Charles University in Prague

Faculty of Mathematics and Physics

RIGOROUS THESIS

Mgr. Ondřej Honzl

On Selected Geometric Properties of Brownian Motion Paths

Mathematical Institute, Charles University

Supervisor of the Doctoral Thesis:	Prof. RNDr. Jan Rataj, CSc.
Study Programme:	Mathematics
Specialization:	4M4 – Probability and Mathematical Statistics

Prague 2012

Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

RIGORÓZNÍ PRÁCE

Mgr. Ondřej Honzl

Vybrané geometrické vlastnosti trajektorií Brownova pohybu

Matematický Ústav Univerzity Karlovy

Vedoucí disertační práce:	Prof. RNDr. Jan Rataj, CSc.
Studijní program:	Matematika
Specializace:	4M4 – Pravděpodobnost a matematická statistika

Praha 2012

Dedicated to our tomcat Vilém Hynek Jarmil Tell.

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Coll., the Copyright Act, as amended, in particular the fact that the Charles University in Prague has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 paragraph 1 of the Copyright Act.

In Prague 16th January 2013 Mgr. Ondřej Honzl

Název práce: Vybrané geometrické vlastnosti trajektorií Brownova pohybu

Autor:	Mgr. Ondřej Honzl
Emailová adresa:	honzl@karlin.mff.cuni.cz
Katedra:	Katedra pravděpodobnosti a matematické statistiky
Vedoucí disertační práce:	Prof. RNDr. Jan Rataj, CSc.
Emailová adresa:	rataj@karlin.mff.cuni.cz
Katedra:	Matematický ústav UK

Abstrakt:

Práce se zaměřuje na studium geometrických vlastností Brownova pohybu.

Nejprve pojednává o kuželových bodech Brownova pohybu v rovině a jejich souvislosti s kritickými body. Motivace studia kritických bodů je skryta v příjemných vlastnostech distanční funkce mimo tyto body. Je dokázána věta o neexistenci dvou $\pi+$ kuželových bodů na pevné přímce. Toto tvrzení nás vede k hypotéze, že kritických bodů Brownova pohybu v rovině je nejvýše spočetně.

Dále se práce zabývá studiem asymptotických vlastností povrchu r-okolí Brownova pohybu zvaného Wienerova klobása. Za užití vlastností Kneserovy funkce je dokázáno tvrzení o vztahu Minkowského objemu a S-objemu. Jako důsledek dostáváme limitní chování povrchu Wienerovy klobásy skoro jistě v dimensích $d \geq 3$.

Nakonec je studována asymptotika počtu souvislých komponent doplňku Wienerovy klobásy v rovině. Motivací se nám stala otázka z článku [19] týkající se střední hodnoty Eulerovy charakteristiky Wienerovy klobásy v rovině. Dokážeme větu o limitním chování počtu souvislých komponent doplňku Wienerovy klobásy v závislosti na jejím poloměru.

Klíčová slova:

Brownův pohyb, kuželové body, kritické body, povrch Wienerovy klobásy, Eulerova charakteristika Wienerovy klobásy v rovině.

Title: On Selected Geometric Properties of Brownian Motion Paths

Author:	Mgr. Ondřej Honzl
E-mail Address:	honzl@karlin.mff.cuni.cz
Department:	Department of Probability and Mathematical Statistics
Supervisor:	Prof. RNDr. Jan Rataj, CSc.
E-mail Address:	rataj@karlin.mff.cuni.cz
Department:	Mathematical Institute, Charles University

Abstract:

Our thesis is focused on certain geometric properties of Brownian motion paths.

Firstly, it deals with cone points of Brownian motion in the plane and we show some connections between cone points and critical points of Brownian motion. The motivation of the study of critical points is provided by a pleasant behavior of the distance function outside of the set of these points. We prove the theorem on a non-existence of $\pi+$ cone points on fixed line. This statement leads us to the conjecture that there are only countably many critical points of the Brownian motion path in the plane.

Next, the thesis discusses an asymptotic behavior of the surface area of r-neighbourhood of Brownian motion, which is called Wiener sausage. Using the properties of a Kneser function, we prove the claim about the relation of the Minkowski content and S-content. As the consequence, we obtain a limit behavior of the surface area of the Wiener sausage almost surely in dimension $d \geq 3$.

Finally, we study the asymptotic number of the connected components of the complement of a Wiener sausage in a plane. We found the motivation for this investigation in the article [19] where the authors ask the question concerning the mean Euler characteristic of the Wiener sausage. We prove a theorem on the limit behavior of the number of the connected components of the complement of a Wiener sausage with dependance on its radius.

Keywords:

Brownian motion, cone points, critical points, surface area of the Wiener sausage, Euler characterization of the Wiener sausage in a plane.

Contents

List of Frequently Used Notation

$\mathbf{1}_M(\cdot)$	indicator function of $M \subset \mathbb{R}^d$, i.e. $\mathbf{1}_M(x) = 1$ if $x \in M$ and 0 otherwise
$\|x\|$	Euclidean norm of $x \in \mathbb{R}^d$, i.e. $\|x\| = \left(\sum_{i=1}^d x_i^2\right)^{1/2}$
$\|M\|$	Lebesgue measure of $M \subset \mathbb{R}^d$
$\mathrm{Int}(M)$	interior of $M \subset \mathbb{R}^d$
M^c	complement of $M \subset \mathbb{R}^d$
$\overline{M}$	closure of $M \subset \mathbb{R}^d$
∂M	boundary of $M \subset \mathbb{R}^d$
$\mathrm{conv}(M)$	convex hull of $M \subset \mathbb{R}^d$
$\mathrm{dist}(x, M)$	distance from point $x \in \mathbb{R}^d$ to $M \subset \mathbb{R}^d$, i.e. $\mathrm{dist}(x, M) = \inf\{\|x - z\| : z \in M\}$
$\mathrm{diam}(M)$	diameter of $M \subset \mathbb{R}^d$, i.e. $\mathrm{diam}(M) = \sup\{\|x - y\| : x, y \in M\}$
$b(x, r)$	open ball of radius $r > 0$ centered in $x \in \mathbb{R}^d$, i.e. $b(x, r) = \{y \in \mathbb{R}^d : \|x - y\| < r\}$
$M \oplus N$	Minkowski sum of $M, N \subset \mathbb{R}^d$, i.e. $M \oplus N = \{x + y : x \in M, y \in N\}$
$M \ominus N$	Minkowski difference of $M, N \in \mathbb{R}^d$, i.e. $M \ominus N = (M^c \oplus N)^c$
$B = \{B(t) : t \geq 0\}$	standard Brownian motion in $\mathbb{R}^d$
$B[a, b]$	trajectory of B on time interval $[a, b]$
$S_r[0, t]$	Wiener sausage, $S_r[0, t] = B[0, 1] \oplus b(o, r)$
$T_r(z)$	time of the first entry of B to the ball $b(z, r)$, i.e. $T_r(z) = \inf\{t > 0 : \|B(t) - z\| = r\}$
$R_{a,b}(z)$	exit time of B from ball $b(z, b)$, $b > a > 0$ after $T_a(z)$, i.e. $R_{a,b}(z) = \inf\{t > T_a(z) : \|B(t) - z\| = r\}$
$W(\alpha, \xi)$	cone in $\mathbb{R}^2$ with angle of size α and orientation ξ
$C_t(\alpha)$	set of α-cone point times
$C_p^+(\alpha)$, $C_p^-(\alpha)$	set of one-side α-cone points
$C_p(\alpha)$	set of α-cone points
$C_{ap}(\alpha, \xi, \gamma, \varepsilon)$	set of α-approximative cone points with direction ξ and radii $\varepsilon > \gamma > 0$
$\mathrm{Crit}(M)$	set of critical points of $M \subset \mathbb{R}^d$
$\mathcal{D}_k$	set of half-open dyadic cubes, $\mathcal{D}_k = \{D_{ij},\ i, j = 1, 2, \ldots, 2^k\}$, $D_{ij} = [(i-1)2^{-k}, i2^{-k}] \times [(j-1)2^{-k}, j2^{-k}]$

$\mathcal{H}^k(M)$	k-dimensional Hausdorff measure of $M \subset \mathbb{R}^d$
$\dim_H M$	Hausdorff dimension of $M \subset \mathbb{R}^d$
$V(r,t)$	volume of the Wiener sausage up to time t and radius r, i.e. $V(r,t) = \mathcal{H}^d(S_r[0,t])$
$S(r,t)$	surface area of the Wiener sausage up to time t and radius r, i.e. $S(r,t) = \mathcal{H}^{d-1}(\partial S_r[0,t])$
M_r	r-neighbourhood of $M \subset \mathbb{R}^d$
$V_M(r)$	volume of M_r
$S_M(r)$	surface area of M_r
$\mathcal{S}^k(M)$	k-dimensional S-content of $M \subset \mathbb{R}^d$
$\mathcal{M}^k(M)$	k-dimensional Minkowski content of $M \subset \mathbb{R}^d$
c.c.	the abbreviation of "connected component"
$C_{a,b}(y)$	the c.c. of $\mathbb{R}^2 \setminus B[T_a(y), R_{a,b}(y)]$ which contains y
$C_{a,b}(y,\gamma)$	the c.c. of $\mathbb{R}^2 \setminus S_\gamma[T_a(y), R_{a,b}(y)]$ which contains y
$C(y,\gamma)$	the c.c. of $\mathbb{R}^2 \setminus S_\gamma[0,1]$ which contains y
$N[u,v)$	number of c.c. of $\mathbb{R}^2 \setminus B[0,1]$ with area in $[u,v)$
$N_\gamma[u,v)$	number of c.c. of $\mathbb{R}^2 \setminus S_\gamma[0,1]$ with area in $[u,v)$
$\chi(M)$	Euler characteristic of $M \subset \mathbb{R}^d$
$\stackrel{d}{=}$	equality in the distribution
ω_k	volume of the unit ball in $\mathbb{R}^k$, i.e. $\omega_k = \frac{\pi^{k/2}}{\Gamma(k/2+1)}$
κ	Euler constant, $\kappa \doteq 0.5772$
$\Gamma(z)$	gamma function, i.e. $\Gamma(z) = \int_0^\infty t^{z-1}\, \mathrm{e}^{-t}\, \mathrm{d}t$
$J_\nu(\cdot)$	Bessel function of the first kind
$Y_\nu(\cdot)$	Bessel function of the second kind
$I_\nu(\cdot)$	modified Bessel function of the first kind

Preface

You are currently reading the doctoral thesis which was created as the final work of my Ph.D. study at the Charles University in Prague between years 2008 and 2012.

First, I would like to thank my supervisor Prof. RNDr. Jan Rataj, CSc. for his never-ending moral support, his scientific leadership and a lot of helpful advice during writing of this work. I could hardly reach such an advanced topic concerning Brownian motion paths without his patient help.

I spent the spring semester in 2011 in Russian Federation at the Saint Petersburg State University. I would like to thank the International contract between the Ministry of Education, Youth and Sports in the Czech Republic and the Ministry of Education and Science in the Russian Federation for financial support. I improved my language abilities there and I saw one of the world cradles of Probability.

I wish to thank grants SVV 265315/2012, GAČR 201/09/H012 and GAČR 201/10/J039 for their financial support.

Finally, I would like to thank my parents, my wife and all of my friends for their care, patience and moral support.

In Prague 16^{th} September 2012 Mgr. Ondřej Honzl

Introduction

Brownian motion is undoubtedly a basal component of numerous processes that are examined in the nature, physics, economics and other scientific disciplines. The apparent simplicity of the definition of Brownian motion can make wrong presumption that there is nothing to study, but the opposite is true. The behavior of its trajectories is so irregular that it is rather difficult to imagine it. Although a lot is known about Brownian motion, the geometric properties of the non-typical points of its path are considerably difficult to determine, especially in dimension two and three, there are many open problems.

The recent investigation of the geometry of the Brownian motion path is mostly focused on the so-called "exceptional" (or "non-typical") points. We can mention e.g. cut points, slow points, fast times, cone points, critical points etc. Our thesis also deals with some of them. We divided this work into four main parts. The particular chapters are not closely related and they can be read separately but for more fluent understanding we recommend to preserve the established order.

Chapter 1 starts with basic definitions and claims. We recall standard definitions and we set other notations, which are essential for techniques of proofs in the next chapters. An advanced reader who is familiar with Brownian motion, Hausdorff measure, Bessel process and the standard notation may skip this part.

In Chapter 2, we analyze some properties of α-cone points in the plane. We define the set of $\pi+$ cone points to be an intersection of the sets of $(\pi + \delta)$-cone points over all positive δ. A point $x \in \mathbb{R}^2$ is a critical point of planar Brownian motion if x lies in the convex hull of its closest points in $B[0,1] = \{B(t) : t \in [0,1]\}$. We show a connection between $\pi+$ cone points and critical points. Using results concerning approximative cone points we prove that there are almost surely no two $\pi+$ cone points with reverse orientation laying on any fixed line. In accordance with listed statement, we conjecture that there exist at most countable many critical points of $B[0,1]$ almost surely. The countability of critical points remains, up to our knowledge, an open problem.

We can ask: "Why should we study the critical points of the Brownian path?"

A brief answer is the following: Whenever the boundary of an r-neighbourhood of $B[0,1]$ (the so-called Wiener sausage) does not contain any critical point then it is regular. More precisely, let r be a regular value of the distance function of $B[0,1]$ then the boundary of the Wiener sausage $S_r[0,1]$ is a Lipschitz manifold and closure of the complement of $S_r[0,1]$ is the set of positive reach. Hence, the curvature measure of the Wiener sausage (see [19]) can be defined. For more detail see work of Joseph Howland Guthrie Fu [7] and for an additional general approach to critical points we refer the reader to Steve Ferry's article [6]. Moreover, almost all values of the distance function of any set in $\mathbb{R}^d$ are regular, whenever $d = 2$ or 3, but it is not true in higher dimension.

In the planar case, the set of all critical values of any set $A \subset \mathbb{R}^2$ has the Hausdorff dimension smaller or equal to 1/2. We guess that the set of the critical points of the trajectory of Brownian motion has zero Hausdorff dimension. Our main Theorem 2.13 does not prove this conclusion, but it gives a related result for a fixed direction.

The behavior of the surface area of the Wiener sausage is discussed in Chapter 3. In the first section, we introduce the known formulae which are related to the asymptotic behavior of the volume of Wiener sausage. The mean volume $\mathbb{E}\,\mathcal{H}^d(S_r[0,t])$ was derived by Berezhkovski et al. [1]. The almost sure asymptotic formulae of the volume $\mathcal{H}^d(S_r[0,t])$ are known due to F. Spitzer [22] and J.-F. Le Gall [12].

$$\mathcal{H}^d(S_r[0,1]) = \frac{\pi}{|\log r|} + o\left(\frac{1}{|\log r|}\right) \qquad d=2,$$
$$\mathcal{H}^d(S_r[0,1]) = (d-2)\pi\omega_{d-2}\, r^{d-2} + o\left(r^{d-2}\right), \quad d \geq 3$$

where $\omega_d = \pi^{d/2}/\Gamma(d/2+1)$.

The second section deals with the surface area of the Wiener sausage. In [18] and [20] it was shown that the mean surface area of the Wiener sausage satisfies

$$\mathbb{E}\,\mathcal{H}^{d-1}(\partial S_r[0,1]) = \frac{d}{dr}\,\mathbb{E}\,\mathcal{H}^d(S_r[0,1]), \quad r>0.$$

We show the almost sure asymptotic behavior of the surface area of the Wiener sausage. In particular, the following formula holds

$$\mathcal{H}^{d-1}(\partial S_r[0,1]) = (d-2)^2\pi\omega_{d-2}\, r^{d-3} + o\left(r^{d-3}\right). \tag{1}$$

We have used the general approach using Kneser function for proving a relation between Minkowski and S-content (see Theorem 3.11). As a corollary, we have obtained an almost sure asymptotic formula for the surface area of the Wiener sausage in (1).

We present a counterexample where these techniques fail, so they can not be used for the asymptotic of the surface area of the Wiener sausage in dimension $d=2$.

The topic of the surface area of the Wiener sausage is not exhausted at all and the future research can be directed on the "fluctuations" of the area of the boundary of $S_r[0,1]$.

Chapter 4 is a bit different in comparison with the previous ones. We extend there known results on Brownian trajectories to Wiener sausage, which can be found in [13] and [16]. In particular, we deal with the asymptotic number of the connected components of the complement of a Wiener sausage in the plane. We were also motivated by the article [19] where it is proved that the mean Euler characteristic of the Wiener sausage is finite, but the asymptotic behavior of it is still open.

More precisely, let $N_\gamma[u,v)$ be a number of the connected components of the complement of a Wiener sausage with area in $[u,v)$, $0 \le u < v \le \infty$ and let $\chi(\gamma) = \chi(S_\gamma[0,1])$ be an Euler characteristic of the Wiener sausage. By virtue of the connectivity of the Wiener sausage we have:

$$\chi(\gamma) = 1 - N_\gamma[0,\infty). \tag{2}$$

We prove (Theorem 4.15) that

$$\lim_{u\to 0} u(\log u)^2 N_\gamma[u,\infty) = 2\pi \quad \text{a.s.} \tag{3}$$

uniformly for

$$0 < \gamma < \left(\frac{u}{\pi}\right)^b, \; b > 1/2.$$

It is not difficult to see that $N_\gamma[0,\infty) \ge N_\gamma[u,\infty)$, $u > 0$, hence (3) gives the estimation of (2). But we expect that the limit behaviour of $N_\gamma[0,\infty)$ and $N_\gamma[u,\infty)$ are asymptotically equal. This hypothesis is supported by the numerical simulation study in [19].

We hope that the Theorem 4.15 will help to find the answer in the future.

1. Preliminaries

This chapter contains definitions and claims which are used in the next chapters. We introduce d-dimensional Brownian motion and its basic properties, some theorems concerning Bessel process, a definition of Hausdorff measure and Hausdorff dimension and other useful definitions and notations.

1.1 Brownian Motion

Definition 1.1. *Let $B = \{B(t) : t \geq 0\}$ be a real-valued stochastic process starting at $x \in \mathbb{R}$. We call it the (linear)* Brownian motion *if the following conditions are fulfilled.*

- *$B(0) = x$,*
- *for any $n \in \mathbb{N}$ and $0 \leq t_1 \leq \dots < t_n < \infty$ the increments $B(t_2) - B(t_1), \dots, B(t_n) - B(t_{n-1})$ are independent (i.e. the process has* independent increments*),*
- *for all $t \geq 0$ and $h > 0$, the increment $B(t+h) - B(t)$ is Gaussian distributed with zero mean and variance h,*
- *trajectories are continuous almost surely, i.e. the function $t \mapsto B(t)$ is continuous almost surely.*

If $B(0) = 0$ we say that B is the standard Brownian motion *and we often leave out the word* standard *when it is not necessary.*

*Let $B = (B_1, \dots, B_d)$, $d \in \mathbb{N}$, be a vector of independent Brownian motions starting at $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Then, we call B the d-*dimensional Brownian motion *starting at x.*

Let $0 \leq a < b < \infty$. We write $B[a, b]$ for the trajectory of $\{B(t) : t \in [a, b]\}$ and similarly, we use the notation $B[A]$ for the set $\{B(t) : t \in A\}$, $A \subset [0, \infty)$.

We will use the symbol $\mathbb{P}_x$ (resp. $\mathbb{E}_x$) to denote the probability measure (resp. the expectation) associated with the Brownian motion started at $x \in \mathbb{R}^d$. To shorten notation, we write only $\mathbb{P}$ instead of $\mathbb{P}_o$ ($o = (0, \dots, 0) \in \mathbb{R}^d$).

Remark 1.2. *The existence of Brownian motion is nontrivial and we refer the reader to [15], Theorem 1.3 with a nice constructive prove.*

Definition 1.3. *(Filtrations) Let $\mathcal{F} = \{\mathcal{F}(t) : t \geq 0\}$ be a filtration, i.e. a family of σ-algebras such that $\mathcal{F}(s) \subset \mathcal{F}(t)$ for all $s < t$.*

A random variable τ defined on a probability space with filtration $\mathcal{F}$ is called a stopping time with respect to $\mathcal{F}$ if the event $\{\tau \leq t\}$ belongs to $\mathcal{F}$.

Let $\{B(t) : t \geq 0\}$ be a Brownian motion. Then we define the natural filtration $\{\mathcal{F}^0(t) : t \geq 0\}$ by $\mathcal{F}^0(t) = \sigma\{B(s) : 0 \leq s \leq t\}$ and we set $\mathcal{F}^+(t) = \bigcap_{s>t} \mathcal{F}^0(s)$.

Now, we summarize some important properties of a Brownian motion. We present the following two lemmas without proofs. For more details, we refer to [15], Chapter 1.

Lemma 1.4. *Let* $\{B(t) : t \geq 0\}$ *be a standard Brownian motion. Then, the following processes* X_i*,* $i = 1, \ldots, 4$*, are also standard Brownian motions.*

(i) Scaling invariance*:*

$$X_1 = \left\{ \frac{1}{\sqrt{a}} B(at) : t \geq 0 \right\}, \ a > 0.$$

(ii) Time inversion*:*

$$X_2 = \left\{ \mathbf{1}_{\{t>0\}} \, tB(1/t) : t \geq 0 \right\}.$$

(iii) Strong Markov property*: For every stopping time* τ *we set*

$$X_3 = \{B(\tau + t) - B(t) : t \geq 0\}$$

and the process X_3 *is independent of* $\mathcal{F}^+(\tau)$*.*

(iv) Symmetry*:*

$$X_4 = \{-B(t) : t \geq 0\}.$$

Lemma 1.5. *(Maximum of Brownian Motion) The distribution function of the maximum of an one-dimensional standard Brownian motion is*

$$\mathbb{P}\left[\max_{t\in[0,T]} B(t) \leq x \right] = \frac{2}{\sqrt{T}} \Phi\left(\frac{x}{T}\right) - 1, \tag{1.1}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{u^2}{2}} \, \mathrm{d}u$$

is the distribution function of a standard normal variable.

Definition 1.6. *(Hitting Times) Let* $\{B(t) : t \geq 0\}$ *be a d-dimensional Brownian motion. We define the first hitting time of the set* $M \subset \mathbb{R}^d$ *by Brownian motion as*

$$T_M = \inf\{t \geq 0 : B(t) \in M\}.$$

We will often use the first hitting time of a ball $b(y, r)$*,* $y \in \mathbb{R}^d$*,* $r > 0$ *and we denote it by*

$$T_r(y) = \inf\{t \geq 0 : |B(t) - y| = r\}. \tag{1.2}$$

We usually write only T_r *instead of* $T_r(o)$*.*

Next, we will often use the exit time from $b(y, a)$ *which happens after stopping time* $T_b(y)$*, thus we set the following abbreviation*

$$R_{a,b}(y) = \inf\{t > T_b(y) : |B(t) - y| = a\}. \tag{1.3}$$

All random variables defined in the previous definition are also the stopping times with respect to the natural filtration (see Definition 1.3).

The following lemma deals with the limit behavior of the probability that a Brownian motion visits ball $b(o, \varepsilon)$ before time 1. Originally, this lemma is a synthesis of Lemma 2.1 in [11] and Lemma 1 in [23].

Lemma 1.7. *([13, Lemma 1.2]) The following limit holds for any $y \in \mathbb{R}^2 \setminus (0,0)$*

$$\lim_{\varepsilon \to 0+} |\log \varepsilon| \;\; \mathbb{P}[T_\varepsilon(y) \leq 1] = \pi \int_0^1 \frac{1}{2\pi s} \cdot \mathrm{e}^{-\frac{|y|^2}{2s}} \, \mathrm{d}s. \tag{1.4}$$

Moreover, there exists a constant $K > 0$ such that for any $y, z \in \mathbb{R}^2 \setminus (0,0)$, $y \neq z$, and any $\varepsilon \in (0, 1/2)$, we have

$$|\log \varepsilon| \cdot \mathbb{P}[T_\varepsilon(y) \leq 1] \;\; \leq \;\; G(y), \tag{1.5}$$

$$|\log \varepsilon|^2 \cdot \mathbb{P}[T_\varepsilon(y) \leq 1, \, T_\varepsilon(z) \leq 1] \;\; \leq \;\; G(y)G(z-y) - G(z)G(y-z), \tag{1.6}$$

where

$$G(y) = G(|y|) = K(1 + \max\{0, -\log|y|\}) \, \mathrm{e}^{-\frac{|y|^2}{16}}.$$

1.2 Bessel Function and Bessel Process

In the sequel text we will need a definition of Bessel functions and some results concerning the Bessel process of the second order.

Definition 1.8. *(Bessel Functions) The Bessel function of the first kind $J_\nu(x)$ is*

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu + n + 1)} \left(\frac{x}{2}\right)^{2n+\nu}, \; x \in \mathbb{R} \setminus \{0\}$$

where ν is a real number except of negative integers. The function J_{-n}, $n \in \mathbb{N}$ is defined by

$$J_{-n}(x) = (-1)^n J_n(x).$$

The Bessel function of the second kind $Y_\nu(x)$ is

$$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)},$$

for non-integer ν.

If ν is an integer, then Y_ν is defined by the following limit

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x).$$

The modified Bessel function (or the hyperbolic Bessel function) of the first kind I_ν is defined by

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n + \nu + 1)} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{1.7}$$

Definition 1.9. *(Bessel process) Let B be a standard d-dimensional Brownian motion. Then the non-negative process $X_{(d)}(t) = |B(t)|$ is called the Bessel process of order d. We shall deal with a planar Brownian motion and with an appropriate Bessel process of order 2, i.e.*

$$X_{(2)}(t) = \sqrt{(B_1(t))^2 + (B_2(t))^2}.$$

Lemma 1.10. *([3]) Let $X_{(2)}$ be a Bessel process of the second order and let $H(z)$ be a stopping time of the Bessel process defined by*

$$H(z) = \inf\{s : X_{(2)}(s) = z\}, \quad z \in [0, \infty).$$

Then

$$\mathbb{P}_x\left[\inf_{0 \le s \le H(z)} X_{(2)}(s) > y\right] = \begin{cases} \frac{\log \frac{x}{y}}{\log \frac{z}{y}} & \text{for } y \le x \le z, \\ 1 & \text{for } y \le z \le x. \end{cases}$$

Lemma 1.11. *([3]) The probability that a Bessel process of the second order is located in the Borel set A at time $t > 0$ is the following*

$$\mathbb{P}_x[X_{(2)}(t) \in A] = \int_A \frac{z}{t} \; \mathrm{e}^{-\frac{x^2+z^2}{2t}} \; I_0\left(\frac{xz}{t}\right) \mathrm{d}z,$$

where $I_0(\cdot)$ is defined in (1.7).

1.3 Hausdorff Measure

For the sake of studying the volume and the surface area in Chapter 3 we introduce the Hausdorff measure and dimension here.

Definition 1.12. *(Hausdorff Measure) Let X be a metric space. For $E \subset X$ we call $\mathcal{H}^s(E)$ the s-*dimensional Hausdorff measure *of E if*

$$\mathcal{H}^s(E) = \lim_{\delta \to 0_+} \mathcal{H}^s_\delta(E), \;\; s \ge 0,$$

where

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} \omega_s \left(\frac{\mathrm{diam}(E_i)}{2}\right)^s : \, E \subseteq \bigcup_{i=1}^{\infty} E_i, \, \mathrm{diam}(E_i) \le \delta\right\}, \;\; \delta > 0$$

and

$$\omega_s = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}+1\right)}.$$

Note that ω_s is the volume a unit ball in $\mathbb{R}^s$ if s is an integer.

Remark 1.13. *Let $X = \mathbb{R}^d$. Then the s-dimensional Hausdorff measure is a Borel regular outer measure for any $s \ge 0$,*

- *Every Borel set* $B \subset \mathbb{R}^d$ *is* $\mathcal{H}^s$*-measurable, i.e. for every* $A \subset \mathbb{R}^d$*:* $\mathcal{H}^s(A) = \mathcal{H}^s(A \cap B) + \mathcal{H}^s(A \setminus B)$.

- *For every* $A \subset \mathbb{R}^d$ *exists a Borel set* $B \subset \mathbb{R}^d$ *such that* $A \subset B$ *and* $\mathcal{H}^s(A) = \mathcal{H}^s(B)$.

Moreover, the Hausdorff measure is translation and rotation invariant. It also coincides with Lebesgue measure in the case $s = d$*. For more details see [17].*

Definition 1.14. *(Hausdorff Dimension) Let* X *be a metric space and* $E \subset X$*. We define the* Hausdorff dimension *of a set* E *as*

$$\dim_H E = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\}.$$

Remark 1.15. *([17] Proposition 1.4.) The Hausdorff dimension can by equivalently defined as*

$$\begin{aligned} \dim_H E &= \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \inf\{s \geq 0 : \mathcal{H}^s(E) < \infty\} \\ & \sup\{s \geq 0 : \mathcal{H}^s(E) > 0\} = \sup\{s \geq 0 : \mathcal{H}^s(E) = \infty\}. \end{aligned}$$

and the following two statements hold

- $s < \dim_H E \;\Rightarrow\; \mathcal{H}^s(E) = \infty$,

- $s > \dim_H E \;\Rightarrow\; \mathcal{H}^s(E) = 0$.

1.4 Other Useful Definitions

In the last short section of Preliminaries, we mention a definition of Minkowski summation, "Big O" (the so-called Bachmann-Landau notation) and "Little o" notation.

Definition 1.16. *(Minkowski Summation) Let* $M, N \subset \mathbb{R}^d$*. We denote*

$$M \oplus N = \{m + n : m \in M, n \in N\}$$

and

$$M \ominus N = (M^c \oplus N)^c$$

Definition 1.17. *(Big O notation) We write* $f(x) = O(g(x))$ *for* $x \to \infty$*, whenever there exist a positive constants* K *and* x_0 *such that*

$$f(x) \leq K|g(x)| \text{ for all } x > x_0$$

and we will write $f(x) = O(g(x))$ *for* $x \to a \in \mathbb{R}$ *if and only if there exist* $K > 0$ *and* $\delta > 0$ *such that*

$$f(x) \leq K|g(x)|, \text{ for all } |x - a| < \delta.$$

Definition 1.18. *(Little o notation) We write* $f(x) = o(g(x))$ *for* $x \to \infty$*, whenever for all* $\varepsilon > 0$ *there exist a positive constant* x_0 *such that*

$$f(x) \leq \varepsilon |g(x)| \text{ for all } x > x_0$$

and similarly we write $f(x) = o(g(x))$ *for* $x \to a \in \mathbb{R}$ *if for any* $\varepsilon > 0$ *exists* $\delta > 0$ *such that*

$$f(x) \leq \varepsilon |g(x)|, \text{ for all } |x - a| < \delta.$$

2. Cone Points and Critical Points

Our focus on a standard planar Brownian motion and research of its trajectories is manifested in this chapter. The subjects of our interest become the so-called "non-typical" points of the Brownian motion paths. This means points which have zero Hausdorff dimension.

First, we summarize some results regarding α-cone points of Brownian motion, i.e. the points of the Brownian motion where the path locally stays in a cone (with angle α) with vertex laying on the trajectory of Brownian motion. We recall results concerning the Hausdorff dimension of α-cone points.

Further, we investigate critical points of the distance function of Brownian motion. These points may not lie on the Brownian motion path but they belong to the convex hull of their nearest points to the Brownian trajectory. The distance of a critical point to the Brownian motion path is called the critical value. The non-critical values are called regular values and the boundary of the Wiener sausage with regular radius has very pleasant properties (see [6]).

We will show some connection of cone points and critical points of Brownian motion and we prove that there almost surely do not exist two $\pi+$ cone points ($(\pi + \delta)$-cone points for arbitrary small $\delta > 0$) laying on a fixed line.

This theorem leads us to write the conjecture of the countability of the critical points of Brownian motion.

2.1 α-cone Points of Brownian Motion

It is a well-known that planar Brownian motion performs infinite number of full windings in both directions around any points of its trajectory almost surely. The asymptotic law of windings number is described by the famous Spitzer's law (see [23]). Let $\{\theta(t),\, t \geq 0\}$ be the argument of planar Brownian motion, then the distribution of $2\theta(t)/\log t$ is asymptotically a standard symmetric Cauchy distribution.

Nevertheless, there still exist some exceptional random points of the Brownian motion path such that Brownian motion performs finite number of orbits or no orbit around them. This property is fulfilled e.g. by points of the boundary of the convex hull of $B[0, 1]$ or points with more specifical behavior, and which are in our interest, the α-cone points.

The alternative view of cone points offers [2], where the points of $B[0, 1]$ which locally stays in both-sided cones are studied.

At the beginning of this section, we define one-sided and both-sided α-cone points and we summarize some results on the Hausdorff dimension.

Definition 2.1. *(Cones) For any* $\alpha \in (0, 2\pi]$ *and* $\xi \in [0, 2\pi)$*, we define the* α-cone *as the following set:*

$$W(\alpha, \xi) = \left\{ (r\cos(\theta + \xi), r\sin(\theta + \xi)) \in \mathbb{R}^2 : \ r \geq 0, \ |\theta| \leq \frac{\alpha}{2} \right\}.$$

We write $x + W(\alpha, \xi)$ *for the cone shifted to* $x \in \mathbb{R}^2$ *instead of* $x \oplus W(\alpha, \xi)$*. The cone* $x + W(2\pi - \alpha, \xi + \pi)$ *is the dual cone of* $x + W(\alpha, \xi)$*.*

The vector $(\cos\xi, \sin\xi)$ *is the direction of the axis of* $W(\alpha, \xi)$*, we abbreviate this unit vector by* $\vec{\xi}$*.*

Definition 2.2. *(One-sided cone points) Let* $B = \{B(t) : \ t \geq 0\}$ *be a standard planar Brownian motion. We will consider* $C_t^+(\alpha)$ *(resp.* $C_t^-(\alpha)$*) to be set of following times:*

$$\begin{aligned} C_t^+(\alpha) &= \bigcup_{\xi \in [0,2\pi)} \{t \in [0,1); \ \exists h > 0 : \ B(s) \in B(t) + W(\alpha, \xi), \ \forall s \in [t, t+h]\}, \\ C_t^-(\alpha) &= \bigcup_{\xi \in [0,2\pi)} \{t \in (0,1]; \ \exists h > 0 : \ B(s) \in B(t) + W(\alpha, \xi), \ \forall s \in [t-h, t]\}. \end{aligned}$$

The corresponding points of the Brownian trajectory are called (one-sided) α*-cone points and we use an abbreviation* $C_p^+(\alpha)$ *(resp.* $C_p^-(\alpha)$*):*

$$\begin{aligned} C_p^+(\alpha) &= B[C_t^+(\alpha)], \\ C_p^-(\alpha) &= B[C_t^-(\alpha)]. \end{aligned}$$

The set $C_p(\alpha) = C_p^+(\alpha) \cap C_p^-(\alpha)$ *is called the set of (both-sided)* α*-cone points.*

But we prefer using another equivalent definition of α-cone points.

Definition 2.3. *(Cone points) A point* $x \in \mathbb{R}^2$ *is an* α*-cone point if* x *is a point of the trajectory* $B[0,1]$ *and there exists* $\varepsilon > 0$ *and* $\xi \in [0, 2\pi)$ *such that*

$$(B[0,1] \cap b(x, \varepsilon)) \subset x + W(\alpha, \xi).$$

For an illustration of α-cone points see Figure 2.1.

It is a well-known property of Brownian motion that for a fixed $t \geq 0$ and for any cone $W(\alpha, \xi)$, $\alpha \in (0, 2\pi)$, $\xi \in [0, 2\pi)$ and for all $\delta > 0$, there exists a time $s \in (t - \delta, t + \delta)$ such that $B(s) \notin B(t) + W(\alpha, \xi)$ with the probability one. Therefore, it is easy to see that the event $\{t \in C_t^+(\alpha)\}$ (resp. $\{t \in C_t^-(\alpha)\}$) has zero probability.

Now, we want to demonstrate for which angles of cones there exist the cone points and even more that the exact Hausdorff dimension of the set of α-cone points is known.

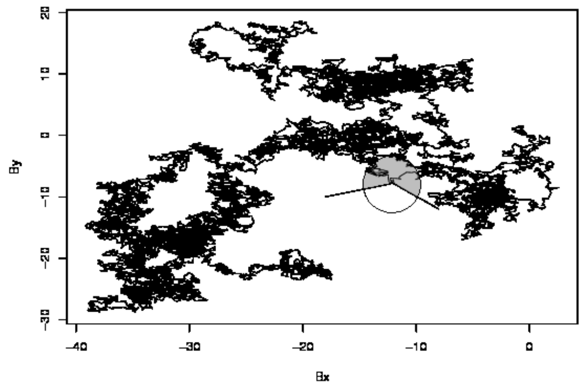

Figure 2.1: An example of α-cone point, $\alpha > \pi$.

Theorem 2.4. *([5], Evans 1985) There is no α-cone point for $\alpha \in (0, \pi)$ almost surely. But for $\alpha \in [\pi, 2\pi)$, α-cone points exist a.s. and the following formula for their Hausdorff dimension holds:*

$$\dim_H(C_p(\alpha)) = 2 - \frac{2\pi}{\alpha} \quad \text{a.s.}$$

The following formula holds for one-sided cone points:

$$\dim_H(C_p^+(\alpha)) = \dim_H(C_p^-(\alpha)) = 2 - \frac{\pi}{\alpha} \quad \text{a.s.}$$

and

$$\dim_H(C_p^+(\alpha) \cap C_p^-(\beta)) = 2 - \frac{\pi}{\alpha} - \frac{\pi}{\beta} \quad \text{a.s.}$$

Similar results hold for the "cone" times of cone points:

$$\dim_H(C_t^+(\alpha)) = \dim_H(C_t^-(\alpha)) = 1 - \frac{\pi}{2\alpha} \quad \text{a.s.}$$

and

$$\dim_H(C_t^+(\alpha) \cap C_t^-(\beta)) = 1 - \frac{\pi}{2\alpha} - \frac{\pi}{2\beta} \quad \text{a.s.}$$

We make the convention, in the previous formulae, whenever right-hand side is negative, then the considered set on the left-hand side is empty.

Proof of this theorem can be found in [5]. For a modern proof see [15] Chapter 10.4. The upper bound is based on so-called *approximative cone points* which allow to use the strong Markov property. We will deal with them below. Proof of the lower bound uses Frostman's lemma which is often used in the theory of fractal dimensions.

Remark 2.5. *A famous result of McKean [14] says:*

$$\dim_H(B[A]) = \min\{2\dim_H(A), d\}, \text{ for any closed set } A \subset [0, \infty).$$

For more details see [15], Chapter 4. The previous theorem gives us that almost every point of the $B[0,1]$ *is a* 2π*-cone point.*

2.2 Critical Points of Brownian Motion

In this section we introduce the critical points and critical values of Brownian motion. We present some of their important properties and their relations. We show almost sure non-existence of two $\pi+$ cone points with the opposite orientation laying on the fixed line (Theorem 2.13). Finally, we formulate the conjecture of countability of critical points of Brownian motion.

Definition 2.6. *Let* M *be a subset of* $\mathbb{R}^d$*. For* $x \in \mathbb{R}^d \setminus \overline{M}$*, we denote by* $N(x, M)$ *the set of points from* $\overline{M}$ *which are the closest to* x*. It means that*

$$N(x, M) = \{y \in \overline{M} : \operatorname{dist}(x, M) = \operatorname{dist}(y, x)\}.$$

If $x \in \mathbb{R}^d \setminus \overline{M}$ *is contained in* $\operatorname{conv}(N(x, M))$ *then we call* x *a critical point (of distance function) of* M *and the value of* $\operatorname{dist}(x, M)$ *is a critical value.*

Let us denote by $\operatorname{Crit}(M)$ *the set of all critical points of* M*.*

Steve Ferry (see [6]) presents the following lemma with a nice and clear proof.

Lemma 2.7. *([6, Proposition 1.5.]) Let* x *and* y *be different critical points of* $M \subset \mathbb{R}^d$*. Then*

$$|\operatorname{dist}(x, M)^2 - \operatorname{dist}(y, M)^2| \le |x - y|^2.$$

Proof. Throughout the proof $x \cdot y$ denotes the scalar product of x, $y \in \mathbb{R}^d$.

Let x be a critical point of M. Since $x \in \operatorname{conv}(N(x, M))$, there exist $x_i \in N(x, M)$ and $t_i > 0$, $\sum t_i = 1$, such that $x = \sum x_i t_i$. Thus

$$\operatorname{dist}(x, M)^2 = |x - x_i|^2 = |x|^2 - 2x \cdot x_i + |x_i|^2.$$

Let $y \in \operatorname{Crit}(M)$, $y \ne x$. Then

$$|y - x_i|^2 = |y|^2 - 2y \cdot x_i + |x_i|^2 \ge \operatorname{dist}(y, M)^2.$$

Subtracting these two equations we obtain

$$\operatorname{dist}(y, M)^2 - \operatorname{dist}(x, M)^2 \le |y|^2 - 2x_i \cdot (y - x) - |x|^2$$

and after multiplying it by t_i and summing over i, we have

$$\operatorname{dist}(y, M)^2 - \operatorname{dist}(x, M)^2 \le |x - y|^2.$$

If we change the roles of x and y, we obtain, what we need. □

The following theorem deals with the countability of critical points, we put forward the proof of it for its very natural and clear character. This can be found in [7].

Theorem 2.8. *(Fu 1985) Let M be a closed subset of $\mathbb{R}^2$. Then the following set is at most countable:*

$$\{x \in \mathrm{Crit}(M) : \ \mathrm{Int}(\mathrm{conv}(N(x, M))) \neq \emptyset\}.$$

Proof. An easy application of Lemma 2.7 gives the proof of the statement. If $x, y \in Z$, $x \neq y$, then $N(x, M)$ and $N(y, M)$ lie in the opposite half-planes which are determined by intersection of $b(x, \mathrm{dist}(x, M))$ and $b(y, \mathrm{dist}(y, M))$. It implies that

$$\mathrm{Int}(\mathrm{conv}(N(x, M))) \cap \mathrm{Int}(\mathrm{conv}(N(y, M))) = \emptyset$$

and a set of disjoint open subsets of $\mathbb{R}^2$ is at most countable. □

Hence, if we focus on the countability of critical points of M, we can restrict our research to those critical points which are at the center of a segment with ending points in their two closest points in the $\overline{M}$. For simplicity of notation, we will continue to write $\mathrm{Crit}_2(M)$ for the subset of $\mathrm{Crit}(M)$ such that its critical points have only two the nearest points to the set M, it means

$$\mathrm{Crit}_2(M) = \{x \in \mathrm{Crit}(M) : \ \#N(x, M) = 2\}.$$

If we focus on the countability of $\mathrm{Crit}(B[0, 1])$, we can restrict to $\mathrm{Crit}_2(B[0, 1])$, due to Theorem 2.8.

Figure 2.2 shows the two discussed types of critical points of $B[0, 1]$.

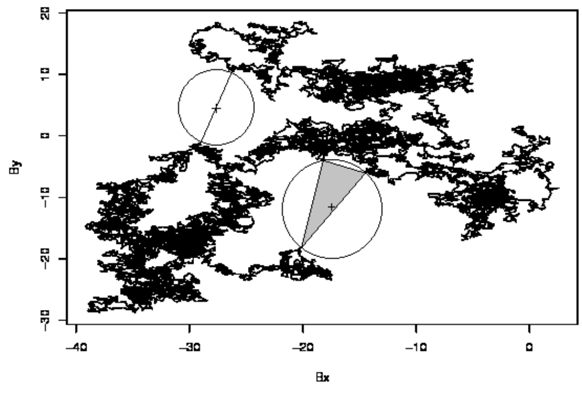

Figure 2.2: Two different types of critical points of $B[0, 1]$.

Now, we show an obvious relation between α-cone points and critical points of Brownian motion. If we consider $x \in \mathrm{Crit}(B[0, 1])$, then the points of $N(x, B[0, 1])$ are $(\pi + \delta)$-cone points for every $\delta > 0$. Let us denote this set by $C_p(\pi+) = \bigcap_{\delta>0} C_p(\pi + \delta)$. We summarize this paragraph in the following observation.

Observation 2.9. *The set* $\{y \in N(x, B[0,1]) : \ x \in \mathrm{Crit}_2(B[0,1])\}$ *is a subset of* $C_p(\pi+)$.

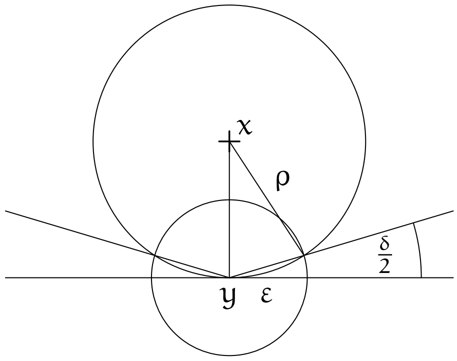

Figure 2.3:

Proof. Let y be a point of $N(x, B[0,1])$. We need to prove that $y \in C_p(\pi+)$. Fix $\delta > 0$ and consider the cone $y + W(\pi + \delta, \xi)$, where $\vec{\xi}|x - y| = y - x$. Let $\varepsilon > 0$ be the distance from y to the intersection of the boundary of $y + W(\pi + \delta, \xi)$ and $\partial b(x, \rho)$, $\rho = \mathrm{dist}(x, M)$. For an illustration see Figure 2.3. Therefore, there is no point of $B[0,1]$ in $(y + W(\pi + \delta, \xi)) \cap b(y, \varepsilon)$, hence $y \in C_p(\pi + \delta)$. Since $\delta > 0$ was arbitrary, $y \in \bigcap_{\delta>0} C_p(\pi + \delta) = C_p(\pi+)$. $\square$

By virtue of Theorem 2.4, the Hausdorff dimension of $(\pi + \delta)$-cone points is

$$2 - \frac{2\pi}{\pi + \delta} = \frac{2\delta}{\pi + \delta},$$

hence the set $C_p(\pi+)$ has the Hausdorff dimension zero.

In [8], we present the following conjecture.

Conjecture 2.10. *There are at most countably many critical points of* $B[0,1]$ *almost surely.*

Remark 2.11. *It is a well-known fact (see [7]) that the Hausdorff dimension of the set of critical values of any compact set in* $\mathbb{R}^2$ *is less or equal to* $1/2$. *If Conjecture 2.10 holds, then the set of critical values of* $B[0,1]$ *is countable and thus its Hausdorff dimension is zero.*

Conjecture 2.10 is supported by Theorem 2.13.

Notation 2.12. $C_p(\alpha, \xi)$ *stands for the set of all* α*-cone points with a fixed cone* $W(\alpha, \xi)$, *and we define*

$$C_p(\pi+, \xi) = \bigcap_{\delta>0} C_p(\pi + \delta, \xi).$$

It is consistent with our previous notation because $C_p(\pi+) = \bigcup_{\xi \in [0, 2\pi)} C_p(\pi+, \xi)$.

Theorem 2.13. *([8, Proposition 1]) Let $\vec{\xi}$ be a fixed direction. Then*

$$\mathbb{P}\left[\exists x \in C_p(\pi+, \xi)\, \exists y \in C_p(\pi+, \xi+\pi),\, \exists c > 0 : y - x = c\vec{\xi}\right] = 0. \qquad (2.1)$$

For the proof, we will use a method using approximative cone points which were introduced in [15].

Definition 2.14. *(Approximative cone points) Let $\alpha \in (0, 2\pi)$ and $\xi \in [0, 2\pi)$ be fixed angles. We call $x \in \mathbb{R}^2$ a (γ, ε)-approximative cone point ($\varepsilon > \gamma > 0$) if*

$$B[T_\varepsilon(x), T_\gamma(x)] \cup B[T_{\frac{\gamma}{2}}(x), R_{\frac{\gamma}{2},\varepsilon}(x)] \subset x + W(\alpha, \xi).$$

We denote by $C_{ap}(\alpha, \xi, \gamma, \varepsilon)$ the set of all (γ, ε)-approximative cone points with cone $W(\alpha, \xi)$.

Remark 2.15. *This definition of approximative cone point differs from the original one in [15], because we replaced zero by stopping time $T_\varepsilon(x)$. We need this change because we deal with two (γ, ε)-approximative cone points with cones $W(\pi + \delta, \xi)$ and $W(\pi + \delta, \xi + \pi)$ respectively. Therefore, it is necessary for us to work with the whole trajectory of B on interval $[0, T_\gamma(x))$.*

As a preparation of the proof of Theorem 2.13, we introduce here a few auxiliary lemmas.

Lemma 2.16. *([15, Lemma 10.40 (a)]) Let $\xi \in [0, 2\pi)$ be fixed, then there exists a constant $C > 0$ depending only on $\alpha \in (0, 2\pi)$ such that for every $0 < \gamma < \varepsilon$ and for all x, $z \in \mathbb{R}^2$, $|x - z| = \gamma/2$, the following formula holds*

$$\mathbb{P}_x\left[B[0, T_\varepsilon(z)] \in z + W(\alpha, \xi)\right] \leq C\left(\frac{\gamma}{\varepsilon}\right)^{\frac{\pi}{\alpha}}.$$

In the cited literature the proof of this lemma was left on the reader as an exercise. We perform the proof applying the following theorem.

Theorem 2.17. *([15, Theorem 7.25]) Let $\alpha \in (0, 2\pi]$. Then for $r > 1$ the following inequality holds*

$$\mathbb{P}_{(1,0)}\left[B[0, T_r(0)] \subset W(\alpha, 0)\right] = \frac{2}{\pi}\arctan\left(\frac{2r^{\frac{\pi}{\alpha}}}{r^{\frac{2\pi}{\alpha}} + 1}\right).$$

Proof of lemma 2.16. We denote by z' the intersection of the boundary of $z + W(\alpha, \xi)$ with the line $x + t\vec{\xi}$, $t \in \mathbb{R}$.

It is obvious that $z + W(\alpha, \xi) \subseteq z' + W(\alpha, \xi)$ and that there exists a constant $C_1 = C_1(\alpha) \in (0, 1)$ depending only on α such that the following arrangement of balls holds

$$b(z, \gamma/2) \subseteq b(z', C_1\varepsilon)$$

and

$$b(z', C_1\varepsilon) \subseteq b(z, \varepsilon).$$

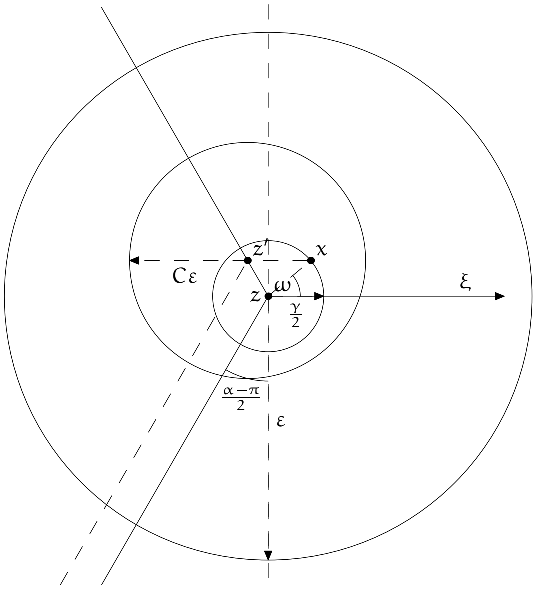

Figure 2.4: The situation of the proof of Lemma 2.16, $\alpha > \pi$.

The situation is shown in Figure 2.4. More precisely, C_1 has to fulfill

$$\begin{aligned} \frac{\gamma}{2} + |z - z'| &< C_1\varepsilon, \\ C_1\varepsilon + |z - z'| &< \varepsilon. \end{aligned}$$

Thus

$$\mathbb{P}_x\left[B[0, T_\varepsilon(z)] \subset z + W(\alpha, \xi)\right] \leq \mathbb{P}_x\left[B[0, T_{C_1\varepsilon}(z')] \subset z' + W(\alpha, \xi)\right]. \tag{2.2}$$

The distance x from z' can be computed exactly:

$$|x - z'| = \frac{\gamma}{2}\left(\cos\omega - \frac{\sin\omega}{\tan(\alpha/2)}\right),$$

where $\omega \in [0, \alpha/2)$ denotes the angle of vectors $x - z$ and $\vec{\xi}$. There exists a constant $c_2 = c_2(\alpha) > 0$ such that

$$|x - z'| = c_2\gamma.$$

Now, we can continue by estimating (2.2). After a suitable shift, rotation and scaling, we can apply Theorem 2.17:

$$\begin{aligned} \mathbb{P}_{(1,0)}\left[B[0, T_{\frac{C_1\varepsilon}{|x-z'|}}(0)] \subset W(\alpha, 0)\right] &= \mathbb{P}_{(1,0)}\left[B[0, T_{\frac{C_1\varepsilon}{c_2\gamma}}(0)] \subset W(\alpha, 0)\right] \\ &= \frac{2}{\pi}\arctan\frac{2\left(\frac{C_1\varepsilon}{c_2\gamma}\right)^{\frac{\pi}{\alpha}}}{\left(\frac{C_1\varepsilon}{c_2\gamma}\right)^{\frac{2\pi}{\alpha}} + 1} \leq \frac{4}{\pi}\left(\frac{C_1\varepsilon}{c_2\gamma}\right)^{-\frac{\pi}{\alpha}}, \end{aligned}$$

which completes the proof. □

The following lemma is a tiny adaptation of Lemma 10.41 in [15].

Lemma 2.18. *Let $\xi \in [0, 2\pi]$ be fixed. Then there exists a constant $C_0 > 0$ depending only on $\alpha \in (0, 2\pi)$ such that for any $z \in \mathbb{R}^2$, it holds that*

$$\mathbb{P}\Big[z \in C_{ap}(\alpha, \xi, \gamma, \varepsilon)\Big] \le C_0 \left(\frac{\gamma}{\varepsilon}\right)^{\frac{2\pi}{\alpha}}.$$

Proof. We apply the strong Markov property at stopping time $T_{\gamma/2}(z)$ to obtain

$$\begin{aligned} \mathbb{P}\Big[z \in C_{ap}(\alpha, \xi, \gamma, \varepsilon)\Big] &\le \mathbb{P}\Big[B[T_\varepsilon(z), T_\gamma(z)] \subset z + W(\alpha, \xi)\Big] \\ &\quad \cdot \mathbb{P}_{B(T_{\gamma/2}(z))}\big[B[0, R_{\gamma/2,\varepsilon}(z)] \subset z + W(\alpha, \xi)\big] \\ &\le C^2 \left(\frac{\gamma}{\varepsilon}\right)^{\frac{2\pi}{\alpha}}, \end{aligned}$$

where we have applied Lemma 2.16 with $C_0 := C^2$. □

The following lemma uses the technics performed in the proof of Lemma 10.42 in [15].

Lemma 2.19. *Let $\mathcal{D}_k$ be the collection of squares in the diadic partition of $[0,1]^2$, $\mathcal{D}_k = \{D_{ij} : i, j = 1, 2, \ldots, 2^k\}$, where $D_{ij} = [(i-1)2^{-k}, i2^{-k}) \times [(j-1)2^{-k}, j2^{-k})$. Let $\xi \in [0, 2\pi)$ be fixed. Then there exists $k_0 \in \mathbb{N}$ and C_1, $C_2 > 0$ which depend only on $\alpha \in (0, 2\pi)$ such that for all $k \ge k_0$*

$$\mathbb{P}\Big[\exists z \in D,\, D \in \mathcal{D}_k,\, z \in C_{ap}(\alpha, \xi, C_1 2^{-k}, \varepsilon)\Big] \le C_2 \cdot 2^{-\frac{2k\pi}{\alpha}}. \tag{2.3}$$

Proof. For given $D \in \mathcal{D}_k$ let $s \in D$ be the intersection of its diagonals. We can find $x = x(D)$ (in [15], it has been called a *focal point* of D) satisfying that it is the tip of cone $x + W(\alpha, \xi)$ such that

- If $\alpha \le \pi$, then the boundary halflines of the cone $x + W(\alpha, \xi)$ are tangent to $b(s, (1 + \sqrt{2})2^{-k})$,
- If $\alpha > \pi$, then the boundary halflines of the dual cone of $x + W(\alpha, \xi)$ are tangent to $b(s, (1 + \sqrt{2})2^{-k})$.

It is not hard to observe, that for every $\varepsilon > 0$ and $\alpha \in (0, 2\pi)$ there exists $k_1 \in \mathbb{N}$ such that for all $k \ge k_1$ and any $D \in \mathcal{D}_k$, we have

$$b(x, \varepsilon/2) \subset b(y, \varepsilon) \quad \text{for all } y \in D.$$

Further, $y + W(\alpha, \xi) \subset x + W(\alpha, \xi)$. There exist constants $C_1 > c_1 > 0$ depending only on α which fulfill:

- $b(y, C_1 2^{-k}) \subset b(x, C_1^2 2^{-k})$,

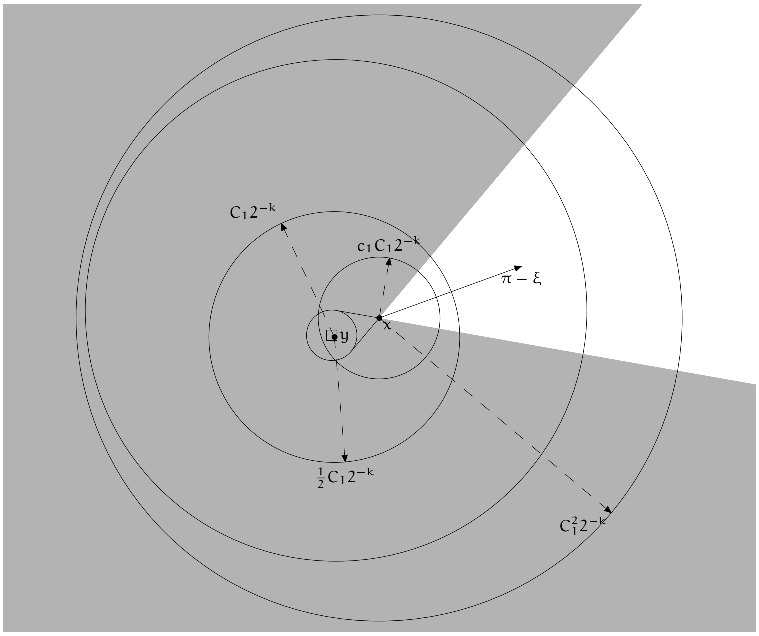

Figure 2.5: The positions of x, y and the corresponding circles, $\alpha > \pi$.

- $b(y, C_1 2^{-k-1}) \supset b(x, c_1 C_1 2^{-k})$,
- $|x - y| < c_1 C_1 2^{-k}$.

See Figure 2.5.

These conditions imply that for k large enough, it holds that whenever $D \in \mathcal{D}_k$ contains an approximative cone point from $C_{ap}(\alpha, \xi, C_1 2^{-k}, \varepsilon)$, then $x(D)$ has to satisfy

$$\left(B[0, T_{C_1^2 2^{-k}}(x)] \cup B[T_{c_1 C_1 2^{-k}}(x), R_{c_1 C_1 2^{-k-1}, \varepsilon/2}(x)]\right) \subset x + W(\alpha, \xi).$$

It means that whenever $D \in \mathcal{D}_k$ (k large enough provided) contains an approximative cone point from $C_{ap}(\alpha, \xi, C_1 2^{-k}, \varepsilon)$, then $x(D)$ belongs to $C_{ap}(\alpha, \xi, c_1 C_1 2^{-k}, \varepsilon/2)$.

Hence, using Lemma 2.18 we obtain

$$\mathbb{P}\left[\exists z \in D,\ D \in \mathcal{D}_k,\ z \in C_{ap}(\alpha, \xi, C_1 2^{-k}, \varepsilon)\right] \leq C_0 \left(\frac{C_1 2^{-k}}{\varepsilon}\right)^{\frac{2\pi}{\alpha}}. \tag{2.4}$$

For given ε we can find $k_0 \geq k_1$ such that the right-hand side of (2.4) is smaller than $C_2 2^{-2k\pi/\alpha}$, which gives (2.3). □

Now, we can finish the proof of Theorem 2.13.

Proof of Theorem 2.13. First, we remind that

$$C_p(\pi+) \subseteq \bigcup_{\xi \in [0, 2\pi)} \bigcap_{\delta > 0} \bigcup_{\varepsilon > 0} \bigcap_{k=1}^{\infty} C_{ap}(\pi + \delta, \xi, C_1 2^{-k}, \varepsilon) \text{ for some } C_1 > 0.$$

Hence for some $C_1 > 0$ the following estimate (for any $\delta > 0$) holds:

$$\mathbb{P}\left[\exists x \in C_p(\pi+\delta,\xi),\ \exists y \in C_p(\pi+\delta,\xi+\pi),\ \exists c>0:\ y-x=c\vec{\xi}\right]$$

$$\leq \mathbb{P}\Big[\exists \varepsilon>0,\ \forall k\in\mathbb{N},\ \exists x\in C_{ap}(\pi+\delta,\xi,C_1 2^{-k},\varepsilon),$$

$$\exists y\in C_{ap}(\pi+\delta,\xi+\pi,C_1 2^{-k},\varepsilon),\ y-x=|y-x|\vec{\xi}\Big].$$

Without loss of generality, we can restrict only to the unit square $[0,1]^2$ and we can split up it into the small squares D_k as we had already done above. We can, without loss of generality, assume that $\xi = (1,0)$.

We look at the event in (2.1) restricted on $[0,1]^2$. It has to be contained in:

$$\bigcup_{\varepsilon>0}\bigcap_{k=1}^{\infty} A_k^{\varepsilon}$$

where

$$A_k^{\varepsilon} = \Big\{\exists x\in C_{ap}(\pi+\delta,\xi,C_1 2^{-k},\varepsilon)\cap[0,1]^2,$$

$$\exists c>0:\ x+c\vec{\xi}\in C_{ap}(\pi+\delta,\xi+\pi,C_1 2^{-k},\varepsilon)\cap[0,1]^2\Big\},$$

for any $\delta > 0$.

We can estimate

$$\mathbb{P}\left[\bigcap_{k=1}^{\infty} A_k^{\varepsilon}\right] \leq \sum_{i=1}^{2^k}\sum_{j=1}^{2^k}\sum_{l=j+1}^{2^k}\mathbb{P}\big[\exists x\in D_{ij},\ x\in C_{ap}(\pi+\delta,\xi,C_1 2^{-k},\varepsilon),$$

$$\exists y\in D_{il},\ y\in C_{ap}(\pi+\delta,\xi+\pi,C_1 2^{-k},\varepsilon)\big]$$

$$= \sum_{i=1}^{2^k}\sum_{j=1}^{2^k}\sum_{l=j+1}^{2^k}\mathbb{P}\big[\exists x\in D_{ij}, x\in C_{ap}(\pi+\delta,\xi,C_1 2^{-k},\varepsilon)\big]\ \cdot$$

$$\cdot\ \mathbb{P}\big[\exists y\in D_{il},\ y\in C_{ap}(\pi+\delta,\xi+\pi,C_1 2^{-k},\varepsilon)\,|\,\exists x\in D_{ij},$$

$$x\in C_{ap}(\pi+\delta,\xi,2^{-k},\varepsilon)\big]. \quad (2.5)$$

The upper bound of (2.5) is easy now. By the Lemma 2.19, the first factor in (2.5) is directly bounded by

$$C_2 2^{-\frac{2k\pi}{\pi+\delta}}. \quad (2.6)$$

The second one can be bounded by using the same inequality but first, we have to apply the strong Markov property. In particular, let τ be the exit time of B from $\bigcap_{x\in D_{ij}} b(x,\varepsilon)$. It is easy to see that $\tau \geq S_{C_1 2^{-k-1},\varepsilon}(z)$ whenever

$$z\in D_{ij}\cap C_{ap}(\pi+\delta,\xi,C_1 2^{-k},\varepsilon).$$

Let z_{ij} be a point with the smallest first coordinate such that D_{ij} is contained in the cone $z_{ij} + W(\pi + \delta, \xi)$. Then we observe that

$$B(\tau) \in z_{ij} + W(\pi + \delta, \xi)$$

and now we can apply strong Markov property at τ, i.e. $B(\tau + t) - B(\tau)$ is Brownian motion independent of $B[0, \tau]$. Thus, the second factor in (2.5) can be estimated again by (2.6). Therefore, we obtain

$$\begin{aligned} \mathbb{P}\left[\bigcap_{k=1}^{\infty} A_k^{\varepsilon}\right] &\leq C_2^2 \cdot 2^{3k} \cdot 2^{-k\,\frac{4\pi}{\pi+\delta}} \\ &= C_2^2 \cdot 2^{-k\,\frac{\pi-3\delta}{\pi+\delta}} \overset{k\to\infty}{\longrightarrow} 0 \end{aligned}$$

and it easily leads to the claim of the theorem.

□

3. Volume and Surface Area of Wiener Sausage

The Brownian motion path has a rather complicated structure therefore its r-neighbourhood is often subject to examination. This neighbourhood is called the Wiener sausage. Let us start with its definition.

Definition 3.1. *Let $\{B(t) : t \geq 0\}$ be a standard d-dimensional Brownian motion. For given radius $r > 0$ and for $t > 0$, we define Wiener sausage by*

$$S_r[0,t] = B[0,t] \oplus b(o,r).$$

It is an easy consequence of scaling invariance of Brownian motion that

$$\begin{aligned} S_r[0,t] &= B[0,t] \oplus b(o,r) \\ &\overset{d}{=} \sqrt{t}B[0,1] \oplus b(o,r) \\ &= \sqrt{t}\left(B[0,1] \oplus b(o,r/\sqrt{t})\right) \\ &\overset{d}{=} \sqrt{t}S_{r/\sqrt{t}}[0,1]. \end{aligned}$$

Therefore, we will consider only Wiener sausage up to time $t = 1$ in the sequel.

In the first section of this chapter, we present results concerning the volume of the Wiener sausage. The second section contains asymptotic behavior of the mean surface area of the Wiener sausage. Then we show almost sure limit behavior of the surface area of the Wiener sausage in dimension $d \geq 3$, i.e.

$$\mathcal{H}^{d-1}(\partial S_r[0,1]) = (d-2)^2 \pi \omega_{d-2} r^{d-3} + o\left(r^{d-3}\right), \quad r \to 0_+ \text{ a.s.} \tag{3.1}$$

A general geometric approach is used to prove that whenever the q-dimensional Minkowski content of the set $A \in \mathbb{R}^d$ exists and equals a positive number a, then also q-dimensional S-content of A exists and is equal to a. We use properties of the volume function derived by L. Stachó [24] to receive this result. In fact, we show a result on asymptotic behavior of Kneser function.

3.1 Volume of Wiener Sausage

Now we summarize some results concerning the volume of the Wiener sausage. Let us denote by $V(r,t)$ the volume of $S_r[0,t]$.

The exact mean volume of the Wiener sausage was derived in [1], using the known distributions of exit times of balls for Brownian motion.

$$\mathbb{E}\, V(r,t) = \omega_d r^d + \frac{4d\omega_d r^d}{\pi^2} \int_0^\infty \frac{1 - \exp\left(-\frac{x^2 t}{2r^2}\right)}{x^3 (J_\nu^2(x) + Y_\nu^2(x))} \,\mathrm{d}x + \mathbf{1}_{\{d \geq 3\}} \frac{d(d-2)}{2} \omega_d t r^{d-2}, \tag{3.2}$$

where $\omega_d = \pi^{d/2}/\Gamma(d/2+1)$ is the volume of a d-dimensional unit ball, $\nu = \frac{d-2}{2}$ and $J_\nu(\cdot)$ and $Y_\nu(\cdot)$ are the Bessel J-function and Y-function (for more details see Definition 1.8).

That leads to

$$d = 1: \qquad \mathbb{E}\, V(r,t) = 2r + \frac{2\sqrt{2}}{\sqrt{\pi}}\sqrt{t}, \tag{3.3}$$

$$\begin{aligned} d = 2: \qquad & \mathbb{E}\, V(r,t) = \frac{2\pi t}{\log t} + \frac{2\pi t}{(\log t)^2}(1+\kappa-\log 2+2\log r) + O\left(\frac{t}{(\log t)^2}\right) \\ & \text{for } t/r^2 \to \infty, \\ d = 3: \qquad & \mathbb{E}\, V(r,t) = 2\pi r t + 4r^2\sqrt{2\pi t} + \frac{4}{3}\pi r^3. \end{aligned} \tag{3.4}$$

The first and the third formula can be found in [1], the second one is due to F. Spitzer [22], $\kappa \doteq 0.5772$ is the Euler constant.

Remark 3.2. *Due to the scaling invariance, we have*

$$\mathbb{E}\, V(r,t) = t^{d/2}\, \mathbb{E}\, V\left(\frac{r}{\sqrt{t}}, 1\right) = r^d\, \mathbb{E}\, V\left(1, \frac{t}{r^2}\right).$$

Remark 3.3. *The expression* (3.3) *corresponds to the property of the mean of a maximum of Brownian motion on time interval* $[0,1]$ *enlarged by* r *and multiplied by two. The mean of a maximum of Brownian motion can be easily computed using* (1.1)*. The case of dimension* $d = 1$ *is geometrically trivial and thus it is not interesting at all.*

In dimension $d \geq 3$, we have the following limit formula (see [4])

$$\lim_{r^2/t \to 0+} \frac{\mathbb{E}\, V(r,t)}{\omega_{d-2} r^{d-2} t} = (d-2)\pi.$$

The study of almost sure limits is much more difficult. First results on the almost sure asymptotic behavior of the volume are due to F. Spitzer [22] – (3.5) and the further expansions and central limit theorems were obtained by J.-F. Le Gall [12] – (3.6).

$$\mathcal{H}^2(S_r[0,1]) = \frac{\pi}{\log\frac{1}{r}} + o\left(\frac{1}{\log\frac{1}{r}}\right), \tag{3.5}$$

$$\mathcal{H}^d(S_r[0,1]) = (d-2)\pi\omega_{d-2}\, r^{d-2} + o\left(r^{d-2}\right), \quad d \geq 3, \tag{3.6}$$

where r tends to zero, almost surely.

3.2 Surface Area of Wiener Sausage

Recently it was shown (see [18] and [20]) that the mean surface area of the Wiener sausage satisfies

$$\mathbb{E}\,\mathcal{H}^{d-1}(\partial S_r[0,t]) = \frac{\mathrm{d}}{\mathrm{d}r}\,\mathbb{E}\,\mathcal{H}^{d}(S_r[0,t]), \quad r>0$$

and exact mean surface area of the Wiener sausage was proved in [18].

For simplicity of notation we write $S(r,t)$ instead of $\mathcal{H}^{d-1}(\partial S_r[0,t])$.

Theorem 3.4. *([18, Corollary 2.3]) Let $d \geq 2$. Then for almost all radii $r>0$ it holds that*

$$\mathbb{E}\,S(r,t) = d\omega_d r^{d-1}\left(1+\frac{t(d-2)^2}{2r^2}+\frac{4d}{\pi^2}\int_0^\infty \frac{\varphi_d\left(\frac{x^2 t}{2r^2}\right)}{x^3(J_\nu^2(x)+Y_\nu^2(x))}\,\mathrm{d}x\right), \qquad (3.7)$$

where $\varphi_d(y) = 1-(1+2y/d)\,\mathrm{e}^{-y}$, $\nu=\frac{d-2}{2}$ and $J_\nu(\cdot)$ and $Y_\nu(\cdot)$ are defined in the Definition 1.8. Moreover, in the case $d=2,3$ formula (3.7) holds for all $r>0$ and especially for $d=3$, we get

$$\mathbb{E}\,S(r,t) = 4\pi r^2 + 8r\sqrt{2\pi t}+2\pi t.$$

We refer to [4] for direct calculation of the following asymptotic behavior of the surface area of the Wiener sausage

$$\lim_{r\to 0_+}\mathbb{E}\,S(r,t) = \begin{cases} \infty & \text{if } d=2\\ 2\pi t & \text{if } d=3\\ 0 & \text{if } d>3\end{cases}$$

Using the same technics as in [4], we can obtain

$$\lim_{r\to 0_+}\mathbb{E}\,S(r,t) = \frac{\pi}{r\left(\log\frac{1}{r}\right)^2} + o\left(\frac{1}{r\left(\log\frac{1}{r}\right)^2}\right), \quad r\to 0,\ d=2$$

and

$$\begin{aligned}\mathbb{E}\,S(r,t) &= (d-2)^2\omega_d\, r^{d-1}\tau + O(\tau)\\ &= 2(d-2)^2\pi\omega_{d-2}\, r^{d-1}\tau + O(\tau), \quad d\geq 3\end{aligned}$$

where $\tau = \frac{t}{2r^2}$ tends to infinity.

As a consequence of our result, we obtain the following almost sure result (see Corollary 3.13):

$$\lim_{r\to 0}\frac{S(r,1)}{(d-2)^2\pi\omega_{d-2}r^{d-3}} = 1 \quad \text{a.s.}, \quad d\geq 3.$$

To prove this formula, we use a general approach with the so-called Kneser function.

Definition 3.5. *Let f be a continuous non-negative real function defined on $[0,\infty)$. We say that f is a Kneser function of order d if for all $0 < a \le b < \infty$ and $\lambda \ge 1$,*

$$f(\lambda b) - f(\lambda a) \le \lambda^d (f(b) - f(a)).$$

We list here some properties of Kneser functions proved by L. Stachó.

Lemma 3.6. *([24, Lemma 2, Theorem 1]) Let f be Kneser function of order d. Then*

(i) *f is absolutely continuous,*

(ii) *$f'(t)$ exists for all $t > 0$ up to a countable set,*

(iii) *left and right hand side derivatives of f (f'_- and f'_+) exist for every $t > 0$, and $f'_- \ge f'_+$,*

(iv) *f'_- and f'_+ are continuous from the left and from the right, respectively,*

(v) *the function $f'_+(t)t^{1-d}$ is decreasing, $t > 0$.*

Notation 3.7. *Let $A \subset \mathbb{R}^d$ be a bounded set. For given $r > 0$ we denote by*

$$A_r := \{z \in \mathbb{R}^d : \operatorname{dist}(z, A) \le r\} = A \oplus b(o, r)$$

the r-parallel neighbourhood of A. Further, we denote by $V_A(r) = \mathcal{H}^d(A_r)$ the volume and $S_A(r) = \mathcal{H}^{d-1}(\partial A_r)$ the surface area of A_r.

The fact that the volume function $V_A(\cdot)$ has the property of Definition 3.5 is due to Martin Kneser.

Proposition 3.8. *([10]) For any bounded set $A \subset \mathbb{R}^d$, V_A is a Kneser function of order d.*

The one-sided derivatives of the volume and the surface area of parallel sets are related as follows.

Proposition 3.9. *([20, Corollary 2.5.]) Let $A \subset \mathbb{R}^d$ be bounded. Then*

$$(V_A)'_+(r) \le S_A(r) \le (V_A)'_-(r) \quad \text{for all } r > 0.$$

Moreover, $(V_A)'(r)$ exists and equals $S_A(r)$ for all $r > 0$ up to a countable set.

Definition 3.10. *The q-dimensional Minkowski content of a set $A \subset \mathbb{R}^d$ is defined as*

$$\mathcal{M}^q(A) = \lim_{r\to 0} \frac{V_A(r)}{\omega_{d-q}\, r^{d-q}},$$

whenever the limit exists.

Analogously, if $q < d$, the q-dimensional S-content of A was introduced in [20] as

$$\mathcal{S}^q(A) = \lim_{r\to 0} \frac{S_A(r)}{(d-q)\omega_{d-q}\, r^{d-q-1}},$$

whenever the limit exists.

It was shown in [20] that if $\mathcal{S}^q(A)$ exists, then $\mathcal{M}^q(A)$ exists as well and they are equal.

The following theorem looks like reverse l'Hôspital rule.

Theorem 3.11. *([9, Theorem 3.1]) Let f be a Kneser function of order $d \geq 2$. If*

$$\lim_{r \to 0_+} \frac{f(r)}{r^p} = a \tag{3.8}$$

for some $p \in [1, d]$ and $0 < a < \infty$, then

$$\lim_{r \to 0_+} \frac{f'_+(r)}{p\, r^{p-1}} = a.$$

Proof of Theorem 3.11. First, we prove that

$$\liminf_{r \to 0_+} \frac{f'_+(r)}{pr^{p-1}} \geq a. \tag{3.9}$$

Let $\delta > 0$ be given. We shall find an $\eta > 0$ such that

$$\frac{f'_+(r)}{pr^{p-1}} > a - \delta \tag{3.10}$$

whenever $0 < r < \eta$. Assume that (3.10) does not hold for some $r > 0$. Using Lemma 3.6 (v), we get $f'_+(t) \leq f'_+(r)\frac{t^{d-1}}{r^{d-1}}$, $t \geq r$, thus

$$f'_+(t) \leq h(t) := \frac{a-\delta}{r^{d-p}} \cdot pt^{d-1}.$$

We denote $s = \tau r$, where $\tau = \left(\frac{a}{a-\delta}\right)^{\frac{1}{d-1}} > 1$.

Using the absolute continuity of f, we can obtain now an upper bound for the difference

$$f(s) - f(r) = \int_r^s f'_+(t)\mathrm{d}t \leq \int_r^s h(t)\mathrm{d}t = \frac{pr^p(\tau^d - 1)}{d}(a - \delta). \tag{3.11}$$

On the other hand, from the assumption (3.8), it follows that for any $\varepsilon > 0$ there exists $r_0 > 0$ such that whenever $0 < r < r_0$ then

$$(a - \varepsilon)r^p < f(r) < (a + \varepsilon)r^p,$$

thus

$$f(s) - f(r) > a(s^p - r^p) - \varepsilon(s^p + r^p).$$

If $\tau r < r_0$, and using the notation above we get:

$$f(s) - f(r) > ar^p(\tau^p - 1) - \varepsilon r^p(\tau^p + 1). \tag{3.12}$$

Putting (3.11) and (3.12) together, we get the following inequality:

$$ar^p(\tau^p - 1) - \varepsilon r^p(\tau^p + 1) < (a-\delta)\frac{p}{d}r^p(\tau^d - 1),$$

hence

$$\begin{aligned} \varepsilon &> \frac{1}{\tau^p + 1}\left(a(\tau^p - 1) - \frac{p(\tau^d - 1)}{d}(a-\delta)\right) \\ &= \frac{p(a-\delta)}{\tau^p+1}\left(\frac{\tau^{d-1+p} - \tau^{d-1}}{p} - \frac{\tau^d - \tau^0}{d}\right). \end{aligned} \tag{3.13}$$

Since $\tau > 1$, the function $u \mapsto \tau^u$ is convex increasing and, as $d - 1 + p \geq d$ by assumption, the right-hand side of (3.13) is positive. Let us denote it by ε_0. Thus, if $0 < \varepsilon < \varepsilon_0$, then (3.10) must be true for all $r < \varepsilon_0/\tau$. This proves (3.9).

In the second part of the proof, we show that

$$\limsup_{r\to 0_+} \frac{f(r)}{pr^{p-1}} \leq a. \tag{3.14}$$

The procedure is similar as in the first part. Let

$$\frac{f'_+(r)}{pr^{p-1}} \geq a + \delta \tag{3.15}$$

for some fixed $\delta > 0$ and $r > 0$. We use Lemma 3.6 (v) to show that $f'_+(t) \geq g(t)$ for $0 < t < r$, with

$$g(t) = \frac{a+\delta}{r^{d-p}} \cdot pt^{d-1}.$$

We denote $v = \rho r$ with $\rho = \left(\frac{a}{a+\delta}\right)^{\frac{1}{d-1}} < 1$. Similarly to (3.11), we obtain

$$f(r) - f(v) = \int_v^r f'_+(t)\mathrm{d}t \geq \frac{pr^p(1-\rho^d)}{d}(a+\delta). \tag{3.16}$$

Let $\varepsilon > 0$ be given. Then, by (3.8), we have

$$f(r) - f(p) < ar^p\,(1-\rho^p) + \varepsilon\, r^p\,(1+\rho^p) \tag{3.17}$$

for sufficiently small r. Putting (3.16) and (3.17) together, we get

$$\begin{aligned} \varepsilon &> \frac{1}{1+\rho^p}\left((a+\delta)\frac{p}{d}(1-\rho^p) - a(1-\rho^p)\right) \\ &= \frac{p(a+\delta)}{1+\rho^p}\left(\frac{1-\rho^d}{d} - \frac{\rho^{d-1} - \rho^{d-1+p}}{p}\right). \end{aligned} \tag{3.18}$$

Now $\rho < 1$, the function $u \mapsto \rho^u$ is convex decreasing. Hence, the right-hand side of the last equality is strictly positive again. So if we choose $\varepsilon > 0$ smaller than 3.18, then (3.15) cannot hold for r arbitrarily small. This proves (3.14) and the proof is finished. □

Corollary 3.12. *Let $A \subset \mathbb{R}^d$ be bounded. If the q-dimensional Minkowski content exists and $0 < \mathcal{M}^q(A) < \infty$ for some $q \le d-1$, then the q-dimensional S-content exists and we have $\mathcal{S}^q(A) = \mathcal{M}^q(A)$.*

Proof. V_A is a Kneser function of order d by Proposition 3.8. So, we can apply Theorem 3.11 with $p = d - q$ and $a := \mathcal{M}^q(A) \in (0, \infty)$. Moreover, by Proposition 3.9 is $S_A(r) \ge (V_A)'_+(r)$ and thus

$$\liminf_{r\to 0} \frac{S_A(r)}{(d-q)\omega_{d-q} r^{d-q-1}} \ge a.$$

For the opposite inequality, we use Lemma 3.6 and Proposition 3.9 to obtain

$$S_A(r) \le (V_A)'_-(r) = \lim_{t\to r_-} (V_A)'_-(t) = \lim_{\mathcal{R}\ni t\to r_-} (V_A)'_+(t),$$

where $\mathcal{R}$ denotes the set of all $t > 0$ where the derivative $(V_A)'(t)$ exists. It follows now from Theorem 3.11 that

$$\limsup_{r\to 0} \frac{S_A(r)}{(d-q)\omega_{d-q} r^{d-q-1}} \le a,$$

which gives the statement. $\square$

Now we will apply the presented results to the Brownian motion path. The Minkowski content of $B[0,1]$ is

$$\mathcal{M}^2(B[0,1]) = \lim_{r\to 0} \frac{V(r,1)}{\omega_{d-2} r^{d-2}} = (d-2)\pi \text{ a.s.} \tag{3.19}$$

Using Corollary 3.12, we obtain S-content for $B[0,1]$.

Corollary 3.13. *Let $d \ge 3$. Then*

$$\mathcal{S}^2(S_r[0,1]) = \lim_{r\to 0} \frac{S(r,1)}{(d-2)\omega_{d-2} r^{d-3}} = (d-2)\pi \text{ a.s.}$$

Remark 3.14. *Recently, the extended version ($0 < p < 1$) of our Theorem 3.11 was proved in the article [21] – Proposition 3.6. More general approach using the "gauge" function there was considered. The mentioned extension of Theorem 3.11 is the following:*

Let f be a Kneser function of order $d \ge 1$. If

$$\lim_{r\to 0} \frac{f(r)}{r^p g(r)} = C$$

for some non-negative non-decreasing function and p, $C \in (0, \infty)$, then

$$\lim_{r\to 0} \frac{f'_+(r)}{p r^{p-1} g(r)} = \lim_{r\to 0} \frac{f'_-(r)}{p r^{p-1} g(r)} = C.$$

The proof is based on the similar technics as we have used here. Since it is not our result and the article [21] waits for publication, we present our original proof of Theorem 3.11.

Unfortunately, the given statement does not say anything about the asymptotic of the surface area of the Wiener sausage in the plane.

3.3 Counterexample and Conjecture

Here we present a counterexample showing that the above technique cannot be applied for general gauge functions. In particular we cannot prove an almost sure asymptotic formula for the Brownian motion in dimension $d = 2$.

The Minkowski dimension of B is 2 again, but the volume has the following asymptotic behavior

$$\lim_{r\to 0} \frac{\mathcal{H}^2(S_r[0,1])}{\frac{1}{|\log r|}} = \lim_{r\to 0} \mathcal{H}^2(S_r[0,1]) \cdot |\log r| = \pi \text{ a.s.}$$

Our conjecture is the following formula

$$\lim_{r\to 0} \frac{\mathcal{H}^1(\partial S_r[0,1])}{\frac{1}{r\log^2 r}} = \lim_{r\to 0} \mathcal{H}^1(\partial S_r[0,1]) \cdot \left(r\log^2 r\right) = \pi \text{ a.s.}$$

but the general approach similar to Theorem 3.11 with general "gauge" function can not be true as the example below shows. For additional results with general gauge function see [21] Chapter 3.

Example 3.15. *([9, Example 4.1]) For given $a \in (0,\infty)$, there exists a Kneser function f of order 2 such that*

$$\lim_{r\to 0} f(r)|\log r| = a,$$

but

$$\lim_{r\to 0} f'_+(r)\, r\log^2 r \text{ does not exist.}$$

Proof. We define a continuous function $f : (0,1] \to [0,\infty)$ by

$$f(x) = a_n x^2 + c_n, \quad x \in [2^{-n}, 2^{-(n-1)}],$$

where

$$a_n = \frac{a\, 2^{2n}}{3n(n-1)\log 2} \text{ and } c_n = \frac{a(3n-4)}{3n(n-1)\log 2}.$$

Note that f can be extended to a Kneser function of order 2 and that

$$f(2^{-n}) = \frac{a}{n\log 2}, \quad n = 1, 2, \ldots.$$

We define functions f_1 and f_2:

$$\begin{aligned} f_1(x) &= f(2^{-n}), \quad x \in [2^{-n}, 2^{-(n-1)}), \\ f_2(x) &= f(2^{-(n-1)}), \quad x \in (2^{-n}, 2^{-(n-1)}]. \end{aligned}$$

Then $f_1(x) \le f(x) \le f_2(x)$. Since

$$\begin{aligned} \lim_{x\to 0_+} f_1(x)|\log x| &= \lim_{n\to\infty} \frac{a}{n\log 2} n\log 2 = a, \\ \lim_{x\to 0_+} f_2(x)|\log x| &= \lim_{n\to\infty} \frac{a}{(n-1)\log 2} n\log 2 = a \end{aligned}$$

we get (3.15).

The derivative of f fulfills

$$f'(x) = 2a_n x, \quad x \in (2^{-n}, 2^{-(n-1)})$$

and we have

$$\begin{aligned}\liminf_{x\to 0+} f'_+(x)x(\log x)^2 &= \lim_{n\to\infty} f'_+(2^{-n})\,2^{-n}\left(\log 2^{-n}\right)^2\\ &= \lim_{n\to\infty} \frac{2a\,2^{2n}}{3n(n-1)\log 2}\,2^{-n}\,2^{-n}n^2(\log 2)^2\\ &= \frac{2}{3}\,a\log 2\end{aligned}$$

and, similarly,

$$\begin{aligned}\limsup_{x\to 0+} f'_-(x)x(\log x)^2 &= \lim_{n\to\infty} f'_-(2^{-(n-1)})\,2^{-(n-1)}\left(\log 2^{-(n-1)}\right)^2\\ &= \lim_{n\to\infty} \frac{2a\,2^{2n}}{3n(n-1)\log 2}\,2^{-(n-1)}\,2^{-(n-1)}(n-1)^2(\log 2)^2\\ &= \frac{8}{3}\,a\log 2.\end{aligned}$$

This shows that (3.15) holds. □

We have showed that there exists the Kneser function f such that the limit $f(r)r\log^2 r$, $r \to 0$ does not exist. More difficult question is to find the set $A \in \mathbb{R}^2$ such that the distance function of A is the Kneser function, which we introduced in Example 3.15 and it is an open problem.

The last conjecture in this chapter will be related to the "fluctuation" of the surface area of the Wiener sausage. J.-F. Le Gall in [12] claims

$$\begin{aligned}\lim_{r\to 0} \frac{V(r,1) - 2\pi r}{4\pi r^2\left(\log\frac{1}{r}\right)^{1/2}} &= \mathcal{N},\ d = 3,\\ \lim_{r\to 0} \frac{V(r,1) - (d-2)\pi\omega_{d-2}r^{d-2}}{r^{d-1}} &= A\cdot\mathcal{N},\ d \geq 4,\end{aligned}$$

where the previous convergences hold in distribution, $\mathcal{N}$ is standardly normally distributed and

$$A^2 = \lim_{r\to 0} \frac{\operatorname{var} V(r,1)}{r^{2d-2}}.$$

We conjecture that the following limits in distribution hold

$$\begin{aligned}\lim_{r\to 0} \frac{S(r,1) - 2\pi}{8\pi r\left(\log\frac{1}{r}\right)^{1/2}} &= \mathcal{N},\ d = 3,\\ \lim_{r\to 0} \frac{S(r,1) - (d-2)^2\pi\omega_{d-2}r^{d-3}}{r^{d-2}} &= B\cdot\mathcal{N},\ d \geq 4,\end{aligned}$$

where

$$B^2 = \lim_{r\to 0} \frac{\operatorname{var} S(r,1)}{r^{2d-4}}.$$

4. Connected Components of the Complement of Wiener Sausage

4.1 Introduction to the Problem

This chapter is devoted to the study of the asymptotic number of connected components of the complement of a Wiener sausage in the plane. The Hausdorff dimension of the trajectory of a planar Brownian motion is 2 (see Remark 2.5).

We were inspired by the question concerning the mean Euler characteristic of the Wiener sausage in the plane. Almost all values of the distance function of $B[0,1]$ are regular and in [19] there was proved that the mean Euler characteristic of the Wiener sausage is finite. The Wiener sausage is connected, hence the Euler characteristic is equal to 1 minus the number of the connected components of its complement.

Some technics of the proofs in this chapter can be found in articles of T. S. Mountford [16] and J.-F. Le Gall [13]. There were proved asymptotic formulae for $N[u,v)$ − the number of connected components of $\mathbb{R}^2 \setminus B[0,1]$ with area in interval $[u,v)$, $0 < u < v \leq \infty$.

T. S. Mountford [16] found the following limit in probability

$$\lim_{x\to 0} x^2(\log x)^2 N\left[\pi(\lambda x^2), \pi x^2\right) = \frac{1-\lambda^2}{2\lambda^2} \quad \text{for any } \lambda \in (0,1)$$

and J.-F. Le Gall refined this result to the following theorem.

Theorem 4.1. *([13, Theorem 1.1]) For any $\varepsilon > 0$ it holds that*

$$\lim_{u\to 0} \sup_{v\in[(1+\varepsilon,\infty)]} \left| \frac{(\log u)^2 N[u,v)}{\frac{1}{u} - \frac{1}{v}} - 2\pi \right| = 0$$

almost surely. In particular

$$\lim_{u\to 0} u(\log u)^2 N[u,\infty) = 2\pi, \text{ a.s.}$$

We have decided to try to extend these results to the Wiener sausage. How we managed it is written in Section 4.4 − Main Results.

The main subject of our examination in this chapter is undoubtedly the connected component of $\mathbb{R}^2 \setminus S_\gamma[0,1]$. Thus, we have decided to abbreviate words "the connected component" by "c.c.". We will use it especially in the formulae to shorten their length.

Definition 4.2. *Let $B = \{B(t) : t \geq 0\}$ be a planar Brownian motion starting at $x \in \mathbb{R}^2$. For $0 < a < b < \infty$, $\gamma > 0$ and $y \in \mathbb{R}^2$, we call $C_{a,b}(y)$ the connected component of the complement of $B[T_a(y), R_{a,b}(y)]$ that contains y:*

$$C_{a,b}(y) \;=\; \text{c.c. of } \mathbb{R}^2 \setminus B[T_a(y), R_{a,b}(y)] \text{ that contains } y$$

It is not difficult to observe that $y \in \mathbb{R}^2$ lies in exactly one connected component of $\mathbb{R}^2 \setminus B[T_a(y), R_{a,b}(y)]$ almost surely.

Similarly, we call $C_{a,b}(y,\gamma)$ the connected component of the complement of $S_\gamma[T_a(y), R_{a,b}(y)]$ that contains y:

$$C_{a,b}(y,\gamma) = \begin{cases} \text{c.c. of } \mathbb{R}^2 \setminus S_\gamma[T_a(y), R_{a,b}(y)] \text{ that contains } y \\ \qquad \text{for } y \notin S_\gamma[T_a(y), R_{a,b}(y)], \\ \emptyset \quad \text{for } y \in S_\gamma[T_a(y), R_{a,b}(y)]. \end{cases}$$

For the definition of the stopping time T_a (resp. $R_{a,b}$) see Preliminaries (1.2) *(resp.* (1.3)*).*

It is easy to see that for $y \notin S_\gamma[T_a(y), R_{a,b}(y)]$, after using scaling invariance of Brownian motion, we have (for $b > a > 0$)

$$\begin{aligned} |C_{a,b}(y)| &\overset{d}{=} \left| a \cdot \left(\text{c.c. of } \mathbb{R}^2 \setminus 1/a \cdot B[T_a(y), R_{a,b}(y)] \text{ that contains } y/a\right)\right| \\ &\overset{d}{=} a^2 \cdot \left|\text{c.c. of } \mathbb{R}^2 \setminus B[T_a(y)/a^2, R_{a,b}(y)/a^2] \text{ that contains } y/a\right| \\ &\overset{d}{=} a^2 \cdot \left|\text{c.c. of } \mathbb{R}^2 \setminus B[T_1(y/a), R_{1,b/a}(y/a)] \text{ that contains } y/a\right| \\ &= a^2 \cdot |C_{1,b/a}(y/a)|, \end{aligned}$$

because

$$\begin{aligned} \frac{T_a(y)}{a^2} &= \frac{1}{a^2} \inf\{a^2\tau \geq 0 : \, |B(a^2\tau) - y| = a\} \\ &= \inf\{\tau \geq 0 : \, |B(a^2\tau) - y| = a\} \\ &\overset{d}{=} \inf\{\tau \geq 0 : \, |a(B(\tau) - y/a)| = a\} \\ &= T_1(y/a) \end{aligned}$$

and, similarly, $R_{a,b}(y) \overset{d}{=} R_{1,b/a}(y/a)$.

The same scaling arguments we have just showed, can be rewritten with replacing Brownian motion by Wiener sausage. Only the diameter changes in addition. Hence, we obtain (for $b > a > 0$)

$$|C_{a,b}(y,\gamma)| \overset{d}{=} \left|C_{1,b/a}(y/a, \gamma/a)\right| \cdot a^2.$$

4.2 Initial Estimates

In this section, we formulate preliminary statements as a preparation for deeper estimations and claims which we use to prove our main results.

First, we present a well-known result on the asymptotic volume of the Wiener sausage (see e.g.[12])

Proposition 4.3. *([12, p. 991, (1.b)])*

$$\lim_{r\to 0} |\log r| \cdot |S_r[0,1]| = \pi, \text{ a.s.}$$

Due to J.-F. Le Gall the following lemma is known. It bounds the probability of the existence of the unbounded component of the complement of the stopped Brownian motion such that this component contains zero.

Lemma 4.4. *([13, Lemma 2.1]) There exists $\beta > 0$ such that for any $R > 1$, it holds that*

$$\mathbb{P}\Big[|C_{1,R}(0)| = \infty\Big] \le R^{-\beta}.$$

We present an analogue of the lemma where the Brownian motion is replaced by the Wiener sausage.

Lemma 4.5. *There exists $\beta > 0$ such that for any $R > 1$ and $0 < \gamma < 1$ the following inequality holds*

$$\mathbb{P}\Big[|C_{1,R}(0,\gamma)| = \infty\Big] \le R^{-\beta}.$$

Proof. Let $\rho > 1$. Then

$$\begin{aligned} \mathbb{P}\Big[|C_{1,\rho^2}(0,\gamma)| = \infty\Big] &\le \mathbb{P}\Big[|C_{1,\rho}(0,\gamma)| = \infty,\ |C_{\rho,\rho^2}(0,\gamma)| = \infty\Big] \\ &= \mathbb{P}\Big[|C_{1,\rho}(0,\gamma)| = \infty\Big] \cdot \mathbb{P}\Big[|C_{1,\rho}(0,\gamma/\rho)| = \infty\Big] \\ &\le \Big(\mathbb{P}[|C_{1,\rho}(0,\gamma/\rho)| = \infty]\Big)^2. \end{aligned}$$

We can perform the same procedure for arbitrary $n \in \mathbb{N}$, $n \ge 2$, thus we get

$$\mathbb{P}\Big[|C_{1,\rho^n}(0,\gamma)| = \infty\Big] \le \left(\mathbb{P}\left[\left|C_{1,\rho}\left(0, \frac{\gamma}{\rho^{n-1}}\right)\right| = \infty\right]\right)^n =: a^n.$$

It is not hard to see that $0 < a < 1$.

For $\rho^n \le R$ the following inequality holds

$$\mathbb{P}\Big[\ |C_{1,R}(0,\gamma)| = \infty\Big] \le \mathbb{P}\Big[\ |C_{1,\rho^n}(0,\gamma)| = \infty\Big] \le a^n.$$

Hence for $\beta > 0$ such that

$$0 < \beta \le -\frac{\log a}{\log \rho},$$

we have

$$R^{-\beta} \ge R^{\frac{\log a}{\log \rho}} \ge R^{\frac{n \log a}{\log R}} = a^n, \quad \text{for any } R > 1,$$

which completes the proof. □

The main idea of the generalization of results concerning $B[0,1]$ to the statements containing Wiener sausage lies in considering the following fact:

"When B enters into $b(o, r+\gamma)$, then at the same time S_γ enters into $b(o,r)$."

From now on, we write only estimates for the Wiener sausage in the text below. Original technics can be found in [13] and [16].

We set a notation for the probabilities that the area of the connected component $C_{1+\gamma,R}(0,\gamma)$ is appropriately bounded and we look at their asymptotic behavior.

Definition 4.6. *For $0 < r+\gamma < 1$ and $R > 1$, we define*

$$\Gamma(r,R,\gamma) = \mathbb{P}\left[|C_{1+\gamma,R}(0,\gamma)| \leq \pi r^2\right].$$

Let $r+\gamma < 1$. For $|x| = 1+\gamma$, we have

$$\begin{aligned}\Gamma(r,R,\gamma) &\leq \mathbb{P}_x\left[\inf_{s\in(0,T_R)} |B(s)| < r+\gamma\right] \\ &= 1 - \frac{\log\left(\frac{1+\gamma}{r+\gamma}\right)}{\log\left(\frac{R}{r+\gamma}\right)} \\ &\leq 1 + \frac{\log\left(\frac{r+\gamma}{1+\gamma}\right)}{\log R} + \frac{\left(\log\left(\frac{r+\gamma}{1+\gamma}\right)\right)^2}{1-\frac{\log(r+\gamma)}{\log R}}\end{aligned}$$

where we used Lemma 1.10.

We make the additional assumption

$$\left|\log\left(\frac{r+\gamma}{1+\gamma}\right)\right| < (\log R)^\alpha \quad \text{for some } 0 < \alpha < 1 \tag{4.1}$$

for receiving

$$\Gamma(r,R,\gamma) \leq 1 + \frac{\log\left(\frac{r+\gamma}{1+\gamma}\right)}{\log R} + O\left((\log R)^\alpha\right) \tag{4.2}$$

with $R \to \infty$.

To compute the corresponding lower bound of $\Gamma(r,R,\gamma)$, we condition it by the event

$$\{T_{(r+\gamma)\cdot\eta} < T_R\} \tag{4.3}$$

where $0 < \eta < 1$. Hence we get

$$\Gamma(r,R,\gamma) \geq \mathbb{P}\left[|C_{1+\gamma,R}(0,\gamma)| \leq \pi r^2 | T_{(r+\gamma)\cdot\eta} < T_R\right] \cdot \mathbb{P}\left[T_{(r+\gamma)\cdot\eta} < T_R\right]. \tag{4.4}$$

Let us consider the event that Brownian motion starting at x, $|x| = 1$ performs a closed loop around $b(o,\gamma)$ between times $T_{(r+\gamma)\eta}$ and $R_{(r+\gamma)\eta,r+\gamma}$. Conditioning the previous event by (4.3) we obtain the lower bound of the first factor in (4.4). We expand it a little more here

$$\begin{aligned}
\mathbb{P}\left[|C_{1+\gamma,R}(0,\gamma)| \le \pi r^2 | T_{(r+\gamma)\cdot\eta} < T_R\right] &\ge \mathbb{P}\left[|C_{(r+\gamma)\cdot\eta,r+\gamma}(0,\gamma)| \le \pi r^2\right] \\
&\ge \mathbb{P}\left[|C_{(r+\gamma)\cdot\eta,r+\gamma}(0,\gamma)| < \infty\right] \\
&= \mathbb{P}\left[\left|C_{1,1/\eta}\left(0, \frac{\gamma}{\eta(r+\gamma)}\right)\right| < \infty\right] \\
&\ge 1-\eta^\beta.
\end{aligned}$$

We have used Lemma 4.5 in the last inequality.

Thus we have (for $|x| = 1+\gamma$)

$$\begin{aligned}
\Gamma(r,R,\gamma) &\ge (1-\eta^\beta)\cdot \mathbb{P}\left[T_{(r+\gamma)\eta} < T_R\right] \\
&= (1-\eta^\beta)\cdot \mathbb{P}_x\left[\inf_{s\in(0,T_R)} |B(s)| < (r+\gamma)\eta\right] \\
&= (1-\eta^\beta)\cdot \left(1 + \frac{\log\left(\frac{r+\gamma}{1+\gamma}\right) + \log\eta}{\log R + |\log(r+\gamma)| - \log\eta}\right).
\end{aligned}$$

We used Lemma 1.10 again and, choosing

$$\eta^\beta = (\log R)^{-M},\ M > 2$$

(i.e. $\log\eta = -M/\beta \log\log R$), we get for $R\to\infty$

$$\Gamma(r,R,\gamma) \ge \Big(1 + O\left((\log R)^\alpha\right)\Big)\cdot\left(1 + \frac{\log\left(\frac{r+\gamma}{1+\gamma}\right) + O(\log\log R)}{\log R + |\log(r+\gamma)| + O(\log\log R)}\right).$$

It leads, with the assumption (4.1), to

$$\Gamma(r,R,\gamma) \ge 1 + \frac{\log\left(\frac{r+\gamma}{1+\gamma}\right)}{\log R} + O\left((\log R)^{2\alpha-2} + \frac{\log\log R}{\log R}\right) \tag{4.5}$$

for $R\to\infty$.

In the last lemma of this section, we present a claim which deals with the probability of the connected component $C_{1+\gamma,R}(0)$ whose area lies in $[\pi(\lambda r)^2, \pi r^2]$ for some $0<\lambda<1$.

Notation 4.7. *Let $\lambda\in(0,1)$. We set*

$$Q(r,R,\gamma) = \mathbb{P}\left[\pi(\lambda r)^2 \le |C_{1+\gamma,R}(0,\gamma)| \le \pi r^2\right].$$

Lemma 4.8. *There exists $K > 0$ such that for $R \to \infty$*

$$Q(r, R, \gamma) = \frac{|\log \lambda|}{\log R} + O\left(\frac{\log \log R}{(\log R)^{3/2}}\right),$$

uniformly for

$$K \log \log R \leq |\log r| \leq (\log R)^{1/2}, \;\; 0 < \gamma < r^a, \quad \textit{for some} \;\; a > 0.$$

Proof. We assume

$$K \log \log R \leq |\log r| \leq (\log R)^{\alpha}, \;\; 0 < \gamma < r^a, \tag{4.6}$$

for some $a > 0$ and for some fixed $0 < \alpha < 1$.

In virtue of (4.6) we have

$$\lim_{r \to 0} \frac{\log\left(\frac{r+\gamma}{1+\gamma}\right)}{\log r} = 1,$$

hence the asymptotic inequalities (4.2) and (4.5) can be written as

$$\Gamma(r, R, \gamma) \leq 1 + \frac{\log r}{\log R} + O\left((\log R)^{2\alpha - 2}\right) \tag{4.7}$$

and

$$\Gamma(r, R, \gamma) \geq 1 + \frac{\log r}{\log R} + O\left((\log R)^{2\alpha - 2} + \frac{\log \log R}{\log R}\right). \tag{4.8}$$

Let $N \in \mathbb{N}$, we use (4.7) and (4.8) to obtain

$$\begin{aligned} \sum_{k=0}^{N-1} Q(\lambda^k r, R, \lambda^k \gamma) &= \Gamma(r, R, \gamma) - \Gamma(\lambda^N r, R, \lambda^N \gamma) \\ &= \frac{N|\log \lambda|}{\log R} + O\left((\log R)^{2\alpha - 2} + \frac{\log \log R}{\log R}\right). \end{aligned} \tag{4.9}$$

For the purpose of the proof, we set the following notation

$$\begin{aligned} A_{r,\gamma} &= \left\{\pi(\lambda r)^2 \leq |C_{1+\gamma, R}(0, \gamma)| \leq \pi r^2\right\}, \\ B_{r,\gamma} &= \left\{\pi(\lambda^2 r)^2 \leq |C_{1+\lambda\gamma, \lambda R}(0, \lambda\gamma)| \leq \pi(\lambda r)^2\right\}, \end{aligned}$$

therefore

$$\begin{aligned} \mathbb{P}[A_{r,\gamma}] &= Q(r, R, \gamma), \\ \mathbb{P}[B_{r,\gamma}] &= Q(\lambda r, \lambda R, \lambda\gamma). \end{aligned}$$

We apply previous estimates of Γ and Lemma 4.5 to receive

$$\mathbb{P}[A_{r,\gamma}] = \mathbb{P}[A_{r,\gamma} \cap \{C_{1+\gamma, R}(0, \gamma) \cap b(o, 1)^c = \emptyset\}] + O\left((\log R)^{-K_1}\right) \tag{4.10}$$

for $K_1 > 0$ large enough. In the following text it is sufficient to assume that $K_1 > 5$.

Equation (4.10) holds because the event

$$\Big\{A_{r,\gamma} \cap \{C_{1+\gamma,R}(0,\gamma) \cap b(o,1)^c \neq \emptyset\}\Big\}$$

is included in $\{T_{r+\gamma} < T_R\}$, hence the component $\mathbb{R}^2 \setminus S_\gamma[T_{r+\gamma}, R_{r+\gamma,1+\gamma}]$ which contains zero has a finite area. Thus Lemma 4.5 implies that

$$\mathbb{P}\Big[A_{r,\gamma} \cap \{C_{1+\gamma,R}(0,\gamma) \cap b(o,1)^c \neq \emptyset\}\Big] \leq O\left(\left(\frac{r+\gamma}{1+\gamma}\right)^\beta\right).$$

The assumption $0 < \gamma < r^a$, for some $a > 1$, gives that

$$O\left(\left(\frac{r+\gamma}{1+\gamma}\right)^\beta\right) \leq O\left(r^\beta\right)$$

and after using the left inequality in (4.6), we have $r^\beta < (\log R)^{-K\beta}$. We can assume that $K_1 = K\beta$ is sufficiently large.

As well as in (4.10), we can observe that

$$\mathbb{P}[B_{r,\gamma}] = \mathbb{P}\Big[B_{r,\gamma} \cap \{C_{1+\lambda\gamma,\lambda R}(0,\lambda\gamma) \cap b(o,\lambda)^c = \emptyset\}\Big] + O\left((\log R)^{-K_1}\right). \quad (4.11)$$

Now, we will deal with the first summand of the right side of (4.11). It is equal to

$$\begin{aligned}
& \mathbb{P}[T_{\lambda(1+\gamma)} < T_R] \cdot \mathbb{P}[B_{r,\gamma} \cap \{C_{1+\lambda\gamma,\lambda R}(0,\lambda\gamma) \cap b(o,\lambda)^c = \emptyset\} | \, T_{\lambda(1+\gamma)} < T_R] \\
= \; & \mathbb{P}[T_{\lambda(1+\gamma)} < T_R] \cdot \mathbb{P}[B_{r,\gamma} \cap \{C_{\lambda(1+\gamma),\lambda R}(0,\lambda\gamma) \cap b(o,\lambda)^c = \emptyset\}] \\
= \; & \mathbb{P}[T_{\lambda(1+\gamma)} < T_R] \cdot \mathbb{P}[\pi(\lambda^2 r)^2 \leq |C_{\lambda(1+\gamma),\lambda R}(0,\lambda\gamma)| \leq \pi(\lambda r)^2; \\
& \qquad\qquad C_{\lambda(1+\gamma),\lambda R}(0,\lambda\gamma) \cap b(o,\lambda)^c = \emptyset] \\
= \; & \mathbb{P}[T_{\lambda(1+\gamma)} < T_R] \cdot \mathbb{P}[\pi(\lambda r)^2 \leq |C_{1+\gamma,R}(0,\gamma)| \leq \pi r^2; \, C_{1+\gamma,R}(0,\gamma) \cap b(o,1)^c = \emptyset] \\
= \; & \mathbb{P}[T_{\lambda(1+\gamma)} < T_R] \cdot \mathbb{P}[A_{r,\gamma} \cap \{C_{1+\gamma,R}(0,\gamma) \cap b(o,1)^c = \emptyset\}] \qquad (4.12)
\end{aligned}$$

where we use the Markov property at $T_{\lambda(1+\gamma)}$ and the fact that

$$C_{1+\lambda\gamma,\lambda R}(0,\lambda\gamma) \overset{d}{=} C_{\lambda(1+\gamma),\lambda R}(0,\lambda\gamma)$$

if we are on the set $\{C_{1+\lambda\gamma,\lambda R}(0,\lambda\gamma) \subseteq b(o,\lambda)\}$.

Let us look at probability how much $A_{\lambda r,\lambda\gamma}$ differs from $B_{r,\gamma}$. By analogous technics as we performed above (using Lemma 4.5), we can choose $K_2 > 0$ such that $C_{1+\lambda\gamma,\lambda R}(0,\lambda\gamma)$ is not contained in $b(o,(\log R)^{K_2})$ with probability smaller than $O((\log R)^{-K_1})$. For simplicity of notation, we write $\xi := (\log R)^{K_2}$. Hence we have

$$\mathbb{P}\Big[A_{\lambda r,\lambda\gamma} \setminus B_{r,\gamma}\Big] = O\left((\log R)^{-K_1}\right) + \mathbb{P}\Big[A_{\lambda r,\lambda\gamma} \setminus B_{r,\gamma}; \, C_{1+\lambda\gamma,\lambda R}(0,\lambda\gamma) \subseteq b(o,\xi)\Big]$$

$$\leq O\left((\log R)^{-K_1}\right) + \mathbb{P}\left[\pi(\lambda^2 r)^2 \leq |C_{1+\lambda\gamma,R}(0,\lambda\gamma)|;\ \inf_{s\in[T_{\lambda R},T_R]} |B(s)| \leq \xi\right],$$

then we use twice the Markov property, first at stopping time

$$R_\lambda = \inf\{t \geq T_{\lambda R} :\ |B(t)| = \xi\}$$

and then at $T_{\lambda R}$ to obtain

$$\begin{aligned}
\mathbb{P}\left[A_{\lambda r,\lambda\gamma} \setminus B_{r,\gamma}\right] &\leq O\left((\log R)^{-K_1}\right) + \mathbb{P}\left[\pi(\lambda^2 r)^2 \leq |C_{1+\lambda\gamma,\lambda R}(0,\lambda\gamma)|\right] \\
&\quad \cdot \mathbb{P}\left[\inf_{s\in[T_{\lambda R},T_R]} |B(s)| \leq \xi\right] \cdot \mathbb{P}\left[R_\lambda < T_R\right] \\
&\leq O\left((\log R)^{-K_1}\right) + \left(1 - \Gamma(\lambda^2 r, \lambda R, \lambda\gamma)\right) \cdot O\left(\frac{1}{(\log R)^2}\right).
\end{aligned}$$

We use there Lemma 1.10, i.e.

$$\mathbb{P}\left[\inf_{s\in[T_{\lambda R},T_R]} |B(s)| \leq \xi\right] = \frac{|\log\lambda|}{\log R - K_2 \log\log R} = O\left(\frac{1}{\log R}\right)$$

and the same holds for

$$\mathbb{P}\left[R_\lambda < T_R\right] = \frac{|\log\lambda|}{\log R - K_2 \log\log R} = O\left(\frac{1}{\log R}\right).$$

We can use (4.8) to estimate

$$1 - \Gamma(\lambda^2 r, \lambda R, \lambda\gamma) \leq -\frac{\lambda^2 r}{\log R} + O\left((\log R)^{2\alpha-2} + \frac{\log\log R}{\log R}\right)$$

and in view of $|\log r| \leq (\log R)^\alpha$ from the assumption (4.6), we have

$$1 - \Gamma(\lambda^2 r, \lambda R, \lambda\gamma) \leq O\Big((\log R)^{\alpha-1}\Big).$$

Therefore, we get

$$\mathbb{P}\left[A_{\lambda r,\lambda\gamma} \setminus B_{r,\gamma}\right] \leq O\left((\log R)^{\alpha-3}\right). \tag{4.13}$$

We can bound the probability of $B_{r,\gamma} \setminus A_{\lambda r,\lambda\gamma}$ outside a set of probability $O\left((\log R)^{-K_1}\right)$ using the same technics as above. Hence

$$\begin{aligned}
\mathbb{P}\left[B_{r,\gamma} \setminus A_{\lambda r,\lambda\gamma}\right] &\leq O\left((\log R)^{-K_1}\right) + \mathbb{P}\left[B_{r,\gamma}\right] \cdot \mathbb{P}\left[\inf_{s\in[T_{\lambda R},T_R]} |B(s)| \leq \xi\right] \\
&\leq O\left((\log R)^{-K_1}\right) + \mathbb{P}\left[B_{r,\gamma}\right] \cdot O\left(\frac{1}{\log R}\right) \qquad (4.14) \\
&\leq O\left((\log R)^{-K_1}\right) + \left(1 - \Gamma(\lambda^2 r, \lambda R, \lambda\gamma)\right) \cdot O\left(\frac{1}{\log R}\right) \\
&\leq O\left((\log R)^{\alpha-2}\right). \qquad (4.15)
\end{aligned}$$

Consequently, combining formulae (4.13) and (4.15), we obtain

$$\begin{aligned}\mathbb{P}\left[B_{r,\gamma}\right] &= \mathbb{P}\left[A_{\lambda r,\lambda\gamma}\right]+\mathbb{P}\left[B_{r,\gamma}\setminus A_{\lambda r,\lambda\gamma}\right]-\mathbb{P}\left[A_{\lambda r,\lambda\gamma}\setminus B_{r,\gamma}\right]\\ &= \mathbb{P}\left[A_{\lambda r,\lambda\gamma}\right]+O\left((\log R)^{\alpha-2}\right). \qquad (4.16)\end{aligned}$$

Now, we join formulae (4.10), (4.11), (4.12) and (4.16) to

$$\begin{aligned}\mathbb{P}\left[B_{r,\gamma}\right] &= \mathbb{P}\left[T_{\lambda(1+\gamma)}<T_R\right]\cdot\mathbb{P}\left[A_{r,\gamma}\right]+O\left((\log R)^{\alpha-2}\right)\\ &= \left(1+\frac{\log\lambda}{\log R-\log(\lambda(1+\gamma))}\right)\cdot\mathbb{P}\left[A_{r,\gamma}\right]+O\left((\log R)^{\alpha-2}\right)\\ &= \left(1+\frac{\log\lambda}{\log R}+O\left((\log R)^{-2}\right)\right)\cdot\mathbb{P}\left[A_{r,\gamma}\right]+O\left((\log R)^{\alpha-2}\right) \qquad (4.17)\end{aligned}$$

where we use Lemma 1.10.

Using the notation of Q, (4.17) can be written in the following way

$$Q(\lambda r,R,\lambda\gamma)=Q(r,R,\gamma)\left(1-\frac{|\log\lambda|}{\log R}\right)+O\left((\log R)^{\alpha-2}\right)$$

and this formula can be iterated

$$Q(\lambda^k r,R,\lambda^k\gamma)=Q(r,R,\gamma)\left(1-\frac{|\log\lambda|}{\log R}\right)^k+O\left(k(\log R)^{\alpha-2}\right),\quad \text{for } k\in\mathbb{N}.$$

After summing this equation over k, provided that $N|\log\lambda|<(\log R)^{1-\delta_1}$, $\delta_1>0$ we obtain

$$\begin{aligned}\sum_{k=0}^{N-1}Q(\lambda^k r,R,\lambda^k\gamma) &= Q(r,R,\gamma)\left(\frac{\log R}{|\log\lambda|}\right)\cdot\left(1-\left(1-\frac{|\log\lambda|}{\log R}\right)^N\right)\\ &\quad +O\left(N^2(\log R)^{\alpha-2}\right)\\ &= Q(r,R,\gamma)\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\frac{|\log\lambda|^{k-1}}{(\log R)^{k-1}}\\ &\quad +O\left(N^2(\log R)^{\alpha-2}\right).\end{aligned}$$

Due to

$$O\left(\binom{N}{k}\frac{|\log\lambda|^k}{(\log R)^k}\right)<O\left(\binom{N}{k-1}\frac{|\log\lambda|^{k-1}}{(\log R)^{k-1}}\right),\ k=2,\dots,N,$$

the sum can be written as

$$\sum_{k=0}^{N-1}Q(\lambda^k r,R,\lambda^k\gamma)=Q(r,R,\gamma)N\left(1+O\left(N(\log R)^{-1}\right)\right)+O\left(N^2(\log R)^{\alpha-2}\right). \qquad (4.18)$$

Now, if we join (4.9) with (4.18), we obtain the following asymptotic formula for Q

$$\left(1+O\left(\frac{N}{\log R}\right)\right) Q(r,R,\gamma) = \frac{|\log\lambda|}{\log R} + O\left(N(\log R)^{\alpha-2} + \frac{1}{N}(\log R)^{2\alpha-2} + \frac{\log\log R}{N\log R}\right). \quad (4.19)$$

There is an unpleasant expression $N(\log R)^{\alpha-2}$ in (4.19). Fortunately, if we choose $N = \lceil(\log R)^{\delta_2}\rceil$ for some small $\delta_2 > 0$, we get that for any $\delta_3 \in (0,\delta_2)$ the asymptotic formula holds

$$Q(r,R,\gamma) = \frac{|\log\lambda|}{\log R} + O\left((\log R)^{-(\delta_3+1)}\right) \quad (4.20)$$

uniformly for $K\log\log R \leq |\log r| \leq (\log R)^{\alpha}$, $0 < \alpha \leq 1/2$, $0 < \gamma < r^a$, $a > 1$, whenever $R \to \infty$.

For proving (4.20), we had to consider the following cases:

1. $Q(r,R,\gamma)\cdot O\left((\log R)^{-1+\delta_2}\right) < O\left((\log R)^{-(\delta_3+1)}\right)$,
2. $O\left((\log R)^{\alpha-2+\delta_2}\right) < O\left((\log R)^{-(\delta_3+1)}\right)$,
3. $O\left((\log R)^{2\alpha-2-\delta_2}\right) < O\left((\log R)^{-(\delta_3+1)}\right)$,
4. $O\left(\log\log R\,(\log R)^{-1-\delta_2}\right) < O\left((\log R)^{-(\delta_3+1)}\right)$.

The first condition leads, after using previous asymptotic of Γ, to the inequality $\delta_3 < 2 - 2\alpha - \delta_2$. The second one is equivalent to $\delta_3 < 1 - \alpha - \delta_2$, the third one gives the assumption $\delta_3 < 1 - 2\alpha + \delta_2$ and in the fourth condition we have to assume $\delta_3 < \delta_2$. Thus we have to choose $\delta_2 > 0$ such that $\delta_3 \in (0,\delta_2)$ with the restriction $\alpha \leq 1/2$.

Now, we can substitute (4.20) into (4.16), thus we have

$$\mathbb{P}\left[B_{r,\gamma}\right] = \frac{|\log\lambda|}{\log R} + O\left((\log R)^{-1-\delta_3}\right) \quad (4.21)$$

and after applying (4.21) in (4.14), we receive a better estimate of the probability of $B_{r,\gamma} \setminus A_{\lambda r,\lambda\gamma}$:

$$\begin{aligned}\mathbb{P}\left[B_{r,\gamma} \setminus A_{\lambda r,\lambda\gamma}\right] &\leq O\left((\log R)^{-K_1}\right) + \mathbb{P}\left[B_{r,\gamma}\right]\cdot(\log R)^{-1} \\ &\leq \frac{|\log\lambda|}{\log R} + O\left((\log R)^{-2}\right)\end{aligned}$$

and thus

$$\mathbb{P}\left[B_{r,\gamma}\right] = \mathbb{P}\left[A_{\lambda r,\lambda\gamma}\right] + O\left((\log R)^{-2}\right).$$

Step by step as above, we obtain instead of (4.19) the following formula

$$\left(1 + O\left(\frac{M}{\log R}\right)\right) Q(r, R, \gamma) = \frac{|\log \lambda|}{\log R} + O\left(M(\log R)^{-2} + \frac{1}{M}(\log R)^{2\alpha-2} + \frac{\log\log R}{M \log R}\right)$$

where $M \in \mathbb{N}$.

Finally, the proof of Lemma 4.8 ends by setting

$$\alpha = 1/2 \quad \text{and} \quad M = \left\lceil K_3(\log R)^{1/2} \right\rceil \quad \text{for some} \quad K_3 > 0.$$

□

4.3 Preliminary Statements

Now, let us focus on connected components of the complement of a Wiener sausage in general view. We will use the estimates proved in the previous section as a preparation to the main results. We will consider B starting at zero again in this part.

Definition 4.9.

$$C(y, \gamma) = \begin{cases} \text{c.c. of } \mathbb{R}^2 \setminus S_\gamma[0,1] \text{ which contains } y & \text{for } y \notin S_\gamma[0,1], \\ \emptyset \quad \text{for } y \in S_\gamma[0,1]. \end{cases}$$

Given $\varepsilon > 0$ and $\lambda \in (0,1)$, we set

$$U(\varepsilon, \gamma) = \left\{ y \in \mathbb{R}^2 \setminus S_\gamma[0,1];\ \pi(\lambda\varepsilon)^2 \le |C(y,\gamma)| \le \pi\varepsilon^2 \right\}.$$

The random set $U(\cdot,\cdot)$, especially its mean and variation, plays an essential role in calculation of the asymptotic number of connected components of the complement of a Wiener sausage.

Theorem 4.10. *Let $0 < \gamma < \varepsilon^a$, $a > 1$. Then for ε tending to zero the following holds:*

$$\mathbb{E}\,|U(\varepsilon,\gamma)| = \frac{\pi|\log\lambda|}{|\log\varepsilon|^2} + O\left(\frac{\log|\log\varepsilon|}{|\log\varepsilon|^{5/2}}\right).$$

Proof. First, we define $\delta = \delta(\varepsilon) > \varepsilon$ by the formula

$$\delta \exp\left(-|\log\delta|^{1/4}\right) = \varepsilon. \tag{4.22}$$

It is difficult now to see why we have chosen δ in this way. This choice should be clarified by the equation

$$\left|\log\frac{\varepsilon}{\delta}\right| = |\log\delta|^{1/4}.$$

For abbreviation we denote

$$R_\nu(y) = \inf\{s > T_{\delta+\gamma}(y) : |B_s - y| > \nu\},$$

where $\nu = \nu(\delta) > \delta$ will be specified later.

Let $|y| \geq \delta+\gamma$. In the proof, we denote $\widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)$ the connected component of $\mathbb{R}^2 \setminus S_\gamma[T_{\delta+\gamma}(y), R_\nu(y)]$ that contains y.

Noticing that

$$\{y \in U(\varepsilon,\gamma)\} \subset \{T_{\delta+\gamma}(y) \leq 1\},$$

the key idea of the proof is the estimate of the following difference

$$\left|\mathbb{P}\left[y \in U(\varepsilon,\gamma)\right] - \mathbb{P}\left[T_{\delta+\gamma}(y) \leq 1;\ \pi(\lambda\varepsilon)^2 \leq |\widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)| \leq \pi\varepsilon^2\right]\right| \tag{4.23}$$

$$\leq \mathbb{P}\left[T_{\delta+\gamma}(y) \leq 1;\ \pi(\lambda\varepsilon)^2 \leq |\widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)| \leq \pi\varepsilon^2;\ C(y,\gamma) \neq \widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)\right] \tag{4.24}$$

$$+\ \mathbb{P}\left[T_{\delta+\gamma}(y) \leq 1;\ \pi(\lambda\varepsilon)^2 \leq |C(y,\gamma)| \leq \pi\varepsilon^2;\ C(y,\gamma) \neq \widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)\right]. \tag{4.25}$$

Firstly, we evaluate (using the Markov property at stopping time $T_{\delta+\gamma}(y)$) the subtrahend in (4.23)

$$\mathbb{P}\left[T_{\delta+\gamma}(y) \leq 1;\ \pi(\lambda\varepsilon)^2 \leq |\widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)| \leq \pi\varepsilon^2\right] = \mathbb{P}\left[T_{\delta+\gamma}(y) \leq 1\right] \cdot Q\left(\frac{\varepsilon}{\delta}, \frac{\nu}{\delta}, \frac{\gamma}{\delta}\right). \tag{4.26}$$

To be able to use Lemma 4.8, we have to assume that

$$0 < \frac{\gamma}{\delta} < \left(\frac{\varepsilon}{\delta}\right)^a,\ a > 1$$

and

$$K \log\log\left(\frac{\nu}{\delta}\right) \leq \left|\log\left(\frac{\varepsilon}{\delta}\right)\right| \leq \left(\log\left(\frac{\nu}{\delta}\right)\right)^{1/2}.$$

The first condition holds trivially due to the assumption of the theorem. The second one is fulfilled for any $\delta \in (0,\delta_0)$ for some $\delta_0 > 0$ and setting (e.g.) $\nu = \frac{1}{|\log\delta|^4}$. Hence (4.26) turns, using Lemma 4.8, to

$$\mathbb{P}\left[T_{\delta+\gamma}(y) \leq 1\right] \cdot \left(\frac{|\log\lambda|}{|\log\delta|} + O\left(\frac{\log|\log\delta|}{|\log\delta|^{3/2}}\right)\right)$$

for $\delta \to 0$.

Our next step is to bound (4.24) by the following sum

$$\mathbb{P}\left[T_{\delta+\gamma}(y) \leq 1 \leq R_\nu(y)\right] \tag{4.27}$$

$$+\ \mathbb{P}\left[T_{\delta+\gamma}(y) \leq R_\nu(y) \leq 1; \pi(\lambda\varepsilon)^2 \leq |\widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)| \leq \pi\varepsilon^2;\right.$$

$$\left.(B[0,T_{\delta+\gamma}(y)] \cup B[R_\nu(y),1]) \cap C_{\delta+\gamma,\nu}(y,\gamma) \neq \emptyset\right] \tag{4.28}$$

The probability in (4.27) is not difficult to estimate using Lemma 1.11:

$$\begin{aligned}
\mathbb{P}\left[T_{\delta+\gamma}(y) \le 1 \le R_\nu(y)\right] &\le \mathbb{P}\left[|B(1) - y| \le \nu\right] = \mathbb{P}_y\left[|B(1)| \le \nu\right] \\
&= \int_0^\nu z\, \mathrm{e}^{-\frac{y^2+z^2}{2}}\, I_0(yz)\, \mathrm{d}z \\
&\le \frac{1}{2}\int_0^\nu z\, \mathrm{e}^{-\frac{y^2+z^2}{2}}\, \mathrm{d}z \\
&= \mathrm{e}^{-\frac{y^2}{2}}\left(1 - \mathrm{e}^{-\frac{\nu^2}{2}}\right) \le \mathrm{e}^{-\frac{y^2}{2}} \cdot \nu^2 \\
&= O\left(|\log\delta|^{-8}\right). \qquad (4.29)
\end{aligned}$$

Let us look at (4.28). It is bounded by

$$\begin{aligned}
&\mathbb{P}\left[T_{\delta+\gamma}(y) \le 1;\ \pi(\lambda\varepsilon)^2 \le |\widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)| \le \pi\varepsilon^2;\ \widetilde{C}_{\delta+\gamma,\nu}(y,\gamma) \subsetneq b(y,\delta+\gamma)\right] \\
&\quad + \mathbb{P}\Big[T_{\delta+\gamma}(y) \le 1;\ B[R_\nu(y), R_\nu(y)+1] \cap b(y,\delta+\gamma) \ne \emptyset; \\
&\qquad\qquad \pi(\lambda\varepsilon)^2 \le |\widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)| \le \pi\varepsilon^2\Big]. \qquad (4.30)
\end{aligned}$$

We can use Lemma 4.5, as we had already done in the preliminary lemmas, to get

$$\begin{aligned}
&\mathbb{P}\left[\pi(\lambda\varepsilon)^2 \le |\widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)| \le \pi\varepsilon^2;\ \widetilde{C}_{\delta+\gamma,\nu}(y,\gamma) \subsetneq b(y,\delta+\gamma)\right] \\
&\le \mathbb{P}\left[|\widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)| = \infty\right] \le \mathbb{P}\left[|C_{1+\gamma/\delta,\nu/\delta}(y/\delta,\gamma/\delta))| = \infty\right] \\
&\le \left(\frac{\nu}{\delta}\right)^{-\beta} = \left(\delta|\log\delta|^4\right)^\beta \qquad (4.31)
\end{aligned}$$

and $(\delta|\log\delta|^4)^\beta \le O(|\log\delta|^{-K_1})$ for any $K_1 > 0$, $\delta \to 0$.

We can apply the Markov property at $R_\nu(y)$ and then at $T_{\delta+\gamma}(y)$ in the second summand in (4.30). Thus we have, due to (4.31) and (4.29), the following estimate of (4.24):

$$\begin{aligned}
&\mathbb{P}\left[T_{\delta+\gamma}(y) \le 1\right]\Big(O\left(|\log\delta|^{-K_1}\right) + Q\left(\frac{\varepsilon}{\delta}, \frac{\nu}{\delta}, \frac{\gamma}{\delta}\right) \cdot \\
&\qquad \cdot \mathbb{P}[B[R_\nu(y), R_\nu(y)+1] \cap b(y,\delta+\gamma) \ne \emptyset]\Big) \qquad (4.32)
\end{aligned}$$

The probability of the event

$$\left\{B[R_\nu(y), R_\nu(y)+1] \cap b(y,\delta+\gamma) \ne \emptyset\right\}$$

is equivalent to the probability that the Brownian motion starting at x, $|x| = \nu$, visits $b(y,\delta+\gamma)$ before time 1. It can be asymptotically estimated according to Lemma 1.7 (using (1.4)) by

$$O\left(\frac{\log|\log\delta|}{|\log\delta|}\right).$$

Thus (4.24) can be bounded by

$$\begin{aligned} &\mathbb{P}\left[T_{\delta+\gamma}(y) \le 1\right] \left(O\left(|\log\delta|^{-K_1}\right) + Q\left(\frac{\varepsilon}{\delta}, \frac{\nu}{\delta}, \frac{\gamma}{\delta}\right) \cdot O\left(\frac{\log|\log\delta|}{|\log\delta|}\right)\right) \\ \le\ &\mathbb{P}\left[T_{\delta+\gamma}(y) \le 1\right] \left(\frac{|\log\lambda|}{\log R} + O\left(\frac{\log|\log\delta|}{|\log\delta|^{3/2}}\right)\right) O\left(\frac{\log|\log\delta|}{|\log\delta|}\right) \\ =\ &\mathbb{P}[T_{\delta+\gamma}(y) \le 1] \cdot O\left(\frac{\log|\log\delta|}{|\log\delta|^2}\right), \end{aligned}$$

and finally this is bounded due to Lemma 1.7 – (1.5) by

$$G(y) \cdot O\left(\frac{\log|\log\delta|}{|\log\delta|^3}\right), \tag{4.33}$$

where $G(\cdot)$ is an integrable function.

The next step is concerned to find a bound for (4.25). It is easy to see that

$$\begin{aligned} &\mathbb{P}\left[T_{\delta+\gamma}(y) \le 1;\ \pi(\lambda\varepsilon)^2 \le |C(y,\gamma)| \le \pi\varepsilon^2;\ C(y,\gamma) \ne \widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)\right] \\ \le\ &\mathbb{P}[T_{\delta+\gamma}(y) \le 1 \le R_\nu] \qquad (4.34) \\ &+ \mathbb{P}\left[T_{\delta+\gamma}(y) \le 1;\ B[T_{\delta+\gamma}(y), R_\nu(y)] \cap b(y, \delta+\gamma) \ne \emptyset;\right. \\ &\qquad \left.\pi(\lambda\varepsilon)^2 \le |\widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)|;\ C(y,\gamma) \ne \widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)\right] \qquad (4.35) \\ &+ \mathbb{P}\left[T_{\delta+\gamma}(y) \le 1;\ \pi(\lambda\varepsilon)^2 \le |\widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)|;\ B[R_\nu(y), R_\nu(y)+1] \cap b(y,\delta+\gamma) \ne \emptyset\right]. \\ &\qquad (4.36) \end{aligned}$$

An upper bound for (4.34) has already been found in (4.29). The third summand, i.e. (4.36), can be bounded by

$$\begin{aligned} &\mathbb{P}\left[T_{\delta+\gamma}(y) \le 1\right] \left(1 - \Gamma\left(\frac{\lambda\varepsilon}{\delta}, \frac{\nu}{\delta}, \frac{\gamma}{\delta}\right)\right) \cdot O\left(\frac{\log|\log\delta|}{|\log\delta|}\right) \\ =\ &\mathbb{P}[T_{\delta+\gamma}(y) \le 1] \left(\frac{\left|\log\frac{\lambda\varepsilon}{\delta}\right|}{\left|\log\frac{\nu}{\delta}\right|} + O\left(\frac{\log|\log\delta|}{|\log\delta|}\right)\right) \cdot O\left(\frac{\log|\log\delta|}{|\log\delta|}\right). \end{aligned} \tag{4.37}$$

Due to the choice of δ in (4.22), $|\log(\gamma/\delta)|$ is asymptotically equivalent to $O(|\log\delta|^{1/4})$ and the choice of ν gives us

$$\left|\log\left(\frac{\nu}{\delta}\right)\right| = O\Big(|\log\delta|\Big).$$

Thus the formula (4.37) is smaller than

$$\mathbb{P}\left[T_{\delta+\gamma}(y) \le 1\right] \cdot O\left(\frac{\log|\log\delta|}{|\log\delta|^{7/4}}\right) \le G(y) \cdot O\left(\frac{\log|\log\delta|}{|\log\delta|^{11/4}}\right).$$

Therefore, it remains only to estimate (4.35). If we restrict ourselves to the set

$$\left\{\widetilde{C}_{\delta+\gamma,\nu}(y,\gamma) \subseteq b(y,\delta+\gamma)\right\},$$

then $\{B[T_{\delta+\gamma}(y), R_\nu(y)] \cap b(y,\delta+\gamma) \neq \emptyset\}$ holds except on a set with probability $O\left(|\log\delta|^{-K_2}\right)$ for $K_2 > 0$ sufficiently large. It runs similarly as in (4.31) or like similar restrictions which we performed in the proof of Lemma 4.8.

Hence, we can bound (4.35) by

$$\begin{aligned}
& \mathbb{P}\left[T_{\delta+\gamma}(y) \leq 1;\ \pi(\lambda\varepsilon)^2 \leq |\widetilde{C}_{\delta+\gamma,\nu}(y,\gamma)|;\ B[R_\nu(y), R_\nu(y)+1] \cap b(y,\delta+\gamma) \neq \emptyset\right] \\
& \quad + \mathbb{P}\left[T_{\delta+\gamma}(y) \leq 1\right] O\left(|\log\delta|^{-K_2}\right) \\
= \ & \mathbb{P}\left[T_{\delta+\gamma}(y) \leq 1\right] \cdot O\left(\frac{\log|\log\delta|}{|\log\delta|^{7/4}}\right).
\end{aligned}$$

Then we apply Lemma 1.7 again, i.e. there exists an integrable function G such that (4.25) is bounded by

$$\left(\frac{\log|\log\delta|}{|\log\delta|^{11/4}}\right) G(y).$$

Plugging the obtained estimates (4.24) and (4.25) into the formula (4.23), we obtain

$$\left|\mathbb{P}\left[y \in U(\varepsilon,\gamma)\right] - \mathbb{P}\left[T_{\delta+\gamma}(y) \leq 1\right] Q\left(\frac{\varepsilon}{\delta}, \frac{\nu}{\delta}, \frac{\gamma}{\delta}\right)\right| \leq \left(\frac{\log|\log\delta|}{|\log\delta|^{11/4}}\right) G(y).$$

Finally, we can finish the proof by the following easy observation.

$$\begin{aligned}
\mathbb{E}\,|U(\varepsilon,\gamma)| &= \int \mathbb{P}\left[y \in U(\varepsilon,\gamma)\right] \mathrm{d}y \\
&= \left(\int \mathbb{P}\left[T_{\delta+\gamma}(y) \leq 1\right] \mathrm{d}y\right) Q\left(\frac{\varepsilon}{\delta}, \frac{\nu}{\delta}, \frac{\gamma}{\delta}\right) + O\left(\frac{\log|\log\delta|}{|\log\delta|^{11/4}}\right) \\
&= \mathbb{E}\left[|S_{\delta+\gamma}[0,1]|\right] Q\left(\frac{\varepsilon}{\delta}, \frac{\nu}{\delta}, \frac{\gamma}{\delta}\right) + O\left(\frac{\log|\log\delta|}{|\log\delta|^{11/4}}\right) \\
&= \left(\frac{\pi}{|\log(\delta+\gamma)|} + O\Big(|\log(\delta+\gamma)|^{-2}\Big)\right) \\
&\quad \cdot \left(\frac{|\log\lambda|}{|\log\delta|} + O\left(\frac{\log|\log\delta|}{|\log\delta|^{3/2}}\right)\right) + O\left(\frac{\log|\log\delta|}{|\log\delta|^{11/4}}\right) \\
&= \frac{\pi|\log\lambda|}{|\log\delta|^2} + O\left(\frac{\log|\log\delta|}{|\log\delta|^{5/2}}\right)
\end{aligned}$$

where we use Lemma 4.8 and the known asymptotic behavior of the Wiener sausage from Proposition 4.3. The substitution of δ by ε is the last step of the proof of Theorem 4.10. □

The next statement shows an upper bound of the variation of $|U(\varepsilon,\gamma)|$.

Lemma 4.11. *There exist $K > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $0 < \gamma < \varepsilon^a$, $a > 1$, the following inequality holds*

$$\operatorname{var} |U(\varepsilon, \gamma)| \le K |\log \varepsilon|^{-11/2}.$$

We postpone the proof of Lemma 4.11 after setting some notation and an auxiliary lemma.

Notation 4.12. *Let B^1, B^2 be two independent standard planar Brownian motions, we denote by S^1, S^2 the corresponding Wiener sausages and we set the following notation*

$$C'(y, \gamma) = \begin{cases} \text{c. c. of } \mathbb{R}^2 \setminus (S^1_\gamma[0,1/2] \cap S^2_\gamma[0,1/2]) \text{ that contains } y \\ \quad \text{for } y \notin S^1_\gamma[0,1/2] \cap S^2_\gamma[0,1/2], \\ \emptyset \quad \text{for } y \in S^1_\gamma[0,1/2] \cap S^2_\gamma[0,1/2]. \end{cases}$$

Let $0 < \lambda < 1$. We denote

$$U'(\varepsilon, \gamma) = \{y \in \mathbb{R}^2 \setminus (S^1_\gamma[0,1/2] \cap S^2_\gamma[0,1/2]) : \pi(\lambda\varepsilon)^2 \le |C'(y,\gamma)| \le \pi\varepsilon^2\}.$$

Similarly for $j = 1, 2$, we use the notation

$$C^j(y, \gamma) = \begin{cases} \text{c. c. of } \mathbb{R}^2 \setminus S^j_\gamma[0,1/2] : \text{ that contains } y \\ \quad \text{for } y \notin S^j_\gamma[0,1/2], \\ \emptyset \quad \text{for } y \in S^j_\gamma[0,1/2]. \end{cases}$$

For $0 < \lambda < 1$, we write

$$U^j(\varepsilon, \gamma) = \{y \in \mathbb{R}^2 \setminus S^j_\gamma[0,1/2] : \pi(\lambda\varepsilon)^2 \le |C^j(y,\gamma)| \le \pi\varepsilon^2\}, \ j = 1, 2.$$

Now, we formulate a lemma concerning the difference of the area of connected components formed by one Wiener sausage and areas of connected components formed by two independent Wiener sausages on half time interval.

Lemma 4.13. *The following asymptotic formula holds for ε tending to zero and $0 < \gamma < \varepsilon^a$, $a > 1$,*

$$\mathbb{E}\Big[|U'(\varepsilon,\gamma)| - \big|U^1(\varepsilon,\gamma)\big| - \big|U^2(\varepsilon,\gamma)\big| \Big]^2 = O\left(|\log \varepsilon|^{-11/2}\right).$$

Proof. It is not difficult to observe that

$$\Big| |U'(\varepsilon,\gamma)| - \big|U^1(\varepsilon,\gamma)\big| - \big|U^2(\varepsilon,\gamma)\big| \Big| \le$$

$$\begin{aligned} &\big|U'(\varepsilon,\gamma) \setminus \left(U^1(\varepsilon,\gamma) \cup U^2(\varepsilon,\gamma)\right)\big| + \big|U^1(\varepsilon,\gamma) \setminus U'(\varepsilon,\gamma)\big| \\ &+ \big|U^2(\varepsilon,\gamma) \setminus U'(\varepsilon,\gamma)\big| + \big|U^1(\varepsilon,\gamma) \cap U^2(\varepsilon,\gamma)\big|. \end{aligned} \tag{4.38}$$

Obviously, it is sufficient to bound only the second moment of last four summands. The first of them can be written as

$$\mathbb{E}\left[\left|U^1(\varepsilon,\gamma)\setminus U'(\varepsilon,\gamma)\right|\right]^2 = \mathbb{E}\left[\int \mathbf{1}_{\{U^1(\varepsilon,\gamma)\setminus U'(\varepsilon,\gamma)\}}(y)\,\mathbf{1}_{\{U^1(\varepsilon,\gamma)\setminus U'(\varepsilon,\gamma)\}}(z)\mathrm{d}y\mathrm{d}z\right].$$

Let $A_1 = U^1(\varepsilon,\gamma)\setminus U'(\gamma,\gamma)$. We set $\delta=\delta(\varepsilon)>0$ as we had already done in the proof of Theorem 4.10 and we bound $\mathbb{E}\,|A_1|^2$ by

$$\begin{aligned} &\mathbb{E}\left[\int \mathbf{1}_{\{C^1(y,\gamma)\not\subseteq b(y,\delta);\,C^1(z,\gamma)\not\subseteq b(z,\delta)\}}\,\mathbf{1}_{A_1}(y)\,\mathbf{1}_{A_1}(z)\mathrm{d}y\mathrm{d}z\right] \\ +\;&\mathbb{E}\left[\int \mathbf{1}_{\{C^1(y,\gamma)\subseteq b(y,\delta);\,C^1(z,\gamma)\subseteq b(z,\delta)\}}\,\mathbf{1}_{A_1}(y)\,\mathbf{1}_{A_1}(z)\mathrm{d}y\mathrm{d}z\right]. \end{aligned}$$

Let us observe that if $A\subset\mathbb{R}^2$ is such that $|A|\le\pi r^2$ then $|A\ominus b(o,r)|=0$. Moreover, if ∂A is a subset of $B[0,t]$, $t>0$, then $|A|\le|S_r[0,t]|$. Consequently, if $A\subset\mathbb{R}^2$ is a set whose boundary is the subset of $S_\gamma[0,t]$, $t>0$ and we know that $|A|\le\pi r^2$ then we have $|A|\le|S_{\gamma+r}[0,t]|$. Thus, the first summand of (4.39) can be bounded by

$$\begin{aligned} &\mathbb{E}\left[\int \mathbf{1}_{\{C^1(y,\gamma)\not\subseteq b(y,\delta)\}}\,\mathbf{1}_{U^1(\varepsilon,\gamma)}(y)\,\mathbf{1}_{U^1(\varepsilon,\gamma)}(z)\mathrm{d}y\mathrm{d}z\right] \\ \le\;&\int\mathbb{E}\left[\mathbf{1}_{\{C^1(y,\gamma)\not\subseteq b(y,\delta)\}}\,\mathbf{1}_{U^1(\varepsilon,\gamma)}(y)\left|S^1_{\varepsilon+\gamma}[0,1/2]\right|\right]\mathrm{d}y \\ \le\;&\int\mathbb{P}\left[y\in U^1(\varepsilon,\gamma);\,C^1(y,\gamma)\not\subseteq b(y,\delta)\right]^{1/2}\mathrm{d}y\cdot\left(\mathbb{E}\left|S^1_{\varepsilon+\gamma}[0,1/2]\right|^2\right)^{1/2} \\ =\;&O\left(|\log\varepsilon|^{-K_1}\right)\cdot O\left(|\log\varepsilon|^{-1}\right) \end{aligned} \tag{4.39}$$

where we used the Hölder inequality in the second inequality and then we applied Lemma 4.5 in the estimate of the first term (similarly as in the proof of Theorem 4.10) and the second one was bounded directly by the Proposition 4.3.

Therefore, after substituting it into (4.39), we have

$$\begin{aligned} &\mathbb{E}\left[\left|U^1(\varepsilon,\gamma)\setminus U'(\varepsilon,\gamma)\right|\right]^2 \\ \le\;&O\left(|\log\varepsilon|^{-K_1}\right)+\mathbb{E}\left[\int \mathbf{1}_{\{C^1(y,\gamma)\subseteq b(y,\delta);C^1(z,\gamma)\subseteq b(y,\delta)\}}\,\mathbf{1}_{A_1}(y)\,\mathbf{1}_{A_1}(z)\mathrm{d}y\mathrm{d}z\right]. \end{aligned} \tag{4.40}$$

We have to remark here that if $C^1(y,\gamma)\subseteq b(y,\delta)$ and $y\in U^1(\varepsilon,\gamma)\setminus U'(\varepsilon,\gamma)$, then

$$S^2_\gamma[0,1/2]\cap b(y,\delta)\neq\emptyset,$$

thus we can continue in estimating of (4.40). We obtain

$$
\begin{aligned}
& \mathbb{E}\left[\,\left|U^1(\varepsilon,\gamma)\setminus U'(\varepsilon,\gamma)\right|\,\right]^2 \\
\le\;& O\left(|\log\varepsilon|^{-K_1}\right) + \mathbb{E}\left[\int \mathbf{1}_{S^2_{\gamma+\delta}[0,1/2]}(y)\,\mathbf{1}_{S^2_{\gamma+\delta}[0,1/2]}(z)\,\mathbf{1}_{U^1(\varepsilon,\gamma)}(y)\,\mathbf{1}_{U^1(\varepsilon,\gamma)}(z)\mathrm{d}y\,\mathrm{d}z\right] \\
=\;& O\left(|\log\varepsilon|^{-K_1}\right) + \int \mathbb{P}\left[y\in U^1(\varepsilon,\gamma),\, z\in U^1(\varepsilon,\gamma)\right] \\
&\qquad\qquad\qquad \cdot\,\mathbb{P}\left[y\in S^2_{\delta+\gamma}[0,1/2],\, z\in S^2_{\delta+\gamma}[0,1/2]\right]\mathrm{d}y\,\mathrm{d}z. \quad (4.41)
\end{aligned}
$$

Now, we want to bound $\mathbb{P}\left[y\in U^1(\varepsilon,\gamma),\, z\in U^1(\varepsilon,\gamma)\right]$ in (4.41). We can, due to the symmetry, consider only the set $\{T^1_{\delta+\gamma}(y)\le T^1_{\delta+\gamma}(z)\}$ (the upper index ^{1} means again that in the definition of the stopping time, B^1 is used instead of B).

We will suppose that $|y-z|\ge|\log\varepsilon|^{-3}$. It is not restricting assumption because the integral in (4.41) out of the set

$$\{(y,z):\ |y-z|\ge|\log\varepsilon|^{-3}\}$$

is negligible for small $\varepsilon>0$.

Hence with $\varepsilon>0$ small enough the following holds

$$
\begin{aligned}
& \mathbb{P}\left[y\in U^1(\varepsilon,\gamma),\, z\in U^1(\varepsilon,\gamma),\, T^1_{\delta+\gamma}(y)\le T^1_{\delta+\gamma}(z)\right] \\
\le\;& \mathbb{P}\left[T^1_{\delta+\gamma}(y)\le T^1_{\delta+\gamma}(z)\le 1/2;\, \pi(\lambda\varepsilon)^2\le|C^1(y,\gamma)|,\, \pi(\lambda\varepsilon)^2\le|C^1(z,\gamma)|\right] \\
& + \mathbb{P}\left[T^1_{\delta+\gamma}(z)\le 1/2\le R^1_\nu(z)\right], \qquad (4.42)
\end{aligned}
$$

where

$$R^1_\nu(z)=\inf\{s>T^1_{\delta+\gamma}(z):\ |B^1(s)-z|>\nu\}$$

and $\nu=\nu(\delta)=|\log\delta|^{-4}$ is chosen as in the proof of Theorem 4.10.

The second term in (4.42) can be bounded by $O\left(|\log\delta|^{-8}\right)$ by using Lemma 1.11 (see (4.29)) and the first one can be expanded and bounded (using the Markov property at stoping time $T^1_{\delta+\gamma}(z)$) by

$$
\begin{aligned}
& \mathbb{P}\left[T^1_{\delta+\gamma}(y)\le T^1_{\delta+\gamma}(z)\le 1/2;\, \pi(\lambda\varepsilon)^2\le|\widetilde{C}^1(y,\gamma)|\right]\left(1-\Gamma\left(\frac{\lambda\varepsilon}{\delta+\gamma},\frac{\nu}{\delta+\gamma},\frac{\gamma}{\delta+\gamma}\right)\right) \\
&\qquad \le K|\log\delta|^{-3/4}\,\mathbb{P}\left[T^1_{\delta+\gamma}(y)\le T^1_{\delta+\gamma}(z)\le 1/2;\, \pi(\lambda\varepsilon)^2\le|\widetilde{C}^1(y,\gamma)|\right] \qquad (4.43)
\end{aligned}
$$

for some $K>0$ where $\widetilde{C}^1(y,\gamma)$ is the connected component of $\mathbb{R}^2\setminus S^1_\gamma[0,R^1_\nu(y)]$ containing y.

Then we use the Markov property at the stopping time $R^1_\nu(y)$ and Lemma 1.7 – (1.5) to obtain an upper estimate of (4.43)

$$K|\log\delta|^{-3/4}\,\mathbb{P}\left[T^1_{\delta+\gamma}(y)\le 1/2;\, \pi(\lambda\varepsilon)^2\le|\widetilde{C}^1(y,\gamma)|\right]\frac{G\left(\frac{z-y}{2}\right)}{|\log(\delta+\gamma)|}$$

and then we use the Markov property at $T^1_{\delta+\gamma}(y)$ to get the following

$$K_2\, G(y)G\left(\frac{z-y}{2}\right)|\log\delta|^{-7/2} \tag{4.44}$$

for $\delta>0$ small enough and some $K_2>0$.

The equation (1.6) in Lemma 1.7 claims that

$$\mathbb{P}\left[y\in S^2_{\delta+\gamma}[0,1/2],\, z\in S^2_{\delta+\gamma}[0,1/2]\right]\le|\log(\delta+\gamma)|^{-2}\big(G(y)G(z-y)+G(z)G(y-z)\big).$$

If we plug it together with (4.44) into (4.41), then we obtain

$$\mathbb{E}\left[\,\left|U^1(\varepsilon,\gamma)\setminus U'(\varepsilon,\gamma)\right|\,\right]^2=O\left(|\log\varepsilon|^{-11/2}\right).$$

We have the same upper bound for

$$\mathbb{E}\left[\,\left|U^2(\varepsilon,\gamma)\setminus U'(\varepsilon,\gamma)\right|\,\right]^2$$

and also for

$$\mathbb{E}\left[\,\left|U^1(\varepsilon,\gamma)\cap U^2(\varepsilon,\gamma)\right|\,\right]^2,$$

because

$$\begin{aligned}&\mathbb{E}\left[\,\left|U^1(\varepsilon,\gamma)\cap U^2(\varepsilon,\gamma)\right|\,\right]^2\\ \le\;&\mathbb{E}\left[\int \mathbf{1}_{U^1(\varepsilon,\gamma)}(y)\,\mathbf{1}_{S^2_{\varepsilon+\gamma}[0,1/2]}(y)\,\cdot\,\mathbf{1}_{U^1(\varepsilon,\gamma)}(z)\,\mathbf{1}_{S^2_{\varepsilon+\gamma}[0,1/2]}(z)\mathrm{d}y\mathrm{d}z\right]\end{aligned} \tag{4.45}$$

which we have bounded above.

Thus, we need to estimate the mean of $|U'(\varepsilon,\gamma)\setminus(U^1(\varepsilon,\gamma)\cup U^2(\varepsilon,\gamma))|^2$ which is equal to

$$\mathbb{E}\left[\int \mathbf{1}_{U'(\varepsilon,\gamma)\setminus(U^1(\varepsilon,\gamma)\cup U^2(\varepsilon,\gamma))}(y)\,\mathbf{1}_{U'(\varepsilon,\gamma)\setminus(U^1(\varepsilon,\gamma)\cup U^2(\varepsilon,\gamma))}(z)\mathrm{d}y\mathrm{d}z\right]. \tag{4.46}$$

We can, once more, restrict our examination to the set

$$\{C'(\cdot,\gamma)\subseteq b(\cdot,\delta)\}$$

because the event

$$\{C'(\cdot,\gamma)\cap b(\cdot,\delta)\ne\emptyset\}$$

is negligible again. Therefore, we have

$$\begin{aligned}&\mathbb{E}\left[\,\left|U'(\varepsilon,\gamma)\setminus\big(U^1(\varepsilon,\gamma)\cup U^2(\varepsilon,\gamma)\big)\right|\,\right]^2\\ \le\;&\mathbb{E}\Bigg[\int \mathbf{1}_{U'(\varepsilon,\gamma)\setminus(U^1(\varepsilon,\gamma)\cup U^2(\varepsilon,\gamma))}(y)\,\mathbf{1}_{U'(\varepsilon,\gamma)\setminus(U^1(\varepsilon,\gamma)\cup U^2(\varepsilon,\gamma))}(z)\\ &\qquad\cdot\mathbf{1}_{\{C'(y,\gamma)\subseteq b(y,\delta)\}}\,\mathbf{1}_{\{C'(z,\gamma)\subseteq b(z,\delta)\}}\,\mathrm{d}y\mathrm{d}z\Bigg]+O\left(|\log\varepsilon|^{-K_1}\right).\end{aligned} \tag{4.47}$$

Let us examine the indicators

$$\begin{aligned} & \mathbf{1}_{\{C'(y,\gamma)\subseteq b(y,\delta)\}}\cdot \mathbf{1}_{U'(\varepsilon,\gamma)\setminus(U^1(\varepsilon,\gamma)\cup U^2(\varepsilon,\gamma))}(\,\cdot\,) \\ \le\ & \mathbf{1}_{S^1_{\delta+\gamma}[0,1/2]}(\,\cdot\,)\cdot \mathbf{1}_{S^2_{\delta+\gamma}[0,1/2]}(\,\cdot\,)\cdot \mathbf{1}_{\{\pi(\lambda\varepsilon)^2\le |C^1(y,\gamma)|\}}\cdot \mathbf{1}_{\{\pi(\lambda\varepsilon)^2\le |C^2(y,\gamma)|\}} \end{aligned}$$

to obtain a bound of (4.47)

$$O\left(|\log\varepsilon|^{-K_1}\right) + \int \Bigg(\mathbb{P}\Big[y\in S^1_{\delta+\gamma}[0,1/2],\ z\in S^1_{\delta+\gamma}[0,1/2], \\ \pi(\lambda\varepsilon)^2\le |C^1(y,\gamma)|,\ \pi(\lambda\varepsilon)^2\le |C^1(z,\gamma)|\Big]\Bigg)^2 \mathrm{d}y\mathrm{d}z. \qquad (4.48)$$

The integrant in (4.48) has been already estimated above, i.e.

$$\begin{aligned} & \mathbb{P}\Big[y\in S^1_{\delta+\gamma}[0,1/2],\ z\in S^1_{\delta+\gamma}[0,1/2],\ \pi(\lambda\varepsilon)^2\le |C^1(y,\gamma)|,\ \pi(\lambda\varepsilon)^2\le |C^1(z,\gamma)|\Big] \\ & \le K_2|\log\varepsilon|^{-7/2}\left(G(y)G(z-y)+G(z)G(z-y)\right) + O\left(|\log\varepsilon|^{-8}\right). \end{aligned}$$

Thus

$$\mathbb{E}\Big[\big|U'(\varepsilon,\gamma)\setminus\left(U^1(\varepsilon,\gamma)\cup U^2(\varepsilon,\gamma)\right)\big|\Big]^2 = O\left(|\log\varepsilon|^{-7}\right).$$

It was the last estimate of (4.38) and it finishes the proof of Lemma 4.13. □

Therefore, we can finish proving process of Lemma 4.11.

Proof of Lemma 4.11. It is easy to see that

$$|U^1(\varepsilon,\gamma)| \stackrel{d}{=} |U^2(\varepsilon,\gamma)|.$$

Moreover, $|U^j(\varepsilon,\gamma)|$, $j=1,2$ are independent and obviously

$$\operatorname{var}|U(\varepsilon,\gamma)| = \operatorname{var}|U'(\varepsilon,\gamma)|.$$

After the scaling, we have

$$\big|U^1(\varepsilon,\gamma)\big| \stackrel{d}{=} \frac{1}{2}\left|U\left(\varepsilon\sqrt{2},\gamma\sqrt{2}\right)\right|.$$

Due to Lemma 4.13, we have for $\varepsilon\to 0$, $0<\gamma<\varepsilon^a$, $a>1$, the following inequality

$$\begin{aligned} \operatorname{var}|U(\varepsilon,\gamma)| &\le \operatorname{var}\Big(|U^1(\varepsilon,\gamma)|+|U^2(\varepsilon,\gamma)|\Big) + O\Big(|\log\varepsilon|^{-11/2}\Big) \\ &= \frac{1}{2}\operatorname{var}\big|U(\sqrt{2}\varepsilon,\sqrt{2}\gamma)\big| + O\Big((|\log\varepsilon)|^{-11/2}\Big). \end{aligned}$$

Denoting

$$f(\varepsilon) = \operatorname{var}\big|U(\varepsilon,\gamma)\big|$$

the following holds (for $\varepsilon>0$ small enough)

$$f(\varepsilon)\le \frac{1}{2}f(\varepsilon\sqrt{2}) + O(|\log\varepsilon|^{-11/2}).$$

After short analysis, it can be shown that there exist $\varepsilon_0>0$ such that $|\log\varepsilon|^{11/4}f(\varepsilon)$ is bounded on the interval $(0,\varepsilon_0)$. Hence, the proof is finished. □

4.4 Main Results

In this section, we present our main result on the asymptotic behavior of the number of connected components of the complement of a Wiener sausage. We perform its proof based on the statements from the previous preliminary sections.

Definition 4.14. *Let $0 < u < v \le \infty$ and $\gamma > 0$. We denote by $N_\gamma[u,v)$ the following quantity*

$$N_\gamma[u,v) = \text{ the number of c. c. of } \mathbb{R}^2 \setminus S_\gamma[0,1] \text{ with area in } [u,v).$$

Theorem 4.15. *The following limit holds almost surely*

$$\lim_{u\to 0} u(\log u)^2 N_\gamma[u,\beta u) = 2\pi\,\frac{\beta-1}{\beta}$$

uniformly for

$$0<\gamma<\left(\frac{u}{\pi}\right)^b,\ b>1/2 \quad \text{and} \quad \beta\in(1,\infty).$$

Hence, we obtain with β tending to infinity that

$$\lim_{u\to 0} u(\log u)^2 N_\gamma[u,\infty) = 2\pi$$

uniformly for

$$0<\gamma<\left(\frac{u}{\pi}\right)^b,\ b>1/2.$$

Proof. During the proof, we modify the notation of $U(\cdot,\cdot)$ because the crucial role plays here different "lambdas". Let us denote

$$U_\lambda(\varepsilon,\gamma) = \big\{y\in\mathbb{R}^2\setminus S_\gamma[0,1];\ \pi(\lambda\varepsilon)^2 \le |C(y,\gamma)| \le \pi\varepsilon^2\big\}.$$

Firstly, we use the strong law of large numbers. Due to Lemma 4.11, we can see that

$$\begin{aligned}\sum_{n=1}^{\infty} \operatorname{var}\Big[(\log\lambda^n)^2|U_\lambda(\lambda^n,\gamma)|\Big] &\le K\sum_{n=1}^{\infty}|\log\lambda^n|^{(-11/2+4)}\\ &= K|\log\lambda|^{-3/2}\sum_{n=1}^{\infty} n^{-3/2} < \infty\end{aligned}$$

uniformly for $0<\gamma<\lambda^{na}$, $a>1$. Hence

$$\lim_{n\to\infty}(\log\lambda^n)^2|U_\lambda(\lambda^n,\gamma)| = \pi|\log\lambda| \quad \text{a.s.} \tag{4.49}$$

where we use Theorem 4.10.

Let $\bar N_\gamma[u,v)$ be the number of the connected components of $\mathbb{R}^2\setminus S_\gamma[0,1]$ with area in $[\pi u^2,\pi v^2)$, i.e.

$$\bar N_\gamma[u,v) = N_\gamma[\pi u^2,\pi v^2)$$

or equivalently

$$\bar{N}_\gamma[\sqrt{u/\pi}, \sqrt{v/\pi}) = N_\gamma[u, v).$$

Due to an easy observation, we get

$$\frac{|U_\lambda(\lambda^n, \gamma)|}{\pi\lambda^{2n}} \leq \bar{N}_\gamma[\lambda^{n+1}, \lambda^n) \leq \frac{|U_\lambda(\lambda^n, \gamma)|}{\pi\lambda^{2(n+1)}}.$$

Now, we choose a fixed $k \in \mathbb{N}$ and decompose $\bar{N}$ to

$$\bar{N}_\gamma[\lambda^{n+1}, \lambda^n) = \sum_{j=0}^{k-1} \bar{N}_\gamma[\lambda^{n+(j+1)/k}, \lambda^{n+j/k}).$$

The previous observation leads to

$$\frac{|U_{\lambda^{1/k}}(\lambda^{n+j/k}, \gamma)|}{\pi\lambda^{2(n+j/k)}} \leq \bar{N}_\gamma[\lambda^{n+(j+1)/k}, \lambda^{n+j/k}) \leq \frac{|U_{\lambda^{1/k}}(\lambda^{n+j/k}, \gamma)|}{\pi\lambda^{2(n+(j+1)/k)}}$$

for $j = 0, \dots, k-1$ and

$$\begin{aligned} \lambda^{-\frac{2j}{k}} \frac{1}{\pi} |U_{\lambda^{1/k}}(\lambda^{n+j/k}, \gamma)| \left(\log \lambda^{n+j/k}\right)^2 &\leq \lambda^{2n} \left(\log \lambda^{n+j/k}\right)^2 \bar{N}_\gamma[\lambda^{n+(j+1)/k}, \lambda^{n+j/k}) \\ \lambda^{2n} \left(\log \lambda^{n+j/k}\right)^2 \bar{N}_\gamma[\lambda^{n+(j+1)/k}, \lambda^{n+j/k}) &\leq \lambda^{-\frac{2(j+1)}{k}} \frac{1}{\pi} |U_{\lambda^{1/k}}(\lambda^{n+j/k}, \gamma)| \left(\log \lambda^{n+j/k}\right)^2. \end{aligned} \tag{4.50}$$

If we consider that

$$\lim_{n\to\infty} \log \lambda^{n+j/k} = \lim_{n\to\infty} \log \lambda^n$$

an if we sum ($j = 0, \dots, k-1$) the middle terms in (4.50), we obtain

$$\begin{aligned} \left|\log \lambda^{1/k}\right| \sum_{j=0}^{k-1} \lambda^{-\frac{2j}{k}} &\leq \liminf_{n\to\infty} \lambda^{2n} (\log \lambda^n)^2 \bar{N}_\gamma[\lambda^{n+1}, \lambda^n) \\ &\leq \limsup_{n\to\infty} \lambda^{2n} (\log \lambda^n)^2 \bar{N}_\gamma[\lambda^{n+1}, \lambda^n) \leq \left|\log \lambda^{1/k}\right| \sum_{j=0}^{k-1} \lambda^{-\frac{2(j+1)}{k}}. \end{aligned} \tag{4.51}$$

The left-hand side is equal to

$$|\log \lambda| \; \frac{\lambda^{-2} - 1}{k(\lambda^{-2/k} - 1)}$$

and for k tendind to infinity converges to

$$|\log \lambda| \; \frac{1 - \lambda^2}{2\lambda^2 |\log \lambda|} = \frac{1 - \lambda^2}{2\lambda^2}.$$

The right-hand side is equal to

$$|\log\lambda|\ \frac{\lambda^{-2/k}(\lambda^{-2}-1)}{k(\lambda^{-2/k}-1)}$$

and it has the same limit as the left-hand side.

Therefore, we have

$$\lim_{n\to\infty}\lambda^{2n}(\log\lambda^n)^2\bar{N}_\gamma[\lambda^{n+1},\lambda^n)=\frac{1-\lambda^2}{2\lambda^2}\quad\text{a.s.}\tag{4.52}$$

We have to be careful for which γ this limit holds. We have assumed that

$$0<\gamma<\lambda^{(n+j/k)a},\ a>1,\ j=0,\ldots,k-1.$$

Thus, we can restrict on

$$0<\gamma<\lambda^{(n+1)a},\ a>1.$$

If we substitute $x=\lambda^{n+1}$ and $\alpha=\lambda^{-1}$ in (4.52), we get

$$\lim_{x\to 0}x^2(\log x)^2\bar{N}_\gamma[x,\alpha x)=\frac{\alpha^2-1}{2\alpha^2}\quad\text{a.s.}$$

uniformly for $0<\gamma<x^a$, $a>1$, $\alpha\in(1,\infty)$.

After reverse changing of notation from $\bar{N}$ to N, we obtain

$$\lim_{u\to 0}u(\log u)^2N_\gamma[u,\beta u)=2\pi\,\frac{\beta-1}{\beta},\ \text{for }\beta\in(1,\infty)\quad\text{a.s.}$$

uniformly for $0<\gamma<(\sqrt{u/\pi})^a$, $a>1$ and $\beta\in(1,\infty)$. Sending β to the infinity, we obtain the claim. □

Remark 4.16. *Let $\chi(S_\gamma[0,1])$ be the Euler characteristic of the Wiener sausage. As we have already mentioned in Introduction and in the beginning of Chapter 4, the following holds*

$$\chi(\gamma)=1-N_\gamma[0,\infty).$$

We know that $N_\gamma[0,\infty)\geq N_\gamma[u,\infty)$ for any $u>0$. Theorem 4.15 determines only a upper bound for the Euler characteristic of the Wiener sausage because there occurs the radius of Wiener sausage depending on the area of holes in the limit. If we could succeed in eliminating this dependance, we would have the limit formula for the Euler characteristic of the Wiener sausage.

We expect that the limit behaviour of $N_\gamma[0,\infty)$ and $N_\gamma[u,\infty)$ are asymptotically equal. This hypothesis is supported by the numerical simulation study in [19]. However it remains open for now.

Bibliography

[1] Berezhkovskii A. M., Makhnovskii Yu. A., Suris R. A., *Wiener sausage volume moments*, J. Stat. Phys. **57** (1989), 333 – 346.

[2] Burdzy K., *Brownian motion and cones*, Ann. Probab. **13**, 3 (1985), 1006 – 1010.

[3] Borodin A. N., Salminen P., *Handbook of Brownian Motion – Facts and Formulae*, 2nd ed. Birkhauser, Basel 2002.

[4] Černý R., *On Geometrical Properties of the r-neighborhood of Brownian Motion and Related Random Structures*, Doctoral thesis MFF UK, Prague 2007.

[5] Evans S. N., *On the Hausdorff dimension of Brownian conepoints*, Math. Proc. Camb. Phil. Soc. **98** (1985), 343 – 353.

[6] Ferry S., *When ε-boundaries are manifolds*, Fundamenta Mathematicae **3** (1976), 199 – 210.

[7] Fu J. H. G., *Tubular neighborhoods in Euclidean spaces*, Duke Math. J. Vol. **52**, 4 (1985), 1025 – 1046.

[8] Honzl O., *Cone Points and Critical Points of Brownian Motion*, WDS'09 Proceedings of Contributed Papers, Part I (2009), 119 – 124.

[9] Honzl O., Rataj J., *Almost sure asymptotic behaviour of the r-neighbourhood surface area of Brownian paths*, Czechoslovak Math. J. Vol. **62**, No. 1 (2012), 67 – 75.

[10] Kneser M., *Über den Rand von Parallelkörpern*, Math. Nachr. **5** (1951), 251 – 258.

[11] Le Gall J.-F., *Sur la sausisse de Wiener et les points multiples du mouvement brownien*, Ann. Probab. **14** (1986), 1219 − 1244.

[12] Le Gall J.-F., *Fluctuation results for the Wiener sausage*, Ann. Probab. **16** (1988), 991 − 1018.

[13] Le Gall J.-F., *On the Connected Components of the Complement of a Two-Dimensional Brownian Path*, Prog. Probab. **28** (1991), 323 – 338.

[14] McKean H. P. Jr., *Hausdorff-Besicovitch dimension of Brownian motion paths*, Duke Math. J. Vol. **22**, No. 2 (1955), 229 – 234.

[15] Mörters P., Peres Y., *Brownian Motion*, Cambridge University Press, (2010).

[16] Mountford T. S., *On the Asymptotic Number of Small Components Created by Planar Brownian Motion*, Stochastics and Stochastics Reports **28** (1989), 177 – 188.

[17] Rataj J., *Three Lectures on Geometric Measure Theory*, the lecture notes from Winter school in Sandberg 2007, http://www.karlin.mff.cuni.cz/ rataj/GMT_Sandbjerg_2007.pdf

[18] Rataj J., Schmidt V., Spodarev E., *On the expected surface area of the Wiener sausage*, Math. Nachr. **282** (2009), 591 – 603.

[19] Rataj J., Spodarev E., Meschenmoser D., *Approximations of the Wiener sausage and its curvature measures*, Ann. Appl. Probab. Vol. **19**, No. 5 (2009), 1840 – 1859.

[20] Rataj J., Winter S., *On volume and surface area of parallel sets*, Indiana Univ. Math. J. **59**, No.5 (2010), 1661 – 1685.

[21] Rataj J., Winter S., *Characterization of Minkowski measurability in terms of surface area*, arXiv:1111.1825

[22] Spitzer F., *Electrostatic capacity, heat flow and Brownian motion*, Z. Wahrscheinlichkeitstheorie verw. Geb. **3** (1964), 110 – 121.

[23] Spitzer F., *Some theorems concerning two-dimensional Brownian motion*, Trans. Amer. Math. Soc. Vol. **87**, No. 1 (1958), 187 – 197.

[24] Stachó L. L., *On the volume function of parallel sets*, *Acta Sci. Math.* **38** (1976), 365 – 374.