Master's thesis report:

# P. Korcsok *Minimal counterexamples to flow conjectures*

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The thesis *Minimal counterexamples to flow conjectures* provides a succinct digest of a series of papers by Kochol [7, 8, 9] and has the virtue of a clear expository style. Additionally, the material content of [12, 10] falls under the umbrella of these papers, too, and might be included among the references to bolster support for the statement that the thesis is a "comprehensive view" on Kochol's method. The thesis serves as an effective introduction to these papers, isolating well the key ideas, especially in Section 2.1.1 concerning the "forbidden networks" of Kochol. The thesis title ought, however, to be more specific: "Minimal counterexamples to Tutte's 5-flow conjecture."

The author also achieves the notable success of improving the latest lower bound obtained by Kochol on the girth of a minimal counterexample from 11 to 12 (Theorem 2.10). The implementation in Sage is described at the end of Section 2.1, with the help of some explanatory code snippets, and presented in full as an appendix and on the author's website. The effect is to make Kochol's work more available for inspection and development and the author is to be commended in this regard.

Theorem 2.13 provides an example graph $H'$ (Figure 2.7, non-crossing perfect matching of even cycle) in Kochol's extended replacement method that is different to Kochol's own choice (Figure 2.6) and which is derived from the vertex Splitting Lemma (stated as Theorem 2.11). However, the significance of Theorem 2.13 is not made clear, with just a comparative table of matrix ranks given after it and no further structural constraints on a minimal counterexample to the 5-flow conjecture being deduced. A related $H'$ (non-crossing matching plus triangle of odd cycle) is remarked to be awaiting consideration.

A thesis by Li [15] also revisits Kochol's method, a few words about which might be included in a wider review of the literature.

The task of the author is specified as a study of the papers [8, 9], an implementation of algorithms from them, and an attempt at transferring techniques to related conjectures such as the Cycle Double Cover Conjecture. In the first two he has succeeded admirably. The third has clearly not met with success as yet, but there remains potential for doing so given more time to elaborate and apply the methodology. I hope the author continues his researches in [redacted]

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## Small comments

**Abstract, and Page 1** Has Kochol been asked for his implementation? Rather than "Kochol has not shared his implementation" better might be "Kochol's implementation has not been made publicly available."

**Page 2** The standard terminology for *$k$-connected* as defined on page 2 is *$k$-edge-connected*. The former is commonly understood to refer to vertex connectivity. Likewise for cyclically $k$-edge-connected.

**Page 3** A *$k$-flow* is most commonly used to refer to a (nowhere-zero) integer flow taking values in absolute value less than $k$. Tutte's fundamental result that the existence of nowhere-zero $k$-flows is equivalent to the existence of nowhere-zero $\mathbb{Z}_k$-flows ought to be included, and then "has a $k$-flow" can be justifiably be used interchangeably with "has a $\mathbb{Z}_k$-flow"

**Page 5** "In 1952, Tutte studied some polynomials counting the numbers of various colourings of graphs" is somewhat vague and does not illuminate why Tutte formulated the quoted conjectures. Tutte's paper [19] introduces the flow polynomial ($\phi(G,n)$ in the notation of the paper) as dual to the chromatic polynomial (more precisely to the tension polynomial $\theta(G,n)$ in the notation of the paper) and it is in this paper that he introduces the *dichromate* of a graph (now called the Tutte polynomial).

**Page 10** and so $F_{I,W}(s) = 0$ for $s \in S_H$. (Similarly on page 11.)
Vector space over $\mathbb{Q}$. "Numbers" $y_1, \ldots, y_r$: rationals (or integers?).

**Page 12** Example illustrating the "main advantage of the method"? Alert reader to that in Section 2.2 (Figure 2.7)?

## Remarks

There are three celebrated unsolved conjectures dealing with nowhere-zero flows, all due to Tutte. The first is the 5-flow conjecture of [19], that every bridgeless graph admits a nowhere-zero 5-flow. The 4-flow conjecture [20] asserts that if a bridgeless graph does not contain a subgraph contractible to the Petersen graph, then it has a nowhere-zero 4-flow. Finally, the 3-flow conjecture is that if a graph is 4-edge-connected (does not have an edge cutset of size less than 4), then it has a nowhere-zero 3-flow.

Cubic graphs without edge-3-colorings, with girth at least 5, and cyclical edge-connectivity at least 4 (i.e., deleting fewer than 4 edges does not disconnect them into components two or more of which contain a circuit) are called *snarks*. Snarks do not have a nowhere-zero 4-flow and candidates for minimal counterexamples to the 5-flow conjecture are snarks.

**Other flow conjectures** The interesting question of whether the techniques of Kochol might be transferred to other flow problems (including the Cycle Double Cover Conjecture and Fulkerson's Conjecture in their formulation as $B$-flow problems by Jaeger [4]) is left to a single mention in the last line (Conclusion, page 19). A short account of what has been considered here (which flow problems, what obstacles were encountered to a trouble-free translation of techniques) would be useful, as would a short section recalling to the reader some of the flow conjectures that might be potentially tackled by a similar approach to that adumbrated in this thesis.

Kochol in [14] applies his techniques to 4-flows, and in [13] applies them to the flow polynomial more generally (not just its evaluation at 5).

**Minimal counterexample** How far does establishing increasingly tight constraints on a putative counterexample in terms of its girth (minimum size of a circuit) and edge-connectivity (minimum size of a cocircuit or cutset) bring us closer to resolving the 5-flow conjecture? Can for example the 6-flow theorem of Seymour be proved by a minimal counterexample argument? Kochol [11], refuting a conjecture of Jaeger and Swart that any snark has girth at most 6, showed that in fact there is a cyclically 5-edge-connected snark of any given girth. A short paragraph assessing the current state of knowledge regarding "the hunting of the snark" might be useful as a way of indicating how the constraints on girth and edge-connectivity for any snark that might refute the 5-flow conjecture lend increasing credence in there being no such counterexample. To widen the context to other flow conjectures, Goddyn [3] showed that a minimal counterexample to the Cycle Double Cover Conjecture must be a snark of girth at least 8. Is there any suggestive overlap of technique in Goddyn's proof?

**Vertex splitting** The Vertex Splitting Lemma (quoted as Theorem 2.12) is used to establish the fundamental result that a minimal counterexample to the 5-flow conjecture must be cubic [1, 17]: perhaps this ought to be included in the first section of Chapter 2 alongside the illuminating example of why $C_3$ and $C_4$ cannot be subgraphs of a minimal counterexample to the 5-flow conjecture. Most conjectures about flows can be easily reduced to the case of cubic graphs by such splitting arguments (including the cycle double cover conjecture). This technique does not apply to the 4-flow conjecture though, since splitting a vertex may introduce a Petersen minor. (The weaker conjecture of Tutte that restricts the 4-flow conjecture to cubic graphs was proved by Robertson, Seymour and Thomas: cubic graphs without a Petersen minor are indeed edge 3-colorable [16].)

To which other of Jaeger's $B$-flow problems [4] does the argument of Theorem 2.1 apply? What similar structural constraints are known for minimal counterexamples to the 4-flow and 3-flow conjectures?

Kochol, for example, uses a construction related to that in the Vertex Splitting Lemma that he calls *superposition* [6] to show that minimal counterexamples to the 3-flow conjecture must be 5-edge-connected [5].

**Open edges and gluing** The *simple networks* of Kochol – graphs with a labelled subset of terminals (degree 1 vertices) – may alternatively be viewed as the graphs with labelled outgoing (open) edges of Szegedy [18]. The gluing operation that marries like-labelled open edges corresponds to the merging operation that glues to networks together to make a graph. Kochol in [6] reverses the process of cutting a graph into two simple networks across an edge cutset by superimposing two networks in order to deduce structual properties of 5-snarks.

An inverse view is to consider cutting in half the edges of a cutset of a graph $G$ to make two networks (the open ends of the half-edges are asisigned terminal vertices) that when glued back together recover $G$. The dual notion would then be to cut in half the edges of a circuit of a graph to make two graphs each with virtual circuits composed of half-edges (rather than virtual cutsets of half edges, corresponding to the terminal edges in a simple network). Can this dual notion be used to prove properties of minimal graphs with respect to having a nowhere-zero tension? (i.e., colour-critical graphs $G$, that have $\chi(H) < \chi(G)$ for every proper subgraph $H$ of $G$.)

This notion of a virtual circuit composed of half-edges, dual to a virtual cutset composed of half-edges (open edges, terminals in simple network), is itself related to the labelled graphs of the graph algebras of e.g. Freedman, Lovász and Schrijver [2]. Can the linear algebras of "quantum graphs" of Lovász et al. be successfully applied to flow conjectures?

**Relation of flow conjectures to known results about planar graphs** Having gone to the trouble of explaining the relationship between flows and face colourings of plane graphs in Section 1.1, the opportunity might be taken to use

(1) Grötzsch's Theorem that triangle-free loopless planar graphs are 3-colourable, equivalent by duality to the statement that every 4-edge-connected planar graph has a nowhere-zero 3-flow to motivate the 3-flow conjecture, which asserts that the assumption of planarity can be dropped here;

(2) the Four Color Theorem, equivalent to the assertion that bridgeless planar cubic graphs have a nowhere-zero 4-flow, to motivate the weak 4-flow conjecture (now theorem [16]) that bridgeless cubic graphs have a nowhere-zero 4-flow, and the 4-flow conjecture that bridgeless graphs with no Petersen minor have a nowhere-zero 4-flow – the Petersen has no edge 3-colouring, while $K_{3,3}$ does. [this covered partially p. 5];

(3) the (easy) Five Colour Theorem for planar graphs to appreciate the sweeping generalization that is Tutte's 5-flow conjecture.

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