Charles University in Prague

Faculty of Social Sciences
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RIGOROUS THESIS

Multivariate Dependence Modeling using Copulas

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Declaration of Authorship

The author hereby declares that he compiled this thesis independently, using only the listed resources and literature.

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Signature

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Abstract

Multivariate volatility models, such as DCC MGARCH, are estimated under assumption of multivariate normal distribution of random variables, while this assumption has been rejected by empirical evidence. Therefore, the estimated conditional correlation may not explain the whole dependence structure, since under non-normality the linear correlation is only one of the dependency measures.

The aim of this thesis is to employ a copula function to the DCC MGARCH model, as copulas are able to link non-normal marginal distributions to create a corresponding multivariate joint distribution. The copula-based MGARCH model with uncorrelated dependent errors permits to model conditional correlation by DCC-MGARCH and dependence by the copula function, separately and simultaneously. In other words the model aims to explain additional dependence not captured by traditional DCC MGARCH model due to assumption of normality. In the empirical analysis we apply the model on datasets consisting primarily of stocks of the PX Index and on the pair of S&P500 and NASDAQ100 in order to compare the copula-based MGARCH model to traditional DCC MGARCH in terms of capturing the dependency structure.

JEL Classification	C3, C5, G0
Keywords	Copula, MGARCH, Dependency, Non-normal multivariate distribution

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Abstrakt

Problémem vícerozměrných modelů pro volatility časových řad, jako je DCC MGARCH model, je jejich předpoklad vícerozměrného normálního rozdělí zkoumaných řad. Mnohé empirické studie však popírají předpoklad normálního rozdělení akcií na finančních trzích. Z toho důvodu mohou být odhadnuté podmíněné korelace zavádějící, jelikož nemusí vysvětlovat celou strukturu závislosti mezi zkoumanými veličinami. Je známé, že korelace je jen z jedním z nástrojů měření závislosti nenormálně rozdělených dat.

Cílem této práce je integrace copula funkcí do tradičního DCC MGARCH modelu, protože právě copula funkce umožnují vytvoření vícerozměrného rozdělení náhodných veličin pro více marginálních rozdělení i v případě, kdy nejsou normálně rozdělená. Takzvaný Copula-based MGARCH model s nekorelovanými závislými rezidui dovoluje modelovat jak korelaci mezi náhodnými veličinami (pomocí DCC MGARCH), tak i závislost mezi nimi (pomocí copula funkce), obojí odděleně avšak simultánně. Jinými slovy, model je schopen vysvětlit dodatečnou závislost, která nebyla zachycena DCC MGARCH modelem kvůli jeho předpokladu normálního rozdělení. V empirické analýze aplikujeme tento model na různé data, zejmená na akcie českého PX index a dále na pár likvidních amerických index; S&P500 a NASDAQ100 abychom mohli srovnat MGARCH založený na copula funkci s tradičním DCC MGARCH. Zaměříme se zejména na výkon modelů při vysvětlovaní závislosti.

Klasifikace JEL	C3, C5, G0
Klíčová slova	Copula, MGARCH, závislost náhodných veličin, nenormální vícerozměrné rozdělení

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Acronyms

ARCH Autoregressive Conditional Heteroscedasticity

GARCH General Autoregressive Conditional Heteroscedasticity

MGARCH Multivariate General Autoregressive Conditional Heteroscedasticity

C-MGARCH Copula-based Multivariate General Autoregressive Conditional Heteroscedasticity

DCC Dynamic Conditional Correlation

CCC Constant Conditional Correlation

BEKK Baba, Engle, Kraft, Kroner

MLE Maximum likelihood Estimation

PSE Prague Stock Exchange

Master Thesis Proposal

Author	Bc. Marek Klaus
Supervisor	PhDr. Boril Šopov, MSc., LL.M.
Proposed topic	Multivariate Dependence Modeling using Copulas

Topic characteristics Since the autoregressive conditional heteroscedasticity (ARCH) model was introduced (Engle 1982), many academic papers have been denoted to univariate volatility modeling using numerous extensions of ARCH and stochastic volatility (SV) models; in other words, the volatility modeling of returns was the centre of attention. Nevertheless, employing multivariate volatility modeling into practice and thus understanding the co-movements of financial returns is of a great practical importance (Silvennoinen & Teräsvirta 2009).

The multivariate GARCH (MGARCH) models allow to understand and predict the temporal dependence in the second-order moment of asset returns, which could be important form many financial econometrics issues. Adding the fact that financial volatilities move together over time across assets and markets, it is obvious multivariate modeling open new additional possibilities in comparison to univariate models. In other words it opens the door to better decision tools in various areas, such as asset pricing, portfolio selection, option pricing, hedging and risk management. Speaking in general, the MGARCH can be applied to study the relations between the volatilities and co-volatilities of several markets (Bauwens *et al.* 2006).

Furthermore, dependency modeling can be difficult as the financial returns are usually following a complicated dynamics; stock market returns are non-normal and it is often simply impossible to specify the multivariate distribution relating return series (Jondeau & Rockinger 2006). In this context, we shall also employ the copula- based MGARCH models with uncorrelated dependent errors, denoted as C-MGARCH, introduced by (Lee & Long 2009).

The aim of this paper is to provide an empirical application of different MGARCH and C-MGARCH models on the data set containing daily returns of Prague Stock Exchange (PX Index) and New York Stock Exchange (NY Composite Index) and subsequently determine:

- what model specification yields better results in explaining the data set
- whether C-MGARCH models estimates give better output than the MGARCH models on the data set

For the purpose of multivariate modeling the following types of MGARCH are going to be used:

- BEKK-GARCH model, where the conditional covariance matrix (CCM) is modeled directly
- DCC-GARCH models, which tend to model the conditional variances and correlation instead of CCM Both of the models can be used as copula-based as introduced by (Lee & Long 2009)

Hypotheses

- DCC-GARCH model is more suitable for explaining our data set than the BEKK-GARCH model
- Copula-based MGARCH are giving better results in explaining the dependencies among the volatilities and co-volatilities comparing to the corresponding normal MGARCH models

Methodology The theoretical part will provide, after a brief introduction of the univariate GARCH, a theoretical discussion of the MGARCH model types and will overview the BEKK and DCC specification proposed by literature. Next part shall introduce the copula functions, their different types and define the copula-based GARCH for the case of BEKK and DCC volatility models. The empirical part will fit the two types of MGARCH and C-MGARCH on our data set. First, we shall evaluate the BEKK and the DCC model based on the parsimony and on the ability to represent the dynamics of the conditional variances and covariances. We focus on the estimated correlations, on model errors testing against iid, and whether the model is correctly specified, i.e. if describes the data set. Moreover, we take in consideration the Akaike's information criterion (AIC) and the likelihood-based ranking. Second, we employ the copula-based GARCH models and compare the results to the normal MGARCH models. Determination criteria shall be similar to the previous part.

Proposed tests:

- The White test – tests for heteroskedasticity
- Augmented Dickey-Fuller Test – unit root testing (stationarity of a time series)
- Ljung Box test – tests, whether any group of autocorrelation is different from zero
- General misspecification tests (proposed by Ling and Li) – testing, whether the model is adequately specified (for a MGARCH model)

Outline

1. Introduction - Why Multivariate Modeling Matters?
2. Literature Survey (univariate and multivariate volatility models, DCC-MGARCH, BEKK-MGARCH and introduction to the copula-based MGARCH models)
3. Data Overview (PX Index, NYSE Composite Index)
4. Empirical Analysis (In-Sample Performance, Out-of-Sample Performance)
5. Models Comparison
6. Conclusion

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Author Supervisor

Chapter 1

Introduction

Modeling of volatility and correlation among financial assets has been of much attention in recent decades and is crucial for modern portfolio theory. Both the Capital Asset Pricing Model (CAPM) and the Arbitrage Pricing Theory (APT) aims to determine an optimal portfolio selection assuming financial returns to follow multivariate normal probability distribution. This assumption allows them to use linear correlation coefficient as dependence measure between different financial instruments.

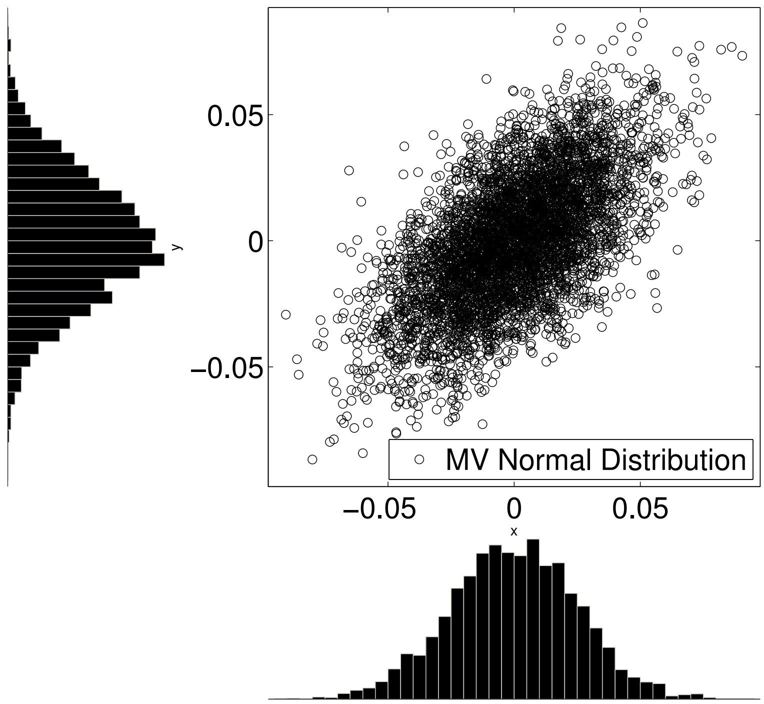
Nevertheless, there is an overwhelming empirical evidence of non-normality of financial assets returns. This non-normality became a source of financial issues such as "heavy-tailedness", which has been currently discussed by both risk managers and regulators, or of "volatility smile" that has been commonly used by traders to define their strategies (Cherubini *et al.* 2004). To put it differently, in the real world, actors do not take the risk of overlooking non-normality.

On the other hand, there is still an important issue when estimating correlation – its interpretation under non-normality. It is well-known that correlation is only one particular measure of dependence and in case of non-elliptical distribution it does not capture the whole dependency structure among random variables.

Simply, non-normality of financial assets returns does not only bring a problem to usage of CAPM or APT but it also complicates usage of multivariate volatility models that aim to estimate dependence (which is crucial for portfolio theory) among multiple assets. If we consider the autoregressive conditional heteroscedasticity (MGARCH) class of multivariate models, which is particularly represented by the VEC MGARCH (Bollerslev *et al.* 1988) and its direct extension the BEKK MGARCH (Engle & Kroner 1995) or by the constant conditional correlation (CCC) MGARCH model proposed by Bollerslev (1990) or its improved "version" the dynamic conditional correlation (DCC) (Engle 2002), it must be noted that these models have a common drawback – they built on multivariate normality. However, as we show below in motivation, this is in conflict with empirical evidence.

To go even further, the estimated correlation by the MGARCH models does not have to capture the whole dependence structure between financial assets. In other words, there might be additional dependence not explained by estimated correlation. For a better interpretation of stated problems, we present the following motivation:

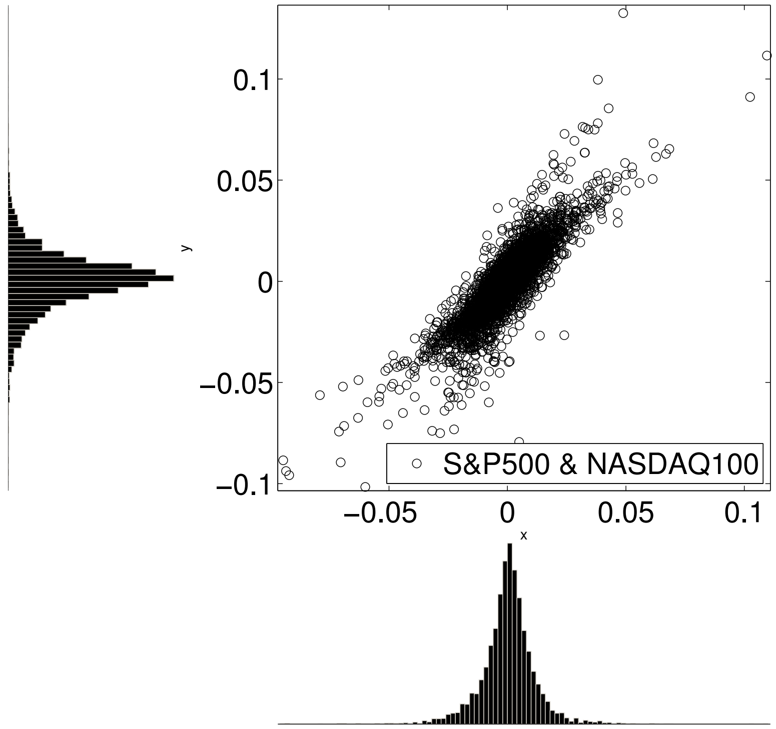
Figure 1.1: Scatter Plot of Multivariate Normal Distribution



Source: author's computations.

Motivation 1. We present two scatter plots in Figure 1.1 and in Figure 1.2 to demonstrate non-normality of financial returns; the upper plot is a simulation of multivariate normal distribution for five thousand points with correlation 0.6. On contrary the lower plot is a scatter of SP500 and NASDAQ100 stock indices for the last two decades. As we can see, according to the histograms, real data are not elliptically distributed, the histograms signal excessive kurtosis and heavy tails.

Figure 1.2: Scatter Plot of S&P500 and NASDAQ100

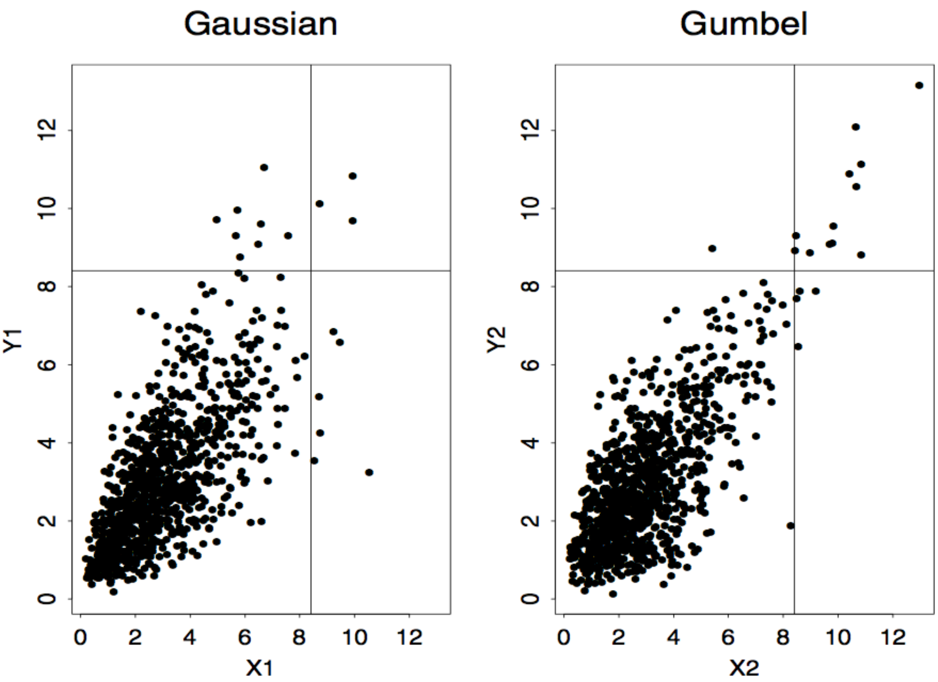


Source: author's computations.

Motivation 2. Following Embrechts *et al.* (2002) we show scatter plots of one thousand bivariate realizations of two different probability models of two random variables. Both models have identical gamma marginal distributions $Gamma(3, 1)$ and the same linear correlation $\rho = 0.7$. However, these two models differ in dependence between the random variables. From Figure 1.3 it is evident, that the models on the right side display much higher tail dependence. In an empirical application on financial markets that would mean financial instruments show a higher probability of extreme losses that have a tendency to happen together.

For the case of univariate modeling, effective answers to non-elliptical distribution of financial returns have been given in terms of risk management or in pricing. Nevertheless, under the multivariate case overlooking the non-normality may be problematic as we shown in motivation 2. On the one hand, asset management such as hedge funds builds up on the non-normality and makes it rather a investment tool than a econometric problem. On the other hand, portfolio management (in terms of diversification) needs a correct estimation of dependence between financial assets, i.e. it accounts for co-movements among variables that are not normally distributed Cherubini *et al.* (2004).

Figure 1.3: Simulation of bivariate distribution with the same correlation but different dependence



Source: Embrechts *et al.* (2002).

Since multivariate analysis does not offer various probability distributions in comparison to univariate case, many academic papers have recently proposed a solution for multivariate modeling under non-normality – that lies in integration of a *copula* function into the models. Copula can be interpreted as a tool adopted from statistics, that allows to link N marginal (i.e. univariate) probability distributions into a corresponding multivariate joint distribution. Additionally, copulas carry all the dependence information between N variables. Therefore, if we apply a copula function in the MGARCH models, we will be able to model non-elliptical multivariate dataset and to explain all the dependence structure as well.

The goal of this thesis is to apply copula-based multivariate GARCH (C-MGARCH) model with uncorrelated dependent errors (Lee & Long (2009)) on real datasets consisting primarily of the PX Index (and its main stocks) traded on Prague Stock Exchange and of a pair of american stock indices; S&P500 and NASDAQ100. The model is able to estimate both the correlation (by MGARCH) and dependency (by a copula) separately and simultaneously. Then, the results shall be compared to traditional DCC MGARCH without a copula. There are three copulas for which we constructed the C-MGARCH;

Gaussian, Clayton and Gumbel. We aim to show the copula-based models reveal additional dependence, that was not captured by estimated correlation of traditional GARCH models. Moreover, we assume that models with copulas dominate the common MGARCH in terms of goodness of fit since copula-based models build up on an assumption of non-elliptical distribution of financial returns. We focus on four bivariate datasets; PX Index & CEZ, Erste Group Bank & Komercni banka, Unipetrol & Telefonica C.R. and on mentioned S&P500 & NASDAQ100.

The structure of this thesis is as follows: Chapter 2 starts with an overview of related literature and presents the evolution of univariate and multivariate volatility models as well as models with copulas. In Chapter 3 we set up a theoretical background for models used in the empirical analysis. Chapter 4 introduces copula functions and shows their importance, additionally copula-based multivariate GARCH models are defined. In Chapter 5 we present needed methodology and procedure for models estimation and comparison of results, moreover we introduce datasets and *ex-ante* checking. Chapter 6 shows results of estimated models and compare performance of copula-based and traditional multivariate GARCH models focusing on correlation and dependence. In Chapter 3 we summarise our results and findings and conclude.

The Rigorous Thesis Extension

The Rigorous Thesis, as a direct extension of the Diploma Thesis submitted in June 2012 [1], enhances the original thesis in the following areas:

- Introduction of the EGARCH and the GJR–GARCH models as an enhanced specification of the conditional variance model for the univariate (the first step) modelling of the multivariate datasets.

- Adding the definition of the Student's t-distribution as an option to the univariate GARCH model specification

- Introduction of new hypothesis; the results of the copula-based MGARCH models improve as the univariate GARCH model (the first step of the estimation) is enhanced - either by employment of the improved specification of the model such as asymmetric GARCH models, or by employment of

[1] the Institute of Economic Studies (IES), Charles University in Prague - submitted under the same title.

the Student's t-distribution instead of the Gaussian distribution. Empirical analysis (comparison) of the models provided in Section 6.5.

- All the data samples have been extended; the SP500–Nasdaq100 dataset has been added with 1792 observation. The PX Index and its components have been slightly improved (one month of observations added). Reestimation of the models, improvement of the source (MATLAB code), slight adjustments to the results - in the Rigorous Thesis the copula-based models unambiguously outperform the traditional DCC model on the tested datasets.
- Covering the opponents remarks, e.g. note about dividends and stock split on the datasets

Chapter 2

Literature Overview

In this section we briefly introduce the related literature on volatility modeling using the ARCH/GARCH family of models. However there exist different types of volatility models, such as Stochastic volatility, we focus only on the GARCH class of models in this thesis.

We divide the topic overview into the following categories; univariate GARCH models, multivariate GARCH models and copula-based multivariate GARCH models.

2.1 Univariate GARCH Models

Before the Engle (1982)'s introduction of *autoregressive conditional heteroscedasticity* (ARCH) in his seminal paper, traditional econometric models, such as autoregressive moving average (ARMA) (Box & Jenkins 1970), were assuming a constant, one-period forecast variance. In other words, the variance did not depend upon the past information, i.e. we discuss an unconditional variance. Moreover, the variance was assumed to be homoscedastic, i.e. with constant mean.

Yet the ARCH model assumes the conditional variance to be dependent on past errors and variances at time t and is designed to model time varying volatility, to put it differently, this model is designed to model heteroscedastic volatility - a volatility changing over time, hence non-constant. Additionally, the model aims to explain a typical financial market feature; the volatility clustering - "large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes" (Mandelbrot & Taylor 1967))".

Considering the mentioned attributes, the ARCH model and its generalizations are applied to modeling interest rates, exchange rates, stock and stock index returns and many others (Silvennoinen & Teräsvirta 2009).

The General ARCH (GARCH) was proposed by Bollerslev (1986) as an extension of the ARCH model to allow for both a longer memory and a more flexible lag structure. It has been shown in empirical applications that a relatively long lag in the conditional equation of the ARCH model is often used - Engle (1982), Engle (1983) and Engle & Kraft (1983) proposed a fixed lag structure to avoid problems with negative variance parameters estimates. To rephrase, the unconditional autocorrelation function of squared residuals of ARCH model decays too rapidly compared to a typical time series behavior, unless a large lag q is employed. On the contrary, GARCH allows for slow decay, though exponential.

Subsequently, many variations to GARCH have been proposed; EGARCH, FIGARCH, TGARCH, QARCH and many others. Yet, discussing these models in not our purpose.

2.2 Multivariate GARCH Models

Multivariate GARCH models were initially developed in the late 1980s as a direct extension of the univariate models.

A typical feature of the first class of MGARCH models is the direct modeling of the conditional covariance matrix H_t, i.e. matrix that contains conditional covariances of random variables on diagonal and conditional variances on off-diagonals. The first model proposed is *VEC* by Bollerslev *et al.* (1988), which models each element of H_t as a linear combination of the lagged squared errors and cross-products of errors and lagged values of the elements of H_t (Bauwens *et al.* 2006). However, the VEC model suffers two main problems; no assurance of positive definiteness of H_t (i.e. no guarantee of positive values for variances and covariances) and a high number of parameters. To correct the latter drawback, Bollerslev *et al.* (1988) introduce the *Diagonal VEC* model, which has H_t positive definite for every t by assuming diagonality of each matrix in the model's form (see Bauwens *et al.* (2006) among others). Subsequently, Engle & Kroner (1995) introduce the *BEKK*[1] model, which is basically a restricted

[1] The name is an acronym of all the contributors to the multivariate model; Baba, Engle, Kroner and Kraft

version of the VEC model, with properties ensuring positive definiteness of the correlation matrix.

The second class of MGARCH, the Factor Models, proposes parameterization of the conditional covariance matrix using the following idea; the comovements of the stock returns are driven by a small number of common underlying variables, the factors (Bauwens *et al.* 2006). The class is represented by the *factor GARCH* (F-GARCH) model (Bollerslev & Engle 1993) and its variation, the *full-factor GARCH* (FF-GARCH) model, proposed by (Vrontos *et al.* 2003).

Recently, the multivariate GARCH models decomposing H_t, i.e. the third class, has gained a lot of attention and become very popular approach to multivariate volatility - to summarize the members of this class; first, Bollerslev (1990) presents *Constant Conditional Correlation* (CCC) model. Second, Engle (2002) improves the model and introduces the *Dynamic Conditional Correlation* (DCC). In the same year, Tse & Tsui (2002) present *Varying Correlation* (VC). Subsequently, various extension of the DCC are proposed; the *Flexible Dynamic Conditional Correlation* (FDCC) of Billio *et al.* (2006), the *Asymmetric Generalized Dynamic Conditional Correlation* (AG-DCC) model of (Cappiello *et al.* 2006), *Generalized Dynamic Conditional Correlation* (GDCC) of (Hafner & Franses 2009). Nevertheless, the empirical part of this paper applies only DCC-MGARCH model and its variations with copula functions.

2.3 Copula Based Multivariate GARCH

The following paragraphs summarize an extension of the multivariate class of GARCH models; we shall briefly overview proposed volatility models incorporating multivariate GARCH with a copula.

Distribution of the portfolio return depends on the univariate distribution of each of the assets and of the dependency between each of them. This class suggests to capture the dependency by a function called a *copula*. The name of the function comes from a latin word for "linked" and was proposed by Sklar (1959) who in his theorem showed, that any N-dimensional joint distribution function may be decomposed into its N marginal distributions, and a copula function that completely describes the dependence between N variables. Nelsen (1999) in his introductory monograph for copula functions starts with the following statement; "from one point a view, copulas are functions that join or 'couple' multivariate distribution functions to their one-dimensional marginal distribution functions. Alternatively, copulas are multivariate distribution functions whose one-dimensional margins are uniform on the interval (0,1)."

Employing copulas in finance has been subjected to enormous interest in recent years. In 1999 a working paper of Embrechts *et al.* (2002) had been already circulation, suggesting using copulas in finance as one of the first and most influential papers of this topic. Following by comprehensive and readable introduction to copulas and their mathematical and statistical properties published by Nelsen (1999) and Joe (1997), oped new opportunities to copula usage not only in finance, but also in macroeconomics, microeconomics and in developing the estimation and valuation theory required for these application (Patton 2009). Considering our topic, we shall focus on application of copulas on financial time series;

It has been known that returns of financial time series are, as well as residuals obtained from a univariate GARCH model, generally non-normal, i.e. fat-tailed distributions for errors have been introduced. However, specification of a multivariate distribution relating to N non-normal univariate distribution is not straightforward, yet it is often impossible. Based on that, copula based GARCH models propose to use GARCH-type models for obtaining the univariate distributions and then the joining distribution by a copula function.

It should be noted that copulas have been used for both univariate and multivariate GARCH models; in the univariate analysis of time series, copulas characterize the dependence between a sequence of observations of a scalar time series process. A conditional cross-sectional dependence among time series is subjected to multivariate modeling.

Chapter 3

The GARCH Class of Models

Working with the multivariate GARCH family of models is conditional upon understanding of univariate GARCH models, therefore we shall start with a brief summary of the ARCH and GARCH models. Subsequently we introduce the multivariate GARCH, define it and describe its variations that are used in the empirical part.

3.1 Univariate Volatility Modeling

For the multivariate estimation we first need an univariate estimation for each series to get univariate residuals that are subsequently used for multivariate modeling and copula building. The complete procedure is presented in Chapter 5. Therefore in this section we define the ARCH and GARCH processes. Even though the ARCH process in not directly used in the empirical part in Chapter 6 we start with its definition in the sake of clarity before introducing the GARCH model.

3.1.1 Autoregressive Conditional Heteroscedasticity

To define the ARCH model proposed by Engle (1983), assume ϵ_t to be a random variable with a mean and a variance conditional on the information set ψ_{t-1} (the σ-field generated by $\epsilon_{t-1}, j \geq 1$), then ARCH model of ϵ_t follows these properties; (1) conditional mean is described as $E\{\epsilon_t|\psi_{t-1}\} = 0$ and (2) the conditional variance $h_t = E\{\epsilon_t^2|\psi_{t-1}\}$ is a nontrivial positive-valued parametric function of ψ_{t-1}. The sequence $\{\epsilon_t\}$ can be a innovation sequence of an econometric model as shown in the equation below (Silvennoinen & Teräsvirta

2009):

$$\epsilon_t = y_t - \mu_t(y_t) \tag{3.1}$$

where y_t is an observable random variable and $\mu_t(y_t) = E\{y_t|\psi_{t-1}\}$ is the conditional mean of y_t on ψ_{t-1}. In financial application ϵ_t is considered to be a vector of log-returns of N assets. According to Engle, ϵ_t can be decomposed as shown in the following equation:

$$\epsilon_t = z_t h_t^{1/2} \tag{3.2}$$

where the sequence z_t has zero mean and unit variance and is independent, identically distributed (*iid*) random variables. Then the ARCH(q) process is defined by following equations:

$$y_t|\psi_{t-1} \sim N(0, h_t) \tag{3.3}$$

$$h_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j \epsilon_{t-j}^2 \tag{3.4}$$

where q is the order of the ARCH process and $\alpha_0 > 0, \alpha_j \geq 0, j = 1,\ldots,q - 1, \alpha_q > 0$, and where N stands for a normal distribution, yet a non-normal distribution, such as student's t or general error distribution can be applied.

3.1.2 General Autoregressive Conditional Heteroscedasticity

Knowing how the ARCH model is defined, it is not demanding to introduce the GARCH model (Bollerslev 1986), since this model is a direct extension of the ARCH model. Assume ϵ_t^2 to be a discrete-time stochastic process and ψ_{t-1} the σ-field information set, then GARCH is given as:

$$y_t|\psi_{t-1} \sim N(0, h_t) \tag{3.5}$$

$$h_t = \alpha_0 + \sum_{j=1}^{q} \alpha_j \epsilon_{t-j}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} \tag{3.6}$$

where $p \geq 0$, $q > 0$, $\alpha_0 > 0$, $\alpha_j \geq 0, j = 1, \ldots, q$, $\beta_j \geq 0, i = 1 \ldots p$. If $p = 0$ than GARCH(p, q) would be equal to the ARCH(q) process and if $p = q = 0$ then ϵ_t^2 is a white noise. This model is preferred to simpler ARCH process since it is a more parsimonious model of the conditional variance than a high-order ARCH.

3.1.3 Exponential GARCH

For the purposes of the empirical part of the thesis we need to introduce the exponential GARCH model originally introduced by Nelson (1991). As Teräsvirta (2009) mentioned in his survey of the univariate class of the conditional heteroscedasticity models; Nelson criticised the GARCH models due to the following points:

- Parameters restrictions ensure positivity of the conditional variance at any point of time
- GARCH does not allow for asymmetric response to shocks
- if the sum of the *alpha* and the *beta* parameters is equal to 1 (the so called integrated GARCH model) then the model is strong but not weak stationary and heavy persistent

As a response to these issues, the EGARCH has been presented in the following form:

$$\ln h_t = \alpha_0 + \sum_{j=1}^{q} g_j(z_{t-j}) + \sum_{j=1}^{p} \beta_j \ln h_{t-j} \tag{3.7}$$

where $g_j(z_{t-j}) = \alpha_j z_{t-j} + \gamma_j(|z_{t-1}| - \mathrm{E}|z_{t-j}|)$ for $j = 1 \ldots q$. Due to the logaritmization the parameters need not to be restricted. The parameter γ_j controls the asymmetric effect, i.e. if $\gamma_j = 0$ then the model treats both the positive and negative shocks equally. Contrary when the parameter is $\gamma_j < 0$ ($\gamma_j > 0$ respectively) than e.g. the negative shock has a larger (respectively smaller) impact on conditional variance than a positive shock.

3.1.4 GJR-GARCH

Similarly to the previous subsection, the model proposed by Glosten *et al.* (1993) is the extension of the GARCH that accounts for asymmetry shocks assuming that the response of the variance to a shock is dependent of the sign of the shock (Teräsvirta 2009). The GJR-GARCH [1] model has the form:

$$h_t = \alpha_0 + \sum_{j=1}^{q} \{\alpha_j + \delta_j I(\epsilon_{t-j} > 0)\}\epsilon_{t-j}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} \tag{3.8}$$

[1] The Glosten-Jagannathan-Runkle GARCH model

Where

$$I(\epsilon_{t-j} > 0) = \begin{cases} 1, & \text{when the argument is true} \\ 0, & \text{otherwise.} \end{cases}$$

3.1.5 Student's t-distribution

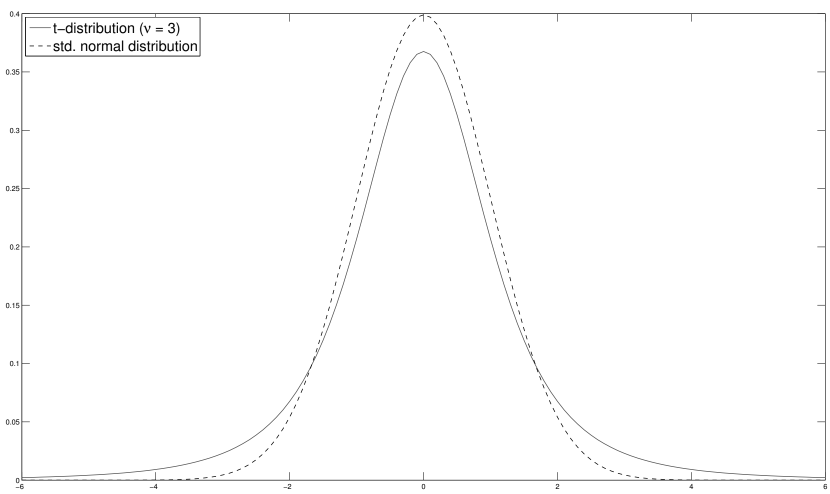
Engle (1982) in his original work assumed the random variable z_t from the Equation 3.2. However, as was reported by many authors (see, for example, Chapter 4 or as we show in Chapter 6 the financial results are not normal, yet leptocurtic and skewed, therefore we show the specification of the Student's t-distribution, that was - among others - proposed by Bollerslev (1986) when introducing the GARCH model.

Compared to the ordinary normal distribution, the t-distribution accounts with thick tails that are controlled by degrees of freedom ν. The distribution becomes identical to normality when $\nu \to \infty$. The density function of the Student's t-distribution is characterised as follows:

$$F(z_t, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi(\nu-2)}}(1 + \frac{z_t^2}{\nu - 2})^{-\frac{\nu+1}{2}} \tag{3.9}$$

Where $\Gamma(\nu) = \int_0^\infty e^{-x}x^{\nu-1}dx$ is the gamma function. In the Figure 3.1 we demonstrate the difference between densities of the standardised normal and Student's t-distribution.

Figure 3.1: Student's t-distribution and std. normal distribution comparison



Source: author's computations.

3.2 Multivariate Volatility Modeling

Recently many studies have focused on extension of the univariate GARCH model to the multivariate case, i.e. allowing for employing of multiple assets returns for volatility modeling. The multivariate modeling framework brings new possibilities to better decision tools in various areas, such as asset pricing, portfolio selection, option pricing, hedging and risk management Bauwens *et al.* (2006). Nevertheless, generalisation of these models face two main drawbacks. First, the number of parameters increases rapidly with the number of assets. Second, it is not effortless to ensure positive definiteness of the covariance matrix Tsay (2006). Considering the mentioned difficulties, many different models have been introduced. The aim of this subsection is to provide an overview of the multivariate GARCH models used in the empirical analysis.

Before defining MGARCH models, it is crucial to redefine the estimated model and its disturbances, ϵ_t, as it differs from the univariate models stated in the previous subsection. For the multivariate modeling, we assume y_t to be a stochastic process of dimension $N \times 1$ and θ to be a finite vector of parameters, then:

$$y_t = \mu_t + \epsilon_t \tag{3.10}$$

where the μ_t is the conditional mean vector and

$$\epsilon_t = H_t^{1/2} z_t \tag{3.11}$$

where the positive definite $N \times N$ matrix $H_t^{1/2}$ is the conditional covariance matrix of returns ϵ_t and z_t. In other words, returns ϵ_t are conditionally homoscedastic on ψ_{t-1}. Analogously to univariate models, ψ_{t-1} is the σ-field information set generated by ϵ_t. Furthermore, z_t has zero mean and unit variance, i.e.:

$$\begin{aligned} E(z_t) &= 0 \\ var(z_t) &= I_N \end{aligned} \tag{3.12}$$

where I_N is identity matrix of order N.

In the following subsections we define MGARCH models with Directly Modeled H_t, which are subsequently used in Chapter 6.

3.2.1 MGARCH Models with Conditional Variances and Correlations

Models presented in this section do not model the conditional correlation matrix H_t directly, but decompose the matrix into conditional standard deviation and correlation. Since these models are nonlinear combination of univariate GARCH models, one can model individual conditional variances and conditional correlation matrix separately.

Generally, showing theoretical results on stationarity, moments and ergodicity is more demanding in comparison to models in previous subsection. On the other hand, this kind of models does not face the problem of too many parameters, hence are easier to apply.

Constant Conditional Correlation Multivariate GARCH

First, we start with the simplest multivariate correlation model, crucial for other models in this class. Bollerslev (1990) introduces *Constant Conditional Correlation* (CCC-) MGARCH, where conditional correlation matrix is time-invariant, constant, hence the conditional covariances are proportional to the product of the corresponding conditional standard deviations, which reduces the number of unknown parameters Bauwens *et al.* (2006). Bollerslev defines H_t in the following way:

$$H_t = D_t R D_t = (\rho_{ij}\sqrt{h_{iit}h_{jjt}}) \tag{3.13}$$

where

$$D_t = diag(\sqrt{h_{11t}}, \ldots, \sqrt{h_{NNt}}) \tag{3.14}$$

The non-diagonal elements of the conditional covariance matrix are defined:

$$[H_t]_{ij} = \sqrt{h_{it}}\sqrt{h_{jT}}\rho_{ij}, \qquad i \neq j \tag{3.15}$$

and $R = [\rho_{ij}]$ is a symmetric, positive definite matrix with $\rho_{ii} = 1$ for $i = 1, \ldots, N$. Any univariate GARCH model can be applied to define h_{iit}. Nevertheless, the CCC model is the usually modeled as the GARCH(p,q) model, therefrom the conditional variance has the following form:

$$h_t = \omega + \sum_{j=1}^{q} A_j r_{t-j}^{(2)} + \sum_{j=1}^{p} B_j h_{t-j} \tag{3.16}$$

where ω is a $N \times 1$ vector, A_j and B_j are diagonal $N \times N$ matrices, and $r_t^{(2)} = r_t \odot r_t$. The model contains $N(N+5)/2$ free parameters. The positive definiteness of H_t is satisfied if and only if R is positive definite and all of the N conditional variances is positive.

The quasi maximum likelihood method is used to estimate the model, assuming conditional normality. Drawback of the CCC model is the constant conditional correlation assumption, which may be a problem for a practical usage Ledoit *et al.* (2003).

Dynamic Conditional Correlation Multivariate GARCH

Second, we discuss another class of MGARCH proposed by Engle (2002) as a generalization of Bollerslev (1990)'s CCC model presented above. *Dynamic Conditional Correlation* (DCC-) MGARCH aims to utilize the GARCH models flexibility but in a combination with parsimonious parametric models for the correlations. These models are estimated with a two step method based on the likelihood function, i.e. the two-stage maximum likelihood estimation. Engle (2002) showed DCC model performs reasonably well in empirical applications and gives sensible empirical results.

The DCC proposed the following form for H_t:

$$H_t = D_t R_t D_t \tag{3.17}$$

i.e. the difference lies in the allowing of R parametrization, in other words R is time-varying in the model contrary to the CCC model. The correlation matrix H_t is positive definite if the conditional correlation matrix R_t is positive definite and the conditional variances h_{it}, $i = 1, \ldots, N$ are well-defined.

Chapter 4

Copula Function and Copula-based Multivariate Model

The structure of this chapter is as follows; first, we introduce basic theory needed for understanding a copula function and the idea behind employing copulas in financial modeling. Second, we present two measures of dependence; traditional correlation and an alternative dependence measure based on a copula function, again, we provide the nature behind copula employment. Third, we introduce two classes of copulas and their particular representatives. Finally, we establish the crucial model of the empirical part, the copula-based MGARCH and show its main properties to highlight how it differs from traditional MGARCH models.

4.1 Copulas

Many empirical studies showed that the financial time series are not normal, they are skewed, leptokurtic and asymmetrically dependent (see, for example, Patton (2006)), therefore we cannot expect a financial dataset consisting of multiple time series to be multivariate normal. Recently, not only as a reaction to the issues of multivariate normality, many papers have proposed employing a copula function in finance and modeling.

Copula is a function that links together univariate marginal distributions in order to create corresponding joint multivariate distribution function. Additionally, following Patton (2009), if all of the variables are continuously distributed, then the copula function is their joint multivariate distribution function with uniformly distributed, i.e. $U(0,1)$ uniform margins.

This section shall present a formal definition of a copula function and the Sklar's Theorem. Subsequently, it defines two different dependency measures, one traditional and one based on copula functions. Next, the elliptical and the Archimedean class of copulas are presented. Finally, at the end of this section, particular copulas and their properties will be introduced.

4.1.1 Copula Definition and Sklar's Theorem

Let's start with the formal definition of a copula function:

Definition 4.1 (Copula Function). A function $C : [0,1]^2 \rightarrow [0,1]$ is a copula if it satisfies:

- $C(u_1, u_2) = 0$ for $u_1 = 0$ or $u_2 = 0$
- $\sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^{i+j} C(u_{1,i}, u_{2,j}) \geq 0$ for all $(u_{1,i}, u_{2,j})$ in $[0,1]^2$ with $u_{1,1} < u_{1,2}$ and $u_{2,1} < u_{2,2}$
- $C(u_1, 1) = u_1$, $C(1, u_2) = u_2$ for all u_1, u_2 in $[0,1]$

Next, we need to introduce a crucial theorem proposed by Sklar (1959) to define the relationship between joint distribution and a copula function:

Sklar's Theorem

Theorem 4.1 (Sklar's Theorem). Let $F_{1,2}$ be a joint distribution function with margins F_1 and F_2. Then there exists a copula C such that for all η_1, η_2,

$$F_{12}(\eta_1, \eta_2) = C(F_1(\eta_1), F_2(\eta_2)) = C(u_1, u_2) \tag{4.1}$$

Conversely, if C is a copula and F_1 and F_2 are marginal distribution functions, then the functions F_{12} defined above is a joint distribution function with margins F_1 and F_2.

The equation (4.1) can be rewritten:

$$F_{12}(\eta_1, \eta_2) = C(F_1(\eta_1), F_2(\eta_2); \theta) = C(u_1, u_2; \theta) \tag{4.2}$$

where θ is the copula parameter of dependency.

Copula is sometimes referred as a “dependency function” since it contains all of the dependence information between F_1 and F_2, because according to **??** the joint distribution F_{12} contains all the univariate and multivariate information, while F_1 and F_2 contains only all the uniform information.

The theorem is very fundamental for a copula application in time series modeling; it says given any marginal distribution $(F_1, F_2, \ldots, F_n)$ and a copula function C then the theorem can be used to obtain the joint distribution function F. Or, on contrary, any joint distribution function can be decomposed into marginal distributions and a copula function.

Suppose the joint distribution function F_{12} to be *n-times* differentiable, then the following equations denote the 2^{nd} cross-partial derivative of equation (4.1):

$$\begin{aligned} f_{12} &= \frac{\partial^2 F_{12}(\eta_1, \eta_2)}{\partial \eta_1 \partial \eta_2} = \\ &= \frac{\partial^2 C(u_1, u_2)}{\partial \eta_1 \partial \eta_2} \cdot \frac{\partial F_1(\eta_1)}{\partial \eta_1} \cdot \frac{\partial F_2(\eta_2)}{\partial \eta_2} = \\ &= c(F_1(\eta_1), F_2(\eta_2)) \cdot f_1(\eta_1) \cdot f_2(\eta_2) \end{aligned} \tag{4.3}$$

where $c(u_1, u_2) = \frac{\partial^2 C(u_1, u_2)}{\partial \eta_1 \partial \eta_2}$ is the cumulative density function (CDF) of copula. To give a simple example, the *independent copula* would be $C(u_1, u_2) = u_1 u_2$ and $c(u_1, u_2) = 1$.

4.1.2 Dependence Measures

Following McNeil *et al.* (2005) we briefly introduce two kinds of dependence measures; the traditional Pearson linear correlation, and the rank correlation. The latter measure is based on a copula function, however both of the mentioned calculates scalar measurement for a pair of random variables. Nevertheless, the specification and the nature behind each of the them varies. This part of the thesis aims to show, how these dependency measures are constructed and what they indicate.

The idea behind incorporating copulas, when a traditional correlation measure exists lies in ability of Pearson correlation to explain the relation among random variables only under a case of multivariate normality, or more generally, under elliptical distribution. These distribution are fully described by by a vector of mean values, a covariance matrix and a characteristic generator function, i.e. since means and variances are products of univariate margins, copulas for elliptical distribution are characterized by the correlation matrix and a generator function. Hence, we first present correlation and show its shortcomings when applied in non-elliptical models and, second, we present the rank correlation and define its representative, the Kendall's tau rank correlation.

Linear Correlation

According to Embrechts *et al.* (2003) we assume X and Y to be random variables with non-zero finite variances, than the linear correlation between than is given as:

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} \tag{4.4}$$

where $\mathrm{Cov} = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)$ denotes covariance of X and Y, and $\mathrm{Var}(X)$ and $\mathrm{Var}(X)$ are the variances of X and Y.

This well-known measure of correlation has the following properties:

(i) $\rho(X,Y) \in \langle -1, 1\rangle$,

(ii) $\rho(X,Y) = |1|$ means perfect correlation, positive or negative,

(iii) $\rho(X,Y) = 0$ indicates no correlation between the random variables

However, it needs to be stated that if two random variables are not correlated, it does not mean they must be independent. No correlation indicates no dependency only under normality. Correlation is considered to be only one particular measure of stochastic dependence among many others (Embrechts *et al.* 2002).

Another drawback of linear correlation is its assumption of finite variances of X and Y. This could be a problem when we apply this measure on a heavy-tailed distributions (which are typical for financial returns), where the variance of random variables may not exist.

Though linear correlation is a popular measure of dependence, it is often misinterpreted. Its popularity is given by the ease with which it can be computed, moreover it is a natural scalar measure of dependence for elliptical distributions (for example for multivariate normal distribution). Nevertheless, as far as we know most of the joint, hence multivariate, distributions are not elliptically distributed. Then, interpreting linear correlation as a measure of dependence would produce misleading results (Embrechts *et al.* 2003).

Rank Correlation

Yet we have presented linear correlation and stated its pitfalls. In this section we aim to briefly introduce the first of copula-based measures of dependence considered in this thesis, the rank correlation. Unlike linear correlation, rank correlation do not depend on marginal distribution but only on bivariate copula. We introduce one of the two important measures of dependence, which is known as Kendall's tau. The other one, Spearman's rho, is not used in the thesis.

According to Cherubini *et al.* (2004), these measures are probably the best alternatives to linear correlation when a dependence of a non-elliptical distribution is measured, as we mentioned above, linear correlation coefficient would not be appropriate and would provide a misleading information.

The name for this class comes from its standard empirical estimator that can be calculated by looking at the *ranks* of the data alone, i.e. the ordering of the sample for each variable is crucial for the coefficient (McNeil *et al.* 2005). The rank correlation used in Chapter 6 is Kendall's tau; it is a measure of concordance for bivariate random vectors.

Assume we have two vectors (X_1, Y_1) and (X_2, Y_2) both coming from the same distribution and the copula function, then we say they are *concordant* if $X_1 > X_2$ whenever $Y_1 > Y_2$ and $X_1 < X_2$ whenever $Y_1 < Y_2$. When opposite, we say they are *discordant*.

Hence, we define Kandall's tau as follows:

$$\tau(X_1, X_2) = P((X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) > 0) - P((X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) < 0) \quad (4.5)$$

It should be noted it is a symmetric dependency measure and takes values $\tau \in \langle -1, 1 \rangle$, where -1 signals a perfect negative correlation, 1 displays a perfect positive correlation and 0 shows no correlation, however, similarly to linear correlation, it does not mean independency. Additional reason to use Kendall's to is to compare parameters of different copulas, since they restricted on different areas as it is shown in subsequent subsection.

Elliptical copulas are simply the copulas of elliptically contoured (or elliptical) distributions. The most commonly used elliptical distributions are the multivariate normal and Student-t distributions. The key advantage of elliptical copulas is that one can specify different levels of correlation between the marginals.

4.1.3 Elliptical Copulas

Here we present a class of copulas constructed of elliptically countered distributions. A typical representative of this class, which we employ into the DCC

MGARCH model, is the Gaussian (normal) copula constructed of multivariate normal distribution. Another important representative would the Student's copula, nevertheless it is not employed in the empirical part, yet not discussed in this thesis.

Gaussian Copula

Assume R to be the symmetric, positive definite correlation matrix, than the Gaussian copula has the following probability distribution function:

$$c^{Gaussian}(u_1, u_2) = \frac{1}{|R|^{\frac{1}{2}}} \cdot \exp(-\frac{1}{2}\eta'(R^{-1} - \mathbf{I})\eta) \tag{4.6}$$

where $\eta = (\Phi^{-1}(u_1), \Phi^{-1}(u_2))'$ and $\Phi^{-1}(\cdot)$ is the inverse of the univariate normal CDF. Then the bivariate Gaussian copula has the following form:

$$C^{Gaussian}(u_1, u_2, R) = \Phi_R(\Phi^{-1}(u_1), \Phi^{-1}(u_2)). \tag{4.7}$$

4.1.4 Archimedean Copulas

This section presents Archimedean class of copulas used for the C-MGARCH model. Nelsen (2006) discuss several reasons why these copulas are popular in various applications: (1) a typical feature of these copulas is the ease of their construction; (2) the class is represented by various different copulas; and (3) there are many nice properties of Archimedean copulas.

Definition 4.2. Let ϕ be a continuous, strictly decreasing function from $\mathbf{I}$ to $\langle 0, \infty\rangle$ such that $\phi(1) = 0$. The *pseudo-inverse* of ϕ is the function $\phi^{[-1]}$ with Dom $\phi^{[-1]} = \langle 0, \infty\rangle$ and Ran $\phi^{[-1]} = \mathbf{I}$ given by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t), & 0 \leq t \leq \phi(0) \\ 0, & \phi(0) \leq t \leq \infty. \end{cases}$$

Note that $\phi^{[-1]}$ is continuous and non increasing on $\langle 0, \infty\rangle$, and strictly decreasing on $\langle 0, \phi(0)\rangle$. Furthermore, $\phi^{[-1]}(\phi(u)) = u$ on $\mathbf{I}$, and

$$\phi(\phi^{[-1]}(t)) = \begin{cases} t, & 0 \leq t \leq \phi(0) \\ \phi(0), & \phi(0) \leq t \leq \infty \end{cases} = \min(t, \phi(0)).$$

Finally, if $\phi(0) = \infty$, then $\phi^{[-1]} = \phi^{-1}$.

Knowing the definition, we present a necessary lemma for construction of a copula from the Archimedean class.

Lemma 4.1. *Let ϕ be a continuous, strictly decreasing function from $\mathbf{I}$ to $\langle 0, \infty \rangle$ such that $\phi(1) = 0$, and let $\phi^{[-1]}$ be the pseudo-inverse of ϕ defined above. Let C be the function from $\mathbf{I^2}$ to $\mathbf{I}$ given by*

$$C(u, v) = \phi^{[-1]}(\phi(u) + \phi(v)). \tag{4.8}$$

Then C satisfies the following boundary conditions for a copula:

$$C(u, 0) = 0 = C(0, v), \tag{4.9}$$

$$C(u, 1) = u, \tag{4.10}$$

$$C(1, v) = v \tag{4.11}$$

The function ϕ is called generator. In this paper, we consider two different copulas, the Gumbel (G) Copula and the Clayton (C) Copula. Next paragraphs defines the copulas and explains differences between them.

Clayton Copula. The Clayton's copula was proposed by Clayton (1978) with the following generator

$$\phi_\theta(t) = \frac{t^{-\theta} - 1}{\theta} \tag{4.12}$$

The copula's PDF is

$$C^{Clayton}(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{1}{\theta} - 2} \tag{4.13}$$

And the CDF is defined as

$$c^{Clayton}(u_1, u_2; \theta) = \frac{(1 + \theta)(u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta - 2}}{(u_1 u_2)^{\theta + 1}} \tag{4.14}$$

The copula parameter θ is defined on the range $(0, \infty)$. We say the marginals are independent when $\theta = 0$ and more the dependent as θ goes to infinity. The Clayton's copula is specific with it's asymmetric tail dependency treatment, i.e. copula is characterized by strong left dependence and, on contrary, with a weak right tail dependence. Moreover, it accounts only for positive dependency.

Gumbel Copula. Opposed to the Clayton's copula the Gumbel's copula

(Gumbel 1960) has a strong right tail dependence and weak left tail dependence. Otherwise the properties of θ are similar; Gumbel's copula considers only positive dependence and its parameter θ is restricted at the range $\langle 1, \infty)$. The generator has the following form

$$\phi_\theta(t) = (-\ln t)^{-1} \tag{4.15}$$

The PDF function is defined

$$C^{Gumbel}(u_1, u_2; \theta) = \exp\{-[(-\ln u_1)^\theta + (-\ln u_2)^\theta]^{\frac{1}{\theta}}\} \tag{4.16}$$

And the CDF is following

$$c^{Gumbel}(u_1, u_2; \theta) =$$

$$= \frac{C^{Gumbel}(u_1, u_2; \theta)(\ln u_1 \ln u_2)^{\theta-1}\{[(-\ln u_1)^\theta + (-\ln u_2)^\theta]^{\frac{1}{\theta}}\} + \theta - 1}{u_1 u_2[(-\ln u_1)^\theta + (-\ln u_2)^\theta]^{2-\frac{1}{\theta}}} \tag{4.17}$$

For better understanding, Figure 4.1 below depicts iso-probability contour plots [1] of two normal marginal distributions connected with Gaussian, Student's, Clayton's or Gumbel's copula to demonstrate their tail dependency for copula parameters corresponding to Kendall's $\tau = 0.33$ for each copulas, i.e.

$$\rho_{Gaussian} = 0.5, \tag{4.18}$$

$$\rho_{Student's} = 0.5, \quad \nu = 3, \tag{4.19}$$

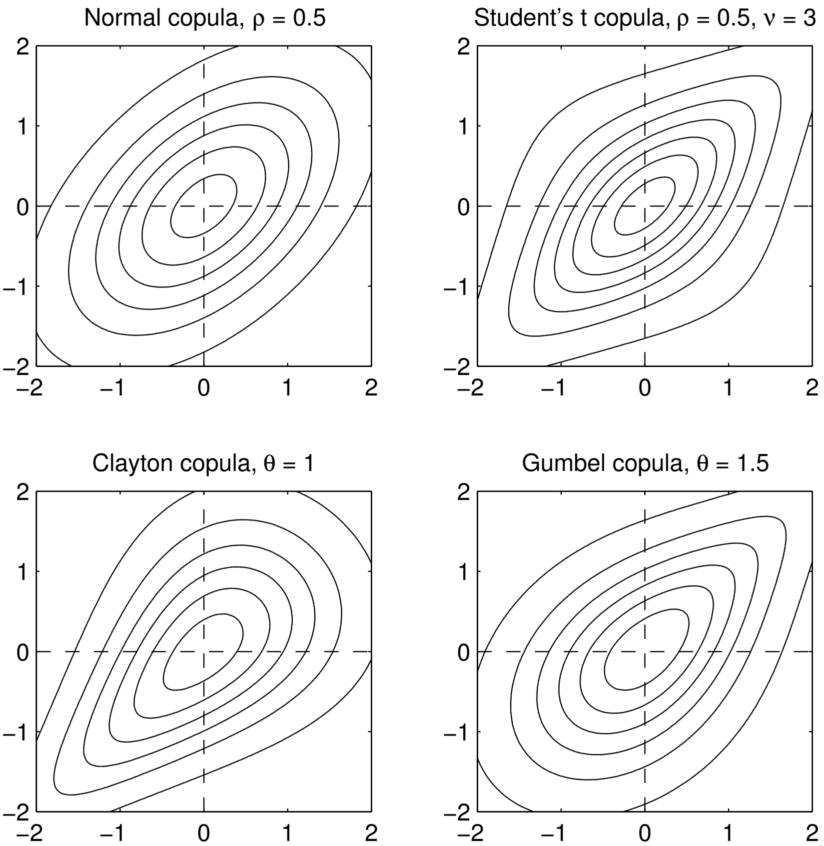
$$\theta_{Clayton} = 1, \tag{4.20}$$

$$\theta_{Gumbel} = 1.5 \tag{4.21}$$

Moreover, in Figure 4.2 we simulated one thousand points of Clayton's and Gumbel's uniform marginal distributions to demonstrate differences in their tail-dependence treatment. As for copula parameters we used $\theta = 3$ for both copulas.

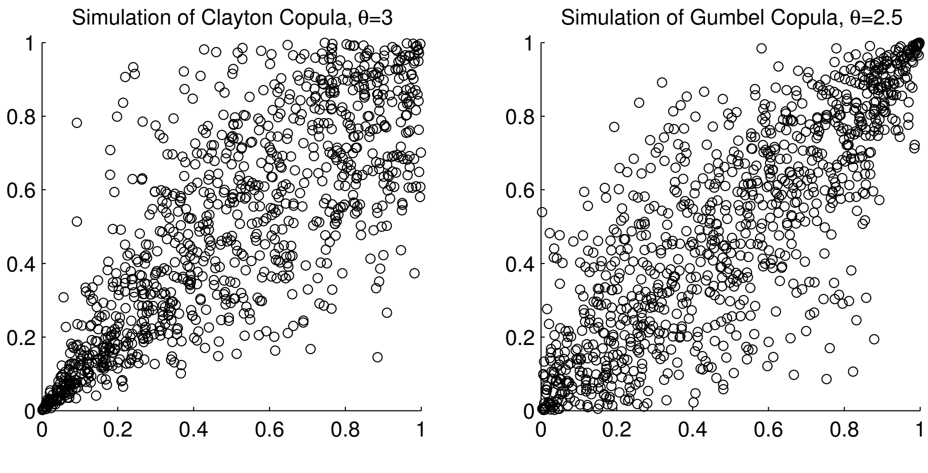
[1] The source code of the copula contour plots for the MatLab software, created for the purposed of research in Patton (2009), can be downloaded from A.J.Patton's website: http://public.econ.duke.edu/ ap172/code.html

Figure 4.1: Gaussian, Student's t, Clayton and Gumbel Contour Plots



Source: author's replication of Patton (2009), page 770.

Figure 4.2: Simulation of Clayton and Gumbel Uniform Margins



Source: author's computations.

4.2 Copula-based Multivariate GARCH models

The multivariate models presented in the previous subsections assume the conditional multivariate normality - the extension of the normal (Gaussian) univariate distribution to higher dimensions. Since a highly significant evidence of non-normality in both marginal and joint distributions of stock returns and market-model residuals have been reported in related literature, the multivariate normal distribution cannot be justified (Richardson & Smith 1993).

The aim of this paper is to employ Copula-based Multivariate GARCH (C-MGARCH) model proposed by Lee & Long (2009), which allows for modeling conditional correlation (by MGARCH) and dependence (by a copula function) separately and simultaneously for non-normal distributions. In other words, the model removes linear correlation from the dependent variable and forms uncorrelated dependent errors which are controlled by copula, while the correlation is controlled by MGARCH. The following part defines the model and explain the differences from the traditional MGARCH.

For the sake of simplicity, we shall use the model for two assets, hence $m = 2$. Then, for returns $r_t = (r_{1,t}, r_{2,t})'$, MGARCH standardized errors $e_t = (e_{1,t}, e_{2,t})'$ and for uncorrelated dependent errors $\eta = (\eta_{1,t}, \eta_{2,t})'$ the C-MGARCH is defined as:

$$\eta|\psi_{t-1} \sim F_{12}(\eta_{1,t}, \eta_{2,t}; \theta_t), \tag{4.22}$$

$$e_t = \Sigma_t^{-1/2}\eta_t, \tag{4.23}$$

$$r_t = H_t^{1/2} e_t, \tag{4.24}$$

where $E(e_t|\psi_{t-1}) = \mathbf{0}$, $E(e_t e_t'|\psi_{t-1}) = \mathbf{I}$, $E(e_t|\eta_{t-1}) = \mathbf{0}$ and $E(\eta_t \eta_t'|\psi_{t-1}) = \mathbf{\Sigma} = (\sigma_{ij,t})$. By the Sklar's theorem

$$F_{12}(\eta_{1,t}, \eta_{2,t}; \theta_t) = C(F_1(\eta_{1,t}; \theta_{1,t}), F_2(\eta_{2,t}; \theta_{2,t}); \theta_{3,t}), \tag{4.25}$$

where $C(\cdot, \cdot)$ is the copula function.

The approach and the main contribution of proposed C-MGARCH model consists of modeling a dependent copula for η_t keeping e_t uncorrelated, i.e.

$$C(u_1, u_2) \neq u_1 u_2 \tag{4.26}$$

In other words the model estimates conditional correlation and dependence structure separately and simultaneously. On contrary, conventional approach of the multivariate GARCH models is to assume independent normality for η_t, hence $C(u_1, u_2) = u_1 u_2$ and $\sigma_{12} = 0$.

The conditional covariance matrix of η_t, Σ_t, is modeled using the Hoeffding's lemma (1940) (reprinted in Hoeffding *et al.* (1994)) as an integral of the covariance between two (or more) random variables in terms of the difference between their marginal distributions F_1 and F_2, and joint distribution F_{12}.

Hoeffding's Lemma

Lemma 4.2 (Hoeffding's Lemma). *Let η_1 and η_2 be random variables with the marginal distributions F_1 and F_2 and the joint distribution F_{12}. If the first and second moments are finite, then*

$$\sigma_{12}(\theta) = \int\int_{\mathbb{R}^2} [F_{12}(\eta_1, \eta_2; \theta) - F_1(\eta_1; \theta) F_2(\eta_2; \theta)] d\eta_1 d\eta_2 \tag{4.27}$$

Putting Hoeffding's Lemma and Sklar's Theorem together, the off-diagonal element of Σ_t, i.e. the covariance between η_1 and η_2 at time t can be expressed as:

$$\begin{aligned}\sigma_{12,t}(\theta_t) = \int\int_{\mathbb{R}^2} \quad & [C(F_1(\eta_1; \theta_{1,t}), F_2(\eta_2; \theta_{2,t}); \theta_{3,t}) \\ & - F_1(\eta_1; \theta_{1,t}) F_2(\eta_2; \theta_{2,t})] d\eta_1 d\eta_2 \end{aligned} \tag{4.28}$$

Furthermore, Lee and Long (2008) assume, for simplicity, the marginal standard normal distribution (for which θ_1 and θ_2 are known) and the copula parameter θ_3 is not time-varying: $\theta_t \equiv \theta = \theta_3$, hence $\sigma_{12,t}(\theta_t) \equiv \sigma_{12}(\theta)$ and $\Sigma_t(\theta_t) \equiv \Sigma(\theta)$.

4.2.1 The C-MGARCH Model Description

The main C-MGARCH contribution lies in introduction of an additional step in separating the remaining dependence from the correlation and model it both simultaneously. Put differently, the model first removes correlation from errors, i.e. the model first creates uncorrelated dependent error by transformation $r_t = H_t^{1/2} e_t$ and then the extra step consists of further transformation, $e_t = \Sigma^{-1/2} \eta_t$, to explain the remaining dependency not captured by the conditional correlation matrix H_t. More precisely, the uncorrelated dependent errors $e_t = (e_{1,t} e_{2,t})'$ and the transformed errors $\eta = (\eta_{1,t} \eta_{2,t})'$ have the following properties:

- under the normal distribution assumption e_t are conditionally correlated, i.e. $E(e_{1,t}e_{2,t}|\psi_{t-1}) = 0$
- under the non-normal distribution assumption e_t should stay conditionally correlated, i.e. $E(e_{1,t}e_{2,t}|\psi_{t-1}) = 0$, however e_t can be dependent
- the transformed errors $\eta = (\eta_{1,t}\eta_{2,t})'$ can be, on contrary, correlated and dependent, i.e. $E(\eta_{1,t}\eta_{2,t}|\psi_{t-1}) \neq 0$ and $F_{12}(\eta_1, \eta_2) \neq F_1(\eta_1)F_2(\eta_2)$ respectively

Considering the properties, the covariance of η_1 and η_2 does not necessary have to be equal to zero. Due to non-normality of $F_{12}(\eta_1, \eta_2)$ and $r_t = H_t^{1/2}\Sigma^{-1/2}\eta_t$ we can say returns $r_t = (r_{1,t}r_{2,t})'$ are non-normally distributed, yet can be modeled by the presented copula-based MGARCH.

Knowing that $e_t = \Sigma^{-1/2}\eta_t$ and assuming the copula parameter θ_3 to be time invariant, hence $\Sigma_t^{-1/2} \equiv \Sigma^{1/2} = (\sigma_{ij})$, the uncorrelated dependent errors can be expressed as:

$$e_{1,t} = \sigma_{11}\eta_{1,t} + \sigma_{12}\eta_{2,t} \quad (4.29)$$

$$e_{2,t} = \sigma_{12}\eta_{1,t} + \sigma_{22}\eta_{2,t} \quad (4.30)$$

In other words, e_t are a linear combination of two dependent random variables η_1 and η_2. According to Lee & Long (2009) even if the errors $\eta_{1,t}$ and $\eta_{2,t}$ were normally distributed, since they are dependent (i.e. $\sigma_{12} \neq 0$), the marginal distribution of $e_{1,t}$ and $e_{2,t}$ would not be normal.

The authors of the model also show, that the proposed C-MGARCH model includes each of existing MGARCH models as a special case; more specifically let's assume the independent copula for η_t (i.e. $C(u_1u_2) = u_1u_2$), then there is no dependency between η_1 and η_2 and Σ is diagonal. Under this scenario, e_t and η_t would be the same. In addition, let's assume η_t margins are normally distributed, then the C-MGARCH would be identical with the traditional MGARCH with bivariate normally distributed returns r_t. The C-MGARCH model can be created for each of the MGARCH model class.

The C-MGARCH Model Contribution

To summarize the model and its contribution, we briefly note, how it solves the two problems this thesis discusses, i.e. the non-normality and additional dependence:

i) The model is constructed to estimate dependencies and volatilities on multivariate data under non-elliptical distribution. Meanwhile the univariate class of GARCH models can be applied to model volatility of non-normally distributed returns of a financial assets, the traditional representatives of the multivariate class, such as DCC, are not able to cope with the non-normality, since they assume multivariate normal or student's t distribution. Thus, integration of a copula function into the MGARCH models, based on Equation 4.1 in Sklar's Theorem, solves the problem of multivariate non-normality.

ii) By employing a copula the model explains additional dependence between assets with the Σ matrix according to Equation 4.28. The traditional DCC estimates conditional correlations with errors e_t, which are uncorrelated, however, may be still dependent (under non-normality).

The C-MGARCH aims to estimate the whole dependence structure among e_t by the following procedure; first the model estimates univariate GARCH on each series to get standardized univariate residuals. Second, the residuals are used for the DCC MGARCH estimation of parameters and of the conditional correlation. Given the correlation matrix, the DCC residuals are transformed to uncorrelated. Third, simultaneously to the previous step, a copula function is employed into Hoeffding's lemma to determine Σ which shows the remaining dependence among variables and to proceed the innovative transformation $e_t = \Sigma^{-1/2}\eta_t$ to separate the remaining dependence from correlation.

Chapter 5

Methodology

This section introduces data used for the empirical analysis in Chapter 6 and the methodology needed for model parameter estimation. Additionally, we present diagnostics for *ex-ante* data checking as well as for *ex post* comparison of models. Finally, log-likelihoods functions for estimation of DCC and copula-based DCC MGARCH are presented.

5.1 Data Description

The main goal of the empirical analysis is to show that copula-based MGARCH models explain additional errors dependency not captured by common multivariate GARCH models. For this purpose we present several different datasets to demonstrate their correlation and the mentioned additional dependency when C-MGARCH applied. The datasets are primarily extracted from the PX index traded in the Czech Republic. Since we build our analysis on the bivariate specification of copulas, our datasets are presented as pairs of two financial time-series;

- PX Index (PX) & CEZ (CEZ)
- Erste Group Bank (ERS) & Komercni Banka (KB)
- Unipetrol (UNIP) & Telefonica C.R. (TEL)

CEZ has the biggest weight among others index shares representing for than 20%[1], therefore we focus on estimating its correlation and dependency with the PX index applying the PX–CEZ dataset. Next, dependence between

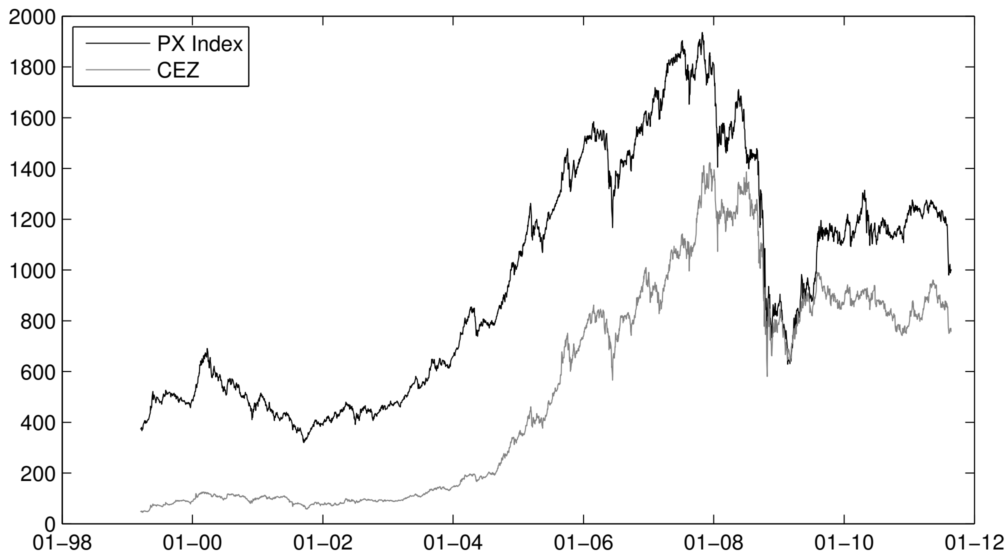
[1] According to BCPP actual on 5^{th} April 2012

the stocks of the two banks traded on the Prague Stock Exchange (PSE) shall be determined based on ERS–KB. These banks together weights more than 35% of the PX index. The UNIP–TEL dataset will be used for determination of the relation between Telefonica C.R and Unipetrol, these two stocks stand for more than 20% weight of the index. The PX Index is the price index and dividends have no impact on its value.

Additionally, we present one pair of two stock exchange indices traded in the United States. Since stocks of both the indices are frequently traded equities, we refer this dataset as a reference liquid pair;

- S&P500 (SP) & NASDAQ 100 (NAS)

Figure 5.1: PX–CEZ Daily Returns



Source: author's computations.

NASDAQ 100 is a capitalization-weighted index, which lists the top 100 stocks of non-financial institutions traded on the National Association of Security Dealers Automated Quotation system (NASDAQ). On contrary SP 500 lists 500 large-cap[2] common stocks traded on New York Stock Exchange (NYSE), American Stock Exchange (AMEX) and on the NASDAQ, and is free-float capitalization based. Both the indices measure market price changes and exclude dividends payment. There was a 2 for 1 stock split on 3th January 1994 in

[2]Over $10 billion

the NASDAQ 100 index, which would affect the first 2089 observation of the index, thus we use the adjusted closing price of both the indices to ensure comparability. In other words, all the observations prior to the stock split date are multiplied by $\frac{1}{2}$ within the NASDAQ 100 index - for a graphical illustration see Figure A.2.

Evolution of the PX–CEZ pair is depicted in Figure A.2 to give a basic notion of daily prices evolution on the PSE. Figures of daily plots of the other datasets are shown in the Appendix. Nevertheless, the datasets need to be transformed to take the following form:

$$R_t = \log(\frac{P_t}{P_{t-1}}) \tag{5.1}$$

where P_t stands for the daily closing prices at time t and R_t for logarithmic returns (log-returns). The transformation is necessary since the evolution of closing prices is usually non-stationary, yet log-returns eliminate the source of non-stationarity. To demonstrate the stationarity we present Table 5.1.

Table 5.1: ADF and ARCH-LM Test

Test:	ADF Test	ARCH-LM Test	
Variable	*t-stat.*	*t-stat.*	*p-value*
PX	-28.8588	445.4734	0.0000
CEZ	-29.5795	434.2074	0.0000
ERS	-28.533	252.6865	0.0000
KB	-29.1993	117.1587	0.0000
UNI	-29.0697	211.9228	0.0000
TEL	-29.0093	86.3726	0.0000
SP	-28.8255	380.1273	0.0000
NAS	-28.0124	302.3798	0.0000

Source: author's computations.

The *ADF Test* column show *t-statistic* of ADF unit root test for each of the series. Since the value of *t-statistic* from which one can reject the null hypothesis of presence of unit root at 1% level of significance, i.e. non of non-stationarity, is equal to -2.56, it is clear that logarithm of first differencing leads to stationarity for each series.

The other column of the Table 5.1 shows *t-statistic* of ARCH-LM test of each univariate dataset. Since none of the presented series has a higher p-value than the 1% significance level, we can reject the null hypothesis of no presence of the ARCH effect. Outputs from the Ljung-Box Q test for serial autocorrelation are

Table 5.2: Datasets Overview

Dataset	*Starting Date*	*Ending Date*	*Sample*
PX—CEZ	17-Mar-1999	30-Sep-2011	3139
ERS—KB	03-Oct-2002	30-Sep-2011	2263
UNI—TEL	16-Mar-1999	30-Sep-2011	3135
SP—NAS	02-Oct-1985	08-Feb-2013	6899

Source: author's computations.

stated in Table 5.3 displaying serial correlation in the TEL returns, the other series can reject the null hypothesis at the 1% level of significance.

Table 5.3: Ljung-Box Q and Jarque-Bera Test

Test:	Ljung-Box Q		Jarque-Bera	
Variable	*t-stat.*	*p-value*	*t-stat.*	*p-value*
PX	61.7040	0.0000	20404.8823	0.0000
CEZ	51.2654	0.0001	13365.2449	0.0000
ERS	89.4340	0.0000	16061.8068	0.0000
KB	72.4577	0.0000	5453.2186	0.0000
UNI	79.2851	0.0000	13941.2106	0.0000
TEL	23.4485	0.2673	19089.1791	0.0000
SP	108.3363	0.0000	10092.6039	0.0000
NAS	88.1333	0.0000	3794.2819	0.0000

Source: author's computations.

Next, corresponding date-sets and sample ranges are showed in the Table 5.2. To summarize the descriptive statistics for each univariate dataset, we present Table 5.4, where stylized facts of the assets returns, such as mean, skewness, kurtosis, are presented. Additional, looking at results of Jarque-Bera test results in Table 5.3 reveals that the series are not normal, yet they are leptokurtic with fat tails, i.e. a typical financial returns feature.

5.2 Testing Data for Multivariate Normality

The idea of testing multivariate dataset (in our case a bivariate one) is based on a well-known implication; if returns are multivariate normally distributed, then they have to be univariate normally distributed. However, this implication is not valid *vice versa*. Following that, one can reject multivariate normality if univariate normality is rejected. It comes from the results of Jarque-Bera test

Table 5.4: Descriptive Statistics

Returns	*Mean*	*Min*	*Max*	*Std. Dev.*	*Kurtosis*	*Skewness*
PX	0.0003	-0.1619	0.1236	0.0154	14.7559	-0.5326
CEZ	0.0009	-0.2383	0.1990	0.0214	15.2719	-0.5326
ERS	0.0001	-0.2510	0.1783	0.0259	16.1034	-0.4659
KB	0.0003	-0.1898	0.1422	0.0226	10.5919	-0.4659
UNI	0.0004	-0.2145	0.2175	0.0254	12.2333	-0.2131
TEL	0.0000	-0.1553	0.1467	0.0207	9.4003	-0.2131
SP	0.0003	-0.7001	0.1720	0.0195	30.3145	-1.2866
NAS	0.0003	-0.2290	0.1096	0.0119	248.4599	-6.7863

Source: author's computations.

that none of our univariate datasets matches a normal distribution, i.e. we can reject multivariate normality for the bivariate datasets.

However, the multivariate normality can be checked through a test proposed by Doornik & Hansen (2008); results are displayed in Table 5.5. *P-values* for each of the pairs rejects the null hypothesis of multivariate normality at the level of significance of 1%.

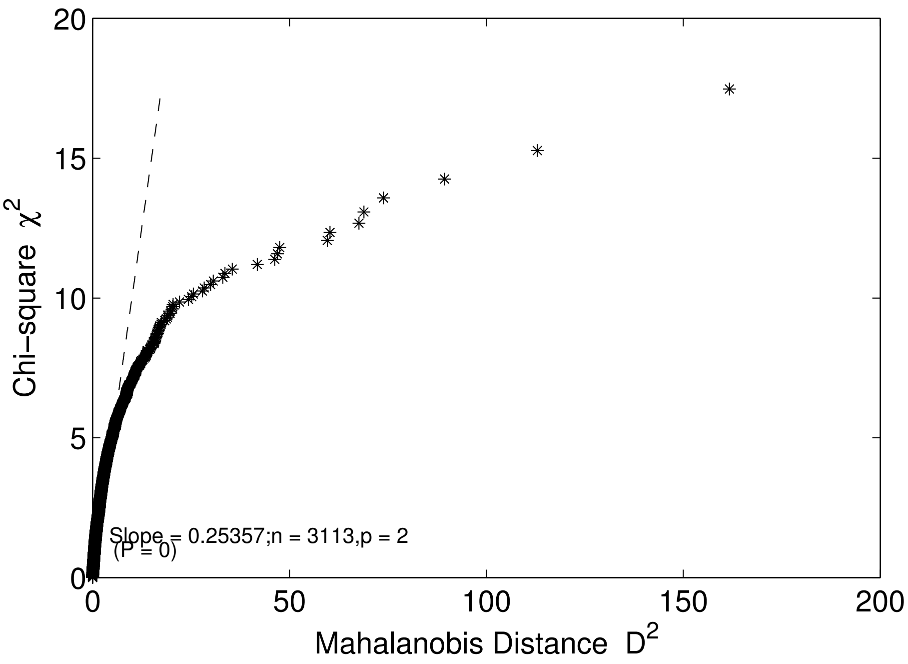
Table 5.5: Doornik & Hansen (2008) Test

Dataset	*Test Stat.*	*p-value*
PX–CEZ	897.1425	0.0000
ERS–KB	679.4775	0.0000
UNI–TEL	15890.5207	0.0000
SP-NAS	93.8506	0.0000

Source: author's computations.

Additionally, in the Figure Figure A.3, we provide a visual interpretation of a test proposed by Wichern & Johnson (1992) applied on PX–CEZ dataset. The output shows a deviation from multivariate normality for these returns, since they do not follow the depicted discontinuous line but significantly deviates from it. The line represents a combination of Mahalanobis distance and χ^2 distribution for which data are considered multivariate normal. Figures for other datasets are depicted in the Appendix.

Figure 5.2: Multivariate Normality Visual Test for PX–CEZ



Source: author's computations.

5.3 Estimation

The aim of this section is to summarize the estimation method, the two-stage maximum likelihood estimator for the following two models; the DCC MGARCH model and DCC C-MGARCH model.

Source codes for both the traditional DCC MGARCH and copula-based MGARCH with Gaussian, Clayton and Gumbel copulas have been written in the MatLab software. Most of it is for the purpose of the empirical study in Chapter 6 and is available upon demand.

5.3.1 The DCC Model Estimation

We estimate the model using the log likelihood estimator proposed by Engle (2002):

$$r_t|\psi_{t-1} \sim N(0, H_t) \tag{5.2}$$

$$
\begin{aligned}
L &= -\frac{1}{2}\sum_t (n\log(2\pi) + \log|H_t| + \epsilon_t' H_t^{-1}\epsilon_t) \\
L &= -\frac{1}{2}\sum_t (n\log(2\pi) + \log|D_t R_t D_t| + \epsilon_t' D_t^{-1} R_t^{-1} D_t^{-1}\epsilon_t) \\
L &= -\frac{1}{2}\sum_t (n\log(2\pi) + 2\log|D_t| + \log|R_t| + \epsilon_t' R_t^{-1}\epsilon_t) \qquad (5.3)
\end{aligned}
$$

According to Engle Equation 5.3 can be simply maximized over the models' parameters. Additionally more estimation methods are proposed to ensure easier model estimation for a case of a large covariance matrix. Following (Engle 2002), the likelihood estimator can be decomposed into two step procedure:

$$
L(\theta,\psi) = L_C(\psi,\theta) + QL_U(\theta) + \sum_t \epsilon_t'\epsilon_t/2 \qquad (5.4)
$$

where

$$
\begin{aligned}
L_C(\psi,\theta) &= -\frac{1}{2}\sum_t (\log(1-\rho_t^2) + \frac{(\epsilon_{1,t}^2 + \epsilon_{2,t}^2 - 2\rho_t\epsilon_{1,t}\epsilon_{2,t})}{(1-\rho_t^2)}) \qquad (5.5) \\
QL_U(\theta) &= -\frac{1}{2}\sum_t (n\log(2\pi) + \sum_{i=1}^{n}(\log(h_{i,t} + \frac{r_{i,t}^2}{h_{i,t}}))) \qquad (5.6)
\end{aligned}
$$

using the correlation estimator:

$$
\rho_{i,j,t} = \frac{q_{i,j,t}}{\sqrt{q_{i,i,t}q_{j,j,t}}} \qquad (5.7)
$$

Equation 5.5 assumes a two-dimensional case, for a higher-dimension case one would use $L_C(\psi,\theta) = -\frac{1}{2}\sum_t(log|R_t| + \epsilon_t' R_t^{-1}\epsilon_t)$ instead. These equations denote the parameters in D as θ and in R as ψ, and furthermore assume that consistent estimates of θ can be found.

Equation 5.6 is the univariate quasi-likelihood function, which is the sum of the QL_U for each individual asset; parameters of these assets can differ, hence need to be estimated as univariate models to assure the standard QMLE properties are hold. "Thus consistent estimates of all the parameters can be obtained by estimating the univariate models and then using these models to define the standardized residuals and finally using one of the listed methods to estimate the parameters of the correlation process" (Engle 2002).

To obtain the maximum likelihood, we use the *fmincon* optimizer in MatLab. The initial points for optimization are set to $\alpha_0 = 0.01$ are $\beta_0 = 0.97$, where α_0 and β_0 represent the corresponding DCC parameters α and β, respectively.

5.3.2 The Copula-based DCC MGARCH Model Estimation

An important property of copulas is their invariance to the increasing and continuous transformation, hence to obtain a log-likelihood function for $\{\eta_t\}_{t=1}^n$ used for parameters estimation, one can transform Equation 4.3:

$$\begin{aligned}\mathcal{L}^\eta(\theta) &= \sum_{t=1}^{n} \log f_{12}(\eta_{1,t}, \eta_{2,t}, \theta) \\ &= \sum_{t=1}^{n} \log f_1(\eta_{1,t}) + \log f_2(\eta_{2,t}) + \log(c(F_1(\eta_{1,t}), F_2(\eta_{2,t}); \theta))\end{aligned} \quad (5.8)$$

Log-likelihood function for returns $\{r_t\}_{t=1}^n$ has the following form:

$$\mathcal{L}^r(\theta, \alpha) = \mathcal{L}^\eta(\theta) + \sum_{t=1}^{n} \log|\Sigma^{1/2}(\theta) H_t^{-1/2}(\alpha)|, \quad (5.9)$$

where $|\Sigma^{1/2} H_t^{-1/2}|$ is the Jacobian of the transformation from η_t to r_t, and, generally, α is the parameter vector in the MGARCH model for H_t, particularly in this paper, α stands for parameter of the DCC MGARCH model.

Similarly to DCC MGARCH, we set the initial values for optimization of the model to $\alpha_0 = 0.01$, $\beta_0 = 0.97$ and additionally to $\theta_{Clayton} = 0.01$ or $\theta_{Gumbel} = 1.01$, where $\theta's$ represent corresponding copulas, Clayton and Gumbel, respectively. The values of copulas parameters are set according to each parameter's restriction. The normal copula does not contain θ, however we consider conditional correlation matrix R as its parameter (see Equation 4.6 and Equation 4.7).

5.4 Models Selection Criteria

When estimated parameters are presented, we shall decide which model performs better in fitting on a real data. For this purpose we introduce information criteria which are considered in the empirical part to determine performance of each model.

Information Criterions

Here we define selection criteria for comparison among the copula-based and traditional MGARCH models; the value of log-likelihood function (LogL) and the Akaike (AIC) and Schwarz Information Criteria (SIC). Suppose that

$$\begin{aligned}(\hat{\theta}_n, \hat{\alpha}_n) &= \arg\max_{\theta,\alpha} \mathcal{L}^r(\theta, \alpha) \\ &= \sum_{t=1}^{n} \log f(\eta_t; \theta) + \log |\Sigma^{1/2}(\theta) H_t^{-1/2}(\alpha)| \end{aligned} \tag{5.10}$$

where k is the number of parameters in each model and $\hat{\theta}_n$ and $\hat{\alpha}_n$ are MLE of θ_n and α_n, respectively. Furthermore, assume n to be the length of the estimated dataset. Than the selection criteria are defined following way:

$$\log \mathrm{L} = \mathcal{L}^r(\hat{\theta}_n, \hat{\alpha}_n) \tag{5.11}$$

$$\mathrm{AIC} = -2\log \mathrm{L} + 2k/n \tag{5.12}$$

$$\mathrm{SIC} = -2\log \mathrm{L} + k\log(n)/n \tag{5.13}$$

The information criteria shall provide a better comparison tool then the ordinary value of the log-likelihood function, since they consider number of parameters in the models as well as number of observation in the dataset. Model (in comparison with others on a same dataset) with the highest LogL and the lowest AIC and SIC should be chosen as the most proper one.

Chapter 6

Empirical Results

Empirical application of the DCC MGARCH and DCC C-MGARCH is presented in this chapter. We apply the models on our datasets presented in Section 5.1 and report the estimates. All the results were obtained through the multi-stage maximum likelihood estimation presented in Section 5.3

The following sections report on results of different models and copulas employed on our datasets. We present the results on a single dataset across different models to emphasise differences among them. Focus is paid to results of copula-based MGARCH, since we expect the models show additional dependence between the time series not captured by traditional MGARCH type of models.

Additionally, we shall report Kendall's τ for each C-MGARCH model to transparently compare copula parameters θ. Otherwise it would be difficult to state which C-MGARCH model presents higher dependence, since the copula parameters are not restricted on the same range. Discussion covering copulas and their parameters is held in Chapter 4. Furthermore, as we discussed, Clayton's copula focuses on the lower-tail dependence, meanwhile Gumbel's copula on the higher-tail, therefore according to Kendall's τ we determine whether the tested dataset has rather lower or higher-tail dependence, i.e. which of the two copulas imply a higher dependency.

For tables in this section, we use denotation as follows:

- α, β are the parameters of the estimated multivariate model
- θ is the copula parameter
- τ is the Kendall's τ, the rank measure of dependence

- σ is the additional covariance between the two time-series of a dataset implied by the corresponding copula
- $LogL$ stands for the value of log-likelihood function in the optimum
- AIC, SIC are Akaike and Schwarz Information Criteria of the multivariate model, respectively

It should be clarified that we expect low values for additional dependence σ implied by a copula function and for both the copula parameter θ and Kendall's τ for each dataset, since these parameters control the remaining dependence in uncorrelated errors.

6.1 The PX Index & CEZ Dataset

This section reports results on dataset consisting of the PX Index and CEZ stocks (as a sub-element of the index). As we reported in the Section 5.1, CEZ is the most influential stock among others within the index, therefore a strong positive correlation is expected. Table 6.1 summarizes models output.

Table 6.1: Results for $PX - CEZ$

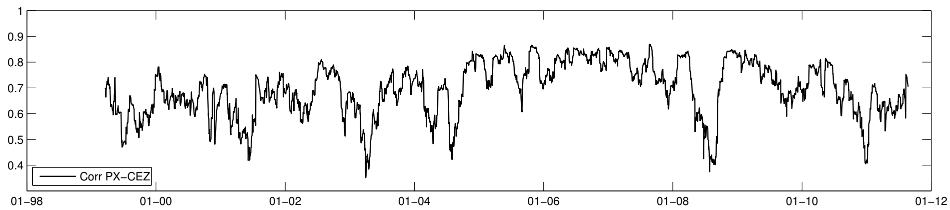
PX–CEZ	DCC	DCC Gaussian	DCC Clayton	DCC Gumbel
α	0.0311	0.0361	0.0394	0.0323
se(α)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
β	0.9609	0.9552	0.9306	0.9496
se(β)	(0.0002)	(0.0002)	(0.0002)	(0.0001)
θ	-	-	0.1111	1.0109
se(θ)	-	-	(0.0006)	(0.0002)
τ	-	-	0.0526	0.0107
σ	-	-	0.0851	0.0177
$LogL$	-19,526	-18,322	-18,324	**-18,319**
AIC	39,053	36,645	36,648	**36,638**
SIC	39,053	36,645	36,648	**36,638**

Source: author's computations.

Additionally, Figure 6.1 shows the conditional correlation estimates computed by DCC MGARCH to demonstrate the correlation between the PX Index and CEZ. The figure shows the correlation reached more than 90% in the years 2006 and 2008, yet a significant drop could be seen during the financial crisis in the end of 2008. Both the index and CEZ decreased in values, however

PX faced a considerable decrease relative to CEZ, thus the striking drop in correlation in the chart between 2008 and 2009. Although the conditional correlation is on a slightly lower level in the last years, the PX–CEZ dataset can be considered as a highly correlated one.

Figure 6.1: PX-CEZ DCC Conditional Correlation



Source: author's computations.

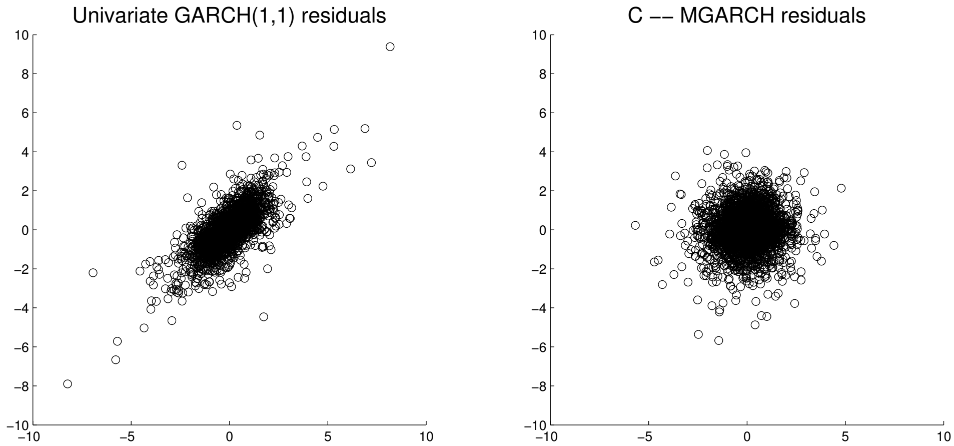
Both of the copula-based MGARCH models demonstrates additional dependence not explained by DCC MGARCH, since the σ's are equal to 0.0851 and 0.0177 for Clayton and Gumbel copula, respectively. As we expected, σ of the Clayton copula is higher than the σ of the Gumbel's - since the prior copula focuses on explanation of lower tail dependence and the latter on the upper tail, we can state the dataset of PX–CEZ has a higher lower tail dependency, which is consistent with empirical findings (see for example Longin & Solnik (2001)).

Looking at he values of log-likelihood function and on the information criteria, the copula-based MGARCH models clearly outperform the traditional DCC model. The model with the Gumbel copula yields the lowest information criteria, although differences among these criteria of the copula-based models are negligible. For the presented dataset, the C-MGARCH models showed additional dependence not captured by the ordinary DCC and additionally, the results show a better relative goodness of fit.

The table also states the sum of estimates of parameters $\hat{\alpha} + \hat{\beta}$ is close to 1. That signals a high persistence in correlation between the two series, i.e. a long-run average of the correlation can be pushed away for a considerably long period by shocks. The standard errors of the parameters and of θ's are very small, which corresponds to the large number of observations for each dataset.

Finally, we can graphically demonstrate the performance of the copula-based MGARCH model with clayton copula; Figure 6.2 show bivariate residuals of the univariate GARCH(1,1) and of the Clayton C-MGARCH. Univariate residuals are considerably correlated, i.e. we can see many points showing correlation in extreme values. Opposed to that, scatter plot of the uncorrelated errors presents much denser concentration around mean value, yet the the *sigma* parameter reveals some dependence in the data.

Figure 6.2: Scatter Plots of Residuals for PX–CEZ



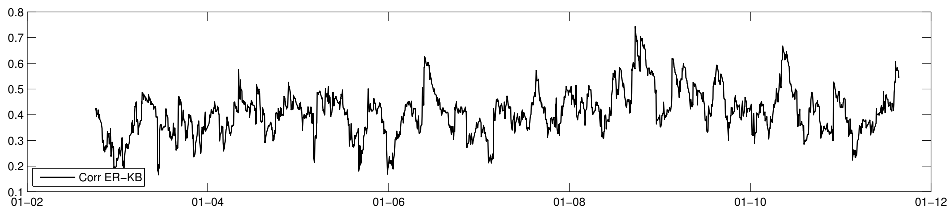
Source: author's computations.

6.2 ERS & KB Dataset

Our second dataset consists of shares of the two banks traded on the Prague Stock Exchange. Again, similarly to the previous dataset, we do expect a higher correlation between these two stocks. Table 6.2 display the estimation and selection criteria.

As Figure 6.3 shows, the conditional correlation estimated by the traditional DCC is oscillating around 0.4, the overall trend of the correlation on the dataset is slightly growing.

Figure 6.3: ERS-KB DCC Conditional Correlation



Source: author's computations.

Table 6.2: Results for $ERS - KB$

ERS–KB	DCC	DCC Gaussian	DCC Clayton	DCC Gumbel
α	0.0301	0.0295	0.0278	0.0258
se(α)	(0.0002)	(0.0002)	(0.0002)	(0.0022)
β	0.9230	0.9263	0.9422	0.9517
se(β)	(0.0006)	(0.0005)	(0.0005)	(0.0011)
θ	-	-	0.0763	1.0114
se(θ)	-	-	(0.0005)	(0.0001)
τ	-	-	0.0368	0.0112
σ	-	-	0.0595	0.0186
LogL	-11,530	-11,504	-11,490	**-11,489**
AIC	23,061	23,009	22,981	**22,978**
SIC	23,061	23,009	22,981	**22,978**

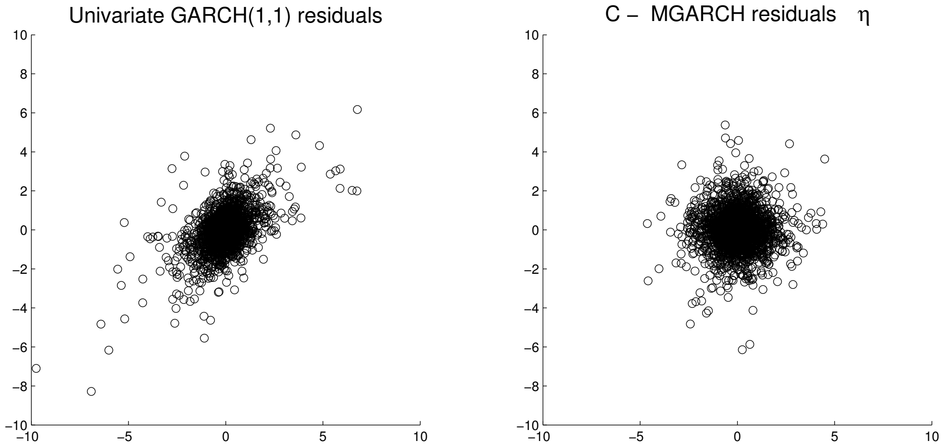
Source: author's computations.

Both AIC and SIC propose the C-MGARCH with Gumbel copula with higher relative goodness of fit. Again , similarly to previous dataset, both Archimedean copula-based MGARHC models indicates additional dependency, since their σ's are 0.0368 for Clayton and 1.0112 for Gumbel. Additionally, $\tau_{Clayton}$ is higher than τ_{Gumbel}, i.e. Clayton copula reveals a higher dependency and, since it focuses on lower-left quadrant, lower-tail dependence. It is also worth mentioning than each of the C-MGARCH models dominates traditional DCC MGARCH model on the ERS–KB dataset, yet the difference among particular log-likelihoods is significantly smaller than it was on the PX–CEZ dataset. We assume that this is given due to the smaller correlation between the two stocks - and as Lee & Long (2009) state in their study the non-copula DCC models are a special case of the copula-based MGARCH models, thus if no additional (uncaptured) dependence among the series, that the copula-based model acts as a non-copula one.

Similarly to other datasets we can see a high persistency of $\hat{\beta}$.

The scatter plots of univariate residuals of GARCH(1,1) for both ERS and KB and of residuals of DCC C-MGARCH with clayton copula are displayed in Figure 6.4 to demonstrate the correlation of univariate residuals and the dependence among uncorrelated errors of the copula-based model.

Figure 6.4: Scatter Plots of Residuals for ERS–KB



Source: author's computations.

6.3 UNIP & TEL dataset

Results of estimation on the dataset of UNI–TEL are shown in Table 6.3. It is obvious that the traditional DCC-MGARCH has the highest *LogL* and the lowest AIC and SIC and, in this case, it outperforms the C-MGARCH models. The outcome is a consequence of very low correlation between the two series. We can see the information criteria to be on a very similar level.

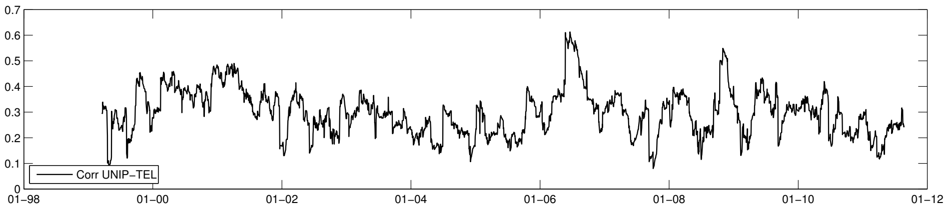
Table 6.3: Results for $UNIP - TEL$

UNIP–TEL	DCC	DCC Gaussian	DCC Clayton	DCC Gumbel
α	0.0208	0.0321	0.0703	0.0623
se(α)	(0.0001)	(0.0002)	(0.0006)	(0.0004)
β	0.9593	0.9375	0.8574	0.8824
se(β)	(0.0003)	(0.0005)	(0.0003)	(0.0012)
θ	-	-	0.0923	1.0169
se(θ)	-	-	(0.0001)	(0.0002)
τ	-	-	0.0441	0.0166
σ	-	-	0.0714	0.0273
LogL	**-15,346**	-15,960	-15,991	-15,988
AIC	**30,693**	31,919	31,982	31,976
SIC	**30,693**	31,919	31,982	31,976

Source: author's computations.

We compared the most influential stock CEZ and the overall index in Section 6.1 and the two banks traded on Prague Stock Exchange in Section 6.2, both of the pairs significantly correlated, however this section show models behaviour on dataset with considerable lower correlation as can be seen in Figure 6.5, where the conditional correlation fluctuates approximately around 0.3 and seams to be very stable.

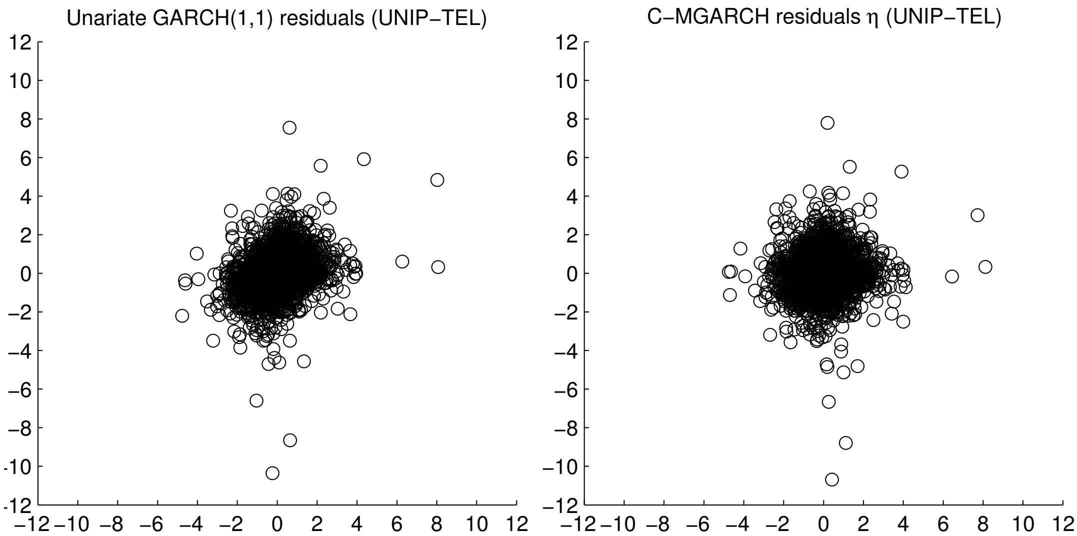
Figure 6.5: UNIP-TEL DCC Conditional Correlation



Source: author's computations.

At the end of this section we show Figure 6.6 to graphically illustrate dependence and performance of C-MGARCH. Even the univariate errors demonstrate a small amount of correlation, nevertheless the uncorrelated residuals present positive *sigma* dependence.

Figure 6.6: Scatter Plots of Residuals UNIP-TEL



Source: author's computations.

6.4 SP & NAS dataset

Results reported in this section relates to SP-NAS dataset, i.e. we are comparing two american popular indices. Estimated correlation of the DCC MGARCH model in Figure 6.7 demonstrates a high correlation between these two indices. The sample starts in 1992 and till Q2 2004 it evinces very stable estimated conditional correlation around 0.9. Nevertheless, the subsequent period shows a much higher jumps in correlation, yet at the end of 2011 the correlation is still significantly high. The overall trend is rather increasing.

Table 6.4: Results for $SP - NAS$

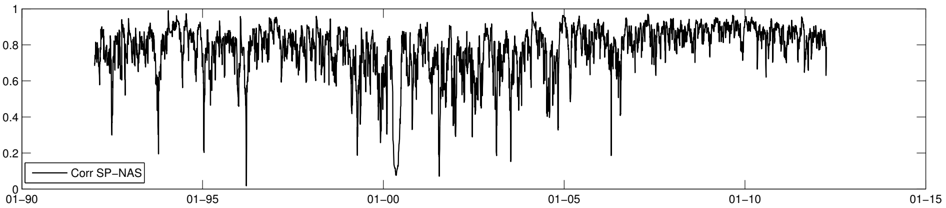
SP-NAS	DCC	DCC Gaussian	DCC Clayton	DCC Gumbel
α	0.1321	0.0967	0.1067	0.1003
se(α)	(0.0001)	(0.0000)	(0.0005)	(0.0003)
β	0.8583	0.8971	0.8586	0.8932
se(β)	(0.0001)	(0.0000)	(0.0004)	(0.0004)
θ	-	-	0.6343	1.0184
se(θ)	-	-	(0.0006)	(0.0003)
τ	-	-	0.2408	0.1962
σ	-	-	0.3716	0.3070
LogL	-48,006	-43,556	-43,329	**-43,321**
AIC	96,013	87,112	86,658	**86,647**
SIC	96,013	87,112	86,658	**86,647**

Source: author's computations.

Estimations presented in Table 6.4 show additional dependency as the estimated coefficients σ's take values of 0.3716 and 0.3070 for Clayton and Gumbel copulas respectively. According to the selection criteria, the C-MGARCH with Gumbel copula yields the best estimates. Furthermore, lower tail dependence is evident since $\tau_{Clayton}$ is higher than τ_{Gumbel}. Both of the Archimedean copula-based MGARCH dominates the traditional DCC in terms of the information criteria.

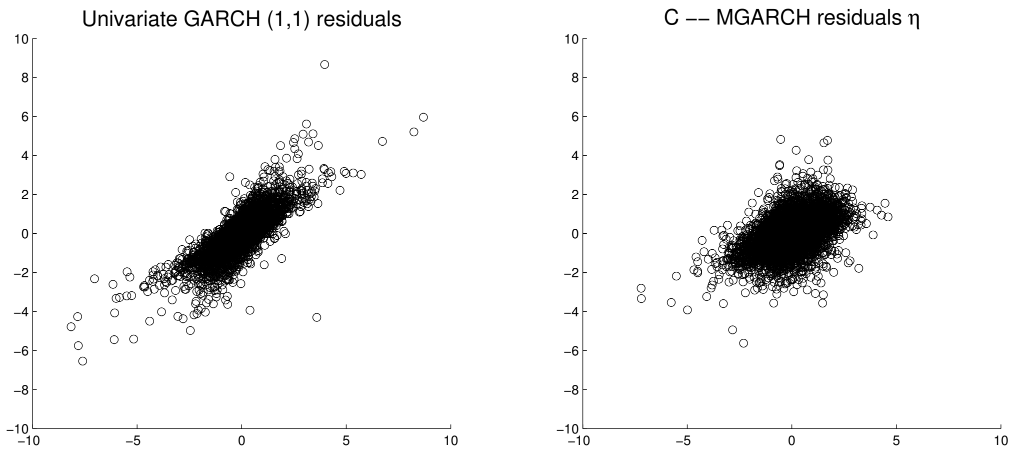
Finally, in Figure 6.8 we can see the scatterplots; univariate residuals are extensively correlated and after uncorrelation they still show dependency. The figure corresponds to the models results, we can see quite significant uncuptered dependence in the data.

Figure 6.7: SP-NAS DCC Conditional Correlation



Source: author's computations.

Figure 6.8: Scatter Plots of Residuals SP-NAS



Source: author's computations.

6.5 Comparison of the Models Performance on Different Univariate GARCH Models

The aim of this section is to investigate whether the specification of GARCH and of univariate error distribution play role in the subsequent results of multivariate GARCH models. We expect the information criteria of the multivariate GARCH model to improve when an enhanced, asymmetric specification of the GARCH model is applied for the estimation of the univariate series - hence for obtaining the univariate standardised residuals that shall be an input parameter of a (copula-based) MGARCH model.

Table 6.5: Univariate types of the GARCH model on the PX–CEZ with Clayton copula MGARCH

Distribution: *Model*:	Gaussian GARCH	Gaussian EGARCH	Gaussian GJR	t GARCH	t EGARCH	t GJR
α	0.0394	0.0314	0.0332	0.0303	0.0301	0.0483
se(α)	(0.0001)	(0.0012)	(0.0007)	(0.0001)	(0.0000)	(0.0006)
β	0.9306	0.9455	0.9449	0.9542	0.9504	0.9238
se(β)	(0.0002)	(0.0012)	(0.0000)	(0.0001)	(0.0000)	(0.0000)
θ	0.1111	0.1152	0.1024	0.1047	0.1161	0.0785
se(θ)	(0.0006)	(0.0157)	(0.0003)	(0.0004)	(0.0000)	(0.0038)
τ	0.0526	0.0545	0.0487	0.0498	0.0549	0.0378
σ	0.0851	0.0881	0.0788	0.0805	0.0888	0.0611
LogL	-18,324	**-18,301**	-18,318	-18,349	-18,337	-18,349
AIC	36,648	**36,602**	36,636	36,699	36,675	36,698
SIC	36,648	**36,602**	36,636	36,699	36,675	36,698

Source: author's computations.

We decided to use the multivariate GARCH model with Clayton copula and the PX–CEZ dataset for which we run the following conditional volatility models:

- GARCH(1,1)
- EGARCH(1,1)
- GJR-GARCH(1,1)

with the following distributions of z_t:

- Gaussian distribution
- Student's t-distribution

Table 6.5 summarises the results of the MGARCH model with Clayton copula on the PX–CEZ dataset. We can see a minor improvement of both the log-likelihood function and the information criteria for the EGARCH model with $p = 1$ and $q = 1$ with normally distributed z_t. The other specifications of the Student's t-distribution do not yield an improvement of the overall MGARCH model goodness-of-fit compared to the univariate GARCH(1,1) with normal distribution of z_t. The copula parameter θ and σ, the dependence of uncorrelated errors, increased. On this bivariate dataset, the improved univariate model specification increased the revealed dependence.

Table 6.6: ARMA–EGARCH specification of the univariate models with Gaussian distribution

ARMA lags:	(1,1)	(1,2)	(2,1)	(2,2)
EGARCH lags:	(1,1)	(1,1)	(2,1)	(2,1)
α	0.0314	0.0408	0.0443	0.0416
se(α)	(0.0012)	(0.0004)	(0.0000)	(0.0000)
β	0.9455	0.9373	0.9309	0.9360
se(β)	(0.0012)	(0.0001)	(0.0000)	(0.0000)
θ	0.1152	0.1169	0.1184	0.1205
se(θ)	(0.0157)	(0.0001)	(0.0000)	(0.0000)
τ	0.0545	0.0552	0.0559	0.0568
σ	0.0881	0.0894	0.0904	0.0919
LogL	-18301	-18291	-18299	-18289
AIC	36603	36583	36597	36578
SIC	36603	36583	36597	36578

Source: author's computations.

Additionally, we employed the ARMA-EGARCH model for the univariate estimation. For the next part, we did not use nor the ARMA-GARCH nor the ARMA-GJR-GARCH model since their results were always dominated by the ARMA-EGARCH on the corresponding lags on the PX–CEZ dataset.

In Table 6.6 we present results of such specification of the ARMA-GARCH that yields better information criterion. Different combination of lags were tested (where each lag was of the following set $L = 0, 1, 2$). The table shows that the information criteria lowered, i.e. improved. Similarly to previous comparison, the unexplained dependence σ grows with the improving log-likelihood.

It was shown in the previous subsections that σ is usually small since it should control the remaining dependence after the error uncorrelation. Therefore, the univariate models shall be test for various specification to ensure the best possible goodness of fit - in that case we receive a more precise estimation of σ.

6.6 Summary

To summarize results and compare performance of traditional DCC against copula-based DCC, we provide Table 6.7 to compare estimated additional dependencies in dataset.

Table 6.7: Results Summary of Unexplained Dependence

Dataset	$\sigma_{Clayton}$	σ_{Gumbel}
PX–CEZ	0.0851	0.0177
ERS–KB	0.0595	0.0186
UNIP–TEL	0.0714	0.0273
SP–NAS	0.3716	0.3070

Source: author's computations.

When comparing dependence, we focus on one copula type across different datasets to make the comparison transparent, however, both of the copulas show the same output, yet on a different scale.

We can see C-MGARCH models with Archimedean copulas propose a higher dependence, which was not captured by DCC, in PX–CEZ, ERS–KB and SP–NAS. This result is reasonable, since we expect the amount of unexplained dependence to be directly related to correlation between returns, and as we show, these two datasets have the higher conditional correlation among others in this thesis.

On contrary, the only dataset for which copula-based models did not prove any additional advantage is the UNIP-TEL sample, i.e. the less conditionally correlated one. The reason we opted for such a type of bivariate set is testing the copula models on less correlated data. Although the conditional correlation between these two stocks have been reported very low, it should be noted data still may show some dependence implied by a copula.

Additionally, in Table 6.8 we summary estimated Kendall's tau for Clayton and Gumbel copulas. It is evident $\tau_{Clayton}$ is higher than τ_{Gumbel} for each of our

Table 6.8: Results Summary of Kendall's tau

Dataset	$\tau_{Clayton}$	τ_{Gumbel}
PX–CEZ	0.0526	0.0107
ERS–KB	0.0368	0.0112
UNIP–TEL	0.0441	0.0166
SP–NAS	0.2408	0.1962

Source: author's computations.

datasets, hence we can say, since estimated parameters display a higher dependency reported by Clayton copula, our datasets are rather dependent in the lower left quadrant, i.e. lower-tail dependent. In other words, Gumbel copula, which is designed to create a joint distribution with higher-tailed dependence, yielded lower values of copula parameter θ compared to Clayton copula.

As a next step we evaluate particular models. Table 6.9 summarises AIC of each model across datasets. We choose AIC since it generated very similar outputs to SIC, and it has a better informative value than simple value of LogL, which does not consider number of parameters.

Table 6.9: AIC Results Summary

Dataset	DCC	DCC Gaussian	DCC Clayton	DCC Gumbel
PX–CEZ	39,053	36,645	36,648	**36,638**
ERS–KB	23,061	23,009	22,981	**22,978**
UNIP–TEL	**30,693**	31,919	31,982	31,976
SP–NAS	96,013	87,112	86,658	**86,647**

Source: author's computations.

The only dataset, where traditional DCC outperformed copula-based DCC, is UNIP–TEL. We interpreted the issue as consequence of very low correlation between obviously not-related stocks. However, in all of the other dataset C-MGARCH models clearly dominated common DCC. Reported AIC shows for PX–CEZ, ERS–KB and SP–NAS an obvious superiority of C-MGARCH model no matter what copula employed. To be more specific, for all the PX–CEZ, ERS–KB and SP-NAS datasets the lowest criterions propose C-MGARCH with Gumbel copula.

Chapter 7

Conclusion

In this thesis we focus on applying copula-based multivariate GARCH model (C-MGARCH) with uncorrelated dependent errors Lee & Long (2009) on bivariate datasets consisting of index and its stocks trade on the Prague Stock Exchange, namely PX Index and CEZ, Erste Bank and Komercni banka, and Unipetrol and Telefonica. Additionally, we apply a dataset constructed from American Stock Exchange markets; SP500 and NASDAQ100.

There are two main problems that need to be considered when a multivariate dataset is analysed:

(i) It is well-known that financial returns do not follow normal distribution, yet they are leptokurtic, skewed and dependent. If we consider a multivariate dataset a problem arises as we need to know its joint distribution. Contrary to univariate case, not many multivariate distributions have been proposed since it is not possible to simply extend each univariate distribution to corresponding multivariate case. Although not always valid, multivariate normality is assumed by traditional class of MGARCH models. Under this assumption, results of such a model can be misleading.

(ii) Despite popularity of linear correlation coefficient, one needs to recall that zero correlations between two random variables does not necessarily means no dependency between them. When a common DCC MGARCH model is estimated and outputs conditional correlation matrix, one cannot be sure the matrix explained all the dependency between returns, since correlation is considered to be only a part of dependency. Thus, the second problem lies in detecting this unexplained dependency.

Both of the issues are being solved by employing a copula function into model. Copulas, sometimes referred as dependency functions, can create a corresponding joint distribution function of any two (or more) marginal distribution. Furthermore, copula function applied into the models carry all the dependency information between random variables, therefore can be used to control the mentioned dependence not captured by traditional MGARCH models.

The copula-based DCC MGARCH model was proposed to separately and simultaneously model both conditional correlation matrix (by MGARCH model) and dependency (by a copula). The idea behind the model is to estimate DCC MGARCH conditional correlation and subsequently to remove the correlation from its residuals. These uncorrelated residuals however may be dependent, therefore a copula is used to control the additional dependence. The model is estimated via multistage maximum likelihood estimator.

Following that, the aim of the thesis is was to compare performance of traditional DCC MGARCH and C-MGARCH model on real data. Our goal was to show that copulas reveal additional dependence between returns. Moreover, we expected C-GARCH models to produce better estimates since they assume non-elliptical distribution (again due to a copula function). Thus, we provided results of estimation of each model and focused on interpreting findings. Models were evaluated according to information criterions.

The results of the empirical application show that for each dataset, the copula-based DCC MGARCH models evince dependence not captured by the traditional DCC M-GARCH. Furthermore, for the PX–CEZ, ERS–KB and SP–NAS datasets we show the C-MGARCH clearly outperforms the common model according to the selection criteria. To be more specific, for the mentioned sets the model with Gumbel copula shows the best goodness of fit. On the PX–CEZ dataset the additional dependence is $\sigma_{Clayton} = 0.0177$ and the copula parameter $\theta_{Gumbel} = 1.0109$.

According to selection criteria, the best model for the ERS–KB pair is again the C-MGARCH with Gumbel copula and its parameter $\theta_{Gumbel} = 1.0114$, the additional dependence implied by the copula is $\sigma_{Gumbel} = 0.0186$. Moreover, it shall be emphasized the performance of the C-MGARCH models with Gaussian or Clayton copula; both of them yield better information criteria than DCC.

The last dataset where the C-MGARCH models performed better than the DCC is the SP–NAS pair. The model with Gumbel copula show slightly better performance among the other copula models. It shall be stated that the difference between the copula-based model and the traditional one is noteworthy. Again, both Archimedean copulas reported additional dependence; $\sigma_{Gumbel} = 0.3070$ and $\theta_{Gumbel} = 1.0184$. Conversely, the UNIP–TEL pair is the only dataset on which the DCC without copula outperformed the C-MGARCH models. We explained that by a very small correlation between the two stocks and expect a small unexplained dependence.

The last part of the empirical study consists of analysis of influence of different specification of the univariate volatility models; the analysis observes the sensibility of the results of the copula-based MGARCH model with uncorrelated dependent errors on the selection of the univariate GARCH models for the first step of the estimation process. On sample dataset we show how a proper specification improves the results and affects the estimated dependence structure. The ARMA-EGARCH model with Gaussian distribution performed with the best output among other tested univariate models.

In conclusion, we showed the DCC copula-based MGARCH with uncorrelated dependent errors outperform the tradition DCC MGARCH. Moreover, we demonstrated how the model with a copula is able to explain the whole dependence structure among random variables in multivariate modeling. A real application of these copula-based models may be in modern portfolio theory such as in CAPM for determination of dependence among assets in a portfolio, To give an example, estimation of dependence is crucial for investors seeking for a diversified portfolio or for risk managers, who aim to compute multivariate Value-at-Risk.

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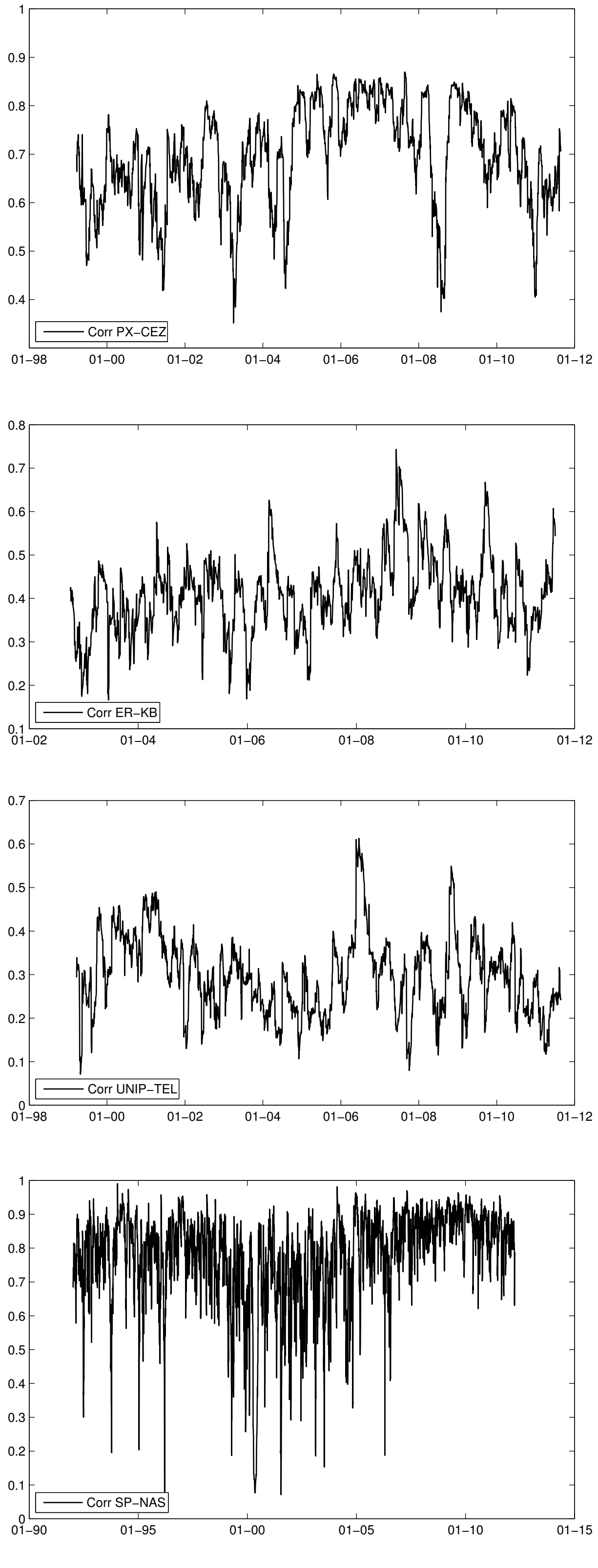
Appendix A

Additional Figures

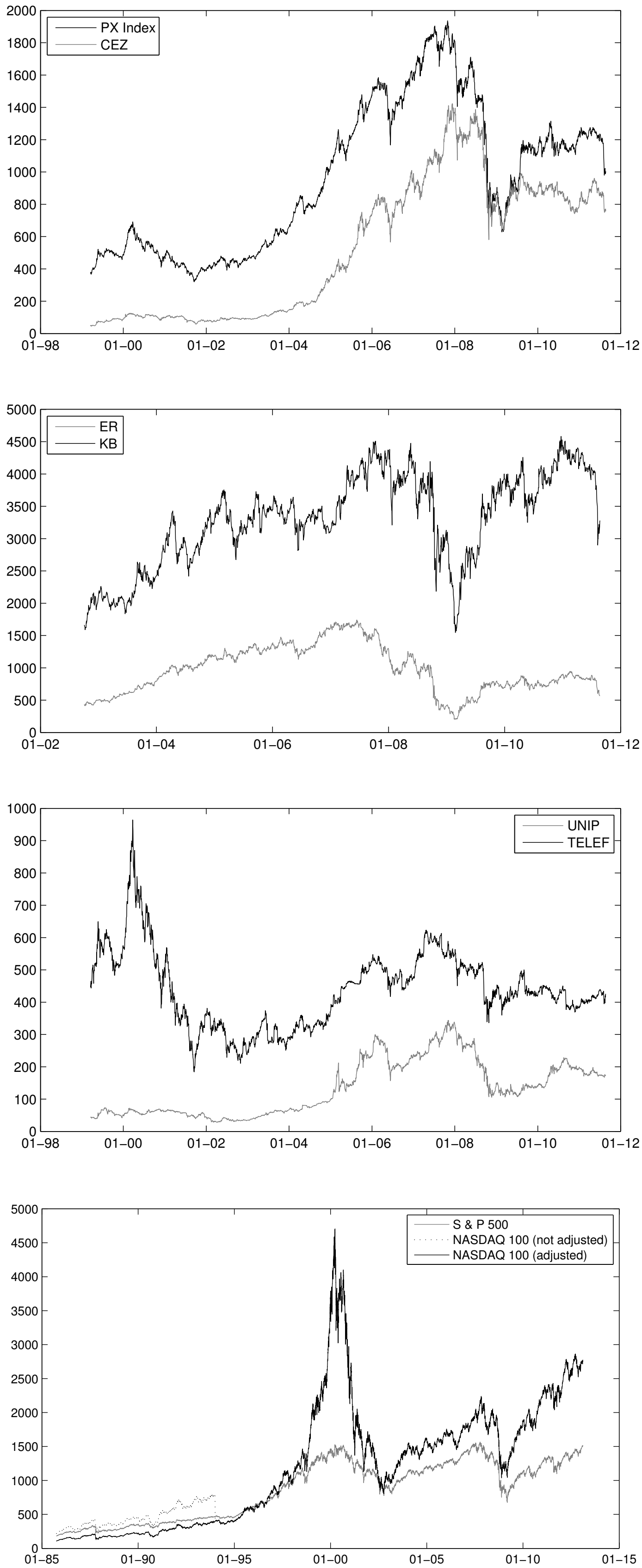
Here we present figures of DCC conditional correlations, daily returns and multivariate visual normality tests for each dataset.

Figure A.1: DCC Conditional Correlations (PX–CEZ, ERS–KB, UNIP–TEL, SP–TEL clockwise ordered)



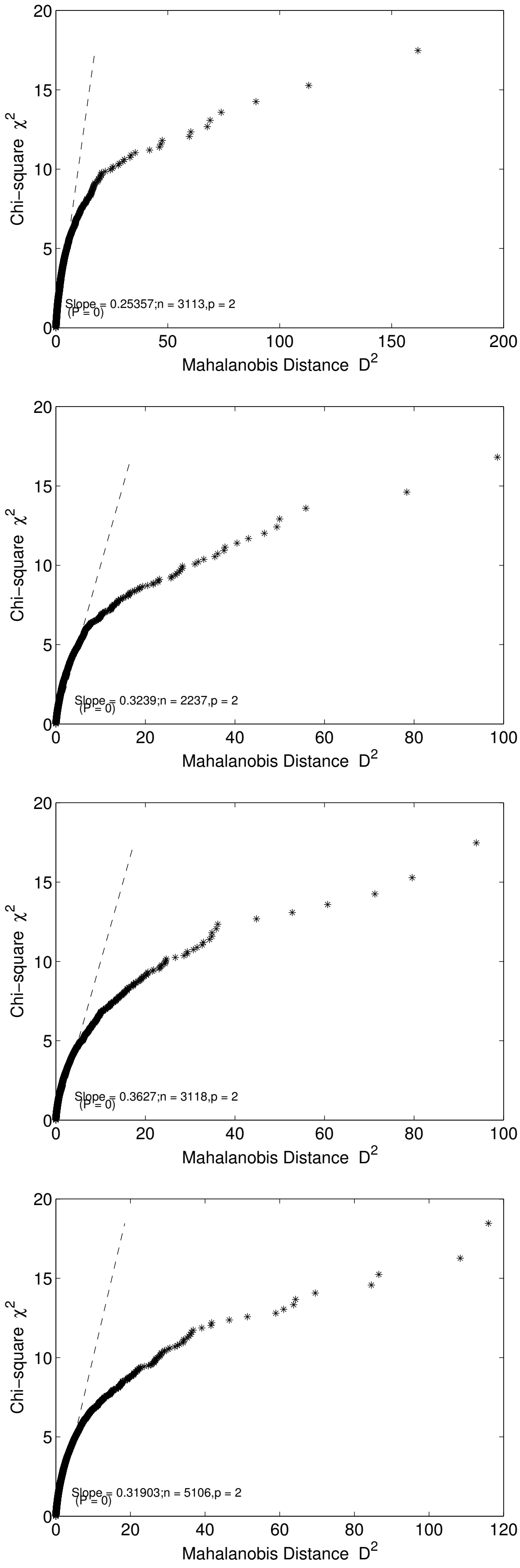
Source: author's computations.

Figure A.2: Daily Returns (PX–CEZ, ERS–KB, UNIP–TEL, SP–TEL top-down ordered)



Source: author's computations.

Figure A.3: Multivariate Normality Visual Test (PX–CEZ, ERS–KB, UNIP–TEL, SP–TEL top-down ordered)



Source: author's computations.