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MASTER THESIS

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Variants of Petersen coloring for some graph classes

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague, May 4, 2015 Hana Bílková

Název práce: Varianty petersenovského obarvení pro některé třídy grafů

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Abstrakt: Normální obarvení – ekvivalentní verze petersenovského obarvení – je speciální dobré hranové obarvení kubických grafů pěti barvami. Každá hrana normálně obarveného grafu je normální, tj. používá spolu se svými čtyřmi sousedy pouze tři barvy nebo všech pět barev. Dle Jaegerovy hypotézy mají všechny kubické grafy bez mostů normální obarvení. Platnost hypotézy by dokázala například hypotézu Cycle double cover. Zde řešíme slabší verzi Jaegerova problému. Hledáme dobré hranové pěti-obarvení takové, že alespoň část hran je normální. Pro obecné hranoly (generalized prisms) ukážeme obarvení s dvěma třetinami normálních hran, pro grafy bez krátkých kružnic obarvení s necelou polovinou normálních hran. Dále navrhneme nový pohled na normální obarvení – řetízky (chains). Pomocí nich dokážeme tvrzení o nemožnosti výskytu právě jedné chyby ve skoro normálním obarvení a také několik tvrzení o řezech v normálně obarveném grafu plynoucí rovněž z nikde-nulového Petersenova toku. Nakonec prozkoumáme čtyřcyklus v normálně obarveném grafu.

Klíčová slova: grafy, cykly, nikde-nulové toky, hranová barvení

Title: Variants of Petersen coloring for some graph classes

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Abstract: Normal coloring — an equivalent version of Petersen coloring — is a special proper 5-edge-coloring of cubic graphs. Every edge in a normally colored graph is normal, i.e. it uses together with its four neighbours either only three colors or all five colors. Jaeger conjectured that every bridgeless cubic graph has a normal coloring. This conjecture, if true, imply for example Cycle double cover conjecture. Here we solve a weakened version of Jaeger's problem. We are looking for a proper 5-edge-coloring such that at least a part of the edges is normal. We show a coloring of generalized prisms with two thirds of the edges normal and a coloring of graphs without short cycles with almost half of the edges normal. Then we propose a new approach to normal coloring — chains. We use chains to prove that there cannot be only one single mistake in an almost normally colored graph. We also prove some statements about cuts in a normally colored graph which also follow from nowhere-zero Petersen flow. Finally, we examine a four-cycle in a normally colored graph.

Keywords: graphs, cycles, nowhere-zero flows, edge colorings

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1. Introduction

We will be concerned with Petersen coloring and primarily with an equivalent version of Petersen coloring — normal coloring. Normal coloring is a special 5-edge-coloring of cubic graphs — graphs with all vertices of degree three.

Jaeger conjectured in article [1] in 1985 that all bridgeles cubic graphs have a normal/Petersen coloring. This conjecture will be essential to us. Proof of Jaeger's conjecture would imply that also famous Cycle double cover conjecture and Berge-Fulkerson conjecture are true.[1] This indicates that Jaeger's problem is probably quite difficult to solve (and, unfortunately, this thesis does not contain a solution to it). Thus we will focus mostly on weakenings of this problem.

Natural weakenings are described at the end of this chapter after we define all crucial terms and formulate the important conjectures. In Chapter 2 we look more closely at normal coloring and some of its properties. We solve some weakenings of Jaeger's problem for some classes of cubic graphs in Chapter 3, the main result of this thesis is Theorem 3.2. Chapter 4 contains a brief summary of Petersen flow — a nowhere-zero flow that all graphs with a Petersen coloring have and vice versa. Petersen flow is a good tool to show some statements about the behaviour of normal coloring in a cut. We prove the statements about cuts and a little bit more using a new technique — chains — in Chapter 5. Chapter 6 is devoted to the study of small cycles, especially of a square, in a normally colored graph.

1.1 Definitions

For the standard definitions and theorems of graph theory see Diestel's book [4]. We start with the definition of Petersen coloring.

Definition. *A Petersen coloring of a cubic graph G is a mapping from the edges of G to the edges of Petersen graph — the graph in Figure 1.1 — such that every triple of adjacent edges in G is mapped to a triple of adjacent edges in Petersen graph. We denote Petersen graph by P.*

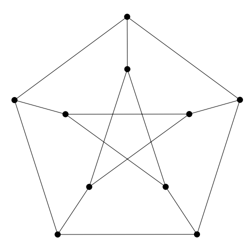

Figure 1.1: Petersen graph.

As Jaeger showed in [1], Petersen coloring is equivalent to normal coloring defined as follows (we prove the equivalence later).

[1] Cycle double cover conjecture was proposed independently by Seymour [7] (1977) and Szekeres [8] (1973). The other conjecture was made independently by Berge and Fulkerson [9] (1971).

Definition. *Consider a cubic graph G with a proper 5-edge-coloring c. We say that an edge is* rich *if the edge and its four neighbours have together five different colors (as many as possible). We call an edge* poor *if the edge with its neighbours uses just three different colors (as few as possible). The coloring c is* normal *coloring if every edge is either rich or poor. We say that an edge is* normal *if it is either rich or poor.*

Petersen graph can be constructed like this:[2] vertices of the graph are all 3-element subsets of a 5-element set. There is an edge between vertices u and v if and only if $|u \cap v| = 1$. If we take for the 5-element set the set of colors $\{1, 2, 3, 4, 5\}$ and color every edge (u, v) of Petersen graph with the color in $u \cap v$, we get a normal coloring n_p of Petersen graph where every edge is rich. See Figure 1.2.

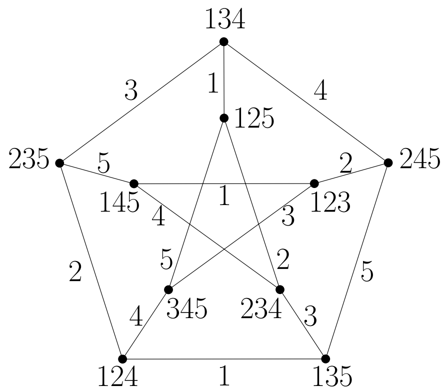

Figure 1.2: Petersen graph (constructed as a Kneser graph) with a normal coloring n_p.

We will use the normal coloring n_p of Petersen graph in the proof of equivalence of normal and Petersen coloring.

Theorem 1.1. (Jaeger) *A graph G has a normal coloring if and only if it has a Petersen coloring.*

Moreover, a normal coloring n of G can be defined from the Petersen coloring p of G as:

$$\forall e : n(e) = n_p(p(e))$$

where n_p is the normal coloring of Petersen graph from Figure 1.2.

And a Petersen coloring p can be defined from a normal coloring n as follows. We set $p(e) = f$ for $e = (u, v)$, $e \in E(G)$, $f \in E(P)$ if

- *e is colored with color c in the coloring n and f is colored with the same color c in the coloring n_p, and if*
- *e has neighbours on one side (adjacent either both to u or both to v) colored with c_1 and c_2 in n and f has neighbours on one side colored with the same colors c_1 and c_2 in n_p.*

[2]The construction originates from Kneser graphs — graphs whose vertices are k-element subsets of a n-element set and there is an edge between u and v iff corresponding subsets are disjoint. Petersen graph is a Kneser graph whose vertices are 2-element subsets of a 5-element set.

Proof. First consider a normal coloring n defined from a Petersen coloring p as above. The proper coloring condition is satisfied in n since every triple of adjacent edges is mapped by p to a triple of adjacent edges in Petersen graph and n_p assign to the adjacent triple three different colors. Consider an edge $e \in G$ and its neighbours. Denote the neighbours as in Figure 1.3.

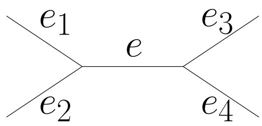

Figure 1.3: Notation for edges.

The triples of adjacent edges (e, e_1, e_2) and (e, e_3, e_4) can be mapped by p either both to the same triple in P, then e is poor in n, or to the neighbouring triples (e_1, e_2, e, e_3, e_4 are mapped to an edge of P and its four neighbours) and then the edge e is rich in n.

Now suppose we have a Petersen coloring p defined from a normal coloring n of G as above. There is exactly one edge $f \in E(P)$ for each $e \in E(G)$ satisfying the conditions from the theorem:

Consider an edge e colored with c and its neighbours colored as in Figure 1.4.

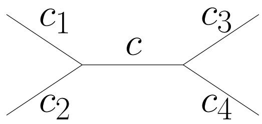

Figure 1.4: Notation for colors.

There is exactly one triple $\{t_1, t_2, t_3\}$ of adjacent edges in Petersen graph colored with $\{c, c_1, c_2\}$ in the coloring n_p and also exactly one triple $\{t'_1, t'_2, t'_3\}$ colored with $\{c, c_3, c_4\}$. If the edge e is poor in the coloring n, then $\{c, c_1, c_2\} = \{c, c_3, c_4\}$ thus $\{t_1, t_2, t_3\} = \{t'_1, t'_2, t'_3\}$ and the edge f for e is unique. If the edge e is rich in n, then $|\{c, c_1, c_2\} \cap \{c, c_3, c_4\}| = 1$ thus $\{t_1, t_2, t_3\}$ and $\{t'_1, t'_2, t'_3\}$ are neighbouring triples and the one edge from $\{t_1, t_2, t_3\} \cap \{t'_1, t'_2, t'_3\}$ is the unique edge f for e. (It does not matter whether we use the neighbours colored with c_1 and c_2 or the ones colored with c_3 and c_4 to find the edge f.)

A triple of adjacent edges colored with some colors c_1, c_2 and c_3 is mapped to a triple of adjacent edges as required — to the triple adjacent to the vertex $c_1c_2c_3$ (vertex of Petersen graph denoted by $c_1c_2c_3$ in Figure 1.2). □

Sometimes it is useful to look at Petersen coloring as a cycle-continuous mapping to Petersen graph. The definition of cycle-continuous mapping and the proof of equivalence of cycle-continuous mapping to Petersen graph and Petersen coloring follows.

Definition. *A cycle continuous mapping from a graph G_1 to a graph G_2 is a mapping from the edges of G_1 to the edges of G_2 such that the preimage of every cycle of G_2 is a union of cycles in G_1.*

Theorem 1.2. *A mapping from the edges of a cubic graph G to the edges of Petersen graph is a cycle-continuous mapping to Petersen graph if and only if it is a Petersen coloring.*

Proof. Suppose we have a graph G with Petersen coloring p. Consider a cycle C in Petersen graph. Preimage of the cycle C cannot contain a vertex v of degree three because the edges adjacent to v are mapped to a triple of adjacent edges in P and the cycle C does not have a vertex of degree three. Similarly the preimage cannot contain a vertex v' of degree one since the triple of edges adjacent to v' is mapped to an adjacent triple in P and there is no vertex of degree one in the cycle C. Thus there can be only vertices of degree either zero of two in the preimage of the cycle so it is a union of cycles. Therefore p is a cycle-continuous mapping.

Consider now a cycle-continuous mapping $m : E(G) \to E(P)$ for a cubic graph G. Assume, for a contradiction, that the triple of edges (e_1, e_2, e_3) adjacent to a vertex $v \in V(G)$ is not mapped to a triple of adjacent edges of P. There is no three-cut in P formed by non-adjacent edges so graph $P' = P \setminus \{m(e_1), m(e_2), m(e_3)\}$ is connected and thus there is a path from one vertex adjacent to $m(e_1)$ to the other one in P'. This path together with the edge $m(e_1)$ creates a cycle whose preimage contains vertex of degree one — v. Therefore the preimage is not a union of cycles. A contradiction. (See Figure 1.5.)

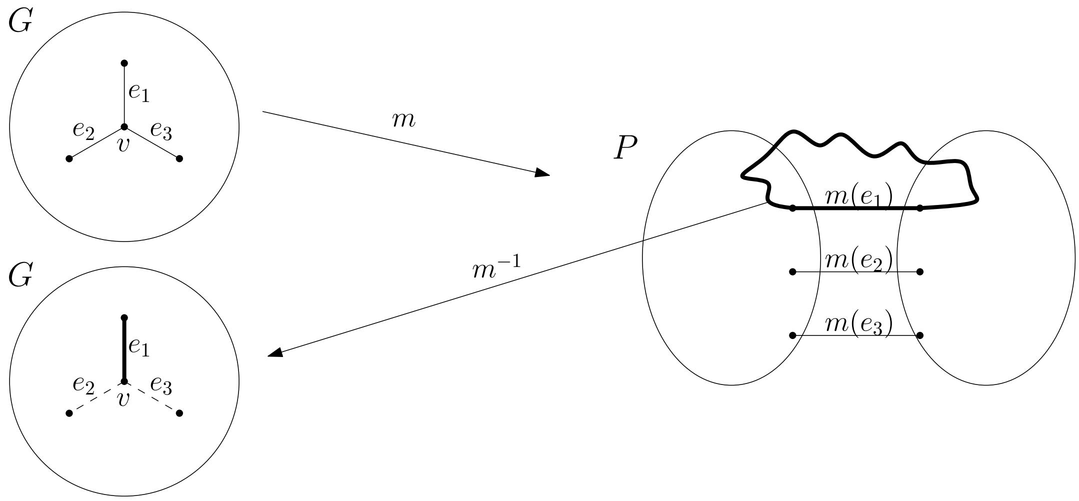

Figure 1.5: A mapping m that is not a Petersen coloring is not a cycle-continuous mapping to Petersen graph either.

□

1.2 Conjectures

In this text we will examine the following problem proposed by Jaeger in [1].

Conjecture 1.3. (Jaeger) *Every bridgeless cubic graph has a Petersen coloring.*

If Jaeger's conjecture was true, it would prove also Cycle double cover and Berge-Fulkerson conjecture.

Conjecture 1.4. (Cycle double cover) *For every bridgeless cubic graph G there is a set of cycles of G such that every edge of G is in exactly two of those cycles.*

Conjecture 1.5. (Berge-Fulkerson) *Every bridgeless cubic graph G has six perfect matchings such that every edge of G is in exactly two of those matchings.*

The idea of the proof of Berge-Fulkerson and Cycle double cover conjecture from Jaeger's conjecture is the following.

Idea of the proof. Let us consider a bridgeless cubic graph G. According to Jaeger's conjecture it has a Petersen coloring and thus also a cycle-continuous mapping c to Petersen graph. We find a set of cycles as in Cycle double cover conjecture for Petersen graph. The preimage of these cycles in the mapping c is a cycle double cover of graph G.

We proceed similarly with Berge-Fulkerson conjecture. We find six perfect matchings as in the conjecture for Petersen graph. Then we use a Petersen coloring of graph G to find the six perfect matchings of graph G that satisfy Berge-Fulkerson conjecture — the preimages of the six perfect matchings of Petersen graph in Petersen coloring are the six perfect matchings of G we are looking for.[3] □

1.3 Natural weakenings of Jaeger's conjecture

As Jaeger's problem 1.3 seems to be quite difficult, we will focus on weakenings of this problem.

In my bachelor thesis [6] I was trying to find for all cubic graphs a normal coloring that uses more than five colors. I.e., the condition that every edge together with its four neighbours uses three (poor edge) or five (rich edge) colors remains but altogether there are more than five colors in the graph. It is easy for ten colors since all cubic graphs have strong 10-edge-coloring in which all edges are rich. For bridgeless cubic graphs we get a normal coloring with seven colors from a theorem that all those graphs have nowhere-zero $\mathbb{Z}_2^3$-flow. In the bachelor thesis I found normal coloring with nine colors for graphs either with a bridge, a two-cut or with a triangle.

Here we will look for 5-edge-colorings where only a certain part of the edges is normal. Our aim is to make the normal part as big as possible. We will show how to color a specific class of cubic graphs — generalized prisms — in such a way that 2/3 of the edges will be normal. For cubic graphs without short cycles we will find a coloring with approximately half of the edges normal.

[3] The complement of a perfect matching of Petersen graph is a union of cycles that go through every vertex. Preimages of those cycles in Petersen coloring is a union of cycles of G that go through every vertex of G and the complement of those cycles is a perfect matching in G.

2. General properties of normal coloring

We start with some properties of normal coloring that we will use later and then we continue with properties that we will not use but that help us to get to know the normal coloring better.

First we look at a basic property of normal coloring that will be useful many times later.

Lemma 2.1. *Suppose there is a colored edge e with three neighbours colored as well, one uncolored neighbour and the proper coloring condition is satisfied. For every such partial coloring there is exactly one color for the last neighbour of e that makes the edge e a normal edge.*

Proof. Denote the color we are looking for by x and the known colors by $c_1, \dots, c_4$ as in Figure 2.1.

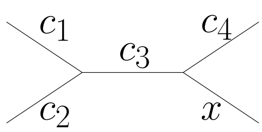

Figure 2.1: Notation for colors.

We distinguish the two following cases.

- If $c_4 \in \{c_1, c_2\}$, then we have to set x to the one color from $\{c_1, c_2\} \setminus c_4$. The edge e is poor.
- If $c_4 \notin \{c_1, c_2\}$, then we have to set x to the color from $\{1, 2, 3, 4, 5\} \setminus \{c_1, c_2, c_3, c_4\}$. In this case e is rich.

□

A particular consequence of Lemma 2.1 is that we cannot change a normal coloring at a single edge, keeping it normal.

For further use we define for every color c the following relation on pairs of colors.

Definition. *Let a, b, c, d, e be some colors. We define a relation $\sim_c$ on unordered pairs of colors. We write that $ab \sim_c de$ iff $c \notin \{a, b, d, e\}$, $a \neq b$, $d \neq e$ and either $\{a, b\} = \{d, e\}$ or $\{a, b\} \cap \{d, e\} = \emptyset$.*

Example 2.2. *We have $12 \sim_3 45$, $12 \sim_3 12$ but $12 \not\sim_3 24$.*

Note 2.3. *An edge colored as in Figure 2.2 is normal iff $ab \sim_c de$.*

Note 2.4. *The relation is equivalence since it is obviously reflexive, transitive and symmetric.*

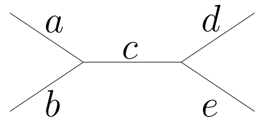

Figure 2.2: A normal edge supposing $ab \sim_c de$.

The next two statements are not important for further text but they help us to get used to normal coloring. The first statement tells us how we can change a normal coloring to obtain another one if there is an uniquely colored cut in the graph.

Theorem 2.5. *Let G be a normally colored graph. Denote the normal coloring by c. Assume that there is a cut $E' \subset E(G)$ of G such that for a color x and every edge $e \in E'$ we have $c(e) = x$ (the cut E' is colored with only one color x). Denote the colors different from x by $x_1, \ldots, x_4$ and the components of $G \setminus E'$ by G_1 and G_2.*

Let us define a new coloring c' of G from coloring c as follows:

- *If $c(e) = x$, then $c'(e) = x$.*
- *If $e \in G_2$, then $c'(e) = c(e)$.*
- *If $e \in G_1$ and*
  - *$c(e) = x_1$, then $c'(e) = x_2$.*
  - *$c(e) = x_2$, then $c'(e) = x_1$.*
  - *$c(e) = x_3$, then $c'(e) = x_4$.*
  - *$c(e) = x_4$, then $c'(e) = x_3$.*

The new coloring c' is also a normal coloring of G.

(I.e., we can pair arbitrarily the colors different from x, switch the colors in these pairs in one component of $G \setminus E'$ and the coloring remain normal.)

Proof. There is no change for edges of G_2 — their colors and colors of their neighbours remain the same — so they are still normal. For edges of G_1 the change is the same as it would be if we permuted colors in whole graph G. Thus the edges of G_1 also remain normal.

Finally let us have a look at the edges in E'. An edge $e \in E'$ has two neighbours e_1, e_2 in G_1 which have both changed its color and two neighbours e_3, e_4 in G_2 whose colors remained the same. Without loss of generality suppose $c(e) = 1$, $c(e_3) = 2$ and $c(e_4) = 3$. Then either $\{c(e_1), c(e_2)\} = \{2, 3\}$ or $\{c(e_1), c(e_2)\} = \{4, 5\}$. Thus $\{c'(e_1), c'(e_2)\}$ is either $\{2, 3\}$ or $\{4, 5\}$ if we switched 2 with 3 and 4 with 5; if we switched 2 with 4 and 3 with 5, then $\{c'(e_1), c'(e_2)\}$ is either $\{4, 5\}$ or $\{2, 3\}$ and the same holds for the case we switched 2 with 5 and 3 with 4. In all cases the edge e remain normal. ◻

The second statement is about the structure of a normally colored graph.

Theorem 2.6. *Consider a normally colored graph G. Let us create a new graph G' from G by removing all vertices that are adjacent to an edge colored with a color c. Every component of G' uses only three colors. (Obviously, there is no edge colored with c in G, but furthermore, none of the components uses all four remaining colors.)*

Proof. All rich edges were removed from the graph G since every rich edge is either colored with c or has an adjacent edge colored with c. And every path of poor edges uses together with all its adjacent edges only three colors so every component of G' uses also only three colors. There is an example of a part of graph with removed vertices adjacent to an edge with color 5 in Figure 2.3.

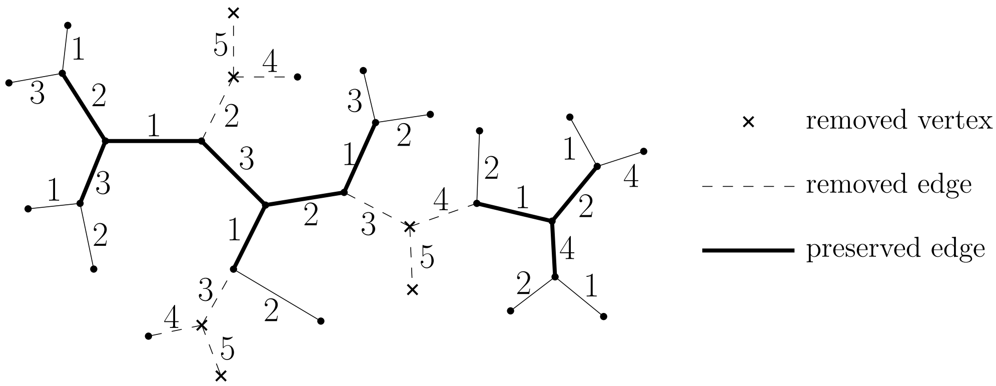

Figure 2.3: An example of a part of a graph with removed vertices adjacent to an edge colored with 5. On the left side there is a component colored with 1, 2 and 3, the component on the right side is colored with 1, 2 and 4.

□

3. Only part of the edges is normal

Let us investigate a weakening of Problem 1.3. What if we did not want every edge to be normal but just part of them?

We can quite easily get a proper edge-coloring of a bridgeless cubic graph G without cycles of length less than five (without a triangle or a square) with 1/3 of edges satisfying the "rich" property. Every bridgeless cubic graph has a perfect matching (this follows from Tutte's Theorem). By contracting the edges in the perfect matching we obtain a 4-regular graph G'. Since G contains neither triangle nor square, there are no multiple edges in G'. Thus we can use Vizing's Theorem to obtain a proper 5-edge-coloring of G'. Now we decontract the edges of the perfect matching and color every one of them with the color that none of its neighbours has. We clearly have a proper coloring where every edge of the perfect matching is rich. The number of edges of a perfect matching is $|E|/3$. The special cases where G has either a triangle or a square can be solved separately and we also get $|E|/3$ normal edges (in this case some of them might be poor).

We show how to obtain 2/3 of edges normal for a special class of graphs — generalized prisms — in the first section of this chapter. In the second section we color graphs without short cycles in such a way that approximately half of the edges is normal.

3.1 Generalized prisms

If we remove edges of a perfect matching from a cubic graph we get a graph consisting just of cycles. If there is only one cycle then it is a Hamiltonian cycle, the graph is 3-edge-colorable and in a 3-edge-coloring all edges are poor. Thus a natural class of cubic graphs for which we can examine Problem 1.3 is formed by generalized prisms.

Definition. *Generalized prism is a cubic graph which consists of two cycles of the same length with a perfect matching between them — see Figure 3.1. Let us call the edges on the two cycles cycle-edges and the edges of the perfect matching matching-edges.*

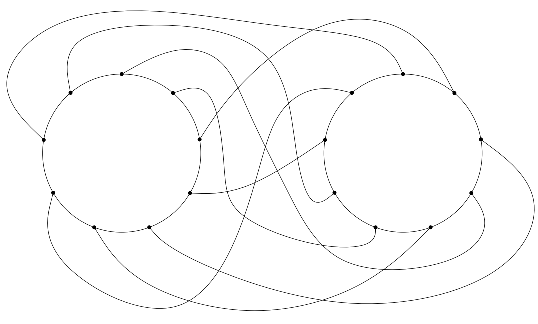

Figure 3.1: A generalized prism.

We will consider only generalized prisms with an odd length of the cycles since a prism with an even length of the cycles is 3-edge-colorable and thus all its edges can be easily poor.

We have already shown how to make all matching-edges rich. Now we prove that we can make all edges on one of the cycles normal (Theorem 2.3 in Šámal's article [3]). That gives us also $|E|/3$ good edges.

Theorem 3.1. (Šámal) *For every proper coloring of the first cycle we can color the rest of the generalized prism in such a way that it is a proper coloring and the edges on the first cycle are all normal.*

Proof. Suppose we have an odd cycle $C_1 = \{e_1, \ldots, e_{2k+1}\}$ and a proper edge-coloring of the cycle $c\colon E(C_1) \to \{1, \ldots, 5\}$. Let us denote the edge adjacent to e_i and e_{i+1} by f_i (and the edge adjacent to e_{2k+1} and e_1 by f_{2k+1}). See Figure 3.2.

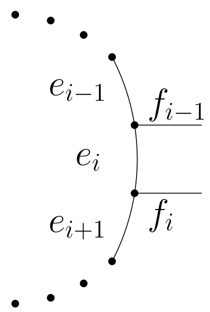

Figure 3.2: Notation for edges.

Note that as soon as we choose a color for f_1, there is (according to Lemma 2.1) only one color for f_2 that makes the edge e_2 normal. After the edge f_2 is colored, there is only one color for f_3 to make e_3 normal, etc. See an example in Figure 3.3.

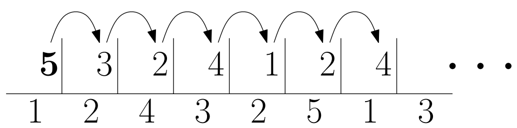

Figure 3.3: After we choose a color for f_1 — here the bold number 5 — the colors for all other f_i are given.

We have three options how to color f_1 — we can use the colors from $\{1, 2, 3, 4, 5\} \setminus \{c(e_1), c(e_2)\}$. Each of them gives us a coloring of $f_2, \ldots, f_{2k+1}$ in which the edges $e_2, \ldots, e_{2k+1}$ are normal. We need to prove that in at least one of them, the edge e_1 is normal too.

Without loss of generality, let us assume that $c(e_1) = 1$ and $c(e_2) = 2$. We create a 5-tuple P_i for every f_i as follows. For f_1 it will be $P_1 = (3, 4, 5, 1, 2)$ — the first three numbers represent colors that can be used for f_1 (so that we do not violate the proper coloring condition), the next two are the remaining colors — the first one is the color of the adjacent edge with an odd index (in this case e_1) the second one is the color of the adjacent edge with an even index (e_2). The other 5-tuples we derive from the previous one as follows.

The first number of P_i will be the color which we have to use for f_i to make the edge e_i normal in the case where f_{i-1} is colored with the first color from P_{i-1}.

Similarly we derive the second and third number of P_i from the second and third number of P_{i-1}.

The last two numbers of P_i will be, as in P_1, $c(e_i)$ and $c(e_{i+1})$ if i is odd, and $c(e_{i+1})$ and $c(e_i)$ if i is even.

An example of deriving 5-tuples is in Figure 3.4.

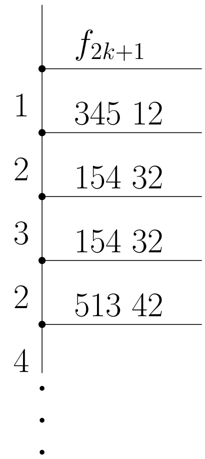

Figure 3.4: An example of deriving 5-tuples.

Now we can color $\{f_1, f_2, \dots, f_{2k+1}\}$ either all with the first color from corresponding 5-tuple or all with the second color or the third color, and the edges $e_2, e_3, \dots, e_{2k+1}$ will all be normal. The edge e_1 will be normal in one of these three colorings iff the hypothetical $(2k+2)$-nd 5-tuple P_{2k+2} derived from P_{2k+1} has one of the first three numbers the same as corresponding number of P_1. (If the first colors are equal then we use the first color for every matching-edge, etc.)

Finally, we prove that there is always a match in these three spots. All the 5-tuples are permutations of $\{1, 2, 3, 4, 5\}$. We will first show that they all have the same sign.

It is sufficient to show that the sign does not change in one step — that the signs of the $(i-1)$-st and the i-th permutation are the same. Let us distinguish two cases.

- If $c(e_{i-1}) = c(e_{i+1})$, then e_i has to be poor and thus the first three numbers of P_{i-1} are the same as that of P_i.

  Furthermore, the last two numbers of P_{i-1} are $c(e_{i-1})$, $c(e_i)$ (or $c(e_i)$, $c(e_{i-1})$ in the case where i is odd) and in P_i it is $c(e_{i+1})$, $c(e_i)$ (or $c(e_i)$, $c(e_{i+1})$) and that is the same. See Figure 3.5.

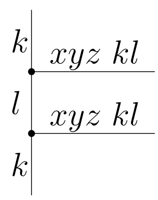

Figure 3.5: In the case $c(e_{i-1}) = c(e_{i+1})$ are permutations for f_{i-1} and f_i equal.

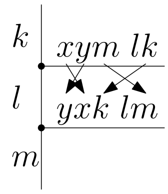

Figure 3.6: In the case $c(e_{i-1}) \neq c(e_{i+1})$ the permutations have still the same sign.

- The case $c(e_{i-1}) \neq c(e_{i+1})$ is in Figure 3.6.

  Without loss of generality $c(e_{i-1}) = k$, $c(e_i) = l$, $c(e_{i+1}) = m$ and x and y are the remaining colors. Putting $c(f_{i-1}) = x$ makes $c(f_i) = y$. If $c(f_{i-1}) = y$, then $c(f_i) = x$. And in the case $c(f_{i-1}) = m$ it is $c(f_i) = k$.

  So the colors x and y are switched in P_{i-1} and P_i. The color l is in the same spot in both 5-tuples (the fourth if i is odd and the fifth if i is even).

  Thus the color k is in the same spot in P_{i-1} as m is in P_i so there is another switch with the colors k and m.

  Therefore, there are two transpositions between these two permutations and the sign is also not changed.

Now P_1 is $(3, 4, 5, 1, 2)$ and the hypothetical $(2k + 2)$-nd 5-tuple P_{2k+2} is $(\_, \_, \_, c(e_{2k+3}), c(e_{2k+2}))$ that is $(\_, \_, \_, c(e_2), c(e_1))$ and that is $(\_, \_, \_, 2, 1)$. Thus there is one transposition in the fourth and fifth place. So there has to be an odd number of transpositions in the first three places. That gives us that P_{2k+2} begins with neither 345, 453 nor 534 and has to begin with either 354, 435 or 543 which gives us a match in either first, third or second place.

For the edges of the other cycle of the generalized prism we proceed greedily in choosing colors — for every edge is good any color that none of its neighbours has. □

But we can do more than $|E|/3$ with generalized prisms. With the following theorem we will have 2/3 of the edges normal.

Theorem 3.2. *Every generalized prism has a proper edge-coloring with all the cycle-edges normal.*

To obtain the result above we first color the matching-edges in such a way that it will be then possible to color the cycle-edges on both cycles. Since we do not want the matching-edges to be normal, there is no influence of the colors of the cycle-edges on the first cycle to the colors of the cycle-edges on the second cycle. Thus we can look at the cycles separately.

Definition. *Consider a graph G consisting of a cycle and an additional edge adjacent to each vertex of the cycle — one cycle and matching-edges (see Figure 3.7). We will call that graph a sun-graph.*

Lemma 3.3. *Suppose we have an edge-coloring c of a sun-graph G such that all cycle-edges are normal and it is a proper edge-coloring. Now we make a new graph*

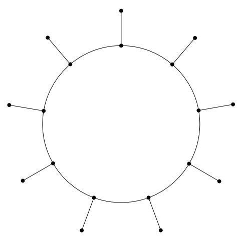

Figure 3.7: A sun-graph.

G' from G by replacing a cycle-edge e with five edges as shown in Figure 3.8 (the new graph has two additional matching-edges and two additional cycle-edges).

For every color $C \neq c(e)$ there is a coloring c' of G' such that it is still a proper edge-coloring with normal cycle-edges, the color of the old edges remains the same, and the two new matching-edges have the color C.

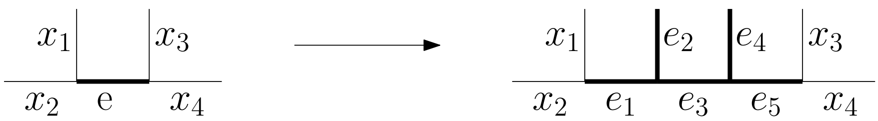

Figure 3.8: Adding new edges.

Proof. We define the new coloring c' as follows. For the old edges:

$$\forall f \in E(G), f \neq e : c'(f) = c(f)$$

as required and for the new edges:

$$c'(e_1) = c'(e_5) = c(e)$$

$$c'(e_2) = c'(e_4) = C(\neq c(e)).$$

We will find a color for the edge e_3 later.

The only problem may appear around the place where we added the new edges. Namely, we have to check whether the edges in Figure 3.8 are normal.

The proper coloring condition is satisfied so far. Edges x_2 and x_4 are still normal as they were before (they "see" the same colors as before). Edge e_3 will be poor as long as we satisfy the proper coloring condition after selecting a color for e_3. We will show that there is a color $c'(e_3)$ for e_3 such that e_1 is normal, the same color is also convenient for edge e_5 and neither $c'(e_3) = c(e)$ nor $c'(e_3) = C$ (which would violate the proper coloring condition).

It follows from Lemma 2.1 that there is a color X that satisfies $c'(x_1)c'(x_2) \sim_{c(e)} CX$. It also holds $c'(x_1)c'(x_2) \sim_{c(e)} c'(x_3)c'(x_4)$ since e was originally normal. And by transitivity of $\sim_{c(e)}$ we get $c'(x_3)c'(x_4) \sim_{c(e)} CX$. So setting $c'(e_3) = X$ is exactly what we need.

□

From this lemma we create "patterns". The basic idea is that we start with a small colored sun-graph and use the lemma to obtain coloring of bigger sun-graphs. Then we glue two colored sun-graphs together (we glue together matching-edges of the same color) and we get a coloring of a generalized prism.

We can use the lemma repeatedly for the sun-graph and we can glue two sun-graphs in different ways. Thus we will be able to color many generalized prisms using this tool. We describe two ways how to choose the original small colored sun-graph and apply Lemma 3.3. This leads to two "patterns"; we will show that we can find one of them in every generalized prism so all generalized prisms can be colored this way.

Pattern 1 is derived from the coloring of the sun-graph with 5 cycle-edges in Figure 3.9.

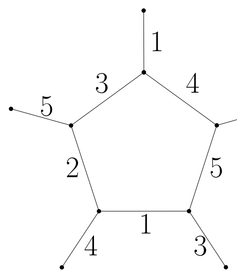

Figure 3.9: Graph for Pattern 1.

We can add any number of pairs of matching-edges colored with 5 to any place except between the matching-edges colored with 2 and 3 (because the cycle-edge here is colored with 5). Note that any of the new cycle-edges — the cycle-edges that are added by the operation from Lemma 3.3 — cannot get color C, in this case 5, so we can really add pairs repeatedly. Therefore we are able to color the cycle-edges in all the cases where matching-edges are colored as in Figure 3.10 (and also in the cases where the colors of matching-edges are as in Figure 3.10 only permuted — we will especially need to permute the colors $\{1, 2, 3, 4\}$).

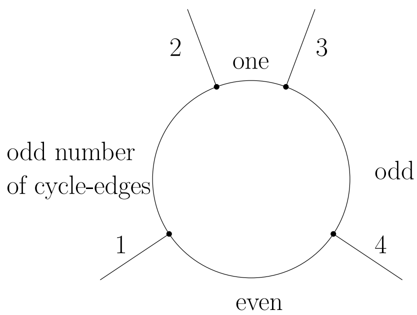

Figure 3.10: One side of Pattern 1 and a coloring of the matching-edges. All matching-edges that are not in the figure are colored with 5.

Now if we found in a generalized prism four matching-edges that satisfy the distance condition from Figure 3.10 on both sides (see Figure 3.11) we would be able to color the whole prism in such a way that the cycle-edges would be normal and it would be a proper edge-coloring.

Pattern 2

Let us proceed similarly with the graph in Figure 3.12.

We can add pairs of matching-edges colored with 5 anywhere here. From that we obtain the pattern in Figure 3.13.

Now we can prove the theorem.

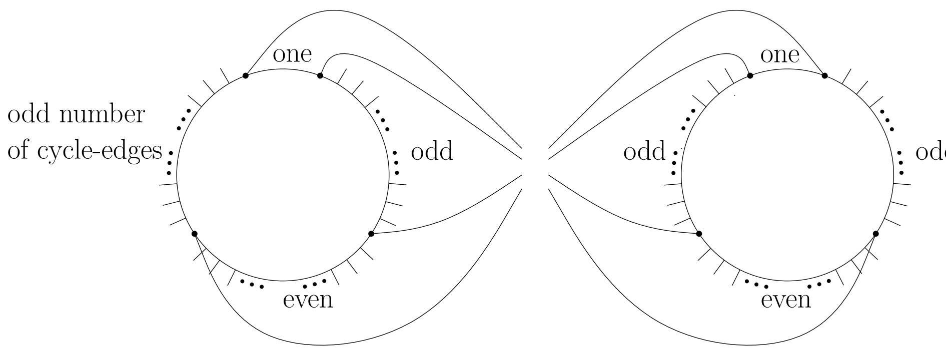

Figure 3.11: Pattern 1. In does not matter what is "in the middle" of the graph — which of the four matching-edges on the left side is which on the right side.

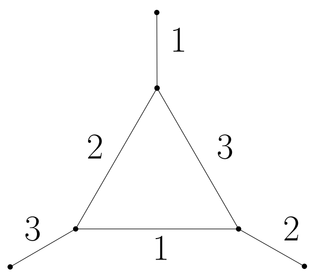

Figure 3.12: Graph for Pattern 2.

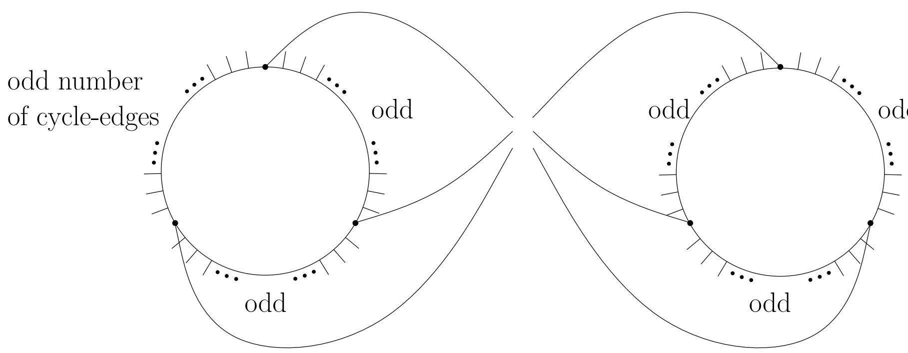

Figure 3.13: Pattern 2.

Proof of Theorem 3.2. We show that we can find either Pattern 1 or Pattern 2 in every generalized prism.

Let us pick any triple of matching-edges that are in a row on the first side of the graph. Note that distances between them on this side are 1, 1, odd[1] and that is odd, odd, odd. We look at the distances between those three edges on the second side. They are either odd, odd, odd — and we found Pattern 2 — or even, even, odd. In this case we will find Pattern 1. We take the edge displayed in Figure 3.14 and we have four matching-edges convenient for Pattern 1 on this side.

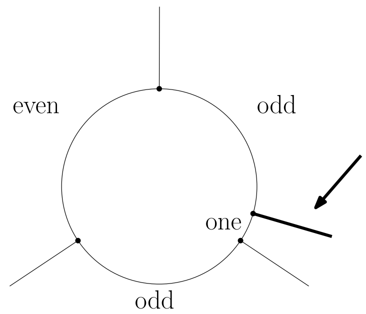

Figure 3.14: The last edge we need for Pattern 1. One of the former sections of even length has been split into section of length one and section of odd length on the right side.

On the other side we have one of the situations from Figure 3.15 which are both also good for Pattern 1 so we really found Pattern 1.

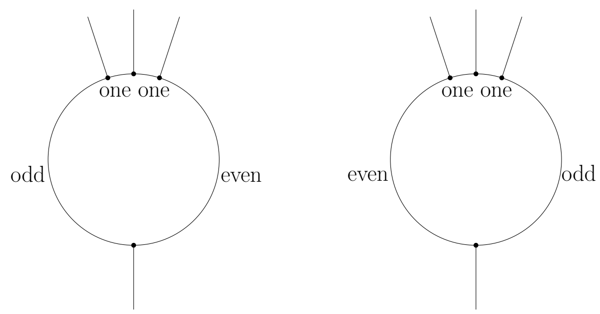

Figure 3.15: The situation on the first side of the graph after the addition of the fourth edge.

□

If we were able to extend this theorem to more than two cycles and allowed diagonals[2] in the graph, we would get a proof of the following conjecture proposed by Šámal (Conjecture 1.3 in [3]). It is a weakening of Jaeger's Conjecture.

[1]The distance still means the number of cycle-edges that are between some matching-edges. If we write about distances of three or more matching-edges we write the numbers of cycle-edges in the gaps between these matching-edges clockwise (or anticlockwise).

[2]Diagonal — a matching-edge that is joining two vertices from one cycle, not vertices from different cycles as in generalized prisms.

Conjecture 3.4. *Consider a bridgeless cubic graph G and a perfect matching of G. There exists a proper 5-edge-coloring of G such that all edges of G that are not in the perfect matching are normal.*

Also the question how to make even the matching-edges normal remains open. We could obtain some normal matching-edges from the previous proof. In the case we found Pattern 1, the three matching-edges that create the pattern are normal and also the thick matching-edges from Figure 3.16 are normal (the ones that are between the matching-edges colored with 1 and 3 on the right side of the prism).

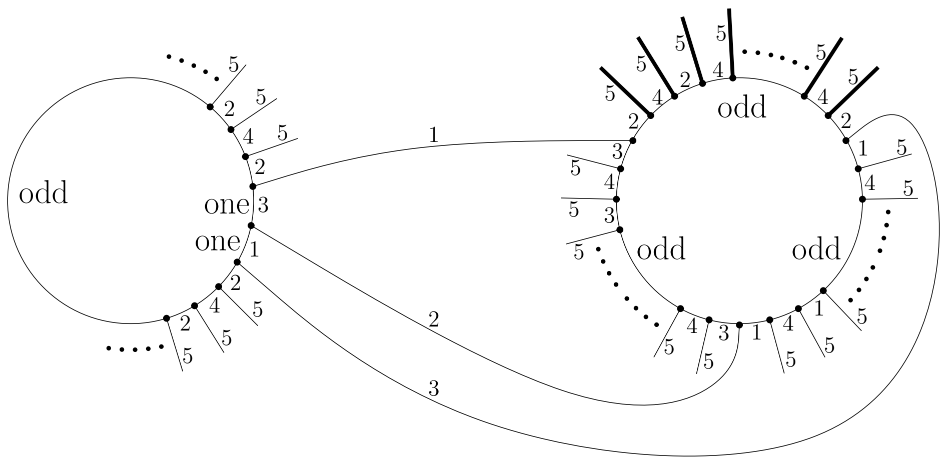

Figure 3.16: In the case we used Pattern 1, the three matching-edges colored with 1, 2, 3 and the thick matching-edges are normal.

However, there does not have to be any edge as the thick ones in Figure 3.16, so only the three normal matching-edges are assured. And in the case we used Pattern 2 we may get even less than three normal matching-edges. But we started the proof by choosing *any* triple of edges that are in a row. As there is as many possibilities as is the number of vertices on one cycle to do that, we can get a lot of proper edge-colorings with all the cycle-edges normal and maybe some of them will have even a lot of matching-edges normal.

3.2 Graphs without short cycles

In graphs with no short cycles we can find a proper 5-edge-coloring that has approximately half of the edges normal.

Lemma 3.5. *Consider a rooted tree T' where the root has three children, all other inner vertices have two children and leaves are all in the same depth d. (The induced subgraph of a cubic graph G without cycle of length $2d+1$ or less made by any vertex of G as a root and all the vertices at the distance up to d from that vertex.)*

We sort the edges of the tree into layers. In the first one there will be edges adjacent to the root, in the second one their children, etc. Denote the tree without the last layer by T.

For every coloring of two last layers of T' satisfying the proper coloring condition, we can color the rest of the tree in such a way that at least $\frac{|E(T)|-9}{2} + 3$ edges in T will be normal. (Edges in the last layer of T' do not have all their neighbours in T' so we cannot say whether they are normal or not.)

Proof. The plan is to color the tree from the leaves to the root, one layer at a time. We will maintain the proper coloring condition throughout the coloring process. We denote the coloring by c.

In the situation from Figure 3.17 (assuming that the colored part does not violate the proper coloring condition) we can always color the $(i-1)$-layer edges in such a way that at least two of the four i-layer edges are normal and the proper coloring rule remain preserved.

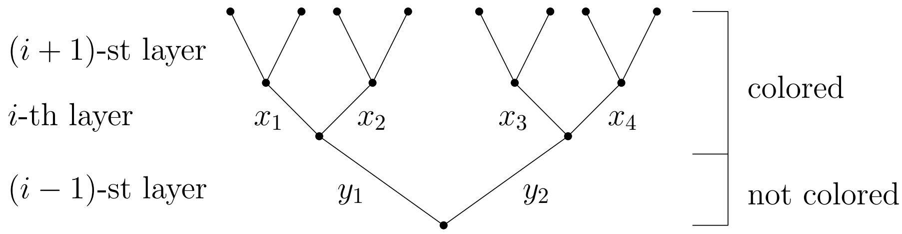

Figure 3.17: Coloring the rooted tree.

According to Lemma 2.1 there is one color c_1 for y_1 such that the edge x_1 is normal. Similarly there is one color c_2 for y_1 to make the edge x_2 normal and also color c_3 (c_4 respectively) for the edge y_2 that makes the edge x_3 (x_4 respectively) normal.

The choice of colors for y_1 and y_2 depends on c_1, c_2, c_3 and c_4.

- If $c_1 \neq c_2$ and $c_3 \neq c_4$, then we pick for y_1 color c_1 and for y_2 a color from $\{c_3, c_4\} \setminus c_1$. Which gives us two of the edges $\{x_1, x_2, x_3, x_4\}$ normal.
- If $c_1 = c_2$ and $c_3 \neq c_4$, we use color c_1 for y_1 and for y_2 a color from $\{c_3, c_4\} \setminus c_1$. That makes three of the edges $\{x_1, x_2, x_3, x_4\}$ normal (x_1, x_2 and either x_3 or x_4).
- We proceed similarly in the symmetric situation $c_1 \neq c_2$ and $c_3 = c_4$.
- Finally, we handle the case $c_1 = c_2$ and $c_3 = c_4$. If moreover $c_1 \neq c_3$, we can make all four edges normal by setting color of y_1 to c_1 and color of y_2 to c_3. If $c_1 = c_3$, then we also color the edge y_1 with c_1 so both x_1 and x_2 are normal. For edge y_2 we choose color from $\{1, 2, 3, 4, 5\} \setminus \{c_1, c(x_3), c(x_4)\}$ just to preserve the proper coloring condition.

We use these rules until the second layer. We will show that the last three uncolored edges in the first layer can be colored in such a way that at least three from last nine edges are normal. That gives us the number $\frac{|E(T)|-9}{2} + 3$ from above.

It is left to show how to select colors for the last three edges.

Let us denote the edges as in Figure 3.18.

All the edges e_i^j are already colored. Our goal is to color f_1, f_2 and f_3 in such a way that the proper coloring condition is satisfied and at least three of the

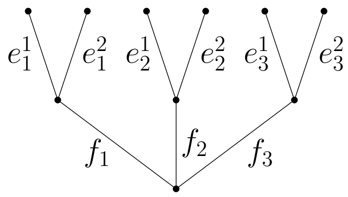

Figure 3.18: Coloring the last three edges — notation.

edges $\{e_1^1, e_1^2, e_2^1, e_2^2, e_3^1, e_3^2, f_1, f_2, f_3\}$ are normal. According to Lemma 2.1 there is exactly one color c_i^j for the edge f_i to make the edge e_i^j normal.

We distinguish the cases according to the number of i for which $c_i^1 = c_i^2$.

- If there is at least one i such that $c_i^1 = c_i^2$ and at least one k that $c_k^1 \neq c_k^2$, then we color f_i with the color c_i^1, f_k with a color from $\{c_k^1, c_k^2\} \setminus \{c_i^1\}$ and the remaining edge with any color that none of its neighbours has. And we have at least three edges normal (e_i^1, e_i^2 and either e_k^1 or e_k^2).

- If for all i we have $c_i^1 \neq c_i^2$, we distinguish whether the three unordered pairs $\{c_i^1, c_i^2\}$ are all the same or not.

  If they are not all the same, then there is a color c_i^x and an index j such that $c_i^x \notin \{c_j^1, c_j^2\}$. Without loss of generality we may assume $c_1^1 \notin \{c_2^1, c_2^2\}$. Then we put $c(f_1) = c_1^1$, $c(f_3) \in \{c_3^1, c_3^2\} \setminus \{c_1^1\}$ and $c(f_2) \in \{c_2^1, c_2^2\} \setminus \{c(f_3)\}$. We get one of edges e_i^1, e_i^2 normal for every i.

  If $\{c_i^1, c_i^2\}$ are the same for all i, then we use color c_1^1 for f_1, c_1^2 for f_2 which makes two of edges $\{e_1^1, e_1^2, e_2^1, e_2^2, e_3^1, e_3^2\}$ normal. And we can made f_3 rich in the following way. Since $\{c_3^1, c_3^2\} = \{c_1^1, c_1^2\}$, it holds $\{c(e_3^1), c(e_3^2)\} \cap \{c_1^1, c_1^2\} = \emptyset$. So we color f_3 with the one color from $\{1, 2, 3, 4, 5\} \setminus \{c_1^1, c_1^2, c(e_3^1), c(e_3^2)\}$ and f_3 is rich.

- Finally, we look at the case where for every i: $c_i^1 = c_i^2$.

  When the colors c_1^1, c_2^1 and c_3^1 are not the same, let us say $c_1^1 \neq c_2^1$ (without loss of generality), then we put $c(f_1) = c_1^1$, $c(f_2) = c_2^1$ and any color for f_3 only to maintain the proper coloring condition. We have the edges e_1^1, e_1^2, e_2^1 and e_2^2 normal.

  If $c_1^1 = c_2^1 = c_3^1$, we can make only two edges from $\{e_1^1, e_1^2, e_2^1, e_2^2, e_3^1, e_3^2\}$ normal so we need one of $\{f_1, f_2, f_3\}$ normal as well. Suppose $c_1^1 = c_2^1 = c_3^1 = 1$, $c(e_1^1) = 2$ and $c(e_1^2) = 3$ (without loss of generality).

  Now there are six cases of coloring of edges e_2^1, e_2^2, e_3^1 and e_3^2 in Figure 3.19. We color these cases as shown in this figure. The thick edges are normal and there are always at least three of them.

  All other cases of coloring of e_2^1, e_2^2, e_3^1 and e_3^2 are somehow equivalent to one of those six.

  - None of edges $\{e_1^1, e_1^2, e_2^1, e_2^2, e_3^1, e_3^2\}$ can be colored with 1.
  - We can suppose that $c(e_i^1) < c(e_i^2)$.

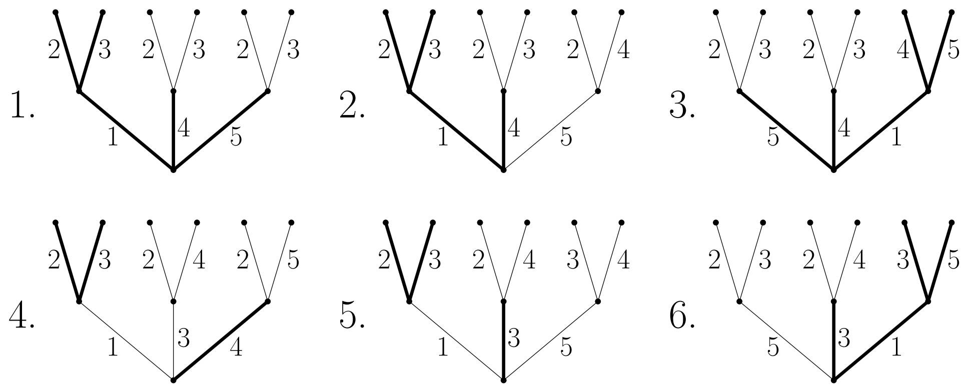

Figure 3.19: Coloring the last three edges—some special cases.

- We can also assume that the three pairs $(c(e_1^1), c(e_1^2))$, $(c(e_2^1), c(e_2^2))$ and $(c(e_3^1), c(e_3^2))$ are sorted lexicographically:

$$(c(e_1^1), c(e_1^2)) <_{lex} (c(e_2^1), c(e_2^2)) <_{lex} (c(e_3^1), c(e_3^2)).$$

- Since $c(e_1^1) = 2$ and $c(e_1^2) = 3$ we can switch 2 and 3 and also 4 and 5 in $c(e_2^1)$ and $c(e_2^2)$ (for example, putting $c(e_2^1) = 2$ and $c(e_2^2) = 4$ is equivalent to $c(e_2^1) = 3$ and $c(e_2^2) = 5$).

That gives us for the pair (e_2^1, e_2^2) only three different pairs of colors that we have to check: $(2, 3)$, $(2, 4)$ and $(4, 5)$.

For each of them there are several possibilities of colors of the pair (e_3^1, e_3^2). In Figure 3.19 there are the cases:

1. $23, 23, 23$ (colors of e_i^j in the format $c(e_1^1)c(e_1^2), c(e_2^1)c(e_2^2), c(e_3^1)c(e_3^2)$)
2. $23, 23, 24$
3. $23, 23, 45$
4. $23, 24, 25$
5. $23, 24, 34$
6. $23, 24, 35$.

The rest that is not excluded by the rules above is:

- $23, 23, 25$ which is equivalent to the case number 2 from above
- $23, 23, 34$ — equivalent to 2 as well
- $23, 23, 35$ — also equivalent to 2
- $23, 24, 24$ — equivalent to 2
- $23, 24, 45$ — equivalent to 6
- $23, 45, 45$ — equivalent to 3.

So the last three edges can be always colored as required.

Note that in the cases 4 and 5 in Figure 3.19 it is not possible to make more than three of the last nine edges normal without recoloring of already colored edges.

□

Theorem 3.6. *Every graph G with minimal cycle length greater than or equal to $2h$ can be colored with five colors in such a way that it is a proper coloring with at least $\frac{|E(G)|}{2} \cdot \frac{2^h-2}{2^h-1}$ edges normal.*

Proof. We start with a proper 5-edge-coloring (we have even a proper 4-edge-coloring from Vizing's Theorem).

For every vertex v of G there is a "rooted almost tree" — induced graph on vertices that are at distance h or less from the vertex v and with v as a root (similar graph to one in Lemma 3.5). Let us denote this graph G_v. The reason for that "almost" is that there could be a cycle in G_v since cycles of length $2h$ are allowed.

We will create for every vertex v rooted trees T_v and T'_v. In the case that G_v does not contain a cycle, we put $T_v = G_v$. Otherwise we derive T_v from G_v as follows. First we delete edges connecting vertices at distance exactly h from v (if there are any). If there is still a cycle, there is also a vertex x in that cycle with two paths from x to v of length h. We duplicate x, put one of the vertices at the end of one path and the other at the end of the second path. (It can also happen that one vertex at distance h is in two cycles as in Figure 3.20. In this case we simply replicate that vertex three-times.)

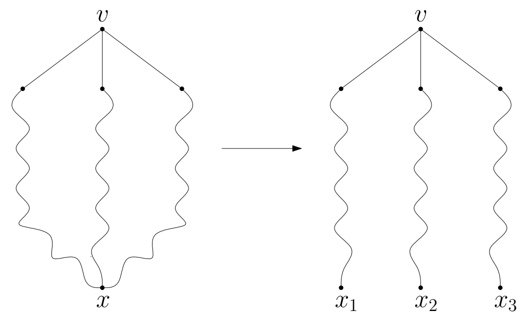

Figure 3.20: We replicate the vertex three-times if it is in more than one cycle.

After we destroy all the cycles, we get rooted tree T_v where the root v has three children, all other inner vertices have two children and the leaves are in depth h. Tree T'_v will be T_v with an extra layer — we add to all leaves of T_v their adjacent edges from G that are not their neighbours in T_v. If some two leaves have the same adjacent edge, we duplicate this edge. If an edge is adjacent to a vertex u in G and the edge is in T_v but it is not adjacent to u in T_v (due to the duplication of vertices before), we duplicate this edge as well. See Figure 3.21.

We color the edges in T_v and T'_v with the same colors as they had in G. All the edges of T_v that were normal in G are now normal in T'_v.

If there is a tree T_v with less than $\frac{|E(T_v)|-9}{2}+3$ normal edges, we use Lemma 3.5 for trees T_v and T'_v to repair that tree. We propagate the changes to G and from G to all other trees T_v and to the trees T'_v.

Note that using the lemma we do not recolor the last layer of T_v (or last two layers of T'_v) and in the previous layers there was no duplication of edges or vertices. So we can do the propagation — the edges that were duplicated for T'_v could not be colored with two different colors. Also the proper coloring condition in G could not be violated because the recolored edges have all their neighbours from G even in T'_v. And all the edges that were or became normal in T'_v are then normal even in G and also in G_v.

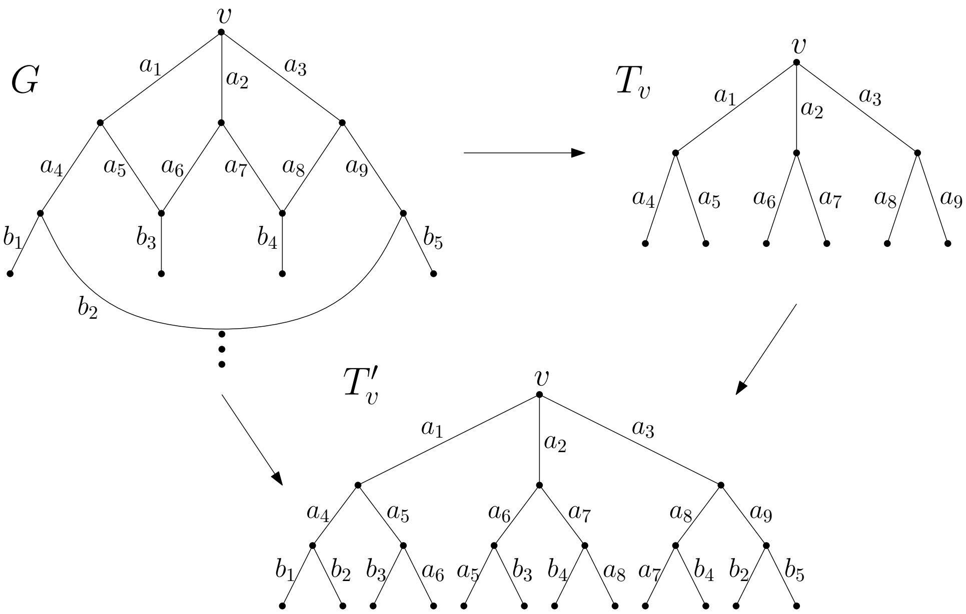

Figure 3.21: Creating trees T_v and T'_v.

We repeat that operation. In every step the number of normal edges increases so at some point there will be no tree with too many bad edges.

We will now compute N — the number of pairs (G_v; normal edge e that is in G_v) in two different ways to obtain the number of normal edges in G.

- The number of pairs is greater than or equal to: *the number of different* G_v $\times$ *the minimal number of normal edges in* G_v.

  The number of G_v is the same as the number of vertices in G which is $\frac{2}{3}|E(G)|$ since G is a cubic graph.

  Minimal number of normal edges is in every G_v after the recoloring greater than or equal to $\frac{|E(G_v)|-9}{2} + 3$ where $|E(G_v)|$ is the number of edges in G_v.

  It holds $|E(G_v)| = |E(T_v)| = 3 + 3 \cdot 2 + 3 \cdot 2^2 + \cdots + 3 \cdot 2^{h-1} = 3(2^h - 1)$.

  From all this we obtain

$$N \geq \frac{2}{3}|E(G)| \cdot \left(\frac{3 \cdot (2^h - 1) - 9}{2} + 3 \right) = \frac{2}{3}|E(G)| \cdot \frac{3 \cdot (2^h - 1) - 3}{2} = |E(G)| \cdot (2^h - 2).$$

- We can compute N also as: *the number of all normal edges in* G $\times$ *the number of different* G_v *that contain one particular edge.*

  Denote the number of normal edges by M — that is what we want to find.

  Every edge is in those G_v whose root is at the distance $(h-1)$ or less from that edge (from closer of the vertices adjacent to that edge). Since the minimal cycle length is $2h$ or more the number of vertices at distance $x \leq h-1$ is $2^{(x+1)}$. (The vertices are arranged as in Figure 3.22. The leaves on the left side can be connected with the leaves on the right side but these are the only cycles that can be on these vertices.)

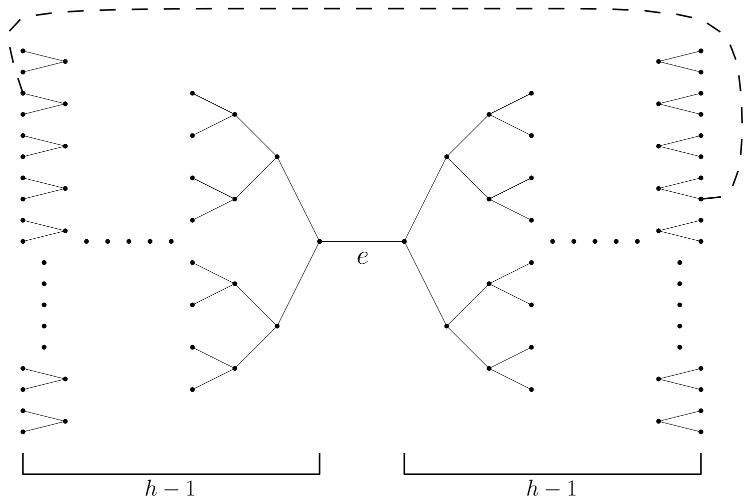

Figure 3.22: Vertices at the distance $h-1$ or less from an edge e.

So the number of all the graphs G_v containing one particular edge is $2^h + 2^{h-1} + \cdots + 2^1 = 2^{h+1} - 2$.

We conclude that

$$N = M \cdot (2^{h+1} - 2).$$

Combining those two computations we obtain

$$M \cdot (2^{h+1} - 2) \geq |E(G)| \cdot (2^h - 2).$$

And finally

$$M \geq \frac{|E(G)|}{2} \cdot \frac{2^h - 2}{2^h - 1}.$$

□

4. Petersen flow

We have already mentioned in Introduction that Petersen coloring and normal coloring are equivalent. Here we write about another property — Petersen flow described by Jaeger in [2] — that is also equivalent to Petersen coloring.

Petersen flow is a nowhere-zero flow. First we define flows and nowhere-zero flows in general.

Definition. *Suppose we have a directed graph G and an Abelian (commutative) group A. A mapping $f : E(G) \to A$ is an A-flow iff for all $V' \subseteq V(G)$:*

$$\sum_{\substack{u \in V' \\ v \in V(G) \setminus V' \\ (u,v) \in E(G)}} f(u,v) - \sum_{\substack{u \in V' \\ v \in V(G) \setminus V' \\ (v,u) \in E(G)}} f(v,u) = 0.$$

Note 4.1. *It is sufficient to require the previous condition only for all vertices $v \in V(G)$ instead of all subsets $V' \in V(G)$. The definition with the condition:*

$$\sum_{(u,v) \in E(G)} f(u,v) - \sum_{(v,u) \in E(G)} f(v,u) = 0$$

for all $v \in V(G)$ is equivalent to the previous one.

Note 4.2. *If G has a bridge, then every A-flow has to have the value 0 on the bridge.*

Definition. *A nowhere-zero flow of a graph G is an A-flow of G that does not use the value 0.*

Note 4.3. *A graph with a bridge cannot have a nowhere-zero flow.*

Sometimes we need to forbid more values in a flow.

Definition. *A B-flow, $B \subseteq A$, is an A-flow that uses only values from B.*

We will be interested in a B-flow for a given $B \subseteq A$ satisfying $B = -B$ and $0 \notin B$. It follows from $B = -B$ that if we have a B-flow for one orientation of G, we can create a B-flow for any orientation of G. Thus we can examine the property of "having a B-flow" on undirected graphs.

Petersen flow is a B-flow with a particular $B \subset \mathbb{Z}_2^6$, $0 \notin B$. The construction of B for Petersen flow will follow.

Note that $B = -B$ as for any element $a \in \mathbb{Z}_2^6$ it holds $a = -a$. And from that we also get that we do not have to consider any orientation of G to get a Petersen flow of G. The flow condition can be modified in this case just to:

$$\forall v : \sum_{(u,v) \in E(G)} f(u,v) = 0.$$

Let us look at the construction of B for Petersen flow. We start with a spanning tree T of Petersen graph. Recall we denoted Petersen graph by P. There are six edges of P that are not in T. Each of those edges corresponds to a cycle of P — edge (u,v) defines a cycle formed by the edge (u,v) and the path from u to v in T. Those six cycles are a basis of the cycle space[1] of Petersen graph. Now consider a matrix M of size 15×6 with rows indexed by the edges of P and columns indexed by the cycles from the basis of the cycle space of Petersen graph. The element in the i-th row and the j-th column $m_{ij} \in \mathbb{Z}_2$ indicates whether the i-th edge is in the j-th cycle from the basis.

Or, in terms of flows, we have six flows $(f_1, \dots, f_6)$ of a basis of the $\mathbb{Z}_2$-flow space of Petersen graph and the rows of the matrix $(m_{i1}, \dots, m_{i6})$ are $(f_1(e_i), \dots, f_6(e_i))$, $e_i \in E(P)$.

The columns of M generate the cycle space P and the rows are the values that are used by Petersen flow.

Definition. *Let $F \subset \mathbb{Z}_2^6$ be a set of the fifteen vectors $(f_1(e), \dots, f_6(e))$, $e \in E(P)$; $(f_1, \dots, f_6)$ is a basis of vector space of $\mathbb{Z}_2$-flows of Petersen graph. Petersen flow is a B-flow with $B = F$.*

Example 4.4. *Let us start with the spanning tree of P from Figure 4.1. On the right side of the figure there is the basis of the cycle space of P created from the edges $\alpha, \dots, \zeta$ outside of the spanning tree.*

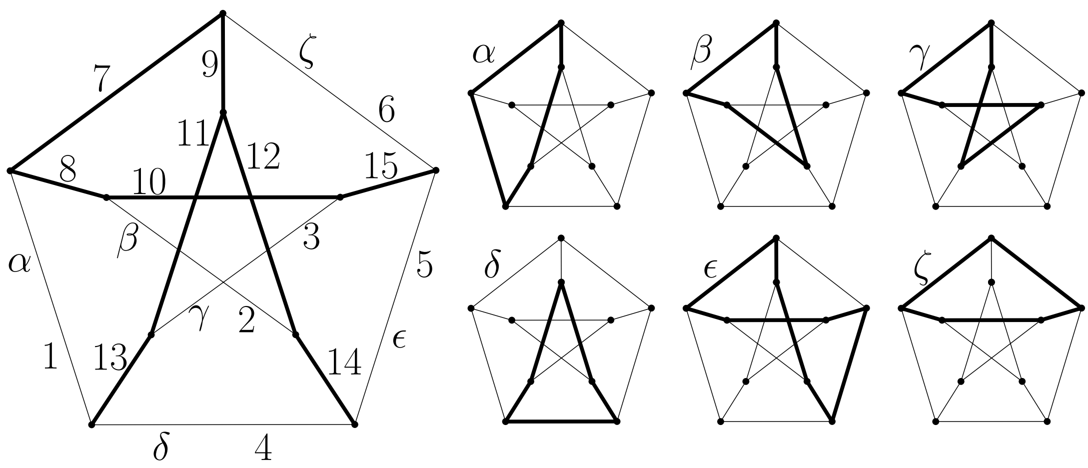

Figure 4.1: A spanning tree and a basis of cycle space of P for an example of construction of Petersen flow.

For the matrix M we order the elements of the basis $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ and all the edges of the Petersen graph $1, \dots, 15$ as in Figure 4.1. The matrix M is the following matrix.

[1] A cycle space of a graph G is a vector space of cycles (unions of cycles) of G. Note that the cycle space of G corresponds to the vector space of $\mathbb{Z}_2$-flows in G.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \end{array} \begin{array}{c} \begin{array}{cccccc} \alpha & \beta & \gamma & \delta & \epsilon & \zeta \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Rows of M create the set F used by Petersen flow.

As Jaeger showed in the article [2], it does not matter which basis $(f_1, \dots, f_6)$ we choose (or with which spanning tree we start) — once a graph has a Petersen flow constructed from one basis, it has a Petersen flow constructed from any basis.

Note 4.5. *All elements in F are different.*

Note 4.6. *If for a set $E' \subseteq E(P)$:*

$$\sum_{e \in E'} (f_1(e), \dots, f_6(e)) = 0,$$

then E' is a cut in Petersen graph. This follows from the fact that cycle-space and cut-space of a graph are orthogonal vector spaces.

It especially means that whenever

$$\sum_{i=1}^{3} (f_1(e_i), \dots, f_6(e_i)) = 0,$$

the edges e_1, e_2 and e_3 are adjacent in P (P is cyclically 4-edge-connected).

Let us now look at the equivalence of Petersen flow and Petersen coloring. The next statement was proved by Jaeger in [2].

Theorem 4.7. *A cubic graph G has a Petersen flow if and only if it has a Petersen coloring.*

Moreover, we can define a Petersen flow f from a Petersen coloring p of graph G as follows. Let $(f_1, \dots, f_6)$ be a basis of the space of $\mathbb{Z}_2$-flows in P. For an edge $e \in E(G)$ we set $f(e)$ to $(f_1(p(e)), \dots, f_6(p(e)))$.

And if we have a Petersen flow f created from a basis $(f_1, \dots, f_6)$ then the mapping $p : E(G) \to E(P)$ such that $p(e) = e'$ if $f(e) = (f_1(e'), \dots, f_6(e'))$ ($e \in E(G)$, $e' \in E(P)$), is a Petersen coloring.

Proof. Suppose we have a Petersen coloring p of a cubic graph G. Let us define a flow f from p as above. We have to check whether for every vertex $v \in V(G)$ is the flow condition satisfied, i.e., whether for every triple of adjacent edges e_1, e_2, e_3 holds: $f(e_1) + f(e_2) + f(e_3) = 0$. Since p is a Petersen coloring, the edges $p(e_1)$, $p(e_2)$ and $p(e_3)$ are adjacent. And $f_1, \ldots, f_6$ are cycles so we have for every i: $f_i(p(e_1)) + f_i(p(e_2)) + f_i(p(e_3)) = 0$ (either zero or two numbers from $\{f_i(p(e_1)), f_i(p(e_2)), f_i(p(e_3))\}$ are equal to 1). Thus the flow condition is satisfied and f is a Petersen flow of G.

Suppose now that G has a Petersen flow f created from a basis $(f_1, \ldots, f_6)$. We show that the mapping p defined from f as in the theorem is a Petersen coloring. For every $e \in E(G)$ there is exactly one $e' \in E(P)$ such that $f(e) = (f_1(e'), \ldots, f_6(e'))$ (follows from Note 4.5). Consider a triple of adjacent edges $e_1, e_2, e_3 \in E(G)$. Suppose $p(e_1) = e'_1$, $p(e_2) = e'_2$ and $p(e_3) = e'_3$. It holds $f(e_1) + f(e_2) + f(e_3) = 0$ which means that

$$\sum_{i=1}^{3} (f_1(e'_i), \ldots, f_6(e'_i)) = 0.$$

From this and Note 4.6 follows that e'_1, e'_2 and e'_3 are adjacent in P thus p is indeed a Petersen coloring of G. □

From the equivalence between Petersen flow and Petersen coloring (and also normal coloring) we obtain immediately the following statements about small cuts in a normally colored graph.

Theorem 4.8. *A normally colored graph does not have a bridge.*

Proof. Normally colored graph has a Petersen flow. But Petersen flow is a nowhere-zero flow and nowhere-zero flows do not exist in graphs with a bridge. A contradiction. □

Theorem 4.9. *The edges in a two-cut in a normally colored graph have the same color.*

Proof. To obtain a contradiction, suppose that the edges e_1 and e_2 in a two-cut of a normally colored cubic graph G have different colors. Denote the normal coloring of G by n. From Theorem 1.1 and the coloring n we get a Petersen coloring p of the graph G such that $p(e_1) \neq p(e_2)$. Then we use Theorem 4.7 and we obtain a Petersen flow of G such that $f(e_1) \neq f(e_2)$. Thus $f(e_1) + f(e_2) \neq 0$ and f is not a flow. A contradiction. □

Note 4.10. *We get from a normal coloring n a Petersen coloring p such that $p(e_1) \neq p(e_2)$ (and from that a contradiction as in the proof before) even in the cases where $n(e_1) = n(e_2) = c$ but the pairs of neighbours of e_1 have not $\sim_c$ equivalent colors as pairs of neighbours of e_2. I.e., if we denote the colors of the neighbours of e_1 and e_2 as in Figure 4.2, then it has to hold $c_1c_2 \sim_c c_3c_4$, otherwise it leads to a contradiction as well.*

Theorem 4.11. *The edges in a three-cut in a normally colored graph G have three different colors.*

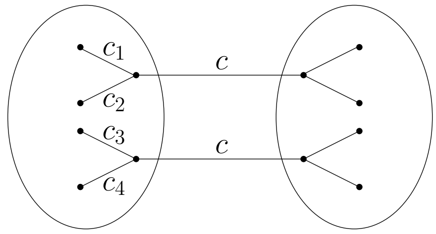

Figure 4.2: Notation for colors in a two-cut.

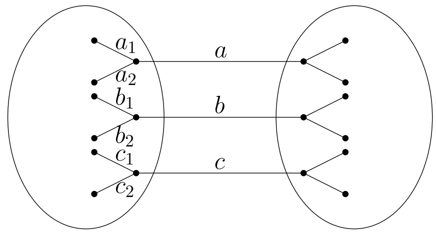

Figure 4.3: Notation for colors in a three-cut.

Proof. Let us denote the normal coloring of G by n, the cut edges by e_1, e_2 and e_3 and the colors of the cut edges and of their neighbours as in Figure 4.3 ($n(e_1) = a$, $n(e_2) = b$ and $n(e_3) = c$).

Suppose that (at least) two colors from $\{a, b, c\}$ are equal. We distinguish two cases.

- There are two edges that are colored with the same color and also the pairs of neighbours of the corresponding edges have equivalent pairs of colors — without loss of generality $a = b$ and $a_1a_2 \sim_a b_1b_2$.
- For every pair of cut edges that have the same color the pairs of colors of their neighbours are not equivalent (i.e., if for example $a = b$, then $a_1a_2 \not\sim_a b_1b_2$ and the same holds for a and c and for b and c).

In the first case we obtain from the coloring n and Theorem 1.1 a Petersen coloring p such that $p(e_1) = p(e_2)$. Thus, by Theorem 4.7, graph G has a Petersen flow f with $f(e_1) = f(e_2)$. But then $f(e_1)+f(e_2) = 0$ and $f(e_1)+f(e_2)+f(e_3) \neq 0$, because $f(e_3) \neq 0$. So f is not a flow. A contradiction.

In the second case we get from n by Theorem 1.1 a Petersen coloring such that $p(e_1)$, $p(e_2)$ and $p(e_3)$ are three different edges and they are not a triple of adjacent edges. Thus in Petersen flow f obtained from p by Theorem 4.7 it is $f(e_1) + f(e_2) + f(e_3) \neq 0$ (follows from Note 4.6). A contradiction. □

Note 4.12. *We obtain $p(e_1)$, $p(e_2)$ and $p(e_3)$ different but not adjacent (and thus we get the same contradiction as in the last case of the proof before) even when a, b, c are different unless also $a_1a_2 \sim_a bc$, $b_1b_2 \sim_a ac$ and $c_1c_2 \sim_a ab$.*

Note 4.13. *In a general cut there can be some pairs of edges that have the same value in Petersen flow and the rest are edges whose values in Petersen flow correspond with edges in Petersen graph that form a cut in Petersen graph (follows from Note 4.6). From that we obtain necessary rules for Petersen coloring and for normal coloring of edges in the cut.*

For example for four-cut we have that the colors of the cut edges in normal coloring are either

- *all the same or*
- *all different or*
- *two and two are the same.*

5. Chains

In this chapter we describe a new approach — chains — to Problem 1.3. The inspiration are Kempe chains[1] that are very useful in graph coloring problems.

The original aim was to find a way how to recolor a properly colored graph locally in such a way that the number of normal edges would increase and the rest of the graph would not be affected. We have already done some local changes in Chapter 3 for graphs without short cycles. But we wish to have a tool that would work even when there were only few edges that are not normal.

An attempt at local recoloring using the chains is at the end of this chapter but it is not as "local" as we would like it to be.

Nevertheless we prove here some of the statements about cuts already shown in Chapter 4 but now we use the chains to do that. And we also prove one new statement that in an almost normally colored graph there cannot be only one edge which is not normal.

5.1 Definition and basic statements

Suppose we have a normal coloring of a cubic graph G. Let us have a look at the subgraph of G made by the edges that are colored either with color 1 or 2. Clearly every vertex in this subgraph has degree at most two so the subgraph is a union of paths and cycles. Contrary to Kempe chains used for the usual edge-coloring, we need to consider all components of this graph together.

Definition. *In a normally colored graph G the* a,b-chain *is a connected component of the subgraph of G made by exactly the edges colored either with color a or b.*

Inner-vertices of the a,b-chains are the vertices with degree two in the chains.

End-vertices of the a,b-chains are the vertices with degree one in the chains.

Outer-vertices of the a,b-chains are vertices that are not in any a,b-chain.

We will refer to an edge from the a,b-chains as a chain-edge *and to an edge not in the a,b-chains as a* non-chain-edge.

We will distinguish the chain-neighbours *and the* non-chain-neighbours *of a vertex. A vertex u is a chain-neighbour of a vertex v iff the edge (u, v) is a chain-edge. Otherwise (if (u, v) is a non-chain-edge), is u a non-chain-neighbour of v.*

There is an example of the chains in Blanuša snark in Figure 5.1.

We can reformulate the definition of normal coloring in terms of chains.

Definition. *We say that a, b-chains satisfy the* neighbour-property *if no inner-vertex is a non-chain-neighbour of an end-vertex and also no outer-vertex is a neighbour of an end-vertex (both cases would imply a not normal edge).*

Figure 5.2 shows general structure of the graph with chains that follows from the neighbour-property.

[1] In an edge-colored graph G an (a,b)-Kempe chain is a component of the subgraph of G made by the edges that are colored with either a or b. For vertex-colored graph it is a maximal path with vertices colored with either a or b. The operation of switching colors a and b in one Kempe chain does not violate the proper coloring condition.

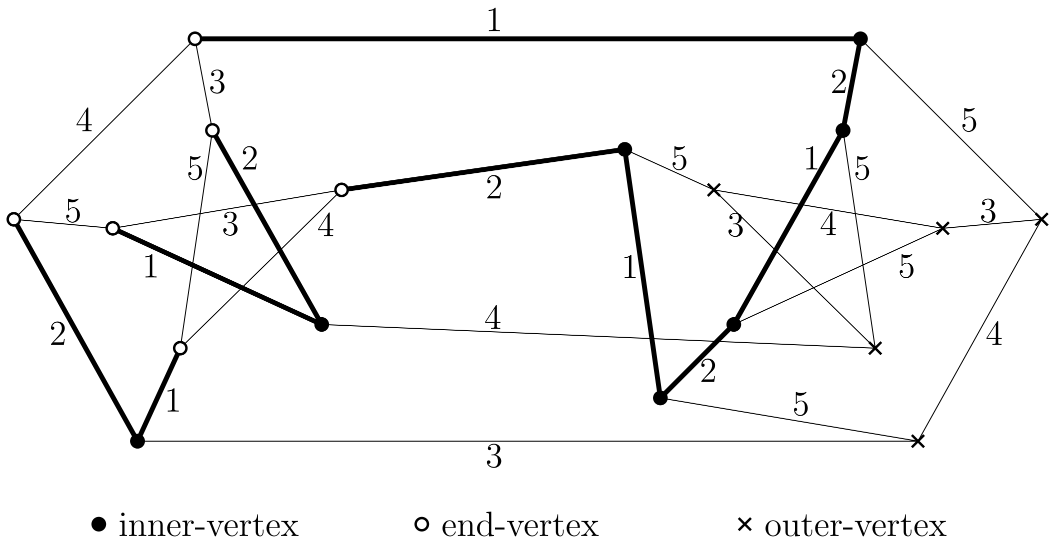

Figure 5.1: The 1,2-chains in Blanuša snark.

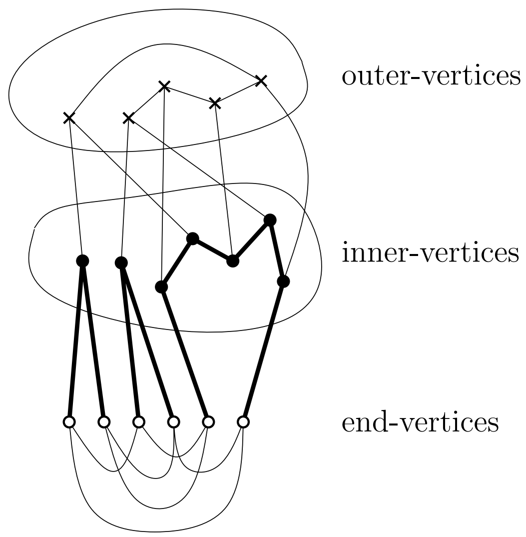

Figure 5.2: The structure of the 1,2-chains in Blanuša snark colored as in Figure 5.1.

Theorem 5.1. *A proper 5-edge-coloring of a cubic graph is a normal coloring iff for every pair of colors a,b the neighbour-property is satisfied.*

Proof. Suppose we have a normal coloring of graph G. Consider the a, b-chains. For a contradiction assume that

- an inner-vertex is a non-chain-neighbour of an end-vertex. Then the edge joining these two vertices has neighbours colored with a and b on one side and a and something else then b (or b and something else then a) on the other side. Thus this edge is not normal. A contradiction.

- an outer-vertex is a neighbour of an end-vertex. The outer-vertex is adjacent to the edges colored with c, d and e ($\{a, b\} \cap \{c, d, e\} = \emptyset$). The end-vertex is adjacent to one edge colored either with either a or b and two edges colored with colors from $\{c, d, e\}$. From that we get that also the edge joining the outer-vertex with the end-vertex is not normal. A contradiction.

Now suppose we have a proper 5-edge-coloring that satisfy the neighbour-property and there is an edge e which is not normal. Without loss of generality we may assume that the edge e has color 1 and its neighbours on one side have colors 2 and 3 and the neighbours on the other side have colors 2 and 4. Consider the 2,3-chains. There are an inner-vertex and an end-vertex joined with the edge e so the neighbour-property for 2,3-chains is violated. A contradiction. □

Now let us have a look at the neighbour-property more closely. The non-chain-neighbours of the end-vertices of the a,b-chains can be only end-vertices as well (for arbitrary a and b). Since every end-vertex has two non-chain-neighbours, there are cycles on the end-vertices formed by the edges that are not in the chains (there are only end-vertices in these cycles and every end-vertex is in such cycle). Furthermore, the cycles has to be all even as we will show in the next theorem. We will call those cycles *the joining cycles*.

Theorem 5.2. *The joining cycles have an even length.*

Proof. Without loss of generality suppose we have 1,2-chains and colors 3, 4, 5 on a joining cycle. Suppose also that there is a vertex u adjacent to chain-edge with color 1 and cycle-edges with colors 3, 4 (clockwise) — we will refer to such vertex as a 134 vertex.

Let us have a look at the next vertex v clockwise on the joining cycle. If the next chain-edge (the chain-edge adjacent to v) is colored with 1, then v is a 143 vertex (because the cycle-edge between u and v has to be poor). If the next chain-edge is colored with 2, then v is a 245 vertex. See Figure 5.3.

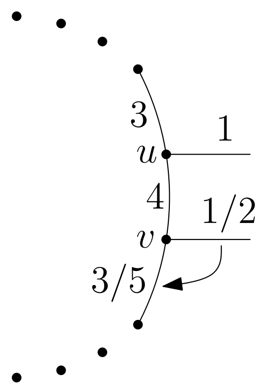

Figure 5.3: The vertex u is a 134 vertex and v is a 143 vertex or a 245 vertex.

The next vertex clockwise after a 143 vertex must be either a 134 vertex or a 235 vertex, etc.

The whole situation is depicted with the finite-state machine in Figure 5.4.

The nodes of the finite-state machine represent vertices on the joining cycle — first number is color of adjacent chain-edge and the other two are colors of adjacent cycle-edges (clockwise). As we are going clockwise the joining cycle vertex by vertex, we change correspondingly state in the finite-state machine. If we are moving to a vertex with adjacent chain-edge colored with 1, we change the state according to the arrow signed by 1 in the finite-state machine. Similarly if we move to vertex with adjacent chain-edge colored with 2.

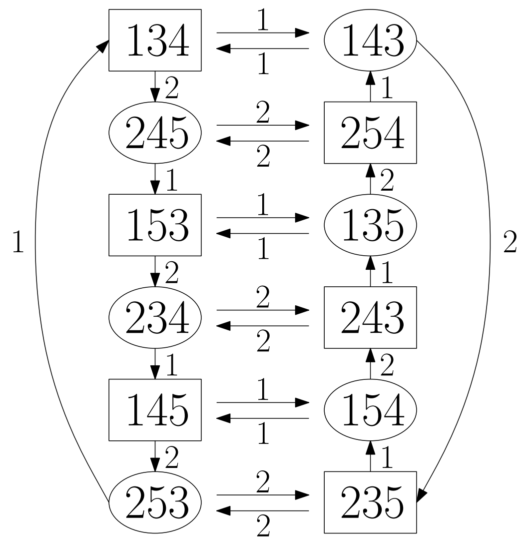

Figure 5.4: Finite-state machine for coloring a joining cycle.

When we return to the vertex where we started on the joining cycle, we should be in the starting state in the finite-state machine. But since in every step we change the state from a circled one to a squared one and vice versa in the finite-state machine, it is not possible to return to the same node after an odd number of steps. Thus the joining cycle has to have an even length. (If it had odd length, then at least one of the edges on the joining cycle would not be normal.) □

5.2 Small cuts

The facts about joining cycles imply the two following statements for cuts. We will consider only the cuts where no pair of edges of the cut is adjacent unless written otherwise.

Lemma 5.3. *Consider a graph G with a cut $E' \subset E(G)$ splitting G into graphs G_1 and G_2. For every pair of colors (a, b) holds the following. The number of end-vertices of a, b-chains that*

- *belong to G_1 (G_2 respectively)*
- *are adjacent to an edge from the cut E' and*
- *have the chain-neighbour in G_1 (G_2 respectively)*

is even. (See Figure 5.5.) We will refer to such end-vertex as an "end-vertex at the cut".

Proof. Every end-vertex is adjacent to one chain-edge and two edges of a joining cycle. Since for every end-vertex at the cut the chain-edge is in G_1 (in G_2 respectively), we have for each such end-vertex one edge from a joining cycle in the cut. And vice versa — every edge from the cut that is in a joining cycle gives us one end-vertex at the cut. Obviously the number of edges in the cut that are in the joining cycles has to be even (the joining cycles are cycles) so the number of end-vertices at the cut has to be even too.

□

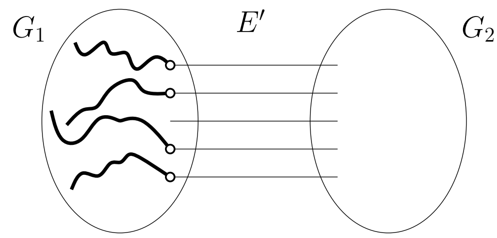

Figure 5.5: The a, b-chains at a cut. The thick lines are the chains. The highlighted end-vertices are the end-vertices at the cut that we are counting in Lemma 5.3.

Lemma 5.4. *Suppose we have a graph G with a cut E' and components of $G \setminus E'$ denoted by G_1 and G_2 one more time. If for some a, b there is no such end-vertex of a,b-chains at the cut as in Lemma 5.3, then there has to be an even number of end-vertices of a,b-chains in G_1 (in G_2 respectively). Which particularly means that in this case there has to be an even number of edges of the cut E' that are in the a,b-chains (i.e., colored with either a or b).*

Proof. If in G_1 (in G_2 respectively) there is no end-vertex at the cut of a,b-chains, then there is no joining cycle in G that contains a cut edge. So every joining cycle is either whole in G_1 or whole in G_2. And since the joining cycles are even (and the vertices of joining cycles are exactly the end-vertices of the a,b-chains), the lemma is true. □

With these tools we can prove some theorems about cuts in a normally colored graph that we have already proved using Petersen flow.

Theorem 5.5. *A cubic graph with a normal coloring does not have a bridge.*

Proof. Let us assume for a contradiction that we have such graph and the bridge is colored with color 1 (without loss of generality). Now if we look at the 1,2-chains (1 and any other color work as well) there is no end-vertex at the cut as in Lemma 5.3. And there is an odd number of end-vertices in one component of G without the bridge — one edge in the cut is colored with either 1 or 2. So we have a contradiction with Lemma 5.4. □

Theorem 5.6. *The edges in a two-cut of a normally colored graph have the same color. Moreover, the pairs of colors of the neighbours of the cut edges are equivalent. (If c is the color of the edges in the cut and $c_1, \ldots, c_8$ are the colors of their neighbours as in Figure 5.6, then $c_1c_2 \sim_c c_5c_6$ (actually it holds $c_ic_{i+1} \sim_c c_jc_{j+1}$ for all $i, j \in \{1, 3, 5, 7\}$ but all these follow from the first equivalence, transitivity of $\sim_c$, and the fact that the edges in the cut are normal).*

Proof. Suppose we have a normally colored graph G with a two-cut colored with different colors — without loss of generality with colors 1 and 2. Denote the components of G without the edges of the two-cut by G_1 and G_2. Consider the 1,3-chains. There is one chain-edge in the cut so there cannot be two end-vertices at the cut as in Lemma 5.3 in G_1. If there is one such end-vertex at the cut, then it contradicts Lemma 5.3. And if there is no end-vertex at the cut in G_1, then we have a contradiction with Lemma 5.4. So the cut edges are colored with the same color.

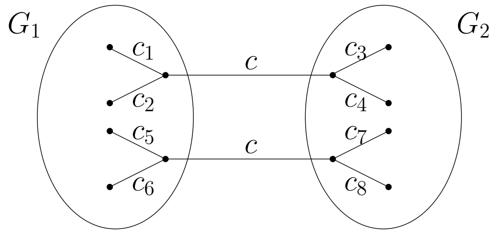

Figure 5.6: Notation for colors in a two-cut.

Assume, without loss of generality, that the cut edges are colored with 1 and suppose that the colors of the neighbours are not equivalent. We may assume $c_1 = 2$, $c_2 = 3$, $c_5 = 2$, $c_6 = 4$ — we can get all the other cases by a permutation of colors. Now consider the 2,3-chains and we have a contradiction with Lemma 5.3 — there is one end-vertex at the cut in G_1 (see Figure 5.7).

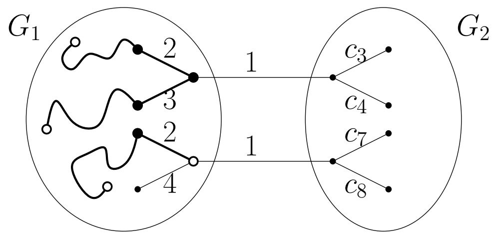

Figure 5.7: The 2,3-chains in the case when the neighbours of cut edges do not have equivalent pairs of colors.

□

Note 5.7. *In a two-cut it cannot happen that the edges in the cut are adjacent, because it would imply that the graph has a bridge and normally colored graphs have to be bridgeless according to Theorem 5.5.*

Theorem 5.8. *The edges in a three-cut have three different colors in a normally colored graph. Furthermore, if we denote the colors of the edges as in Figure 5.8, we get that* $a_1 a_2 \sim_a bc$, $b_1 b_2 \sim_b ac$ *and* $c_1 c_2 \sim_c ab$.

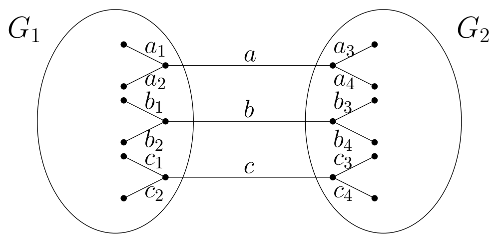

Figure 5.8: Notation for colors in a three-cut.

Proof. Suppose we have a normally colored graph with a three-cut colored with at most two colors. We may assume that these colors are 1 and 2 (or just 1). Then, considering the 1,2-chains, there are no end-vertices at the cut as in Lemma 5.3 and there is an odd number (three) of chain-edges in the cut so we get a contradiction with Lemma 5.4. Thus the colors a, b, c are different.

Assume $a = 1$, $b = 2$, $c = 3$. Suppose for a contradiction that one of the relations $a_1a_2 \sim_a bc$, $b_1b_2 \sim_b ac$ and $c_1c_2 \sim_c ab$ is not true. Without loss of generality $a_1a_2 \not\sim_a bc$ and $a_1 = 2$ and $a_2 = 4$. (If $a_1a_2 \not\sim_a bc$, then one of $\{a_1, a_2\}$ is from $\{b, c\}$, that is $\{2, 3\}$, and the other is not.) Consider the 2,4-chains. See Figure 5.9. Denote the vertices in G_1 adjacent to the cut edges by u, v, w as in this figure.

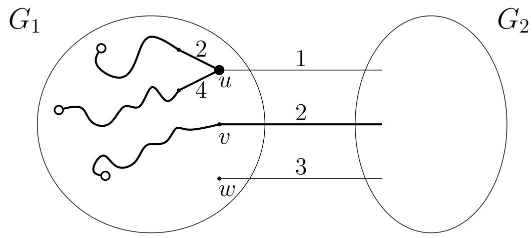

Figure 5.9: The 2,4-chains in a bad coloring of neighbours of a three-cut.

Vertices u and v cannot be end-vertices at the cut as in Lemma 5.3. If w is such end-vertex, then it contradicts Lemma 5.3. Otherwise (if neither of u, v, w is such end-vertex), we have a contradiction with Lemma 5.4. □

Note 5.9. *If there are two adjacent edges in a three-cut, we can obtain a similar result. The two adjacent edges in the three-cut imply a two-cut right next to the three-cut. The situation is depicted in Figure 5.10. Denote the colors of the edges as in this figure.*

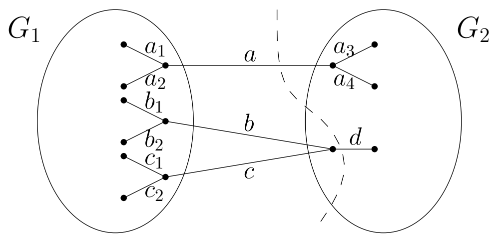

Figure 5.10: Notation for the colors of the edges in a three-cut with two adjacent cut edges. The dashed line indicates the two-cut.

From Theorem 5.6 about two-cut we get $a = d$ and $a_1a_2 \sim_a bc$.

The fact that $a = d$ implies $a \neq b$ and $a \neq c$. Also $b \neq c$ because corresponding edges are adjacent. So the edges of the three-cut have different colors even in this case.

About the colors of neighbours of the cut edges we already know $a_1a_2 \sim_a bc$. Furthermore, $dc \sim_b ac$ and $db \sim_c ab$ because $d = a$. That gives us (together with the fact that the edges colored with b and c are normal) also $b_1b_2 \sim_b ac$ and $c_1c_2 \sim_c ab$.

5.3 Not one mistake alone

So far we have shown statements that had been already proved in Chapter 4. Now we prove one more statement — that there cannot be only one not normal edge in a properly edge-colored graph.

Theorem 5.10. *Suppose we have a cubic graph with a proper 5-edge-coloring and assume there is one edge that is not normal. Then there is at least one another edge that is not normal either.*

Proof. To obtain a contradiction, suppose that there is a properly edge-colored cubic graph G with all edges normal except one. Denote this edge by e and its adjacent vertices by u and v. Without loss of generality we can assume that e is colored with 1 and its neighbours have colors 2 and 3 on one side and 2 and 4 on the other side. Consider the 2,3-chains. See Figure 5.11.

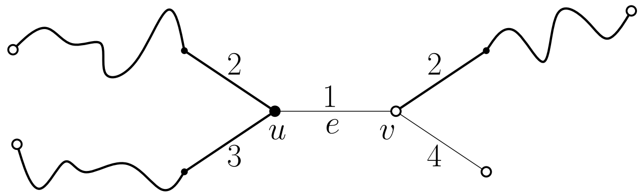

Figure 5.11: The 2,3-chains around a not normal edge e.

The edge e is a non-chain-edge that connects the inner-vertex u and the end-vertex v. Since e is the only not normal edge, this is the only violation of the neighbour-property in 2,3-chains. So for every end-vertex except v holds that the two its non-chain-neighbours are end-vertices as well. And for v only one of its two non-chain-neighbours is an end-vertex. And that is not possible. (The subgraph of G made by the end-vertices of the 2,3-chains and the non-chain-edges joining those vertices would have one vertex of degree one and all other vertices of degree two.) A contradiction. ☐

5.4 An attempt at local recoloring

In a properly edge-colored graph we can switch the colors in one Kempe chain and the coloring remain proper.

Unfortunately, in the case of normal coloring it does not work that way. If we switch the colors in only one a,b-chain, we get also a normal coloring only if the a,b-chain is a cycle. We can switch the colors on all a,b-chains, but that is just a permutation of colors which is not very helpful when we want to do some local changes in the coloring. But if we are lucky, we will be able to switch colors in some a,b-chains but not in all of them.

Let us start with switching colors in one a,b-chain (that is not a cycle) and see which edges might stop being normal. See Figure 5.12 for the notation.

We have to check the chain-edges whose color has been changed — $c_1, \dots, c_n$, and the neighbours of them — the edges $f_1, \dots, f_4, g_1, \dots, g_{n-1}$.

The edges $g_1, \dots, g_{n-1}$ are still normal. They have neighbours colored still with a and b on one side and on the other side with two colors that are $\sim_{c(g_i)}$ equivalent to ab.

Also $c_1, \dots, c_n$ remain normal:

- c_1 and c_n were rich and it holds for different colors r, s, t, u, v: $rs \sim_t uv$ iff $ts \sim_r uv$,

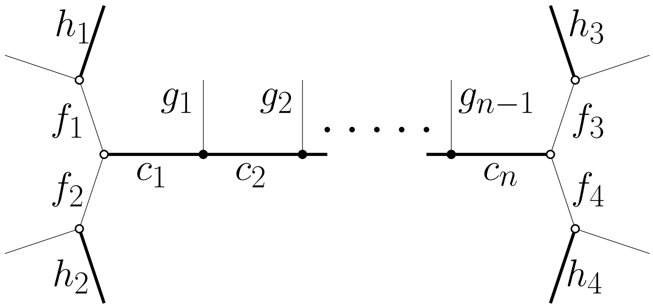

Figure 5.12: Notation for edges in and around a chain.

- $c_2, \ldots, c_{n-1}$ were poor and it holds for any colors x, y, z: $xy \sim_z xy$ iff $zy \sim_x zy$.

But $f_1, \ldots, f_4$ are not normal now because from the five edges — f_i and its four neighbours — only one has changed its color (for every i).

The edges $f_1, \ldots, f_4$ are all joining two end-vertices so they are all adjacent to one more chain-edge different from c_1 and c_n. In Figure 5.12 are these chain-edges denoted by $h_1, \ldots, h_4$. If we switch colors even in the a,b-chains beginning with the edges $h_1, \ldots, h_4$, the edges $f_1, \ldots, f_4$ became normal again. However, this can make another non-chain edges adjacent to end-vertices of recolored chains not normal. (Similar edges for each chain as $f_1, \ldots, f_4$ are for the chain in Figure 5.12 can stop being normal.) So we switch another a,b-chains until all edges are normal.

We can quite easily end up with all a,b-chains switched this way. This recoloring can be useful in the cases where there is a proper subset J of the set of joining cycles, with a property that all the chains which start with a (end-)vertex that is in a joining cycle from J, also end with a vertex of a joining cycle from J. Then we can recolor only these chains to get from one normal coloring another normal coloring. See Figure 5.13.

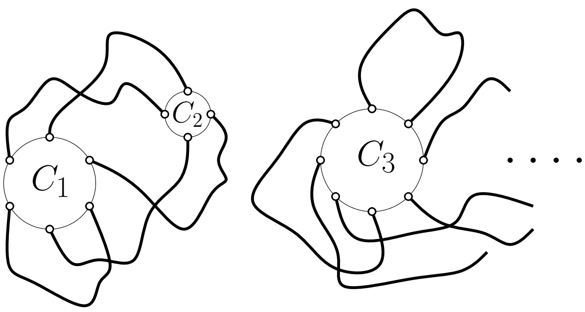

Figure 5.13: The a,b-chains and their joining cycles in a graph where we can actually switch colors in only some a,b-chains. Note that there are not all edges in the figure — for example the non-chain-edges adjacent to inner-vertices of the chains are missing; the graph is not disconnected. The joining cycles C_1 and C_2 can create a proper subset of joining cycles J as described in the text, and we can switch colors in the chains that start/end with a vertex of these cycles.

There are cases when switching colors in chains might also help us to repair some mistakes in an almost normal coloring. A very artificial example: if we switch colors in some a,b-chains in a normally colored graph and get a not normal coloring, then, of course, switching it back will repair the coloring. But we can expect that the real cases will not be that friendly.

6. Small cycles in a normally colored graph

For many conjectures in graph theory we have theorems of the form: the hypothetical counterexample does not contain a particular small graph (such as small cycle, a path etc.).

For Jaeger's problem Hägglund and Steffen showed in [5] that minimal counterexample does not contain graph $K^*_{3,3}$ as a subgraph (see Figure 6.1).

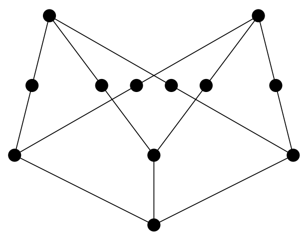

Figure 6.1: Graph $K^*_{3,3}$.

Here we will look at small cycles. It is easy to show that minimal counterexample does not have a triangle — for every normally colored graph we can add a triangle to any vertex and expand the coloring to the new edges as in Figure 6.2. Which means that if we had a graph with a triangle that did not have a normal coloring, then this graph without the triangle — a smaller graph — would not have normal coloring either. We can also get from a normally colored graph with a triangle a normal coloring of that graph with the triangle contracted.

Theorem 6.1. *Consider cubic graphs G and G' such that G' contains a triangle and G is created from G' by contraction of the edges of the triangle.*

Suppose we have a normal coloring of G. Then we can provide a normal coloring of the graph G'.

It is also possible to find a normal coloring of the graph G if we have a normal coloring of the graph G'.

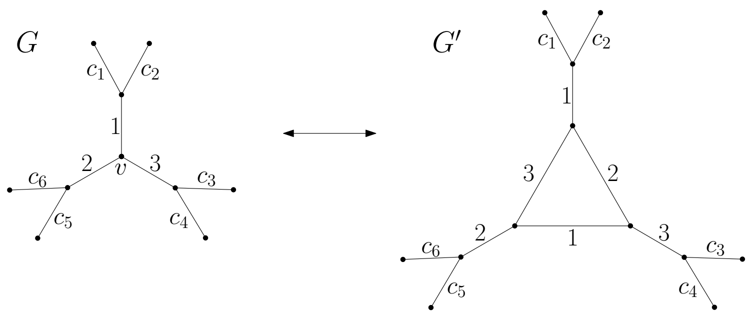

Figure 6.2: Adjustment of coloring for graph with an additional triangle/for graph with a contracted triangle.

Proof. We may assume, without loss of generality, that the edges adjacent to v are colored with 1, 2 and 3. We can expand the coloring as in Figure 6.2 no matter what the colors $c_1, \ldots, c_6$ are.

It is not possible for an edge of a triangle in a normally colored graph to be rich thus it has to be poor. Therefore we may assume that the edges of the triangle and the adjacent edges are colored as in Figure 6.2 and we can use in G the same colors as in G' (also in Figure 6.2). □

However, a similar elimination of square seems to be, if possible, much bigger task. Following theorems tell us at least how does the normal coloring around the square look like and show how we can add a square to a normally colored graph in some cases.

Theorem 6.2. *Assume we have a normally colored graph G' containing a square. Then the four edges adjacent to the square are colored either all with the same color or with two colors — one edge with an edge that is not opposite from it have one color and the other two have the other color. See Figure 6.3.*

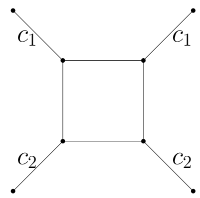

Figure 6.3: The only possible coloring of the four edges adjacent to a square in a normally colored graph. It might be $c_1 = c_2$.

Proof. Let us denote the edges as in Figure 6.4.

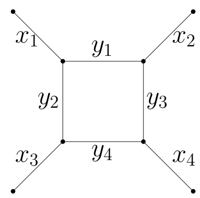

Figure 6.4: Notation for the edges.

We will rule out all the possible colorings of edges $x_1, \ldots, x_4$ except the one above. Let c denote a normal coloring of G'.

- Suppose $c(x_1), \ldots, c(x_4)$ are all different. Without loss of generality, we may assume that $c(x_1) = 1$, $c(x_2) = 2$, $c(x_3) = 3$ and $c(x_4) = 4$.

  If y_1 is poor, then $c(y_2) = 2$ and $c(y_3) = 1$. For the edge y_4 there remains only the color 5, otherwise the coloring would not be proper. Now it should be $c(y_1) = 4$ so that the edge y_2 is normal, but also it should be $c(y_1) = 3$ so that y_3 is normal. A contradiction. See Figure 6.5.

  If y_1 is rich, then $c(y_2) \in \{4, 5\}$ and $c(y_3) \in \{3, 5\}$. It cannot be $c(y_2) = 5$ because neither with $c(y_3) = 3$ nor with $c(y_3) = 5$ would y_5 be normal. So it is $c(y_2) = 4$ and that imply $c(y_3) = 3$. Which means that y_4 is poor and we have a symmetric situation to the one where y_1 was poor.

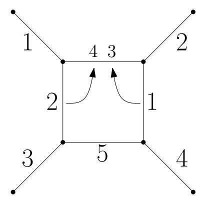

Figure 6.5: The case where the colors $c(x_1), \ldots, c(x_4)$ are different and the edge y_1 is poor.

- The edges $x_1, \ldots, x_4$ use together three different colors and the edges with the same color are next to each other (not opposite). Without loss of generality, we put $c(x_1) = 1$, $c(x_2) = 1$, $c(x_3) = 2$ and $c(x_4) = 3$. The edge y_1 has to be poor so $c(y_2) = c(y_3)$. But as $c(x_3) \neq c(x_4)$, the edge y_4 cannot be normal. See Figure 6.6.

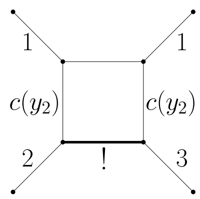

Figure 6.6: The case where $c(x_1) = c(x_2)$ and $c(x_3)$, $c(x_4)$ are different.

- The edges $x_1, \ldots, x_4$ use together three different colors and the edges with the same color are opposite each other. Without loss of generality assume that $c(x_1) = 1$, $c(x_2) = 2$, $c(x_3) = 3$ and $c(x_4) = 1$. All the edges $y_1, \ldots, y_4$ have to be rich, otherwise the proper coloring condition is violated. This imply that all $c(y_1), \ldots, c(y_4)$ are either 4 or 5. So either $c(y_1) = c(y_4) = 4$ and $c(y_2) = c(y_3) = 5$ or the other way around. However, in neither of these cases is any of the edges $y_1, \ldots, y_4$ normal.
- The edges $x_1, \ldots, x_4$ use together two colors but the same colored edges are opposite each other. Without loss of generality, $c(x_1) = c(x_4) = 1$ and $c(x_2) = c(x_3) = 2$. Also in this case all edges $y_1, \ldots, y_4$ have to be rich. That means that $c(y_1), \ldots, c(y_4) \in \{3, 4, 5\}$. We may assume, without loss of generality, that $c(y_1) = 3$ and $c(y_3) = 4$. To make y_1 rich, we need $c(y_2) = 5$. Now it should $c(y_4) = 4$ so that y_2 would be rich, but also $c(y_4) = 5$ so that y_3 would be rich. A contradiction. See Figure 6.7.

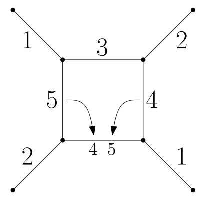

Figure 6.7: The case where $c(x_1) = c(x_4)$ and $c(x_2) = c(x_3)$ but $c(x_1) \neq c(x_2)$.

Note that the other cases are possible — see Figure 6.8.

□

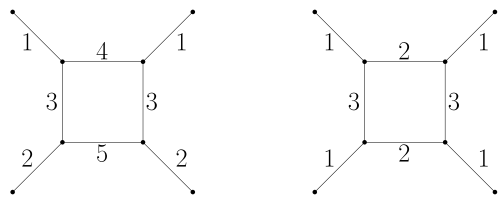

Figure 6.8: The good cases where it is possible to color the edges of the square.

Now we show how to modify the coloring after removing a square.

Theorem 6.3. *Consider a normally colored graph G' that contains a square. Denote the color of one pair of adjacent edges by c_1 and the color of the other pair by c_2 as in Figure 6.3. If we make the operation from Figure 6.9 with the square, we get a smaller graph G that is also normally colored.*

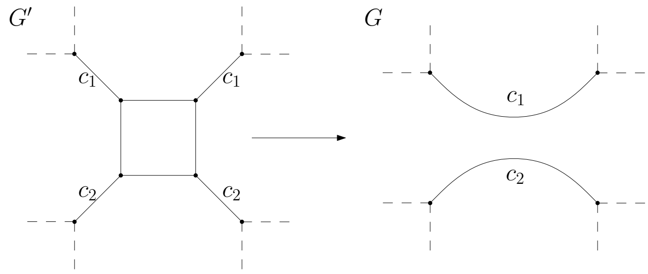

Figure 6.9: Removing a square.

Proof. We only have to check if the two new edges are normal as the other edges have their neighbours colored in G with the same colors as in G'. Denote the edges as is suggested on the left side of Figure 6.10.

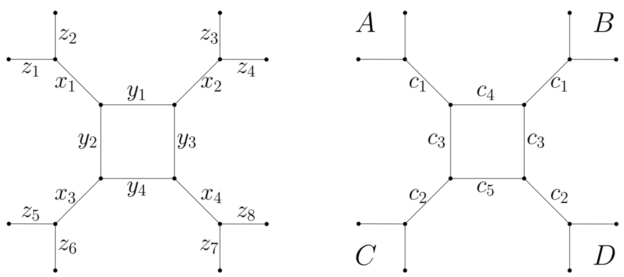

Figure 6.10: On the left side is the notation for edges, on the right side the notation for colors. The capital letters A, B, C, D represents the pairs of colors of pairs of edges z_1z_2, z_3z_4, etc.

The edges x_1 and x_2 are colored both with the color c_1 so the edge y_1 is poor. Thus the edges y_2 and y_3 have the same color.

Denote the colors as on the right side of Figure 6.10. The pair of colors of edges z_1, z_2 is denoted by A. Also B, C and D represents pairs of colors of corresponding edges.

Since x_1 and x_2 are normal, it holds $A \sim_{c_1} c_3c_4$ and $c_3c_4 \sim_{c_1} B$. By transitivity of $\sim_{c_1}$ we obtain $A \sim_{c_1} B$. Similarly we get $C \sim_{c_2} D$. Thus the new edges in G are normal. ☐

However, such a simple modification of coloring does not seem possible when we are adding a square. At least not for all the colorings of the former graph. We will show which are the good colorings and how to modify them after an addition of a square. First we color uncolored edges in a special situation that helps us to color the square later.

Lemma 6.4. *Suppose that we have a situation as in Figure 6.11. The thick lines — $a_1, \dots, a_6$ — are colored and the rest — $b_1, \dots, b_3$ — are not. We would like to color the uncolored edges in such a way that the edges a_1, b_3 and a_2 would be normal. This is possible for any coloring of the colored edges satisfying the proper coloring condition, except the case: $a_1 = a_2$, $a_3a_4 \not\sim_{a_1} a_5a_6$.*

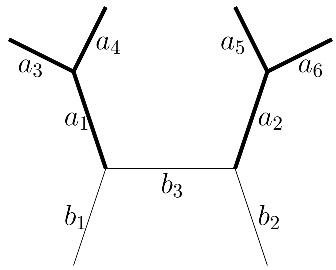

Figure 6.11: Notation for colors.

Proof. Without loss of generality we can assume that $a_1 = 1$, $a_3 = 2$ and $a_4 = 3$. Now the cases where we only switch colors 2 and 3 in a_2, a_5, a_6 are equivalent. Also switching colors 4 and 5 gives us equivalent cases. So all the cases of proper colorings of $a_1, \dots, a_6$ (except those where $a_1 = a_2$ and $a_3a_4 \not\sim_{a_1} a_5a_6$) are equivalent to one of the ten cases in Figure 6.12. And those we color as suggested in this figure.

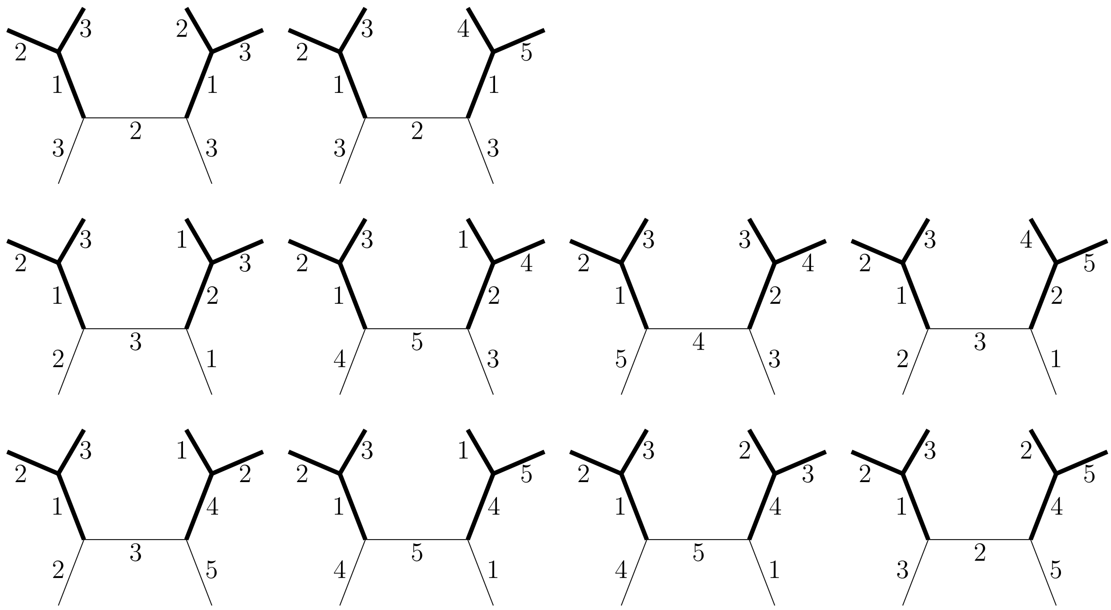

Figure 6.12: Coloring of the three uncolored edges in the situation from Lemma 6.4.

□

Note 6.5. *The first two cases in Figure 6.12 have more solutions but the rest have only the one in this figure.*

Note 6.6. *The case $a_1 = a_2$ and $a_3a_4 \not\sim_{a_1} a_5a_6$ really does not have a solution. We may assume, without loss of generality, that $a_1 = a_2 = 1$, $a_3 = 2$, $a_4 = 3$, $a_5 = 2$ and $a_6 = 4$.*

The middle edge (the one colored with b_3) has to be poor so $b_1 = b_2$. Thus $b_1b_3 \sim_1 b_2b_3$. But it is $23 \sim_1 b_1b_3$ and $24 \sim_1 b_2b_3$ and by transitivity of $\sim_1$ we get $23 \sim_1 24$. A contradiction.

Theorem 6.7. *Consider a graph G with a normal coloring c. We create a graph G' from G by adding a square to two edges e and f (inverse operation to the one in Figure 6.9).*

There is a normal coloring c' of G' whenever:

- *$c(e) \neq c(f)$ or*
- *$c(e) = c(f) = x$ and the pairs of colors of neighbours of e and f are $\sim_x$ equivalent.*

Proof. We color every edge that is both in G and G' with the same color as in the coloring c. The four edges adjacent to the square get colors according to $c(e)$ and $c(f)$ — see Figure 6.13.

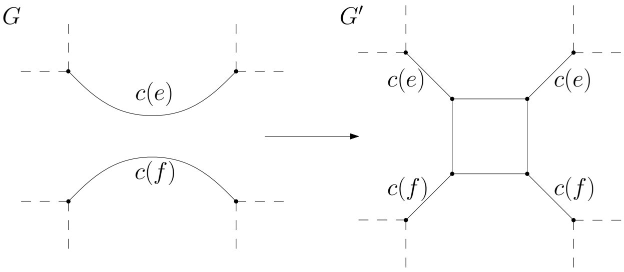

Figure 6.13: Adding a square.

That makes all the edges that are both in G and G' normal.

Now we need to find colors for the edges of the square in such a way that the new edges will be normal.

We split the square into two symmetric parts as in Figure 6.14 and color first the left part using Lemma 6.4.

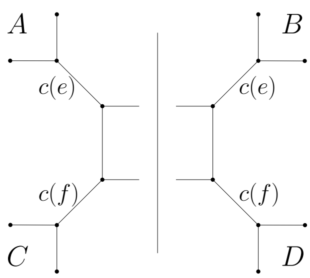

Figure 6.14: Splitting the square. The capital letters represent the pairs of colors one more time.

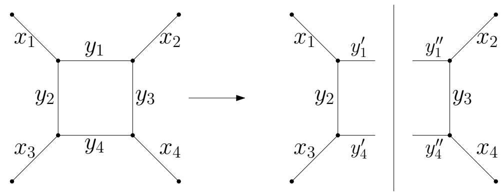

Figure 6.15: Notation for edges.

For the notation for edges see Figure 6.15. The notation for pairs of colors of edges adjacent to $x_1, \ldots, x_4$ is in Figure 6.14.

According to Lemma 6.4 we can color the edges y_1', y_2 and y_4' in such a way that x_1, y_2 and x_3 are normal unless $c(e) = c(f)$ and $A \not\sim_{c(e)} C$ which is exactly the case we have excluded in this theorem. So we color y_1', y_2 and y_4' using Lemma 6.4.

It holds $A \sim_{c(e)} B$ and $C \sim_{c(f)} D$ so coloring y_1'' with the came color as y_1', y_3 as y_2 and y_4'' as y_4' makes the edges x_2, y_3 and x_4 normal. And finally, if we use these colors in the former — not split — square (for y_1 and y_4 we use the colors of y_1' and y_4'), the edges $x_1, \ldots, x_4, y_2, y_3$ will be still normal and y_1 and y_4 will be poor. □

Conclusion

We examined some weakenings of Jaeger's conjecture of the form: for all graphs in a specific class of cubic graphs there is a proper 5-edge-coloring with at least a given number of edges normal.

We showed how to color all generalized prisms in such a way that 2/3 of the edges are normal. Namely, the edges of both cycles of generalized prisms are normal in our coloring. There arise several possibilities of improving this result. We can extend the class of generalized prisms by allowing diagonals or adding one or even more cycles. Natural improvement is also increasing the number of normal edges.

Then we colored graphs without cycles of length less than $2h$ in such a way that at least $\frac{|E(G)|}{2} \cdot \frac{2^h-2}{2^h-1}$ edges were normal. This is less that the $\frac{2}{3}|E(G)|$ we proved for the generalized prisms, but we do not require any special structure of the graph besides the absence of short cycles.

Also, we proposed a different way to look at normal coloring — chains. We proved some statements about cuts in a normally colored graph, which also follow from Petersen flow, and another statement, that in an almost normally colored graph there cannot be only one not normal edge. We hope that chains may help with further progress in Jaeger's problem.

Finally, we investigated a square in a normally colored graph and we outlined obstacles in a way to prove that the minimal counterexample to Jaeger's conjecture does not contain a square.

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