can write the events

$$\{X_{i_1+l} = x_1, \ldots, X_{i_m+l} = x_m\}$$

resp.

$$\{Y_{j_1+k} = y_1, \ldots, Y_{j_n+k} = y_n\}$$

as

$$T^{-l}(X_{i_1} = x_1, \ldots, X_{i_m} = x_m)$$

resp.

$$T^{-k}(Y_{j_1} = y_1, \ldots, Y_{j_n} = y_n).$$

Remark 9. *Consider $n \in \mathbb{N}$, $r \in (0, \infty)$ and sets D_j for $j \in \mathbb{N}$. For simplicity, we use notation*

$$\bigcup_{j=n}^{r} D_j = \bigcup_{j=n}^{\lfloor r \rfloor} D_j$$

for $r \geq n$ and

$$\bigcup_{j=n}^{r} D_j = \emptyset$$

for $r < n$.

3.4 Main result

Main theorem 1. *Let $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ be a skew-product such that $EY_1^2 < \infty$. If one of the following conditions is satisfied:*

(A) $EY_1 > 0$,

(B) E is finite and $EY_1 = 0$,

then there exists a sequence of sets $\widetilde{C}_n \subseteq C_n$ such that the following two conditions hold:

- $\lim\limits_{n\to\infty} \mu\big(\bigcup\limits_{D_n \in \widetilde{C}_n} D_n\big) = 0$,

- *for all sequences* $\{D_n\}_{n=1}^{\infty}$, *where* $D_n \in C_n \setminus \widetilde{C}_n$ *and for all* $t \in [0, \infty)$,

$$\lim_{n\to\infty} F_{D_n}(t) = 1 - e^{-t}.$$

Proof is based on several lemmas and propositions introduced later. In Section 4 we show some properties which do not depend on conditions (A) and (B). In Section 5 we continue with properties of skew-product under condition (A) resp. in Section 6 with properties under condition (B). The proof of Main theorem follows now.

Proof. First assume that the condition (A) holds. Take the sequence of sets $\widetilde{C}_n$ from Lemma 10, then by this Lemma

$$\lim_{n\to\infty} \mu(\bigcup_{D_n \in \widetilde{C}_n}) = 0.$$

We shall verify assumptions of Proposition 4 for every sequence of cylinders $\{D_n\}$, where $D_n \in C_n \setminus \widetilde{C}_n$. By definition of C_n,

$$\mu(D_n) > 0 \ \ and \ \ \mu(D_n) \leq q^n \to 0.$$

We verify assumptions of Lemma 9 for events W_{D_n} from Definition 11, where we denote $W_{D_n}(t) = W_{D_n}$ for arbitrary $t > 0$. By Lemma 11, $W_{D_n}(t)$ is independent of D_n and $\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n) \cap W_{D_n}(t)$ is independent of D_n. By Lemma 12, we have

$$\lim_{n\to\infty} \mu(W^c_{D_n}(t)) = \lim_{n\to\infty} \mu(W^c_{D_n}) = 0.$$

Furthermore, by Lemma 9, we get $\lim_{n\to\infty} |F_{D_n}(t) - \widetilde{F}_{D_n}(t)| = 0$ for every $t > 0$ and Proposition 4 concludes the proof. Now we provide the proof of Main theorem 1 under condition (B). Let us take sequence of sets $\widetilde{C}_n$ from Lemma 13, then by this Lemma

$$\lim_{n\to\infty} \mu(\bigcup_{D_n \in \widetilde{C}_n}) = 0.$$

Again we verify assumptions of Proposition 4 for every sequence of cylinders $\{D_n\}$, where $D_n \in C_n \setminus \widetilde{C}_n$. By definition of C_n,

$$\mu(D_n) > 0 \ \ and \ \ \lim_{n\to\infty} \mu(D_n) = 0.$$

By Lemma 14, $W_{D_n}(t)$ is independent of D_n and $\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n) \cap W_{D_n}(t)$ is independent of D_n. Furthermore, by Lemma 6, we obtain

$$\lim_{n\to\infty} \mu(W^c_{D_n}(s)) = 0$$

for every $t > 0$. Then the assumptions of Lemma 9 are satisfied and therefore $\lim_{n\to\infty} |F_{D_n}(t) - \widetilde{F}_{D_n}(t)| = 0$. Proposition 4 concludes the proof. □

4 Properties of skew-products

Definition 9. *Let* $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ *be a skew-product and* $D_n \in C_n$ *be a cylinder of length* n *with the representation* $A_n \cap B_n$. *For each* $k \in \mathbb{N}_0$, *we define a random variable* $\widetilde{\tau}_{B_n}(k)$ *on the probability space* $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu)$ *recursively by*

$$(\widetilde{\tau}_{B_n}(0))(x,y) = n,$$

$$(\widetilde{\tau}_{B_n}(k))(x,y) = \inf\{j > \widetilde{\tau}_{B_n}(k-1)(x,y) \mid (x,y) \in T^{-j}(B_n)\}.$$

Lemma 5. *For every* $k \in \mathbb{N}$ *is* $\widetilde{\tau}_{B_n}(k)$ $\mathcal{G}_n^{\infty}$*-measurable and for any* $l > 0$

$$\lim_{n\to\infty} \mu(\widetilde{\tau}_{B_n}(k) \le n^l) = 0$$

and for every $n \in \mathbb{N}$

$$\mu(\widetilde{\tau}_{B_n}(k) < \infty) = 1.$$

Proof. $\widetilde{\tau}_{B_n}(k)$ is $\mathcal{G}_n^{\infty}$-measurable by the definition. Clearly,

$$\mu(\widetilde{\tau}_{B_n}(k) \le n^l) \le \mu(B_n) n^l \le q^n n^l.$$

Define $(\lambda(0))(x,y) = 0$ and for $k \in \mathbb{N}$

$$(\lambda(k))(x,y) = \inf\{j > (\lambda(k-1))(x,y) \mid (x,y) \in T^{-nj}(B_n)\},$$

then $\lambda(1)$ have the geometric distribution with parameter $\mu(B_n) > 0$ and therefore $\lambda(k)$ is finite almost surely. Finally, $\widetilde{\tau}_{B_n}(k) \le n\lambda(k)$. $\square$

4.1 Properties of cylinders

Definition 10. *Let* $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ *be a skew-product and* λ *be a constant such that* $\lambda \in (0,1)$. *We say that a cylinder* D_n *of length* n *is* λ***-self-repelling****, if*

$$\mu(B_n \cap T^{-k}(B_n)) = 0 \quad for \quad k \in \{1, \dots, \lfloor \lambda n \rfloor\}.$$

We denote by $\widetilde{D_n}$ *the set of all* λ*-self-repelling cylinders of length* n *and by* $\widetilde{D_n}^c$ *the set of all cylinders of length* n*, which are not* λ*-self-repelling.*

Lemma 6. *Let $\widetilde{D}_n^c$ be the set of all cylinders of length n, which are not λ-self-repelling. Then*

$$\lim_{n\to\infty} \mu(\bigcup_{D_n \in \widetilde{D}_n^c} D_n) = 0.$$

Proof. For $k \in \{1, \ldots, \lfloor \lambda n \rfloor\}$, we denote

$$\widetilde{D}_n^k = \{D_n \in C_n \mid \mu(B_n \cap T^{-k}(B_n)) > 0\}$$

and $l = \lfloor \frac{n}{k} \rfloor - 1 + (n \mod k)$. Then for every cylinder $D_n \in \widetilde{D}_n^k$ resp. for its representation $A_n \cap B_n$, we get

$$B_n = \{Y_0 = y_0, \ldots Y_{k-1} = y_{k-1}, Y_k = y_0, \ldots, Y_{2k-1} = y_{k-1}, \ldots Y_{n-1} = y_l\}$$

and thus

$$\mu(B_n) \leq \mu(Y_0 = y_0, \ldots Y_{k-1} = y_{k-1}) q^{n-k}.$$

For $z = (z_0, \ldots, z_{n-1}) \in \mathbb{Z}^n$, we denote by

$$C(z) = \{D_n \in C_n \mid B_n = \{Y_0 = z_0, \ldots, Y_{n-1} = z_{n-1}\}\}.$$

the system of all cylinders with fixed B_n. Further, we define

$$o : \mathbb{Z}^k \to \mathbb{Z}^n$$

$$o((y_0, y_1, \ldots, y_{k-1})) = (y_0, \ldots, y_{k-1}, y_0, \ldots, y_{k-1}, \ldots y_l).$$

Then for $y = (y_0, y_1, \ldots, y_{k-1}) \in \mathbb{Z}^k$

$$\begin{aligned}\mu(\bigcup_{D_n \in C(o(y))} D_n) &= \sum_{(x_1,\ldots,x_m)\in E^m} \mu(X_{i_1} = x_1, \ldots, X_{i_m} = x_m)\mu(B_n) \\ &\leq q^{n-k}\mu(Y_0 = y_0, \ldots, Y_{k-1} = y_{k-1}),\end{aligned}$$

and hence

$$\begin{aligned}\mu(\bigcup_{D_n \in \widetilde{D}_n^k} D_k) &= \sum_{y=(y_0,\ldots,y_{k-1})\in M^k} \mu(\bigcup_{D_n \in C(o(y))} D_n) \\ &\leq q^{n-k} \sum_{y=(y_0,\ldots,y_{k-1})\in M^k} \mu(Y_0 = y_0, \ldots, Y_{k-1} = y_{k-1}) \\ &\leq q^{n-k}.\end{aligned}$$

Finally,

$$\mu(\bigcup_{D_n \in \widetilde{D}_n^c} D_n) = \mu(\bigcup_{k=1}^{\lfloor \lambda n \rfloor} \bigcup_{D_n \in \widetilde{D}_n^k} D_n)$$
$$\leq \sum_{k=1}^{\lfloor \lambda n \rfloor} q^{n-k} \leq c q^{n(1-\lambda)}.$$

□

Lemma 7. *Let* $\{D_n\}_{n=1}^{\infty}$ *be a sequence of* λ*-self-repelling cylinders of length* n*. Then there exists a constant* $c > 0$ *such that*

$$\sum_{j=1}^{n} \frac{\mu(D_n \cap U^{-j}(D_n))}{\mu(D_n)} \leq c q^{\lambda n}.$$

Proof. Since

$$U^{-k}(D_n) \subseteq T^{-k}(B_n)$$

for every $D_n \in \widetilde{D_n}$, we have

$$\sum_{j=1}^{n} \frac{\mu(D_n \cap U^{-j}(D_n))}{\mu(D_n)} \leq \sum_{j=1}^{n} \frac{\mu(A_n \cap B_n \cap T^{-j}(B_n))}{\mu(A_n)\mu(B_n)}$$
$$\leq \sum_{j=\lfloor \lambda n \rfloor + 1}^{n} \frac{\mu(B_n \cap T^{-j}(B_n))}{\mu(B_n)}$$
$$\leq \sum_{j=\lfloor \lambda n \rfloor + 1}^{n} \frac{\mu(\bigcap_{i=0}^{n-1}(Y_i = Y_{i+j} = y_i))}{\mu(Y_0 = y_0, \dots Y_{n-1} = y_{n-1})}$$
$$\leq \sum_{j=\lfloor \lambda n \rfloor + 1}^{n} \mu(Y_n = y_{n-j+1}, \dots, Y_{n+j-1} = y_{n-1})$$
$$\leq \sum_{j=\lfloor \lambda n \rfloor + 1}^{n} q^j$$
$$= q^{\lfloor \lambda n \rfloor + 1} \frac{1 - q^{n - \lfloor \lambda n \rfloor}}{1 - q} \leq c q^{\lambda n}.$$

□

4.2 Relations between distribution functions

Lemma 8. *Let* $(\Omega, \mathcal{A}, \mu, T)$ *be a measure-preserving dynamical system,* $A \in \mathcal{A}$*,* $\mu(A) > 0$*,* τ_A *be the hitting time to* A*,* $\widetilde{F}_A(t)$ *be the distribution function of normalized return time and* $F_A(t)$ *be the distribution function of normalized hitting time. Then*

$$0 \leq \int_0^t (1 - \widetilde{F}_A(s)) ds - F_A(t) \leq \mu(A)$$

for all $t \geq 0$.

Proof. The proof can be find in [6]. ☐

Lemma 9. *Let* $\{D_n\}_{n=1}^{\infty}$ *be sequence of* λ*-self-repelling cylinders of length* n*. Suppose that for every* $t \in (0, \infty)$ *and* $n \in \mathbb{N}$*, there are events* $W_{D_n}(t)$ *in* $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu)$ *such that the following conditions hold:*

- $\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n) \cap W_{D_n}(t)$ *is independent of* D_n*,*
- $W_{D_n}(t)$ *is independent of* D_n*,*
- $\lim_{n\to\infty} \mu(W_{D_n}(t)) = 1$.

Then for every $t > 0$

$$\lim_{n\to\infty} |F_{D_n}(t) - \widetilde{F}_{D_n}(t)| = 0.$$

Proof. Let $t > 0$ and $\{D_n\}_{n=1}^{\infty}$ be given. For simplicity, we denote $F_n(t) = F_{D_n}(t)$ and $\widetilde{F}_n(t) = \widetilde{F}_{D_n}(t)$. For n big enough, $t > n\mu(D_n)$ and we write

$$|F_n(t) - \widetilde{F}_n(t)| \leq K_n^1(t) + K_n^2(t) + K_n^3(t) + K_n^4(t) + K_n^5(t),$$

where

$$K_n^1(t) = |\mu(\tau_{D_n} \leq \frac{t}{\mu(D_n)}) - \mu(\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n))|,$$

$$K_n^2(t) = |\mu(\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n)) - \mu(\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n) \cap W_{D_n}(t))|,$$

$$K_n^3(t) = |\mu(\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n) \cap W_{D_n}(t)) - \frac{\mu(\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n) \cap D_n \cap W_{D_n}(t))}{\mu(D_n)}|,$$

$$K_n^4(t) = |\frac{\mu(\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n) \cap D_n \cap W_{D_n}(t))}{\mu(D_n)} - \frac{\mu(\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n) \cap D_n)}{\mu(D_n)}|,$$

$$K_n^5(t) = |\frac{\mu(\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n) \cap D_n)}{\mu(D_n)} - \widetilde{F}_n(t)|.$$

We find an upper bound for $K_n^{(i)}(t)$ for each $i \in \{1, \dots, 5\}$. For $K_n^{(1)}(t)$, we obtain

$$\begin{aligned} K_n^{(1)}(t) &= \mu(\tau_{D_n} \leq n) \\ &\leq n\mu(D_n) \\ &\leq nq^n. \end{aligned}$$

By definition of $K_n^{(2)}(t)$,

$$K_n^{(2)}(t) = \mu(\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n) \cap (W_{D_n}(t))^c)$$
$$\leq \mu((W_{D_n}(t))^c).$$

Events $\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n) \cap W_{D_n}(t)$ and D_n are independent and therefore

$$K_n^{(3)}(t) = 0.$$

Since $W_{D_n}(t)$ is independent of D_n, we get for $K_n^{(4)}(t)$

$$\begin{aligned} K_n^{(4)}(t) &= \frac{\mu(\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n) \cap D_n \cap (W_{D_n}(t))^c)}{\mu(D_n)} \\ &\leq \frac{\mu(D_n \cap (W_{D_n}(t))^c)}{\mu(D_n)} \\ &= \mu((W_{D_n}(t))^c). \end{aligned}$$

For $K_n^{(5)}(t)$, by Lemma 7, we obtain

$$\begin{aligned} K_n^{(5)}(t) &= \frac{\mu(\{\tau_{D_n} \leq n\} \cap D_n)}{\mu(D_n)} \\ &\leq \sum_{j=1}^{n} \frac{\mu(D_n \cap U^{-j}(D_n))}{\mu(D_n)} \\ &\leq cq^{\lambda n}. \end{aligned}$$

By assumptions on $W_{D_n}(t)$,

$$|F_n(t) - \widetilde{F}_n(t)| \leq nq^n + cq^{\lambda n} + 2\mu((W_{D_n}(t))^c) \to 0.$$

□

Previous two lemmas lead us to the key proposition.

Proposition 4. *Let* $(\Omega, \mathcal{A}, \mu, T)$ *be a measure-preserving dynamical system. Let* $\{D_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ *be a sequence of events such that* $\mu(D_n) > 0$ *for every* $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} \mu(D_n) = 0.$$

Further, assume that for every $t > 0$

$$\lim_{n\to\infty} |F_{D_n}(t) - \widetilde{F}_{D_n}(t)| = 0.$$

Then for every $r > 0$

$$\lim_{n\to\infty} F_{D_n}(r) = \lim_{n\to\infty} \widetilde{F}_{D_n}(r) = 1 - e^{-r}.$$

Proof. For simplicity, denote $F_n(t) = F_{D_n}(t)$, $\widetilde{F}_n(t) = \widetilde{F}_{D_n}(t)$ and $f_n(t) = \widetilde{F}_n(t) - F_n(t)$ for every $t > 0$. Then by assumption

$$\lim_{n\to\infty} |f_n(t)| = 0$$

and if

$$\lim_{n\to\infty} F_{D_n}(t) = 1 - e^{-t},$$

then also

$$\lim_{n\to\infty} \widetilde{F}_{D_n}(t) = 1 - e^{-t}.$$

Let $r > 0$ and $\{D_n\}_{n=1}^{\infty}$ be given. For every $t \geq 0$, by Lemma 8, we have

$$\int_0^t (1 - \widetilde{F}_n(s))ds = F_n(t) + g_n(t), \tag{3}$$

where $|g_n(t)| \leq \mu(D_n)$. Substitute $f_n(s) + F_n(s)$ for $\widetilde{F}_n(s)$ in (3) to obtain

$$\int_0^t (1 - F_n(s) - f_n(s))ds = F_n(t) + g_n(t). \tag{4}$$

We multiply equation by e^t and integrate from 0 to r, we get

$$\int_0^r \int_0^t e^t(1 - F_n(s) - f_n(s))dsdt = \int_0^r e^t(F_n(t) + g_n(t))dt.$$

By Fubini's theorem,

$$\int_0^r \int_s^r e^t(1 - F_n(s) - f_n(s))dtds = \int_0^r e^t F_n(t)dt + \int_0^r e^t g_n(t)dt$$

$$\int_0^r (1 - F_n(s) - f_n(s))(e^r - e^s)ds = \int_0^r e^t F_n(t)dt + \int_0^r e^t g_n(t)dt$$

$$e^r \int_0^r (1 - F_n(s) - f_n(s))ds - \int_0^r e^s(1 - F_n(s) - f_n(s))ds = \int_0^r e^t F_n(t)dt + \int_0^r e^t g_n(t)dt.$$

We substitute from (4),

$$e^r(F_n(r) + g_n(r)) - (e^r - 1) + \int_0^r e^s f_n(s)ds = \int_0^r e^s g_n(s)ds.$$

Expressing $F_n(r)$ from the equation, we get

$$F_n(r) = 1 - e^{-r} - g_n(r) - e^{-r}(\int_0^r e^s f_n(s)ds - \int_0^r e^s g_n(s)ds).$$

Further, we know

$$|-g_n(r) - e^{-r}\int_0^r e^s f_n(s)ds + e^{-r}\int_0^r e^s g_n(s)ds| \leq$$
$$\mu(D_n) + |e^{-r}\int_0^r e^s f_n(s)ds| + |e^{-r}\int_0^r e^s g_n(s)ds|.$$

We estimate the third part with

$$\begin{aligned} |e^{-r}\int_0^r e^s g_n(s)ds| &\leq e^{-r}\int_0^r e^s |g_n(s)|ds \\ &\leq e^{-r}\int_0^r e^s \mu(D_n)ds \\ &\leq \mu(D_n). \end{aligned}$$

Since $|f_n(s)| \leq 2$, for every $s \in [0, r]$, and by dominated convergence theorem, we obtain

$$|e^r \int_0^r e^s f_n(s)ds| \leq e^{-r}\int_0^r e^s |f_n(s)|ds \to 0.$$

Define

$$K_n(r) = e^{-r}\int_0^r e^s |f_n(s)|ds,$$

then

$$F_n(r) \in [1 - e^{-r} - 2\mu(D_n) - K_n(r), 1 - e^{-r} + 2\mu(D_n) + K_n(r)].$$

□

5 Non-symmetrical motion

In this section we will consider the skew-product which satisfied assumptions of Main theorem with condition (A). For this section, we denote by $\mathcal{B}$ this skew-product.

5.1 Construction of a suitable subset of cylinders

Lemma 10. *Let $\mathcal{B}$ be a skew-product. Let $\widetilde{D}_n$ be set of all λ-self-repelling cylinders of length n. Denote*

$$\widetilde{D}'_n = \{D_n \in C_n \mid \forall (x,y) \in D_n : \max_{0 \le i \le n} |S_i(y) - iEY_1| \ge \frac{n}{2}\}$$

and $\widetilde{C}_n = \widetilde{D}_n^c \cup \widetilde{D}'_n$. Then

$$\lim_{n\to\infty} \mu(\bigcup_{D_n \in \widetilde{C}_n} D_n) = 0.$$

Proof. By Lemma 2, for $\widetilde{D'}_n$,

$$\begin{aligned} \mu(\bigcup_{D_n \in \widetilde{D'}_n} D_n) &\le \mu(\max_{0\le i\le n}\{|S_i(Y) - iEY_1|\} \ge \frac{n}{2}) \\ &\le c\frac{E(S_n - nEY_1)^2}{n^2} \\ &= c\frac{EY_1^2}{n} \to 0. \end{aligned} \tag{5}$$

By Lemma 6 and (5), we obtain

$$\lim_{n\to\infty} \mu(\bigcup_{D_n \in \widetilde{C}_n} D_n) = 0.$$

□

Corollary 5. *Define systems of cylinders*

$$\widetilde{D}''_n = \{D_n \in C_n \mid \forall (x,y) \in D_n : \max_{0\le i\le n}\{S_i(y)\} - \min_{0\le i\le n}\{S_i(y)\} \ge n(1 + EY_1)\}$$

and

$$\widetilde{D}'''_n = \{D_n \in C_n \mid \forall (x,y) \in D_n : |S_n(y)| \ge n(1 + EY_1)\}.$$

Then

$$\widetilde{D}''_n \cup \widetilde{D}'''_n \subseteq \widetilde{C}_n.$$

Proof. We have

$$\max_{0\le i\le n}\{S_i\} - nEY_1 - \min_{0\le i\le n}\{S_i\}$$
$$\le \max_{0\le i\le n}\{S_i - iEY_1\} - \min_{0\le i\le n}\{S_i - iEY_1\}$$
$$\le 2\max_{0\le i\le n}\{|S_i - iEY_1|\}$$

and therefore $\widetilde{D}''_n \subseteq \widetilde{D}'_n$. Similarly, we have

$$\begin{aligned} |S_n| - nEY_1 &= |S_n| - |nEY_1| \\ &\le ||S_n| - |nEY_1|| \\ &\le |S_n - nEY_1| \\ &\le \max_{0\le i\le n}\{|S_i - iEY_1|\} \end{aligned}$$

and therefore $\widetilde{D}'''_n \subseteq \widetilde{D}'_n$. ◻

5.2 Construction of W_{D_n}

Definition 11. *Let $\mathcal{B}$ be a skew-product and let D_n be a cylinder of length n. Define*

$$W_{D_n} = \bigcap_{k=1}^{\infty}(S_n^{\widetilde{\tau}_{B_n}(k)} > 4n(1 + EY_1)),$$

where $\widetilde{\tau}_{B_n}(k)$ is from Definition 9.

Property of independence for events W_{D_n}

Lemma 11. *Let $\widetilde{C}_n$ be from Lemma 10, $D_n \in C_n \setminus \widetilde{C}_n$, W_{D_n} be events from the previous definition and $t > 0$. Then the following two conditions are satisfied:*

- *W_{D_n} is independent of D_n,*
- *$\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} S^{-j}(D_n) \cap W_{D_n}$ is independent of D_n.*

Proof. Since $B_n \in \mathcal{G}_0^{n-1}$, $A_n \in \mathcal{A}$ and

$$\bigcap_{k=1}^{\infty}(S_n^{\widetilde{\tau}_{B_n}(k)} > 4n(1 + EY_1)) \in \mathcal{G}_n^{\infty},$$

we get

$$\begin{aligned} \mu(W_{D_n} \cap D_n) &= \mu(A_n \cap B_n \cap \bigcap_{k=1}^{\infty}(S_n^{\widetilde{\tau}_{B_n}(k)} > 4n(1 + EY_1))) \\ &= \mu(D_n)\mu(W_{D_n}). \end{aligned}$$

Denote $c = \lceil (1 + EY_1) \rceil$ and $u = \lfloor \frac{t}{\mu(D_n)} \rfloor$. Since $D_n \in C_n \setminus \widetilde{C}_n$ and by Corollary 5, we obtain

$$\max_{0 \le i \le n} \{S_i(Y)\} - \min_{0 \le i \le n} \{S_i(Y)\} \le cn \quad on \quad D_n$$

and

$$A_n \in \mathcal{A}_{-cn}^{cn}.$$

Further, let us take $l \in \mathbb{N}$, then

$$T^{-l}(A_n) \in \mathcal{A}_{-cn+l}^{\infty}.$$

If we take $l > 2nc$, we get

$$T^{-l}(A_n) \in \mathcal{A}_{cn+1}^{\infty}$$

and it follows that $\mu(A_n \cap T^{-l}(A_n)) = \mu(A_n)\mu(T^{-l}(A_n))$. The statement of lemma is clearly true for $u < n$. Suppose $u \ge n$, then $W_{D_n} \in \mathcal{G}_n^{\infty}$ and for $u \ge j \ge n$ clearly $T^{-j}(B_n) \in \mathcal{G}_n^{\infty}$. Since $D_n \in C_n \setminus \widetilde{C}_n$ and by Corollary 5, we have

$$|S_n| \le nc \quad on \quad B_n$$

and by definition of W_{D_n}

$$S_n^j \ge 4nc \quad on \quad T^{-j}(B_n) \cap W_{D_n}.$$

Together, we get

$$S_j = S_n^j + S_n \ge S_n^j - |S_n| \ge 3nc \quad on \quad T^{-j}(B_n) \cap W_{D_n} \cap B_n. \tag{6}$$

Further D_n is fixed and therefore B_n is fixed. Suppose $z = (z_0, \ldots, z_{n-1})$ such that

$$B_n = \{Y_0 = z_0, \ldots, Y_{n-1} = z_{n-1}\}.$$

Define

$$H = W_{D_n} \cap \bigcup_{j=n}^{u} T^{-j}(B_n),$$

$$\Omega = \{(y_n, \ldots, y_{u+n-1}) \in M^u \mid \{Y_n = y_n, \ldots, Y_{u+n-1} = y_{u+n-1}\} \cap H \neq \emptyset\}$$

and for $y \in \Omega$, define

$$P(y) = W_{D_n} \cap \{Y_n = y_n, \ldots, Y_{u+n-1} = y_{u+n-1}\},$$

$$L(y) = \bigcup_{j=n}^{u} (T^{-j}(B_n) \cap T^{-S_n^j(y) - S_n(z)}(A_n)) \cap B_n \cap P(y).$$

Then by (6), we get $L(y) \in \sigma(\mathcal{G} \cup \mathcal{A}_{2nc}^{\infty})$ for every $y \in \Omega$. Hence also

$$\bigcup_{j=n}^{u} U^{-j}(D_n) \cap W_{D_n} \cap B_n = \bigcup_{y \in \Omega} L(y) \in \sigma(\mathcal{G} \cup \mathcal{A}_{2nc}^{\infty}).$$

But $\sigma(\mathcal{G} \cup \mathcal{A}_{2nc}^{\infty})$ is independent of $\mathcal{A}_{-cn}^{cn}$ and therefore

$$\begin{aligned} \mu(\bigcup_{j=n}^{u} U^{-j}(D_n) \cap W_{D_n} \cap D_n) &= \mu(\bigcup_{y \in \Omega} L(y) \cap A_n) \\ &= \mu(\bigcup_{y \in \Omega} L(y))\mu(A_n). \end{aligned}$$

For $y \in \Omega$, define

$$L'(y) = \bigcup_{j=n}^{u} (T^{-j}(B_n) \cap T^{-S_n^j(y) - S_n(z)}(A_n)) \cap P(y).$$

Since $L'(y) \in \sigma(\mathcal{G}_n^{\infty} \cup \mathcal{A})$ and $L(y) = L'(y) \cap B_n$, we obtain

$$\begin{aligned} \mu(\bigcup_{y \in \Omega} L(y))\mu(A_n) &= \mu(\bigcup_{y \in \Omega} L'(y) \cap B_n)\mu(A_n) \\ &= \mu(\bigcup_{y \in \Omega} L'(y))\mu(B_n)\mu(A_n). \end{aligned}$$

The last step is to show that

$$\mu(\bigcup_{y \in \Omega} L'(y)) = \mu(\bigcup_{j=n}^{u} U^{-j}(D_n) \cap W_{D_n}).$$

For $y \in \Omega$, denote by

$$N_{B_n}(y) = card\{n \leq i \leq u \mid y_i = z_0, \ldots, y_{n+i-1} = z_{n-1}\},$$

the number of hitting times to B_n in y and by $(\tau(j))(y)$ the j-th hitting time to B_n in y. For $z' \in M^n$, define

$$Q(z') = \{Y_0 = z'_0, \ldots, Y_{n-1} = z'_{n-1}\}.$$

Then

$$L'(y) = \bigcup_{j=1}^{N_{B_n}(y)} T^{-S_n^{(\tau(j))(y)}(y) - S_n(z)}(A_n) \cap P(y)$$

and

$$\begin{aligned}
\mu(L'(y)) &= \mu(P(y))\mu(\bigcup_{j=1}^{N_{B_n}(y)} T^{-S_n^{(\tau(j))(y)}(y)-S_n(z)}(A_n)) \\
&= \mu(P(y))\mu(T^{-S_n(z)}(\bigcup_{j=1}^{N_{B_n}(y)} T^{-S_n^{(\tau(j))(y)}(y)}(A_n))) \\
&= \mu(P(y))\mu(\bigcup_{j=1}^{N_{B_n}(y)} T^{-S_n^{(\tau(j))(y)}(y)}(A_n)) \\
&= \sum_{z'\in M^n} \mu(P(y)\cap Q(z'))\mu(\bigcup_{j=1}^{N_{B_n}(y)} T^{-S_n^{(\tau(j))(y)}(y)-S_n(z')}(A_n)) \\
&= \mu(\bigcup_{z'\in M^n} (P(y)\cap Q(z')\cap \bigcup_{j=1}^{N_{B_n}(y)} T^{-S_n^{(\tau(j))(y)}(y)-S_n(z')}(A_n))).
\end{aligned}$$

Finally, we get

$$\begin{aligned}
\mu(\bigcup_{y\in\Omega} L'(y)) &= \mu(\bigcup_{y\in\Omega}\bigcup_{z'\in M^n} (P(y)\cap Q(z')\cap \bigcup_{j=1}^{N_{B_n}(y)} T^{-S_n^{(\tau(j))(y)}(y)-S_n(z')}(A_n))) \\
&= \mu(\bigcup_{j=n}^{u} U^{-j}(D_n)\cap W_{D_n}).
\end{aligned}$$

□

Limit of measure of W_{D_n}

Lemma 12. *Let $\mathcal{B}$ be a skew-product and $C_n \setminus \widetilde{C}_n$ from Lemma 10. Then, for any sequence $\{D_n\}$, where $D_n \in C_n \setminus \widetilde{C}_n$,*

$$\lim_{n\to\infty} \mu(W_{D_n}^c) = 0.$$

Proof. Let $\{D_n\}$ be given. We denote

$$c = 4(1+EY_1)\ ,\ c' = \frac{EY_1}{2} \ \textit{ and } \ \widetilde{\tau}_{B_n} = \widetilde{\tau}_{B_n}(1).$$

By Corollary 1 we have

$$\lim_{n\to\infty} \mu(\sum_{j=n}^{n^2+n} Y_j \ge c'n^2) = \lim_{n\to\infty} \mu(\sum_{j=0}^{n^2} Y_j \ge c'n^2) = 1.$$

We denote $k_n^{(1)} = \mu(\sum_{j=0}^{n^2} Y_j < c'n^2)$. Further, we define $k_n^{(2)} = (n^2+2n)q^n$, then clearly

$$\mu(\widetilde{\tau}_{B_n} < n^2+2n) \le k_n^{(2)} \to 0$$

and

$$k_n^{(1)} \to 0.$$

We have

$$\begin{aligned}\mu(W_{D_n}^c) &\le \mu(W_{D_n}^c \cap \{\widetilde{\tau}_{B_n} \ge n^2+2n\} \cap \{S_n^{n^2+n} \ge c'n^2\}) \\ &\quad + \mu(\widetilde{\tau}_{B_n} < n^2+2n) + \mu(S_n^{n^2+n} < c'n^2) \\ &\le \mu(W_{D_n}^c \cap \{\widetilde{\tau}_{B_n} \ge n^2+2n\} \cap \{S_n^{n^2+n} \ge c'n^2\}) + k_n^{(1)} + k_n^{(2)}.\end{aligned}$$

Hence, it is enough to show that

$$\lim_{n\to\infty} \mu(\bigcup_{k=1}^{\infty} \{S_n^{\widetilde{\tau}_{B_n}(k)} \le cn\} \cap \{\widetilde{\tau}_{B_n} \ge n^2+2n\} \cap \{S_n^{n^2+n} \ge c'n^2\}) = 0.$$

It is easy to see that for

$$\forall (x,y) \in \bigcup_{k=1}^{\infty} \{S_n^{\widetilde{\tau}_{B_n}(k)} \le cn\} \cap \{\widetilde{\tau}_{B_n} \ge n^2+2n\} \cap \{S_n^{n^2+n} \ge c'n^2\}$$

there exists $l \in \mathbb{N}$ such that

$$S_{n^2+n}^{n^2+2n+l}(y) \le 0.$$

We have

$$\bigcup_{k=1}^{\infty} \{S_n^{\widetilde{\tau}_{B_n}(k)} \le cn\} \cap \{\widetilde{\tau}_{B_n} \ge n^2+2n\} \cap \{S_n^{n^2+n} \ge c'n^2\} \subseteq \{\inf_{l\in\mathbb{N}} \{S_{n^2+n}^{n^2+2n+l}\} \le 0\}$$

and therefore

$$\begin{aligned}&\mu(\bigcup_{k=1}^{\infty} \{S_n^{\widetilde{\tau}_{B_n}(k)} \le cn\} \cap \{\widetilde{\tau}_{B_n} \ge n^2+2n\} \cap \{S_n^{n^2+n} \ge c'n^2\}) \\ &\qquad \le \mu(\inf_{l\in\mathbb{N}} \{S_{n^2+n}^{n^2+2n+l}\} \le 0) \\ &\qquad = \mu(\inf_{l\in\mathbb{N}} \{S_{n+l}\} \le 0).\end{aligned}$$

By Lemma 3

$$\lim_{n\to\infty} \mu(\inf_{l\in\mathbb{N}} \{S_{n+l}\} \le 0) = 0.$$

□

6 Symmetrical motion

In this section we will consider the skew-product which satisfied assumptions of Main theorem with the condition (B). For this section, we denote by $\mathcal{B}$ this skew-product.

6.1 Construction of a suitable subset of cylinders

Lemma 13. *Let $\mathcal{B}$ be a skew-product and $\widetilde{D}_n$ be sets of all λ-self-repelling cylinders. For arbitrary $\lambda' \in (\frac{1}{2}, 1)$, denote*

$$\widetilde{D}'_n = \{D_n \in C_n \mid \forall (x,y) \in D_n : (\max_{0 \le i \le n}\{S_i(y)\} - \min_{0 \le i \le n}\{S_i(y)\}) \ge n^{\lambda'}\}$$

and $\widetilde{C}_n = \widetilde{D}^c_n \cup \widetilde{D}'_n$. Then

$$\lim_{n\to\infty} \mu(\bigcup_{D_n \in \widetilde{C}_n} D_n) = 0.$$

Proof. By Corollary 3, we get

$$\begin{aligned} \mu(\max_{0 \le i \le n}\{S_i\} - \min_{0 \le i \le n}\{S_i\} \ge n^{\lambda'}) &\le 2^{2\lambda'} \frac{E(\sum_{j=0}^{n-1} Y_j)^2}{n^{2\lambda'}} \\ &\le \frac{c}{n^{2\lambda' - 1}}. \end{aligned} \tag{7}$$

By (7) and by Lemma 6, we obtain

$$\lim_{n\to\infty} \mu(\bigcup_{D_n \in \widetilde{C}_n} (D_n)) = 0.$$

□

6.2 Construction of W_{D_n}

Definition 12. *Let $\mathcal{B}$ be a skew-product and let D_n be a cylinder of length n. For $t \in (0, \infty)$, define*

$$W_{D_n}(t) = \bigcap_{k=1}^{\infty} (\{|S_n^{\widetilde{\tau}_{B_n}(k)}| > 3n\} \cup \{\widetilde{\tau}_{B_n}(k) > \lfloor \frac{t}{\mu(D_n)} \rfloor\}).$$

Property of independence for events W_{D_n}

Lemma 14. *Let $\widetilde{C}_n$ be from Lemma 13, $D_n \in C_n \setminus \widetilde{C}_n$, $t > 0$ and $W_{D_n}(t)$ from the previous definition. Then the following two conditions are satisfied:*

- *$W_{D_n}(t)$ is independent of D_n,*
- *$\bigcup_{j=n}^{\frac{t}{\mu(D_n)}} U^{-j}(D_n) \cap W_{D_n}(t)$ is independent of D_n.*

Proof. Let $t > 0$ and D_n be fixed. Since $B_n \in \mathcal{G}_0^{n-1}, A_n \in \mathcal{A}$, $W_{D_n}(t) \in \mathcal{G}_n^{\infty}$, we get

$$\mu(W_{D_n}(t) \cap D_n) = \mu(W_{D_n}(t))\mu(D_n).$$

Since $D_n \in C_n \setminus \widetilde{C}_n$, $\lambda' < 1$ and by Lemma 13, we obtain

$$\max_{0 \le i \le n}\{S_i\} - \min_{0 \le i \le n}\{S_i\} \le n^{\lambda'} \quad on \quad D_n$$

and

$$A_n \in \mathcal{A}_{-n^{\lambda'}}^{n^{\lambda'}} \subseteq \mathcal{A}_{-n}^{n}.$$

Consider $l \in \mathbb{Z}$, $|l| > 2n$, we have $T^{-l}(A_n) \in \mathcal{A}_{n+1}^{\infty}$ for $l > 2n$ (resp. $T^{-l}(A_n) \in \mathcal{A}_{-\infty}^{-n-1}$ for $l < 2n$) and $\mu(A_n \cap T^{-l}(A_n)) = \mu(A_n)^2$. Denote $u = \lfloor \frac{t}{\mu(D_n)} \rfloor$. We show that

$$\mu(\bigcup_{j=n}^{u} U^{-j}(D_n) \cap W_{D_n}(t) \cap D_n) = \mu(\bigcup_{j=n}^{u} U^{-j}(D_n) \cap W_{D_n}(t))\mu(D_n).$$

The equality clearly holds for $u < n$. Suppose $u \ge n$, since $D_n \in C_n \setminus \widetilde{C}_n$,

$$|S_n| \le \max_{0 \le i \le n}\{S_i\} - \min_{0 \le i \le n}\{S_i\} \le n^{\lambda'} \le n \quad on \quad B_n.$$

Further, take $n \le j \le u$, then

$$|S_n^j| > 3n \quad on \quad T^{-j}(B_n) \cap W_{D_n}(t)$$

and

$$|S_j| = |S_n^j + S_n| > 2n \quad on \quad T^{-j}(B_n) \cap W_{D_n}(t) \cap B_n. \tag{8}$$

Suppose $z = (z_0, \ldots, z_{n-1})$ such that

$$B_n = \{Y_0 = z_0, \ldots, Y_{n-1} = z_{n-1}\}.$$

Define

$$H = W_{D_n}(t) \cap \bigcup_{j=n}^{u} T^{-j}(B_n),$$

$$\Omega = \{(y_n, \dots, y_{u+n-1}) \in M^u \mid \{Y_n = y_n, \dots, Y_{u+n-1} = y_{u+n-1}\} \cap H \neq \emptyset\}$$

and for $y \in \Omega$, define

$$P(y) = W_{D_n}(t) \cap \{Y_n = y_n, \dots, Y_{u+n-1} = y_{u+n-1}\},$$

$$L(y) = \bigcup_{j=n}^{u} (T^{-j}(B_n) \cap T^{-S_n^j(y) - S_n(z)}(A_n)) \cap B_n \cap P(y).$$

Then by (8), we have $L(y) \in \sigma(\mathcal{G} \cup \mathcal{A}_{n+1}^{\infty} \cup \mathcal{A}_{-\infty}^{-n-1})$ for every $y \in \Omega$. Hence also

$$\bigcup_{j=n}^{u} U^{-j}(D_n) \cap W_{D_n}(t) \cap B_n = \bigcup_{y \in \Omega} L(y) \in \sigma(\mathcal{G} \cup \mathcal{A}_{n+1}^{\infty} \cup \mathcal{A}_{-\infty}^{-n-1}).$$

But $\sigma(\mathcal{G} \cup \mathcal{A}_{n+1}^{\infty} \cup \mathcal{A}_{-\infty}^{-n-1})$ is independent of $\mathcal{A}_{-n}^{n}$ and therefore

$$\begin{aligned} \mu(\bigcup_{j=n}^{u} U^{-j}(D_n) \cap W_{D_n}(t) \cap D_n) &= \mu(\bigcup_{y \in \Omega} L(y) \cap A_n) \\ &= \mu(\bigcup_{y \in \Omega} L(y)) \mu(A_n). \end{aligned}$$

The rest of the proof may be obtained in the same way as in Lemma 11. □

Limit of measure of W_{D_n}

Proposition 6. *Let $\mathcal{B}$ be a skew-product and $W_{D_n}(t)$ be from Definition 12. Then for all $t \in (0, \infty)$ and $\{D_n\}_{n=1}^{\infty}$, where $D_n \in C_n \setminus \widetilde{C}_n$,*

$$\lim_{n \to \infty} \mu(W_{D_n}^c(t)) = 0.$$

Proof. Let $\{D_n\}_{n=1}^{\infty}$ be sequence of cylinders such that $D_n \in C_n \setminus \widetilde{C}_n$ and $t > 0$. For simplicity, denote $\widetilde{\tau}_n(k) = \widetilde{\tau}_{B_n}(k)$. Consider only $n \in \mathbb{N}$ such that $t \geq nq^n$. We denote

$$c_n' = \frac{tq^{\frac{-n}{10}}}{\mu(A_n)}, \qquad c_n = \lfloor c_n' \rfloor, \qquad d(n,k) = 3n^6k^4,$$

$$Q_n = \{\mu(B_n)\widetilde{\tau}_n(c_n) \leq \frac{t}{\mu(A_n)}\}$$

and

$$R_n = \bigcup_{k=1}^{c_n} \{\widetilde{\tau}_n(k) - \widetilde{\tau}_n(k-1) \leq d(n,k)\}.$$

The event Q_n^c and the constant c_n control number of hitting times to B_n until the time $\frac{t}{\mu(D_n)}$ and the event R_n^c assures that there is enough space between two hitting times. Since $c'_n \geq tq^{\frac{-n}{10}}$,

$$\lim_{n\to\infty} c'_n = \infty \quad and \quad \lim_{n\to\infty} c_n = \infty.$$

Further,

$$\begin{aligned} W_{D_n}^c(t) \cap Q_n^c &= \bigcup_{k=1}^{\infty}(\{|S_n^{\widetilde{\tau}_n(k)}| \leq 3n\} \cap \{\widetilde{\tau}_n(k) \leq \lfloor\frac{t}{\mu(D_n)}\rfloor\}) \cap \{\widetilde{\tau}_n(c_n) > \frac{t}{\mu(D_n)}\} \\ &\subseteq \bigcup_{k=1}^{c_n}(\{|S_n^{\widetilde{\tau}_n(k)}| \leq 3n\} \cap \{\widetilde{\tau}_n(c_n) > \frac{t}{\mu(D_n)}\}), \end{aligned}$$

and hence

$$\begin{aligned} \mu(W_{D_n}^c(t)) &\leq \mu(W_{D_n}^c(t) \cap Q_n) + \mu(W_{D_n}^c(t)) \cap Q_n^c) \\ &\leq \mu(Q_n) + \mu(\bigcup_{k=1}^{c_n}\{|S_n^{\widetilde{\tau}_n(k)}| \leq 3n\} \cap Q_n^c) \\ &\leq \mu(Q_n) + \mu(R_n) + \mu(\bigcup_{k=1}^{c_n}\{|S_n^{\widetilde{\tau}_n(k)}| \leq 3n\} \cap Q_n^c \cap R_n^c). \end{aligned} \tag{9}$$

In the first step we prove that $\lim_{n\to\infty} \mu(R_n) = 0$. For every $D_n \in C_n \setminus \widetilde{C}_n$,

$$\mu(B_n) \leq q^n$$

and since E is finite, we have

$$p = \min_{e\in E}\{\mu(X_0 = e)\} > 0$$

and

$$\mu(A_n) \geq p^{n^{\lambda'}}.$$

Easy calculation shows that

$$\begin{aligned} \mu(R_n) &= \mu(\bigcup_{k=1}^{c_n}(\widetilde{\tau}_n(k) - \widetilde{\tau}_n(k-1) \leq d(n,k))) \\ &\leq \sum_{k=1}^{c_n} \mu(\widetilde{\tau}_n(k) - \widetilde{\tau}_n(k-1) \leq d(n,k)) \\ &\leq \sum_{k=1}^{c_n} d(n,k)\mu(B_n) \\ &= \sum_{k=1}^{c_n} 3n^6k^4\mu(B_n) \\ &\leq 3n^6c_n^{4+1}\mu(B_n) \\ &= 3n^6\frac{t^5q^{(\frac{-5n}{10})}}{(\mu(A_n))^5}\mu(B_n) \\ &\leq 3n^6t^5q^{(\frac{-n}{2})}q^np^{-5n^{\lambda'}} \to 0. \end{aligned} \tag{10}$$

Now we show that $\lim_{n\to\infty} \mu(Q_n) = 0$. We distinguish two cases. The first case is that there exist c_n occurrences of B_n such that they do not overlap each other. And the second case is that there exists at least two occurrences of B_n that overlap. In other words, we can write $\mu(Q_n)$ as follows,

$$\mu(Q_n \cap (\bigcap_{k=2}^{c_n} \{\tilde{\tau}_n(k) - \tilde{\tau}_n(k-1) > n\} \cup \bigcup_{k=2}^{c_n} \{\tilde{\tau}_n(k) - \tilde{\tau}_n(k-1) \leq n\})). \tag{11}$$

The cylinder D_n is λ-self-repelling and therefore

$$T^{-k}(B_n) \cap T^{-k-j}(B_n) = \emptyset$$

for every $k \in \mathbb{N}$ and $j \in \{1, \dots \lfloor \lambda n \rfloor\}$. We obtain

$$\begin{aligned}
\mu(Q_n \cap \bigcup_{k=2}^{c_n} (\tilde{\tau}_n(k) - \tilde{\tau}_n(k-1)) \leq n) &\leq \mu(\bigcup_{k=2}^{c_n} (\tilde{\tau}_n(k) - \tilde{\tau}_n(k-1)) \leq n) \\
&\leq \mu(\bigcup_{k=2}^{c_n} \bigcup_{j=\lfloor \lambda n \rfloor+1}^{n} T^{-k}(B_n \cap T^{-j}(B_n))) \\
&\leq \sum_{k=2}^{c_n} \sum_{j=\lfloor \lambda n \rfloor+1}^{n} \mu(B_n \cap T^{-j}(B_n)) \\
&\leq \sum_{k=2}^{c_n} \sum_{j=\lfloor \lambda n \rfloor+1}^{n} q^{n+j} \\
&\leq c' c_n q^n \\
&= c' \frac{t q^{-\frac{n}{10}}}{\mu(A_n)} q^n \leq t c' q^{cn} p^{-n^{\lambda'}}
\end{aligned} \tag{12}$$

where c, c' are suitable positive constants. There are less then $\binom{\frac{t}{\mu(D_n)}}{c_n}$ combinations how to place B_n on set $\{n, \dots, \frac{t}{\mu(D_n)}\}$ c_n-times. In the case they do not overlap, each c_n placements of B_n results in probability smaller than $\mu(B_n)^{c_n}$. We get

$$\mu(Q_n \cap \bigcap_{k=2}^{c_n} (\tilde{\tau}_n(k) - \tilde{\tau}_n(k-1)) > n) \leq \binom{\frac{t}{\mu(D_n)}}{c_n} (\mu(B_n))^{c_n}.$$

In general for $l \geq k$, we have

$$\begin{aligned}
\binom{l}{k} &= \frac{l!}{(l-k)!k!} \\
&= \frac{l(l-1)\dots(l-k+1)}{k!} \\
&\leq \frac{l^k}{k!}.
\end{aligned}$$

By Stirling's approximation,

$$k^k e^{-k}\sqrt{k2\pi} \leq k!$$

and therefore

$$\frac{l^k}{k!} \leq \frac{l^k}{k^k e^{-k}\sqrt{k2\pi}}.$$

For $l = \frac{t}{\mu(D_n)}$ and $k = c_n$, we get

$$\begin{aligned}\binom{\frac{t}{\mu(D_n)}}{c_n}(\mu(B_n))^{c_n} &\leq \frac{(\frac{t}{\mu(D_n)})^{c_n}}{c_n!}(\mu(B_n))^{c_n} \\ &\leq \frac{(\frac{t}{\mu(A_n)})^{c_n}}{\sqrt{2c_n\pi}(\frac{c_n}{e})^{c_n}} \\ &\leq \frac{(eq^{\frac{n}{10}})^{c_n}}{\sqrt{2\pi c_n}}.\end{aligned}$$

Since $\lim_{n\to\infty} c_n = \infty$,

$$\mu(Q_n) \leq \frac{(eq^{\frac{n}{10}})^{c_n}}{\sqrt{2\pi c_n}} + tc'q^{cn}p^{-n^{\lambda'}} \to 0. \tag{13}$$

The last step is to prove

$$\lim_{n\to\infty} \mu(\bigcup_{k=1}^{c_n}\{|S_n^{\widetilde{\tau}_n(k)}| \leq 3n\} \cap Q_n^c \cap R_n^c) = 0.$$

Easy calculation shows that

$$\begin{aligned}\mu(\bigcup_{k=1}^{c_n}\{|S_n^{\widetilde{\tau}_n(k)}| \leq 3n\} \cap Q_n^c \cap R_n^c) &\leq \mu(\bigcup_{k=1}^{c_n}\{|S_n^{\widetilde{\tau}_n(k)}| \leq 3n\} \cap R_n^c) \\ &\leq \sum_{k=1}^{c_n}\mu(\{|S_n^{\widetilde{\tau}_n(k)}| \leq 3n\} \cap R_n^c).\end{aligned} \tag{14}$$

For $k \in \{1, \ldots, c_n\}$ denote

$$T_k = \{\widetilde{\tau}_n(k) - \widetilde{\tau}_n(k-1) \geq d(n,k)\}.$$

By definition of R_n, we obtain

$$\begin{aligned}R_n^c &= \bigcap_{j=1}^{c_n}\{\widetilde{\tau}_n(j) - \widetilde{\tau}_n(j-1) > d(n,j)\} \\ &\subseteq \{\widetilde{\tau}_n(k) - \widetilde{\tau}_n(k-1) \geq d(n,k)\} \\ &= T_k\end{aligned}$$

and therefore

$$\mu(\{|S_n^{\widetilde{\tau}_n}(k)| \le 3n\} \cap R_n^c) \le \mu(\{\widetilde{\tau}_n(k) \ge d(n,k)\} \cap T_k \cap \{|S_n^{\widetilde{\tau}_n(k)}| \le 3n\})$$
$$= \sum_{j=d(n,k)}^{\infty} \mu(\{\widetilde{\tau}_n(k) = j\} \cap T_k \cap \{|S_n^j| \le 3n\})$$
$$= \sum_{j=d(n,k)}^{\infty} \sum_{l=-3n}^{3n} \mu(\{\widetilde{\tau}_n(k) = j\} \cap T_k \cap \{S_n^j = l\}). \quad (15)$$

Now we find an upper bound for probability of the event

$$\{\widetilde{\tau}_n(k) = j\} \cap T_k \cap \{S_n^j = l\},$$

(see Figure 1).

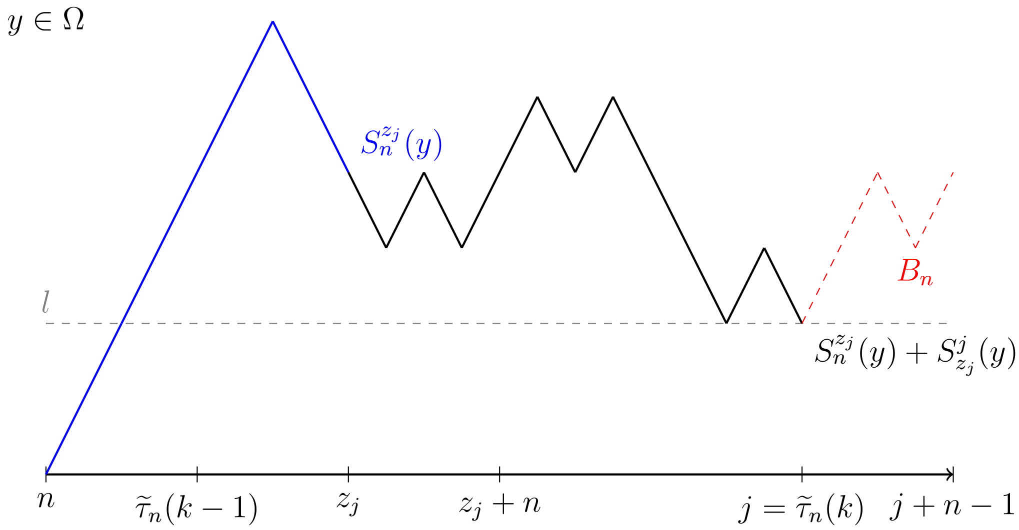

Figure 1: Realization of $\{\widetilde{\tau}_n(k) = j\} \cap T_k \cap \{S_n^j = l\}$

Define

$$z_j = j - \frac{d(n,k)}{3}$$
$$I_j = \{z_j - n, \dots, z_j\}.$$

We show that

$$\mu(\{\widetilde{\tau}_n(k) = j\} \cap T_k \cap \{S_n^j = l\}) \le \mu(\widetilde{\tau}_n(k) \in I_j)\phi(\frac{d(n,k)}{3}),$$

where

$$\phi(i) = \sup_{x \in \mathbb{Z}} \mu(S_i = x).$$

Clearly

$$\{\widetilde{\tau}_n(k) = j\} \cap T_k \cap \{S_n^j = l\} \in \mathcal{G}_n^{j+n-1}$$

and we can this write event as

$$\bigcup_{(y_n,\dots,y_{j+n-1})\in\Omega} \{Y_n = y_n, \dots Y_{j+n-1} = y_{j+n-1}\}$$

for suitable set $\Omega \subset M^j$. Take $(y_n, \dots, y_{j+n-1}) \in \Omega$ and assume that

$$B_n = \{Y_0 = y'_0, \dots, Y_{n-1} = y'_{n-1}\}.$$

By definition of Ω (resp. the event $\{\widetilde{\tau}_n(k) = j\}$),

$$(y_j, \dots, y_{j+n-1}) = (y'_0, \dots, y'_{n-1}).$$

Define permutation π on M^j such that π take last n coordinates and put them between coordinates $z_j - 1$ and z_j i.e.,

$$\pi(y) = (y_n, \dots, y_{z_j-1}, y_j, y_{j+1}, \dots, y_{j+n-1}, y_{z_j}, \dots, y_{j-1})$$

(see Figure 2). Further, denote

$$\Omega' = \pi(\Omega).$$

Since $(Y_n, \dots, Y_{n+j-1})$ is vector of independent identically distributed random variables, for every $y \in \Omega$, we get

$$\mu((Y_n, \dots, Y_{j+n-1}) = y) = \mu((Y_n, \dots, Y_{j+n-1}) = \pi(y)).$$

For $y \in \Omega$, by the definition of π, we know that k-hitting time into B_n in vector $\pi(y)$ happens no later than z_j and by event T_k not earlier than $z_j - n$. Further, since $S^j_n(y) = l$, we get $S^{z_j}_n(\pi(y)) + S^j_{z_j+n}(\pi(y)) = l$. We denote

$$V_j = [\widetilde{\tau}_{B_n}(k) \in I_j].$$

It follows from the previous that

$$\begin{aligned}\mu(\{\widetilde{\tau}_n = j\} \cap T_k \cap \{S^j_n = l\}) &= \mu(\bigcup_{y\in\Omega} \{(Y_n, \dots Y_{j+n-1}) = y\}) \\ &= \mu(\bigcup_{y'\in\Omega'} \{(Y_n, \dots Y_{j+n-1}) = y'\}) \\ &\le \mu(V_j \cap \{S^{z_j}_n + S^{j+n}_{z_j+n} = l\}).\end{aligned}$$

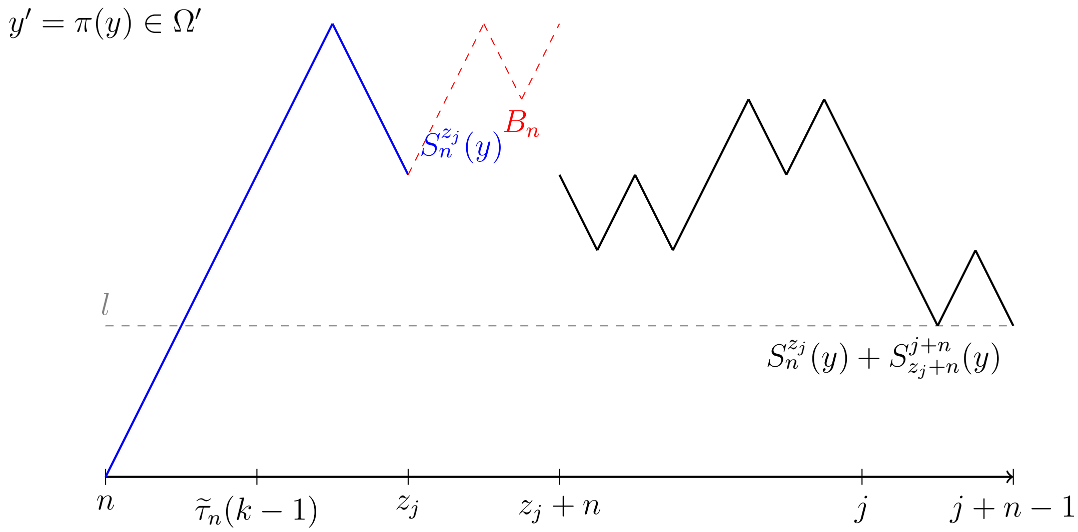

Figure 2: Realization after π-transformation

Furthermore,

$$\begin{aligned}\mu(V_j \cap \{S_n^{z_j} + S_{z_j+n}^{j+n} = l\}) &= \sum_{m=-\infty}^{\infty} \mu(V_j \cap \{S_n^{z_j} = m\} \cap \{S_{z_j+n}^{j+n} = l - m\}) \\ &= \sum_{m=-\infty}^{\infty} \mu(V_j \cap \{S_n^{z_j} = m\})\mu(S_{z_j+n}^{j+n} = l - m) \\ &\leq \sum_{m=-\infty}^{\infty} \mu(V_j \cap \{S_n^{z_j} = m\}) \sup_{x \in \mathbb{Z}} \mu(S_{z_j+n}^{j+n} = x) \\ &\leq \mu(V_j)\phi(\frac{d(n,k)}{3}).\end{aligned}$$

Therefore $\mu(V_j)\phi(\frac{d(n,k)}{3})$ is the upper bound for

$$\mu(\{\tilde{\tau}_n(k) = j\} \cap T_k \cap \{S_n^j = l\}),$$

which does not depend on l. For V_j we obtain

$$\begin{aligned}\sum_{j=d(n,k)}^{\infty} \mu(V_j) &\leq \sum_{j=d(n,k)}^{\infty} \sum_{i=0}^{n-1} \mu(\tilde{\tau}_n(k) = z_j - n + 1 + i) \\ &= \sum_{i=0}^{n-1} \sum_{j=d(n,k)}^{\infty} \mu(\tilde{\tau}_n(k) = j - \frac{d(n,k)}{3} - n + 1 + i) \leq n.\end{aligned}$$

We continue with (15)

$$\begin{aligned}\sum_{j=d(n,k)}^{\infty} \sum_{l=-3n}^{3n} \mu(\{\tilde{\tau}_n(k) = j\} \cap T_k \cap \{S_n^j = l\}) &\leq \sum_{j=d(n,k)}^{\infty} \sum_{l=-3n}^{3n} \mu(V_j)\phi(\frac{d(n,k)}{3}) \\ &\leq \sum_{j=d(n,k)}^{\infty} 7n\mu(V_j)\phi(\frac{d(n,k)}{3}) \\ &\leq 7n^2\phi(\frac{d(n,k)}{3}). \qquad (16)\end{aligned}$$

And now back to (14). From previous and by inequality in (2), we have

$$\begin{aligned}\sum_{k=1}^{c_n} \mu(\{S_n^{\widetilde{\tau_n}(k)} \leq 3n\} \cap R_n^c) &\leq \sum_{k=1}^{c_n} 4n^2 \phi(\frac{d(n,k)}{3}) \\ &\leq 7n^2 \sum_{k=1}^{\infty} \frac{c}{\sqrt{\frac{d(n,k)}{3}}} \\ &\leq 7n^2 \sum_{k=1}^{\infty} \frac{c}{\sqrt{n^6 k^4}} \\ &\leq 7n^2 \frac{1}{n^3} c \to 0. \end{aligned} \tag{17}$$

From (17), (13), (10) and (9), we finally obtain

$$\lim_{n\to\infty} \mu((W_{D_n}(s))^c) = 0.$$

□

7 Mixing measure-preserving dynamical systems

Definition 13. *Suppose* $(\Omega, \mathcal{F}, \mu)$ *is probability space and* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ *are* σ*-algebras. Define the following measures of dependence:*

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}} \sup_{B \in \mathcal{B}} |\mu(A \cap B) - \mu(A)\mu(B)|,$$

$$\phi(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}; \mu(A)>0} \sup_{B \in \mathcal{B}} |\mu(B|A) - \mu(B)|,$$

$$\psi(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}; \mu(A)>0} \sup_{B \in \mathcal{B}; \mu(B)>0} |\frac{\mu(A \cap B)}{\mu(A)\mu(B)} - 1|.$$

Remark 10. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a probability space and* $\mathcal{A}, \mathcal{B}$ *be* σ*-algebras. Then*

$$2\alpha(\mathcal{A}, \mathcal{B}) \leq \phi(\mathcal{A}, \mathcal{B}) \leq \frac{1}{2}\psi(\mathcal{A}, \mathcal{B}).$$

Proof. [3, Theorem 3.11] □

Definition 14. *Suppose* $V = \{V_k, k \geq 0\}$ *is a sequence of random variables on* $(\Omega, \mathcal{F}, \mu)$*. For* $0 \leq n \leq m \leq \infty$*, define the* σ*-algebra* $\mathcal{H}_n^m = \sigma(V_k, n \leq k \leq m)$*. For each positive integer* n*, define the following dependence coefficients:*

$$\alpha(n) = \alpha(n, V) = \sup_{j \in \mathbb{N}} \alpha(\mathcal{H}_0^j, \mathcal{H}_{j+n}^\infty),$$

$$\phi(n) = \phi(n, V) = \sup_{j \in \mathbb{N}} \phi(\mathcal{H}_0^j, \mathcal{H}_{j+n}^\infty),$$

$$\psi(n) = \psi(n, V) = \sup_{j \in \mathbb{N}} \psi(\mathcal{H}_0^j, \mathcal{H}_{j+n}^\infty).$$

The random sequence V *is said to be*

a) ***strongly mixing*** *or* $\boldsymbol{\alpha}$***-mixing*** *if* $\lim_{n\to\infty} \alpha(n) = 0$,

b) $\boldsymbol{\phi}$***-mixing*** *if* $\lim_{n\to\infty} \phi(n) = 0$,

c) $\boldsymbol{\psi}$***-mixing*** *if* $\lim_{n\to\infty} \psi(n) = 0$.

Lemma 15. *Let* $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ *be a skew-product. Denote by* $\mathcal{C}_\infty$ *the smallest* σ*-algebra containing all cylinders. For* $n \in \mathbb{N}$*, denote by* $\mathcal{C}_n$ *the smallest* σ*-algebra containing all cylinders of length* n*. Further, define a sequence of random variables* $\{V_k, k \geq 0\}$ *on* $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$*, where* $V_0(x, y) = (x_0, y_0)$

and $V_k(x,y) = V_0(U^k(x,y))$. *Then*

$$\alpha(l,V) = \sup_{n\geq 1}\sup_{A\in\mathcal{C}_n}\sup_{B\in\mathcal{C}_\infty} |\mu(A\cap U^{-n-l}(B)) - \mu(A)\mu(B)|,$$

$$\phi(l,V) = \sup_{n\geq 1}\sup_{A\in\mathcal{C}_n;\mu(A)>0}\sup_{B\in\mathcal{C}_\infty} = |\mu(U^{-n-l}(B)|A) - \mu(B)|,$$

$$\psi(l,V) = \sup_{n\geq 1}\sup_{A\in\mathcal{C}_n;\mu(A)>0}\sup_{B\in\mathcal{C}_\infty;\mu(B)>0} = |\frac{\mu(A\cap U^{-n-l}(B))}{\mu(A)\mu(B)} - 1|.$$

If V *is* α*(resp.* ϕ*,* ψ*)-mixing we say* $(E^{\mathbb{Z}}\times M^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ *is* α*(resp.* ϕ*,* ψ*)-mixing.*

Proof. We provide proof only for $\alpha(l,V)$. Since $\mathcal{C}_n = \mathcal{H}_0^n$, resp. $\mathcal{C}_\infty = \mathcal{H}_0^\infty$, for every $k\in\mathbb{N}$ and $B\in\mathcal{H}_0^\infty$, we get $U^{-k}(B)\in\mathcal{H}_k^\infty$. Therefore for every $l,n\in\mathbb{N}, A\in\mathcal{C}_n$,

$$\sup_{B\in\mathcal{C}_\infty} |\mu(A\cap U^{-n-l}(B)) - \mu(A)\mu(B)| = \sup_{B\in\mathcal{F}_{n+l}^\infty} |\mu(A\cap B) - \mu(A)\mu(B)|.$$

□

Remark 11. *Let* $\mathcal{G}$ *be* σ*-algebra with finitely or countable many atoms. Denote* $G_1, G_2, \ldots$ *the atoms of* $\mathcal{G}$*. Further, let* $\mathcal{A}$ *be an arbitrary* σ*-algebra, then*

$$\phi(\mathcal{G},\mathcal{A}) = \sup_{i\in\mathbb{N}}\sup_{A\in\mathcal{A}} |\mu(A|G_i) - \mu(A)|.$$

Proof. The proof can be found in [3, Proposition 3.21]. □

7.1 Mixing in skew-products

Example 2. *Let* $(E^{\mathbb{Z}}\times\{0,1\}^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ *be a skew-product, such that for all* $n\in\mathbb{N}$ *there exists* $x\in E$ *such that* $\mu(X_0 = x) < \frac{1}{n}$*. Then* $(E^{\mathbb{Z}}\times\{0,1\}^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ *is* ϕ*-mixing, but is not* ψ*-mixing.*

Proof. First we take two cylinders of length 1,

$$A_1 = \{X_0 = x_0, Y_0 = y_0\}, B_1 = \{X_0 = a_0, Y_0 = b_0\}.$$

Obviously,

$$U^{-1}(B_1) = \{X_{Y_0} = a_0, Y_1 = b_0\}$$

and for $k\in\mathbb{N}$

$$U^{-1-k}(B_1) = \{X_{S_{k+1}} = a_0, Y_{k+1} = b_0\}.$$

We will distinguish three possibilities. First take $y_0 = 0$ and $x_0 \neq a_0$, then

$$U^{-1}(B_1) \cap A_1 = \{\emptyset\}$$

and

$$U^{-k-1}(B_1) \cap A_1 \cap \{S_{k+1} = 0\} = \{\emptyset\}.$$

For $y_0 = 1$, we have

$$U^{-1}(B_1) \cap A_1 = \{X_0 = x_0, X_1 = a_0, Y_0 = 1, Y_1 = b_0\}$$

and therefore

$$\mu(U^{-1}(B_1) \cap A_1) = \mu(A_1)\mu(B_1).$$

By the same argument, for $k \in \mathbb{N}$, we obtain

$$\begin{aligned}\mu(U^{-1-k}(B_1) \cap A_1) &= \mu(X_0 = x_0, X_{1+S_1^{k+1}} = a_0, Y_0 = 1, Y_{1+k} = b_0) \\ &= \mu(A_1)\mu(B_1).\end{aligned}$$

Finally, for $y_0 = 0$ and $x_0 = a_0$,

$$\begin{aligned}\mu(U^{-1}(B_1) \cap A_1) &= \mu(Y_0 = 0, Y_1 = b_0, X_0 = a_0) \\ &= \mu(A_1)\frac{\mu(B_1)}{\mu(X_0 = a_0)}\end{aligned}$$

and

$$\mu(U^{-1-k}(B_1) \cap A_1 \cap \{S_1^{k+1} = 0\}) = \mu(A_1)\frac{\mu(B_1)}{\mu(X_0 = a_0)}(1 - EY_0)^k.$$

Furthermore, for all y_0 and x_0, we get

$$\mu(U^{-1-k}(B_1) \cap A_1 \cap \{S_1^{k+1} > 0\}) = \mu(A_1)\mu(B_1)\mu(S_1^{k+1} > 0).$$

Now we can show that skew-product is not ψ-mixing. For $k \in \mathbb{N}$ we take previous cylinders A_1 and B_1 such that $y_0 = 0$ and $x_0 = a_0$, then

$$\begin{aligned}\psi(k) &\geq |\frac{\mu(U^{-1-k}(B_1) \cap A_1)}{\mu(A_1)\mu(B_1)} - 1| \\ &= |\frac{\mu(U^{-1-k}(B_1) \cap A_1 \cap \{S_1^{k+1} = 0\}) + \mu(U^{-1-k}(B_1) \cap A_1 \cap \{S_1^{k+1} > 0\})}{\mu(A_1)\mu(B_1)} - 1| \\ &= |\frac{(1 - EY_0)^k}{\mu(X_0 = x_0)} + (1 - (1 - EY_0)^k) - 1|.\end{aligned}$$

By assumption, for every $k \in \mathbb{N}$, we can find $x_0 \in E$ such that

$$\frac{(1 - EY_0)^k}{\mu(X_0 = x_0)} > 5$$

and therefore skew-product is not ψ-mixing. Now we prove that skew-product is ϕ-mixing. Cylinders of length n are atoms in $\mathcal{C}_n$, therefore by Remark 11, it is sufficient to show that there exists a sequence $\{a_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} a_k = 0.$$

and

$$|\mu(U^{-n-k}(A)|D_n) - \mu(A)| \leq a_k$$

for every $n \in \mathbb{N}$, D_n cylinder of length n and $A \in \mathcal{C}_\infty$. Easy calculation shows that

$$\begin{aligned}\mu(U^{-n-k}(A) \cap D_n \cap \{S_n^{n+k} = 0\}) &\leq \mu(D_n \cap \{S_n^{n+k} = 0\}) \\ &= \mu(D_n)(1 - EY_0)^k. \end{aligned} \tag{18}$$

Consider $y \in \{0, 1\}^n$ such that for D_n resp. for its representation $A_n \cap B_n$ holds

$$B_n = \{Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}\}.$$

Since $\mathcal{C}_\infty \subseteq \sigma(\mathcal{A}_0^\infty \cup \mathcal{G})$, we have

$$U^{-k-n}(A) \in \sigma(\mathcal{A}_0^\infty \cup \mathcal{G}_{k+n}^\infty)$$

and for every $y' = (y'_n, \ldots, y'_{n+k-1}) \in \{0, 1\}^k$ such that $S_n^{n+k}(y') > 0$,

$$B_n \cap \{Y_n = y'_n, \ldots, Y_{n+k-1} = y'_{n+k-1}\} \cap U^{-n-k}(A) \in \sigma(\mathcal{A}_{S_n(y)+S_n^{n+k}(y')}^\infty \cup \mathcal{G}_0^\infty),$$

$$\{Y_n = y'_n, \ldots, Y_{n+k-1} = y'_{n+k-1}\} \cap U^{-n-k}(A) \in \sigma(\mathcal{A}_0^\infty \cup \mathcal{G}_n^\infty)$$

and since $A_n \in \mathcal{A}_0^{S_n(y)}$, we obtain

$$\mu(U^{-n-k}(A) \cap D_n \cap \{S_n^{k+n} > 0\}) = \mu(D_n)\mu(A)(1 - (1 - EY_0)^k). \tag{19}$$

By (19), we get

$$\begin{aligned}&\frac{\mu(U^{-n-k}(A) \cap D_n)}{\mu(D_n)} = \\ &= \frac{\mu(U^{-n-k}(A) \cap D_n \cap \{S_n^{n+k} > 0\}) + \mu(U^{-n-k}(A) \cap D_n \cap \{S_n^{n+k} = 0\})}{\mu(D_n)} \\ &= \mu(A)(1 - (1 - EY_0)^k) + \frac{\mu(U^{-n-k}(A) \cap D_n \cap \{S_n^{n+k} = 0\})}{\mu(D_n)}\end{aligned}$$

and hence by (18),

$$|\mu(A) - \frac{\mu(U^{-n-k}(A) \cap D_n)}{\mu(D_n)}| \leq \mu(A)(1 - EY_0)^k + (1 - EY_0)^k$$
$$\leq 2(1 - EY_0)^k.$$

We define $a_k = 2(1 - EY_0)^k$ and skew-product is ϕ-mixing. □

Example 3. *$(E^{\mathbb{Z}} \times \{-1, 1\}^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ is not ϕ-mixing.*

Proof. Let us take

$$B = \{X_0 = x_0, Y_0 = y_0\} \in \mathcal{C}_\infty.$$

For every $l \in \mathbb{N}$, we show that

$$\phi(l) \geq \mu(B).$$

Consider $x_1 \neq x_0$ and cylinder A of length $3l$ such that for its representation $C \cap D$ holds

$$C = \{Y_0 = 1, \dots Y_{2l-1} = 1, Y_{2l} = -1, \dots, Y_{3l-1} = -1\}$$

and

$$D = \{X_0 = x_1, \dots, X_{2l} = x_1\},$$

Hence

$$S_{3l} = l \ \ on \ \ C,$$

$$S_{4l} \in [0, 2l] \ \ on \ \ C$$

and also

$$X_{S_{4l}} = x_1 \ \ on \ \ C.$$

Therefore

$$\mu(A \cap U^{-3l-l}(B)) = 0.$$

and skew-product is not ϕ-mixing. □

7.2 Skew-product represented by a random process

Let $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ be a skew-product and let J be an arbitrary countable alphabet such that there is a bijective mapping $f : E \times M \to J$. Define a mapping

$$\varphi : E^{\mathbb{Z}} \times M^{\mathbb{N}_0} \to J$$

$$\varphi(x, y) = j \quad if \ \ (X_0(x, y), Y_0(x, y)) = f^{-1}(j).$$

For $n \in \mathbb{N}_0$ define projections π_n on $J^{\mathbb{N}_0}$

$$\pi_n(j_0, j_1, \ldots) = j_n$$

and a mapping φ_∞

$$\varphi_\infty : E^{\mathbb{Z}} \times M^{\mathbb{N}_0} \to J^{\mathbb{N}_0}$$

$$\pi_n(\varphi_\infty(x, y)) = \varphi(U^n(x, y)).$$

Let $\mathcal{J}$ be a product σ-algebra on $J^{\mathbb{N}_0}$ and

$$\sigma : J^{\mathbb{N}_0} \to J^{\mathbb{N}_0}$$

$$\sigma((j_0, j_1, \ldots)) = (j_1, j_2, \ldots)$$

be the shift on $J^{\mathbb{N}_0}$. For $n, m \in \mathbb{N}_0, n \geq m$ and $j_m, \ldots, j_n \in J$ we naturally define cylinder of rank (m, n) in $(J^{\mathbb{N}_0}, \mathcal{J}, \sigma)$ as a measurable set

$$\begin{aligned}[j_m, \ldots j_n] &= \pi_m^{-1}(j_m) \cap \pi_{m+1}^{-1}(j_{m+1}) \cap \ldots \cap \pi_n^{-1}(j_n) \\ &= \sigma^{-m}(J_{j_m}) \cap \sigma^{-m-1}(J_{j_{m+1}}) \cap \ldots \cap \sigma^{-n}(J_{j_n}),\end{aligned}$$

where $J_{j_i} = \pi_0^{-1}(j_i)$. Clearly, $\mathcal{J}$ is the smallest σ-algebra containing all cylinders. Let

$$D_n = V_{e_0, y_0} \cap U^{-1}(V_{e_1, y_1}) \cap U^{-n+1}(V_{e_{n-1}, y_{n-1}})$$

be a cylinder of length n in $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ and $j_0, \ldots, j_{n-1} \in J$ be such that $j_i = f((e_i, y_i))$. Obviously,

$$\varphi_\infty^{-1}(\sigma^{-k}[j_0, \ldots, j_{n-1}]) = U^{-k}(D_n) \in \mathcal{F} \tag{20}$$

for every $k \in \mathbb{N}_0$. Since set of all cylinders in $(J^{\mathbb{N}_0}, \mathcal{J}, \sigma)$ is closed on finite intersections and

$$\{B \in \mathcal{J} \mid \varphi_\infty^{-1}(B) \in \mathcal{F}\}$$

is Dynkin system, φ_∞ is $(\mathcal{F}, \mathcal{J})$-measurable. We define probability measure ν on $(J^{\mathbb{N}_0}, \mathcal{J})$ as

$$\nu(B) = \mu(\varphi_\infty^{-1}(B))$$

for $B \in \mathcal{J}$. Further, by (20)

$$\begin{aligned}
\nu(\sigma^{-1}[j_0, \ldots, j_{n-1}]) &= \mu(\varphi_\infty^{-1}(\sigma^{-1}[j_0 \ldots, j_{n-1}])) \\
&= \mu(U^{-1}(D_n)) \\
&= \mu(D_n) \\
&= \nu([j_0, \ldots, j_{n-1}])
\end{aligned}$$

and hence $(J^{\mathbb{N}_0}, \mathcal{J}, \nu, \sigma)$ is measure-preserving dynamical system. Furthermore, $\varphi_\infty^{-1}(B) \in \mathcal{C}_n$ for $B \in \mathcal{J}_0^n$, where $\mathcal{J}_0^n$ is the smallest σ-algebra containing all cylinders of rank $(0, n)$. Once again, for every $n \in \mathbb{N}_0$, $k \in \mathbb{N}_0$,

$$\{A \in \mathcal{J} \mid \exists A' \in \mathcal{F} \text{ such that } \varphi_\infty^{-1}(\sigma^{-k-n}A) = U^{-n-k}A'\}$$

is Dynkin system which contains all cylinders. Therefore, for every $k, n \in \mathbb{N}_0$ and $A \in \mathcal{J}$, there exist $A' \in \mathcal{F}$ such that

$$\varphi_\infty^{-1}(\sigma^{-k-n}A) = U^{-n-k}(A').$$

Hence

$$\nu(B \cap \sigma^{-k-n}A) = \mu(\varphi_\infty^{-1}(B) \cap U^{-n-k}(A')).$$

It follows that if $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ is α-mixing, then $(J^{\mathbb{N}_0}, \mathcal{J}, \nu, \sigma)$ is α-mixing and for every $n \in \mathbb{N}$ is $\alpha(n)$ in $(E^{\mathbb{Z}} \times M^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ greater or equal then $\alpha(n)$ in $(J^{\mathbb{N}_0}, \mathcal{J}, \nu, \sigma)$.

7.3 Discussion

We proved some mixing properties for particular skew-products, now we would like to discuss the relation of our work with some known results. We will show relation between our result and [2, Theorem 1]. First, [2, Theorem 1] in is formulated for the dynamical systems with shift. As we have seen in previous, this is not significant restriction. Further, [2, Theorem 1] assume that dynamical system is α-mixing. It is not easy to see that skew-product is at least α-mixing in general. Therefore we can not use it for general skew-product. Further, our proof of the limit distribution follows different approach, therefore we do not get the exactly same result.

Example 4. *We continue with Example 2. By Main theorem 1, for every* $t > 0$ *and for a sequence of cylinders* $\{D_n\}_{n=1}^{\infty}$ *such that* $D_n \in C_n \setminus \widetilde{C}_n$*,*

$$\lim_{n\to\infty} \mu(\tau_{D_n} > \frac{t}{\mu(D_n)}) = e^{-t}.$$

We can find similar result in [2]. More precisely consider $(J^{\mathbb{N}_0}, \mathcal{J}, \nu, \sigma)$ *constructed in previous section for skew-product defined in Example 2. Since* $(E^{\mathbb{Z}} \times \{0,1\}^{\mathbb{N}_0}, \mathcal{F}, \mu, U)$ *is* ϕ*-mixing, it is also* α*-mixing by Remark 10 and therefore* $(J^{\mathbb{N}_0}, \mathcal{J}, \nu, \sigma)$ *is* α*-mixing. By [2, Theorem 1], for* $(J^{\mathbb{N}_0}, \mathcal{J}, \nu, \sigma)$*, the following result holds:*

For any sequence $\{A_n\}_{n=1}^{\infty}$ *such that* $A_n \in \mathcal{A}_0^{n-1}$*,* $\nu(A_n) > 0$ *and*

$$\lim_{n\to\infty} \nu(\tau_{A_n} \le n) = 0,$$

there exist normalizing constants $\lambda(A_n)$ *such that*

$$\lim_{n\to\infty} \sup_{t\ge 0} |\nu(\lambda(A_n)\nu(A_n)\tau_{A_n} > t) - \exp(-t)| = 0$$

and $\limsup \lambda(A_n) \le 1$*.*

As we see from previous example we only provide a pointwise convergence of distribution functions while the convergence in [2, Theorem 1] is uniform. On the other hand, we do not need a rescaling constant $\lambda(A_n)$.

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