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Spontaneous symmetry breaking in strong and electroweak interactions

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Declaration of originality

This dissertation contains the results of research conducted at the Nuclear Physics Institute of the Academy of Sciences of the Czech Republic in Řež, in the period between fall 2002 and spring 2006. With the exception of the introductory Chapter 2, and unless an explicit reference is given, it is based on the published papers whose copies are attached at the end of the thesis. I declare that the presented results are original. In some cases, they have been achieved in collaboration as indicated by the authorship of the papers.

Prague, April 3, 2006 Tomáš Brauner

Contents

Chapter 1

Introduction

The principle of spontaneous symmetry breaking underlies much of our current understanding of the world around us. Although it has been introduced and developed in full generality in particle physics, its applications also cover a large part of condensed matter physics, including such fascinating phenomena as superconductivity, superfluidity, and Bose–Einstein condensation.

Ever since the very birth of science, philosophers, and later physicists, admired the beauty of the laws of nature, one of their most appealing features always being the *symmetry*. Indeed, it was symmetry considerations that lead Einstein to the creation of his theory of gravity, the general relativity, and it is symmetry that is the basic building block of the modern theories of the other fundamental interactions as well as all attempts to reconcile them with Einstein's theory.

Symmetry is not only aesthetic, it is also practical. It provides an invaluable guide to constructing physical theories and once applied, imposes severe constraints on their structure. This philosophy has, in particular, lead to the development of methods that allow us to exploit the symmetry content of the system even if we actually cannot solve the equations of motion. The theory of groups and their representations was first applied in quantum mechanics to the problem of atomic and molecular spectra, and later in quantum field theory, starting from the quark model and current algebra and evolving to the contemporary gauge theories of strong and electroweak interactions, and the modern concept of effective field theory.

There are many physical systems that, at first sight, display asymmetric behavior, yet there is a reasonable hope that they are described by symmetric equations of motion. Such a belief may be based, for instance, on the existence of a normal, symmetric phase, like in the case of superconductors and superfluids. Another nice example was provided by the historical development of the standard model of electroweak interactions. By the sixties, it was known that the only renormalizable quantum field theories including vector bosons were of the Yang–Mills type. It was, however, not clear how to marry the non-Abelian gauge invariance of the Yang–Mills theory with the requirement enforced by experiment, that the vector bosons be massive.

All these issues are resolved by the ingenious concept of a *spontaneously broken symmetry*. The actual behavior of the physical system is determined by the solution of the equations of motion, which may violate the symmetry even though the action itself is symmetric.

The internal beauty of the theory is thus preserved and, moreover, one is able to describe simultaneously the normal phase and the symmetry-breaking one. Just choose the solution which is energetically more favorable under the specified external conditions.

This thesis presents a modest contribution to the physics of spontaneous symmetry breaking within the standard framework for the strong and electroweak interactions and slightly beyond. The core of the thesis is formed by the research papers whose copies are attached at the end. Throughout the text, these articles are referred to by capital roman numbers in square brackets, while the work of others is quoted by arabic numbers. The calculations performed in the published papers are not repeated. We merely summarize the results and provide a guide for reading these articles and, to some extent, their complement.

The thesis is a collection of works on diverse topics, ranging from dynamical electroweak symmetry breaking to color superconductivity of dense quark matter and Goldstone boson counting in dense relativistic systems. Rather than giving an exhaustive review of each of them, we try to keep clear the unifying concept of spontaneous symmetry breaking and emphasize the similarity of methods used to describe such vastly different phenomena.

Of course, such a text cannot (and is not aimed to) be self-contained, and the bibliography cannot cover all original literature as well. In most cases, only those sources are quoted that were directly used in the course of writing. For sake of completeness we quote several review papers where the original references can also be found. The less experienced reader, e.g. a student or a non-expert in the field, is provided with a couple of references to lecture notes on the topics covered.

The thesis is organized as follows. The next chapter contains an introduction to the physics of spontaneous symmetry breaking. We try to be as general as possible to cover both relativistic and nonrelativistic systems. The following three chapters are devoted to the three topics investigated during the PhD study. Chapter 3 elaborates on the general problem of the counting of Goldstone bosons, in particular in relativistic systems at finite density. The electroweak interactions are considered in Chapter 4 and an alternative way of dynamical electroweak symmetry breaking is suggested. Finally, in Chapter 5 we study dense matter consisting of quarks of a single flavor and propose a novel mechanism for quark pairing, leading to an unconventional color-superconducting phase. After the summary and concluding remarks, the full list of author's publications as well as other references are given. The reprints of the research papers published in peer-reviewed journals, forming an essential and inseparable part of the thesis, are attached at the end.

Chapter 2

Spontaneous symmetry breaking

In this chapter we review the basic properties of spontaneously broken symmetries. First we discuss the general features, from both the physical and the mathematical point of view. To illustrate the rather subtle technical issues associated with the implementation of the broken symmetry on the Hilbert space of states, a simple example is worked out in some detail – the Heisenberg ferromagnet.

After the general introduction we turn our attention to the methods of description of spontaneously broken symmetries. We start with a short discussion of the model-independent approach of the effective field theory, and then recall two particular models that we take up in the following chapters – the linear sigma model and the Nambu–Jona-Lasinio model.

An extensive review of the physics of spontaneous symmetry breaking is given in Ref. [1]. A pedagogical introduction with emphasis on the effective-field-theory description of Goldstone bosons may be found in the lecture notes [2, 3, 4, 5].

2.1 General features

We shall be concerned with spontaneously broken *continuous internal* symmetries, that one meets in physics most often. The reason for such a restriction is twofold. First, this is exactly the sort of symmetries we shall deal with in the particular applications to the strong and electroweak interactions. Second, on the general ground, spontaneous breaking of discrete symmetries does not give rise to the most interesting existence of Goldstone bosons, while spacetime symmetries are more subtle, see Ref. [6].

As already noted in the Introduction, a symmetry is said to be spontaneously broken, if it is respected by the dynamical equations of motion (or, equivalently, the action functional), but is violated by their particular solution.[1] In quantum theory we use, however, operators and their expectation values rather than solutions to the classical equations of motion. Since virtually all information about a quantum system may be obtained with the knowledge of its ground state, it is only necessary to define spontaneous breaking of a symmetry in the ground state or, the vacuum [9].

[1]For a nice introductory account as well as several classical examples see Refs. [7, 8].

2.1.1 Realization of broken symmetry

Consider the group of symmetry transformations generated by the charge Q. If the symmetry were a true, unbroken one, it would be realized on the Hilbert space of states by a set of unitary operators. In such a case, their existence is guaranteed by the Wigner theorem [10] and we speak of the *Wigner–Weyl realization* of the symmetry. The vacuum is assumed to be a discrete, *nondegenerate* eigenstate of the Hamiltonian. Consequently, it bears a one-dimensional representation of the symmetry group, and therefore also is an eigenstate of the charge Q. The excited states are organized into multiplets of the symmetry, which may be higher-dimensional provided the symmetry group is non-Abelian.

By this heuristic argument we have arrived at the definition of a spontaneously broken symmetry: *A symmetry is said to be spontaneously broken if the ground state is not an eigenstate of its generator Q.* A very clean physical example is provided by the ferromagnet. Below the Curie temperature, the electron spins align to produce spontaneous magnetization. While the Hamiltonian of the ferromagnet is invariant under the SU(2) group of spin rotations (not to be mixed up with spatial rotations – see Section 2.2 for more details), this alignment clearly breaks all rotations except those about the direction of the magnetization.

Note that as a necessary condition for symmetry breaking it is usual to demand just that the generator Q does not annihilate the vacuum. Such a criterion, however, does not rule out the possibility that the ground state is an eigenstate of Q with nonzero eigenvalue. On the other hand, the vacuum charge can always be set to zero by a convenient shift of the charge operator.

A distinguishing feature of broken symmetry is that the vacuum is infinitely degenerate. In the case of the ferromagnet, the degeneracy corresponds to the choice of the direction of the magnetization. In general, the ground states are labeled by the values of a symmetry-breaking order parameter. Formally, the various ground states are connected by the broken-symmetry transformations.

With this intuitive picture in mind a natural question arises, whether a physical system actually chooses as its ground state one of those with a definite value of the order parameter, or their superposition. To find the answer, we go to finite volume and switch on a weak external perturbation (such as a magnetic field). The degeneracy is now lifted and there is a unique state with the lowest energy. This mechanism is called *vacuum alignment.*

After we perform the infinite volume limit and let the perturbation go to zero (in this order), we obtain the appropriate ground state. In order for this argument to be consistent, however, the resulting set of physically acceptable vacua should not depend on the choice of perturbation. Indeed, it follows from the general principles of causality and cluster decomposition that there is a basis in the space of states with the lowest energy such that all observables become diagonal operators in the infinite volume limit [11].

We have thus come to the conclusion that the correct ground state is one in which the order parameter has a definite value. The superpositions of such states do not survive the infinite volume limit and therefore are not physical. Moreover, transitions between individual vacua are not possible. This means that rather than being a set of competing ground states within a single Hilbert space, each of them constitutes a basis of a Hilbert space of its own, all bearing inequivalent representations of the broken symmetry. This is called the *Nambu–Goldstone realization* of the symmetry.

To summarize, when a symmetry is spontaneously broken, the vacuum is infinitely degenerate. The individual ground states are labeled by the values of an order parameter. In the infinite volume limit they give rise to physically inequivalent representations of the broken symmetry. Transitions between different spaces are only possible upon switching on an external perturbation. This lifts the degeneracy and by varying it smoothly, one can adiabatically change the order parameter.

This procedure can again be exemplified on the case of the ferromagnet. To change the direction of the magnetization, one first imposes an external magnetic field in the original direction of the magnetization. The magnetic field is next rotated, driving the magnetization to the desired direction, and afterwards switched off.

The issue of inequivalent realizations of the broken symmetry has rather subtle mathematical consequences [1], which we now shortly discuss and later, in Section 2.2, demonstrate explicitly on the case of the ferromagnet. As already mentioned, the Hilbert spaces with different values of the order parameter are connected by broken-symmetry transformations. The reason why they are called inequivalent is that these broken-symmetry transformations are not represented by unitary operators. They merely provide formal mappings between the various Hilbert spaces. By the same token, the generator Q is not a well defined operator in the infinite volume limit. What is well defined is just its commutators with other operators, which generate infinitesimal symmetry transformations.

Since the broken symmetry is not realized by unitary operators, it is also not manifested in the multiplet structure of the spectrum. This is determined by the unbroken part of the symmetry group. Let us, however, stress the fact that the broken symmetry is by no means similar to an approximate, but spontaneously unbroken one. Even though it does not generate multiplets in the spectrum, it still yields *exact constraints* which must be satisfied by, e.g., the Green's functions of the theory.

2.1.2 Goldstone theorem

One of the most striking consequences of spontaneous symmetry breaking is the existence of soft modes in the spectrum, ensured by the celebrated Goldstone theorem [12, 13]. In its most general setting applicable to relativistic as well as nonrelativistic theories, it can be formulated as follows: *If a symmetry is spontaneously broken, there must be an excitation mode in the spectrum of the theory whose energy vanishes in the limit of zero momentum.* In the context of relativistic field theory this, of course, means that the so-called *Goldstone boson* is a massless particle.

Several remarks to the Goldstone theorem are in order. First, in the general case it does not tell us how many Goldstone modes there are. Anyone who learned field theory in the framework of particle physics knows that in Lorentz-invariant theories, the number of Goldstone bosons is equal to the number of broken-symmetry generators [11]. In the nonrelativistic case, however, the situation is more complex and there is in fact no completely general counting rule that would tell us the exact number of the Goldstone modes. This issue will be discussed in much more detail in Chapter 3.

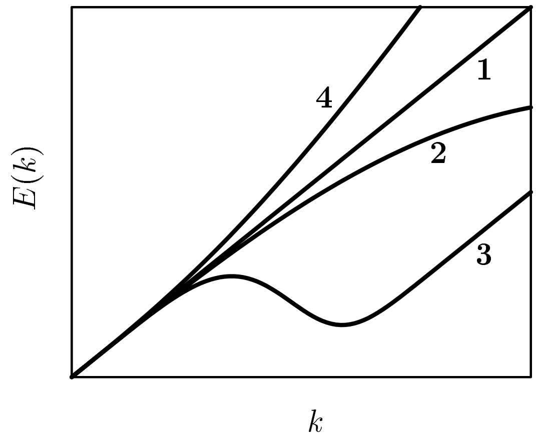


Figure 2.1: Dispersion relations of the Goldstone bosons in four physically distinct systems, conveniently normalized to have the same slope at the origin. 1. The Goldstone boson in a relativistic field theory. 2. The acoustic phonon in a solid. 3. The phonon-roton excitation in the superfluid helium. 4. The phonon in the relativistic linear sigma model at finite chemical potential (see Chapter 3).

Second, there are technical assumptions which, in some physically interesting cases, may be avoided, thus invalidating the conclusions of the Goldstone theorem. A sufficient condition for the theorem to hold is the causality which is inherent in relativistic field theories. The nonrelativistic case is, again, more complicated. In general, the Goldstone theorem applies if the potential involved in the problem decreases fast enough towards the spatial infinity. An example in which this condition is not satisfied is provided by the superconductors where the long-range Coulomb interaction lifts the energy of the low-momentum would-be Goldstone mode, producing a nonzero gap [14].

Third, the Goldstone theorem gives us information about the low-momentum behavior of the dispersion relation of the Goldstone boson. In the absence of other gapless excitations, the long-distance physics is governed by the Goldstone bosons and can be conveniently described by an effective field theory. This does not tell us, however, anything about the high-energy properties of the Goldstone bosons. At high energy, the dispersion relation of the Goldstone mode is strongly affected by the details of the short-distance physics. It is thus not as simple and universal as the low-energy limit, but at the same time not uninteresting, as documented by Fig. 2.1.

Let us now briefly recall the proof of the Goldstone theorem. The starting assumption is the existence of a conserved current, $j^\mu(x)$. From its temporal component, the charge operator generating the symmetry is formed,

$$Q(t) = \int \mathrm{d}^3\mathbf{x}\, j^0(\mathbf{x}, t).$$

The domain of integration is not indicated in this expression. The charge operator itself is well defined only in finite volume, but as long as its commutators with other operators are considered, the integration may be safely extended to the whole space [1].

Now the broken-symmetry assumption about the ground state $|0\rangle$ is that a (possibly composite) operator Φ exists such that

$$\langle 0|[Q,\Phi]|0\rangle \neq 0. \tag{2.1}$$

Note that this immediately yields our previous intuitive definition of broken symmetry: The vacuum cannot be an eigenstate of the charge Q. This vacuum expectation value is precisely what we called an order parameter before.

Inserting a complete set of intermediate states into Eq. (2.1) and assuming the translation invariance of the vacuum, one arrives at the representation

$$\langle 0|[Q,\Phi]|0\rangle = \sum_n (2\pi)^3\delta(\mathbf{k}_n)\left[e^{-\mathrm{i}E(\mathbf{k}_n)t}\langle 0|j^0(0)|n\rangle\langle n|\Phi|0\rangle - e^{\mathrm{i}E(\mathbf{k}_n)t}\langle 0|\Phi|n\rangle\langle n|j^0(0)|0\rangle\right]. \tag{2.2}$$

Using the current conservation one can show that the Goldstone commutator in Eq. (2.1) is time-independent provided the surface term which comes from the integral,

$$\int \mathrm{d}^3\mathbf{x}\,[\nabla\cdot\mathbf{j},\Phi],$$

vanishes. This is the central technical assumption which underlies the requirements of causality or fast decrease of the potential mentioned above.

Once this condition is satisfied, the time independence of the Goldstone commutator forces the right-hand side of Eq. (2.2) to be time-independent as well. This is, however, not possible unless there is a mode in the spectrum such that $\lim_{\mathbf{k}\to 0} E(\mathbf{k}) = 0$, which is the desired Goldstone boson.

2.2 Toy example: Heisenberg ferromagnet

The general statements about spontaneous symmetry breaking will now be demonstrated on the Heisenberg ferromagnet. Consider a cubic lattice with a spin-$\frac{1}{2}$ particle at each site. The dynamics of the spins is governed by the Hamiltonian

$$H = -J\sum_{\text{pairs}} \mathbf{s}_i\cdot\mathbf{s}_j, \tag{2.3}$$

which is invariant under simultaneous rotations of all the spins, that form the group SU(2).

For simplicity we choose the *nearest-neighbor interaction* so that the sum in Eq. (2.3) runs only over the pairs of neighboring sites. The coupling constant J is assumed positive so that the interaction favors parallel alignment of the spins. In finite volume we shall take up the periodic boundary condition in order to preserve the (discrete) translation invariance of the Hamiltonian (2.3).

2.2.1 Ground state

The scalar product of two neighboring spin operators may be simplified to

$$\mathbf{s}_i\cdot\mathbf{s}_j = \tfrac{1}{2}\left[(\mathbf{s}_i+\mathbf{s}_j)^2 - (\mathbf{s}_i^2+\mathbf{s}_j^2)\right] = \tfrac{1}{2}(\mathbf{s}_i+\mathbf{s}_j)^2 - \tfrac{3}{4}.$$

It is now clear that the state with the lowest energy will be one in which all pairs of spins will be arranged to have total spin one. The scalar product $\mathbf{s}_i \cdot \mathbf{s}_j$ then reduces to $\frac{1}{4}$. In a three-dimensional cubic lattice with N sites in total, there are altogether $3N$ such pairs so that the ground-state energy of the ferromagnet is

$$E_0 = -\tfrac{3}{4}NJ.$$

As we learned in the course of our general discussion of broken symmetries, the ground state is infinitely degenerate. The individual states may be labeled by the direction of the magnetization, a unit vector $\mathbf{n}$. All spins are aligned to point in this direction, which means that the ground state vector $|\Omega(\mathbf{n})\rangle$ is a direct product of one-particle states, the eigenvectors of the operators $\mathbf{n}\cdot\mathbf{s}_i$ with eigenvalue one half,

$$|\Omega(\mathbf{n})\rangle = \prod_{i=1}^{N} |i,\mathbf{n}\rangle, \quad \text{where} \quad (\mathbf{n}\cdot\mathbf{s}_i)|i,\mathbf{n}\rangle = \tfrac{1}{2}|i,\mathbf{n}\rangle.$$

The one-particle states may be expressed explicitly in terms of the two spherical angles θ, φ in the basis of eigenstates of the third component of the spin operator,

$$|\mathbf{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{\mathrm{i}\varphi}\sin\frac{\theta}{2} \end{pmatrix}. \tag{2.4}$$

The two vectors $|i,\mathbf{n}\rangle$ and $|i,-\mathbf{n}\rangle$ form an orthonormal basis of the one-particle Hilbert space $\mathcal{H}_i$. The products of these vectors then constitute a basis of the full Hilbert space of the ferromagnet, $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$.

In finite volume N, states with all possible directions $\mathbf{n}$ can be accommodated within a single Hilbert space. Two one-particle bases $\{|\mathbf{n}_1\rangle, |-\mathbf{n}_1\rangle\}$ and $\{|\mathbf{n}_2\rangle, |-\mathbf{n}_2\rangle\}$ are, as usual, connected by the unitary transformation corresponding to the rotation that brings the vector $\mathbf{n}_1$ to the vector $\mathbf{n}_2$. Likewise, the two corresponding product bases of the full Hilbert space $\mathcal{H}$ are connected by the induced unitary rotation on this product space.

Let us now calculate the scalar product of the ground states assigned to two directions $\mathbf{n}_1$ and $\mathbf{n}_2$. By exploiting the rotational invariance of the system, we may rotate one of the vectors, say $\mathbf{n}_1$, to the z-axis. The explicit expression for the eigenvectors (2.4) then yields $\langle\mathbf{n}_1|\mathbf{n}_2\rangle = \cos\frac{\theta_{\mathbf{n}_1,\mathbf{n}_2}}{2}$, where $\theta_{\mathbf{n}_1,\mathbf{n}_2}$ is the angle between the two unit vectors.

The scalar product of the two ground-state vectors is then given by

$$\langle\Omega(\mathbf{n}_1)|\Omega(\mathbf{n}_2)\rangle = \left(\cos\frac{\theta_{\mathbf{n}_1,\mathbf{n}_2}}{2}\right)^N$$

and it apparently goes to zero as $N\to\infty$ unless $\mathbf{n}_1$ and $\mathbf{n}_2$ are (anti)parallel.

Using a slightly different formalism we shall now construct the whole Hilbert space $\mathcal{H}(\mathbf{n})$ above the ground state $|\Omega(\mathbf{n})\rangle$ and show that, in fact, any two vectors, one from $\mathcal{H}(\mathbf{n}_1)$ and the other from $\mathcal{H}(\mathbf{n}_2)$, are orthogonal in the limit $N\to\infty$.

Recall that the two-dimensional space of spin $\frac{1}{2}$ may be viewed as the Fock space of the fermionic oscillator. One defines an annihilation operator $a(\mathbf{n})$ and a creation operator $a^\dagger(\mathbf{n})$ so that

$$a(\mathbf{n})|\mathbf{n}\rangle = 0 \quad \text{and} \quad \{a(\mathbf{n}), a^\dagger(\mathbf{n})\} = 1.$$

These are actually nothing else than the lowering and raising operators familiar from the theory of angular momentum. In addition to the identities above, they satisfy

$$[a(\mathbf{n}), a^\dagger(\mathbf{n})] = 2\mathbf{n}\cdot\mathbf{s}, \quad \text{so that} \quad \mathbf{n}\cdot\mathbf{s} = -a^\dagger(\mathbf{n})a(\mathbf{n}) + \tfrac{1}{2}.$$

When $\mathbf{n} = (0,0,1)$, these operators are just $a = s_x + \mathrm{i}s_y$, $a^\dagger = s_x - \mathrm{i}s_y$, and in the general case they can be found explicitly by the appropriate unitary rotation.

The Hilbert space $\mathcal{H}(\mathbf{n})$ is set up as a Fock space above the vacuum $|\Omega(\mathbf{n})\rangle$. In the ground state all spins point in the direction $\mathbf{n}$, while the excited states are obtained by the action of the creation operators $a_i^\dagger(\mathbf{n})$ that flip the spin at the i-th lattice site to the opposite direction.[2] The basis of the space $\mathcal{H}(\mathbf{n})$ contains all vectors of the form $a_{i_1}^\dagger(\mathbf{n})a_{i_2}^\dagger(\mathbf{n})\cdots|\Omega(\mathbf{n})\rangle$ where a *finite* number of spins are flipped.

It is now obvious that in the infinite-volume limit, all basis vectors from the space $\mathcal{H}(\mathbf{n}_1)$ are orthogonal to all basis vectors from the space $\mathcal{H}(\mathbf{n}_2)$ that is, these two spaces are completely orthogonal.

To put it in yet another way, at finite N any vector from the space $\mathcal{H}(\mathbf{n}_1)$ may be expressed as a linear combination of the basis vectors of the space $\mathcal{H}(\mathbf{n}_2)$, and thus these two spaces may be identified. This is, however, no longer true as $N\to\infty$, for the linear combination in question then contains an infinite number of terms, and is divergent. There is no other way out than treating the spaces $\mathcal{H}(\mathbf{n}_1)$ and $\mathcal{H}(\mathbf{n}_2)$ as distinct, orthogonal ones.

To summarize, in the limit $N\to\infty$ one has a continuum of mutually orthogonal separable Hilbert spaces $\mathcal{H}(\mathbf{n})$ labeled by the direction of the magnetization $\mathbf{n}$. In the absence of explicit symmetry breaking no transition between different spaces is possible and one has to choose the vector $\mathbf{n}$ once for all and work within the space $\mathcal{H}(\mathbf{n})$. Operators representing the observables are then constructed from the annihilation and creation operators $a_i(\mathbf{n})$ and $a_i^\dagger(\mathbf{n})$.

The symmetry transformations are formally generated by the operator of the total spin, $\mathbf{S} = \sum_i \mathbf{s}_i$. It is now evident that those transformations that change the direction of the magnetization $\mathbf{n}$, i.e. the spontaneously broken ones, are not realized by unitary operators since they do not operate on the Hilbert space $\mathcal{H}(\mathbf{n})$. The only operator that does is the projection of the total spin on the direction of the magnetization, $\mathbf{n}\cdot\mathbf{S}$. This generates the unbroken subgroup. (Yet, this operator is unbound for $N\to\infty$, but it can be normalized by dividing by N to yield the spin density, which is already finite.)

It is worth emphasizing, however, that physically all directions $\mathbf{n}$ are equivalent. Measurable effects can only arise from the change of the direction of $\mathbf{n}$.

2.2.2 Goldstone boson

One may now ask where is the Goldstone boson associated with the spontaneous breakdown of the SU(2) symmetry of the Hamiltonian (2.3). In the general discussion of the Goldstone theorem we assumed full translation invariance, while this lattice system has only a discrete one. Fortunately, this is not a problem in the infinite-volume limit, where

[2] Note that, in this setting, annihilation and creation operators at different lattice sites *commute* rather than anticommute as usual. The change of sign induced by the interchange of two distinguishable fermions is, however, merely a convention.

there is still a continuous momentum variable $\mathbf{k}$ to label one-particle states. The only difference is that only a finite domain of momentum, the Brillouin zone, should be used. We shall therefore assume that $-\pi/\ell \leq k_x, k_y, k_z \leq +\pi/\ell$, where ℓ is the lattice spacing.

As we emphasized above, all directions of $\mathbf{n}$ are physically equivalent, so we shall from now on set $\mathbf{n} = (0,0,1)$. The scalar product of two neighboring spins may be rewritten in terms of the annihilation and creation operators,

$$\begin{aligned}\mathbf{s}_i \cdot \mathbf{s}_j = \tfrac{1}{4}(a_i + a_i^\dagger)(a_j + a_j^\dagger) - \tfrac{1}{4}(a_i - a_i^\dagger)(a_j - a_j^\dagger) + (-a_i^\dagger a_i + \tfrac{1}{2})(-a_j^\dagger a_j + \tfrac{1}{2}) = \\ = -\tfrac{1}{2}(a_i^\dagger - a_j^\dagger)(a_i - a_j) + a_i^\dagger a_i a_j^\dagger a_j + \tfrac{1}{4}. \quad (2.5)\end{aligned}$$

Note that the Hamiltonian preserves the 'particle number' that is, the number of flipped spins generated by the operator $\sum_i a_i^\dagger a_i$. This is of course, up to irrelevant constants, nothing but the third component of the total spin, which is not spontaneously broken and thus can be used to label physical states. We shall restrict our attention to the 'one-particle' space, spanned on the basis $|i\rangle = a_i^\dagger|\Omega(\mathbf{n})\rangle$. The physical reason behind this restriction is that the sought Goldstone boson turns out to be the spin wave – a traveling perturbation induced by flipping a single spin.

On the one-particle space, the second term on the right hand side of Eq. (2.5) gives zero while the constant $\frac{1}{4}$ may be dropped. The one-particle Hamiltonian thus reads

$$H_{1\mathrm{P}} = \frac{J}{2}\sum_{\text{pairs}}(a_i^\dagger - a_j^\dagger)(a_i - a_j),$$

and acts on the basis states as[3]

$$H_{1\mathrm{P}}|i\rangle = -\frac{J}{2}\big(|i+1\rangle - 2|i\rangle + |i-1\rangle\big). \quad (2.6)$$

The discrete translation invariance is apparently not broken in the ground state. That means that the stationary states are simultaneously the eigenstates of the shift operator, $T : |i\rangle \to |i+1\rangle$. The eigenvalues of the shift operator are of the form $e^{\mathrm{i}k\ell}$. Eq. (2.6) implies that the one-particle Hamiltonian is diagonalized in the basis of eigenstates of T. The corresponding energies are

$$E(k) = \frac{J}{2}\left(2 - e^{\mathrm{i}k\ell} - e^{-\mathrm{i}k\ell}\right) = 2J\sin^2\frac{k\ell}{2}, \quad (2.7)$$

and in three dimensions we would analogously find $E(\mathbf{k}) = 2J\big(\sin^2\frac{k_x\ell}{2} + \sin^2\frac{k_y\ell}{2} + \sin^2\frac{k_z\ell}{2}\big)$.

We have thus found our Goldstone boson, in the case of the ferromagnet it is called the *magnon*. We stress the fact that we used no approximation, so Eq. (2.7) is the exact dispersion relation of the magnon, and the eigenstate $\sum_j e^{\mathrm{i}jk\ell}|j\rangle$ is the exact eigenstate of the full Hamiltonian (2.3).

Note also that there is just one Goldstone mode even though two symmetry generators, S_x and S_y, are spontaneously broken. This may be intuitively understood by acting

[3] As the low-energy dynamics of the Goldstone boson is isotropic, we work without lack of generality in one space dimension. The index i now refers to the linear ordering of the spin chain.

with either broken generator on the vacuum $|\Omega(\mathbf{n})\rangle$. We find $S_x|\Omega(\mathbf{n})\rangle = \frac{1}{2}\sum_j |j\rangle$ and $S_y|\Omega(\mathbf{n})\rangle = \frac{\mathrm{i}}{2}\sum_j |j\rangle$ that is, both operators create the same state, which formally corresponds to the zero-momentum magnon. This fact appears to be tightly connected to the dispersion relation of the magnon, which is quadratic at low momentum. The phenomenon is quite general and its detailed discussion is deferred to Chapter 3.

Having found the exact dispersion relation, it is suitable to comment on the issue of finite vs. infinite volume. Strictly speaking, there is no spontaneous symmetry breaking in finite volume. All the effects such as the unitarily inequivalent implementations of the symmetry and the existence of a gapless excitation appear only in the limit of infinite volume. Real physical systems are, on the other hand, always finite-sized. They are, however, large enough compared to the intrinsic microscopic scale (here the lattice spacing ℓ) of the theory so that the infinite-volume limit is both meaningful and practical.

In particular, when the ferromagnet lattice is of finite size N, the periodic boundary condition requires the momentum k to be quantized, the minimum nonzero value being $k_{\text{min}}\ell = 2\pi/N$. The energy gap in the magnon spectrum is then $E_{\text{min}} \approx 2\pi^2 J/N^2$, which is small enough for any macroscopic system to be to set to zero.

2.3 Description of spontaneous symmetry breaking

So far we have been discussing the very general features of spontaneously broken symmetries. To investigate a physical system in more detail, one next has to fix the Lagrangian. Before going into particular models we shall make an aside and mention the very important concept of effective field theory.

The method of effective field theory relies on the fact that, in the absence of other gapless excitations, the long-distance physics of a spontaneously broken symmetry is governed by the Goldstone bosons.[4] One then constructs the most general effective Lagrangian for the Goldstone degrees of freedom, compatible with the underlying symmetry [11].

The chief advantage of this approach is that it provides a model-independent description of the broken symmetry. The point is that by exploiting the underlying symmetry, it essentially yields the most general parametrization of the observables in terms of a set of low-energy coupling constants.

From the physical point of view, a disadvantage of effective field theory is that it tells us nothing about the origin of symmetry breaking – one simply has to assume a particular form of the symmetry-breaking pattern.

To show that the symmetry is broken at all and to specify the symmetry-breaking pattern, one has to find an appropriate order parameter. It is therefore not surprising that the issue of finding a suitable order parameter is of key importance, and considerable difficulty, for the description of spontaneous symmetry breaking.

In the following, we recall two particular models of spontaneous symmetry breaking. The operator whose vacuum expectation value provides the order parameter is an elementary

[4] Quite generally, the effective field theory approach may be applied whenever there are two or more energy scales in the system which can be treated separately. It is thus not special only to spontaneous symmetry breaking. This philosophy is emphasized in the lecture notes by Kaplan [3] and Manohar [4].

Figure 2.2: The Mexican-hat potential in the everyday life – the Italian pasta *sombreroni*.

field in the first case, and a composite object in the second one. In both cases, an approximation is made such that the quantum fluctuations of the order parameter are neglected.

2.3.1 Linear sigma model

Perhaps the most popular and universal approach to spontaneous symmetry breaking is to construct the Lagrangian so that it already contains the order parameter. This is very much analogous to the Ginzburg–Landau theory of second-order phase transitions. One introduces a scalar field[5] and adjusts the potential so that it has a nontrivial minimum. The result is the paradigmatic Mexican hat, see Fig. 2.2.

The great virtue of this method is that the order parameter is provided by the vacuum expectation value of an elementary scalar field, which may be chosen conveniently to achieve the desired symmetry-breaking pattern. As a particular example we shall now review the simplest model with Abelian symmetry.

Starting with a pure scalar theory, we define the Lagrangian for a complex scalar field ϕ as

$$\mathcal{L}_\phi = \partial_\mu \phi^\dagger \partial^\mu \phi + M^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2. \tag{2.8}$$

This Lagrangian is invariant under the phase transformations $\phi \to \phi e^{\mathrm{i}\theta}$ that form the Abelian group U(1). At tree level, the ground state is determined by the minimum of the static part of the Lagrangian, which is found at $\phi^\dagger\phi = v^2/2 = M^2/2\lambda$ so that the symmetry is spontaneously broken. As explained in Section 2.1.1, there is a continuum of solutions to this condition (distinguished by their complex phases) and the physical vacuum may be chosen as any one of them, but not their superposition. This is the reason why the following classical analysis actually works.

It is customary to choose the order parameter real and positive i.e., we set $\langle\phi\rangle = v/\sqrt{2}$. The scalar field is next shifted to the minimum and parametrized as $\phi = (v + H + \mathrm{i}\pi)/\sqrt{2}$. Upon this substitution the Lagrangian becomes

$$\mathcal{L}_\phi = \tfrac{1}{2}(\partial_\mu H)^2 + \tfrac{1}{2}(\partial_\mu \pi)^2 + \tfrac{1}{4}M^2 v^2 - M^2 H^2 - \lambda v H^3 - \tfrac{1}{4}\lambda H^4 - \tfrac{1}{2}\lambda H \pi^2 - \tfrac{1}{4}\lambda H^2 \pi^2 - \tfrac{1}{4}\lambda \pi^4.$$

The first three terms represent the kinetic terms for H and π and minus the vacuum energy

[5] The order parameter has to be a scalar unless one wants to break the space-time symmetry [15].

density, respectively. There is also the mass term for H, while the field π is massless – this is the Goldstone boson.

It is instructive to evaluate the U(1) Noether current in terms of the new fields,

$$j^{\mu} = \mathrm{i}(\phi^{\dagger}\partial^{\mu}\phi - \partial^{\mu}\phi^{\dagger}\phi) = -v\partial^{\mu}\pi + (\pi\partial^{\mu}H - H\partial^{\mu}\pi). \tag{2.9}$$

We can see that the Goldstone boson is annihilated by the broken-symmetry current, as predicted by the Goldstone theorem. The corresponding matrix element is given by $\langle 0|j^{\mu}(0)|\pi(\mathbf{k})\rangle \propto vk^{\mu}$, the constant of proportionality depending on the normalization of the one-particle states.

In the standard model of electroweak interactions, the scalar field is in fact added just for the purpose of breaking the gauge and global symmetries of the fermion sector. The same may be done in our toy model. We start with a free massless Dirac field whose Lagrangian, $\mathcal{L}_{\psi} = \bar{\psi}\mathrm{i}\not{\partial}\psi$, is invariant under the $\mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}$ chiral group. The mass term of the fermion violates the axial part of the symmetry and thus can be introduced only after this is broken.

To that end, we add the scalar field Lagrangian $\mathcal{L}_{\phi}$ and an interaction term $\mathcal{L}_{\phi\psi} = y(\bar{\psi}_L\psi_R\phi + \bar{\psi}_R\psi_L\phi^{\dagger})$. The full Lagrangian, $\mathcal{L} = \mathcal{L}_{\psi} + \mathcal{L}_{\phi} + \mathcal{L}_{\phi\psi}$, remains chirally invariant provided the scalar ϕ is assigned a proper axial charge. The nontrivial minimum of the potential in Eq. (2.8) now breaks the axial symmetry spontaneously and, upon the reparametrization of the scalar field, the fermion acquires the mass $m = vy/\sqrt{2}$.

2.3.2 Nambu–Jona-Lasinio model

In contrast to the phenomenological linear sigma model stands the idea of dynamical spontaneous symmetry breaking. Here, one does not introduce any artificial degrees of freedom in order to break the symmetry by hand but rather tries to find a symmetry-breaking solution to the quantum equations of motion.

Physically, this is the most acceptable and ambitious approach. Unfortunately, it is also much more difficult than the previous one. The reason is that one often has to deal with strongly coupled theories and, moreover, the calculations always have to be nonperturbative. As a rule, it is usually simply assumed that a symmetry-breaking solution exists and after it is found, it is checked to be energetically more favorable than the perturbative vacuum.

By this sort of a variational argument, one is able to prove that the symmetric perturbative vacuum is not the true ground state. On the other hand, it does not follow that the found solution is, which might be a problem in complex systems where several qualitatively different candidates for the ground state exist [16].

As an example, we shall briefly sketch the model for dynamical breaking of chiral symmetry invented by Nambu and Jona-Lasinio [17, 18, 19]. As the same model will be used in Chapter 5 to describe a color superconductor [20], we shall take up this opportunity to introduce the mean-field approximation that we later employ.

The Lagrangian of the original Abelian NJL model reads

$$\mathcal{L} = \bar{\psi}\mathrm{i}\not{\partial}\psi + G\left[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2\right]. \tag{2.10}$$

Its invariance under the Abelian chiral group $\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_\mathrm{A}$ is most easily seen when the interaction is rewritten in terms of the chiral components of the Dirac field, $\mathcal{L} = \bar{\psi}\mathrm{i}\not{\partial}\psi + 4G|\bar{\psi}_R\psi_L|^2$.

Following the original method due to Nambu and Jona-Lasinio, we anticipate spontaneous generation of the fermion mass by the interaction and split the Lagrangian into the *massive* free part and an interaction, $\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$, where

$$\mathcal{L}_{\text{free}} = \bar{\psi}(\mathrm{i}\not{\partial} - m)\psi, \quad \mathcal{L}_{\text{int}} = m\bar{\psi}\psi + G\left[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2\right].$$

At this stage already, we are making the choice of the ground state by introducing the mass term and requiring that m be real and positive. The general parametrization of the mass term would be $\bar{\psi}(m_1 + \mathrm{i}m_2\gamma_5)\psi$ with real m_1, m_2. The physical mass of the fermion would then be $\sqrt{m_1^2 + m_2^2}$.

The actual value of the mass m is determined by the condition of self-consistency, that it receives no one-loop radiative corrections. This gives rise to the gap equation

$$1 = 8\mathrm{i}G\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2 - m^2}. \tag{2.11}$$

The same result may be obtained with a method due to Hubbard and Stratonovich, which keeps the symmetry of the Lagrangian manifest at all stages of the calculation. One adds to the Lagrangian a term $-|\phi - 4G\bar{\psi}_R\psi_L|^2/4G$. In the path integral language, this amounts to an additional Gaussian integration over ϕ that merely contributes an overall numerical factor. Eq. (2.10) then becomes

$$\mathcal{L} = \bar{\psi}\mathrm{i}\not{\partial}\psi - \frac{1}{4G}(\phi_1^2 + \phi_2^2) + \bar{\psi}(\phi_1 + \mathrm{i}\phi_2\gamma_5)\psi, \tag{2.12}$$

the ϕ_1, ϕ_2 being the real and imaginary parts of ϕ, respectively.

The Lagrangian is now bilinear in the Dirac field so that this may be integrated out, yielding an effective action for the scalar order parameter ϕ,

$$\mathcal{S}_{\text{eff}} = -\frac{1}{4G}\int \mathrm{d}^4x\,(\phi_1^2 + \phi_2^2) - \mathrm{i}\log\det\left[\mathrm{i}\not{\partial} + (\phi_1 + \mathrm{i}\phi_2\gamma_5)\right] \tag{2.13}$$

With this effective action one can evaluate the partition function, or the thermodynamic potential, in the saddle-point approximation. This means that we have to replace the dynamical field ϕ with a constant determined as a solution to the stationary-point condition,

$$\frac{\delta\mathcal{S}_{\text{eff}}}{\delta\phi_1} = \frac{\delta\mathcal{S}_{\text{eff}}}{\delta\phi_2} = 0.$$

Looking back at Eq. (2.12) we see that the constant *mean field* ϕ yields precisely the effective mass of the fermion, and the stationary-point condition,

$$1 = 8\mathrm{i}G\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2 - \phi^\dagger\phi},$$

is identical to the gap equation (2.11).

In the Nambu–Jona-Lasinio model, the Goldstone boson required by the Goldstone theorem is a bound state of the elementary fermions. In the simple case of the Lagrangian (2.10) it is a pseudoscalar and may be revealed as a pole in the two-point Green's function of the composite operator $\bar{\psi}\gamma_5\psi$ [17].

Chapter 3

Goldstone boson counting in nonrelativistic systems

This chapter is devoted to a detailed discussion of the issue raised in Section 2.1.2: How many Goldstone bosons are there, given the pattern of spontaneous symmetry breaking? As already mentioned, in Lorentz-invariant theories the situation is very simple: The number of Goldstone bosons is equal to the number of the broken-symmetry generators. In nonrelativistic systems, however, these two numbers may differ.

We have already met an example where this happens – the ferromagnet. Historically, this was perhaps the first case in which the 'abnormal' number of Goldstone bosons was reported, and it still remains the only textbook one. Nevertheless, the same phenomenon has recently been studied in some relativistic systems at finite density [21, 22, 23, 24] as well as in the Bose–Einstein condensed atomic gases [25, 26], and it is therefore desirable to analyze the problem of the Goldstone boson counting on a general ground.

We start with a review of the general counting rule by Nielsen and Chadha [27] and some other partial results. The main body of this chapter then consists of the discussion of the Goldstone boson counting in the framework of the relativistic linear sigma model at finite chemical potential. The presented results are based on the paper [III], where the details of the calculations may be found.

3.1 Review of known results

3.1.1 Nielsen–Chadha counting rule

Following closely the treatment of Nielsen and Chadha [27], we consider a continuous symmetry, some of whose generators, Q_a, are spontaneously broken. The broken-symmetry assumption (2.1) now generalizes to

$$\det\langle 0|[Q_a, \Phi_i]|0\rangle \neq 0, \quad a, i = 1, \dots, \#\text{ of broken generators.}$$

In addition, it is assumed that the translation invariance is not entirely broken and that for any two local operators $A(x)$ and $B(x)$ a constant $\tau > 0$ exists such that

$$|\langle 0|[A(\mathbf{x}, t), B(0)]|0\rangle| \to e^{-\tau|\mathbf{x}|} \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{3.1}$$

It is then asserted that there are two types of Goldstone bosons – type-I, for which the energy is proportional to an odd power of momentum, and type-II, for which the energy is proportional to an even power of momentum in the long-wavelength limit. *The number of Goldstone bosons of the first type plus twice the number of Goldstone bosons of the second type is always greater or equal to the number of broken generators.*

The difference between the two types of Goldstone bosons is nicely demonstrated on the contrast between the ferromagnet and the antiferromagnet. In the ferromagnet, there is a single Goldstone boson (the magnon). The Nielsen–Chadha counting rule then enforces that it must be of type II and indeed, its dispersion relation is quadratic at low momentum, see Section 2.2.2. In the antiferromagnet, on the other hand, there are two distinct magnons with different polarizations. Their dispersion relation is linear.

Note that the result of Nielsen and Chadha does not restrict in any way the power of momentum to which the energy is proportional. As far as the counting of the Goldstone bosons is concerned, it only matters whether this power is an odd or an even number. It seems, however, that there are in fact no systems of physical interest where the power is greater than two.

It is also worthwhile to mention that the Nielsen–Chadha counting rule is formulated as an *inequality*, in most cases of physical interest this inequality is, however, saturated. This happens not only for the ferromagnet and the antiferromagnet. To the best of the author's knowledge, all exceptions where a sharp inequality occurs, happen at a phase boundary of the theory [22, 28]. Later in this chapter we shall see a generic class of such exceptions: The phase transition to the Bose–Einstein condensed phase of the theory, at which the phase velocity of the superfluid phonon vanishes and the phonon thus becomes a type-II Goldstone boson.

It is natural to ask what is the difference between the ferromagnet and the antiferromagnet that causes such a dramatic discrepancy in their behavior. The answer lies in the nonzero net magnetization of the ferromagnet. In general, it is nonzero vacuum expectation values of some of the charge operators that distinguish the type-II Goldstone bosons from the type-I ones. At a very elementary level, one can say that nonzero charge densities break time reversal invariance and thus allow for the presence of odd powers of energy in the effective Lagrangian for the Goldstone bosons [2]. The issue of charge densities, however, deserves more attention because they are usually easier to determine than the Goldstone boson dispersion relations.

3.1.2 Other partial results

As we have just shown, the issue of Goldstone boson counting is tightly connected to densities of conserved charges. We thus deal with three distinct features of spontaneously broken symmetries that are related to each other: The Goldstone boson counting, the charge densities in the ground state, and the dispersion relations of the Goldstone bosons.

The connection between the Goldstone boson counting and the dispersion relations is enlightened by the Nielsen–Chadha counting rule. In general, little is known about the direct relation of the Goldstone boson counting and the charge densities. There is a partial (in fact, only negative) result of Schaefer et al. [22] who proved that *the number of Goldstone bosons is usual i.e., equal to the number of broken generators, provided the* *commutators of all pairs of broken generators have zero density in the ground state.*

A necessary condition for an abnormal number of Goldstone bosons is thus a nonvanishing vacuum expectation value of a commutator of two broken generators. The value of this result is that it shows that the pattern of symmetry breaking must involve the non-Abelian structure of the symmetry group. For instance, the Goldstone boson counting is usual in all color-superconducting phases of QCD in which only the net baryon number density is nonzero. The reason is that the baryon number corresponds to a U(1) factor of the global symmetry group and therefore does not give rise to an order parameter for spontaneous symmetry breaking.

Intuitively, the necessity to modify the counting of the Goldstone bosons in the presence of charge densities can be understood as follows [III]. Assume that the commutator of the charges Q_a and Q_b develops nonzero ground-state expectation value. We may then in Eq. (2.2) set $Q = Q_a$ and take the charge density $j^0_b(x)$ in place of the interpolating field for the Goldstone boson, Φ. We find

$$\mathrm{i}f_{abc}\langle 0|j^0_c(0)|0\rangle = \langle 0|[Q_a, j^0_b(x)]|0\rangle = 2\mathrm{i}\,\mathrm{Im}\sum_n (2\pi)^3\delta(\mathbf{k}_n)\langle 0|j^0_a(0)|n\rangle\langle n|j^0_b(0)|0\rangle, \qquad (3.2)$$

where f_{abc} are the set of structure constants of the symmetry group. Two points here deserve a comment. First, it is again clear that a non-Abelian symmetry group is needed. Only then may the vacuum charge density be treated as an order parameter for spontaneous symmetry breaking. Second, it follows from the right hand side of Eq. (3.2) that a single Goldstone boson couples to two broken currents, j^μ_a and j^μ_b. We have already seen in Section 2.2 that this happens in the case of the ferromagnet. This suggests the way how the counting rule for the Goldstone bosons should be modified once nonzero density of a non-Abelian charge is involved. Nevertheless, it still remains to turn this heuristic argument into a more rigorous derivation of the proper counting rule.

Finally, the connection between the charge densities and the Goldstone boson dispersion relations was provided by the work of Leutwyler [29]. Leutwyler analyzed spontaneous symmetry breaking in nonrelativistic translationally and rotationally invariant systems. He determined the leading-order low-energy effective Lagrangian for the Goldstone bosons as the most general solution to the Ward identities of the symmetry. His results show that when a non-Abelian generator develops nonzero ground-state density, a term with a single time derivative appears in the effective Lagrangian. The time reversal invariance is then broken and the leading-order Lagrangian is of the Schrödinger type, resulting in the quadratic dispersion relation of the Goldstone boson. It should perhaps be stressed that when this happens, the effective Lagrangian is invariant with respect to the prescribed symmetry only up to a total derivative.

We shall now give a simple argument, also due to Leutwyler, explaining how such a single-time-derivative term in the Lagrangian affects the Goldstone boson counting. The effective Lagrangian is constructed on the coset space of the broken symmetry. Consequently, the number of independent *real fields* appearing in the Lagrangian is always equal to the number of broken generators.

Now if the single-time-derivative term is absent in the Lagrangian, the Goldstone boson dispersion relation is linear and comes, at tree level, in the form $E^2 \propto \mathbf{k}^2$. This equation has both positive and negative energy solutions which may be combined into a single real scalar field (similar to the Klein–Gordon field). There is therefore a one-to-one correspondence between the Goldstone bosons and the fields in the Lagrangian.

On the other hand, if there is a term with a single time derivative in the Lagrangian, the Goldstone boson dispersion relation is quadratic and appears as $E \propto \mathbf{k}^2$. This equation has, of course, only positive energy solutions, very much like the Schrödinger equation. As a result, the type-II Goldstone boson is to be described with a *complex field* or, equivalently, with a pair of real fields. This shows why the type-II Goldstone bosons have to be counted twice, when comparing their number to the number of broken generators.

Now and again, this intuitive picture easily accommodates only the Goldstone bosons with linear or quadratic dispersion. The question of the existence of Goldstone bosons with energy proportional to higher powers of momentum remains open as well as the possibility of their description in terms of a low-energy effective Lagrangian. Note that to achieve the appropriate power of momentum in the dispersion law, one would have to get rid of the standard bilinear kinetic term in the Lagrangian, which would invalidate the conventional perturbation expansion as well as the power-counting scheme.

3.2 Linear sigma model at finite chemical potential

The rest of this chapter is devoted to the study of a particular class of Lorentz-noninvariant systems – relativistic theories at finite density. The microscopic dynamics of such systems is Lorentz-invariant, Lorentz symmetry being violated only at the macroscopic level, by medium effects. This suggests that much more could be said about the patterns of symmetry breaking and properties of the Goldstone bosons than the Nielsen–Chadha theorem does, by exploiting the underlying Lorentz invariance.

In the following, we shall stay in the framework of the relativistic linear sigma model and derive an exact correspondence between the Goldstone boson counting, charge densities, and the Goldstone boson dispersion laws. The discussion of the possible extension of the achieved results is postponed to the Conclusions.

3.2.1 $\mathrm{SU}(2) \times \mathrm{U}(1)$ invariant sigma model

We start with a simple example: The linear sigma model with an $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry, which has been used as a toy model for kaon condensation in the Color-Flavor-Locked phase of QCD [21, 22]. All essential steps leading to the final counting rule for the Goldstone bosons will be first demonstrated within this model, then within a more complicated one with an $\mathrm{SU}(3) \times \mathrm{U}(1)$ symmetry, and afterwards generalized to the sigma model with arbitrary symmetry.

The model is defined by the Lagrangian,

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - M^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2, \tag{3.3}$$

where the scalar ϕ is a complex doublet. Nonzero density of the $\mathrm{U}(1)$ charge is implemented in terms of the chemical potential μ, which enters the Lagrangian through the covariant derivative, $D_0 \phi = (\partial_0 - \mathrm{i}\mu)\phi$.

In the absence of the chemical potential, the Lagrangian (3.3) is invariant under the extended group $\mathrm{SU}(2)\times\mathrm{SU}(2)\simeq\mathrm{SO}(4)$. The chemical potential breaks it explicitly down to $\mathrm{SU}(2)\times\mathrm{U}(1)$. In the context of the CFL phase with the kaon condensate, the SU(2) group corresponds to the isospin and the U(1) to the strangeness. The field ϕ is just the (charged or neutral) kaon doublet.

The chemical potential contributes a term $\mu^2\phi^\dagger\phi$ to the static part of the Lagrangian. When $\mu>M$, the perturbative vacuum $\phi=0$ becomes unstable and a new, nontrivial minimum appears – the $\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetry is spontaneously broken down to its U(1) subgroup. This is the relativistic Bose–Einstein condensation.

To reveal the physical content of the model in the spontaneously broken phase, we proceed in the standard manner i.e., calculate the minimum of the potential, shift the scalar field, and expand the Lagrangian about the new ground state. The scalar field is reparametrized as

$$\phi=\frac{1}{\sqrt{2}}e^{\mathrm{i}\pi_k\tau_k/v}\begin{pmatrix}0\\ v+H\end{pmatrix},\quad\text{where}\quad v^2=\frac{\mu^2-M^2}{\lambda},$$

τ_k being the Pauli matrices. The three 'pion' fields π_k would, in the absence of the chemical potential, correspond to the three Goldstone bosons of the coset $[\mathrm{SU}(2)\times\mathrm{U}(1)]/\mathrm{U}(1)$.

The excitation spectrum is determined by the bilinear part of the Lagrangian,

$$\mathcal{L}_{\mathrm{bilin}}=\frac{1}{2}(\partial_\mu\pi_k)^2+\frac{1}{2}(\partial_\mu H)^2-v^2\lambda H^2+\mu(\pi_1\partial_0\pi_2-\pi_2\partial_0\pi_1)+\mu(H\partial_0\pi_3-\pi_3\partial_0 H).\quad(3.4)$$

The presence of the chemical potential apparently leads to nontrivial mixing of the fields which cannot be removed by a global unitary transformation. To find the dispersion laws of the four degrees of freedom, it is therefore more appropriate to look for the poles of the propagators. It turns out [21, 22] that the mixing of π_1 and π_2 gives rise to *one* Goldstone boson with the low-momentum dispersion law $E(\mathbf{k})=\mathbf{k}^2/2\mu$, while the other mode is gapped, $E(\mathbf{k})=2\mu+\mathcal{O}(\mathbf{k}^2)$. On the other hand, the sector (π_3,H) produces one gapless excitation with $E(\mathbf{k})=\sqrt{\frac{\mu^2-M^2}{3\mu^2-M^2}}|\mathbf{k}|+\mathcal{O}(|\mathbf{k}|^3)$, and a massive radial mode with a gap $\sqrt{3\mu^2-M^2}$.

In conclusion, there are two Goldstone bosons, one with a linear dispersion law (the phonon) and one with a quadratic dispersion law. This is in accord with the Nielsen–Chadha counting rule since the vacuum expectation value $\langle\phi\rangle$ carries nonzero isospin. To see in more detail how this fact affects the structure of the bilinear Lagrangian (3.4), note that

$$\mu(\pi_1\partial_0\pi_2-\pi_2\partial\pi_1)=-\frac{\mu}{v^2}\pi_k\partial_0\pi_l\,\mathrm{Im}\big\langle[\tau_k,\tau_l]\big\rangle.$$

In this form it is obvious how the nonzero density of the commutator of two broken charges (3.2) enters the Lagrangian and thus gives rise to the existence of a single type-II Goldstone boson instead of two type-I ones.

To understand more deeply the nature of the type-II Goldstone boson, we shall now investigate the corresponding plane-wave solution of the classical equation of motion. Note first that the unbroken U(1) group is generated by the matrix $\frac{1}{2}(1+\tau_3)$. In order to keep this U(1) symmetry manifest, we combine π_1 and π_2 into one complex field, $\psi=\frac{1}{\sqrt{2}}(\pi_2+\mathrm{i}\pi_1)$. In fact, ψ is nothing but the upper component of the original doublet ϕ, expanded to first order in π.

As far as the quadratic Goldstone boson is concerned, we may drop the fields π_3 and H and rewrite the Lagrangian (3.4) in terms of ψ,

$$\mathcal{L}_\psi = 2\mathrm{i}\mu\psi^\dagger\partial_0\psi + \partial_\mu\psi^\dagger\partial^\mu\psi.$$

The field ψ annihilates the type-II Goldstone and the corresponding classical plane-wave solution is given by $\psi = \psi_0 e^{-\mathrm{i}k\cdot x}$, with the exact (tree-level) dispersion relation

$$E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + \mu^2} - \mu.$$

The $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry gives rise to four conserved currents which, in terms of the doublet ϕ, read

$$j^\mu_k = -2\,\mathrm{Im}\,\phi^\dagger\tau_k\partial^\mu\phi + 2\mu\delta^{\mu 0}\phi^\dagger\tau_k\phi, \quad j^\mu = -2\,\mathrm{Im}\,\phi^\dagger\partial^\mu\phi + 2\mu\delta^{\mu 0}\phi^\dagger\phi.$$

For the quadratic Goldstone plane wave we find

$$j^\mu_1 = +(k^\mu + 2\delta^{\mu 0}\mu)v\sqrt{2}\,\mathrm{Re}\,\psi, \quad j^\mu_2 = -(k^\mu + 2\delta^{\mu 0}\mu)v\sqrt{2}\,\mathrm{Im}\,\psi.$$

We can immediately see that the isospin density rotates in the isospin plane $(1,2)$ i.e., the plane wave is circularly polarized. In this way, a single Goldstone boson exploits two broken-symmetry generators, as suggested by the general form of the commutator (3.2). It is notable that the plane wave with the opposite circular polarization corresponds to the gapped excitation in the sector (π_1, π_2).

The remaining two currents are conveniently expressed in the rotated basis, explicitly separating the unbroken and broken generator,

$$\begin{aligned}
\tfrac{1}{2}(1+\tau_3): &\quad j^\mu = 2(k^\mu + \delta^{\mu 0}\mu)|\psi|^2,\\
\tfrac{1}{2}(1-\tau_3): &\quad j^\mu = \delta^{\mu 0}\mu v^2.
\end{aligned}$$

It is seen that the isospin wave is associated with a uniform current of the unbroken symmetry that is, the Goldstone boson carries the unbroken charge. This seems to be a generic feature of type-II Goldstone bosons.

Finally, the broken generator $\frac{1}{2}(1-\tau_3)$ gives rise just to nonzero charge density and, moreover, is independent of the amplitude and momentum of the isospin wave. It is therefore to be interpreted as just a background on which the isospin waves propagate.

3.2.2 Linear sigma model for SU(3) sextet

As a nontrivial example of a spontaneously broken symmetry with nonzero charge densities the linear sigma model for an SU(3) sextet scalar field will now be investigated.

The Lagrangian reads

$$\mathcal{L} = \mathrm{Tr}(D_\mu\Phi^\dagger D^\mu\Phi) - M^2\,\mathrm{Tr}\,\Phi^\dagger\Phi - a\,\mathrm{Tr}(\Phi^\dagger\Phi)^2 - b(\mathrm{Tr}\,\Phi^\dagger\Phi)^2, \tag{3.5}$$

and is invariant under the global $\mathrm{SU}(3) \times \mathrm{U}(1)$ symmetry that transforms the scalar field Φ as $\Phi \to U\Phi U^{\mathrm{T}}$. A U(1) chemical potential is introduced so that the covariant derivative is $D_0\Phi = (\partial_0 - 2\mathrm{i}\mu)\Phi$.

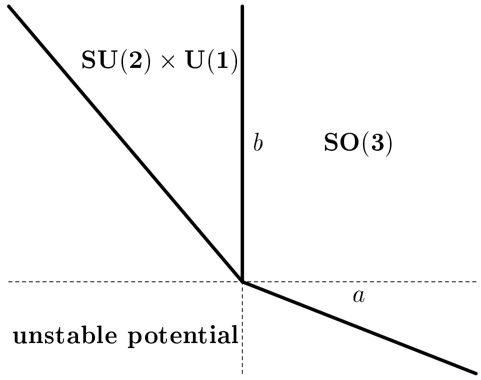


Figure 3.1: Phase diagram of the model defined by the Lagrangian (3.5). The ordered phases are labeled by the symmetry of the ground state. The 'unstable potential' region marks a domain of parameters where the tree-level potential is not bounded from below.

This model provides a phenomenological description of the color-superconducting phase of QCD with a color-sextet pairing of quarks of a single flavor, which was proposed in Ref. [I]. The global SU(3) symmetry is what remains of the color gauge invariance after the gluons have been 'integrated out', while the U(1) corresponds to the baryon number. The scalar field Φ is an effective composite field for the quark Cooper pairs.

It turns out that this theory has two different ordered phases, with different symmetry-breaking patterns and excitation spectra, see Fig. 3.1. The Bose–Einstein condensation sets at $\mu = M/2$. All phase transitions, between the normal and an ordered phase as well as between the ordered phases, are of second order.

In general, the excitations are grouped into multiplets of the unbroken symmetry. This means that the more of the original SU(3) $\times$ U(1) symmetry is spontaneously broken, the more complicated the structure of the spectrum is. Both phases will now be treated separately.

The $a > 0$ phase

The static part of the Lagrangian (3.5) is minimized by a scalar field proportional to the unit matrix i.e., $\Phi = \Delta\mathbb{1}$. The SU(3) $\times$ U(1) symmetry is thus spontaneously broken to its SO(3) subgroup.

With this symmetry-breaking pattern in mind, the scalar field Φ is parametrized as

$$\Phi(x) = e^{2\mathrm{i}\theta(x)}V(x)[\Delta\mathbb{1} + \varphi(x)]V^{\mathrm{T}}(x).$$

Here θ is the Goldstone boson of the spontaneously broken U(1) and $V = e^{\mathrm{i}\pi_k\lambda_k}$, $k = 1, 3, 4, 6, 8$, contains the 5-plet of Goldstone bosons of the coset SU(3)/SO(3). The real symmetric matrix φ represents six heavy 'radial' modes.

Using the notation $\Pi = \pi_k\lambda_k$, the excitation spectrum is determined by the bilinear

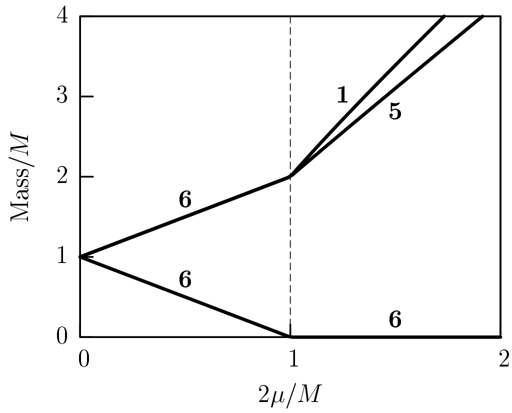


Figure 3.2: Masses of the excitations as a function of the chemical potential in the SO(3)-symmetric phase. Degeneracies of the excitation branches are indicated by the numbers. The numerical data were obtained with $a = b = 1$.

Lagrangian,

$$\mathcal{L}_{\text{bilin}} = 12\Delta^2(\partial_\mu\theta)^2 + 4\Delta^2\,\mathrm{Tr}(\partial_\mu\Pi)^2 + \mathrm{Tr}(\partial_\mu\varphi)^2 - \\ - 4\Delta^2\left[a\,\mathrm{Tr}\,\varphi^2 + b(\mathrm{Tr}\,\varphi)^2\right] - 16\mu\Delta\left[\partial_0\theta\,\mathrm{Tr}\,\varphi + \mathrm{Tr}(\varphi\partial_0\Pi)\right].$$

We find that there are six Goldstone bosons, all with linear dispersion relation. Since there are six broken generators as well, this result is in accord with the Nielsen–Chadha counting rule. All excitations fall into irreducible representations of the unbroken SO(3) group. In particular, there is a Goldstone singlet and a gapped singlet in the sector $(\theta, \mathrm{Tr}\,\varphi)$. In addition, there are two 5-plets, a gapless and a gapped one, stemming from mixing of Π with the traceless part of φ, see Fig. 3.2.

The $a < 0$ phase

In this case the minimum of the static potential can be recast to the diagonal form with a single nonzero entry, $\Phi = \mathrm{diag}(0, 0, \Delta)$. The symmetry-breaking pattern is now $\mathrm{SU}(3) \times \mathrm{U}(1) \to \mathrm{SU}(2) \times \mathrm{U}(1)$. The scalar sextet is conveniently parametrized as

$$\Phi(x) = e^{\mathrm{i}\Pi(x)} \begin{pmatrix} \sigma(x) & \\ & \Delta + H(x) \end{pmatrix} e^{\mathrm{i}\Pi^{\mathrm{T}}(x)}.$$

The matrix field Π is again given by the linear combination of the broken generators, $\Pi = \pi_k\lambda_k$, $k = 4, 5, 6, 7, 8$, σ is a complex symmetric 2×2 matrix, and H is a real scalar. The bilinear part of the Lagrangian is

$$\mathcal{L}_{\text{bilin}} = \mathrm{Tr}(\partial_\mu\sigma^\dagger\partial^\mu\sigma) + (\partial_\mu H)^2 + 2\Delta^2(\partial_\mu\Pi\partial^\mu\Pi)_{33} + 2\Delta^2(\partial_\mu\Pi_{33})^2 - \\ - 4\Delta^2(a+b)H^2 + 2\Delta^2 a\,\mathrm{Tr}\,\sigma^\dagger\sigma - 16\mu\Delta H\partial_0\Pi_{33} - 4\mu\Delta^2\,\mathrm{Im}[\Pi, \partial_0\Pi]_{33} - 4\mu\,\mathrm{Im}\,\mathrm{Tr}\,\sigma^\dagger\partial_0\sigma.$$

The SU(2) singlets H and π_8 mix, giving a Goldstone boson with linear dispersion law and a massive 'radial' mode. The fields $\pi_4, \pi_5, \pi_6, \pi_7$ altogether form a complex doublet of

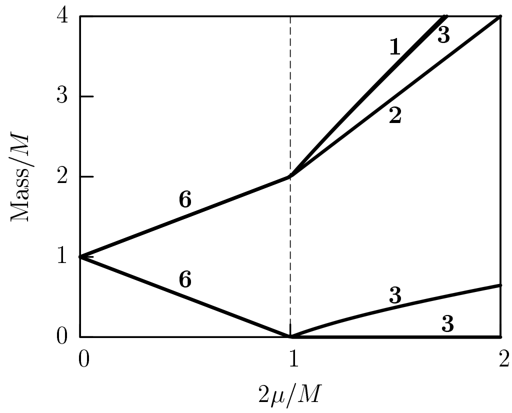


Figure 3.3: Masses of the excitations as a function of the chemical potential in the SU(2) × U(1) symmetric phase. Degeneracies of the excitation branches are indicated by the numbers. The numerical data were obtained with $a = -0.5$ and $b = 1$.

SU(2). They yield a doublet of gapped modes and a doublet of type-II Goldstone bosons with a quadratic dispersion relation. Finally, the complex matrix σ contains two real triplets of massive particles. For summary see Fig. 3.3.

Note that there are now only three Goldstone bosons even though five generators are spontaneously broken. This is, however, again in agreement with the Nielsen–Chadha rule since two of the Goldstones are of the second type. Their existence is connected with the fact that in this case, the generator λ_8 develops nonzero ground-state density. The modified Goldstone boson counting suggested by Eq. (3.2) thus applies.

Phase boundary

At the boundary between the two ordered phases the model displays quite remarkable properties. The Lagrangian (3.5) is then invariant under an extended SU(6) × U(1) symmetry under which Φ transforms as a fundamental sextet. The minima of the potential corresponding to the two phases are now degenerate and both leave unbroken the SU(5) × U(1) subgroup meaning that there are altogether eleven broken generators.

This enhanced symmetry must, of course, be reflected in the number and type of the Goldstone bosons [28]. Indeed, by properly performing the limit $a \to 0$ it can be shown on both sides of the phase transition that there are six Goldstone bosons. One is an SU(5) singlet and has a linear dispersion law – this is the superfluid phonon. The other five transform as the fundamental SU(5) 5-plet and all have a quadratic dispersion that is, are type-II. The Nielsen–Chadha counting is thus saturated as expected.

3.2.3 General analysis

The results achieved so far by the study of linear sigma models with particular symmetries will now be extended to the general case. We start with the formulation and a short discussion of our main result: *Nonzero vacuum density of a commutator of two broken* *generators implies the existence of one type-II Goldstone boson with a quadratic dispersion law.*

The existence of a single Goldstone boson corresponding to two broken generators, whose commutator has nonzero density, has been expected on the basis of Eq. (3.2). Here we explicitly prove the missing piece that is, the Goldstone boson is type-II as it must be in order to satisfy the Nielsen–Chadha counting rule. We shall also see that the statement formulated above holds strictly speaking only when a convenient basis of broken generators is chosen.

In a sense, this result is converse to the theorem by Schaefer et al. [22]. While they prove that zero density of commutators of broken charges implies usual counting of the Goldstone bosons, here we show that nonzero densities, on the contrary, lead to the existence of type-II Goldstones and thus modified counting.

Let us consider the linear sigma model with chemical potential assigned to one or more generators of the internal symmetry group. In general, the chemical potential for a conserved charge Q is introduced by replacing the Hamiltonian H with $H - \mu Q$. The key observation is that, as far as *exact* symmetry is concerned, the chemical potential is always assigned to a U(1) factor of the symmetry group that is, the charge Q commutes with all generators of the exact symmetry group. The reason is that even if the charge Q is originally a part of some larger non-Abelian symmetry group, by adding it to the Hamiltonian we explicitly break all generators that do not commute with it.

The Lagrangian for the general linear sigma model is defined as

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - V(\phi). \tag{3.6}$$

The scalar field ϕ transforms under a given representation of the global symmetry group G and $V(\phi)$ is the most general G-invariant renormalizable potential. Finally the chemical potential enters the Lagrangian through the covariant derivative $D_\mu \phi = (\partial_\mu - \mathrm{i}A_\mu)\phi$ [30], A_μ being the constant external gauge field which is eventually set to $A_\mu = (\mu Q, 0, 0, 0)$ or the sum of similar terms, when more chemical potentials are present.

The presence of the chemical potential destabilizes the perturbative ground state, $\phi = 0$, and eventually leads to spontaneous symmetry breaking by the Bose–Einstein condensation. We assume that the new minimum ϕ_0 breaks the global symmetry group of the Lagrangian, G, to its subgroup H. All generators, both broken and unbroken, are then classified by irreducible representations of H.

In the spontaneously broken phase the scalar field is parametrized as

$$\phi(x) = e^{\mathrm{i}\Pi(x)}\left[\phi_0 + H(x)\right]. \tag{3.7}$$

The matrix Π is a linear combination of the broken generators while H contains the massive (Higgs) fields. Upon expanding the Lagrangian (3.6) in terms of the field components, its bilinear part becomes

$$\begin{aligned}\mathcal{L}_{\text{bilin}} = \partial_\mu H^\dagger \partial^\mu H - V_{\text{bilin}}(H) - 2\,\mathrm{Im}\,H^\dagger A^\mu \partial_\mu H + \\ + \phi_0^\dagger \partial_\mu \Pi \partial^\mu \Pi \phi_0 - 4\,\mathrm{Re}\,H^\dagger A^\mu \partial_\mu \Pi \phi_0 - \mathrm{Im}\,\phi_0^\dagger A^\mu [\Pi, \partial_\mu \Pi]\phi_0.\end{aligned} \tag{3.8}$$

Here V_{bilin} is the bilinear part of the potential, which involves only the 'radial' field H, due to the used parametrization (3.7).

Eq. (3.8) is the main result which contains essentially all information about the spectrum of the sigma model. To understand better its consequences, we resort for a moment to a simple bilinear Lagrangian with just two scalar fields,

$$\mathcal{L}_{\text{bilin}} = \frac{1}{2}(\partial_\mu \pi)^2 + \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}f^2(\mu)h^2 - g(\mu)h\partial_0\pi. \tag{3.9}$$

One of the fields, h, possibly has a mass term and there is also a single-derivative mixing term, both depending explicitly on the chemical potential. This is the generic form of the bilinear Lagrangian we met in the two particular examples in the preceding sections.

A simple calculation reveals that the Lagrangian (3.9) describes a (massive) particle with dispersion relation $E^2(\mathbf{k}) = f^2(\mu) + g^2(\mu) + \mathcal{O}(\mathbf{k}^2)$, and a gapless mode with dispersion

$$E^2(\mathbf{k}) = \frac{f^2(\mu)}{f^2(\mu)+g^2(\mu)}\mathbf{k}^2 + \frac{g^4(\mu)}{[f^2(\mu)+g^2(\mu)]^3}\mathbf{k}^4 + \mathcal{O}(\mathbf{k}^6). \tag{3.10}$$

If $f(\mu) = 0$ that is, if both π and h are Goldstone fields mixed by the single-derivative term, we arrive at one type-II Goldstone boson. The expansion of its energy in powers of momentum starts at the order $\mathbf{k}^2$. On the other hand, when $|f(\mu)| > 0$, the field h represents a massive mode. The mixing of h and π then results in a type-I Goldstone boson with linear dispersion relation.

We can now understand the content of Eq. (3.8). There are kinetic terms for both the radial fields H and the Goldstones Π, and the mass term for H, essentially given by the curvature of the static potential at the minimum ϕ_0. Finally, there are three mixing terms with a single derivative, proportional to the external field A^μ.

The analysis of the model Lagrangian (3.9) tells us that mixing of a radial field with a Goldstone field gives rise to one type-I Goldstone boson. The mixing of two Goldstone fields, on the other hand, produces one type-II Goldstone boson. A short glance at the last term on the right hand side of Eq. (3.8) shows that the Goldstone–Goldstone mixing term is, as expected, proportional to the ground-state expectation value of a commutator of two broken generators. We have thus established the desired result that nonzero density of a commutator of two broken generators gives rise to a single type-II Goldstone boson.

In order for the conclusions just reached to be reliable, we have to show that the results of the analysis of the simple Lagrangian (3.9) are applicable to the much more complicated case of Eq. (3.8). A detailed proof may be found in Ref. [III] and will not be repeated here. Instead, we limit our discussion to a simplified version where, nevertheless, all the essential steps are provided.

The crucial observation regarding the charge densities is that one may always choose a basis of broken generators so that all generators with nonzero vacuum expectation value mutually commute. We give a simple proof of this statement for the case of unitary symmetries [31, 32]. The set of vacuum expectation values $\langle 0|Q_a|0\rangle$ of the generators may by regarded as a vector v_a in the space of the adjoint representation of the Lie algebra $\mathfrak{g}$ of the group G. In the fundamental representation of the unitary group, the generators Q_a are realized by Hermitian matrices, say T_a. Now v_aT_a is also a Hermitian matrix and as such can be diagonalized by a proper unitary transformation. After this transformation v_aT_a is a linear combination of just the diagonal generators of the symmetry group that all mutually commute i.e., span the Cartan subalgebra of $\mathfrak{g}$.

We can now take up the generators that have nonzero density in the ground state and complement them to the Cartan subalgebra of $\mathfrak{g}$. The rest of the generators is grouped according to the standard root decomposition of Lie algebras [33]. The point is that within this basis, for any generator there is a unique generator such that their commutator lies in the Cartan subalgebra. It is now proved that the broken generators participate in the last term of Eq. (3.8) in pairs and the simple two-field analysis of Eq. (3.9) is therefore applicable.

It should, of course, also be proved that the same conclusion is true for the mixing of the Goldstone fields with the radial ones, and of the radial ones with themselves. Omitting the details, we just note that this follows from the Wigner–Eckart theorem upon a proper decomposition of the matrix fields Π and H into irreducible representations of the unbroken subgroup H.

Chapter 4

Dynamical electroweak symmetry breaking

The standard model of electroweak interactions has been one of the most successful achievements of modern physics. Within a simple and elegant framework, it perfectly describes essentially all experimental data collected so far. It is, however, somewhat disturbing that its only ingredient that has not been experimentally verified yet, the Higgs boson, is crucial for the mechanism of symmetry breaking of the $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$ gauge invariance and thus also for the generation of the masses of the elementary particles. Anyway, arguments based on the naturalness principle suggest that the standard model is just a low-energy limit of some more fundamental theory, and that new physics is most likely to be found at the energies accessible already to the upcoming LHC machine at CERN.

In this Chapter we shall take a different point of view of the standard model. In Section 2.3.1 we explained how a phenomenological Lagrangian of the Ginzburg–Landau type may be used to induce spontaneous symmetry breaking. We have, however, emphasized that such an approach is physically unsatisfactory since it does not give an answer to the basic question about the origin of symmetry breaking.

This happens exactly in the standard model, where the scalar sector is introduced for sake of breaking the gauge symmetry. Attempts at replacing the conventional Higgs mechanism with a dynamical model of electroweak symmetry breaking appeared soon after the construction of the standard model itself [34, 35]. The introduction to the idea of dynamical electroweak symmetry breaking may be found in the lecture notes [36, 37], while a more detailed review is provided by Refs. [38, 39].

The technicolor scenarios dispose with the elementary Higgs and, instead of its vacuum expectation value, generate the order parameter for symmetry breaking by a fermion–antifermion condensate. This is bound together by a new strong gauge interaction.

Here we propose a different idea for dynamical electroweak symmetry breaking. We retain the elementary scalar, but with a positive mass squared so that the usual particle interpretation is preserved even in the absence of interactions. Our basic assumption is the existence of a strong Yukawa interaction between the scalar and the massless fermions. We show that, provided the Yukawa coupling is large enough, the fermion masses may be generated spontaneously as a self-consistent solution of the Schwinger–Dyson equations.

In other words, no strong gauge force is needed. The strong Yukawa interaction breaks spontaneously the chiral symmetry, allowing for nonzero fermion masses. Only after then, the $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$ gauge interaction is switched on perturbatively, resulting in the same symmetry-breaking pattern as in the Higgs mechanism.

In order to make the proposed mechanism more transparent, we first demonstrate it on the dynamical breaking of a global Abelian chiral symmetry, following our paper [II]. The concluding section is devoted to the discussion of the extension to the full $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$ gauge symmetry [V]. This model, as well as the Abelian one with the axial symmetry gauged, are, however, still being worked on.

4.1 Toy model: Global Abelian chiral symmetry

We consider a model of two Dirac fermions and a complex scalar defined by the Lagrangian,

$$\begin{aligned}\mathcal{L} = \sum_{j=1,2} \left(\bar{\psi}_{jL} \mathrm{i}\partial\!\!\!/ \psi_{jL} + \bar{\psi}_{jR} \mathrm{i}\partial\!\!\!/ \psi_{jR}\right) + \partial_\mu \phi^\dagger \partial^\mu \phi - M^2 \phi^\dagger \phi - \lambda(\phi^\dagger\phi)^2 + \\ + y_1 \left(\bar{\psi}_{1L}\psi_{1R}\phi + \bar{\psi}_{1R}\psi_{1L}\phi^\dagger\right) + y_2 \left(\bar{\psi}_{2R}\psi_{2L}\phi + y_2\bar{\psi}_{2L}\psi_{2R}\phi^\dagger\right). \quad (4.1)\end{aligned}$$

The Yukawa couplings y_1, y_2 are, without lack of generality, assumed to be real. Note that this Lagrangian has a global $\mathrm{U}(1)_{\mathrm{V1}} \times \mathrm{U}(1)_{\mathrm{V2}} \times \mathrm{U}(1)_{\mathrm{A}}$ symmetry. The vector U(1)'s correspond to independent phase transformations of the two Dirac spinors ψ_1, ψ_2. The axial U(1) consists of simultaneous transformations of all the fields concerned,

$$\psi_1 \to e^{+\mathrm{i}\theta\gamma_5}\psi_1, \quad \psi_2 \to e^{-\mathrm{i}\theta\gamma_5}\psi_2, \quad \phi \to e^{-2\mathrm{i}\theta}\phi.$$

Note that the scalar field ϕ carries the axial charge. It plays a crucial role in the proposed mechanism of chiral (or axial) symmetry breaking. Also, the axial charges of the fermions are opposite in order to remove the anomaly in the axial current. It should be stressed that, as far as global symmetry is concerned, the axial anomaly is nothing disastrous and, in fact, gives rise to physical effects such as the $\pi^0 \to \gamma\gamma$ decay in QCD. However, having in mind the future application to electroweak interactions where the symmetry is gauged, we choose to remove the anomaly from the very beginning.

4.1.1 Ward identities: general

The first step in the investigation of the model (4.1) is the analysis of the symmetry. In quantum field theory, this is encoded into a set of Ward identities for the Green's functions. Since the existence of a Goldstone boson is a robust prediction of the Goldstone theorem, we show that the Ward identities alone provide a lot of information about the Goldstone boson properties. We work them out without any further dynamical assumption so that we are later able to compare dynamical symmetry breaking with the conventional Higgs mechanism as presented in Section 2.3.1.

The $\mathrm{U}(1)_{\mathrm{V1}} \times \mathrm{U}(1)_{\mathrm{V2}} \times \mathrm{U}(1)_{\mathrm{A}}$ symmetry of the Lagrangian implies the existence of three conserved currents, two vector and one axial, given by

$$\begin{aligned} j^\mu_{V1} = \bar{\psi}_1\gamma^\mu\psi_1, \quad j^\mu_{V2} = \bar{\psi}_2\gamma^\mu\psi_2, \\ j^\mu_A = \bar{\psi}_1\gamma^\mu\gamma_5\psi_1 - \bar{\psi}_2\gamma^\mu\gamma_5\psi_2 + 2\mathrm{i}\left[(\partial^\mu\phi)^\dagger\phi - \phi^\dagger\partial^\mu\phi\right]. \end{aligned} \quad (4.2)$$

Of all the correlation functions of these currents, we shall consider the three-point ones, with a single current and a pair of fermions or scalars. The vector currents do not couple to the scalar, so there are just two non-trivial Green's functions, $G^{\mu}_{V1}(x,y,z) = \langle 0|T\{j^{\mu}_{V1}(x)\psi_1(y)\bar{\psi}_1(z)\}|0\rangle$ and $G^{\mu}_{V2}(x,y,z) = \langle 0|T\{j^{\mu}_{V2}(x)\psi_2(y)\bar{\psi}_2(z)\}|0\rangle$. The corresponding proper vertex functions $\Gamma^{\mu}_{V1,2}$ satisfy the usual Ward identities,

$$q_\mu \Gamma^{\mu}_{V1,2}(p+q,p) = S^{-1}_{1,2}(p+q) - S^{-1}_{1,2}(p),$$

$S_{1,2}$ being the full fermion propagators.

In contrast to the vector currents, the axial current j^{μ}_{A} contains a contribution from the scalar ϕ. As will become clear later, it is convenient to construct a formal scalar doublet,

$$\Phi = \begin{pmatrix} \phi \\ \phi^\dagger \end{pmatrix},$$

and use it instead of the original scalar field ϕ. We now introduce three Green's functions, $G^{\mu}_{A\psi_1}(x,y,z) = \langle 0|T\{j^{\mu}_{A}(x)\psi_1(y)\bar{\psi}_1(z)\}|0\rangle$, $G^{\mu}_{A\psi_2}(x,y,z) = \langle 0|T\{j^{\mu}_{A}(x)\psi_2(y)\bar{\psi}_2(z)\}|0\rangle$, and $G^{\mu}_{A\phi}(x,y,z) = \langle 0|T\{j^{\mu}_{A}(x)\Phi(y)\Phi^\dagger(z)\}|0\rangle$. The corresponding Ward identities read

$$\begin{aligned} q_\mu \Gamma^{\mu}_{A\psi_1}(p+q,p) &= S_1^{-1}(p+q)\gamma_5 + \gamma_5 S_1^{-1}(p), \\ q_\mu \Gamma^{\mu}_{A\psi_2}(p+q,p) &= -S_2^{-1}(p+q)\gamma_5 - \gamma_5 S_2^{-1}(p), \\ q_\mu \Gamma^{\mu}_{A\phi}(p+q,p) &= -2D^{-1}(p+q)\Xi + 2\Xi D^{-1}(p). \end{aligned} \tag{4.3}$$

Here $\mathrm{i}D(x-y) = \langle 0|T\{\Phi(x)\Phi^\dagger(y)\}|0\rangle$ is the matrix propagator of the scalar doublet and Ξ is the diagonal matrix in the scalar doublet space, $\Xi = \mathrm{diag}(1,-1)$.

Ward identities for the Higgs mechanism

The Ward identities (4.3) must hold whether the symmetry is spontaneously broken or not. Also, they do not depend on the particular dynamical way the symmetry is broken. As a warmup, we shall therefore show how they fit the tree-level analysis of the Higgs mechanism discussed in Section 2.3.1 i.e., we assume for a moment that $M^2 < 0$ in Eq. (4.1).

Upon the expansion of the scalar field, $\phi = (v + H + \mathrm{i}\pi)/\sqrt{2}$, the Yukawa interaction becomes

$$\mathcal{L}_{\text{Yukawa}} = \sum_{j=1,2} \left(m_j \bar{\psi}_j \psi_j + \frac{m_j}{v}\bar{\psi}_j\psi_j H \right) + \frac{\mathrm{i}}{v}\left(m_1\bar{\psi}_1\gamma_5\psi_1 - m_2\bar{\psi}_2\gamma_5\psi_2 \right)\pi, \tag{4.4}$$

where $m_{1,2} = v y_{1,2}/\sqrt{2}$ are the generated fermion masses.

We shall exemplify the saturation of the axial Ward identity on the case of a fermion, say ψ_1. The right hand side of Eq. (4.3) then becomes

$$S_1^{-1}(p+q)\gamma_5 + \gamma_5 S_1^{-1}(p) = (\not{p} + \not{q} + m_1)\gamma_5 + \gamma_5(\not{p} + m_1) = \not{q}\gamma_5 + 2m_1\gamma_5. \tag{4.5}$$

The proper three-point vertex function consists, at the tree level, of two contributions – the bare coupling of the fermion to the axial current and a pion pole term, see Fig. 4.1.

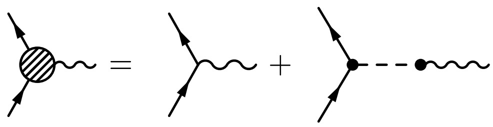


Figure 4.1: The axial three-point vertex function in the case of the Higgs mechanism. The second Feynman graph on the right hand side contains a pole due to the Goldstone boson.

The direct coupling of the fermion to the axial current accounts for the $\not{q}\gamma_5$ term on the right hand side of Eq. (4.5). With the help of Eqs. (2.9) and (4.4) we see that the pion pole contribution becomes

$$q_\mu \left[-\frac{m_1}{v}\gamma_5 \times \frac{\mathrm{i}}{q^2} \times (2\mathrm{i}vq^\mu) \right] = 2m_1\gamma_5.$$

Note that the last factor is $2\mathrm{i}vq^\mu$ instead of $-\mathrm{i}vq^\mu$, as Eq. (2.9) would suggest, because of a different normalization of the scalar contribution to the axial current (4.2).

We have thus verified that the axial Ward identity (4.3) is indeed satisfied. Moreover, it is now clear that, in order to compensate for the symmetry-breaking (mass) term in Eq. (4.5), *there must be a massless pole in the broken current correlation function due to the propagation of the Goldstone boson.*

This observation will be crucial for the analysis of our model of dynamical symmetry breaking. While in the Higgs mechanism (where the Goldstone boson corresponds to an elementary field in the Lagrangian) the Ward identities serve merely as a check of consistency, here they will be used to predict the properties of the composite Goldstone boson.

4.1.2 Spectrum of scalars

From now on we shall assume that $M^2 > 0$ in the Lagrangian (4.1), i.e., in the absence of interactions the scalar field ϕ annihilates a complex particle of mass M. Our goal is to show that once a sufficiently strong Yukawa interaction is introduced, the axial $\mathrm{U}(1)_\mathrm{A}$ symmetry is spontaneously broken and fermion masses are generated.

Our strategy will be as follows: We shall *assume* that fermion masses or more precisely, chirality-changing self-energies, are somehow generated. Plugging them into the Schwinger–Dyson equations for the Green's functions of the theory we later show that a nontrivial solution actually does exist. This is a standard philosophy in dealing with dynamical symmetry breaking – one simply has to make a proper ansatz that incorporates one's expectations as to the form of the solution.

The fermions, however, interact with the scalar ϕ, so it is natural to ask, and investigate prior to any calculation, what is the impact of chiral symmetry breaking on the spectrum in the scalar sector.

The answer lies in the fact that the scalar field carries nonzero axial charge. Once the axial $\mathrm{U}(1)_\mathrm{A}$ is spontaneously broken, the scalar field carries no conserved quantum number and nothing prevents the appearance of the 'anomalous'[1] Green's function $\langle 0|T\{\phi\phi\}|0\rangle$.

[1]The word 'anomalous' has nothing to do with the axial anomaly. This terminology is taken over from

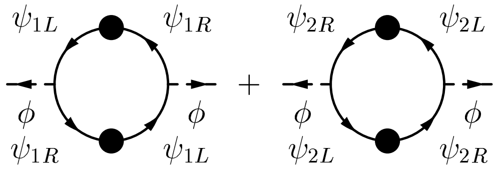


Figure 4.2: One-loop contributions to the anomalous scalar proper self-energy. The solid blobs denote the full chirality-changing fermion self-energies.

In the language of the Feynman graphs, this corresponds to diagrams with two external scalar legs, both pointing outwards. Such graphs may only arise in the presence of nonzero fermion masses, see Fig. 4.2. We can thus see that the breaking of the chiral symmetry in the fermion sector (i.e., fermion masses) is tightly connected to the breaking in the scalar sector.

The effect of the anomalous Green's function on the scalar spectrum may be roughly understood by assuming that it is momentum-independent, and neglecting all other radiative corrections to the scalar propagator. The scalar spectrum is then determined by the bilinear Lagrangian

$$\mathcal{L}^{(0)}_{\text{scalar}} = \partial_\mu \phi^\dagger \partial^\mu \phi - M^2 \phi^\dagger \phi - \frac{1}{2}\mu^{2*}\phi\phi - \frac{1}{2}\mu^2 \phi^\dagger \phi^\dagger ,$$

the parameter μ^2 corresponding to the anomalous correlation function in question. It turns out that such a Lagrangian describes two *real* scalar particles with masses $M^2_{1,2} = M^2 \pm |\mu|^2$. The anomalous correlation function thus amounts to the splitting of the spectrum in the scalar sector.

4.1.3 Ward identities for dynamically broken symmetry

As noted above, the vertex function of a broken current possesses a massless pole due to the corresponding Goldstone boson. With our assumption that the axial symmetry is spontaneously broken, there must be such a Goldstone boson coupled to the axial current. Unlike the Higgs mechanism, however, now it is a composite particle i.e., a bound state of the elementary fermions and scalars. It again gives rise to a pole in the vertex function, but now due to quantum loops, see Fig. 4.3.

As the Goldstone boson is composite, its interaction vertices cannot be inferred directly from the Lagrangian. They can, however, be determined with the help of the Ward identities (4.3), in terms of the fermion and scalar propagators [40, 41]. Denoting the proper vertex functions as P_{ψ_1}, P_{ψ_2}, P_ϕ (quite analogously to the Γ^μ's, just the axial current is replaced with the Goldstone boson), the resulting formulas read

$$\begin{aligned} P_{\psi_1}(p+q,p) &= \frac{1}{N}\left[S_1^{-1}(p+q)\gamma_5 + \gamma_5 S_1^{-1}(p) - \not{q}\gamma_5\right] , \\ P_{\psi_2}(p+q,p) &= -\frac{1}{N}\left[S_2^{-1}(p+q)\gamma_5 + \gamma_5 S_2^{-1}(p) - \not{q}\gamma_5\right] , \\ P_{\phi}(p+q,p) &= -\frac{2}{N}\left[D^{-1}(p+q)\Xi - \Xi D^{-1}(p) - q\cdot(2p+q)\Xi\right] , \end{aligned} \tag{4.6}$$

condensed-matter physics, where it is used e.g. for superconductors. There, the particle-number-violating Green's function appears because of the Cooper pairing [14].

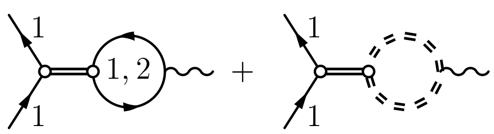
$\Gamma^{\mu}_{A\psi_1,\mathrm{pole}} =$ 

Figure 4.3: The pole part of the proper vertex function of the axial current and the fermion pair $\psi_1\bar{\psi}_1$. The double solid line represents the Goldstone boson and the empty circles its effective vertices with the fermion and the scalar, respectively. The double dashed line stands for the propagator of the doublet scalar Φ. Both ψ_1 and ψ_2 can circulate in the closed fermion loop. The graphs for the other two vertex functions of the axial current are analogous.

the normalization factor N will be specified later. Note that these effective vertices are unambiguous only up to order $\mathcal{O}(q)$ since only the pole parts of the axial current vertex functions were kept in the Ward identities [42].

To determine the Goldstone interactions more concretely, the knowledge of the full fermion and scalar propagators is necessary. It is, however, obvious that the most important are their symmetry-breaking parts. In order to be able to write down analytic expressions for the vertices (4.6), we make the following simplifications.

We neglect the symmetry-preserving renormalization of the fermion and scalar propagators and assume that the sheer effect of quantum corrections is to generate the symmetry breaking so that the propagators acquire the form

$$S^{-1}_{1,2}(p) = \not{p} - \Sigma_{1,2}(p), \quad D^{-1}(p) = \begin{pmatrix} p^2 - M^2 & -\Pi(p) \\ -\Pi^*(p) & p^2 - M^2 \end{pmatrix}. \tag{4.7}$$

Here $\Sigma_{1,2}(p)$ are the chirality-changing proper self-energies of the fermions while $\Pi(p)$ is the anomalous proper self-energy of the scalar field ϕ.

The effective vertices (4.6) now become

$$\begin{aligned} P_{\psi_1}(p+q,p) &= -\frac{1}{N}\left[\Sigma_1(p+q) + \Sigma_1(p)\right]\gamma_5, \quad P_{\psi_2}(p+q,p) = \frac{1}{N}\left[\Sigma_2(p+q) + \Sigma_2(p)\right]\gamma_5, \\ P_{\phi}(p+q,p) &= -\frac{2}{N}\begin{pmatrix} 0 & \Pi(p+q) + \Pi(p) \\ -\Pi^*(p+q) - \Pi^*(p) & 0 \end{pmatrix}. \end{aligned} \tag{4.8}$$

The normalization factor N is given by $N = \sqrt{J_{\psi_1}(0) + J_{\psi_2}(0) + J_{\phi}(0)}$, the loop integrals $J_{\psi_1}(q^2)$, $J_{\psi_2}(q^2)$ and $J_{\phi}(q^2)$ being defined as

$$\begin{aligned} -\mathrm{i}q^{\mu}J_{\psi_{1,2}}(q^2) &= 8\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k-q)^{\mu}\Sigma_{1,2,k}}{k^2 - \Sigma^2_{1,2,k}}\frac{\Sigma_{1,2,k} + \Sigma_{1,2,k-q}}{(k-q)^2 - \Sigma^2_{1,2,k-q}}, \\ -\mathrm{i}q^{\mu}J_{\phi}(q^2) &= 8\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(2k-q)^{\mu}(k^2 - M^2)}{(k^2 - M^2)^2 - |\Pi_k|^2}\frac{\mathrm{Re}\left[\Pi^*_{k-q}\big(\Pi_k + \Pi_{k-q}\big)\right]}{\left[(k-q)^2 - M^2\right]^2 - |\Pi_{k-q}|^2}. \end{aligned}$$

4.1.4 Spectrum of fermions

So far we have simply assumed that axial symmetry is spontaneously broken, giving rise to nonzero proper self-energies $\Sigma_{1,2}(p)$ and $\Pi(p)$. Now we have to close the chain of arguments by demonstrating that this is indeed the case.

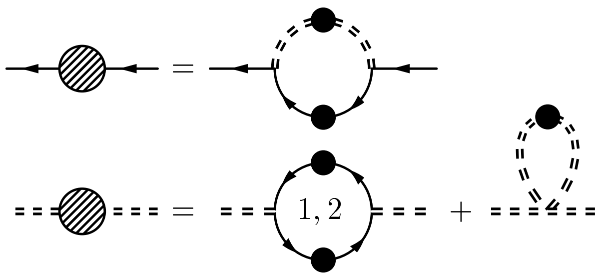


Figure 4.4: The one-loop Schwinger–Dyson equations for the fermion and scalar propagators. The first line applies equally to ψ_1 and ψ_2. The proper self-energies are denoted by the dashed blobs, while the full propagators are represented by the solid blobs.

To that end, we consider the Schwinger–Dyson equations for the Green's functions of our model. Having in mind that we are looking for spontaneous symmetry breaking in the propagators, we neglect for simplicity all vertex corrections. The propagators are then found by a self-consistent solution of the one-loop equations that are depicted in Fig. 4.4.

With the ansatz (4.7), we arrive at the set of three coupled integral equations,

$$\begin{aligned}
\Sigma_{1,p} &= \mathrm{i}y_1^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\Sigma_{1,k}}{k^2-\Sigma_{1,k}^2} \frac{\Pi_{k-p}}{\left[(k-p)^2-M^2\right]^2-|\Pi_{k-p}|^2}, \\
\Sigma_{2,p} &= \mathrm{i}y_2^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\Sigma_{2,k}}{k^2-\Sigma_{2,k}^2} \frac{\Pi_{k-p}^*}{\left[(k-p)^2-M^2\right]^2-|\Pi_{k-p}|^2}, \\
\Pi_p &= -\sum_{j=1,2} 2\mathrm{i}y_j^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\Sigma_{j,k}}{k^2-\Sigma_{j,k}^2} \frac{\Sigma_{j,k-p}}{(k-p)^2-\Sigma_{j,k-p}^2} + \mathrm{i}\lambda \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\Pi_k}{(k^2-M^2)^2-|\Pi_k|^2}.
\end{aligned} \tag{4.9}$$

For sake of numerical solution of the Schwinger–Dyson equations (4.9), further simplifying assumptions are made. First, since the symmetry-preserving quantum corrections have been neglected, we also abandon the λ interaction in the last of Eqs. (4.9). The reason is that it merely provides a counterterm in the one-loop effective Lagrangian, whereas the spontaneous breaking itself is induced by the Yukawa interaction.

Second, the Yukawa couplings y_1, y_2 are set equal so that the set of equations (4.9) reduces to two equations for $\Sigma = \Sigma_1 = \Sigma_2$ and Π. This conclusion is justified as long as the scalar self-energy Π is real, since the discrete symmetry of the Lagrangian, $\psi_1 \leftrightarrow \psi_2$ and $\phi \leftrightarrow \phi^\dagger$, is then not spontaneously broken.

The numerical results of the calculations in Euclidean space are displayed in Fig. 4.5. It is notable that a nontrivial solution seems to exists only when the Yukawa interaction is strong enough. A preliminary analysis shows that the critical value for spontaneous breaking of the chiral symmetry is $y_{\text{crit}} \approx 30$.[2]

[2]Very recently, we have discovered an error in the original numerical code. Our new computations, to be published, suggest that the critical value of the Yukawa coupling might be significantly larger, about 80. The qualitative behavior of the self-energies, however, does not change.

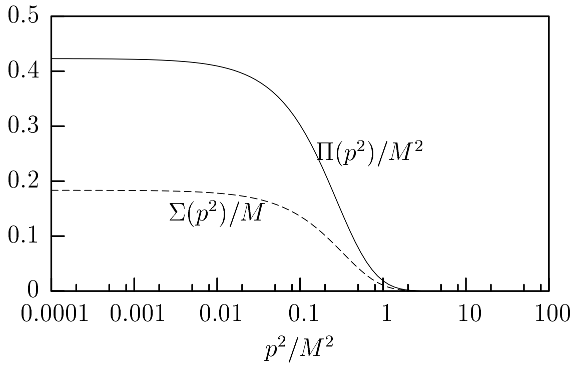


Figure 4.5: Numerical results for the fermion and scalar proper self-energies Σ and Π, respectively. The author is grateful to Petr Beneš for doing the numerical computation and providing this figure.

4.2 Extension to $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ gauge symmetry

Having demonstrated how fermion masses may be generated dynamically by a strong Yukawa interaction, we now turn our attention to the case of utmost physical importance, the spontaneous breaking of the electroweak $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ gauge symmetry. This case has not been investigated in full detail yet, including the numerical solution of the Schwinger–Dyson equations. Therefore, we just sketch the main idea as done in our paper [V]. Further work on this model is in progress.

The basic strategy is the same as in Section 4.1. The only difference is that now all formulas are more complicated because of the isospin and flavor structure of the standard model. The particle content is identical to that of the standard model with two exceptions. First, in the fermion sector, we introduce N_f neutrino right-handed isospin singlets ν_R with zero weak hypercharge in order to account for the nonzero neutrino masses.

Second, in the scalar sector, we introduce two complex doublets, $S = (S^{(+)}, S^{(0)})$ and $N = (N^{(0)}, N^{(-)})$, with weak hypercharges $Y_S = +1$ and $Y_N = -1$ and different ordinary masses M_S and M_N, respectively. It will become clear later that they serve to generate the masses of the lower and upper components of the fermion isospin doublets.

The Lagrangian of our model differs from that of the standard model by the presence of two scalar quartic self-couplings, λ_S and λ_N, and by the Yukawa interaction

$$\mathcal{L}_{\text{Yukawa}} = \bar{\ell}_L y_e e_R S + \bar{\ell}_L y_\nu \nu_R N + \bar{q}_L y_d d_R S + \bar{q}_L y_u u_R N + \text{H.c.}, \tag{4.10}$$

where the Yukawa couplings y_e, y_ν, y_d, y_u are to be treated as $N_f \times N_f$ complex matrices in the flavor space.

4.2.1 Particle spectrum

As in the simple Abelian model (4.1), the assumed fermion mass terms give rise to ‘anomalous’ self-energies in the scalar sector, mixing different modes. At one-loop, the neutral components $S^{(0)}$ and $N^{(0)}$ develop nonzero two-point correlation functions breaking the

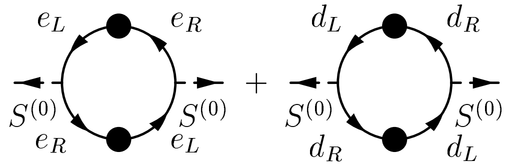


Figure 4.6: One-loop contributions to the anomalous proper self-energy of the neutral scalar $S^{(0)}$. The solid blobs denote the chirality-changing parts of the full fermion propagators. The same graphs apply to $N^{(0)}$ upon replacing e, d with ν, u.

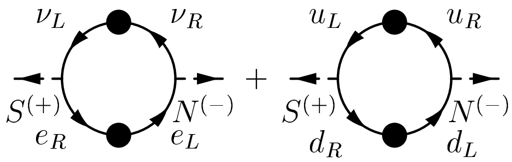


Figure 4.7: One-loop mixing of the charged scalars induced by the dynamically generated fermion masses.

particle number, see Fig. 4.6. As a result, there are four real particles, two with masses split around M_S, and the other two around M_N. It should also be noted that at higher orders, all these four modes mix with one another since there is no conserved quantum number that would prevent them from doing so. The charged components of the scalar doublets also mix, as shown in Fig. 4.7. Due to the conservation of electric charge, there are now two charged scalars, being the orthogonal mixtures of $S^{(+)}$ and $N^{(-)\dagger}$.

Leaving aside the details of the calculations that may be found in the paper [V], we just note that the fermion and scalar self-energies are determined as a solution to the truncated Schwinger–Dyson equations, very much analogous to Eqs. (4.9).

The essential difference is that now the $\mathrm{SU(2)_L \times U(1)_Y}$ chiral symmetry is gauged i.e., the chiral currents are coupled to dynamical vector gauge fields. As a consequence, the three Goldstone bosons of the coset $[\mathrm{SU(2)_L \times U(1)_Y}]/\mathrm{U(1)_Q}$ become the longitudinal components of the three massive vector bosons $W^{\pm}, Z$. The Ward identities enable us to calculate the couplings of the Goldstone modes to the gauge bosons. Due to the propagation of the intermediate Goldstone boson, the self-energy of the gauge field acquires a massless pole. Upon neglecting the finite contributions to the polarization tensor, the gauge boson mass squared is equal to the residue at this pole [42]. The new feature of the proposed model is that the couplings of the Goldstones to the gauge bosons, and hence also the gauge boson masses, are expressed through one-loop graphs containing the symmetry-breaking self-energies of the fermions and scalars. The gauge boson masses are therefore tied to the masses of the other particles by certain sum rules [40].[3]

Finally, let us note that we have so far not dealt with the Majorana masses of the neutrinos. It turns out that once a hard Majorana mass term is introduced for the right-handed neutrinos, the left-handed neutrino Majorana masses are generated as a one-loop effect. Together, they also produce a new contribution to the anomalous self-energy of $N^{(0)}$. In conclusion, it is perhaps more appropriate to treat all the masses, both Dirac and

[3] In fact, in the paper [V] we omitted the scalar contribution to the gauge boson masses. Now that we have gained some experience by the study of the Abelian model of Section 4.1, the application of the idea to the electroweak symmetry breaking is being revised.

Majorana, on the same footing that is, self-consistently. The spectrum then contains $2N_f$ massive Majorana neutrinos, presumably with a seesaw-like hierarchy of masses.

4.2.2 Phenomenological constraints

Several constraints apart from reproducing the fermion and gauge boson mass spectrum must be met before our model may be accepted as an alternative to the standard model of electroweak interactions. Since we have not yet reached the stage of solving the Schwinger–Dyson equations numerically, we shall discuss these constraints only qualitatively.

First, since the Goldstone bosons of the spontaneously broken symmetry are bound states of the elementary fermions and scalars, all the elementary scalars remain in the spectrum of physical states, unlike in the standard model. Consequently, the neutral ones mediate flavor-changing processes, thus contributing to the flavor-changing neutral currents. Since these are highly suppressed in the standard model, the scalar masses M_S, M_N must be large enough in order to avoid experimental bounds.

Second, it is well known that pair production of longitudinally polarized massive vector bosons violates tree unitarity at high energies, rendering the theory nonrenormalizable [36]. In order for the growth of the scattering amplitude to be cut off at high energies, it is necessary that there be new particles at the energy scale of order 1 TeV.

Chapter 5

Quantum chromodynamics at nonzero density

The physics of hot and/or dense matter is described by the phase diagram of QCD. While the region of low net baryon density and high temperature is being explored experimentally in heavy ion collisions, the cold and very dense nuclear matter seems to exist only in the neutron stars.

It has been known for a long time that at sufficiently high density quarks are no longer confined[1] and may undergo the Cooper pairing very much analogous to that in ordinary superconductors [43]. However, only in the past decade has the phenomenon of color superconductivity attracted considerable attention due to the discovery that it may appear already at densities attainable in the neutron stars [44, 45].

Since then, the subject has been investigated to great detail and several qualitatively different phases have been found. Extensive reviews are given in Refs. [16, 20, 46, 47]. An introduction to the physics of cold dense quark matter may be found in the lecture notes [48, 49].

Despite the amount of energy devoted to the study of the QCD phase diagram, there is still a controversy regarding the structure of the ground state at moderate baryon density. It seems that we are only confident that at very high densities the quark matter resides in the *Color-Flavor-Locked* phase [50]. This is supported by the weak-coupling calculations from first principles, which are applicable due to the asymptotic freedom of QCD.

On the other hand, the knowledge of the moderate-density region of the phase diagram is rather weak. Usually, either the weak-coupling results are directly extrapolated just by running the QCD coupling, or the structure of the interaction is taken over from the high-density regime and used as an input to the phenomenological models such as that of Nambu and Jona-Lasinio.

This chapter consists of two main parts. In the first one, we introduce an alternative mechanism for generating the effective four-quark interaction and show that it leads to an unconventional pairing in the color-sextet channel. This is based on our paper [I].

The second part, based on the recent paper [IV], deals with a different approach to the QCD phase diagram. Inasmuch as we cannot attack the problem of the QCD phase

[1] In fact, the term *quark confinement* loses its sense once the mean distance between quarks is much smaller than the confinement scale. The quarks then do not feel the long-distance strong attraction and provide the appropriate degrees of freedom to describe the highly squeezed matter.

diagram at moderate density directly and current lattice techniques fail in that region as well, it is plausible to study theories similar to QCD which are amenable to both analytical and lattice calculations. We describe a simple case of such a theory – the two-color QCD with two quark flavors – and provide a new setting for its low-energy description in terms of the chiral perturbation theory.

5.1 Single-flavor color superconductor with color-sextet pairing

It is most common to describe the quark matter at moderate baryon density within the Nambu–Jona-Lasinio model [20]. In this approach, the crucial point is the choice of the model interaction. The color and flavor structure of the interaction are usually taken over from the weak-coupling regime – either perturbative (the one-gluon exchange) or nonperturbative (the instanton-mediated interaction). Both these interactions share the common feature that they are attractive in the color-antisymmetric channel and repulsive in the color-symmetric one. It should, however, be stressed that the arguments based on the weakly coupled QCD merely provide an *evidence*. There is *no proof* that the strongly coupled QCD at moderate density inevitably leads to the same behavior. It is therefore worth exploring the alternatives.

In this section we shall investigate the behavior of dense quark matter under the assumption that the quarks pair in the color-symmetric (sextet) channel. We shall for simplicity consider a homogeneous phase of a single-flavor quark matter. The physical reasoning behind this assumption is the following. The color, flavor and spin structures of the Cooper pair are connected by the requirement that the Pauli exclusion principle be satisfied. This means that, as long as the orbital momentum is zero, the total spin of the color-sextet Cooper pair of quarks of a single flavor must be zero. On the contrary, in the color-antitriplet channel the Pauli principle requires total spin one.

The point is that the spin and orbital momentum effects dramatically reduce the energy gap i.e., the binding energy of the Cooper pair. Indeed, while – in the color-antitriplet channel – the gap of the two-flavor spin-zero superconductor at moderate density is roughly estimated as tens MeV, the gap of the one-flavor spin-one superconductor is only tens or a hundred keV [51]. In the latter case, the color-sextet pairing might prevail even if the pairing interaction is quite weak.

It is well known that while at very high density the CFL phase is the stable ground state of the three-flavor quark matter, at moderate density the CFL pairing is disfavored by the strange quark mass and the resulting mismatch of the Fermi momenta. The $2+1$ pairing scheme is more likely. The up and down flavors are bound by the strong attractive interaction in the color-antitriplet channel. The strange quarks then pair with themselves and we suggest here that the pairing be in the *color-sextet spin-zero* channel rather than the color-antitriplet spin-one channel favored by the one-gluon exchange interaction.

We first *assume* the particular form of the pairing and explore its impact on the symmetry of the theory. It is only later that we provide a physical motivation for the attraction in the color-symmetric channel and work out the description within the Nambu–Jona-Lasinio model.

5.1.1 Kinematics of color-sextet condensation

Suppose that the superconducting phase is described by the order parameter Φ which transforms in the **6** representation of the color SU(3) group. It is best represented by a complex symmetric 3×3 matrix upon which the $\mathrm{SU}(3) \times \mathrm{U}(1)$ transformations[2] act as $\Phi \to U\Phi U^{\mathrm{T}}$. The assumption that the ordered phase be homogeneous translates to the requirement that Φ be a spacetime-independent constant. Note that in the Nambu–Jona-Lasinio model Φ_{ij} will correspond to the vacuum expectation value of the bilinear operator $\psi_{\alpha i}(C\gamma_5)_{\alpha\beta}\psi_{\beta j}$, but for now this interpretation is not needed.

The crucial observation is that any complex symmetric matrix Φ may be brought by a suitable $\mathrm{SU}(3) \times \mathrm{U}(1)$ transformation to a special form Δ which is diagonal, real and positive [52]. We shall denote its diagonal entries as $\Delta_1, \Delta_2, \Delta_3$. These cannot be changed by a unitary transformation since they are the eigenvalues of the positive Hermitian matrix $(\Phi^\dagger\Phi)^{1/2}$, and thus represent *three independent order parameters* of the phase.

The presence of three order parameters makes the phase structure of the color-sextet superconductor quite rich. Depending on the relative values of the order parameters, several symmetry-breaking patterns may be distinguished:

1. *All Δ's are different and nonzero.* This is the most general as well as intriguing possibility. The continuous $\mathrm{SU}(3) \times \mathrm{U}(1)$ symmetry is completely broken, only a discrete $(Z_2)^3$ is left.

2. *Two Δ's are equal and nonzero.* In this case, there is a residual O(2) symmetry in the corresponding 2×2 block of Φ.

3. $\Delta_1 = \Delta_2 = \Delta_3 \neq 0$. Quite similar to the previous case, but now the enhanced symmetry of the ground state is O(3).

4. *Some of the Δ's are zero.* According to the number of vanishing order parameters, there is a residual U(1) or U(2) invariance, simply meaning that the corresponding colors do not participate in the pairing.

It will turn out in the following that the possibility of most interest is the O(3)-symmetric phase. Since this results in the same number of broken color generators as the breaking $\mathrm{SU}(3) \to \mathrm{SU}(2)$ by the standard color antitriplet, it is worthwhile to comment on the difference between these two symmetry-breaking patterns.

The structure of the spectrum is always determined by the unbroken subgroup. Now the breaking $\mathrm{SU}(3) \to \mathrm{SO}(3)$ is isotropic so that all five broken generators fall into a single (5-plet) representation of SO(3). On the other hand, in the $\mathrm{SU}(3) \to \mathrm{SU}(2)$ case four of the broken generators form a complex SU(2) doublet while the remaining one is a singlet.

5.1.2 Ginzburg–Landau description

To determine which of the possible symmetry-breaking patterns are actually realized, one has to employ a particular model to calculate the order parameter Φ. Ignoring

[2] Recall from Section 3.2.2 that the U(1) here represents the baryon number.

for the moment the fluctuations of the order parameter(s), we have to write down the most general $\mathrm{SU}(3) \times \mathrm{U}(1)$ invariant potential, whose minimum determines the ground state. Such a potential can always be written in terms of a certain set of algebraically independent invariants. In our case there are three of them, namely $\mathrm{Tr}\,\Phi^\dagger\Phi$, $\det\Phi^\dagger\Phi$ and $\mathrm{Tr}(\Phi^\dagger\Phi)^2$.

Restricting to quartic polynomials of the Ginzburg–Landau type, the most general potential reads

$$V(\Phi) = -a\,\mathrm{Tr}\,\Phi^\dagger\Phi + b\,\mathrm{Tr}(\Phi^\dagger\Phi)^2 + c(\mathrm{Tr}\,\Phi^\dagger\Phi)^2.$$

Such a potential was already investigated in Section 3.2.2. It was shown that the nature of the global minimum depends on the sign of the parameter b. When $b > 0$, the order parameter Δ is proportional to the unit matrix so that the ground state has the $\mathrm{SO}(3)$ symmetry. When $b < 0$, the minimizing configuration is such that Δ has a single nonzero diagonal entry, corresponding to the symmetry breaking pattern $\mathrm{SU}(3) \times \mathrm{U}(1) \to \mathrm{SU}(2) \times \mathrm{U}(1)$.

We stress the fact that the parameters a, b, c are unknown at this stage so that we cannot decide which of the ordered phases is actually realized. It is, however, possible to derive the Ginzburg–Landau functional from the underlying microscopic model, either QCD or Nambu–Jona-Lasinio [53].

To account for the fluctuations of the order parameter Φ, the Ginzburg–Landau functional has to be enriched with derivative terms. The lowest-order Lagrangian reads

$$\mathcal{L} = \alpha_e\,\mathrm{Tr}\,\partial_0\Phi^\dagger\partial^0\Phi + \alpha_m\,\mathrm{Tr}\,\partial_i\Phi^\dagger\partial^i\Phi - V(\Phi). \tag{5.1}$$

The coefficients α_e and α_m are in general different since Lorentz invariance is broken by medium effects. Note that the "kinetic term" of Φ is not canonically normalized – this is because Φ represents a composite object, the Cooper pair of quarks [54, 55].

Treating the dense quark matter at moderate baryon density as a BCS-type superconductor, one may next switch on the colored gauge fields perturbatively. Within the effective Lagrangian (5.1), this amounts to replacing the ordinary derivatives with the covariant ones,

$$\partial_\mu\Phi \to D_\mu\Phi = \partial_\mu\Phi - \mathrm{i}gA^a_\mu\left(\tfrac{1}{2}\lambda_a\Phi + \Phi\tfrac{1}{2}\lambda^{\mathrm{T}}_a\right),$$

and adding the Yang–Mills kinetic term for the gluons. As a result of the usual Higgs mechanism, both electric and magnetic gluons acquire nonzero masses – the Debye and the Meissner ones, respectively. At zero temperature, the coefficients are roughly $\alpha_{e,m} \sim \mu^2/\Delta^2$ so that both Debye and Meissner masses are found to be of order $g\mu$ (for detailed results and their discussion see Ref. [I]).

However, as pointed out by Rischke [55], the gauged lowest-order Lagrangian (5.1) does not reproduce correctly the mass ratios of the gluons of different adjoint colors. The reason is the restriction to operators of dimension four or less we employed to construct the Lagrangian (5.1). For a more proper treatment, higher-order operators like $\left|\mathrm{Tr}(\Phi^\dagger D_i\Phi)\right|^2$ have to be included, which also contribute to the gluon masses.

5.1.3 Nambu–Jona-Lasinio model

We shall now develop the description using the elementary quark fields. Here we come to the point of the proper choice of the four-fermion interaction. As already mentioned above,

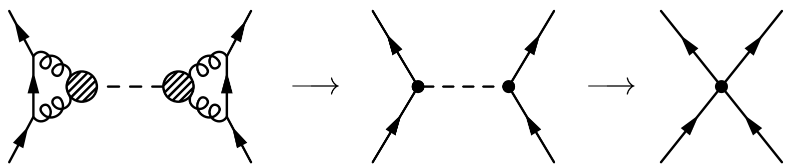


Figure 5.1: Effective four-quark interaction induced by the exchange of the scalar color-octet glueball.

we do not take up any of the interactions commonly used in literature, but rather follow a different approach. Our motivation goes back two decades to the paper by Hansson et al. [56]. These authors investigated the possibility of the existence of the bound states of two gluons and classified the strength of the QCD-induced force by the total spin and color content of the gluon pair.

They discovered that apart from the colorless glueball, the most strongly bound state is that of total spin zero which transforms as a color octet. Such a state, of course, cannot exist as an excitation of the QCD vacuum. It might, however, be a well defined degree of freedom in the dense deconfined phase. Now if it really exists, it certainly interacts with the quarks and its exchange leads to the effective fermionic Lagrangian (see Fig. 5.1),

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\not{\partial} - m + \mu\gamma_0)\psi + G(\bar{\psi}\boldsymbol{\lambda}\psi)^2, \tag{5.2}$$

with $G > 0$. (The color and spin indices are suppressed.) The proposed interaction is attractive in the color-sextet channel and provides the basis for the following analysis.

We use the method outlined in Section 2.3.2. Anticipating the color-sextet condensate, we split the full Lagrangian (5.2) in such a way that the free part, which determines the propagator, reads

$$\mathcal{L}_{\text{free}} = \bar{\psi}(\mathrm{i}\not{\partial} - m + \mu\gamma_0)\psi + \frac{1}{2}\bar{\psi}\Delta(C\gamma_5)\bar{\psi}^{\mathrm{T}} - \frac{1}{2}\psi^{\mathrm{T}}\Delta^{\dagger}(C\gamma_5)\psi.$$

Here Δ stands for the diagonal matrix of the order parameters.

This Lagrangian is conveniently diagonalized with the help of the Nambu–Gorkov notation,

$$\Psi = \begin{pmatrix} \psi \\ \bar{\psi}^{\mathrm{T}} \end{pmatrix}.$$

We find, for each color i, two types of fermionic quasiparticles – a quark-like and an antiquark-like – whose dispersion relations are

$$E_{i\pm}^2(\mathbf{k}) = \left(\sqrt{\mathbf{k}^2 + m^2} \pm \mu\right)^2 + |\Delta_i|^2.$$

In the mean-field approximation the gaps Δ_i are determined by the requirement of the cancelation of the one-loop corrections. We obtain three separate but identical gap equations. Integrating over the frequency and regulating the three-dimensional integral with a cutoff Λ they read, at finite temperature T,

$$1 = \frac{2}{3}G\int^{\Lambda}\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left(\frac{1}{E_+(\mathbf{k})}\tanh\frac{E_+(\mathbf{k})}{2T} + \frac{1}{E_-(\mathbf{k})}\tanh\frac{E_-(\mathbf{k})}{2T}\right).$$

Several remarks to this result are in order. First, in its derivation we have not been entirely self-consistent. We compared the terms of the same structure, $\bar{\psi}\Delta(C\gamma_5)\bar{\psi}^{\mathrm{T}}$, in the free Lagrangian and the one-loop correction. The presence of the chemical potential induces, however, a similar term $\bar{\psi}\Delta(C\gamma_5)\gamma_0\bar{\psi}^{\mathrm{T}}$ at one loop, and this has been neglected. Since the full Lorentz invariance is broken by the chemical potential, it is natural that such a term appears. To be fully self-consistent, we would have to include such operators into our Lagrangian from the very beginning and solve a coupled set of gap equations for their coefficients. Such an analysis was done in Ref. [57].

Second, note that we derived three identical gap equations for the order parameters $\Delta_1, \Delta_2, \Delta_3$. Since the integrands in the gap equation are monotonic in Δ, there is obviously only one nonzero solution and thus all the gaps acquire the same value. This means that the four-quark interaction we chose prefers the SO(3) symmetric phase discussed above. This might, however, be just an artifact of the mean-field approximation. Indeed, the separation of the three colors occurs only at the one-loop level. The physical picture is such that the quarks of any individual color generate a mean field which is in turn felt only by the quarks of the same color. It is then not surprising that all the three gaps have equal size. At two or more loops the colors start to mix and this might lead to lifting the degeneracy and splitting of the gaps. As shown above, if this happens the color SU(3) invariance is completely broken. A definite answer may be given only after a more sophisticated approximation is employed.

5.2 Two-color QCD: Chiral perturbation theory

We have already mentioned that realistic QCD calculations from first principles are not available at moderate baryon density because of the large coupling constant. The trouble is that neither are the lattice simulations. The reason is that the Euclidean Dirac operator, $\mathcal{D} = \gamma_\nu(\partial_\nu - A_\nu) + m - \mu\gamma_0$, is complex at nonzero baryon chemical potential μ.

This gave rise to interest in QCD-*like* theories that do not have the sign problem [58]. There are two distinguished classes of such theories – QCD with quarks in the adjoint representation of SU(3) and two-color QCD [59]. In the following, we shall consider the latter case.

It turns out that the determinant of the Euclidean Dirac operator of two-color QCD, defining the path-integral measure for the gauge bosons, is in general just real. In order for it to be positive, there must be an even number of quarks with the same quantum numbers [60]. Therefore, the case of an even number of flavors is usually studied.

5.2.1 Symmetry

The key feature of the two-color QCD is the pseudoreality of the gauge group generators, the Pauli matrices, $T_k^* = -T_2T_kT_2$. Assuming the quarks in the fundamental (doublet) representation of the gauge SU(2), the right-handed component of the Dirac spinor, ψ_R (color and flavor indices are suppressed), may be traded for the left-handed spinor $\tilde{\psi}_R = \sigma_2T_2\psi_R^*$, the Pauli matrices σ_k acting in the Dirac space. The conjugate left-handed spinor has the same transformation properties as ψ_L and is used to replace the conventional Dirac spinor with

$$\Psi = \begin{pmatrix} \psi_L \\ \tilde{\psi}_R \end{pmatrix}.$$

The Euclidean Lagrangian of massive two-color QCD at finite chemical potential thus becomes

$$\mathcal{L} = \mathrm{i}\Psi^\dagger \sigma_\nu (D_\nu - \Omega_\nu)\Psi - m\left[\tfrac{1}{2}\Psi^{\mathrm{T}}\sigma_2 T_2 M \Psi + \mathrm{H.c.}\right]. \tag{5.3}$$

Now D_ν is the gauge-covariant derivative and Ω_ν is the constant external field that accounts for the effects of the chemical potential. Finally, M is the block matrix in the Ψ space,

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Using the new spinor Ψ it is easily seen that instead of the naively expected chiral $\mathrm{SU(N_f)_L} \times \mathrm{SU(N_f)_R}$ symmetry, the Lagrangian (5.3) is, in the chiral limit $m = 0$ and at $\Omega_\nu = 0$, invariant under an extended group $\mathrm{SU(2N_f)}$. At zero chemical potential, this symmetry is broken by the standard chiral condensate down to its $\mathrm{Sp(2N_f)}$ subgroup [59].

In the Ψ notation, the standard chiral transformations correspond to independent unitary rotations of the upper and lower components ψ_L and $\tilde{\psi}_R$, respectively. The new transformations in the extended group $\mathrm{SU(2N_f)}$ mix these and thus break the baryon number. In terms of the order parameters, these transformations rotate the chiral condensate $\langle\bar{\psi}\psi\rangle$ into the diquark condensate $\langle\psi\psi\rangle$.

It is therefore not surprising that the chemical potential term breaks the $\mathrm{SU(2N_f)}$ down to the conventional chiral subgroup $\mathrm{SU(N_f)_L} \times \mathrm{SU(N_f)_R} \times \mathrm{U(1)_B}$. The reason is that it lifts the degeneracy between the particles and antiparticles, and the transformations breaking the baryon number $\mathrm{U(1)_B}$ therefore no longer leave the Lagrangian invariant.

Unlike the case of the real, three-color QCD, the two-color QCD has the remarkable property that two quarks may form a color-singlet state. This is again connected to the pseudoreality of the fundamental representation of the gauge group. It follows that the ordered phase with quarks Cooper-paired should not be called color-superconducting, but rather just superfluid.

On the technical level, this fact has the far-reaching consequence that the superfluidity of two-color QCD[3] may be investigated within the framework of the chiral perturbation theory. The effective Lagrangian is constructed on the coset space $\mathrm{SU(2N_f)/Sp(2N_f)}$. This effective theory has been investigated to great detail, including both the loop [61] and finite temperature [62] effects.

The Goldstone bosons are, as usual, generated from the ground state by spacetime-dependent symmetry transformations. In this case, they are parametrized by an antisymmetric unimodular unitary matrix Σ. The leading-order low-energy effective Lagrangian reads

$$\mathcal{L}_{\mathrm{eff}} = \frac{F^2}{2}\mathrm{Tr}(\nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger) - G\,\mathrm{Re}\,\mathrm{Tr}(J\Sigma), \tag{5.4}$$

[3] One should carefully distinguish the Bose–Einstein condensation of Goldstone bosons with the quantum numbers of the diquark, from the Cooper pairing of quarks near the Fermi sea. Both effects result in the baryon number superfluidity, but while the former occurs in the confined phase, the latter arises from the pairing interaction between deconfined quarks. The nice feature of two-color QCD is that the diquark condensate may be used as an order parameter in both the confined and the deconfined regime.

where the ∇'s denote the covariant derivatives,

$$\nabla_\nu\Sigma = \partial_\nu\Sigma - (\Omega_\nu\Sigma + \Sigma\Omega_\nu^{\mathrm{T}}), \quad \nabla_\nu\Sigma^\dagger = \partial_\nu\Sigma^\dagger + (\Sigma^\dagger\Omega_\nu + \Omega_\nu^{\mathrm{T}}\Sigma^\dagger),$$

and J is a source field for Σ. When the quark mass is included, the Goldstone bosons acquire nonzero mass m_π which is related to the quark mass m by the Gell-Mann–Oakes–Renner relation

$$mG = F^2 m_\pi^2.$$

In the following we shall concentrate on the simplest case $\mathrm{N_f} = 2$. Here one can take advantage of the Lie algebra isomorphisms SU(4) $\simeq$ SO(6) and Sp(4) $\simeq$ SO(5). We shall argue that it is more convenient to describe the low-energy effective theory on the coset space SO(6)/SO(5).

5.2.2 Coset space

The coset SU(4)/Sp(4) is parametrized by the antisymmetric unimodular unitary matrix Σ, while the coset SO(6)/SO(5) corresponds to the unit sphere S^5 i.e., it is described by a unit vector $\mathbf{n}$ in the six-dimensional Euclidean space. The mapping between these two formalisms is provided by the relation

$$\Sigma = n_i\Sigma_i,$$

where Σ_i are a set of six conveniently chosen matrices, satisfying the identity $\Sigma_i^\dagger\Sigma_j + \Sigma_j^\dagger\Sigma_i = 2\delta_{ij}$. One particular realization of the basis matrices is given by

$$\Sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 0 & \mathrm{i}\tau_1 \\ -\mathrm{i}\tau_1 & 0 \end{pmatrix},$$
$$\Sigma_4 = \begin{pmatrix} \mathrm{i}\tau_2 & 0 \\ 0 & -\mathrm{i}\tau_2 \end{pmatrix}, \quad \Sigma_5 = \begin{pmatrix} 0 & \mathrm{i}\tau_2 \\ \mathrm{i}\tau_2 & 0 \end{pmatrix}, \quad \Sigma_6 = \begin{pmatrix} 0 & \mathrm{i}\tau_3 \\ -\mathrm{i}\tau_3 & 0 \end{pmatrix}.$$

This particular choice of the basis is not accidental. The first three matrices have been used in literature to denote the chiral, diquark, and isospin condensate, respectively [59, 60]. The physical nature of the individual matrices is made more transparent by assigning to them quark bilinears,

$$\Sigma \to \tfrac{1}{2}\Psi^T\sigma_2 T_2\Sigma\Psi + \text{H.c.},$$

that provide the interpolating fields for the Goldstone bosons correspondingly.

Concretely, we find that Σ_2 and Σ_4 are real and imaginary parts of an isospin singlet with baryon number +1, the diquark. Further, $\Sigma_3, \Sigma_5, \Sigma_6$ form an isospin triplet with no baryon charge – the pion. Finally, Σ_1 corresponds to the isospin singlet with no baryon charge i.e., the σ field,

$$\Sigma_2 \to -\tfrac{1}{2}\psi^T C\gamma_5 T_2\tau_2\psi + \text{H.c.}, \quad \Sigma_4 \to -\tfrac{1}{2}\mathrm{i}\psi^T C\gamma_5 T_2\tau_2\psi + \text{H.c.},$$
$$\Sigma_3 \to -\mathrm{i}\bar\psi\tau_1\gamma_5\psi, \quad \Sigma_5 \to \mathrm{i}\bar\psi\tau_2\gamma_5\psi, \quad \Sigma_6 \to -\mathrm{i}\bar\psi\tau_3\gamma_5\psi,$$
$$\Sigma_1 \to \bar\psi\psi.$$

5.2.3 Effective Lagrangian

We shall now rewrite the effective Lagrangian in terms of the unit vector $\mathbf{n}$. The baryon number chemical potential μ is incorporated in terms of the external field $\Omega_\nu = \delta_{\nu 0}\mu B$, where the baryon number generator is represented by the block matrix

$$B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Adjusting the source J to reproduce the quark mass effect, the leading-order Lagrangian (5.4) becomes

$$\mathcal{L}_{\text{eff}} = 2F^2(\partial_\nu \mathbf{n})^2 + 4\mathrm{i}F^2\mu(n_2\partial_0 n_4 - n_4\partial_0 n_2) - 2F^2\mu^2(n_2^2 + n_4^2) - 4F^2 m_\pi^2 n_1. \qquad (5.5)$$

To determine the spectrum of the theory for a particular value of the chemical potential, one has to find the ground state by minimizing the static part of the Lagrangian, and then expand the Lagrangian about the minimum to second order in the fields.

Normal phase

For $\mu < m_\pi$ the static Lagrangian is minimized by the conventional chiral condensate i.e., $\mathbf{n} = (1, 0, 0, 0, 0, 0)$. The five independent degrees of freedom may be identified with $n_2, \dots, n_6$, and the resulting dispersion relations are

$$\begin{aligned} E(\mathbf{k}) &= \sqrt{\mathbf{k}^2 + m_\pi^2} && \text{pion triplet } n_3, n_5, n_6, \\ E(\mathbf{k}) &= \sqrt{\mathbf{k}^2 + m_\pi^2} - \mu && \text{diquark } n_2 + \mathrm{i}n_4, \\ E(\mathbf{k}) &= \sqrt{\mathbf{k}^2 + m_\pi^2} + \mu && \text{antidiquark } n_2 - \mathrm{i}n_4. \end{aligned}$$

This result is exactly what we would expect. The pion triplet carries no baryon charge so its dispersion relation is not affected at all by the chemical potential. The dispersions of the diquark and antidiquark are split and the gap of the diquark is getting smaller until it eventually vanishes at $\mu = m_\pi$. At this point the Bose–Einstein condensation sets, breaking the baryon number spontaneously. The diquark is the corresponding Goldstone boson.

Bose–Einstein condensation phase

When $\mu > m_\pi$, the vacuum condensate is given by $\mathbf{n} = (\cos\alpha, \sin\alpha, 0, 0, 0, 0)$, where $\cos\alpha = m_\pi^2/\mu^2$. In the excitation spectrum we again find the pion triplet, but now with the dispersion $E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + \mu^2}$. Finally, there are two excitations, the mixtures of the (anti)diquark and σ, whose dispersion relations are

$$E_\pm^2(\mathbf{k}) = \mathbf{k}^2 + \frac{\mu^2}{2}(1 + 3\cos^2\alpha) \pm \frac{\mu}{2}\sqrt{\mu^2(1 + 3\cos^2\alpha)^2 + 16\mathbf{k}^2\cos^2\alpha}.$$

Note that the gap of the '$-$' solution vanishes so that this is the Goldstone boson of the spontaneously broken symmetry. In accordance with the general discussion in Chapter 3, its dispersion relation is linear at low momentum.

Our calculations confirm the results achieved previously in literature [59, 60]. The notable advantage of the SO(6)/SO(5) formalism presented here is that it allows a straightforward physical interpretation of the various modes, being the linear combinations of $n_1, \ldots, n_6$ whose quantum numbers are well known.

In particular, it turns out that the quantum numbers of the Goldstone boson in the Bose–Einstein condensed phase change as the chemical potential increases. Just at the phase transition point, it is the diquark, matching continuously the diquark mode in the normal phase. On the other hand, in the extreme limit $\mu \gg m_\pi$, it is just n_4, the imaginary part of the diquark. It is now the linear combination of the diquark and the antidiquark and thus carries no definite baryon number. This is, of course, hardly surprising since the baryon number is spontaneously broken and hence is not a good quantum number anymore.

Chapter 6

Conclusions

In the preceding three chapters the results achieved during the PhD study have been presented. Full details of the calculations may be found in the research papers [I–IV] that are attached at the end of this thesis. Here we give a short summary and outline the prospects for future work.

In Chapter 3 we investigated the effects of finite chemical potential on the pattern of symmetry breaking in a Lorentz-invariant field theory. With the help of the Goldstone commutator we suggested a connection between the vacuum densities of non-Abelian charges and the counting of the Goldstone bosons. In the framework of the linear sigma model, we were able to formulate, and prove, an exact counting rule.

It should be stressed, however, that we stayed all the time at the tree level. It would be desirable to investigate whether all our conclusions survive when loop corrections are taken into account. In particular, we expect that the leading power-like behavior of the Goldstone boson dispersion relations does not change, up to a possible multiplicative factor, so that the Nielsen–Chadha counting rule is saturated.

On the other hand, we reported that right at the phase transition the phase velocity of the linear Goldstones vanishes, changing their type from I to II. We also emphasized that this is the only generic case where the Nielsen–Chadha inequality is not saturated. Since quantum corrections are expected to be important in the vicinity of the phase transition, this result calls for verification at one loop.

Moreover, the effect of the quartic interaction comes into play only after the quantum corrections are included since at the tree level, the λ term merely serves to stabilize the static Lagrangian. Finally, due to the nonlinear nature of the Goldstone dispersion relations (even those of type I, because of higher orders in the power expansion of the energy), these are kinematically allowed to decay. It is again a matter of the one-loop effective action to determine the corresponding decay rates. We hope that all these issues will be clarified soon by the one-loop calculations currently being done.

Chapter 4 was devoted to dynamical generation of fermion masses. We showed that a sufficiently strong Yukawa interaction with a complex scalar field may result in spontaneous breaking of the chiral symmetry. This general mechanism may find a particular application to the standard model of electroweak interactions – breaking of the chiral symmetry induces breaking of the electroweak gauge invariance – and thus provide an alternative to the conventional Higgs mechanism. The extension of the present Abelian model to the electroweak $SU(2)_L \times U(1)_Y$ symmetry was sketched.

However, for sake of numerical computations, we used quite crude simplifications. We neglected all quantum corrections but the symmetry-breaking ones to the propagators. In particular, we neglected all vertex corrections. Such an approximation is not really consistent with the assumed symmetry i.e., the Ward identities, because the broken symmetry implies that the Yukawa interaction vertex has a pole due to the Goldstone boson. It would be perhaps more appropriate, for instance, to generate the Schwinger–Dyson equations from a symmetric effective action for the full propagators, by the method of Cornwall, Jackiw, and Tomboulis [63].

Our future program is first to gauge the simple Abelian model presented here. As an exercise we plan to work out signatures of the model that distinguish it from the Higgs mechanism. The last step is to promote the idea to the electroweak symmetry breaking. Then we shall, of course, have to deal with the challenges of the phenomenological restrictions. In order to make quantitative predictions to be compared with experimental data, the approximation used here will have to be improved a lot. Even though this seems to be far ahead, we believe that the mechanism we propose may provide a viable alternative to the Higgs mechanism.

The last topic of this thesis, the phase diagram of quantum chromodynamics, is discussed in Chapter 5. We first suggest an unconventional pairing of quarks of a single flavor in the color-symmetric channel. Since the total spin of such pairs is zero, they might provide a rival to the color-antisymmetric spin-one pairing pursued in literature. An evidence is provided that the pairing in the color-sextet channel may arise from the exchange of a color-octet scalar field, a bound state of two gluons.

The second part of Chapter 5 is devoted to the two-color QCD. We propose an alternative low-energy description of the two-color QCD with two quarks flavors, based on the $SO(6)/SO(5)$ coset space. We work out in detail the correspondence with the $SU(4)/Sp(4)$ formalism used in literature and verify the results obtained by other authors.

List of publications

Research papers

[I] T. Brauner, J. Hošek, and R. Sýkora, *Color superconductor with a color-sextet condensate*, *Phys. Rev.* **D68** (2003) 094004, [hep-ph/0303230].

[II] T. Brauner and J. Hošek, *Dynamical fermion mass generation by a strong Yukawa interaction*, *Phys. Rev.* **D72** (2005) 045007, [hep-ph/0505231].

[III] T. Brauner, *Goldstone boson counting in linear sigma models with chemical potential*, *Phys. Rev.* **D72** (2005) 076002, [hep-ph/0508011].

[IV] T. Brauner, *On the chiral perturbation theory for two-flavor two-color QCD at finite chemical potential*, *Mod. Phys. Lett.* **A21** (2006) 559–569, [hep-ph/0601010].

Preprints

[V] T. Brauner and J. Hošek, *A model of flavors*, hep-ph/0407339.

Conference proceedings

[VI] T. Brauner, *Color superconductivity with a color-sextet order parameter*, in *WDS'03 Proceedings of Contributed Papers: Part III* (J. Šafránková, ed.), pp. 544–549, Matfyzpress, 2003.

[VII] T. Brauner, *Color superconductor with a color-sextet condensate*, *Eur. Phys. J.* **C33** (2004) S597–S599.

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Reprints of published papers

The copies of the research papers [I–IV] listed on page 52 are attached below. These articles constitute the essence of the thesis, to which the text below provides a complement and, in some cases, an extension.

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Color superconductor with a color-sextet condensate

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We analyze color superconductivity of one massive flavor quark matter at moderate baryon density with a spin-zero color-sextet condensate. The most general Higgs-type ground-state expectation value of the order parameter implies complete breakdown of the $SU(3)\times U(1)$ symmetry. However, both the conventional fourth-order polynomial effective bosonic description and the Nambu–Jona-Lasinio–type fermionic description in the mean-field approximation favor an enhanced $SO(3)$ symmetry of the ground state. We ascribe this finding to the failure of the mean-field approximation and propose that a more sophisticated technique is needed.

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I. INTRODUCTION

Viewing the low-temperature deconfined QCD matter at moderate baryon densities as a BCS-type color superconductor is based on good assumptions (see [1–3] for original references and [4] for a recent review). First, the only degrees of freedom relevant for the effective field theory description of such a matter are the relativistic colored quark fields with their appropriate Fermi surfaces. The colored gauge fields can be introduced perturbatively, and eventually switched off in the lowest approximation. Second, the quarks interact with each other by an attractive interaction providing for Cooper instability. It is natural to speak of the Higgs phases of QCD [5].

Because of the mere fact that the quarks carry the Lorentz index (spin), color and flavor, the ordered colored-quark phases could be numerous. Which of them is energetically most favorable depends solely upon the numerical values of the input parameters (chemical potentials, and the dimensionful couplings) in the underlying effective Lagrangian. Because there are no experimental data on the behavior of the cold deconfined quark matter available, all generically different, theoretically safe [6] and interesting possibilities should be phenomenologically analyzed. Moreover, one should be prepared to accept the fact that one or both of our assumptions can be invalid. In any case there are the low-temperature many-fermion systems which are not the Landau-Fermi liquids, and which become peculiar superconductors [7].

Recently, all distinct forms of the quasiquark dispersion laws corresponding to different sets of 16 matrices in the Lorentz index were systematically derived [8]. Those exhibiting spontaneous breakdown of the rotational symmetry manifested in the anisotropic form of the dispersion law are particularly interesting. Their possible nodes can yield important physical consequences even if the corresponding gaps are numerically small [9].

To have a complete list of different ordered quantum phases of the quark matter it would be good to know what is the pattern of spontaneous breakdown of the color $SU(3)$ if an effective interaction prefers not the standard quark-quark Cooper pairing in the antisymmetric color antitriplet, but rather in the symmetric color sextet. Such a pairing would influence qualitatively not only the quark, but also the gluon spectrum.

Although the explicit analysis presented in this paper is strictly phenomenological, we describe here briefly a mechanism which, within QCD and under plausible assumptions, can yield the desired color-sextet diquark condensate. Instabilities of the perturbative QCD in the two-gluon channel discussed in [10] justify contemplating several types of effective colored excitations in the deconfined phase at moderate densities with effective (but in practice theoretically unknown) couplings to both quarks and gluons. According to [10], there should be four types of two-gluon collective excitations: spin-zero color-singlet, spin-zero color octet, spin-one color octet, and spin-two color 27-plet. It is easy to show that exchange of a massive color-octet scalar results in a four-quark interaction

$$\mathcal{L}_{\text{int}} = G(\bar{\psi}\vec{\lambda}\psi)^2, \tag{1}$$

with $G>0$, which is necessary for the color-sextet diquark condensation. It is, however, not easy to show which of the exchanges, including the one-gluon one, is eventually the most important. In fact, exchange of the color-singlet scalar would also lead to an attractive interaction in the color-sextet

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quark-quark channel, but as we aim at a phenomenological analysis and do not attempt to evaluate the effective coupling G, we restrict ourselves in the following to the single interaction term (1).

We note that the argument leading to the conjectured colored collective modes excited by two gluon operators is the same as that leading, in the quark sector, to the phenomenologically useful [11] color-antitriplet scalar field with the quantum numbers of a diquark.

The possibility of diquark condensation in the color-symmetric channel has already been investigated in various contexts, for instance, within the color-flavor-locking scheme [12], and as an admixture to the color-antitriplet condensate [13,14]. The algebraic structure of spontaneous symmetry breaking due to an $SU(3)$-sextet condensate is, however, richer than so far discussed in literature, and it is the general characterization of this structure that we focus on here.

The outline of the paper is as follows. In Sec. II we describe the color-sextet superconductivity phenomenologically i.e., in terms of a scalar color-sextet Higgs field. We are not aware of any systematic treatment of the Higgs mechanism with an $SU(3)$ sextet in the literature and, therefore we go quite into detail. In Sec. III we review main ideas of the semi-microscopic approach i.e., a self-consistent BCS-type approximation for a relativistic fermionic second-quantized quark field, and apply it to the case of color-sextet condensation. Section IV contains a summary and a brief discussion of the obtained results and comparison of the two approaches.

II. HIGGS MECHANISM WITH AN $SU(3)$ SEXTET

Simplifying as much as possible we consider the relativistic quark matter of one massive flavor (say s-quark matter) in the deconfined phase at moderate baryon density. We assume that its ground state is characterized by the quark-quark Cooper-pair condensate in the antisymmetric spin-zero state. By Pauli principle this means the symmetric sextet state in $SU(3)$ i.e.,

$$\langle 0|\psi_{\alpha i}(C\gamma_5)_{\alpha\beta}\psi_{\beta j}|0\rangle \propto \langle \Phi_{ij}\rangle_0, \tag{2}$$

where we insert a dimensionful constant of proportionality to make Φ a dimension-one operator. The constant of proportionality can be determined within the mean-field approximation to be $3/2G$, see Sec. III.

Treating the u and d quarks as nearly degenerate in mass and both much lighter than the s quark, such a condensate may provide a complement to the usual picture of u and d pairing in the color-antitriplet channel [15].

In an effective Higgs description Φ_{ij} is a spin-zero color-sextet order parameter which transforms under the color $SU(3)$ as a complex symmetric matrix,

$$\Phi \to U\Phi U^T.$$

The dynamics of Φ is governed by the most general Lagrangian invariant under global $SU(3)\times U(1)$ and space-time transformations. As the full Lorentz invariance is explicitly broken by the presence of a dense medium, we require that the Lagrangian be invariant under spatial rotations only.

Since we aim at an effective description of the superconducting phase, renormalizability is not an issue here, and we have to include all possible interactions built up from the sextet Φ that respect the symmetry of the theory.

In accordance with our assumptions, the gauge interaction can be switched on perturbatively by gauging the global $SU(3)$ color symmetry. Formally, we just replace the ordinary derivative of Φ with the covariant derivative

$$D_\mu\Phi = \partial_\mu\Phi - igA_\mu^a\left(\tfrac{1}{2}\lambda_a\Phi + \Phi\tfrac{1}{2}\lambda_a^T\right), \tag{3}$$

where A_μ^a is the colored gluon field. The effective Lagrangian thus has the form

$$\mathcal{L} = \alpha_e \mathrm{tr}(D_0\Phi)^\dagger D^0\Phi + \alpha_m \mathrm{tr}(D_i\Phi)^\dagger D^i\Phi - V(\Phi) + \dots, \tag{4}$$

where $V(\Phi)$ is the most general $SU(3)\times U(1)$-invariant polynomial in Φ and the ellipses stand for other possible terms that involve covariant derivatives and/or gauge field strength tensors $F_{a\mu\nu}$.

A. $SU(3)$ invariants from a sextet

The ground-state expectation value $\langle\Phi\rangle_0 = \phi$ is at the tree level given by the minimum of the scalar potential $V(\Phi)$. To proceed with our analysis, we have to specify its concrete form.

Note that the group $SU(3)$ has only three algebraically independent invariant tensors, namely, δ_j^i, ε_{ijk}, and ε^{ijk}, the lower and upper indices transforming under the fundamental representation of $SU(3)$ and its complex conjugate, respectively (see, for example, Ref. [16]). As a consequence, the most general $SU(3)\times U(1)$ invariant built up from a single sextet Φ can be constructed from products and sums of $\det(\Phi^\dagger\Phi)$ and $\mathrm{tr}(\Phi^\dagger\Phi)^n$, the symbols "det" and "tr" referring to determinant and trace in the color space, respectively [17].

Of these polynomials, however, only three are algebraically independent. Indeed, express

$$\mathrm{tr}\Phi^\dagger\Phi = \alpha+\beta+\gamma,$$

$$\mathrm{tr}(\Phi^\dagger\Phi)^2 = \alpha^2+\beta^2+\gamma^2,$$

$$\det\Phi^\dagger\Phi = \alpha\beta\gamma,$$

where α,β,γ are the eigenvalues of $\Phi^\dagger\Phi$ [18], and define the symmetric polynomials

$$\pi_1 = \alpha + \beta + \gamma,$$

$$\pi_2 = \alpha\beta + \alpha\gamma + \beta\gamma = \tfrac{1}{2}[-\mathrm{tr}(\Phi^\dagger\Phi)^2 + (\mathrm{tr}\Phi^\dagger\Phi)^2],$$

$$\pi_3 = \alpha\beta\gamma.$$

Note that the values of π_1, π_2, π_3 determine those of α, β, γ uniquely as the three roots of the cubic equation $x^3 - \pi_1 x^2 + \pi_2 x - \pi_3 = 0$. Thus also the values of all $\mathrm{tr}(\Phi^\dagger\Phi)^n = \alpha^n + \beta^n + \gamma^n$ for $n \geqslant 3$ are fixed. Moreover, they can be expressed directly in terms of π_1, π_2, π_3 as the Taylor coefficients of the generating function

$$f(t) \equiv \mathrm{tr}\ln(1 + t\Phi^\dagger\Phi) = \ln\det(1 + t\Phi^\dagger\Phi),$$

which is readily rewritten as

$$f(t) = \ln(1 + \pi_1 t + \pi_2 t^2 + \pi_3 t^3). \tag{5}$$

We have thus shown that the scalar potential $V(\Phi)$ can always be expressed as a function of the three independent invariants $\det(\Phi^\dagger\Phi)$, $\mathrm{tr}(\Phi^\dagger\Phi)$, and $\mathrm{tr}(\Phi^\dagger\Phi)^2$.

B. Symmetry-breaking patterns

We shall now turn to the structure of the ground state. In our effective Higgs approach, the $SU(3)\times U(1)$ symmetry is spontaneously broken by the ground-state expectation value ϕ of the field Φ, which is a constant due to the translation invariance of the ground state. We can exploit the symmetry to give the ϕ as simple form as possible. In fact, as shown by Schur [19], any complex symmetric matrix can always be written as

$$\phi = U\Delta U^T,$$

where U is an appropriate unitary matrix, and Δ is a real, diagonal matrix with non-negative entries. In our case, we set $\Delta = \mathrm{diag}(\Delta_1, \Delta_2, \Delta_3)$.

Consequently, there are several distinct patterns of spontaneous symmetry breaking possible.

a. $\Delta_1 > \Delta_2 > \Delta_3 > 0$. This ordering can always be achieved by the allowed appropriate real orthogonal transformations. The continuous $SU(3)\times U(1)$ symmetry is completely broken [only a discrete $(Z_2)^3$ symmetry is left].

b. Two Δ*'s are equal*, say $\Delta_1 = \Delta_2 \neq \Delta_3$. This implies an enhanced $O(2)$ symmetry in the corresponding 2×2 block of ϕ.

c. $\Delta_1 = \Delta_2 = \Delta_3 \neq 0$. The vacuum remains $O(3)$ symmetric.

d. Some of $\Delta_i = 0$. Then there is a residual $U(1)$ or $U(2)$ symmetry of the vacuum corresponding to the vanishing entry or entries of Δ.

The concrete type of the symmetry breaking pattern is determined by the scalar potential $V(\Phi)$. Note that, having relaxed the renormalizability requirement, we can always choose the potential $V(\Phi)$ so that it yields as its minimum any desired values of $\Delta_1, \Delta_2, \Delta_3$, just take

$$V(\Phi) = \tfrac{1}{2}a_1[\mathrm{tr}\Phi^\dagger\Phi - \pi_1]^2 + \tfrac{1}{2}a_2[\mathrm{tr}(\Phi^\dagger\Phi)^2 - \pi_1^2 + 2\pi_2]^2 + \tfrac{1}{2}a_3[\det\Phi^\dagger\Phi - \pi_3]^2$$

with all a_1, a_2, a_3 positive. The π's here are to be interpreted as vacuum expectation values of the corresponding operators.

C. Higgs mechanism with a quartic potential

Up to now we have repeatedly stressed the fact that we are dealing with an effective theory and therefore we should include in our Lagrangian all possible interactions preserving the $SU(3)\times U(1)$ symmetry.

Nevertheless, under some specific conditions it is plausible to start up with a renormalizable linear sigma model, that is, take a general quartic potential $V(\Phi)$ and neglect all operators of dimension greater than 4. In Sec. IV we will see that this rather restrictive choice is justified when the underlying microscopic interaction is of four-fermion type.

We thus take up a general quartic potential [20],

$$V(\Phi) = -a\,\mathrm{tr}\Phi^\dagger\Phi + b\,\mathrm{tr}(\Phi^\dagger\Phi)^2 + c(\mathrm{tr}\Phi^\dagger\Phi)^2, \tag{6}$$

where the minus sign before a suggests spontaneous symmetry breaking at the tree level. Varying Eq. (6) with respect to $\Phi^\dagger$, we derive a necessary condition for the vacuum expectation value ϕ,

$$-a\phi + 2b\phi\phi^\dagger\phi + 2c\phi\mathrm{tr}(\phi^\dagger\phi) = 0. \tag{7}$$

A simple observation of Eq. (7) reveals that, should the matrix ϕ be non-singular, we can divide by it and arrive at the condition

$$2b\phi^\dagger\phi = a - 2c\mathrm{tr}(\phi^\dagger\phi).$$

Thus, unless $b = 0$, $\phi^\dagger\phi$ and hence also Δ must be proportional to the identity matrix.

Moreover, even when ϕ is singular, it can be replaced with the real diagonal matrix Δ and we see from Eq. (7) that all non-zero entries Δ_i satisfy the equation

$$2b\Delta_i^2 = a - 2c\mathrm{tr}\Delta^2.$$

Thus all non-zero Δ's develop the same value.

Which of the suggested solutions of Eq. (7) represents the absolute minimum of the potential depends on the input parameters a, b, c, which must be inferred from the underlying theory [23]. We therefore stop the Higgs-like analysis here with the simple conclusion that under fairly general circumstances the quartic potential can be minimized by a matrix Δ proportional to the unit matrix, thus leading to an interesting symmetry-breaking pattern (see Secs. II B *c* and II D *c*).

D. Gluon mass spectrum

Let us now switch on the gauge interaction perturbatively. Due to the spontaneous symmetry breaking some of the gluons acquire non-zero masses via the Higgs mechanism. At the lowest order of the power expansion in the effective theory, the mass matrix of the gluons follows from the scalar field kinetic terms in Eq. (4) upon replacing Φ with ϕ.

Now, recalling the particular form of the covariant derivative in Eq. (3), we arrive at the following gluon mass squared matrix:

$$M_{e,m}^2=\alpha_{e,m}g^2\times\begin{bmatrix}(\Delta_1+\Delta_2)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & (\Delta_1-\Delta_2)^2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 2(\Delta_1^2+\Delta_2^2) & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{3}}(\Delta_1^2-\Delta_2^2)\\ 0 & 0 & 0 & (\Delta_1+\Delta_3)^2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & (\Delta_1-\Delta_3)^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & (\Delta_2+\Delta_3)^2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & (\Delta_2-\Delta_3)^2 & 0\\ 0 & 0 & \frac{2}{\sqrt{3}}(\Delta_1^2-\Delta_2^2) & 0 & 0 & 0 & 0 & \frac{2}{3}(\Delta_1^2+\Delta_2^2+4\Delta_3^2)\end{bmatrix}$$

The subscripts e,m distinguish between the temporal (“electric”) and spatial (“magnetic”) components of the gluon field.

Let us briefly comment on the above mentioned four types of symmetry breaking patterns.

a. $\Delta_1>\Delta_2>\Delta_3>0$. The $SU(3)\times U(1)$ symmetry is completely broken; therefore there are nine massless Nambu-Goldstone modes. Eight of them are eaten by the gluons, which thus acquire non-zero unequal masses [with an appropriate diagonalization in the (A^3,A^8) block]. There is one physical Nambu-Goldstone boson corresponding to the broken global $U(1)$ baryon number symmetry of the underlying theory. Going to the unitary gauge, we can transform away eight of the original twelve degrees of freedom and parametrize the sextet field Φ as

$$\Phi(x)=\frac{1}{\sqrt{2}}e^{i\theta(x)}\begin{pmatrix}\Delta_1(x) & 0 & 0\\ 0 & \Delta_2(x) & 0\\ 0 & 0 & \Delta_3(x)\end{pmatrix},$$

the Δ’s representing three massive radial modes and θ the Nambu-Goldstone mode.

b. $\Delta_1=\Delta_2\neq\Delta_3$. One gluon is left massless, corresponding to the Gell-Mann matrix λ_2 which generates the $SO(2)$ symmetry of the ground state.

c. $\Delta_1=\Delta_2=\Delta_3\neq 0$. There are three massless gluons corresponding to the generators λ_2, λ_5, λ_7 of the $SO(3)$ subgroup of $SU(3)$. All other gluons receive equal masses so that the symmetry breaking $SU(3)\rightarrow SO(3)$ is isotropic.

d. Some of $\Delta_i=0$. There is always an unbroken global $U(1)$ symmetry that arises from a combination of the original baryon number $U(1)$ and the diagonal generators of $SU(3)$, hence all Nambu-Goldstone modes that stem from the symmetry breaking are absorbed into the gauge bosons.

E. Interpretation of the results

So far in this section, we have worked out the usual Higgs mechanism for the case that the scalar field driving the spontaneous symmetry breaking transforms as a sextet under the color $SU(3)$. However, one must exercise some care when applying the results to the physical situation under consideration, that is, color superconductivity. In the very origin of possible problems lies the fact that Φ is not an elementary dynamical field but rather a composite order parameter.

Anyway, our analysis of symmetry breaking patterns still holds as for this purpose one can regard Φ as simply a shorthand notation for the condensate in Eq. (2).

The most apparent deviation from the standard Higgs mechanism is the presence of non-trivial normalization constants at the kinetic terms in Eq. (4). This is due to the compositeness of the field Φ [24,25].

Further, the power expansion of the effective Lagrangian (4) can be reliable as long as the expansion parameter is sufficiently small. In the standard Ginzburg-Landau theory, this is only true near the critical temperature. It is, however, plausible to think of a zero-temperature effective field theory for the superconducting phase. We therefore understand our Lagrangian as such an effective expansion in terms of the Nambu-Goldstone modes [26,27] generalized by inclusion of modes of the modulus of the order parameter [25,28]. In ordinary superconductivity, the Nambu-Goldstone mode is the Bogolyubov-Anderson mode, and the modulus mode is the Abrahams-Tsuneto mode [29].

Our last remark points to the above calculated masses of gluons generated by the Higgs mechanism. To specify the scale of the masses one would have to know the normalization coefficients $\alpha_{e,m}$. These are unknown parameters of the effective theory and have to be determined from the matching with the microscopic theory. At zero temperature, they are roughly

$$\alpha_{e,m} \propto \mu^2/\phi^2,$$

and as a result, both electric and magnetic masses are found to be of order $g\mu$, where μ is the baryon chemical potential. Their physical origin is, however, very different. The electric (Debye) mass is non-zero even in the normal state i.e., above the critical temperature, due to polarization effects in the quark medium. On the other hand, the magnetic (Meissner) mass arises purely as a consequence of the spontaneous symmetry breaking. It is thus zero at the critical point and increases as the temperature is lowered, to become roughly equal in order of magnitude to the Debye mass at $T=0$.

Unfortunately, this is not the end of the story. As pointed out by Rischke who calculated the gluon masses microscopically for the two-flavor color superconductor [25], the lowest order kinetic term alone does not give correct ratios of gluon masses of different adjoint colors. It is therefore not of much help to just try to adjust the normalization of the kinetic term. As a remedy to this problem, it is necessary to make use of higher order contributions to the gluon masses.

In the two-flavor color superconductor with a color-antitriplet condensate, there is only one generically different higher order contribution that can change the ratios of the gluon masses from those given by the lowest order kinetic term (see Ref. [25], Eq. (153)). This reflects the symmetry of the problem: the order parameter (conventionally chosen to point in the direction of the third color) leaves unbroken an $SU(2)$ subgroup of the original color $SU(3)$. Under the unbroken subgroup, the gluons of colors 4–7 transform as a complex doublet and thus have to receive equal masses, possibly different from the mass of gluon 8. The most general gluon mass matrix is thus specified by two parameters.

In our case of a color-sextet condensate, the $SU(3)$ symmetry can be completely broken and we thus expect that there are in general no relations among the eight gluon masses. We do not go into details here, but just list the kinetic terms of order 4 in the field Φ, which give gluon mass ratios different from the lowest order values:

$$|\mathrm{tr}(\Phi^\dagger D_i \Phi)|^2,$$
$$\mathrm{tr}[(D_i\Phi)^\dagger (D^i\Phi)\Phi^\dagger\Phi],$$
$$\mathrm{tr}[\Phi^\dagger (D_i\Phi)\Phi^\dagger (D^i\Phi)] + \mathrm{H.c.},$$

and analogously the terms contributing to the electric gluon masses.

In our Lagrangian the $SU(3)\times U(1)$ symmetry is realized linearly and these terms are found "by inspection." It would be appropriate to repeat the analysis using the non-linearly realized effective Lagrangian along the lines of [30] analyzing the color-antitriplet case. The kinetic terms should follow from symmetry considerations, albeit again with theoretically undetermined coefficients.

Finally we note that as the Debye masses of all gluons are non-zero in the normal state, one might expect that in the superconducting phase they remain non-zero even for those gluons which correspond to unbroken symmetries, in contrast to the conclusions of the effective theory discussed. However, as shown by Rischke for the two-flavor color superconductor, the "unbroken" electric gluons have, somewhat surprisingly, zero Debye mass at $T=0$. This is because the quark colors they interact with are bound in the condensate and hence there are no low energy levels to be excited by long-wavelength chromoelectric fields.

This line of reasoning can be easily carried over to our case, since due to the diagonal nature of the matrix Δ, one can immediately check which quark colors participate in the condensate. We thus conjecture that the naive expectation that the Debye masses of the gluons of the unbroken symmetry are zero, is correct at zero temperature, as long as the colors that the gluon interacts with both have non-zero gap Δ_i. This is the case, for instance, for the gluons of the $SO(2)$ and $SO(3)$ ground state symmetries discussed before (see Secs. II D b and II D c).

To provide a waterproof verification of this conjecture, on should carry out a microscopic calculation similar to that of [25].

III. FERMIONIC BCS-TYPE DESCRIPTION

In the previous section we used an effective Higgs-like theory to treat the kinematics of color superconductivity with a color-sextet condensate. The construction of the effective Lagrangian is based solely on the $SU(3)\times U(1)$ symmetry. Such an approach is thus pretty convenient to extract as much information about the kinematics as possible, but fails to explain the very fact of Cooper pair formation. To understand the dynamics of color superconductivity, we need a microscopic description of the quark system.

As is well known from BCS theory of superconductivity, fermions (quarks in our case) will tend to form Cooper pairs if there is an attractive effective two-body interaction between them. As is usual in attempts to describe the behavior of deconfined QCD matter, we employ the Nambu–Jona-Lasinio (NJL) model and look for the diquark condensate as a constant self-consistent solution to the equations of motion.

Because the excitation spectrum of cold strongly coupled deconfined QCD matter at moderate baryon density is not known, the effective quark-quark interaction relevant for color superconductivity can only be guessed. In any case the excitations of such a matter are of two sorts.

(1) Colored quasiparticles excited by the primary quantum fields with modified dispersion laws.

(2) Collective excitations, which can be in principle both colored and colorless, and are excited by the appropriate polynomials of the primary quantum fields.

We want to argue in favor of the possible existence of massive color-octet spin-zero collective modes excited by two gluon operators [10], the exchange of which produces the desired effective four-quark interaction attractive in the color-sextet quark-quark channel. The (naive) point is that the QCD-induced force between two gluons, which can in general be in any of

$$8\otimes 8 = 1\oplus 8\oplus 8\oplus 10\oplus \overline{10}\oplus 27,$$

is attractive in the color-octet spin-zero configuration.

Inspired by this argument, we choose for our analysis a four-quark interaction which mimics the exchange of an intermediate color-octet scalar particle. As we note below, however, we could have as well included interactions with Lorentz vectors or tensors. Nonetheless, the Lorentz structure of the interaction does not play almost any role in our calculation, and we therefore restrict to the single interaction term (1) suggested above.

Our effective Lagrangian for one massive quark flavor thus reads

$$\mathcal{L}=\bar{\psi}(i\partial\!\!\!/-m+\mu\gamma_0)\psi+G(\bar{\psi}\vec{\lambda}\psi)^2, \tag{8}$$

where the arrow over Gell-Mann λ-matrices implies appropriate summation over adjoint $SU(3)$ indices. Otherwise, Lorentz and color indices are suppressed.

We treat the model Lagrangian (8) in the mean-field approximation. As this is a standard way of dealing with NJL-type models, we sketch only the main steps. Detailed account of the techniques used can be found, for example, in the recent paper by Alford *et al.* [8].

To extract the color-sextet condensate, we split our Lagrangian into a free and an interacting part $\mathcal{L}_0'$ and $\mathcal{L}_{\text{int}}'$, respectively,

$$\mathcal{L}_0'=\bar{\psi}(i\partial\!\!\!/-m+\mu\gamma_0)\psi+\tfrac{1}{2}\bar{\psi}\Delta(C\gamma_5)\bar{\psi}^T-\tfrac{1}{2}\psi^T\Delta^\dagger(C\gamma_5)\psi,$$

$$\mathcal{L}_{\text{int}}'=-\tfrac{1}{2}\bar{\psi}\Delta(C\gamma_5)\bar{\psi}^T+\tfrac{1}{2}\psi^T\Delta^\dagger(C\gamma_5)\psi+G(\bar{\psi}\vec{\lambda}\psi)^2,$$

where Δ is the desired gap parameter which, as shown in the preceding section, can be sought in the form of a real diagonal non-negative matrix in the color space. We introduce the standard Nambu-Gorkov doublet notation,

$$\Psi(p)=\begin{pmatrix}\psi(p)\\ \bar{\psi}^T(-p)\end{pmatrix},$$

in which the calculation of the free propagator amounts to inverting a 2×2 matrix,

$$S^{-1}(p)=\begin{pmatrix}p\!\!\!/-m+\mu\gamma_0 & \Delta(C\gamma_5)\\ -\Delta^\dagger(C\gamma_5) & (p\!\!\!/+m-\mu\gamma_0)^T\end{pmatrix}.$$

The explicit form of the propagator has been given by several authors, see, for instance, Refs. [31,32].

In the mean-field approximation, Δ is determined from a single one-loop Feynman graph. Regulating the quadratic divergence with a three-momentum cutoff Λ and evaluating explicitly the Wick-rotated integral over the temporal component of the loop momentum, we finally arrive at the gap equation

$$1=\frac{2}{3}G\int^{\Lambda}\frac{d^3\vec{p}}{(2\pi)^3}\left(\frac{1}{E_+}+\frac{1}{E_-}\right), \tag{9}$$

where $E_\pm$ represent the positive energies given by the dispersion relations of the quasiquark excitations,

$$E_\pm^2=(\sqrt{\vec{p}^2+m^2}\pm\mu)^2+|\Delta|^2.$$

A few remarks to the gap equation (9) are in order. First, in the loop integral we have ignored a term proportional to $\mu\gamma_0$ which generates the operator $\bar{\psi}(C\gamma_5)\gamma_0\bar{\psi}^T$ that breaks Lorentz invariance. In fact, we should have expected such a term to appear, since Lorentz invariance is explicitly broken by the presence of the chemical potential in Lagrangian (8). For our treatment of color superconductivity at non-zero chemical potential to be fully consistent, we would have to include such operators into our Lagrangian from the very beginning and solve a coupled set of gap equations for both Lorentz invariant and non-invariant condensates [33]. Here, for the sake of simplicity, we ignore this difficulty and neglect the secondary effects of Lorentz-invariance breaking induced by the chemical potential.

Second, the gap equation (9) is understood as a matrix equation in the color space. Its matrix structure is, however, trivial. In fact, we get three separated identical equations for the diagonal elements $\Delta_1,\Delta_2,\Delta_3$ of the gap matrix. This means that, at least at the level of the mean-field approximation, our model favors an enhanced $SO(3)$ symmetry of the ground state—the gaps for all three colors are the same. This is apparently not a peculiar consequence of our particular choice of interaction in Eq. (8), but holds for any $SU(3)$-invariant four-fermion interaction. The only effect of adding also the Lorentz vector or tensor channel interactions, for example, would be in the modification of the effective coupling constant G. The Lorentz structure of the interaction does not play any role and the resulting form of the gap equation is a consequence of the identity $\vec{\lambda}\Delta\vec{\lambda}^T=4\Delta/3$, which holds for any diagonal matrix Δ. We will return to the discussion of this point in the next section where we will comment on a correspondence between the bosonic and fermionic approaches.

Third, the extension of the gap equation to non-zero temperatures is easy. We can either first calculate the thermodynamical potential Ω and then minimize it with respect to Δ or, alternatively, proceed in the same manner as before and derive a self-consistency condition for the thermal Green function [34]. Performing the sum over Matsubara frequencies in the last step, the result is

$$1=\frac{2}{3}G\int^{\Lambda}\frac{d^3\vec{p}}{(2\pi)^3}\left(\frac{1}{E_+}\tanh\frac{E_+}{2T}+\frac{1}{E_-}\tanh\frac{E_-}{2T}\right).$$

This gap equation can be used for the study of temperature dependence of the gap and, in particular, for finding the critical temperature at which the $SU(3)$ symmetry is restored [31].

IV. SUMMARY AND DISCUSSION

Let us briefly summarize our results. First we developed the Higgs mechanism for a color sextet and found out that although the underlying symmetry allows for a complete spontaneous breakdown, for a generic quartic scalar potential the pattern $SU(3)\to SO(3)$ is preferred.

Then we used the Nambu–Jona-Lasinio model to calculate the gaps $\Delta_1,\Delta_2,\Delta_3$ self-consistently in the mean-field approximation and our result was in accord with the preceding Higgs-type analysis.

This is, of course, not only a coincidence, but follows from a general correspondence between four-fermion-interaction models and linear sigma models provided by the Hubbard-Stratonovich transformation.

Let us sketch the main idea. In the path integral formalism, one first introduces an auxiliary scalar integration variable which has no kinetic term and couples to the fermion via the Yukawa interaction. The action now becomes bilinear in the fermion variables and one can integrate them out explicitly. The logarithm of the fermion determinant gives rise to a kinetic term of the scalar field and the model hence becomes equivalent to the linear sigma model, up to a choice of the renormalization prescription [36].

In terms of the NJL model the interpretation of the correspondence is a little bit different. Here one cannot carry out the usual renormalization program and the choice of an ultraviolet regulator becomes physically significant. So in the effective scalar field action the operators with dimension 4 or less are dominant since they are generated with divergent coefficients. The quadratic divergences cancel due to the gap equation in the underlying NJL model but the logarithmic ones remain [37].

One thus receives an *a posteriori* justification for the choice of the linear sigma model as the starting point for the Higgs-type analysis in Sec. II C. On the other hand, one should bear in mind that these conclusions are valid only in the mean-field approximation that we employed.

In terms of the effective scalar field Φ, the true vacuum is determined by the absolute minimum of the full quantum effective potential which is no longer restricted to contain operators of dimension 4 or less.

In the NJL model, going beyond the mean-field approximation [38] could destroy the simple structure of the one-loop gap equation (9). Generally, the resulting set of algebraic equations for $\Delta_1,\Delta_2,\Delta_3$ must be permutation invariant since permutations of diagonal elements of the matrix Δ belong to the symmetry group $SU(3)$ of the theory. For four-fermion interactions the $SU(3)$ structure of an arbitrary Feynman graph can be investigated making use of the Fierz identities in the color space. One gets three coupled, but still rather simple, equations for the three gaps. It is then perhaps a matter of numerical calculations to decide whether these equations possess asymmetric solutions and whether they are more energetically favorable than those with $\Delta_1=\Delta_2=\Delta_3$.

We suspect that asymmetric solutions implying a complete breakdown of the $SU(3)\times U(1)$ symmetry can also be obtained from interactions that mimic many-body forces (six-fermion or more). The correspondence with linear sigma model via the Hubbard-Stratonovich transformation is then lost and it could hopefully suffice to stay at the level of the mean-field approximation, thus requiring much less manual work than in the previous case.

Investigations in the two directions mentioned above are already in progress.

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below. It is clear that at least one of these parameters must be positive; positivity of both being, of course, the safest choice. The sizes of the two quartic interaction terms are restricted by the inequalities $\mathrm{tr}(\Phi^\dagger\Phi)^2 \leqslant (\mathrm{tr}\Phi^\dagger\Phi)^2 \leqslant 3\,\mathrm{tr}(\Phi^\dagger\Phi)^2$ where the equality sign in the left and right hand side inequality occurs when only one Δ_i is non-zero and $\Delta_1 = \Delta_2 = \Delta_3$, respectively. The potential is thus bounded from below if and only if b is positive and $c > -b/3$, or c is positive and $b > -c$.
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Dynamical fermion mass generation by a strong Yukawa interaction

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We consider a model with global Abelian chiral symmetry of two massless fermion fields interacting with a complex massive scalar field. We argue that the Schwinger-Dyson equations for the fermion and boson propagators admit ultraviolet-finite chiral-symmetry-breaking solutions provided the Yukawa couplings are large enough. The fermions acquire masses and the elementary excitations of the complex scalar field are the two real spin-zero particles with different masses. As a necessary consequence of the dynamical chiral symmetry breakdown both in the fermion and scalar sectors, one massless pseudoscalar Nambu-Goldstone boson appears in the spectrum as a collective excitation of both the fermion and the boson fields. Its effective couplings to the fermion and boson fields are calculable.

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I. INTRODUCTION

One of the major challenges to current high energy physics is to understand the origin of particle masses [1]. All particle interactions except gravity are, up to energies accessible in to-date experiments, successfully described by the standard model. There is therefore no doubt that the standard model is the correct *effective* theory of particle interactions in the energy range so far explored.

It is well known that the detailed underlying physics manifests itself in the effective theory through the effective coupling constants so that when looking for new physics, we should try to reveal the origin of the parameters in the effective Lagrangian.

In the standard model, the part of the Lagrangian describing the particle interactions is beautifully constrained by the gauge invariance principle. The matter part, however, seems rather ugly. The fermion masses cannot be introduced directly as they are prohibited by the gauge $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetry, whose chiral structure is in turn necessary to provide an explanation for parity violation in weak interactions.

The masses therefore have to be generated by means of spontaneous symmetry breaking. This is achieved by introducing the scalar Higgs field and properly adjusting its potential so that it develops a nonzero vacuum expectation value at tree level. However, the dynamical origin of the "wrong sign" of the Higgs mass squared, which is responsible for the tree-level condensation, remains unclear. Moreover, the fermion masses eventually stem from the Yukawa interaction with the Higgs, and hence there are as many different couplings as there are particle species. It would certainly be desirable to understand the particle masses as the consequence of some, yet unknown, quantum dynamics.

We would like to argue that dynamical mass generation is possible by means of the Yukawa interaction itself, without ever having to change the sign of the Higgs mass squared. This would open a new possibility of the existence of an elementary massive scalar field in the standard model Lagrangian. It should be heavy enough so that current experimental bounds are met, and also due to naturalness arguments, according to which its mass should be shifted by radiative corrections up to the scale of new physics.

Our ambitions in the present paper are much more modest than to cure the standard model from its disease. We disregard the otherwise phenomenologically very important issues such as the hierarchy of particle families and *CP* violation, and concentrate on a simple Abelian model to show that spontaneous chiral symmetry breaking by the Yukawa interaction is viable. We believe that the central idea of the symmetry-breaking mechanism can thus be displayed more transparently. The implementation of this idea to the standard model phenomenology is deferred to future work.

The plan of the paper is as follows. In the next section we introduce our model and investigate, at a rather elementary and pictorial level, the consequences of the assumed spontaneous generation of the fermion mass. We do so for the reader's convenience and to emphasize the robustness of the main idea of the paper. In particular we show that the fermion mass induces an anomalous symmetry-breaking two-point Green's function of the scalar. The spectrum of the system then contains two real spinless particles coupled to the original complex field, and with their masses split.

In the next part of the text, a matrix formalism is developed which allows us to treat both the symmetry-preserving and the symmetry-breaking Green's functions on the same footing. We write down the one-loop Schwinger-Dyson equations and exploit the underlying symmetry by means of the Ward identities. Still *assuming* the spontaneous symmetry breaking to occur in the ground state, we show how the Nambu-Goldstone boson arises and examine its properties.

In the last part we demonstrate that, under reasonable simplifying assumptions, a symmetry-breaking solution to

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the Schwinger-Dyson equations actually does exist, thus establishing a firm ground for the preceding heuristic arguments. We conclude the paper with a discussion of our results and the future perspective.

II. PRELIMINARY CONSIDERATIONS

Our model is defined by the Lagrangian,

$$\mathcal{L} = \sum_{j=1,2} (\bar{\psi}_{jL} i\partial\!\!\!/ \psi_{jL} + \bar{\psi}_{jR} i\partial\!\!\!/ \psi_{jR}) + \partial_\mu \phi^\dagger \partial^\mu \phi - M^2 \phi^\dagger \phi - \frac{1}{2}\lambda(\phi^\dagger\phi)^2 + y_1 \bar{\psi}_{1L}\psi_{1R}\phi + y_1 \bar{\psi}_{1R}\psi_{1L}\phi^\dagger + y_2 \bar{\psi}_{2R}\psi_{2L}\phi + y_2 \bar{\psi}_{2L}\psi_{2R}\phi^\dagger. \quad (1)$$

The Yukawa couplings $y_{1,2}$ are real without lack of generality. Another remark is in order here. In view of the future application of our idea on the electroweak symmetry breaking, it is necessary that the global symmetry to be spontaneously broken is amenable to gauging. With just a single fermion species there would be an axial anomaly present. While anomaly cancellation is automatic in the standard model due to its particle content, here we have to introduce two fermions with opposite axial charges to remove the anomaly in the Abelian axial current. It should be clear, however, that this minor technical complication does not alter at all the underlying idea.

The Lagrangian Eq. (1) enjoys a global Abelian $\mathrm{U}(1)_{V1} \times \mathrm{U}(1)_{V2} \times \mathrm{U}(1)_A$ symmetry. The two vector U(1)'s are associated with separate conservation of the numbers of fermions of the first and second type. The corresponding Noether currents are the well known

$$j^\mu_{V1} = \bar{\psi}_1 \gamma^\mu \psi_1, \qquad j^\mu_{V2} = \bar{\psi}_2 \gamma^\mu \psi_2. \quad (2)$$

The axial U(1) transformations of the fermions are tied by the Yukawa coupling to the scalar field. For the Lagrangian to be invariant, it is necessary that the fields transform as

$$\psi_1 \to e^{+i\theta\gamma_5}\psi_1, \qquad \psi_2 \to e^{-i\theta\gamma_5}\psi_2, \qquad \phi \to e^{-2i\theta}\phi. \quad (3)$$

The axial Noether current has the following form:

$$j^\mu_A = \bar{\psi}_1\gamma^\mu\gamma_5\psi_1 - \bar{\psi}_2\gamma^\mu\gamma_5\psi_2 + 2i[(\partial^\mu\phi)^\dagger\phi - \phi^\dagger\partial^\mu\phi]. \quad (4)$$

Now the standard tree-level mechanism of spontaneous symmetry breaking corresponds to $M^2 < 0$ in Eq. (1). Then the scalar field, $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, develops a nonzero ground-state expectation value, which is conveniently chosen to be real, $v \equiv \langle\phi_1\rangle_0 = (-2M^2/\lambda)^{1/2}$. Consequently, the fermions acquire the masses $m_{1,2} = \frac{1}{\sqrt{2}} v y_{1,2}$, ϕ_2 becomes the massless Nambu-Goldstone boson, and $\phi_1' = \phi_1 - v$ becomes a massive real scalar particle with mass $M_1 = \sqrt{-2M^2}$. The Goldstone boson ϕ_2 interacts with the fermions by the Yukawa coupling $\frac{m_1}{v}\bar{\psi}_1 i\gamma_5\psi_1\phi_2 - \frac{m_2}{v}\bar{\psi}_2 i\gamma_5\psi_2\phi_2$.

From now on we set $M^2 > 0$ and investigate the model of Eq. (1) with respect to the possibility of spontaneous symmetry breaking. The scalar field now possesses an ordinary mass term, and there is therefore no condensation and no symmetry breaking at tree level. Rather, if the symmetry is broken in the ground state, it must be a result of the quantum dynamics of the system.

As spontaneous symmetry breaking is a nonperturbative phenomenon, it cannot be achieved at any finite order of perturbation theory. The nonperturbative method we employ here is to look for self-consistent symmetry-breaking solutions to the Schwinger-Dyson equations. We thus temporarily *assume* that there is a solution for which the full fermion propagators have nonvanishing chirality-changing parts, which means nonzero masses of the fermions.

Now let us observe that such a solution induces mixing of ϕ and $\phi^\dagger$ or, in other words, nonzero correlation function $\langle\phi\phi\rangle$, see Fig. 1. This could have been expected as the field ϕ couples to the axial part of the $\mathrm{U}(1)_{V1} \times \mathrm{U}(1)_{V2} \times \mathrm{U}(1)_A$ symmetry, and once this is broken nothing prevents ϕ and $\phi^\dagger$ from mixing.

The second observation is that the "anomalous" two-point scalar Green's function $\langle\phi\phi\rangle$ in turn enters the one-loop Schwinger-Dyson equations for the fermion propagators, see Fig. 2. The set of Schwinger-Dyson equations for the fermion and scalar propagators are thus intrinsically coupled and must be solved simultaneously if a symmetry-breaking solution is to be found.

Before we switch to a formal description to come in the next section, we would like to give a more physical picture of what is going on here. Let us for simplicity suppose that the one-particle-irreducible part of the anomalous scalar two-point function $\langle\phi\phi\rangle$ is momentum independent and equal to $-i\mu^2$. The spectrum in the scalar sector is then determined by the quadratic part of the renormalized Lagrangian,

$$\mathcal{L}^{(0)}_{\text{scalar}} = \partial_\mu\phi^\dagger\partial^\mu\phi - M^2\phi^\dagger\phi - \frac{1}{2}\mu^{2*}\phi\phi - \frac{1}{2}\mu^2\phi^\dagger\phi^\dagger. \quad (5)$$

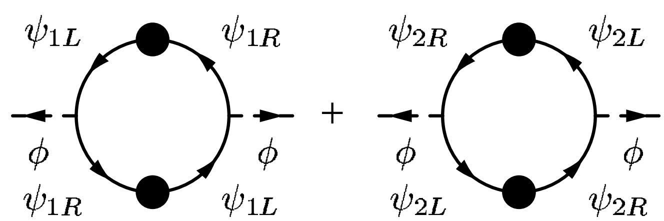

FIG. 1. Mixing in scalar sector induced by the fermion mass terms. The full blobs denote the chirality-changing part of the complete fermion propagator.

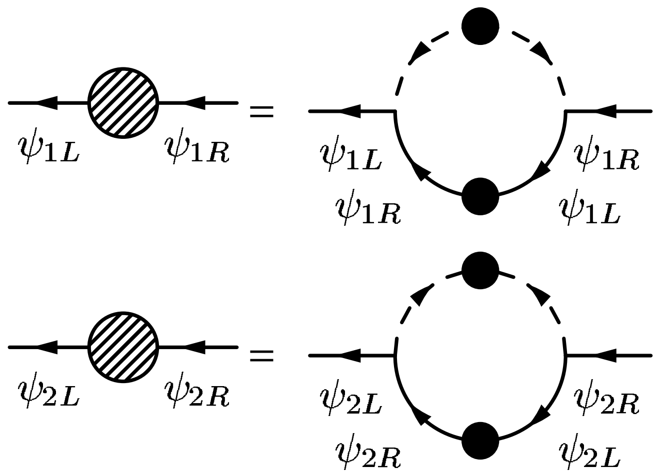

FIG. 2. Chirality changing fermion proper self-energies induced by the scalar mixing.

The nonderivative (mass) part of the Lagrangian is easily diagonalized by a rotation in the $\phi - \phi^\dagger$ space. The spectrum then contains two real spin-0 particles with masses

$$M_{1,2}^2 = M^2 \pm |\mu^2|.$$

The corresponding real fields φ_1 and φ_2 are defined through

$$\phi = \frac{1}{\sqrt{2}} e^{i\alpha}(\varphi_1 + i\varphi_2),$$

where the "mixing angle" α is merely given by the phase of the anomalous mass term, $\tan 2\alpha = \mathrm{Im}\,\mu^2/\mathrm{Re}\,\mu^2$.

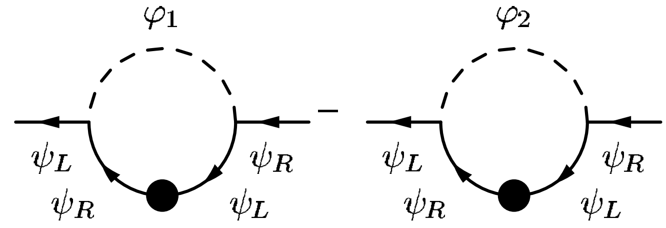

FIG. 3. One-loop SD equation for the fermion self-energy expressed in terms of the physical fields φ_1 and φ_2.

Note also that the anomalous propagator $\langle\phi\phi\rangle$ is now equal to $\frac{1}{2}e^{2i\alpha}(\langle\varphi_1\varphi_1\rangle - \langle\varphi_2\varphi_2\rangle)$; that is, the one-loop graphs in Fig. 2 may be replaced with Fig. 3. The difference of scalar propagators significantly improves the convergence of the Schwinger-Dyson kernel.

III. FORMAL DEVELOPMENTS

A. The scalar Nambu doublet

We have seen in the previous section that once chiral symmetry is spontaneously broken the scalar field ϕ develops a nonzero anomalous propagator mixing it with $\phi^\dagger$. For the sake of simplicity of the general formulas to come, we introduce a doublet field,

$$\Phi = \begin{pmatrix} \phi \\ \phi^\dagger \end{pmatrix},$$

and the matrix propagator which contains as its entries both normal and anomalous two-point functions of the field ϕ,

$$iD(x-y) = \langle 0|T\{\Phi(x)\Phi^\dagger(y)\}|0\rangle = \begin{pmatrix} \langle 0|T\{\phi(x)\phi^\dagger(y)\}|0\rangle & \langle 0|T\{\phi(x)\phi(y)\}|0\rangle \\ \langle 0|T\{\phi^\dagger(x)\phi^\dagger(y)\}|0\rangle & \langle 0|T\{\phi^\dagger(x)\phi(y)\}|0\rangle \end{pmatrix}.$$

Note that this notation is similar to the Nambu formalism frequently used in the theory of superconductivity [2]. It is also quite natural due to its analogy in the fermion sector. There, we could have well introduced two "normal" propagators for the left- and right-handed chiral fields and treated the desired mass term connecting ψ_L with ψ_R as an anomalous part of the propagator. Instead, we work with the Dirac field

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},$$

which incorporates both chiral fields in a single four-component spinor. The chirality-changing part of the fermion propagator is then given by the off-diagonal component of the full Dirac propagator,[1]

$$iS(x-y) = \langle 0|T\{\psi(x)\bar{\psi}(y)\}|0\rangle = \begin{pmatrix} \langle 0|T\{\psi_L(x)\bar{\psi}_L(y)\}|0\rangle & \langle 0|T\{\psi_L(x)\bar{\psi}_R(y)\}|0\rangle \\ \langle 0|T\{\psi_R(x)\bar{\psi}_L(y)\}|0\rangle & \langle 0|T\{\psi_R(x)\bar{\psi}_R(y)\}|0\rangle \end{pmatrix}.$$

[1] We work in the chiral basis of the Dirac γ matrices, in which γ_5 is diagonal, and quite deliberately denote by $\psi_{L,R}$ both the two-component Weyl spinors and the four-component Dirac spinors with just the upper two or the lower two entries nonzero. It should be clear from the context which of these two spinors is actually used.

The propagators of ψ_1 and ψ_2 will be denoted as S_1 and S_2, respectively.

B. The Schwinger-Dyson equations

As already stressed above, spontaneous symmetry breaking cannot be revealed at any finite order of perturbation theory. In order to deal with this nonperturbative effect, we employ the technique of the Schwinger-Dyson equations.

These constitute an infinite system of coupled equations for the Green's functions of the theory. To make them more tractable, it is usual to close the system at a certain order by assuming a convenient ansatz for the higher-point Green's functions. In order to achieve a simple gap equation we neglect all but the two-point connected Green's functions [3]. We are thus left with a self-consistent set of equations for the fermion and scalar propagators, which are depicted diagrammatically in Fig. 4. The double-dashed line represents the Nambu doublet Φ. This symbolic notation is used to stress the fact that the Schwinger-Dyson equations for both the symmetry-preserving and the symmetry-breaking parts of the propagators are represented by Feynman graphs of the same topology and can thus be put in a simple compact form as in Fig. 4.

We do not write down all the formulas that correspond to the Feynman diagrams in Fig. 4. Instead, having in mind our future simplification neglecting all symmetry-preserving radiative corrections, we put explicitly just the expressions for the symmetry-breaking proper self-energies. The upper indices, L or R for the fermion propagators and 1 or 2 for the scalar propagator, specify the matrix elements of the two-by-two matrices for S and D introduced above. The same matrix notation is used for the proper self-energies Σ and Π:

$$\begin{aligned}
\Sigma_1^{LR}(p) &= iy_1^2 \int \frac{d^4k}{(2\pi)^4} S_1^{RL}(k) D^{12}(k-p),\\
\Sigma_2^{LR}(p) &= iy_2^2 \int \frac{d^4k}{(2\pi)^4} S_2^{RL}(k) D^{21}(k-p),\\
\Pi^{12}(p) &= -iy_1^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}[S_1^{LR}(k) S_1^{LR}(k-p)] \\
&\quad - iy_2^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}[S_2^{RL}(k) S_2^{RL}(k-p)] \\
&\quad + i\lambda \int \frac{d^4k}{(2\pi)^4} D^{12}(k).
\end{aligned} \tag{6}$$

Before concluding the discussion of the Schwinger-Dyson equations, let us note that neglecting corrections to the interaction vertices is not all that consistent with the envisaged spontaneous symmetry breaking. In Sec. III C we explain that the coupling of both the fermions and the scalar to the current of the broken symmetry (that is, the axial current) develops a pole due to the intermediate massless Nambu-Goldstone boson state. Now the same is also true for the (pseudo)scalar Yukawa coupling of the fermion and scalar. As a result, the full Φ propagator should possess a massless pole due to the Nambu-Goldstone boson, with the residuum determined by the factor $\langle 0|\phi|\pi(q)\rangle$.

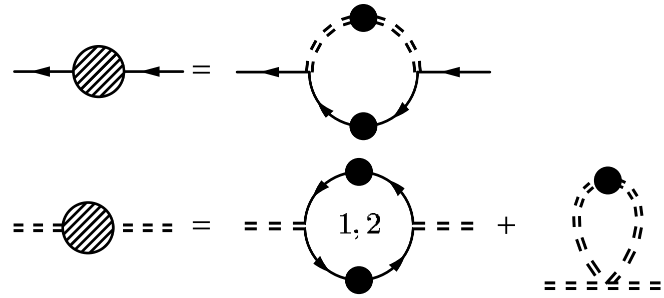

FIG. 4. The diagrammatical representation of the one-loop Schwinger-Dyson equations for the fermion and scalar self-energies. The first line holds for both ψ_1 and ψ_2. The dashed blobs stand for the proper self-energies, while the solid blobs denote the full propagators. The double-dashed line is for the Nambu Φ doublet.

Neglecting the vertex corrections therefore means that we are not going to reproduce correctly the whole analytical structure of the propagators. Our set of Schwinger-Dyson equations amounts to resummation of a certain class of Feynman diagrams which, however, is sufficient to discover spontaneous symmetry breaking. In other words, we are looking for spontaneous symmetry breaking in the spectrum of the elementary excitations of the fields $\psi_{1,2}$ and Φ. The discussion of the collective excitations, which of course also manifest themselves in the full propagators, is deferred to Sec. III C.

C. The Ward identities

We now exploit the symmetry properties of the theory. At the classical level, the $\mathrm{U}(1)_{V1} \times \mathrm{U}(1)_{V2} \times \mathrm{U}(1)_A$ invariance of the Lagrangian II implies the existence of three conserved Noether currents—two vector, see Eq. (2), and one axial, see Eq. (4).

At the level of quantum field theory, the conservation of the vector and axial currents is expressed in terms of the set of Ward identities for the Green's functions containing the current operators. We investigate here the three-point functions with the conserved current and a fermion or scalar pair, respectively.

The vector currents couple only to the fermions, so there is just one nontrivial Ward identity for each, for the vertex functions $G^\mu_{V1}(x,y,z) = \langle 0|T\{j^\mu_{V1}(x)\psi_1(y)\bar\psi_1(z)\}|0\rangle$ and $G^\mu_{V2}(x,y,z) = \langle 0|T\{j^\mu_{V2}(x)\psi_2(y)\bar\psi_2(z)\}|0\rangle$, which have the well-known form

$$\begin{aligned}
q_\mu \Gamma^\mu_{V1}(p+q,p) &= S_1^{-1}(p+q) - S_1^{-1}(p),\\
q_\mu \Gamma^\mu_{V2}(p+q,p) &= S_2^{-1}(p+q) - S_2^{-1}(p).
\end{aligned}$$

The proper vertex functions $\Gamma^{\mu}_{V1,2}$ correspond to $G^{\mu}_{V1,2}$ with full fermion propagators of the external legs cut off.

The axial current, on the other hand, couples to both the fermions and the scalar. There are hence altogether three vertex functions, $G^{\mu}_{A\psi_1}(x, y, z) = \langle 0|T\{j^{\mu}_A(x)\psi_1(y)\bar{\psi}_1(z)\}|0\rangle$, $G^{\mu}_{A\psi_2}(x, y, z) = \langle 0|T\{j^{\mu}_A(x)\psi_2(y)\bar{\psi}_2(z)\}|0\rangle$, and $G^{\mu}_{A\phi}(x, y, z) = \langle 0|T\{j^{\mu}_A(x)\Phi(y)\Phi^{\dagger}(z)\}|0\rangle$. The corresponding Ward identities read

$$\begin{aligned} q_\mu \Gamma^{\mu}_{A\psi_1}(p+q, p) &= S_1^{-1}(p+q)\gamma_5 + \gamma_5 S_1^{-1}(p), \\ q_\mu \Gamma^{\mu}_{A\psi_2}(p+q, p) &= -S_2^{-1}(p+q)\gamma_5 - \gamma_5 S_2^{-1}(p), \\ q_\mu \Gamma^{\mu}_{A\phi}(p+q, p) &= -2D^{-1}(p+q)\Xi + 2\Xi D^{-1}(p). \end{aligned} \tag{7}$$

The matrix Ξ,

$$\Xi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

operates in the $\phi - \phi^{\dagger}$ space and is quite analogous to γ_5 in the fermion sector.

Before closing the general discussion of the Ward identities, let us remark that the identities of Eq. (7) must hold whether the symmetry is spontaneously broken or not. In both cases they strongly constrain the form of the vertex functions but, particularly if the symmetry is broken, they allow us to visualize the massless collective excitation predicted by the Goldstone theorem, as we will now see.

D. The Nambu-Goldstone boson

Once chiral symmetry is broken, there must exist a massless Nambu-Goldstone boson in the spectrum of the theory, which couples to the Noether current of the broken symmetry. It is the axial current which is broken in our case and, as it couples to both the fermions and the scalar, the Nambu-Goldstone boson must be a collective excitation of both fermions and scalars.

General analytical properties of Green's functions imply the existence of a pole corresponding to an intermediate particle, once the total momentum of a proper subset of external legs approaches the mass shell of the particle [4]. This means that the Nambu-Goldstone boson can be seen as a pole in the vertex functions of the axial current as $q^2 \to 0$. Its properties are then expressed in terms of the symmetry-breaking self-energies of the fermions and the scalar, which are obtained by solving the set of Schwinger-Dyson equations stated in Sec. III B.

To proceed further, we approximate the proper vertex functions $\Gamma^{\mu}_{A\psi_{1,2}}$ and $\Gamma^{\mu}_{A\phi}$ by the sum of the bare vertex and the pole contribution. We follow the analysis of Jackiw and Johnson [5].

The bare vertices are determined by the usual rules of perturbation theory. For the sake of later reference and to show how the Φ notation works, we fix them down in Fig. 5.

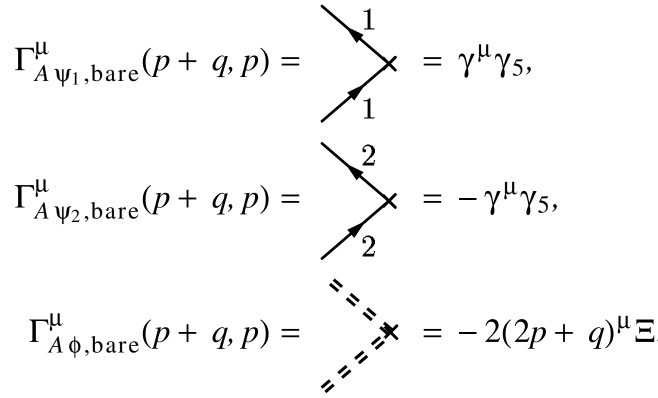

FIG. 5. The bare parts of the proper vertex functions of the axial current. The crosses indicate the axial current. The numbers at the solid lines distinguish between ψ_1 and ψ_2.

The pole of the vertex functions arises from the propagator of the intermediate Nambu-Goldstone boson, see Fig. 6. The yet unknown effective vertices of the Nambu-Goldstone boson with the fermions and the scalar, to be extracted from the Ward identities (7), are denoted by empty circles in the figures and by $P_{\psi_{1,2}}(p+q, p)$ and $P_\phi(p+q, p)$ in the formulas.

We can now write down the formulas for the pole contributions,

$$\begin{aligned} \Gamma^{\mu}_{A\psi_1,\text{pole}} &= P_{\psi_1}(p+q, p)\frac{i}{q^2}[I^{\mu}_{\psi_1}(q) + I^{\mu}_{\psi_2}(q) + I^{\mu}_{\phi}(q)], \\ \Gamma^{\mu}_{A\psi_2,\text{pole}} &= P_{\psi_2}(p+q, p)\frac{i}{q^2}[I^{\mu}_{\psi_1}(q) + I^{\mu}_{\psi_2}(q) + I^{\mu}_{\phi}(q)], \\ \Gamma^{\mu}_{A\phi,\text{pole}} &= P_{\phi}(p+q, p)\frac{i}{q^2}[I^{\mu}_{\psi_1}(q) + I^{\mu}_{\psi_2}(q) + I^{\mu}_{\phi}(q)], \end{aligned} \tag{8}$$

where $I^{\mu}_{\psi_{1,2}}(q)$ and $I^{\mu}_{\phi}(q)$ represent the fermion and scalar

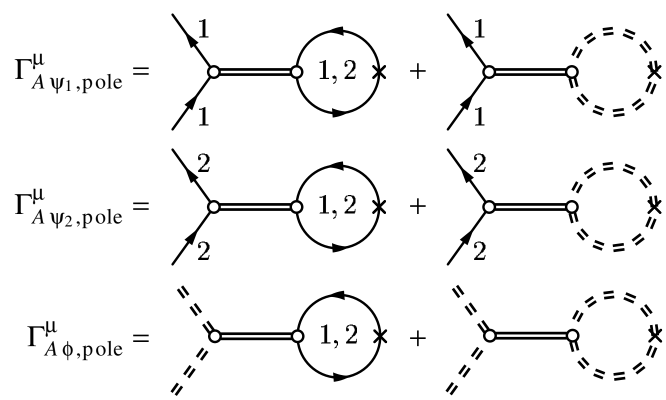

FIG. 6. The pole parts of the proper vertex functions of the axial current. The double solid line represents the Nambu-Goldstone boson and the empty circles its effective vertices with the fermions and the scalar, respectively. Both ψ_1 and ψ_2 can circulate in the closed fermion loops.

loops in Fig. 6. They are given by the equations

$$\begin{aligned} I^{\mu}_{\psi_1}(q) &= -\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}[\gamma^{\mu}\gamma_5 iS_1(k-q) \\ &\quad \times P_{\psi_1}(k-q,k)iS_1(k)], \\ I^{\mu}_{\psi_2}(q) &= -\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}[-\gamma^{\mu}\gamma_5 iS_2(k-q) \\ &\quad \times P_{\psi_2}(k-q,k)iS_2(k)], \\ I^{\mu}_{\phi}(q) &= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}[-2(2k-q)^{\mu}\Xi iD(k-q) \\ &\quad \times P_{\phi}(k-q,k)iD(k)]. \end{aligned} \tag{9}$$

The difference of the integral prefactors here arises from the different nature of the degrees of freedom circulating in the loops. The -1 is the standard fermion loop factor, while the $\frac{1}{2}$ follows from the fact that by introducing Φ instead of ϕ we have effectively doubled the number of degrees of freedom, which must be compensated for when calculating the trace over a closed loop.

As the loop integrals $I^{\mu}_{\psi_{1,2}}(q)$ and $I^{\mu}_{\phi}(q)$ depend only on a single external momentum q, they are forced by Lorentz covariance to have the form

$$I^{\mu}_{\psi_{1,2}}(q) = -iq^{\mu}I_{\psi_{1,2}}(q^2), \qquad I^{\mu}_{\phi}(q) = -iq^{\mu}I_{\phi}(q^2).$$

Inserting this into Eq. (8) and the sum of the bare and the pole contributions into the Ward identities (7), and going onto the Nambu-Goldstone boson mass shell, $q^2 \to 0$, we arrive at the general formula for the effective vertices,

$$\begin{aligned} P_{\psi_1}(p+q,p) &= \frac{1}{N}[S_1^{-1}(p+q)\gamma_5 + \gamma_5 S_1^{-1}(p) - \not{q}\gamma_5], \\ P_{\psi_2}(p+q,p) &= -\frac{1}{N}[S_2^{-1}(p+q)\gamma_5 + \gamma_5 S_2^{-1}(p) - \not{q}\gamma_5], \\ P_{\phi}(p+q,p) &= -\frac{2}{N}[D^{-1}(p+q)\Xi - \Xi D^{-1}(p) \\ &\quad - q\cdot(2p+q)\Xi], \end{aligned} \tag{10}$$

where the normalization factor N is given by $N = I_{\psi_1}(0) + I_{\psi_2}(0) + I_{\phi}(0)$. As noted by Jackiw and Johnson [5], the effective vertices found this way are ambiguous at the order $O(q^2)$, since we have approximated the loop corrections to the full proper vertex functions $\Gamma^{\mu}_{A\psi_{1,2}}$ and $\Gamma^{\mu}_{A\phi}$ by their pole parts, and completely neglected other finite contributions.

To complete the calculation of the vertex functions, we must now plug the expressions (10) back into Eq. (9) and solve the resulting system of equations for $I_{\psi_{1,2}}(0)$ and $I_{\phi}(0)$.

IV. MODEL RESULTS

A. Effective vertices and loop integrals

The results obtained above are fairly general as we have used only very few and weak assumptions like the pole term dominance in the proper vertex functions. On the other hand, it is not easy to draw any concrete results from formulas like Eqs. (9) and (10). To push our conclusions a little bit further, we now make a simplifying assumption that will allow us to finish the calculation.

Since we are looking for spontaneous symmetry breaking, we shall neglect ordinary (symmetry-preserving) renormalization of the fermion and scalar propagators [6] and retain just the symmetry-breaking self-energies. This will enable us to proceed analytically as far as possible. Of course, as soon as one pretends at phenomenological relevance of the obtained results, all radiative corrections must be included, but this is not the aim of the present paper. Here we just wish to *demonstrate* that spontaneous symmetry breaking is possible in a model like Eq. (1).

We thus make the following ansatz for the fermion and scalar propagators:

$$\begin{aligned} S_{1,2}^{-1}(p) &= \not{p} - \Sigma_{1,2}(p), \\ D^{-1}(p) &= \begin{pmatrix} p^2 - M^2 & -\Pi(p) \\ -\Pi^*(p) & p^2 - M^2 \end{pmatrix}, \end{aligned} \tag{11}$$

where $\Sigma_{1,2}(p)$ are the Lorentz-scalar chirality-changing proper self-energies, and $\Pi(p)$ is the anomalous proper self-energy of the scalar field.

With this assumption, the effective vertices (10) become

$$\begin{aligned} P_{\psi_1}(p+q,p) &= -\frac{1}{N}[\Sigma_1(p+q) + \Sigma_1(p)]\gamma_5, \qquad P_{\psi_2}(p+q,p) = \frac{1}{N}[\Sigma_2(p+q) + \Sigma_2(p)]\gamma_5, \\ P_{\phi}(p+q,p) &= -\frac{2}{N}\begin{pmatrix} 0 & \Pi(p+q) + \Pi(p) \\ -\Pi^*(p+q) - \Pi^*(p) & 0 \end{pmatrix}. \end{aligned} \tag{12}$$

We can now go on to evaluate the last missing piece, that is, the normalization factors $I_{\psi_{1,2}}(0)$ and $I_{\phi}(0)$. We substitute for the propagators (11) and the effective vertices (12) in the loop integrals (9), which turn out to be parametrized in terms of the integrals (for the sake of readability, we put the arguments of $\Sigma_{1,2}$ and Π to the lower index),

$$\begin{aligned}
-iq^\mu J_{\psi_1}(q^2) &= 8\int \frac{d^4k}{(2\pi)^4}\frac{(k-q)^\mu \Sigma_{1,k}}{k^2-\Sigma_{1,k}^2}\frac{\Sigma_{1,k}+\Sigma_{1,k-q}}{(k-q)^2-\Sigma_{1,k-q}^2},\\
-iq^\mu J_{\psi_2}(q^2) &= 8\int \frac{d^4k}{(2\pi)^4}\frac{(k-q)^\mu \Sigma_{2,k}}{k^2-\Sigma_{2,k}^2}\frac{\Sigma_{2,k}+\Sigma_{2,k-q}}{(k-q)^2-\Sigma_{2,k-q}^2},\\
-iq^\mu J_{\phi}(q^2) &= 8\int \frac{d^4k}{(2\pi)^4}\frac{(2k-q)^\mu (k^2-M^2)}{(k^2-M^2)^2-|\Pi_k|^2}\frac{\mathrm{Re}[\Pi_{k-q}^*(\Pi_k+\Pi_{k-q})]}{[(k-q)^2-M^2]^2-|\Pi_{k-q}|^2}.
\end{aligned} \tag{13}$$

With these definitions, the expressions for $I_{\psi_{1,2}}(q^2)$ and $I_\phi(q^2)$ read

$$I_{\psi_{1,2}}(q^2) = \frac{J_{\psi_{1,2}}(q^2)}{N}, \qquad I_\phi(q^2) = \frac{J_\phi(q^2)}{N},$$

where the overall normalization factor of the vertices in Eq. (12) is equal to

$$N = \sqrt{J_{\psi_1}(0) + J_{\psi_2}(0) + J_\phi(0)}. \tag{14}$$

Let us finally note that the ansatz (11) also allows us to simplify the Schwinger-Dyson equations (6), whose solution will be the subject of the next section. With the same index notation as above in Eq. (13), Eqs. (6) become

$$\begin{aligned}
\Sigma_{1,p} &= iy_1^2\int \frac{d^4k}{(2\pi)^4}\frac{\Sigma_{1,k}}{k^2-\Sigma_{1,k}^2}\frac{\Pi_{k-p}}{[(k-p)^2-M^2]^2-|\Pi_{k-p}|^2},\\
\Sigma_{2,p} &= iy_2^2\int \frac{d^4k}{(2\pi)^4}\frac{\Sigma_{2,k}}{k^2-\Sigma_{2,k}^2}\frac{\Pi_{k-p}^*}{[(k-p)^2-M^2]^2-|\Pi_{k-p}|^2},\\
\Pi_p &= -\sum_{j=1,2} 2iy_j^2\int \frac{d^4k}{(2\pi)^4}\frac{\Sigma_{j,k}}{k^2-\Sigma_{j,k}^2}\frac{\Sigma_{j,k-p}}{(k-p)^2-\Sigma_{j,k-p}^2} + i\lambda\int\frac{d^4k}{(2\pi)^4}\frac{\Pi_k}{(k^2-M^2)^2-|\Pi_k|^2}.
\end{aligned} \tag{15}$$

B. Solution to the Schwinger-Dyson equations

We now wish to demonstrate that the Schwinger-Dyson equations (15) actually do have a nontrivial solution; that is, our mechanism is capable of generating fermion masses dynamically. To that end note that Eqs. (15) constitute a set of coupled nonlinear integral equations for the unknown functions $\Sigma_{1,p}$, $\Sigma_{2,p}$, and Π_p. This is still too complicated to deal with, and we therefore introduce further simplifications. We keep in mind that we are just attempting to break the chiral symmetry dynamically, and do not intend to produce any phenomenological conclusions at this stage.

First, we abandon the λ-term in the last of Eqs. (15). Formally, the advantage of this step is in the fact that Π is then expressed exclusively in terms of the Σ's. Physically, the symmetry is broken by the strong dynamics of the Yukawa interaction, while the scalar field quartic self-interaction can be later switched on perturbatively. This possibility is in marked contrast with the standard use of a condensing scalar. The $\lambda(\phi^\dagger\phi)$ term in the Lagrangian will become indispensable as a counterterm after including ordinary, symmetry-preserving quantum corrections.

Second, we consider for simplicity just the special case $y_1 = y_2$. The Lagrangian is then invariant under the discrete symmetry $\psi_1 \leftrightarrow \psi_2$, $\phi \leftrightarrow \phi^\dagger$. As far as the induced anomalous scalar self-energy is real, we may assume that this discrete symmetry is not spontaneously broken and the self-energies Σ_1 and Σ_2 are therefore equal. We are thus left with two coupled equations for Π and just one Σ.

The reduced set of equations may now in principle be solved iteratively. We performed a numerical calculation to estimate the order of magnitude of the generated fermion mass. We used the Euclidean approximation, that is, made a formal Wick rotation of the momenta in Eqs. (15). We made use of the fact that, after canceling the λ-term, the right-hand side of the last of Eqs. (15) depends just on the Σ's.

We took an initial ansatz for the Σ and used it to calculate the zeroth approximation for the Π. After then, we solved the two coupled equations iteratively. Our results are summarized by the graphs in Fig. 7.

We can see that the Eqs. (15) indeed possess a nontrivial solution. As far as we were able to check, this solution is unique in the sense that it is independent of the initial ansatz for the $\Sigma's$. The self-energies fall down rapidly once the momentum exceeds the bare scalar mass M, thus verifying our assumptions on the convergence of the loop integrals.

It would be perhaps more appropriate to do all the calculations in the Minkowski space as the physical mass lies in the timelike region of momenta. Such a calculation has been performed in Ref. [7].

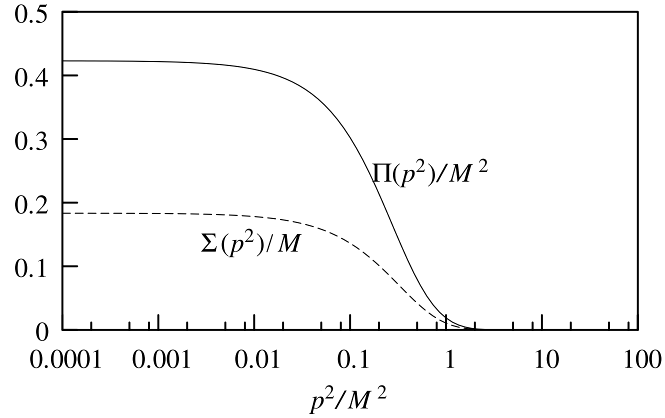

FIG. 7. Results of the numerical computation of the self-energies Σ and Π.

V. DISCUSSION AND CONCLUSION

Within simplifying assumptions stated in the text, we have demonstrated that a strong chirally invariant Yukawa interaction of massless fermions with a massive complex scalar field can generate the fermion masses by genuinely quantum (i.e. nonperturbative) loop effects. By the existence theorem this implies the massless pseudoscalar Nambu-Goldstone boson in the spectrum. In this respect our program is very much the same as that of the renown Nambu–Jona-Lasinio (NJL) paper [8].

We believe that a certain appeal of our suggestion is in the ultraviolet finiteness of nonperturbatively calculated quantities. It can be traced to the necessity of a generic coupling of the Schwinger-Dyson equations for the fermion and the scalar field propagators. It is definitely more subtle than the single Schwinger-Dyson equation for the fermion propagator with chirality conserving vector interactions. Technically the $\phi - \phi^\dagger$ mixing results in the difference of propagators of the scalar mass eigenstates and, consequently, in decent ultraviolet behavior of anomalous (symmetry-breaking) loop integrals.

Vexing assumptions of the present exploratory stage of the development of the model have to be replaced by better ones. In particular, it is desirable to have approximate analytic solutions $\Sigma(p^2)$ and $\Pi(p^2)$ in Minkowski space. The fermion masses $m_{1,2}$ are determined by solving $m_{1,2}^2 = \Sigma_{1,2}^2(p^2 = m_{1,2}^2)$ and by the dimensional argument the solution must have the form

$$m_{1,2} = Mf_{1,2}(y_{1,2}). \tag{16}$$

Preliminary numerical analysis in Euclidean space suggests that $\Sigma_{1,2} \to 0$ for $y_{1,2}$ approaching a (large) critical value. Our numerical calculation gives a rough estimate $y_{\text{crit.}} \approx 35$. The formula (16) is to be compared with $m_{1,2} = \frac{1}{\sqrt{2}} v y_{1,2}$ of the standard tree-level approach with a condensing scalar.

Possible uses of our model, if harshly justified, are also those of NJL in its contemporary interpretation:

(i) The apparently non-BCS-like form of the Schwinger-Dyson equations for the fermion propagator (6) is suggestive for modeling fermionic superfluidity with scalar effective degrees of freedom.

(ii) Non-Abelian generalization and gauging of the NJL model resulted in the past in models of the dynamical mass generation in $SU(2)_L \times U(1)_Y$ gauge-invariant electroweak models [9,10]. With our way of treating scalars we plan to follow the same path [11].

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Goldstone boson counting in linear sigma models with chemical potential

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We analyze the effects of finite chemical potential on spontaneous breaking of internal symmetries within the class of relativistic field theories described by the linear sigma model. Special attention is paid to the emergence of "abnormal" Goldstone bosons with a quadratic dispersion relation. We show that their presence is tightly connected to nonzero density of the Noether charges, and formulate a general counting rule. The general results are demonstrated on an SU(3) × U(1) invariant model with an SU(3)-sextet scalar field, which describes one of the color-superconducting phases of QCD.

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I. INTRODUCTION

Spontaneous symmetry breaking plays an important role in many areas of physics and encounters a host of fascinating phenomena. The most distinguishing feature of spontaneous symmetry breaking is the presence of soft modes, long-wavelength fluctuations of the order parameter(s), guaranteed by the Goldstone theorem [1,2].

For low-energy properties of the spontaneously broken symmetry it is important to know the number of the Goldstone bosons (GBs). While for spontaneously broken internal symmetry (space-time symmetries will not be the subject of this paper; see e.g. Ref. [3]) in a Lorentz-invariant field theory it is always equal to the number of broken-symmetry generators, the original Goldstone theorem predicts the existence of *at least one* GB. Indeed, there are several examples in nonrelativistic physics where the number of GBs is smaller than one would naively expect. The most profound one is perhaps the ferromagnet where the rotational SO(3) symmetry is spontaneously broken down to SO(2), but only one GB (the magnon) exists.

The issue of GB counting in nonrelativistic field theories was enlightened by Nielsen and Chadha [4]. They showed that the defect in the number of GBs is related to the low-momentum behavior of their dispersion relations. GBs with energy proportional to an odd power of momentum are classified as type I, and those with energy proportional to an even power of momentum as type II. The improved counting rule then states that *the number of GBs of type I plus twice the number of GBs of type II is greater or equal to the number of broken generators*.

It should be noted that the form of the dispersion law of the lightest degrees of freedom has important phenomenological consequences, e.g. for the low-temperature thermodynamics of the system. For instance, the heat capacity of a gas of bosons with $E \propto |\mathbf{p}|$ falls down as T^3 for $T \to 0$, while for bosons with $E \propto \mathbf{p}^2$ it is only $T^{3/2}$. If no massless particles are present, the heat capacity is suppressed by factor $e^{-m/kT}$, where m is the mass of the lightest particle.

The interest in the problem of GB counting has been revived recently, mainly thanks to the progress in understanding the phase diagram of quantum chromodynamics. At finite density Lorentz invariance is explicitly broken and GBs with nonlinear (as a matter of fact, generally quadratic) dispersion relations may appear even in a relativistic field theory as a medium effect [5,6]. Their presence turns out to be connected to the fact that some of the broken Noether charges develop nonzero density in the ground state, as has been observed in various color-superconducting phases of QCD [7,8] or in a neutron ferromagnet [9].

Schafer *et al.* [5] have proved the following theorem: if the commutators of all pairs of broken generators have a zero ground-state expectation value, then the number of GBs is equal to the number of broken generators. It is therefore clear that the nonzero charge density itself is not sufficient for a quadratic GB to appear. Indeed, the baryon number density does not cause any harm to the usual linear GBs in the color superconductors. The corresponding generator must rather be a part of a non-Abelian symmetry group. Our main goal is to show that the opposite to the theorem of Schafer *et al.* generally holds: *nonzero density of a commutator of two broken generators implies one GB with quadratic dispersion law.*

The paper is organized as follows. The following section is devoted to preparatory considerations: we explain how the quadratic GB is manifested in the Goldstone commutator and sketch its realization in the linear sigma model. In the next part, an example with an SU(3)-sextet condensation is investigated in detail. The general analysis is performed in the last section.

II. PRELIMINARY CONSIDERATIONS

In this section we shall investigate how the quadratic GBs come about, first at the rather general level of the Goldstone commutator and later more explicitly within the linear sigma model.

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A. Goldstone commutator

Let us briefly recall the proof of the Goldstone theorem. Following Ref. [4], we assume there is a local (possibly composite) field $\Phi(x)$ and a broken Noether charge Q such that $\langle 0|[\Phi(x), Q]|0\rangle \neq 0$. Inserting the complete set of intermediate states into the commutator, one arrives at the representation

$$\begin{aligned}\langle 0|[\Phi(x), Q]|0\rangle = \sum_{n=1}^{l}[&e^{-iE_{\mathbf{k}}t}\langle 0|\Phi(0)|n_{\mathbf{k}}\rangle\langle n_{\mathbf{k}}|j^0(0)|0\rangle \\ &- e^{iE_{-\mathbf{k}}t}\langle 0|j^0(0)|n_{-\mathbf{k}}\rangle\langle n_{-\mathbf{k}}|\Phi(0)|0\rangle] \\ \text{at } \mathbf{k} = 0,\end{aligned} \tag{1}$$

where the index n counts the GBs.

Now assume that we deal with a non-Abelian symmetry group and some of its charges have nonzero density in the ground state. Take as the GB field $\Phi(x)$ the zero component of the Noether current itself, so that $\langle 0|[j_a^0(x), Q_b]|0\rangle = if_{abc}\langle 0|j_c^0(x)|0\rangle$, where f_{abc} is the set of structure constants of the symmetry group. Should this be nonzero, we infer from Eq. (1) that both $\langle 0|j_a^0(0)|n\rangle$ and $\langle n|j_b^0(0)|0\rangle$ must be nonzero for some Goldstone mode n.

The point of the above heuristic argument is that while in Lorentz-invariant theories there is a one-to-one correspondence between the GBs and the broken currents, here a single GB couples to two Noether currents. This explains (not proves, of course) at a very elementary level how the GB counting rule is to be modified in the presence of nonzero charge density.

One should perhaps note that the Nielsen-Chadha counting rule is formulated in terms of the GB dispersion relations rather than charge densities. The connection between these two was clarified by Leutwyler [10], who showed by the analysis of the Ward identities for the broken symmetry that nonzero density of a non-Abelian charge induces a term in the low-energy effective Lagrangian with a single time derivative. The leading order effective Lagrangian is thus of the Schrödinger type and the energy of the GB is proportional to momentum squared.

B. Goldstone bosons within the linear sigma model

In order to elaborate more on the properties of the GBs, we restrict ourselves from now on to the framework of the linear sigma model, that is, a general scalar field theory with quartic self-interaction.

To see how the Goldstone commutator emerges in this language, recall the $SU(2) \times U(1)$ invariant model of Schafer *et al.* [5] and Miransky and Shovkovy [6]. The Lagrangian for the complex doublet field ϕ of mass M in Minkowski space reads

$$\mathcal{L} = D_\mu\phi^\dagger D^\mu\phi - M^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2.$$

Finite density of the statistical system is represented by the chemical potential μ, which enters the Lagrangian in terms of the covariant derivative [11], $D_\mu\phi = (\partial_\mu - i\delta_{0\mu}\mu)\phi$. Upon expanding the covariant derivatives, the Lagrangian becomes

$$\begin{aligned}\mathcal{L} = {}&\partial_\mu\phi^\dagger\partial^\mu\phi - 2\mu\,\mathrm{Im}\phi^\dagger\partial_0\phi + (\mu^2 - M^2)\phi^\dagger\phi \\ &- \lambda(\phi^\dagger\phi)^2.\end{aligned} \tag{2}$$

For $\mu > M$ the static potential develops a nontrivial minimum and the scalar field condenses. To find the spectrum of excitations at tree level we reparametrize it as

$$\phi = \frac{1}{\sqrt{2}}e^{i\pi_k\tau_k/v}\begin{pmatrix}0\\ v+\varphi\end{pmatrix}, \qquad v^2 = \frac{\mu^2 - M^2}{\lambda},$$

and look at the bilinear part of the Lagrangian. The crucial contribution comes from the term in Eq. (2) with one time derivative. Upon expanding the exponentials it yields among others the expression

$$\begin{aligned}-\frac{1}{2}\mu\,\mathrm{Im}&(0 \quad 1)[\pi_k\tau_k, \partial_0\pi_l\tau_l]\begin{pmatrix}0\\1\end{pmatrix} \\ &= \mu(\pi_1\partial_0\pi_2 - \pi_2\partial_0\pi_1).\end{aligned}$$

As will be made clear in the next subsection, it is this term that is responsible for the quadratic dispersion relation of one of the GBs. Its origin from the nonzero density of a commutator of two generators is now made obvious. This is the main idea to be remembered. The necessary technical details will come in the next two sections.

C. Bilinear Lagrangians and dispersion laws

Bilinear Lagrangians with single-time-derivative terms will frequently occur throughout the whole text. It is therefore worthwhile to fix once for all the corresponding excitation spectrum.

The bilinear Lagrangians we will encounter will have the generic form

$$\begin{aligned}\mathcal{L}_{\text{bilin}} = {}&\frac{1}{2}(\partial_\mu\pi)^2 + \frac{1}{2}(\partial_\mu H)^2 - \frac{1}{2}f^2(\mu)H^2 \\ &- g(\mu)H\partial_0\pi.\end{aligned} \tag{3}$$

The notation suggests that H is a massive (Higgs) mode whose mass function $f^2(\mu)$ depends on the chemical potential, while π is the Goldstone mode. The excitation spectrum is found from the poles of the two-point Green functions or, equivalently, by solving the condition

$$\det\begin{pmatrix}E^2 - \mathbf{p}^2 & +iEg(\mu)\\ -iEg(\mu) & E^2 - \mathbf{p}^2 - f^2(\mu)\end{pmatrix} = 0.$$

It turns out there is one massive mode, with dispersion relation

$$E^2 = f^2(\mu) + g^2(\mu) + \mathcal{O}(\mathbf{p}^2), \tag{4}$$

and one massless mode, with dispersion relation

$$E^2 = \frac{f^2(\mu)}{f^2(\mu) + g^2(\mu)}\mathbf{p}^2 + \frac{g^4(\mu)}{[f^2(\mu) + g^2(\mu)]^3}\mathbf{p}^4 + \mathcal{O}(\mathbf{p}^6). \tag{5}$$

Now if $f^2(\mu) > 0$, the Lagrangian (3) indeed describes a massive particle and a GB, whose energy is linear in momentum in the long-wavelength limit. On the other hand, when $f^2(\mu) = 0$, that is, when both π and H would correspond to linear GBs in the absence of the chemical potential, the dispersion relation of the gapless mode reduces to $E = \mathbf{p}^2/|g(\mu)|$. This is the sought quadratic Goldstone.

In conclusion, the term with a single time derivative, in general, mixes the original fields in the Lagrangian. Mixing of a massive mode with a massless one yields one massive particle and one linear GB; mixing of two massless modes results in a massive particle and a quadratic GB [12].

III. LINEAR SIGMA MODEL FOR SU(3)-SEXTET CONDENSATION

As a nontrivial demonstration of the general idea proposed in the previous section, we shall now analyze in detail a particular model of spontaneous symmetry breaking. Consider a scalar field Φ that transforms as a symmetric rank-two tensor under the group SU(3), $\Phi \to U\Phi U^T$. Such a field describes a one-flavor diquark condensate in one of the superconducting phases of QCD [13].

In addition to the SU(3) group, Φ is subject to U(1) transformations corresponding to quark number, $\Phi \to e^{i\theta}\Phi e^{i\theta} = e^{2i\theta}\Phi$. The most general $\mathrm{SU}(3) \times \mathrm{U}(1)$ invariant Lagrangian has the form

$$\mathcal{L} = \mathrm{tr}(D_\mu \Phi^\dagger D^\mu \Phi) - M^2 \,\mathrm{tr}\Phi^\dagger\Phi - a\,\mathrm{tr}(\Phi^\dagger\Phi)^2 - b(\mathrm{tr}\Phi^\dagger\Phi)^2. \tag{6}$$

The quark-number U(1) has been assigned chemical potential μ so that $D_0\Phi = (\partial_0 - 2i\mu)\Phi$. The parameters a, b are constrained by the requirement of boundedness of the static potential [13]. It is necessary that either both are non-negative (and at least one of them nonzero), or $a < 0$ and $b > |a|$, or $b < 0$ and $a > 3|b|$.

A. Minimum of the static potential

We start our analysis with a careful inspection of the static potential,

$$V(\Phi) = -(4\mu^2 - M^2)\mathrm{tr}\Phi^\dagger\Phi + a\,\mathrm{tr}(\Phi^\dagger\Phi)^2 + b(\mathrm{tr}\Phi^\dagger\Phi)^2. \tag{7}$$

A potential of the same type has been analyzed by Iida and Baym [14]. In their case, however, the global symmetry was different, and we therefore provide full details.

When $4\mu^2 - M^2 > 0$, the stationary point $\Phi = 0$ becomes unstable and a new, nontrivial minimum appears [15]. The stationary-point condition reads

$$\Phi(-4\mu^2 + M^2 + 2a\Phi^\dagger\Phi + 2b\,\mathrm{tr}\Phi^\dagger\Phi) = 0. \tag{8}$$

Before going into a detailed solution of this equation we note that by multiplying Eq. (8) from the left by $\Phi^\dagger$ and taking the trace, the stationary-point value of the potential (7) is found to be

$$V_{\mathrm{stat}} = -\frac{1}{2}(4\mu^2 - M^2)\mathrm{tr}\Phi^\dagger\Phi.$$

Any nontrivial stationary point of the potential is thus energetically more favorable than the perturbative vacuum $\Phi = 0$. We are, however, obliged to find a stable ground state, that is, the absolute minimum of the potential.

We now make use of the fact that the field Φ can always be brought by a suitable $\mathrm{SU}(3) \times \mathrm{U}(1)$ transformation to the standard form, which is a real diagonal matrix with non-negative entries [16]. Equation (8) then splits into three conditions and it is easy to see that all nonzero diagonal elements acquire the same value, denoted here by Δ.

Let there be n of them, $n = 1, 2, 3$. Equation (8) implies

$$\Delta^2 = \frac{1}{2}\frac{4\mu^2 - M^2}{a + bn}, \qquad V_{\mathrm{stat}} = -\frac{1}{4}\frac{(4\mu^2 - M^2)^2}{b + \frac{a}{n}}.$$

To find the absolute minimum of the potential, it remains to minimize this expression with respect to n.

For $a > 0$ the minimum occurs at $n = 3$, and Φ is proportional to the unit matrix, $\Phi = \Delta\mathbb{1}$, where

$$\Delta^2 = \frac{1}{2}\frac{4\mu^2 - M^2}{a + 3b}.$$

The $\mathrm{SU}(3) \times \mathrm{U}(1)$ symmetry is broken down to SO(3).

For $a < 0$ the potential is minimized by $n = 1$, that is, Φ is diagonal with a single nonzero entry and is conventionally chosen to be $\Phi = \mathrm{diag}(0, 0, \Delta)$, where now

$$\Delta^2 = \frac{1}{2}\frac{4\mu^2 - M^2}{a + b}.$$

The unbroken subgroup is now $\mathrm{SU}(2) \times \mathrm{U}(1)$.

For $a = 0$ the local minima corresponding to different n are degenerate since in that case, the Lagrangian (6) is invariant under an enhanced $\mathrm{SU}(6) \times \mathrm{U}(1)$ symmetry, treating Φ as a fundamental sextet. A nonzero ground-state expectation value of Φ breaks this symmetry to $\mathrm{SU}(5) \times \mathrm{U}(1)$. As we shall see, such an enhanced symmetry leads to an increased number of GBs with a quadratic dispersion relation [17].

B. Noether currents and charge densities

Having found the vacuum configuration of the scalar field, we are ready to reparametrize it and find the excitation spectrum from the bilinear part of the Lagrangian.

Before doing that, we evaluate the ground-state densities of the Noether charges in order to make *a priori* predictions about the nature of the GBs.

The infinitesimal SU(3) × U(1) transformation of Φ has the generic form $\delta\Phi = i\theta_k(\lambda_k\Phi + \Phi\lambda_k^T)$, where the λ_k stands for the Gell-Mann matrices ($k = 1, \dots, 8$) and the unit matrix ($k = 0$), respectively. The corresponding Noether currents are

$$j_k^\mu = -i\,\mathrm{tr}[D^\mu\Phi^\dagger(\lambda_k\Phi + \Phi\lambda_k^T) - \mathrm{H.c.}].$$

Taking a generic static field configuration to be $\Phi = \mathrm{diag}(\Delta_1, \Delta_2, \Delta_3)$ results in the charge densities

$$j_0^0 = 8\mu(\Delta_1^2 + \Delta_2^2 + \Delta_3^2),$$

$$j_3^0 = 8\mu(\Delta_1^2 - \Delta_2^2),$$

$$j_8^0 = \frac{8}{\sqrt{3}}\mu(\Delta_1^2 + \Delta_2^2 - 2\Delta_3^2).$$

In the SO(3) symmetric phase ($a > 0$), all generators but the U(1) quark number have zero density. As this is an Abelian generator, we expect six linear GBs corresponding to the six broken generators $\mathbb{1}, \lambda_1, \lambda_3, \lambda_4, \lambda_6, \lambda_8$. In the $a < 0$ case, the densities of λ_0 and λ_8 are nonzero. This means that the commutators $[\lambda_4, \lambda_5]$ and $[\lambda_6, \lambda_7]$ have nonzero ground-state density. With regard to the general discussion above, we thus expect two quadratic GBs corresponding to the pairs (λ_4, λ_5) and (λ_6, λ_7), and one linear GB of the generator λ_8.

C. The $a > 0$ case

We shall now proceed to the calculation of the mass spectrum of the $a > 0$ phase. We could do well with just shifting Φ by its vacuum expectation value $\Delta\mathbb{1}$, but this would complicate the identification of the massless modes. It is more convenient, and physical, to find such a parametrization that the GBs disappear from the static potential.

To that end, recall that the field $\Phi(x)$ (now coordinate dependent) can be brought to the diagonal form by a suitable SU(3) × U(1) transformation. In other words, it may be written as

$$\Phi(x) = e^{2i\theta(x)}U(x)D(x)U^T(x),$$

where $U(x) \in \mathrm{SU}(3)$ and $D(x)$ is real, diagonal, and nonnegative. Now the unitary matrix U can be (at least in the vicinity of unity) expressed as a product $U = VO$, $O \in \mathrm{SO}(3)$ being an element of the unbroken subgroup and V being built from the broken generators, $V = e^{i\pi_k\lambda_k}$, $k = 1, 3, 4, 6, 8$. A simple observation that $O(x)D(x)O^T(x)$ is the general parametrization of a real symmetric matrix leads to the final prescription,

$$\Phi(x) = e^{2i\theta(x)}V(x)[\Delta\mathbb{1} + \varphi(x)]V^T(x).$$

The real symmetric matrix φ contains six massive modes, while V contains five GBs. With θ this is altogether 12 degrees of freedom, as it should for Φ is a complex symmetric 3×3 matrix.

It is now straightforward, though somewhat tedious, to plug this parametrization into the Lagrangian (6) and expand to the second order in the fields. Omitting details of the calculations, we just report on the results.

The full static potential (up to a constant term—the vacuum energy density) becomes

$$V(\Phi) = 4\Delta^2[a\,\mathrm{tr}\varphi^2 + b(\mathrm{tr}\varphi)^2] + 4\Delta(a\,\mathrm{tr}\varphi^3 + b\,\mathrm{tr}\varphi\,\mathrm{tr}\varphi^2) + a\,\mathrm{tr}\varphi^4 + b(\mathrm{tr}\varphi^2)^2.$$

The bilinear Lagrangian turns out to be (we use the notation $V = e^{i\Pi}$)

$$\mathcal{L}_{\mathrm{bilin}} = 12\Delta^2(\partial_\mu\theta)^2 + 4\Delta^2\,\mathrm{tr}(\partial_\mu\Pi)^2 + \mathrm{tr}(\partial_\mu\varphi)^2 - 4\Delta^2[a\,\mathrm{tr}\varphi^2 + b(\mathrm{tr}\varphi)^2] - 16\mu\Delta[\partial_0\theta\,\mathrm{tr}\varphi + \mathrm{tr}(\varphi\partial_0\Pi)].$$

The kinetic terms are brought to the canonical form by a simple rescaling of the fields, upon which the spectrum is readily determined from Eqs. (4) and (5).

The excitations fall into irreducible multiplets of the unbroken SO(3) group. There are two singlets, stemming from the mixing of θ and $\mathrm{tr}\varphi$,

massive mode $$E^2 = 24\mu^2 - 2M^2 + \mathcal{O}(\mathbf{p}^2),$$

linear GB $$E^2 = \frac{4\mu^2 - M^2}{12\mu^2 - M^2}\mathbf{p}^2 + \mathcal{O}(\mathbf{p}^4),$$

and two 5-plets, the mixtures of $(\pi_1, \pi_3, \pi_4, \pi_6, \pi_8)$ and the traceless part of φ,

massive modes

$$E^2 = \frac{(24\mu^2 - 2M^2)a + 48\mu^2 b}{a + 3b} + \mathcal{O}(\mathbf{p}^2),$$

linear GBs

$$E^2 = \frac{(4\mu^2 - M^2)a}{(12\mu^2 - M^2)a + 24\mu^2 b}\mathbf{p}^2 + \mathcal{O}(\mathbf{p}^4).$$

It is easily seen from these formulas that the masses of the massive singlet and the massive 5-plet are connected by

$$m_{\mathbf{1}}^2 = m_{\mathbf{5}}^2 + (4\mu^2 - M^2)\frac{6b}{a + 3b} = m_{\mathbf{5}}^2 + 12\Delta^2 b.$$

The singlet is heavier than the 5-plet for $b > 0$ and vice versa.

The excitation spectrum is plotted in Fig. 1 for the case $M^2 > 0$. Below the phase transition to the Bose-Einstein-condensed phase, the medium-modified dispersion relations are simply $E = \sqrt{\mathbf{p}^2 + M^2} \pm 2\mu$. Right at the transition point, there are six modes with mass $2M$ and six

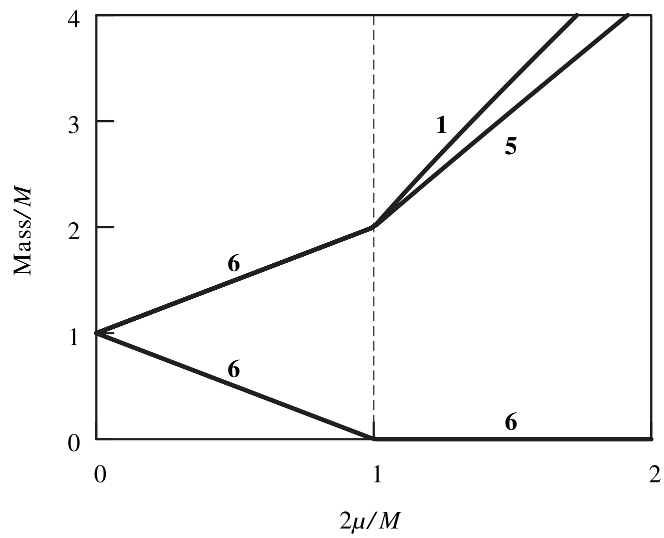

FIG. 1. Mass spectrum as a function of the chemical potential for $a > 0$. The boldface-typed numbers denote the degeneracies of the excitation branches. To obtain numerical results, particular values $a = b = 1$ were chosen.

massless ones with dispersion $E = \mathbf{p}^2/4\mu$. As the phase transition is second order, the dispersion relations of all excitation branches must be continuous functions of μ, that is, *all GBs become quadratic at the transition point*. This is also easily checked on the broken-symmetry side of the transition. As $2\mu \to M+$, the phase velocities of the linear GBs tend to zero, and their dispersions become quadratic.

Note that also for $a = 0$ the dispersion relation of the GB 5-plet becomes quadratic, $E = \mathbf{p}^2/4\mu$. This is in accordance with the enhanced SU(6) × U(1) symmetry of the Lagrangian. There are altogether 11 broken generators of the coset SU(6)/SU(5), one linear GB and five quadratic ones [forming now the 5-plet of the unbroken SU(5)], and the Nielsen-Chadha counting rule is thus satisfied.

D. The $a < 0$ case

We use the same method for parametrization of Φ as in the previous case. This time we write $\Phi(x) = U(x)D(x)U^T(x)$, where $U(x) \in \mathrm{SU}(3) \times \mathrm{U}(1)$. Next we perform the decomposition $U = e^{i\Pi}U'$, where $\Pi = \pi_k\lambda_k$, $k = 4, 5, 6, 7, 8$, and U' belongs to the unbroken subgroup SU(2) × U(1). Since $U'(x)D(x)U'^T(x)$ is block-diagonal with a complex symmetric 2×2 matrix in the upper-left corner, we arrive at the parametrization

$$\Phi(x) = e^{i\Pi(x)}[\mathrm{diag}(0, 0, \Delta) + \Sigma(x)]e^{i\Pi^T(x)},$$
$$\Sigma(x) = \begin{pmatrix} \sigma(x) & \\ & H(x) \end{pmatrix}.$$

Here H is a real field and σ is a complex symmetric 2×2 matrix. These two embody the massive modes that survive in the static potential,

$$\begin{aligned} V(\Phi) = {} & 4\Delta^2(a+b)H^2 - 2\Delta^2 a\,\mathrm{tr}\sigma^\dagger\sigma + 4\Delta(a+b)H^3 \\ & + (a+b)H^4 + 4\Delta bH\,\mathrm{tr}\sigma^\dagger\sigma + 2bH^2\,\mathrm{tr}\sigma^\dagger\sigma \\ & + a\,\mathrm{tr}(\sigma^\dagger\sigma)^2 + b(\mathrm{tr}\sigma^\dagger\sigma)^2. \end{aligned}$$

The bilinear part of the Lagrangian reads

$$\begin{aligned} \mathcal{L}_{\mathrm{bilin}} = {} & \mathrm{tr}(\partial_\mu\sigma^\dagger\partial^\mu\sigma) + (\partial_\mu H)^2 + 2\Delta^2(\partial_\mu\Pi\partial^\mu\Pi)_{33} \\ & + 2\Delta^2(\partial_\mu\Pi_{33})^2 - 4\Delta^2(a+b)H^2 \\ & + 2\Delta^2 a\,\mathrm{tr}\sigma^\dagger\sigma - 16\mu\Delta H\partial_0\Pi_{33} \\ & - 4\mu\Delta^2\,\mathrm{Im}[\Pi, \partial_0\Pi]_{33} - 4\mu\,\mathrm{Im}\,\mathrm{tr}\sigma^\dagger\partial_0\sigma. \end{aligned}$$

The excitations are again organized in multiplets of the unbroken SU(2) × U(1). H and π_8 mix to form two singlets,

$$\text{massive mode} \qquad E^2 = 24\mu^2 - 2M^2 + \mathcal{O}(\mathbf{p}^2),$$

$$\text{linear GB} \qquad E^2 = \frac{4\mu^2 - M^2}{12\mu^2 - M^2}\mathbf{p}^2 + \mathcal{O}(\mathbf{p}^4),$$

and the pairs (π_4, π_5) and (π_6, π_7) give rise to a doublet of massive modes and a doublet of massless ones,

$$\text{massive modes} \qquad E^2 = 16\mu^2 + \mathcal{O}(\mathbf{p}^2),$$

$$\text{quadratic GBs} \qquad E^2 = \frac{\mathbf{p}^4}{16\mu^2} + \mathcal{O}(\mathbf{p}^6).$$

The matrix σ represents two triplets of massive particles. The part of the bilinear Lagrangian containing σ may be rewritten as

$$\mathcal{L}_\sigma = \mathrm{tr}(D_\mu\sigma^\dagger D^\mu\sigma) - (4\mu^2 + 2\Delta^2|a|)\mathrm{tr}\sigma^\dagger\sigma,$$

which immediately implies the dispersion relations

$$E = \sqrt{4\mu^2 + 2\Delta^2|a|} \pm 2\mu + \mathcal{O}(\mathbf{p}^2).$$

The mass spectrum is shown in Fig. 2. The unbroken-phase part of the spectrum is the same as in the $a > 0$ case, since for $2\mu < M$ the *tree-level* masses of the particles do not depend at all on the quartic potential, i.e. the parameters a, b. Also, the same remark about the continuity of the dispersion relations across the phase transition applies.

Again, in the limit $a = 0$, the lighter of the two triplets in σ becomes a triplet of quadratic GBs, and joins the other two quadratic GBs to form the full SU(5) 5-plet.

To summarize our results, the theory described by the Lagrangian (6) has two different ordered phases, both occurring at $4\mu^2 > M^2$, distinguished by the symmetry of the ground state. The corresponding phase diagram in the (a, b) plane is displayed in Fig. 3.

As the excitations above the ordered ground state are grouped into irreducible multiplets of the unbroken symmetry, it is interesting to find out how the structure of these multiplets changes across the phase transition from one

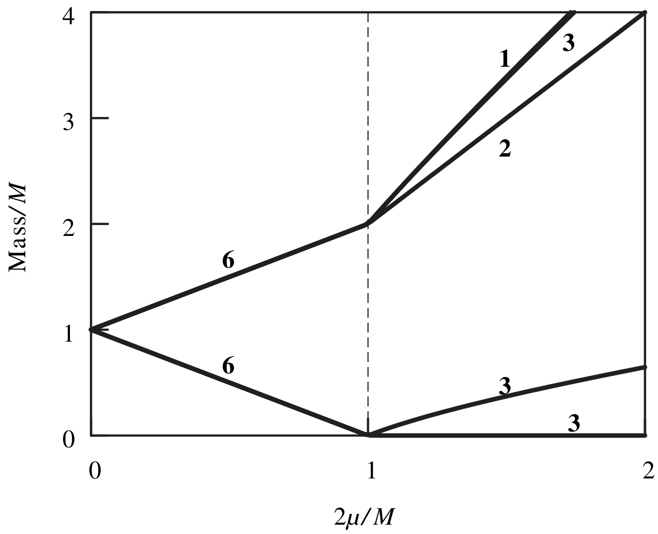

FIG. 2. Mass spectrum as a function of the chemical potential for $a < 0$. The singlet and triplet lines are so close that they almost coincide, but they are not degenerate. The spectrum is plotted for $a = -0.5$ and $b = 1$.

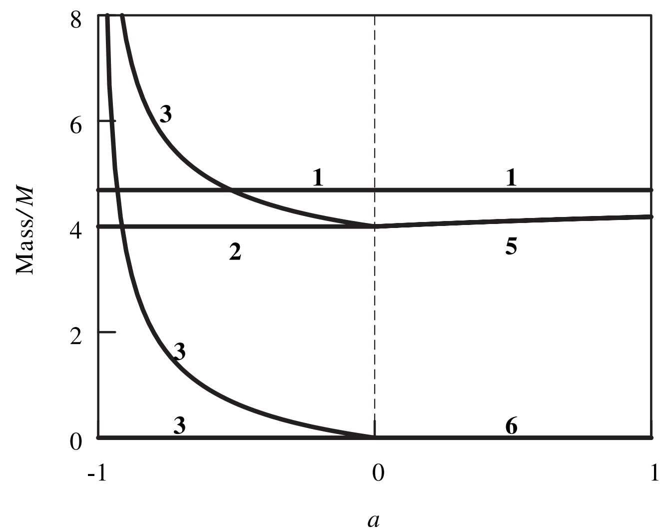

FIG. 4. Mass spectrum as a function of a. The graph is plotted for $\mu = M$ and $b = 1$. The potential is unstable for $a < -1$. The singular behavior of the masses of σ is due to divergence of Δ towards the stability limit of the potential.

ordered phase to the other. In Fig. 4 we show the dependence of the masses on the parameter a at constant chemical potential. The masses are continuous functions of a as the transition is second order.

As a final remark we note that in the original application of Ref. [13], the field Φ represented a diquark condensate and the SU(3) was the color gauge group of QCD. One might wonder whether the usual Higgs mechanism for gauge boson masses survives when there are fewer GBs than the number of broken generators, because of the presence of quadratic GBs. This question was answered affirmatively by Gusynin *et al.* [18], and there is therefore no need to worry about the fate of gluons.

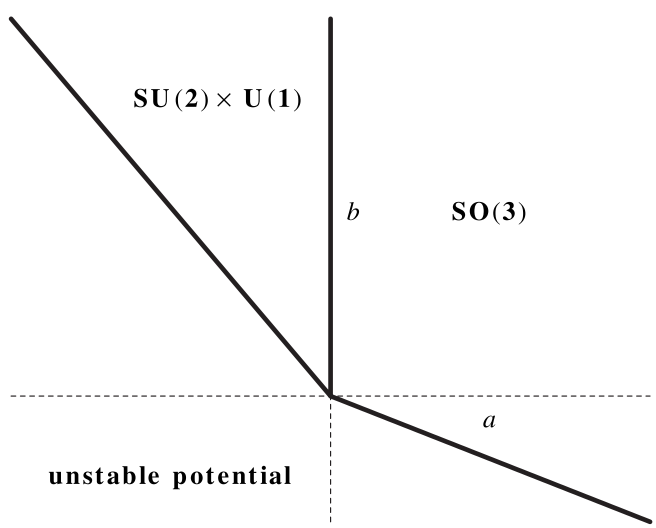

FIG. 3. Phase diagram of the linear sigma model for SU(3)-sextet condensation. The phases are labeled by the symmetry of the ground state. The line of the second order phase transition at $a = 0$, $b > 0$ has SU(5) × U(1) symmetry.

IV. GENERAL ANALYSIS

In this section we shall collect experience gained by solving particular examples and set out for a general analysis. We will find out, with some effort, that the ideas sketched in Sec. II and demonstrated in Sec.III have a straightforward generalization to a whole class of theories. It is understood, however, that we shall all the time stay in the framework of the linear sigma model, and at the tree level. The possibilities of further progress are discussed in the conclusions.

A. Chemical potential and global symmetry

As the starting point we shall address the question, what is the most general symmetry of a theory with nonzero chemical potential.

Let the microscopic theory possess a global continuous symmetry with the corresponding conserved Noether charges. The physical meaning of the chemical potential μ is that we wish to fix the statistical average of a conserved charge, say Q. This is technically achieved by introducing the grand canonical ensemble and replacing the microscopic Hamiltonian H with $H - \mu Q$.

It is now clear that by adding the chemical potential, we break explicitly all Noether charges that do not commute with Q. This is the technical realization of the physically intuitive fact that we cannot keep the values of two noncommuting operators (i.e. incompatible observables) simultaneously fixed.

This simple observation implies that, as far as *exact* symmetry is concerned, chemical potential is always assigned to a generator that commutes with all others, that is, to a U(1) factor of the exact global symmetry group.

Of course, when the symmetry of the microscopic theory is non-Abelian, then adding of the chemical potential generally produces a number of approximately conserved charges (at least for small μ) that generate approximate symmetries. These may also be spontaneously broken, resulting in the corresponding set of pseudo-Goldstone bosons. Throughout this paper we are, however, concerned only with true GBs, and therefore only the exact global symmetry will be considered.

It is also interesting to find out how the Abelian nature of the charge equipped with chemical potential is manifested in the Lagrangian formalism. There, as already mentioned, chemical potential enters the Lagrangian in terms of the covariant derivative of "matter" fields [11].

The Lagrangian can be made formally *gauge invariant* by introducing an external gauge field A_μ. Provided the matter fields ϕ transform under the symmetry group linearly as $\phi \to U\phi$, A_μ transforms as usual as $A_\mu \to UA_\mu U^{-1} + iU\partial_\mu U^{-1}$. Now the exact symmetry is such that the Lagrangian is invariant under the *global* transformation of the matter fields with A_μ *fixed* at $A_\mu = (\mu Q, 0, 0, 0)$. This is possible only when $A_\mu = UA_\mu U^{-1}$. We thus again arrive at the conclusion that the generator being assigned chemical potential must commute with all others.

B. Linear sigma model

Now consider a general linear sigma model defined by the Lagrangian

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - V(\phi). \tag{9}$$

Here ϕ denotes a set of complex [19] scalar fields that form a (possibly reducible) multiplet of the exact global symmetry group G, i.e. span the target space of a (possibly reducible) representation of G, say $\mathcal{R}$. $V(\phi)$ is the most general G-invariant static potential containing terms up to the fourth power of ϕ, and the covariant derivative is given by $D_\mu \phi = (\partial_\mu - iA_\mu)\phi$. A_μ is the constant external field that incorporates chemical potential for one or more U(1) factors of G, and is eventually set to $A_\mu = (\sum_i \mu_i Q_{i\mathcal{R}}, 0, 0, 0)$, where the Q_i's are the U(1) generators, the subscript $\mathcal{R}$ denoting the image in the representation $\mathcal{R}$.

Upon expanding the covariant derivatives Eq. (9) takes the form

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - 2\,\mathrm{Im}\,\phi^\dagger A^\mu \partial_\mu \phi - V_{\mathrm{eff}}(\phi), \tag{10}$$

the effective μ-dependent potential being $V_{\mathrm{eff}}(\phi) = V(\phi) - \phi^\dagger A^\mu A_\mu \phi$.

Spontaneous symmetry breaking occurs when $V_{\mathrm{eff}}(\phi)$ develops a nontrivial minimum at some $\phi = \phi_0$. In order to elucidate the physical content of such a theory, it is necessary to conveniently parametrize the field ϕ.

We stress the generality of the parametrization method suggested and applied in Sec. III. One first writes $\phi(x) = U_{\mathcal{R}}(x)\phi_{\mathrm{std}}(x)$, where ϕ_{std} is a standard form to which the field ϕ can always be brought by a suitable transformation $U \in$ G. Next $U_{\mathcal{R}}$ is factorized as $U_{\mathcal{R}} = e^{i\Pi} U'_{\mathcal{R}}$, where Π is a linear combination of the broken generators (or more precisely, their $\mathcal{R}$ images) and U' belongs to the unbroken subgroup H. The final step is to identify $U'_{\mathcal{R}}(x)\phi_{\mathrm{std}}(x)$ with a certain representation of H and parametrize it linearly as $\phi_0 + H(x)$. $H(x)$ is going to be the multiplet of massive (Higgs) fields. We therefore invoke the parametrization

$$\phi(x) = e^{i\Pi(x)}[\phi_0 + H(x)]. \tag{11}$$

In order to specify the transformation properties of H, recall that the GBs transform linearly in the adjoint representation of the unbroken subgroup [20], i.e. $\Pi \to U'_{\mathcal{R}} \Pi U'^{-1}_{\mathcal{R}}$ for any $U' \in$ H. As a consequence, $H = e^{-i\Pi}\phi - \phi_0$ transforms as $H \to U'_{\mathcal{R}} H$, since ϕ_0 is an H singlet.

To summarize, H transforms in the representation $\mathcal{R}$ truncated to the subgroup H, and the multiplets of the massive modes are therefore found in the decomposition of $\mathcal{R}$ into irreducible representations of H.

For instance, in our case $a > 0$ the symmetric rank-two tensor representation of SU(3) splits under the SO(3) subgroup into a traceless symmetric rank-two tensor and a singlet. On the other hand, in the $a < 0$ case it yields a symmetric rank-two tensor of SU(2) (the field σ) plus a singlet.

As an aside let us remark that the physical spectrum of the theory of course does not depend on the parametrization chosen for the field ϕ. What if we chose e.g. the linear parametrization mentioned (and abandoned) above in Sec. III C? Instead of Eq. (11), we would then have analogously

$$\phi(x) = \phi_0 + H(x) + i\Pi(x)\phi_0. \tag{12}$$

It is easy to see that the *bilinear* terms in the Lagrangian with one or two derivatives come out identical as for the parametrization (11). The reason is that the only difference stemming from the nonlinear structure of $e^{i\Pi}$ could possibly come in the form $\phi_0^\dagger A^\mu \partial_\mu \Pi^2 \phi_0$, but this is real and therefore it drops out of the Lagrangian (10).

The only difficulty with the linear parametrization (12) is that the GBs do not disappear automatically from the static potential. Instead, we have to use explicitly the G invariance to show that Π disappears from the *bilinear* (mass) part of the potential.

Upon the field redefinition as in Eq. (11), the effective potential $V_{\rm eff}$ becomes (up to a constant term)

$$V_{\rm eff}(\phi) = V(\phi_0 + H) - (H^\dagger A^\mu A_\mu H + 2\,{\rm Re}H^\dagger A^\mu A_\mu \phi_0).$$

As we are expanding the potential about its absolute minimum, the additional term linear in H is right enough to cancel a similar term coming from $V(\phi_0 + H)$. We are interested in the bilinear part of the potential, $V_{\rm bilin}(H)$, which determines the mass term for H.

Now we analyze the first two terms of the Lagrangian (10). The two-derivative term yields the bilinear contribution

$$\partial_\mu H^\dagger \partial^\mu H + \phi_0^\dagger \partial_\mu \Pi \partial^\mu \Pi \phi_0 + 2\,{\rm Im}\phi_0^\dagger \partial_\mu \Pi \partial^\mu H. \quad (13)$$

The first two terms in Eq. (13) are the expected kinetic terms for the Higgs and Goldstone fields, respectively. The GB term, however, asks for a check that it is nondegenerate.

Let $\Pi(x) = \pi_k(x) T_k$, T_k being the set of broken generators. The GB kinetic term becomes $\partial_\mu \pi_k \partial^\mu \pi_l \phi_0^\dagger T_k T_l \phi_0 = \frac{1}{2} \partial_\mu \pi_k \partial^\mu \pi_l \phi_0^\dagger \{T_k, T_l\} \phi_0$. The matrix $\phi_0^\dagger \{T_k, T_l\} \phi_0$ is real and symmetric and may be chosen, by taking an appropriate basis of broken generators, diagonal. It is obviously nondegenerate, as necessary in order to have kinetic terms for all the GBs, since otherwise $\phi_0^\dagger T_k T_k \phi_0 = 0$ for some T_k, implying that T_k is in fact not broken.

The third term in Eq. (13) eventually turns out to be zero. Nevertheless, as other terms of a similar structure will be dealt with in the following, we shall analyze it in detail. The crucial point is the way various fields transform under the unbroken subgroup H. Virtually all information about the structure of the bilinear Lagrangian may be obtained by a proper decomposition of the representation $\mathcal{R}$ into irreducible representations of H, and making repeated use of the Wigner-Eckart theorem.

Now when H and Π belong to different representations of H, the Wigner-Eckart theorem immediately tells us that the last term of Eq. (13) vanishes. There is, however, a subtle exception to this argument. As $\mathcal{R}$ is a complex representation, real representations of H are doubled in its decomposition. The reason is that when the set of vectors χ_k constitute the basis of a real representation of H, the vectors $i\chi_k$ form an independent basis of an equivalent representation.

It may be that H and Π (or $\Pi\phi_0$) are such doubles. This happens, for instance, for the two 5-plets in Sec. III C. In such a case, however, $\phi_0^\dagger \partial_\mu \Pi \partial^\mu H$ is real and, again, does not contribute to Eq. (13).

The single-derivative term in Eq. (10) gives, after a short manipulation, the bilinear terms

$$-2\,{\rm Im}H^\dagger A^\mu \partial_\mu H - 4\,{\rm Re}H^\dagger A^\mu \partial_\mu \Pi \phi_0 - {\rm Im}\phi_0^\dagger A^\mu [\Pi, \partial_\mu \Pi] \phi_0. \quad (14)$$

Throughout the calculation we made use of the fact that A^μ is a U(1) generator, and therefore commutes with Π.

Putting together all the pieces of Eqs. (13) and (14), we arrive at our main result—the bilinear Lagrangian for a general linear sigma model,

$$\mathcal{L}_{\rm bilin} = \partial_\mu H^\dagger \partial^\mu H - V_{\rm bilin}(H) - 2\,{\rm Im}H^\dagger A^\mu \partial_\mu H + \phi_0^\dagger \partial_\mu \Pi \partial^\mu \Pi \phi_0 - 4\,{\rm Re}H^\dagger A^\mu \partial_\mu \Pi \phi_0 - {\rm Im}\phi_0^\dagger A^\mu [\Pi, \partial_\mu \Pi] \phi_0. \quad (15)$$

This formula contains all the information about the particle spectrum of the theory, and the rest of the section is therefore devoted to its analysis.

C. Discussion of the results

There are altogether three terms with a single time derivative in Eq. (15). The term ${\rm Im}H^\dagger A^\mu \partial_\mu H$ causes splitting of the masses of the massive modes. The term ${\rm Re}H^\dagger A^\mu \partial_\mu \Pi \phi_0$ mixes massive and massless modes and, according to Sec. II C, produces linear GBs. Finally, the term ${\rm Im}\phi_0^\dagger A^\mu [\Pi, \partial_\mu \Pi] \phi_0$ mixes the Goldstone fields and gives rise to the quadratic Goldstones.

With the Wigner-Eckart theorem at hand it is easy to check that each of the elementary fields appears in at most one of the three single-derivative terms. This fact essentially reduces the analysis of the Lagrangian (15) to the model two-field problem discussed in Sec. II C.

To prove it, note that the mixing term ${\rm Re}H^\dagger A^\mu \partial_\mu \Pi \phi_0$ can be nonzero only when H and Π are the two copies of the doubled real representation of H. Now the real multiplet H gives real $H^\dagger A^\mu \partial_\mu H$, and therefore does not contribute to the mixing of the massive modes. The real multiplet Π analogously does not contribute to ${\rm Im}\phi_0^\dagger A^\mu [\Pi, \partial_\mu \Pi] \phi_0$ as a consequence of the analysis that follows.

As the main concern of this paper is Goldstone boson counting, we shall now concentrate on the last term of Eq. (15), which produces the quadratic GBs.

First, it is clear that our suspicion about the connection between the quadratic GBs and nonzero charge densities was right. For by the very same method as in Sec. III B we derive the Noether current corresponding to the conserved charge T,

$$j_T^\mu = -i(D^\mu \phi^\dagger T \phi - {\rm H.c.}),$$

and the ground-state density of T is

$$j_T^0 = 2\phi_0^\dagger A^0 T \phi_0. \quad (16)$$

The last term of Eq. (15) is therefore indeed proportional to the ground-state density of the commutator of two generators.

We may now, in the general case, proceed as in Sec. III, that is, find the ground state, calculate the Noether charge densities, and make a definite prediction for the particle spectrum. We can, however, do even better, at least a bit.

We need not calculate the charge densities explicitly to say which of the generators may produce quadratic GBs. It is obvious from Eq. (16) that only such a generator T may acquire nonzero density, which is a singlet of the unbroken subgroup H. We therefore just have to decompose the adjoint representation of G into irreducible representations of H and look for spontaneously broken singlets.

As in the examples above, we next choose such a basis that all the generators with nonzero density mutually commute. This ensures that they can be completed to form the Cartan subalgebra of the Lie algebra of G. Following the standard root decomposition of Lie algebras (see e.g. Ref. [21]), the rest of generators group into pairs whose commutator lies in the Cartan subalgebra. They are the lowering and raising operators or their Hermitian linear combinations, and together with their commutator span an SU(2) subalgebra of G.

The point of this procedure is that only pairs of Goldstone fields are then mixed by the single-derivative term $\mathrm{Im}\phi_0^\dagger A^\mu[\Pi, \partial_\mu\Pi]\phi_0$ and the excitation spectrum may be fully described with the help of the simple two-field bilinear Lagrangian (3). Consequently, the quadratic GBs count as one per each pair of generators whose commutator develops nonzero ground-state density.

The feasibility of such a pairing also follows from group theory and the Wigner-Eckart theorem. As the commutator of the two generators is to be an H singlet, they must come from the same irreducible representation of H.

To briefly conclude this section, we once again emphasize the fact that almost all we need to know about the excitation spectrum of the general linear sigma model (9) may be extracted from the bilinear Lagrangian (15) by simple group theory. We decompose the adjoint representation of G with respect to the unbroken subgroup H to determine the multiplet structure of the Goldstones. The remaining H multiplets in the decomposition of the representation $\mathcal{R}$ of the scalar field ϕ are the massive modes.

The quadratic GBs are discovered with the knowledge of the ground-state densities of the broken generators. Without further calculation, we can even determine their dispersion relations. Making use of the continuity of the dispersion relations across the phase transition and the known dispersion relations in the unbroken phase, we may assert that the quadratic GB dispersion relation is generically of the form $E = \mathbf{p}^2/2\mu Q$, where Q is the charge of the GB field under the U(1) subgroup equipped with the chemical potential.

V. CONCLUSIONS

We have analyzed spontaneous breaking of internal symmetries in the framework of the relativistic linear sigma model with finite chemical potential. Our prime motivation was to establish a counting rule for Goldstone bosons in view of the fact that explicit breaking of Lorentz invariance by medium effects may cause the number of GBs to differ from the number of broken-symmetry generators.

Our results confirm the Nielsen-Chadha counting rule. We show that the GBs have either a linear or quadratic dispersion law at low momentum, and that the number of the first plus twice the number of the second gives exactly the number of broken generators.

In addition, we find a criterion which gives in a purely algebraic way the number of quadratic GBs, the only necessary input being the structure of the ground state. *There is one quadratic GB for each pair of generators, whose commutator has nonzero ground-state density.*

However, despite the generality of our results, many open questions still remain. First, we stress the fact that we work all the time at the tree level. It would be interesting to know the effect of radiative corrections on the details of the spectrum. On the other hand, it seems that at least the dispersion relations of the quadratic GBs are rather generic as they depend only on the chemical potential in a very simple way. There might be a more robust, nonperturbative method to determine them, which relies only on the broken symmetry, and does not depend on the details of the dynamics of symmetry breaking.

Second, we worked within the linear sigma model as it is easy to manipulate perturbatively once the scalar field has been properly shifted to its new ground state. It may happen that our results are valid generally for relativistic theories with chemical potential. At least the argument presented in Sec. II A that clarifies the connection between the charge densities and the GB counting suggests such a possibility.

As adding chemical potential breaks Lorentz invariance in a very particular way, it might be possible to strengthen the Nielsen-Chadha counting rule at the cost of limiting its validity to a smaller class of theories. Even such a theorem would, however, find many applications on relativistic many-particle systems. We hope that our future work will help to find the answer to these questions.

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ON THE CHIRAL PERTURBATION THEORY FOR TWO-FLAVOR TWO-COLOR QCD AT FINITE CHEMICAL POTENTIAL

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We construct the chiral perturbation theory for two-color QCD with two quark flavors as an effective theory on the SO(6)/SO(5) coset space. This formulation turns out to be particularly useful for extracting the physical content of the theory when finite baryon and isospin chemical potentials are introduced, and Bose–Einstein condensation sets on.

Keywords: Two-color QCD; chiral perturbation theory; chemical potential.

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1. Introduction

The phase diagram of quantum chromodynamics (QCD) has attracted much attention in recent years. The region of high baryon density and low temperature is relevant for the description of deconfined quark matter, which can be found in the centers of neutron stars. It is expected to exhibit a variety of color-superconducting phases.[1]

Unfortunately, there is only very little firm knowledge concerning the behavior of the cold and dense quark matter. At very high densities, asymptotic freedom of QCD allows one to use weak-coupling methods to determine the structure of the ground state. On the other hand, the phenomenologically interesting region of densities corresponds to the strong-coupling regime where *ab initio* calculations within QCD are not available.

At the same time, current techniques of lattice numerical computations are not able to reach sufficiently high densities, due to the complexity of the fermionic Dirac operator that occurs in the Euclidean path-integral measure. This gave rise to the interest in QCD-like theories that are amenable to lattice simulations, in particular the two-color QCD with fundamental quarks and three-color QCD with adjoint quarks.[2,3]

It turns out that these theories may also be studied by means of a low-energy effective field theory similar to the chiral perturbation theory of QCD, and nontrivial information about their phase diagram thus obtained. Within this approach, the structure of the phase diagram has been investigated in detail, including the effects of finite temperature.[3–6] The model-independent predictions of the low-energy effective field theory have been complemented by lattice computations[7,8] and calculations within several models.[9–12]

The aim of this paper is to provide an alternative low-energy effective formulation of the simplest of this class of theories — the two-color QCD with two quark flavors.[a] While the general description of the whole class is based on the extended $\mathrm{SU}(2N_\mathrm{f})$ chiral symmetry of the underlying Lagrangian with N_f quark flavors, we construct the effective Lagrangian by exploiting the Lie algebra isomorphism $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$. We show that such a picture displays more transparently the physical content of the theory and at the same time allows for an easy determination of the true ground state, which has been sought by a convenient ansatz previously.

The paper is organized as follows. In Sec. 2 we summarize the basic features of two-color QCD to set the stage for the following considerations. Next we work out the mapping between the coset space SO(6)/SO(5) that we use, and the SU(4)/Sp(4) used in the literature. The rest of the paper is devoted to the construction of the effective Lagrangian and its detailed analysis.

2. Two-Color QCD

In this section we recall the basic properties of two-color QCD, following closely the treatment of Kogut *et al.*[3] The distinguishing feature of two-color QCD is the pseudoreality of the gauge group generators, the Pauli matrices, $T_k^* = -T_2 T_k T_2$. Consider now a set of N_f quark flavors in the fundamental representation of the gauge group. As an immediate consequence, we may trade the right-handed component of the quark field, ψ_R (flavor and color indices are suppressed), for the left-handed conjugate spinor $\tilde{\psi}_R = \sigma_2 T_2 \psi_R^*$ (the Pauli matrices σ_k act in the Dirac space).

Instead of the usual Dirac spinor, $\psi = (\,\psi_L \quad \psi_R\,)^\mathrm{T}$, we now work with the left-handed spinor, $\Psi = (\,\psi_L \quad \tilde{\psi}_R\,)^\mathrm{T}$, in terms of which the quark Euclidean Lagrangian of the massive two-color QCD at finite chemical potential becomes

$$\mathcal{L} = i\Psi^\dagger \sigma_\nu (D_\nu - \Omega_\nu)\Psi - m\left[\frac{1}{2}\Psi^\mathrm{T}\sigma_2 T_2 M \Psi + \mathrm{h.c.}\right]. \tag{1}$$

Here D_ν is the gauge-covariant derivative that includes the SU(2) gluon field. Ω_ν is the static uniform external U(1) gauge field that incorporates the chemical potential.[13] In the two-flavor case we shall deal with both the baryon number and the isospin chemical potential, μ_B and μ_I, respectively, so that Ω_ν will eventually

[a] The determinant of the Dirac operator is always real in two-color QCD, it is, however, positive only for an even number of flavors.

be set to $\Omega_\nu = \delta_{\nu 0}(\mu_B B + \mu_I I)$. Here,

$$B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad I = \frac{1}{2}\begin{pmatrix} \tau_3 & 0 \\ 0 & -\tau_3 \end{pmatrix} \tag{2}$$

are the baryon number and isospin generators, respectively. (The Pauli matrices τ_k act in the flavor space.) Finally,

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

denotes the mass matrix in the basis of the spinor Ψ and σ_ν stands for the four-vector of spin matrices, $\sigma_\nu = (-i, \sigma_k)$.

In the chiral limit and the absence of the chemical potential, the Lagrangian Eq. (1) is invariant under the extended global symmetry $\mathrm{SU}(2N_\mathrm{f})$, which includes the naive chiral group $\mathrm{SU}(N_\mathrm{f})_L \times \mathrm{SU}(N_\mathrm{f})_R$ and additional symmetry transformations due to the pseudoreality of the gauge group generators. The global symmetry is spontaneously broken by the standard chiral condensate down to its $\mathrm{Sp}(2N_\mathrm{f})$ subgroup.

The low-energy effective field theory for the Goldstone bosons of the broken symmetry is thus naturally constructed on the coset space $\mathrm{SU}(2N_\mathrm{f})/\mathrm{Sp}(2N_\mathrm{f})$. This is parametrized by an antisymmetric unimodular unitary matrix Σ, in terms of which the leading-order effective Lagrangian reads

$$\mathcal{L}_\mathrm{eff} = \frac{F^2}{2}\,\mathrm{Tr}(\nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger) - G\,\mathrm{Re}\;\mathrm{Tr}(J\Sigma)\,. \tag{3}$$

The ∇'s denote the covariant derivatives,

$$\nabla_\nu \Sigma = \partial_\nu \Sigma - (\Omega_\nu \Sigma + \Sigma \Omega_\nu^\mathrm{T})\,,$$

$$\nabla_\nu \Sigma^\dagger = \partial_\nu \Sigma^\dagger + (\Sigma^\dagger \Omega_\nu + \Omega_\nu^\mathrm{T} \Sigma^\dagger)\,,$$

while J serves as a source field for Σ, and is eventually set to mM. The quark mass m is connected to the Goldstone boson mass squared m_π^2 by the Gell-Mann–Oakes–Renner relation

$$mG = F^2 m_\pi^2\,.$$

It is worth emphasizing that the incorporation of the chemical potential into the effective theory involves no extra free parameters — the way the chemical potential enters the Lagrangian is fixed by the form of the covariant derivatives.

3. The SO(6)/SO(5) Coset Space

From now on we shall restrict our attention to the case $N_\mathrm{f} = 2$. In that case, note the Lie algebra isomorphisms $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$ and $\mathrm{Sp}(4) \simeq \mathrm{SO}(5)$. This allows us to recast the low-energy effective field theory on the SO(6)/SO(5) coset.[14] There are altogether five degrees of freedom, or Goldstone bosons, corresponding to the five independent entries of the antisymmetric unimodular unitary matrix Σ.

3.1. *Matrix basis*

The mapping to the SO(6)/SO(5) coset space is now provided by the formula

$$\Sigma = n_i \Sigma_i \,, \tag{4}$$

where $\mathbf{n}$ is a six-dimensional real unit vector and Σ_i is a convenient set of independent antisymmetric 4×4 matrices. For Σ to be unitary, the basis matrices must satisfy the constraint

$$\Sigma_i^\dagger \Sigma_j + \Sigma_j^\dagger \Sigma_i = 2\delta_{ij} \,. \tag{5}$$

Such a relation is fulfilled for instance by the matrices

$$\Sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}, \qquad \Sigma_3 = \begin{pmatrix} 0 & i\tau_1 \\ -i\tau_1 & 0 \end{pmatrix},$$

$$\Sigma_4 = \begin{pmatrix} i\tau_2 & 0 \\ 0 & -i\tau_2 \end{pmatrix}, \qquad \Sigma_5 = \begin{pmatrix} 0 & i\tau_2 \\ i\tau_2 & 0 \end{pmatrix}, \qquad \Sigma_6 = \begin{pmatrix} 0 & i\tau_3 \\ -i\tau_3 & 0 \end{pmatrix}.$$

This particular set has been chosen to comply with existing literature. In fact, Kogut *et al.*[3] use the notation Σ_c and Σ_d for our Σ_1 and Σ_2, respectively, while Splittorff *et al.*[4] denote our Σ_1, Σ_2 and Σ_3 by Σ_M, Σ_B and Σ_I, respectively.

Let us in addition show a simple argument that suggests how to choose in general a set of matrices satisfying Eq. (5). Recall that six independent antisymmetric *Hermitian* 4×4 matrices generate the real Lie algebra $\mathrm{SO}(4) \simeq \mathrm{SO}(3) \times \mathrm{SO}(3)$. This means that we deal with two sets of three matrices, which can be shown to fulfill the usual anticommutator of Pauli matrices, $\{\tau_i, \tau_j\} = 2\delta_{ij}$. By multiplying the matrices from one of the sets by i, we arrive at three Hermitian matrices, $H_i = \{\Sigma_2, \Sigma_3, \Sigma_6\}$, and three anti-Hermitian ones, $A_i = \{\Sigma_1, \Sigma_4, \Sigma_5\}$. These satisfy the relations

$$\{H_i, H_j\} = 2\delta_{ij} \,, \qquad \{A_i, A_j\} = -2\delta_{ij} \,, \qquad [H_i, A_j] = 0 \,,$$

that are equivalent to Eq. (5).

3.2. *Structure of the coset*

It remains to prove that Eq. (4) provides a one-to-one parametrization of the coset SU(4)/Sp(4). To that end, note that any antisymmetric 4×4 matrix U may be expanded in the basis Σ_i, $U = z_i \Sigma_i$, where z_i are in general complex coefficients. The unitarity of U constrains these coefficients as

$$1 = U^\dagger U = \sum_i |z_i|^2 + i \sum_{i \neq j} (x_i y_j - x_j y_i) \Sigma_i^\dagger \Sigma_j \,,$$

the x_i and y_i being the real and imaginary parts of z_i, respectively.

It is now crucial to observe that the products $i\Sigma_i^\dagger \Sigma_j$ for $i \neq j$ span the set of 15 linearly independent generators of SU(4) so that the unitarity of U requires separately $\sum_i |z_i|^2 = 1$ and $x_i y_j = x_j y_i$ for all pairs of i, j.

The latter condition means that the complex phases of all the z_i's must be equal so that $z_i = n_i e^{i\varphi}$ with real n_i, while the former one requires $\sum_i n_i^2 = 1$. It is a matter of simple algebra to calculate the determinant of U,

$$\det U = e^{4i\varphi} \left(\sum_i n_i^2 \right)^2 = e^{4i\varphi} \,.$$

Since the elements of the coset SU(4)/Sp(4) are unimodular matrices, we are left with two distinct possibilities, $\varphi = 0$ or $\varphi = \pi/2$. (The next solution, $\varphi = \pi$, already corresponds to $\varphi = 0$ with just the sign of all the n_i's inverted.)

In conclusion, every antisymmetric unimodular unitary matrix Σ may be cast in the form (4), where $\mathbf{n}$ is either real, or pure imaginary vector. However, in the standard coset construction of the effective Lagrangian,[15,16] the global symmetry group is required to act transitively on the parameter space of the Goldstone fields that is, the actual coset space must be connected. As the chiral condensate, above which we build our effective theory, is described by the matrix Σ_1 (note that $M = -\Sigma_1$), we have to choose the connected component with real $\mathbf{n}$, as in Eq. (4).

3.3. *Physical content of the basis matrices*

It is instructive to look at the transformation properties of the matrix Σ. This will allow us to classify the Goldstone modes by their baryon and isospin quantum numbers.

Recall that Σ is an antisymmetric tensor under SU(4) that is, it transforms as $\Sigma \to U\Sigma U^{\mathrm{T}}$ for $U \in \mathrm{SU}(4)$. For an infinitesimal transformation generated by the baryon number or the third component of the isospin we get

$$\delta_\varepsilon \Sigma = i\varepsilon(Q\Sigma + \Sigma Q^{\mathrm{T}}) = i\varepsilon\{Q, \Sigma\} \,, \qquad Q = B, I \,.$$

Let Σ be a general block matrix of the form $\left(\begin{smallmatrix} K & L \\ M & N \end{smallmatrix}\right)$. Then

$$\{B, \Sigma\} = \begin{pmatrix} K & 0 \\ 0 & -N \end{pmatrix} , \qquad \{I, \Sigma\} = \begin{pmatrix} \frac{1}{2}\{\tau_3, K\} & \frac{1}{2}[\tau_3, L] \\ -\frac{1}{2}[\tau_3, M] & -\frac{1}{2}\{\tau_3, N\} \end{pmatrix} .$$

The quantum numbers of the particular components of Σ are summarized in Table 1.

Table 1. Quantum numbers of the components of the matrix Σ in the expansion (4).

Σ_2, Σ_4	$B = \pm 1; I = 0$	diquark and antidiquark
$\Sigma_3, \Sigma_5, \Sigma_6$	$B = 0; I = \pm 1, 0$	isospin triplet $\boldsymbol{\pi}$
Σ_1	$B = 0; I = 0$	singlet σ

To gain more insight into the nature of the effective field Σ, let us assign to it a composite field,

$$\Sigma \to \frac{1}{2}\Psi^{\mathrm{T}}\sigma_2 T_2 \Sigma \Psi + \mathrm{h.c.}\,,$$

which corresponds to the form of the mass term in Eq. (1). Such a composite operator may be regarded as an interpolating field for the Goldstone boson.

With the explicit knowledge of the matrices Σ_i it is now straightforward to find the particle content of the corresponding interpolating fields, cf. also Table 1,

$$\Sigma_2 \to -\frac{1}{2}\psi^{\mathrm{T}} C\gamma_5 T_2 \tau_2 \psi + \mathrm{h.c.}\,, \qquad \Sigma_4 \to -\frac{1}{2} i\psi^{\mathrm{T}} C\gamma_5 T_2 \tau_2 \psi + \mathrm{h.c.}\,,$$

$$\Sigma_3 \to -i\bar{\psi}\tau_1\gamma_5\psi\,, \qquad \Sigma_5 \to i\bar{\psi}\tau_2\gamma_5\psi\,, \qquad \Sigma_6 \to -i\bar{\psi}\tau_3\gamma_5\psi\,, \qquad \Sigma_1 \to \bar{\psi}\psi\,.$$

4. Chiral Perturbation Theory

We are now ready to write down the leading-order effective Lagrangian and use it to analyze the phase diagram of the theory. First, we have to minimize the static part of the Lagrangian in order to determine the ground state at nonzero chemical potential.

4.1. *Global minimum of the static Lagrangian*

From Eq. (3) we can immediately infer the static part,

$$\mathcal{L}_{\mathrm{stat}} = -\frac{F^2}{2}\,\mathrm{Tr}[(\Omega_\nu\Sigma + \Sigma\Omega_\nu^{\mathrm{T}})(\Sigma^\dagger\Omega_\nu + \Omega_\nu^{\mathrm{T}}\Sigma^\dagger)] - G\,\mathrm{Re}\,\mathrm{Tr}(J\Sigma)\,. \tag{6}$$

We include the external source J in the general form

$$J = j_i \Sigma_i^\dagger\,,$$

with real j_i. Note that setting $j_1 = m$, we reproduce the quark mass contribution to the effective Lagrangian.

The other sources can be taken as infinitesimally small, since they essentially serve to generate the ground-state condensates,

$$\langle \Sigma_i \rangle = -\frac{\partial \mathcal{L}_{\mathrm{stat}}}{\partial j_i}\,.$$

From the orthogonality property, $\mathrm{Tr}(\Sigma_i^\dagger \Sigma_j) = 4\delta_{ij}$, we find $\mathrm{Re}\,\mathrm{Tr}(J\Sigma) = 4\mathbf{j}\cdot\mathbf{n}$ so that we have

$$\langle \boldsymbol{\Sigma} \rangle = 4G\mathbf{n}\,.$$

It is obvious that the vacuum condensate rotates on a sphere in the six-dimensional space, with coordinates corresponding to the six basis matrices Σ_i. It remains to calculate the vector $\mathbf{n}$ minimizing the static Lagrangian (6).

Note first that in the absence of chemical potential, $\Omega_\nu = 0$, the static Lagrangian is minimal when the condensate is aligned with the external source $\mathbf{j}$.

When baryon and isospin chemical potentials are switched on, we shall for simplicity assume that only the sources j_1, j_2, j_3 are present. It is sufficient to include the quark mass effects and calculate both the diquark and the isospin (pion) condensate. Taking into account the explicit form of the charge matrices, Eq. (2), the static Lagrangian becomes

$$\mathcal{L}_{\text{stat}} = -2F^2[\mu_B^2(n_2^2 + n_4^2) + \mu_I^2(n_3^2 + n_5^2)] - 4G(j_1 n_1 + j_2 n_2 + j_3 n_3)\,.$$

The first term is invariant under $\mathrm{SO}(2) \times \mathrm{SO}(2)$ rotations in the planes $(2,4)$ and $(3,5)$. In the absence of the external sources, this symmetry may be exploited to set $n_4 = n_5 = 0$. The source J breaks the symmetry and, in fact, prefers the solutions with $n_4 = n_5 = 0$. The problem of finding the ground state thus reduces to minimizing the expression,

$$\mathcal{L}_{\text{stat}} = -2F^2(\mu_B^2 n_2^2 + \mu_I^2 n_3^2) - 4G(j_1 n_1 + j_2 n_2 + j_3 n_3)\,, \tag{7}$$

on the sphere $S^5 : \mathbf{n}^2 = 1$.

The Lagrangian now does not depend on n_4, n_5, n_6 so that we are actually looking for a minimum on the ball, $n_1^2 + n_2^2 + n_3^2 \leq 1$. It is clear that at the *global* minimum, both terms on the right-hand side of Eq. (7) are negative (otherwise we could lower the energy by the inversion, $\mathbf{n} \to -\mathbf{n}$). By the same token, the global minimum must lie on the surface of the ball, since if this were not the case, we could lower the energy by scaling up the vector: $\mathbf{n} \to t\mathbf{n}, t > 1$.

We have thus shown that in the global minimum, $n_1^2 + n_2^2 + n_3^2 = 1$ and $n_4 = n_5 = n_6 = 0$, and the ground-state condensate is given by the linear combination $\Sigma = n_1\Sigma_1 + n_2\Sigma_2 + n_3\Sigma_3$. We stress the simplicity of the proof of this fact within the SO(6)/SO(5) coset formulation of the chiral perturbation theory. Indeed, using the standard SU(4)/Sp(4) formalism, Splittorff *et al.*[4] only assumed such a form of Σ, and also did not prove that the minimum thus found was global.

To demonstrate the power of the formalism we have built so far, we shall next rederive the results of Kogut *et al.*[3] for the case of nonzero baryon chemical potential μ_B. We shall thus set $\mu_I = 0$ and $j_i = \delta_{i1} m$. Isospin chemical potential can be introduced along the same lines and the results of Splittorff *et al.*[4] would be easily recovered.

With the assumptions made, the static Lagrangian becomes

$$\mathcal{L}_{\text{stat}} = -2F^2 m_\pi^2 (x^2 \sin^2\alpha + 2\cos\alpha)\,,$$

where $x = \mu_B/m_\pi$ and α parametrizes the minimum, $\Sigma = \Sigma_1 \cos\alpha + \Sigma_2 \sin\alpha$. (The same argument as above tells us that when $\mu_I = 0$ and $j_3 = 0$, then $n_3 = 0$ in the global minimum.)

Now when $x < 1$, the minimum occurs at $\alpha = 0$ — only the chiral condensate is nonzero, this is the normal phase. When, on the other hand, $x > 1$, the Lagrangian is minimized by $\cos\alpha = 1/x^2$. In this case, the chiral condensate rotates into the diquark condensate as the angle α increases. This is the Bose–Einstein condensation phase.

4.2. *Excitation spectrum*

The spectrum of excitations above the ground state is determined by the bilinear part of the Lagrangian. Expanding Eq. (3) in terms of the components n_i, it acquires the form

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= 2F^2(\partial_\nu \mathbf{n})^2 + 4iF^2\mu_B(n_2\partial_0 n_4 - n_4\partial_0 n_2) \\ &\quad - 2F^2\mu_B^2(n_2^2 + n_4^2) - 4F^2 m_\pi^2 n_1 \,. \end{aligned} \tag{8}$$

To proceed, we have to deal separately with the two phases of the theory.

4.2.1. *The normal phase*

When $x < 1$, the ground state expectation values of $\mathbf{n}$ are $n_1 = 1$ and all other components zero. The independent excitations above the ground state may be identified with $n_i, i = 2, \ldots, 6$, while n_1 is expressed in terms of them via the constraint $\mathbf{n}^2 = 1$,

$$n_1 = \sqrt{1 - \sum_{i=2}^{6} n_i^2} = 1 - \frac{1}{2}\sum_{i=2}^{6} n_i^2 + \text{higher order terms}\,.$$

The bilinear part of the Lagrangian (8) becomes

$$\begin{aligned}\frac{\mathcal{L}_{\text{bilin}}}{2F^2} &= \sum_{i=3,5,6} (\partial_\nu n_i)^2 + (\partial_0 N - \mu_B N)(\partial_0 N^\dagger + \mu_B N^\dagger) \\ &\quad + \nabla N \cdot \nabla N^\dagger + m_\pi^2 \sum_{i=2}^{6} n_i^2 \,,\end{aligned}$$

where we have introduced $N = n_2 + in_4$, a complex field that carries baryon number one. This field thus corresponds to the diquark, while $N^\dagger$ describes the antidiquark.

We find the following dispersion relations,

$$\begin{aligned} E &= \sqrt{\mathbf{p}^2 + m_\pi^2} && \text{pion triplet } n_3, n_5, n_6\,, \\ E &= \sqrt{\mathbf{p}^2 + m_\pi^2} - \mu_B && \text{diquark } N\,, \\ E &= \sqrt{\mathbf{p}^2 + m_\pi^2} + \mu_B && \text{antidiquark } N^\dagger\,. \end{aligned}$$

4.2.2. *The Bose–Einstein condensation phase*

For $x > 1$, the chiral condensate alone is no longer the proper ground state and the Bose–Einstein condensation sets. We therefore parametrize the field $\mathbf{n}$ as

$$\mathbf{n} = (\rho\cos\varphi, \rho\sin\varphi, n_3, n_4, n_5, n_6)\,.$$

The ground state corresponds to $\rho = 1$, $\varphi = \alpha$ and $n_3 = n_4 = n_5 = n_6 = 0$. We set $\varphi = \alpha + \theta$ so that the five independent degrees of freedom are now θ and

$n_i, i = 3, \ldots, 6$. The radial parameter ρ is given by

$$\rho = \sqrt{1 - \sum_{i=3}^{6} n_i^2} = 1 - \frac{1}{2}\sum_{i=3}^{6} n_i^2 + \text{higher order terms}.$$

The bilinear Lagrangian reads in this case,

$$\frac{\mathcal{L}_{\text{bilin}}}{2F^2} = \sum_{i=3}^{6}(\partial_\nu n_i)^2 + (\partial_\nu \theta)^2 + 2i\mu_B(\theta\partial_0 n_4 - n_4\partial_0\theta)\cos\alpha$$
$$+ \mu_B^2\left(\sum_{i=3,5,6} n_i^2 + \theta^2 \sin^2\alpha\right).$$

Three of the degrees of freedom, n_3, n_5, n_6, again represent the pion triplet, now with the dispersion relation $E = \sqrt{\mathbf{p}^2 + \mu_B^2}$. The dispersions of the remaining two excitations are obtained by a diagonalization of the inverse propagator in the (θ, n_4) sector. The result is

$$E_\pm^2 = \mathbf{p}^2 + \frac{\mu_B^2}{2}(1 + 3\cos^2\alpha) \pm \frac{\mu_B}{2}\sqrt{\mu_B^2(1 + 3\cos^2\alpha)^2 + 16\mathbf{p}^2\cos^2\alpha},$$

in accord with previous work.[3,4] The masses of these modes are given by

$$m_+^2 = \mu_B^2(1 + 3\cos^2\alpha) = \mu_B^2 + \frac{3m_\pi^2}{\mu_B^2}, \qquad m_-^2 = 0.$$

In contrast to the normal phase, there is always one truly massless Goldstone boson stemming from the exact baryon number U(1) symmetry, which is spontaneously broken by the diquark condensate Σ_2.

It is, however, worth emphasizing that the nature of this Goldstone boson, as well as of the massive mode, changes as the chemical potential increases. There are two reasons — the rotation of the ground state in the (n_1, n_2) plane, and the balance between the mass term in the bilinear Lagrangian and the term with a single time derivative.

In the limit $\alpha \to 0$, the parameter θ is, to the lowest order, equal to n_2 and the parametrization of the case $x < 1$ is recovered. Here the Goldstone boson is the diquark $N = \theta + in_4$.

As the angle α grows, the orientation of θ also changes in the (n_1, n_2) plane so that it is always perpendicular to the direction of the condensate, see Fig. 1. In the limit $\alpha \to \pi/2$, i.e. $\mu_B \gg m_\pi$, the condensate is purely diquark and θ has the quantum numbers of n_1, i.e. the σ field. The Goldstone boson is now n_4. Note also that it is a linear combination of the diquark and the antidiquark so that it has no definite baryon number. This is, of course, not surprising since the baryon number is spontaneously broken and thus it cannot be used to label the physical states.

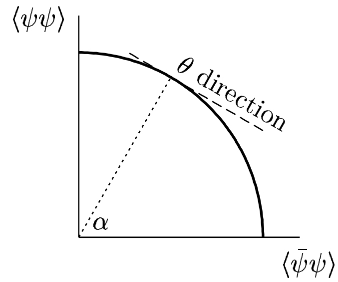

Fig. 1. The orientation of θ is perpendicular to the direction of the ground-state condensate, which is represented by the dotted line. The coordinates n_1, n_2 are labeled schematically by the chiral and the diquark condensate, respectively.

5. Conclusions

We have constructed the chiral perturbation theory for two-color QCD with two quark flavors on the SO(6)/SO(5) coset. We have provided an explicit mapping between this formulation and that used previously in literature, based on the SU(4)/Sp(4) coset space.

The virtue of the present approach is that the orthogonal rotations, in contrast to the unitary symplectic transformations, can be easily visualized and the physical content of the theory thus made manifest. We were also able to give a simple proof of the fact that the condensate taken previously as an ansatz is indeed the true ground state, and we thus justified the assumptions made in the older work.

Since the SO(N)-symmetric nonlinear sigma model is known in great detail, the connection provided here can hopefully lead to the improvement in the understanding of the phase diagram of the two-color QCD at low energies, at least in the two-flavor case.

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