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MASTER'S THESIS

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FUNKCE KONTINUA NA REGULÁRNÍCH KARDINÁLECH ZA PŘÍTOMNOSTI VELKÝCH KARDINÁLŮ

THE CONTINUUM FUNCTION ON REGULAR CARDINALS IN THE PRESENCE OF LARGE CARDINALS

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Abstrakt: V této práci zkoumáme, jak na sebe vzájemně působí velké kardinály a funkce kontinua. Z Eastonova výsledku víme, že funkce kontinua na regulárních kardinálech má v ZFC velkou volnost. Avšak velké kardinály kladou na chování funkce kontinua další omezující podmínky. Vzájemné ovlivňování velkých kardinálů a funkce kontinua se liší pro jednotlivé typy velkých kardinálů. Abychom poukázali na tyto rozdíly, soustředíme se na slabě kompaktní a měřitelný kardinál. Pro srovnání také přezkoumáme nepopsatelné kardinály, na kterých ukážeme, že není snadné přesně určit důvod těchto rozdílů.

Klíčová slova: funkce kontinua, velké kardinály, GCH

Abstract: This thesis examines the interactions between the continuum function and large cardinals. It is know, by a result of Easton, that the continuum function on regular cardinals has great freedom in ZFC. However, large cardinals lay additional constraints to possible behaviour of the continuum function. We focus on weakly compact and measurable cardinal to point out the differences in interactions with the continuum function between various types of large cardinals. We also study the case of indescribable cardinals for the comparison, and the results lead us to conclude that it is not easy to pinpoint the reason for these differences.

Keywords: continuum function, large cardinals, GCH

1 Introduction

The aim of this thesis is to study the properties of the continuum function on regular cardinals and to explore the effects large cardinals and the continuum function have on each other. We work in ZFC, the axiomatic Zermelo–Fraenkel set theory with the axiom of choice, and in its extensions by various large cardinal axioms. Section 1 contains preliminaries and brief history of the continuum function. In Section 2 we present Easton's independence result, which shows that the continuum function have great freedom in ZFC. Section 3 is an introduction of a few types of large cardinals and in Section 4 we study the interactions between these large cardinals and the continuum function. We focus on what we consider the most known types of large cardinals: weakly compact and measurable cardinals. For comparison, we also study indescribable cardinals, an intermediate stage between those two types.

1.1 Preliminaries

Our set-theoretical notation should be standard. We use $\alpha, \beta, \gamma, \ldots$ to denote ordinals and $\kappa, \lambda, \theta, \ldots$ to denote cardinals. ORD, $CARD$ and REG stand for the class of ordinals, cardinals and regular cardinals respectively. We use $\mathcal{P}$ to denote the power set operation and V_α is used for the αth level of the cumulative hierarchy. Sometimes we consider the cumulative hierarchy in a smaller model M, in which case we denote it as V_α^M. We also use ZFC$^-$ for the suitable fragment of ZFC. Usually, ZFC minus the powerset axiom is used, as $H(\kappa)$ (the set of all the sets with transitive closure of size less than κ) is a model of such fragment for κ regular. Since we study the behaviour of the continuum function, we often use statement "GCH holds (fails) at κ" as a convenient way of saying "$2^\kappa = \kappa^+$ ($2^\kappa > \kappa^+$)".

We assume the reader is familiar with forcing as an important technique for obtaining independence results in set theory. This includes working with basic notions of forcing (such as Cohen forcing) as well as with product and iterated forcing, and analyzing forcing properties like chain condition and closure. For those who do no have enough previous experience with forcing, we recommend Kunen's book [Kun80] or alternatively Jech's book [Jec03]. Iterated forcing is treated better in [Bau83] and we base our usage of iterated forcing on this article. We usually use $\mathbb{P}$,$\mathbb{Q}$ to denote notions of forcing, $\mathbb{P} \times \mathbb{Q}$ to denote the product and $\mathbb{P} * \dot{\mathbb{Q}}$ to denote the iteration. When dealing with infinite iterations we use $\mathbb{P}_\kappa$ to denote the iteration of length κ. When we need to split this iteration at some stage μ, we denote the two steps as $\mathbb{P}_\mu$ and $\mathbb{P}_{\mu+1,\kappa}$ (iteration up to and excluding μ and iteration from μ to κ). We often use V as the ground model for forcing instead of explicitly taking countable transitive model M, but we believe this will not cause any confusion. For a forcing notion $\mathbb{P}$ in V, $V[G]$ stands for the extension of V by a $\mathbb{P}$-generic filter G. We often use $V[\mathbb{P}]$ for the extension in the case where we do not need to speak about a particular generic filter. $Add(\kappa, \lambda)$ stands for Cohen forcing adding λ many subsets of κ and while we work only with Cohen forcing, its products and iterations in this thesis, we encourage the reader to get familiar with the wide variety of other forcing notions and their applications.

For the sake of clarification, we present the definitions of the forcing properties we use. Let $\mathbb{P}$ be a forcing notion and κ an infinite cardinal.

Definition 1.1. $\mathbb{P}$ is κ-cc iff every antichain of $\mathbb{P}$ has size less than κ.

$\mathbb{P}$ is κ-Knaster iff for every $X \subseteq \mathbb{P}$ with $|X| = \kappa$ there is $C \subseteq X$, such that $|C| = \kappa$ and all elements of C are pairwise compatible.

$\mathbb{P}$ is κ-closed iff every decreasing sequence of conditions of $\mathbb{P}$ of size less than κ has a lower bound in $\mathbb{P}$.

$\mathbb{P}$ is κ-distributive iff forcing with $\mathbb{P}$ does not add any new sequences of ordinals of size less than κ.

Recall that κ-Knaster implies κ-cc and κ-closed implies κ-distributive. Moreover, if $\mathbb{P}$ is κ-cc then forcing with $\mathbb{P}$ preserves cardinals $\geq\kappa$ and if $\mathbb{P}$ is κ-distributive then it preserves cardinals $\leq\kappa$.

An important tool for establishing chain condition (in fact Knaster property) of a forcing notion is the Δ-system lemma (examples of usage can be found in [Kun80]). System of sets $\mathcal{A}$ is called a Δ-*system* if there exists a fixed set r, called the *root* of the system, such that $a \cap b = r$ whenever a and b are distinct members of $\mathcal{A}$.

Fact 1.2 (Δ-system lemma)**.** *Suppose κ is an infinite cardinal. Let $\theta > \kappa$ be regular cardinal that satisfies $\forall \alpha < \theta\ |^{<\kappa}\alpha| < \theta$. Assume $|\mathcal{A}| \geq \theta$ and $\forall x \in \mathcal{A}\ |x| < \kappa$. Then there is $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| = \theta$ and $\mathcal{B}$ forms a Δ-system.*

We will also explicitly use the following forcing facts:

Fact 1.3. *Assume $\mathbb{P}$ is a forcing notion. If $\mathbb{P}$ is either κ-closed or κ-cc then forcing with $\mathbb{P}$ preserves stationary subsets of κ.*

Fact 1.4 (Product forcing)**.** *Suppose $\mathbb{P}_0$ and $\mathbb{P}_1$ are forcing notions, $G_0 \subseteq \mathbb{P}_0$ and $G_1 \subseteq \mathbb{P}_1$. Then the following are equivalent:*

1. *$G_0 \times G_1$ is $\mathbb{P}_1 \times \mathbb{P}_2$-generic over V,*

2. *G_0 is $\mathbb{P}_0$-generic over V and G_1 is $\mathbb{P}_1$-generic over $V[G_0]$,*

3. *G_1 is $\mathbb{P}_1$-generic over V and G_0 is $\mathbb{P}_0$-generic over $V[G_1]$.*

Furthermore, if these conditions hold, then $V[G_0 \times G_1] = V[G_0][G_1] = V[G_1][G_0]$.

See §1 of Chapter VIII of [Kun80] for the proof.

Fact 1.5. *Assume κ is regular uncountable cardinal, $\mathbb{P}$ is a forcing notion in V and $\mathbb{Q}$ is a forcing notion in $V[\mathbb{P}]$. Then the following are equivalent:*

1. *$\mathbb{P}$ is κ-cc and $\Vdash_{\mathbb{P}} \mathbb{Q}$ is κ-cc.*

2. *$\mathbb{P} * \dot{\mathbb{Q}}$ is κ-cc.*

See Theorem 2.1 of [Bau83] for the proof.

Fact 1.6. *Assume κ is Mahlo cardinal and $\mathbb{P}_\kappa$ is an iteration of length κ. Suppose $|\mathbb{P}_\beta| < \kappa$ for each $\beta < \kappa$ and $\mathbb{P}_\beta$ is a direct limit of $\{\mathbb{P}_\gamma \mid \gamma < \beta\}$ whenever $\beta \leq \kappa$ and β is inaccessible. Then $\mathbb{P}_\kappa$ is κ-cc.*

See Corollary 2.4 of [Bau83] for the proof.

Regarding large cardinals, we assume the reader knows the most famous large cardinal types like inaccessible, weakly compact and measurable. Nevertheless, we give a brief introduction in Section 3. Larger part of this introduction is based on sections of Kanamori's book [Kan08].

Apart from forcing and large cardinal background, we assume some basic knowledge from model theory regarding elementary substructures.

Fact 1.7. *Let $\mathfrak{A}$ be a structure and $\langle \mathfrak{A}_n \mid n < \omega \rangle$ a chain of elementary substructures of $\mathfrak{A}$ (i.e. $\mathfrak{A}_n \prec \mathfrak{A}_{n+1}$ and $\mathfrak{A}_n \prec \mathfrak{A}$). Then the limit of the chain $\langle \mathfrak{A}_n \mid n < \omega \rangle$ is also an elementary substructure of $\mathfrak{A}$.*

Mostowski collapse lemma, an important tool in set theory, is used in a few places of the thesis.

Fact 1.8 (Mostowski collapse lemma)**.** *Suppose $\langle M, E \rangle$ is a (possibly proper class) structure with E a binary relation on M satisfying:*

1. *E is well-founded;*

2. $\langle M, E\rangle$ is extensional, i.e. if $a, b \in M$ and xEa iff xEb for every $x \in M$, then $a = b$;

3. E is set-like, i.e. $\{x \mid xEa\}$ is a set for every $a \in M$.

Then there is a unique isomorphism $\pi\colon \langle M, E\rangle \longrightarrow \langle \bar{M}, \in\rangle$ where $\bar{M}$ is transitive. $\langle \bar{M}, \in\rangle$ is the transitive collapse *of $\langle M, E\rangle$, π is the* collapsing isomorphism *and it is defined by recursion on E: $\pi(x) = \{\pi(y) \mid yEx\}$ for every $x \in M$.*

1.2 Historical Background

The continuum function is the function from $CARD$ to $CARD$ which assigns to each cardinal the size of its powerset (κ is assigned the value 2^κ). The very first question concerning the continuum function, Cantor's continuum hypothesis (CH), rose at the very beginning of set theory; in fact, in many respects it stimulated the birth of set theory. Cantor proved that there is no one-to-one correspondence between the real numbers and the natural numbers, so the set of real numbers has greater size than the set of natural numbers. He then tried to determine if there are any sets of reals with intermediate size. CH is a hypothesis that claims the answer to that question is negative. We can paraphrase it as follows: "There is no set with cardinality strictly between that of the natural numbers and that of the real numbers". In terms of the continuum function, it claims that $2^{\aleph_0} = \aleph_1$. CH was the first problem on the famous Hilbert's list of the mathematical problems for the 20^{th} century. It was later generalized to the generalized continuum hypothesis (GCH) that states that the continuum function does not jump on any infinite cardinal (not only the first one). In terms of the continuum function, it says that $\forall \alpha \in ORD$ $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. An overview of the history of GCH up to Gödel's consistency result can be found in [Moo11].

As it turned out, not much can be proved about the continuum function in ZFC, especially regarding regular cardinals. Only simple statements were proved about its behaviour on the whole domain. The first property the continuum function obviously has is monotonicity. It can be formulated as $\alpha < \beta \rightarrow 2^{\aleph_\alpha} \leq 2^{\aleph_\beta}$. The next property is irreflexivity. Cantor proved that for each set x, the powerset of x has strictly bigger size than x itself. This is known as Cantor's theorem and it restricts the continuum function in the sense that $\forall \alpha\ \aleph_\alpha < 2^{\aleph_\alpha}$. The last property proved follows from König's theorem; it says that $\forall \alpha\ \aleph_\alpha < cf(2^{\aleph_\alpha})$ and we refer to it as the König's inequality. Note that this also subsumes irreflexivity since

$\forall\alpha\ cf(\alpha) \leq \alpha$. These were the only properties set-theorists proved the continuum function has, although many tried to resolve CH one way or the other and this gave rise to many useful and interesting concepts in set theory. The first breakthrough in solving the question of CH was the Gödel's famous construction of the constructible universe L in 1938 [Göd38]. Gödel proved that L is an inner model of ZF and that AC holds in L. In addition, he showed that GCH holds in L, thus showing GCH is consistent with ZFC and resolving the first part of the question of CH. The solution to the second part had to wait another twenty-five years until Cohen's development of new independence technique, forcing, in 1963 [Coh63]. He proved that ¬CH is also consistent with ZFC. Moreover, he showed that any value of $2^{\aleph_0}$ greater than $\aleph_0$ not contradicting the König's inequality is consistent with ZFC.

Since Cohen's introduction of forcing, this technique has been reworked and has proven to be extremely powerful in development of independence results. Not long after Cohen's result, the question of the behaviour of the continuum function on regular cardinals has been completely resolved by Easton ([Eas70]). He showed that any possible behaviour of the continuum function on regular cardinals satisfying the monotonicity and respecting the König's inequality is consistent with ZFC. Moreover, he showed this in a constructive way. Given the desired values for regular cardinals and a model satisfying GCH we can produce a model where the continuum function has the desired values. This result is the main topic of Section 2.

After Easton's result, many believed it would be possible to generalize it also to singular cardinals. However, in 1974 Silver [Sil74], to the surprise of many including himself, proved that for a singular cardinal λ with uncountable cofinality, if $\forall\kappa < \lambda$ $2^\kappa = \kappa^+$ then $2^\lambda = \lambda^+$. Hence, the value of the continuum function on λ is restricted by what happens bellow λ. This can also be viewed as a kind of reflection. If GCH is violated at λ, it already had to be violated somewhere below λ. We shall see this kind of reflection again when dealing with measurable cardinals. There are more results concerning singular cardinals and the continuum function, but we shall focus on the regular cardinals.

When large cardinal are taken into account, the situation complicates a little. Easton forcing, used in the proof of Easton's theorem, can easily kill large cardinal; therefore, it is interesting to study what is necessary in order to manipulate the values of the continuum function while preserving large cardinals.

2 The continuum function on regular cardinals in ZFC

Recall that only basic restrictions on the behaviour of the continuum function have been proved in ZFC:
$\forall \alpha, \beta \in ORD$, $\alpha < \beta$

- $2^{\aleph_\alpha} \leq 2^{\aleph_\beta}$, (monotonicity)
- $\aleph_\alpha < cf(2^{\aleph_\alpha})$. (König's inequality)

Cohen proved these are the only restrictions ZFC puts on the value of $2^{\aleph_0}$. He developed forcing, a new independence technique, with which he was able to blow up the powerset of ω to any given size, as long as it satisfied the restrictions given above. His method can be directly generalized to manipulating the power set of finitely many regular cardinals where you start from the biggest cardinal and continue to the smallest one. The fact that you start with the biggest is essential; thus, to generalize the result for infinitely many cardinals, new technique was needed.

This was done several years later. Easton, building on Cohen's work, used product forcing to manipulate the power set of infinitely many cardinals "at once". In fact, he showed that this can be done for any set of regular cardinals and with some extra work even for proper class of regular cardinals. We will state this more formally later.

2.1 Easton's theorem

Here we formulate and prove Easton's theorem, the main independence result about the behaviour of the continuum function in ZFC. This topic is well described in [Kun80] and we follow closely §4 of Chapter VIII. Before we get to the main proof, we need to prepare a couple of lemmas and definitions. Note that we will try to manipulate the continuum function only on a *set* of regular cardinals. Although it can be done for a *proper class* of regular cardinals, and we discuss this briefly at the end of the section, it involves working with a forcing with proper class of conditions and we do not what to go into details of such construction.

The crucial point in Easton's theorem is the problem of preserving cardinals. This is solved by showing how certain forcing properties are preserved in the extensions.

Lemma 2.1 (Easton's lemma)**.** *Assume $\mathbb{P}, \mathbb{Q} \in V$ are forcing notions, $\mathbb{P}$ is κ-cc and $\mathbb{Q}$ is κ-closed. Then the following holds:*

(a) $1_{\mathbb{Q}} \Vdash \check{\mathbb{P}}$ *is* $\check{\kappa}$*-cc,*

(b) $1_{\mathbb{P}} \Vdash \check{\mathbb{Q}}$ *is* $\check{\kappa}$*-distributive.*

Proof For (a), assume for contradiction that in $M[\mathbb{Q}]$ there exists an antichain $A \subseteq \mathbb{P}$ of size κ. This means there is one-to-one function $f : \kappa \longrightarrow A$ enumerating elements of A, so $A = \{p_\alpha \mid \alpha < \kappa\}$. By forcing theorem, there is a condition $q \in \mathbb{Q}$, such that $q \Vdash \dot{A}$ is an antichain in $\check{\mathbb{P}}$ of size $\check{\kappa}$ and $\dot{f}$ enumerates its elements. We construct, by induction, a decreasing chain of conditions of size κ below q, such that

1. $q_0 = q$,
2. $q_\alpha \leq q_\beta$ for $\beta \leq \alpha$,
3. $q_{\alpha+1} \Vdash \dot{f}(\check{\alpha}) = \check{p}_\alpha$.

At successor steps, we have constructed the chain till q_α. Because q_α is bellow q, it forces that f will be a function from κ to A, especially $q_\alpha \Vdash (\exists x \in \dot{A})(\dot{f}(\check{\alpha}) = x)$. Consequently, there is some condition $q_{\alpha+1}$ bellow q_α and $p_\alpha \in A$ such that $q_{\alpha+1} \Vdash \dot{f}(\check{\alpha}) = \check{p}_\alpha$. At limit steps, just take the lower bound of the chain constructed so far. This bound is guaranteed to exist by the κ-closure.

Now we define $B = \left\{p_\alpha \mid \alpha < \kappa \;\&\; q_\alpha \Vdash \dot{f}(\check{\alpha}) = \check{p}_\alpha\right\}$. B is clearly in M and we claim it is an antichain of $\mathbb{P}$. If not, then there exist $p_\alpha, p_\beta \in B$ that are compatible. Denote $\gamma = \max(\alpha, \beta)$. Then $q_\gamma \Vdash \check{p}_\alpha, \check{p}_\beta \in \dot{A}$ and $\dot{A}$ is an antichain. Since p_α and p_β are compatible in the ground model, it is also forced by every condition $q \in \mathbb{Q}$, especially by q_γ, that they are compatible, but this is a contradiction. It follows that B is an antichain, all $p_\alpha \in B$ are different and B has size κ. Since B is an antichain of $\mathbb{P}$ in the ground model and has size κ, it contradicts the fact that $\mathbb{P}$ is κ-cc.

For (b), we need to show that every sequence of less than κ ordinals in $M[\mathbb{P}][\mathbb{Q}]$ is already in $M[\mathbb{P}]$. Suppose $f \in M[\mathbb{P}][\mathbb{Q}]$ and $f : \lambda \longrightarrow ORD$ where $\lambda < \kappa$. By Fact 1.4, $M[\mathbb{P}][\mathbb{Q}] = M[\mathbb{Q}][\mathbb{P}]$, so in $M[\mathbb{Q}]$ there is a name $\dot{f}$ for f. From (a), we know that $\mathbb{P}$ is κ-cc in $M[\mathbb{Q}]$. We now work in $M[\mathbb{Q}]$. Since $\mathbb{P}$ is κ-cc, for every $\alpha < \lambda$ there are only $<\kappa$ many candidates for the value of $f(\alpha)$. Formally, if $\forall \alpha < \lambda$ we denote $B_\alpha = \left\{\gamma \mid (\exists p \in \mathbb{P})\ (p \Vdash \dot{f}(\check{\alpha}) = \check{\gamma})\right\}$, then $|B| < \kappa$ for $B = \bigcup_{\alpha<\lambda} B_\alpha$. For every $\alpha < \kappa$ and $\gamma \in B_\alpha$ take p_α^γ to be some witness for γ belonging to B_α. Then $p_\alpha^\gamma \Vdash \dot{f}(\check{\alpha}) = \check{\gamma}$ and we can cook up a name for f that has size less than κ:

$$\sigma_f = \{\langle\langle\check{\alpha}, \check{\gamma}\rangle, p_\alpha^\gamma\rangle \mid \alpha < \lambda,\ \gamma \in B_\alpha\}.$$

In M we know that $\mathbb{Q}$ is κ-closed, so σ_f is already in M. Hence f, as the realization of σ_f, is already in $M[\mathbb{P}]$. □

We have showed Easton's lemma at the beginning, as it tells us something about preserving properties of forcing notions in general. Now we can proceed to the development of Easton forcing.

Definition 2.2. An *index function* is a function E, such that $dom(E)$ is a set of regular cardinals.
An *Easton index function* is an index function, such that

(a) $\forall \kappa \in dom(E)\ E(\kappa)$ is a cardinal and $cf(E(\kappa)) > \kappa$,

(b) $\forall \kappa, \kappa' \in dom(E)\ (\kappa < \kappa' \to E(\kappa) \leq E(\kappa'))$.

Note that the conditions in the definition of Easton index function corresponds to those we know ZFC lays on the continuum function.

Definition 2.3. For an index function E, we define *Easton forcing* $\mathbb{P}(E)$ as the set of functions p, such that

1. $dom(p) = dom(E)$,
2. $\forall \kappa \in dom(p)\ (p(\kappa) \in Add(\kappa, E(\kappa))$,
3. $\forall \lambda \in REG\ (|\{\kappa \in \lambda \cap dom(E) : p(\kappa) \neq \emptyset\}| < \lambda)$.

$\mathbb{P}(E)$ is ordered coordinate-wise: $p \leq p'$ iff $\forall \kappa \in dom(E)(p'(\kappa) \subseteq p(\kappa))$.

Notice that since the domain of E consists only of cardinals, condition 3 demands something nontrivial only in case λ is weakly inaccessible. Still, it is necessary for proving that Easton forcing has some nice properties.

Lemma 2.4. *If E is an index function, $dom(E) \subseteq \lambda^+$, $2^{<\lambda} = \lambda$, and λ is regular, then $\mathbb{P}(E)$ is λ^+-cc.*

The idea of the proof is the same as in proving chain condition of $Add(\kappa, \kappa')$.

Proof Consider the set $d(p) = \{\{\kappa\} \times dom(p(\kappa)) \mid \kappa \in dom(E)\}$ for each $p \in \mathbb{P}(E)$. By condition 3 of Definition 2.3, the size of $\{\kappa \mid \kappa \in dom(E) \cap \lambda\}$ is less than λ. It follows that also the size of $\{\{\kappa\} \times dom(p(\kappa)) \mid \kappa \in dom(E) \cap \lambda\}$ is less than λ. In addition, $p(\lambda) \in Add(\lambda, E(\lambda))$, so $|dom(p(\lambda))| < \lambda$. Thus, we obtain $|d(p)| < \lambda$.

Now suppose for contradiction there is an antichain A of size λ^+ in $\mathbb{P}(E)$. Observe that the set $\{d(p) \mid p \in A\}$ satisfies the assumptions of the Δ-system lemma (Fact 1.2). As a result, there is $B \subseteq A$ with $|B| = \lambda^+$, such that $\{d(p) \mid p \in B\}$ forms a Δ-system. Denote r the root of the system. As $|r| < \lambda$, $2^{|r|} \leq \lambda$. This means we can partition B into at most λ many sets, such that p_α, p_β are in the same set if and only if $(\forall \langle \kappa, i\rangle \in r)\ (p_\alpha(\kappa)(i) = p_\beta(\kappa)(i))$. Because of the regularity of λ^+, at least one of these sets has cardinality λ^+. However, all conditions in this set are compatible, contradicting the fact that A is an antichain. $\square$

Lemma 2.5. *If E is an index function and $dom(E) \cap \lambda^+ = \emptyset$, then $\mathbb{P}(E)$ is λ^+-closed.*

Proof We know that $Add(\kappa, \kappa')$ is κ-closed for κ regular. Since $dom(E) \cap \lambda^+ = \emptyset$, every factor of the product is λ^+-closed. A $<\lambda^+$-descending chain in $\mathbb{P}(E)$ gives us a $<\lambda^+$-descending chain in every factor. In each factor there is a lower bound for the descending chain and the sequence of these lower bounds is the lower bound witnessing the λ^+-closure of $\mathbb{P}(E)$. $\square$

We have showed that small Easton forcing $\mathbb{P}(E)$ (i.e. with $dom(E) \subseteq \kappa$) has, with some additional conditions, a good chain condition. On the other hand, big Easton forcing $\mathbb{P}(E)$ (i.e. with $dom(E) \cap \kappa = \emptyset$) has good closure. It is possible to take advantage of both approaches if we view Easton forcing as a product of two factors.

Definition 2.6. If E is an index function, then we define $E^{>\lambda} = E\restriction\{\kappa \mid \kappa > \lambda\}$ and $E^{\leq\lambda} = E\restriction\{\kappa \mid \kappa \leq \lambda\}$

Observation 2.7. *For any cardinal λ, $\mathbb{P}(E)$ is isomorphic to $\mathbb{P}(E^{>\lambda}) \times \mathbb{P}(E^{\leq\lambda})$.*

We now have everything we need to show that cardinals are preserved in the extension by Easton forcing, assuming we start with the right model (i.e. satisfying GCH).

Lemma 2.8. *If E is an index function in the ground model M and GCH holds in M, then forcing with $\mathbb{P} = \mathbb{P}(E)$ preserves cofinalities and hence cardinals.*

Proof Suppose $\mathbb{P}$ does not preserve cofinalities. Then there is a $\mathbb{P}$-generic filter G and some cardinal θ, which is regular in M, but singular in $M[G]$. Take $\lambda = cf(\theta)^{M[G]}$. In M we look at $\mathbb{P}$ as a product of two factors, $\mathbb{P} = \mathbb{P}_0 \times \mathbb{P}_1$ for $\mathbb{P}_0 = \mathbb{P}(E^{\leq\lambda})$ and $\mathbb{P}_1 = \mathbb{P}(E^{>\lambda})$. Then also G can be viewed as a product $G_0 \times G_1$ where $G_0 \subseteq \mathbb{P}_0$ and $G_1 \subseteq \mathbb{P}_1$. By Fact 1.4, G_0 is $\mathbb{P}_0$-generic over M (and also over $M[G_1]$) and G_1 is $\mathbb{P}_1$-generic over M (and also over $M[G_0]$). In addition, by Lemma 2.4, $\mathbb{P}_0$ is λ^+-cc in M because GCH holds in M and λ is regular. Also by Lemma 2.5, $\mathbb{P}_1$ is λ^+-closed in M.

Now take f to be the cofinal map from λ to θ in $M[G]$. Since $\mathbb{P}_1$ is λ^+-closed in M, $\mathbb{P}_0$ is λ^+-cc in $M[G_1]$ by Lemma 2.1. This means that $\forall\alpha < \lambda$ there is a set of candidates of cardinality $\leq\lambda$ for the value of $f(\alpha)$. Formally, there is a function $F : \lambda \to P(\theta)$ such that $(\forall\alpha < \lambda)$ $(f(\alpha) \in F(\alpha))$ and $|F(\alpha)| \leq \lambda$. F is already in M because $\mathbb{P}_1$ is λ^+-closed in M. Now consider $\bigcup_{\alpha<\lambda} F(\alpha)$. Not only is this a cofinal subset of θ, but it also is a union of λ many sets of cardinality at most λ. It follows that its cardinality is λ, contradicting the regularity of θ in M. □

Now we can finally state and prove how does the continuum function behave in the extension by Easton forcing.

Theorem 2.9 (Easton's theorem [Eas70]). *Suppose that in M, E is an Easton index function, $\mathbb{P} = \mathbb{P}(E)$ and GCH holds. If G is $\mathbb{P}$-generic over M, then in $M[G]$ it holds that $(\forall\kappa \in dom(E))(2^\kappa = E(\kappa))$.*

Moreover, if we denote $E'(\theta) = max(\theta^+, sup\{E(\delta) \mid \delta < \theta, \delta \in dom(E)\})$ and define

$$E^\star(\theta) = \begin{cases} E'(\theta) & \text{if } cf(E'(\theta)) > \theta, \\ E'(\theta)^+ & \text{otherwise,} \end{cases}$$

then in $M[G]$ it holds that

$$(\forall\theta \in CARD)(2^\theta = E^\star(\theta)).$$

Proof We showed in Lemma 2.8 that this forcing preserves cardinals, so the definition of $E^\star(\theta)$ is absolute for M and $M[G]$. We proceed to show the equality.

Firstly, we show that $\forall\theta$ $(2^\theta \geq E^\star(\theta))$. From the definition of $E^\star(\theta)$, it is obviously sufficient to prove that $2^\kappa \geq E(\kappa)$ for $\kappa \in dom(E)$. But this is easy, because we defined the forcing to do exactly that. Just notice that the factor $Add(\kappa, E(\kappa))$ forces $E(\kappa)$ new subsets of κ to be added: $\forall\alpha < E(\kappa)$ define

$$A_\alpha = \{\beta < \kappa \mid (\exists p \in G)(p(\kappa)(\langle\beta, \alpha\rangle) = 1)\}.$$

Exactly as in classic Cohen forcing, all A_α are new subsets of κ (i.e. different from any subset of κ in the ground model), and $\alpha \neq \beta \to A_\alpha \neq A_\beta$. So $(2^\kappa \geq E(\kappa))^{M[G]}$.

To show $(\forall\theta)(2^\theta \leq E^\star(\theta))$ we need to work a little more. Firstly, consider the special case when θ is regular. We once more view the forcing as a product. $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_0$, where $\mathbb{P}_1 = \mathbb{P}(E^{>\theta})$ and $\mathbb{P}_0 = \mathbb{P}(E^{\leq\theta})$. In M, we know that $\mathbb{P}_1$ is θ^+-closed and $\mathbb{P}_0$ is θ^+-cc. We can use the well-known technique of counting nice names. For that we need the size of $\mathbb{P}_0$: $\forall\kappa \leq \theta$ it holds that

$$|Add(\kappa, E(\kappa))| \leq |Add(\kappa, E^\star(\theta))| \leq E^\star(\theta)^\theta.$$

Since $\mathbb{P}_0$ is the product of such factors for $\kappa \in dom(E)$ and $\kappa < \theta$, it holds that $|\mathbb{P}_0| \leq (E^\star(\theta)^\theta)^\theta$. As $cf(E^\star(\theta)) > \theta$ and GCH holds, $E^\star(\theta)^\theta = E^\star(\theta)$ in M. So in M, $|\mathbb{P}_0| \leq E^\star(\theta)$.

Now let's review the situation in $M[G_1]$. Since $\mathbb{P}_1$ was θ^+-closed in M, it still holds in $M[G_1]$ that $E^\star(\theta)^\theta = E^\star(\theta)$ and consequently that $|\mathbb{P}_0| \leq E^\star(\theta)$. Also, by Lemma 2.1, $\mathbb{P}_0$ is θ^+-cc in $M[G_1]$. As a consequence, there are at most $E^\star(\theta)^\theta = E^\star(\theta)$ many antichains in $\mathbb{P}_0$ and at most $E^\star(\theta)^\theta = E^\star(\theta)$ nice names for the subsets of $\check{\theta}$. So we have $(2^\theta \leq E^\star(\theta))^{M[G]}$ for the special case, when θ is regular.

To handle the general case, we first show that $(E^\star(\theta)^\lambda = E^\star(\theta))^{M[G]}$ is true for any cardinal θ and $\lambda = cf(\theta)$. Consider any such $f \in {}^\lambda E^\star(\theta) \cap M[G]$. Since $(\mathbb{P}_0 \text{ is } \theta^+\text{-cc})^{M[G_1]}$, it is true in $M[G_1]$ that there are only $\leq\lambda$ many candidates for the value $\dot{f}(\check{\alpha})$ for each $\alpha < \lambda$. Thus, there exists such $F \in {}^{\lambda\times\lambda}E^\star(\theta) \cap M[G_1]$ that $(\forall\alpha < \lambda)(\exists\beta < \lambda)(f(\alpha) = F(\alpha, \beta))$ holds. Since $\mathbb{P}_1$ is λ^+-closed in M, F exists already in M. Moreover, $(E^\star(\theta)^\lambda = E^\star(\theta))^M$ because GCH holds in M. Therefore, there are only $E^\star(\theta)$ many such Fs in $M[G_1]$.

Obviously each f in ${}^\lambda E^\star(\theta) \cap M[G]$ needs such F as a witness. We have just counted the number of possible witnesses, the remaining question is how many fs can have the same witness. For a witness F denote

$$A_F = \left\{f \in {}^\lambda E^\star(\theta) \cap M[G] \mid (\forall\alpha < \lambda)(\exists\beta < \lambda)(f(\alpha) = F(\alpha, \beta))\right\}.$$

We can put an upper bound to the size of A_F, because there is a one-to-one function from A_F to ${}^\lambda\lambda$. It maps each $f \in A_F$ to the function that assigns to each ordinal α the "witnessing" β, i.e. the β for which it holds that $F(\alpha, \beta) = f(\alpha)$. Thus, each such function F can be witness for at most λ^λ many functions f. We have proved earlier that $\lambda^\lambda = 2^\lambda = E^\star(\lambda) \leq E^\star(\theta)$ holds in $M[G]$. Putting it all together, there are at most $E^\star(\theta)$ witnesses and each can witness for at most $E^\star(\theta)$ functions in ${}^\lambda E^\star(\theta) \cap M[G]$. This means that $(E^\star(\theta)^\lambda \leq E^\star(\theta) \times E^\star(\theta) = E^\star(\theta))^{M[G]}$.

Finally, we show that $2^\theta \leq E^\star(\theta)$ for θ singular in $M[G]$. Denote $\lambda = cf(\theta)$ and B the set of all bounded subsets of θ. We have already proved that for every regular $\delta < \theta$ $(|\mathcal{P}(\delta)| = E^\star(\delta) \leq E^\star(\theta))$ is true. Since $B = \bigcup\{\mathcal{P}(\delta) \mid \delta < \theta, \delta \in REG\}$, it holds that $|B| \leq \theta \times E^\star(\theta) = E^\star(\theta)$. In addition, there is an injection from $\mathcal{P}(\theta)$ to ${}^\lambda B$. If we fix f, a cofinal mapping from λ to θ, we can map $x \subseteq \theta$ to a function $g \in {}^\lambda B$, such that $g(\alpha) = x \cap f(\alpha)$ for each $\alpha < \lambda$. As a consequence, we obtain that $\mathcal{P}(\theta) \leq |{}^\lambda B| \leq E^\star(\theta)^\lambda$ and this is equal to $E^\star(\theta)$ as we have proved earlier.

We have shown that in the extension after forcing with Easton forcing defined from Easton index function E, it holds that $(\forall\theta \in CARD)(2^\theta = E^\star(\theta))$. □

Easton actually proved something more general. He proved it is possible to manipulate the value of the continuum function for a proper class of regular cardinals. Besides the fact we have to be more careful when dealing with proper classes, the main obstacle is that in this case, the Easton forcing $\mathbb{P}(E)$ would be a proper class and forcing with a class of conditions does not in general yields model of ZFC. Fortunately, it does in the Easton case and once this is shown, the rest is analogous to the set case. The details can be found in [Eas70].

Easton's theorem completely resolves the problem of the behaviour of the continuum function on regular cardinals in ZFC. If we comply with the restrictions given in (a) and (b) of Definition 2.2, then we are free to have any behaviour of the continuum function we want. For example the first jump of the continuum function can occur at any regular cardinal (recall this is not true for singular cardinals).

On the other hand, if one works in a stronger theory (i.e. ZFC + existence of some large cardinal), Easton forcing can easily kill large cardinal. In the rest of the thesis, we will examine the behaviour of the continuum function in the presence of large cardinals and focus on the possibility of manipulating the continuum function while preserving large cardinals.

3 Introduction to large cardinals

The notion of large cardinals is very interesting and plays an important role in modern set theory. Different types of large cardinals arose from different areas of mathematics, often when set-theorists tried to generalize properties known to be true for ω to some larger cardinals. Today a big hierarchy of large cardinals is known and the study of relations between various types of large cardinals in this hierarchy alone would make an interesting thesis. Therefore, we narrowed our selection to those types which are largely known and sufficient to demonstrate the various connections between the behaviour of the continuum function and large cardinals.

The smallest (weakest) large cardinal is weakly inaccessible cardinal and even the existence of this cardinal cannot be proved in ZFC (see Corollary 4.13, Chapter VI of [Kun80]). Since every large cardinal, to deserve to be called large, has to be at least weakly inaccessible, it follows that ZFC + existence of a large cardinal is a strictly stronger theory than ZFC.

3.1 Inaccessible and Mahlo cardinals

Definition 3.1. Cardinal κ is called *weakly inaccessible* if it is uncountable, regular and limit. Cardinal κ is called *(strongly) inaccessible* if it is uncountable, regular and $\forall\lambda < \kappa\ (2^\lambda < \kappa)$.

Notice that every inaccessible cardinal is also weakly inaccessible and if we removed the word uncountable, ω would satisfy both definitions. Inaccessible cardinal is an important notion because V_κ is a model of ZFC for κ inaccessible.

Mahlo cardinals are slightly stronger than inaccessible cardinals. The definition of Mahlo cardinal is based on another interesting and extremely useful notion, that of stationary sets (see Chapter 8 of [Jec03] for brief introduction of stationary sets).

Definition 3.2. Cardinal κ is called *(weakly) Mahlo*, if it is (weakly) inaccessible and the set of all regular cardinals bellow κ is stationary.

Since the set of all limit cardinals below limit cardinal λ is closed unbounded in λ and the intersection of closed unbounded and stationary subset is again stationary, it follows that the set of (weakly) inaccessible cardinals below (weakly) Mahlo cardinal κ is stationary in κ and thus κ is the κth (weakly) inaccessible cardinal. If GCH holds then notions of weakly inaccessible and inaccessible cardinal (and also the notions of weakly Mahlo and Mahlo) coincide.

3.2 Weakly compact cardinal

The next type of large cardinal we consider is weakly compact cardinal. The notion of weakly compact cardinal is particularly interesting because there are many equivalent definitions of this notion that came from different areas of set theory. The original definition can be tracked down to Tarski and involves generalization of the language of first order logic (see [Kan08], §4 of Chapter 1 for the brief history) In classic language, we have countable many variables, connectives of finite arity and quantification over finitely many variables. If we do not restrict ourselves to ω, we can consider infinitary languages $\mathfrak{L}_{\lambda,\theta}$ that has $max(\lambda, \theta)$ many variables, connectives of arity less than λ and quantification over less than θ variables.

We know that $\mathfrak{L}_{\omega,\omega}$ satisfies the Compactness Theorem. It is possible to generalize the statement to infinitary languages and we will see this leads to a large cardinal notion. We will work only with the weaker version.

Definition 3.3. We say $\mathfrak{L}_{\kappa,\kappa}$ satisfies the *Weak Compactness Theorem* if whenever Σ is a set of sentences in $\mathfrak{L}_{\kappa,\kappa}$ such that $|\Sigma| = \kappa$ and $\forall \Sigma' \subseteq \Sigma$ it holds that if $|\Sigma'| < \kappa$ then Σ' has a model, then Σ has a model.

Definition 3.4. Cardinal κ is called *weakly compact*, if it is inaccessible and the language $\mathfrak{L}_{\kappa,\kappa}$ satisfies the Weak Compactness Theorem.

We present the definition as formulated by Tarski. Satisfying the Weak Compactness Theorem alone does not imply inaccessibility, it needs to be added explicitly. This definition is adopted in [Jec03]. [Kan08] works with slightly altered definition, which does imply inaccessibility.

As we mentioned earlier, several other properties are equivalent to the weak compactness. We present a few we use in this thesis.

Fact 3.5. *The following definitions are equivalent for uncountable cardinal κ:*

(a) *κ is weakly compact.*

(b) *κ has the Extension property, i.e. for all $R \subseteq V_\kappa$ the structure $(V_\kappa, \in, R)$ has a transitive elementary extension $(M, \in, R')$ such that $\kappa \in M$.*

(c) *κ is inaccessible and has the tree property, i.e. each κ-tree has a cofinal branch.*

(d) *κ is Π^1_1 indescribable, i.e. for all $R \subseteq \kappa$ and σ a Π^1_1 sentence if $(V_\kappa, \in, R) \models \sigma$, then there exists $\alpha < \kappa$ such that $(V_\alpha, \in, R \cap V_\alpha) \models \sigma$.*

(e) For every transitive set M with $|M| = \kappa$, $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$ there is an elementary embedding $j : M \longrightarrow N$, where N is transitive, $|N| = \kappa$, ${}^{<\kappa}N \subseteq N$ and $cp(j) = \kappa$.

Characterization (b) is due to Keisler [Kei62] and the proof of (a) $\Leftrightarrow$ (b) can be found in [Kan08], Theorem 4.5. Characterization (d) is due to Hanf and Scott [HS61] and the proof of (d) $\Leftrightarrow$ (b) is in [Kan08], Theorem 6.4. Characterization (e) was presented by Hauser in [Hau91] where the proof of (d) $\Leftrightarrow$ (e) can be found. Finally, the proof of equivalence (a) $\Leftrightarrow$ (c) can be found in [Jec03], Theorem 17.13. We can see from this proof that to prove weak compactness, the tree property for a subset of all κ-trees is sufficient. We only need cofinal branches for the trees consisting of nodes of the form $t\colon \gamma \longrightarrow \{0,1\}$ for some $\gamma < \kappa$.

We show the proof of (a) $\Rightarrow$ (b) and use it to prove that L admits the existence of a weakly compact cardinal. Stated formally, we show that $Con(ZFC + \exists \text{ weakly compact cardinal}) \Rightarrow Con(ZFC + \exists \text{ weakly compact cardinal} + V = L)$.

Proof (a) $\Rightarrow$ (b)
We take for Σ the set of all the sentences that are true in the expanded structure $\mathfrak{V} = (V_\kappa, \in, R, x)_{x \in V_\kappa}$. These are all the formulas true about members of structure $(V_\kappa, \in, R)$ and they will guarantee that the model of Σ will be elementary extension of $(V_\kappa, \in, R)$. In addition, we put to Σ sentences about new constant c:

$$\{c \text{ is an ordinal}\} \cup \{c \neq \alpha \mid \alpha < \kappa\}\,.$$

Now, any $\Sigma' \subset \Sigma$ of size less than κ, obviously has a model, namely $\mathfrak{V}$ with additional constant c where c is interpreted as an ordinal bigger than all the ordinals mentioned in Σ'. By weak compactness, Σ has some model $\mathfrak{A} = (A, E, R', x)_{x \in V_\kappa}$. We can assume without loss of generality that $A \supseteq V_\kappa$, $E \cap V_\kappa = \in$, $R' \cap V_\kappa = R$. $\mathfrak{A}$ is obviously an elementary extension of V_κ and contains some ordinal $\geq \kappa$. To finish the proof, we need to show that this model is well-founded. Then by using Mostowski collapse lemma we obtain a transitive elementary extension with ordinal $\geq \kappa$, so it has to contain κ as well.

To verify well-foundness, observe that we have in Σ the sentence

$$\neg(\exists v_0 \ldots \exists v_n \ldots) \bigwedge_{n \in \omega} (v_{n+1} \in v_n),$$

so every model of Σ is well-founded. $\square$

Lemma 3.6. *Suppose κ is weakly compact cardinal, $A \subseteq \kappa$ and for every $\alpha < \kappa$ ($A \cap \alpha \in L$). Then A is in L.*

Proof Consider the sentence

$$\sigma = \forall\alpha\, \exists x\, (x \in L \,\&\, x = A \cap \alpha).$$

According to the assumption, σ is true in $(V_\kappa, \in, A)$. By the previous proof, there is a transitive elementary extension $(M, \in, A')$ of $(V_\kappa, \in, A)$ with $A' \cap \kappa = A$. σ is true in M by elementarity and $\kappa \in M$, so $\exists x\, (x \in L \,\&\, x = A' \cap \kappa)$ is true in M. Since $A' \cap \kappa = A$, we have $A \in L^M$ and consequently $A \in L$. □

This enables us to prove a nice result that the constructible universe admits the existence of a weakly compact cardinal.

Theorem 3.7. *If κ is weakly compact, then (κ is weakly compact)^L.*

Proof κ is obviously inaccessible in L because it is inaccessible in the universe. We prove it has the tree property in L.

Consider a κ-tree $(T, <_T)$ in L. Recall that without loss of generality we can assume that all nodes $t \in T$ are of the form $t : \gamma \longrightarrow \{0, 1\}$ for some $\gamma < \kappa$. We show that T has a cofinal branch in L. We know T has a cofinal branch B in V. Denote A the subset of κ with characteristic function $\bigcup B$. Notice that A satisfies the assumption of Lemma 3.6. For $\alpha < \kappa$ take a node $t \in B$ such that $dom(t) = \alpha$. For A_α take the subset of α with characteristic function t. Then $A_\alpha = A \cap \alpha$. Since $t \in L$, A_α is also in L.

It follows that $A \in L$ and also $B \in L$ because we can define B from A and T as $\{t \in T \mid \exists\alpha < \kappa\, (t \text{ is a characteristic function of } A \cap \alpha)\}$. As a result, κ has the tree property in L and (κ is weakly compact)^L. □

3.3 Measurable Cardinal

Measurable cardinal has a special place in the hierarchy of large cardinals because it is the first to give rise to a powerful tool, an elementary embedding of the universe. Nice and more detailed introduction to measurable cardinals, can be found in [Kan08], §5 of Chapter 1. We adopt the following definition of measurable cardinal:

Definition 3.8. An uncountable cardinal κ is *measurable* if and only if there is a non-principal κ-complete ultrafilter U over κ.

By non-principal we mean it is not generated by a single subset of κ. It follows that U is uniform. Sometimes we shall call U a *measure* on κ. With U a measure on κ, it is possible to construct ultrapower of the universe V by U over κ. We denote it Ult as all the parameters of the ultrapower construction should be obvious from the context. If we denote j the ultrapower embedding, then for every set x $j(x) = c_x$, where $c_x \colon \kappa \longrightarrow \{x\}$. It follows from Łoś's theorem that Ult is a class model of ZFC and j is an elementary embedding (see Theorem 5.2 of [Kan08] for the version of Łoś's theorem used in our situation). Moreover, since U is κ-complete (in particular σ-complete), Ult is well-founded and thus we can collapse it to a transitive class M using Mostowski collapse lemma. Hence M is an inner model of ZFC. Thus, we have obtained an elementary embedding $j \colon V \longrightarrow Ult \cong M$ for a measure U on κ. We will not make a distinction between Ult and M anymore.

An elementary embedding is *non-trivial* if $\exists \alpha\ j(\alpha) > \alpha$. For a non-trivial elementary embedding j, the critical point of j ($cp(j)$), denotes the least α such that $j(\alpha) > \alpha$. We show the elementary embedding associated with a measurable cardinal κ is non-trivial and its critical point is κ.

Lemma 3.9. *Suppose κ is a measurable cardinal with U a measure on κ and the corresponding elementary embedding $j \colon V \longrightarrow Ult \cong M$. Then $cp(j) = \kappa$.*

Proof First we show $j(\alpha) = \alpha$ for all $\alpha < \kappa$. Suppose this is not true. Take α to be the least ordinal such that $\alpha < j(\alpha)$. Take f to be the function for which the equivalence class $[f]$ is the ordinal α in Ult. So $Ult \vDash \alpha = [f] < j(\alpha) = c_\alpha$ This translates to $\{\beta < \kappa \mid f(\beta) < c_\alpha(\beta)\} = \{\beta < \kappa \mid f(\beta) < \alpha\} \in U$. But $\{\beta < \kappa \mid f(\beta) < \alpha\} = \bigcup_{\gamma<\alpha} \{\beta < \kappa \mid f(\beta) = \gamma\}$ and since $\alpha < \kappa$ and U is κ-complete, $\exists \gamma < \alpha\ \{\beta < \kappa \mid f(\beta) = \gamma\} = \{\beta < \kappa \mid f(\beta) = c_\gamma(\beta)\} \in U$. However, this means that $[f] = [c_\gamma] = j(\gamma) = \gamma < \alpha$, a contradiction.

Next we show that $j(\kappa) > \kappa$. In Ult consider the equivalence class of the identity function on κ, $[id]$. Obviously $\forall \alpha, \beta < \kappa\ (\alpha < \beta \rightarrow c_\alpha(\beta) < id(\beta))$ and $\forall \alpha < \kappa$ $\{\beta < \kappa \mid \beta > \alpha\} \in U$. Hence $Ult \vDash \forall \alpha < \kappa\ \alpha = j(\alpha) < [id]$ and $Ult \vDash \kappa \leq [id]$. On the other hand, $\forall \alpha < \kappa\ (id(\alpha) < c_\kappa(\alpha))$, so $Ult \vDash [id] < j(\kappa)$. Putting it together, we obtain $\kappa \leq [id] < j(\kappa)$.

We have proved that $\forall \alpha < \kappa\ \alpha = j(\alpha)$ and also $\kappa < j(\kappa)$; therefore, κ is the critical point of j. □

An interesting result directly following from Lemma 3.9 is that unlike weakly compact cardinal, a measurable cardinal cannot exist in L.

Corollary 3.10 (Scott [Sco61])**.** *If there is a measurable cardinal, then $V \neq L$.*

Proof Suppose $V = L$ and κ is the least measurable cardinal with the corresponding elementary embedding $j : V \longrightarrow M$. Then $L \subseteq M \subseteq V = L$, for M is an inner model. Since (κ is the least measurable cardinal)^L, by elementarity we get ($j(\kappa)$ is the least measurable cardinal)^M. From $M = L$ we have $\kappa = j(\kappa)$, contradicting the fact κ is the critical point of j. $\square$

Remark. This result was independently derived by Petr Vopěnka [Vop62],0 who later generalized this result together with his student Karel Hrbáček [VH66] to the theorem: "If there is a strongly compact cardinal, then $V \neq L(A)$ for any set A."

The converse of Lemma 3.9 is also true and it was proved by Keisler.

Fact 3.11 (Keisler)**.** *If there is an elementary embedding $j : V \longrightarrow M$ for some inner model M, then its critical point is a measurable cardinal.*

Proof We do not give the complete proof here, only the idea. If we denote $\kappa = cp(j)$ then we can define U by:

$$X \in U \text{ iff } X \subseteq \kappa \;\&\; \kappa \in j(X).$$

With a little work, it is possible to verify that $\kappa > \omega$ and U is κ-complete non-principal ultrafilter over κ. $\square$

Now we take a closer look at M and examine how closely it resembles V. We also put some boundaries to where j can map κ and derive some form of reflection for the measurable cardinal.

Lemma 3.12. *Suppose κ is a measurable cardinal with measure U and the corresponding elementary embedding $j \colon V \longrightarrow Ult \cong M$. Then the following holds:*

1. (a) $V_\kappa^M = V_\kappa$,

   (b) $\forall x \subseteq V_\kappa \; (j(x) \cap V_\kappa = x)$,

   (c) $V_{\kappa+1}^M = V_{\kappa+1}$,

   (d) $(\kappa^+)^M = \kappa^+$,

2. $2^\kappa \leq (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$.

Proof

1. (a) Since $\forall\alpha < \kappa\ j(\alpha) = \alpha$, it follows that $V_\alpha^M = V_\alpha$, and also $V_\kappa^M = V_\kappa$ because κ is limit.

   (b) Suppose $x \subseteq V_\kappa$. If $y \in x$, then $y \in V_\kappa$ and by elementarity we obtain $y = j(y) \in j(x)$. So $y \in j(x) \cap V_\kappa$ and $x \subseteq j(x) \cap V_\kappa$.

   Conversely, if $y \in j(x) \cap V_\kappa$ then $j(y) \in j(x)$ because $y = j(y)$. By elementarity of j, $y \in x$ and $j(x) \cap V_\kappa \subseteq x$.

   (c) Obviously $V_{\kappa+1}^M \subseteq V_{\kappa+1}$. For the converse, take any $x \in V_{\kappa+1}$. Then $x \subseteq V_\kappa$ and by (b), $j(x) \cap V_\kappa = x$. Using (a), we get $j(x) \cap V_\kappa^M = x$, so $x \subseteq V_\kappa^M$ and $x \in V_{\kappa+1}^M$.

   (d) For every ordinal α with $\kappa \leq \alpha < \kappa^+$, there is a well-ordering $<_\alpha$ of κ such that $(\kappa, <_\alpha) \cong (\alpha, \in)$. As every well-ordering of κ is a subset of $\kappa \times \kappa$, it can be coded as a subset of κ. Since each subset of κ is a member of $V_{\kappa+1}$, it follows by (c) that every well-ordering $<_\alpha$ is in M. Hence, $(\kappa^+)^M$ cannot be anything less than κ^+.

2. By 1.(c) $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)$. Since $M \subseteq V$, $2^\kappa \leq (2^\kappa)^M$ follows because every bijection between $\mathcal{P}(\kappa)$ and some ordinal in M is also in V (but not necessarily the other way around).

   The next inequality $(2^\kappa)^M < j(\kappa)$ holds, because $j(\kappa) > \kappa$ and $j(\kappa)$ is, by elementarity of j, inaccessible (even measurable) in M.

   For the final inequality, it is enough to show that $j(\kappa) = \{[f] \mid f \in {}^\kappa\kappa\}$. We know $j(\kappa) = [c_\kappa]$, where c_κ is the constant function that assigns κ to each ordinal $\alpha < \kappa$. If $g : \kappa \longrightarrow \kappa$ then obviously from Łoś's theorem we have $Ult \vDash [g] \in [c_\kappa]$. For the converse, suppose $h\colon \kappa \longrightarrow V$ and $Ult \vDash [h] \in [c_\kappa]$. If we define $A = \{\alpha < \kappa \mid h(\alpha) \in c_\kappa(\alpha) = \kappa\}$, then A is in the ultrafilter. Now define h' as

   $$h'(\alpha) = \begin{cases} h(\alpha) & \text{if } \alpha \in A, \\ 0 & \text{otherwise.} \end{cases}$$

   Obviously $h' : \kappa \longrightarrow \kappa$ and in addition $[h'] = [h]$. So $[h] \in \{[f] \mid f \in {}^\kappa\kappa\}$. Thus, $j(\kappa) = \{[f] \mid f \in {}^\kappa\kappa\}$ and the inequality $j(\kappa) < (2^\kappa)^+$ follows because $|\{[f] \mid f \in {}^\kappa\kappa\}| \leq 2^\kappa$.

□

We have showed that j maps κ to some ordinal strictly between 2^κ and $(2^\kappa)^+$. Next, we examine the reflection of measurable cardinals.

Corollary 3.13. *If κ is measurable and $\forall\lambda < \kappa\ 2^\lambda = \lambda^+$ then $2^\kappa = \kappa^+$.*

Proof Once again, consider $j\colon V \longrightarrow M$ the corresponding elementary embedding. By elementarity of j, $M \vDash \forall\lambda < j(\kappa)\ 2^\lambda = \lambda^+$. This means that $(2^\kappa)^M = (\kappa^+)^M$ because $\kappa < j(\kappa)$. If we combine results 1.(d) and 2. of Lemma 3.12 we obtain that $2^\kappa \leq (2^\kappa)^M = (\kappa^+)^M = \kappa^+$. □

One way to see this result is that measurable cardinal directly puts a restriction on the continuum function; measurable cardinal cannot be the first point where the continuum function jumps.

Observation 3.14. *Notice that Corollary 3.13 can be generalized also to double successor and actually to nth successor for any $n < \omega$, but only in a weaker form. Suppose $1 < n < \omega$. If $\forall\lambda < \kappa\ 2^\lambda \leq \lambda^{+n}$ then $M \vDash 2^\kappa \leq \kappa^{+n}$. Exactly as before, we get $2^\kappa \leq (\kappa^{+n})^M$. Although we do not have $(\kappa^{+n})^M = \kappa^{+n}$ for $n > 1$, we at least have $(\kappa^{+n})^M \leq \kappa^{+n}$ because $M \subseteq V$.*

The assumption in Corollary 3.13 can be weaken a little. It can be proved that the sufficient (though not necessary) condition for GCH to hold on a measurable κ is that in the ultrafilter there is a set of points under κ where GCH holds. However, in this case the ultrafilter has to have an additional property. This property is called normality and leads to the very useful notion of *normal filter*. It can be proved there is a normal ultrafilter on a measurable cardinal and we use this to a more detailed analysis of the ultrapower.

Definition 3.15. Filter F over λ is *normal* iff the diagonal intersection of less than λ many sets from F is also in F. Formally, if $\langle X_\alpha \mid \alpha < \lambda\rangle \in {}^\lambda F$ then $\triangle_{\alpha<\lambda} X_\alpha = \left\{\xi < \lambda \mid \xi \in \bigcap_{\alpha<\xi} X_\alpha\right\} \in F$.

It is easy to prove that if there is a normal filter on λ, then λ is uncountable and regular. In addition, normality subsumes λ-completeness. Normal filters have many good properties that found usages in many areas. For us it is important that normal ultrafilter exists on a measurable cardinal and we can say more about the ultrapower associated with the normal ultrafilter.

Fact 3.16. *If κ is measurable, then*

- *there exists a normal ultrafilter on κ,*
- *κ-complete ultrafilter U over κ is normal iff $[id]_U = \kappa$ in Ult.*

Recall that $[id]_U \geq \kappa$ holds in Ult for any κ-complete ultrafilter. The equality enables us to say something more about the target of the elementary embedding and the reflective properties of κ.

Lemma 3.17. *Assume κ is a measurable cardinal, U is a normal ultrafilter over κ and $j : V \longrightarrow M$ is the corresponding elementary embedding. Then every member of M is of the form $j(f)(\kappa)$ for some f, a function with domain κ. Formally, $M = \{j(f)(\kappa) \mid f : \kappa \longrightarrow V\}$.*

Proof For each $x \in M$, $x = [f]$ for some $f : \kappa \longrightarrow V$ by the definition of the ultrapower. We also know that $j(f) = [c_f]$ and, by normality, $\kappa = [id]$. It follows that $j(f)(\kappa) = [c_f]([id])$. If we translate the equality $[c_f]([id]) = [f]$, we obtain the statement $\{\alpha < \kappa \mid c_f(\alpha)(id(\alpha)) = f(\alpha)\} \in U$, i.e. $\{\alpha < \kappa \mid f(\alpha) = f(\alpha)\} \in U$ and this is obviously true. □

Lemma 3.18. *Assume κ is measurable and U is a normal ultrafilter over κ. If $\left\{\lambda < \kappa \mid 2^\lambda = \lambda^+\right\} \in U$ then $2^\kappa = \kappa^+$.*

Proof $\left\{\lambda < \kappa \mid 2^\lambda = \lambda^+\right\} \in U$ implies GCH holds at $[id]$ in Ult. Since U is normal, $[id] = \kappa$ and we have $Ult \vDash 2^\kappa = \kappa^+$. $V \vDash 2^\kappa = \kappa^+$ now follows exactly as in Corollary 3.13. □

Observation 3.19. *Exactly as in Observation 3.14, this can be generalized in the weaker form for nth successor. Assume $1 < n < \omega$. If $\left\{\lambda < \kappa \mid 2^\lambda \leq \lambda^{+n}\right\} \in U$ then $2^\kappa \leq \kappa^{+n}$.*

Note that Lemma 3.18 gives us one more thing. If GCH is violated on κ, then it is violated on a set of measure one (otherwise the set where it holds would have measure one). On the other hand, if GCH is violated on a set of measure one, it is consistent that GCH still holds on κ, so the situation is not symmetric. We shall return to this issue again in Section 4.

3.4 Indescribable cardinals

Here we present indescribable cardinals, presumably a lesser-known type of large cardinals. We do not go into much details, we mention them because they present an intermediate stage between weakly compact and measurable cardinal and it is interesting to compare their properties with the well-known types of large cardinals. We have already mentioned in Fact 3.5 that weakly compact cardinal can be equivalently characterized via indescribability. This concept can be generalized and leads to another large cardinal notion.

This introduction to indescribable cardinals is taken from [Kan08], §6 of Chapter 1. The relation of indescribable cardinals and the continuum function was examined by Hauser in his Ph.D. thesis [Hau89] with main results also appearing in [Hau91].

Indescribability is connected to the reflection principle in ZFC, which says that each formula of language of set theory true in the universe is already true at some stage V_α of the cumulative hierarchy of sets. Hanf and Scott noticed that one arrives at large cardinal notions if formulas of higher order language are considered.

Remark. By higher order language, we mean it contains variables and quantifiers of type more than 1. We say quantifier is of type m if it quantifies over variables of type m. Variables of higher types can be defined inductively. Variables of type 1 are those ranging over members of the domain $\mathcal{D}$ of a structure, variables of type 2 range over subsets of $\mathcal{D}$, i.e. over $\mathcal{P}(\mathcal{D})$ and inductively variables of type m range over members of $\mathcal{P}^{m-1}(\mathcal{D})$ ($\mathcal{P}^i$ means the powerset operation applied i times).

Definition 3.20. Suppose Q is a class of formulas in higher order language of set theory and extended by some unary predicate. We say ordinal α is Q *indescribable* if for every formula ϕ in Q and every $A \subseteq V_\alpha$ the following holds:

$$\langle V_\alpha, \in, A\rangle \vDash \phi(A) \implies \exists \beta < \alpha \ \langle V_\beta, \in, A \cap V_\beta\rangle \vDash \phi(A \cap V_\beta).$$

The large cardinal notion appears if we take for Q some level of the generalized hierarchy of formulas.

Definition 3.21. We say formula ϕ is Π^m_n (Σ^m_n) if its prenex normal form starts with n alternating blocks of quantifiers of type $m+1$ starting with universal (existential) quantifier, and does not contain any quantifiers of higher type.

This is especially interesting when considering the levels of the cumulative hierarchy of sets as structures $\langle V_\alpha, \in\rangle$. Since $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, V_α can use variables of types more than 1 to speak above sets located in higher levels of the hierarchy.

Above all Π^m_n indescribable cardinals we can consider one more type.

Definition 3.22. We say α is *totally indescribable* if α is Π^m_n indescribable for all $m, n \in \omega$.

Hanf and Scott [HS61] showed that in ZFC Π^1_0 indescribable cardinals are exactly inaccessible cardinals and Π^1_1 are exactly weakly compact cardinals. Moreover, they showed that measurable cardinals are Π^2_1 indescribable. However, measurable cardinal is characterized by an existence of ultrafilter and this can be express in V_κ as $\exists U\phi$ where U is a third order variable and ϕ is a second order formula; thus, measurable cardinal is actually Σ^2_1 describable. On the other hand, a theorem of Vaught (see [Kan08]) states that there are many totally indescribable cardinals below measurable. To be precise, for κ measurable and U a normal measure on κ, the set of all totally indescribable cardinals below κ has measure one.

As a consequence, we obtain that totally indescribable cardinal is strictly weaker in consistency strength than measurable. This also follows from another property of indescribable cardinals; they are consistent with $V = L$.

Theorem 3.23. *Suppose Q is either Π^m_n or Σ^m_n and either $m > 1$ or else $m = 1$ and $n > 0$. Then if κ is Q indescribable then (κ is Q indescribable)^L.*

See [Kan08] for the proof. This generalizes the result for weakly compact cardinal we showed in Theorem 3.7.

Indescribable cardinals can also be defined in terms of elementary embedding using set models of ZFC^-.

Theorem 3.24 ([Hau91])**.** *($m, n \geq 1$). An inaccessible cardinal κ is Π^m_n indescribable iff $\forall M$(M is a transitive model of ZFC^-, $|M| = \kappa$, $\kappa \in M$, ${}^{<\kappa}M \subseteq M \implies \exists j, N$($N$ is transitive, $|N| = |V_{\kappa+m-1}|$, N is Σ^m_{n-1}-correct for κ and $j : M \longrightarrow N$ such that $cp(j) = \kappa$)).*

Here N is Σ^m_{n-1}-correct for κ if ${}^{|V_{\kappa+m-2}|}N \subseteq N$ (${}^{<\kappa}N \subseteq N$ for $m = 1$), and N correctly computes Σ^m_{n-1} facts over V_κ that hold in parameters from $N \cap V_{\kappa+m}$.

Hauser used his characterization via elementary embeddings to obtain some consistency results concerning indescribable cardinals. For example he studied consistency strength of failure of GCH at indescribable cardinal and we shall return to it in Section 4.

He also showed a surprising consistency result concerning the order of the least Π_n^m and Σ_n^m cardinals.

Fact 3.25. *Denote σ_n^m and π_n^m the least Σ_n^m and Π_n^m cardinal, respectively (assuming, of course, such cardinals exist). Then the following holds ([HS61]):*

1. *κ is Σ_1^1 indescribable iff κ is inaccessible; especially σ_1^1 is the least inaccessible cardinal.*
2. *For any $n \in \omega$, κ is Σ_{n+1}^1 indescribable iff κ is Π_n^1 indescribable.*
3. *For $n > 0$ $\pi_n^1 = \sigma_{n+1}^1 < \pi_{n+1}^1$.*
4. *For $m > 1$ and $n > 0$, $\pi_n^m \neq \sigma_n^m$ and $\pi_n^m < \sigma_{n+1}^m, \pi_{n+1}^m$.*

For the relation of π_n^m and σ_n^m, two different consistency results are known.

- *(Moschovakis [Mos76]) If $V = L$, $m > 1$ and $n > 0$ then $\sigma_n^m < \pi_n^m$.*
- *(Hauser [Hau91]) If $m > 1$ and $n > 0$ and there exists a Σ_n^m indescribable cardinal with Π_n^m indescribable below it, then there exists a generic extension where $\sigma_n^m > \pi_n^m$.*

Indescribable cardinals have by definition some kind of reflection, but this reflection differs from the one of a measurable cardinal. In Section 4 we examine if this reflection also influences the behaviour of the continuum function.

4 The continuum function and large cardinals

In this section we study how the continuum function and large cardinals interact. We show some examples of manipulating the continuum function while preserving large cardinals. The key ingredient in such examples is using characterizations of large cardinals in terms of elementary embedding and lifting the embedding to the extended models.

In the case of weakly compact and indescribable cardinals we deal with set models of ZFC^-, in the case of measurable cardinal we deal with inner models. In both cases, closure of the models on long enough sequences is crucial. For this reason, we present several claims that tell us not only when the closure of a model is preserved (when moving to generic extension), but also how to build set models of ZFC^- with sufficient closure. In addition, we need to know how to construct generic filters over such models.

4.1 Preliminaries

Lemma 4.1. *Let κ be an inaccessible cardinal, X a set of size at most κ. Then there exists a set M with $X \subseteq M$, $|M| = \kappa$, $\kappa \in M$, ${}^{<\kappa}M \subseteq M$ and M is a model of ZFC^-.*

Proof We extend the standard construction of Skolem Hull in Löwenheim-Skolem theorem to get elementary substructure of suitable model of ZFC^- that satisfies all the conditions of the lemma. The plan is to take a suitable model of ZFC^-, take its elementary substructure of size κ that contains κ and is a superset of X, and finally iterate to get the desired closure.

We start with $H(\lambda^+)$ for some $\lambda \geq \kappa$ such that $X \in H(\lambda^+)$. Denote it $\mathfrak{A}$. Also denote the Skolem Hull of set Y in $\mathfrak{A}$ as $SH_{\mathfrak{A}}(Y)$. Note that if you start with $M_0^\star = \kappa + 1 \cup X$ and set $M_0 = SH_{\mathfrak{A}}(M_0^\star)$, than M_0 already satisfies all the conditions except perhaps the closure on $<\kappa$-sequences. To ensure this, we build up a chain of elementary substructures of length κ:

- Start with $M_0^\star$ and M_0 as defined above.
- At step $\beta = \alpha + 1$ take $M_\beta^\star = {}^{<\kappa}M_\alpha \cup M_\alpha$ and $M_\beta = SH_{\mathfrak{A}}(M_\beta^\star)$
- At limit step, take M_γ to be the limit of the chain of elementary substructures $\{M_\alpha \mid \alpha < \gamma\}$. This is again an elementary substructure of $\mathfrak{A}$ (Fact 1.7).

Finally, set $M = M_\kappa$ to be the limit of the chain $\{M_\alpha \mid \alpha < \kappa\}$ of elementary structures of $\mathfrak{A}$. We verify all the conditions of the lemma.

- M is a model of ZFC^-. This holds, because M is an elementary substructure of $\mathfrak{A}$.
- M has size κ. It is easy to prove by induction that each step M_α has size κ because κ is inaccessible. It follows M_κ has size κ.
- $\kappa \in M$ and $X \subseteq M$ as this already holds for M_0.
- M is closed under $<\kappa$-sequences. This is the reason for which we had to build the chain of length κ. Suppose for $\beta < \kappa$ $S = \langle m_\alpha \mid \alpha < \beta, m_\alpha \in M\rangle$ is a sequence of elements from M of length less than κ. Since κ is regular limit, there is $\gamma < \kappa$ such that all members of S are already in M_γ. But in the next step we closed M_γ under $<\kappa$-sequences, so $S \in M_{\gamma+1}$ and thus $S \in M$.

□

Lemma 4.2. *Suppose M and N are inner models of ZFC with $M \subseteq N$, $\mathbb{P} \in M$, $N \models {}^{<\kappa}M \subseteq M$ and $N \models \mathbb{P}$ is κ-cc. If G is P-generic over N then $N[G] \models {}^{<\kappa}M[G] \subseteq M[G]$.*

Proof Suppose $\beta < \kappa$ and $S = \langle x_\alpha \mid \alpha < \beta\rangle$, where $S \in N[G]$ and $X \subseteq M[G]$ where $X = \{x_\alpha \mid \alpha < \beta\}$. We show S has a name in N with size less than κ.

Intuitively, since $\mathbb{P}$ is κ-cc, for each $\alpha < \beta$ there are less than κ candidates for the value of $S(\alpha)$ in N.

Formally, there is a condition $p \in G$ and an N-name $\dot{S}$ for S such that $p \Vdash \dot{S}$ is a function from $\check{\beta}$ to $\dot{X}$. Now for each $\alpha < \beta$ denote $B_\alpha = \left\{q \leq p \mid \exists x\; q \Vdash \dot{S}(\check{\alpha}) = \dot{x}\right\}$ and take A_α to be a maximal antichain in B_α, i.e. $A_\alpha \subseteq B_\alpha$ is an antichain but no C with $A_\alpha \subsetneq C \subseteq B_\alpha$ is an antichain. By κ-cc, $|A_\alpha| < \kappa$. Since κ is regular, $|A| < \kappa$ for $A = \bigcup_{\alpha<\beta} A_\alpha$. Now define $S' = \left\{\langle\langle\check{\alpha}, \dot{x}\rangle, q\rangle \mid q \in A_\alpha \text{ and } q \Vdash \dot{S}(\check{\alpha}) = \dot{x}\right\}$, where for the simplicity we take $\langle\check{\alpha}, \dot{x}\rangle$ to be the name for the ordered pair $\langle\alpha, x\rangle$. But from $|A| < \kappa$ we immediately obtain $|S'| < \kappa$.

By our assumption, M is closed under $<\kappa$ sequences, so $S' \in M$. It is obvious from the way we constructed S' that $i_G(S') = i_G(\dot{S}) = S$. It follows that $S \in M[G]$.

□

Lemma 4.3. *Suppose M and N are inner models with $M \subseteq N$. Let $N \vDash \kappa$ is regular uncountable cardinal. Then $N \vDash {}^{<\kappa}M \subseteq M$ iff $N \vDash {}^{<\kappa}On \subseteq M$.*

Proof Implication from left to right is trivial. For the converse, suppose $\alpha < \kappa$ and $S = \langle x_\beta \mid \beta < \alpha \rangle$ is a sequence of M-elements that lives in N. We wish to show it is already in M. We know there exists some γ such that $\forall \beta < \alpha \; x_\beta \in V_\gamma^M$. Also AC holds in M and so $M \models \exists f : V_\gamma^M \longrightarrow \bar{\gamma}$ a bijection between V_γ^M and some ordinal $\bar{\gamma}$. In N denote $S' = \langle f(S(\beta)) \mid \beta < \alpha \rangle$. This is a sequence of ordinals of length less than κ and therefore a member of M. Now $S \in M$ follows from $S' \in M$ and $f \in M$ as $S = \langle f^{-1}(S'(\beta)) \mid \beta < \alpha \rangle$. □

Lemma 4.4. *Suppose M and N are inner models with $M \subseteq N$, $\mathbb{P} \in M$ is a forcing notion, $N \models \mathbb{P}$ is κ-closed and $N \models {}^{<\kappa}M \subseteq M$. Then if G is a $\mathbb{P}$-generic over N then $N[G] \models {}^{<\kappa}M[G] \subseteq M[G]$.*

Proof We first show the $N[G] \models {}^{<\kappa}On \subseteq M[G]$. Suppose $\alpha < \kappa$ and $f \in N[G]$, $f : \alpha \longrightarrow On$. In few steps we conclude that $f \in M[G]$:

1. $f \in N[G]$ & $\mathbb{P}$ is κ-closed $\Rightarrow f \in N$,
2. $f : \alpha \longrightarrow On \in N$ & $N \models {}^{<\kappa}M \subseteq M \Rightarrow f \in M$,
3. $f \in M \Rightarrow f \in M[G]$.

$N[G] \models {}^{<\kappa}M[G] \subseteq M[G]$ now follows from Lemma 4.3. □

Lemma 4.5. *Suppose M and N are inner models of ZFC with $M \subseteq N$, $\mathbb{P} \in M$ and $N \models {}^{<\kappa}M \subseteq M$. If $M \models \mathbb{P}$ is κ-closed, then $N \models \mathbb{P}$ is κ-closed.*

Proof Suppose $\beta < \kappa$ and $\langle p_\alpha \in \mathbb{P} \mid \alpha < \beta \rangle$ is a sequence of conditions in N. By assumptions, this sequence is also in M. $\mathbb{P}$ is κ-closed in M, so there is a lower bound p_β. But then p_β is also a lower bound in N. □

Remark. Actually, M and N does not have to be inner models in the aforementioned claims. The proofs can easily be reworked for the case of set models of ZFC^-.

Lemma 4.6. *Suppose M is a transitive model of ZFC^- such that $|M| = \kappa$ and ${}^{<\kappa}M \subseteq M$. If $\mathbb{P} \in M$ is a forcing notion which is κ-closed in M, then in V there is $\mathbb{P}$-generic filter G over M. In addition, if $p \in \mathbb{P}$ then there exists such generic filter G with $p \in G$.*

Proof This is a simple generalization of the construction in the Rasiowa-Sikorski theorem for countable transitive models. Take $\langle D_\alpha \mid \alpha < \kappa \rangle$ to be an enumeration of all dense subsets of $\mathbb{P}$ that lie in M. This is possible since $|M| = \kappa$, so there are at most κ many of them. We proceed to build a decreasing sequence of conditions of length κ such that $\forall \alpha < \kappa \ (p_{\alpha+1} \in D_\alpha)$.

- Start with $p_0 = p$.
- For $\alpha = \beta + 1$, D_β is a dense subset, so there is a condition q in D_β such that $q \leq p_\beta$. Set $p_\alpha = q$.
- For α limit, take the sequence $\langle p_\beta \mid \beta < \alpha \rangle$. According to assumptions, this sequence is in M. We can use κ-closure of $\mathbb{P}$ in M and obtain a condition r such that $r \leq p_\beta$ for all $\beta < \alpha$. Set $p_\alpha = r$.

Note that the resulting sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$ in not in M, except in trivial cases of $\mathbb{P}$. Finally take $G = \{p \in \mathbb{P} \mid \exists \alpha < \kappa \, p_\alpha \leq p\}$. It is easy to verify that G is a $\mathbb{P}$-generic filter over M. □

This construction can be done also for inner models, if we know the generic filter does not have to meet too many things.

Lemma 4.7. *Suppose M is an inner model and $\mathbb{P} \in M$ is a forcing notion. If, in V, $\mathbb{P}$ is κ-closed and $\mathbb{P}$ has at most κ many maximal antichains that lies in M, then we can build in V a filter G that is $\mathbb{P}$-generic over M. In addition, if $p \in \mathbb{P}$ then there exists such generic filter G with $p \in G$.*

Proof The proof is practically the same as in Lemma 4.6. Let $\langle A_\alpha \mid \alpha < \kappa \rangle$ be an enumeration of the maximal antichains of $\mathbb{P}$ that are in M. We can again build a decreasing sequence of conditions of length κ such that $p_{\alpha+1}$ extends some member of A_α. Start with condition p. In step $\alpha + 1$, there exists $a \in A_\alpha$, such that a and p_α are compatible. Take $a_{\alpha+1}$ to be their common extension. At limit steps, just take the lower bound of the sequence built so far. The lower bound is guaranteed to exists by the κ-closure. Finally, let $G = \{q \in \mathbb{P} \mid \exists \alpha \, p_\alpha \leq q\}$. It is obvious G is $\mathbb{P}$-generic over M and $p \in G$. □

When manipulating the values of the continuum function on large cardinals, lifting of an elementary embedding from the ground model to the extension is necessary. Silver came up with a simple condition that is sufficient for the lifting to be possible.

Lemma 4.8 (Silver's lifting lemma)**.** *Let $j : M \longrightarrow N$ be an elementary embedding between transitive models of ZFC. Let $\mathbb{P} \in M$ be a notion of forcing and G a $\mathbb{P}$-generic filter over M. Let H be $j(\mathbb{P})$-generic over N. Then the following are equivalent:*

1. *$\forall p \in G\ j(p) \in H$.*
2. *There exists an elementary embedding $j^+ : M[G] \longrightarrow N[H]$, such that $j^+ \restriction M = j$ and $j^+(G) = H$.*

Proof Obviously 2. $\Rightarrow$ 1. For the converse, define $j^+(i_G(\dot{\sigma})) = i_H(j(\dot{\sigma}))$. It is an easy exercise to check that j^+ is indeed well-defined elementary embedding from $M[G]$ to $N[H]$, $j^+ \restriction M = j$ and $j^+(G) = H$ (see Proposition 9.1 of [Cum10] for details). $\square$

For the convenience we shall denote the lifted embedding with the same letter as the original one, since this should not cause any confusion. One of the easier way to satisfy Silver's condition is to find a master condition.

Definition 4.9. Let $j : M \longrightarrow N$ be an elementary embedding and $\mathbb{P}$ in M. A *master condition* for j and $\mathbb{P}$ is a condition $q \in j(\mathbb{P})$ such that for every dense set $D \subseteq \mathbb{P}$ with $D \in M$, there is a condition $p \in D$ such that q is compatible with $j(p)$. Master condition is called *strong* if $q \leq j(p)$.

The idea of a strong master condition is used in the following way: Suppose q is a strong master condition for j and $\mathbb{P}$, and consider $G = \{p \in \mathbb{P} \mid q \leq j(p)\}$. Then it is easy to verify that G is $\mathbb{P}$-generic filter over M. Moreover, for any H, an $j(\mathbb{P})$-generic over N such that $q \in H$, $G = j^{-1\prime\prime} H$ and the Silver's condition is obviously satisfied.

4.2 Large cardinals and forcing

Our main goal is to be able to manipulate the continuum function while preserving large cardinals. We study some classes of forcing notions in order to find out what is the relation between large cardinals in the ground model and in the extension. By this we mean destroying and preserving particular large cardinal and even recreating large cardinal if it was destroyed before. Here we present some examples of forcing notions and their properties. Some of them are needed later, some of them are interesting on their own.

The first type are mild forcing notions. These notions does not make any difference between V and $V[G]$ regarding large cardinals (They preserve them both ways).

Fact 4.10. *Let κ be an infinite cardinal and let X denote one of the following property: inaccessible, Mahlo, weakly compact, measurable. Suppose $\mathbb{P}$ is a notion of forcing with $|\mathbb{P}| < \kappa$ and G is a $\mathbb{P}$-generic filter over V. Then κ is X in V iff κ is X in $V[G]$.*

For the proof, see [Jec03], §21.

In contrast, the first and simplest forcing notion that adds subsets to a weakly compact cardinal can already make a difference.

Lemma 4.11. *Suppose κ is weakly compact and $\mathbb{P} = Add(\kappa, 1)$. Then forcing with $\mathbb{P}$ may destroy weak compactness of κ.*

Proof Start with ground model satisfying $V = L$ and force with $\mathbb{P}$. Then in the extension there is a new subset of κ. The new subset A is not constructible. Yet $\mathbb{P}$ is κ-closed, so it does not add new sequences of size less than κ. This means that all $A \cap \alpha$, $\alpha < \kappa$, are in the ground model and hence constructible. By Lemma 3.6, κ cannot be weakly compact in the extension. $\square$

We will use this to show that classic Easton product cannot be used to add subsets to weakly compact cardinal. But this lemma alone is not enough. If we destroy the weak compactness, we need to make sure, it will not be restored later in the product. We get that from the next lemma and return to it later.

Lemma 4.12. *If $\mathbb{P}$ is κ-Knaster and T is a κ-tree then forcing with $\mathbb{P}$ does not add any new cofinal branches to T.*

Proof Suppose for contradiction that for some generic filter G, there is a new branch B in T in $V[G]$. This means in V there is a condition $p \in \mathbb{P}$ such that $p \Vdash \dot{B}$ is a cofinal branch of $\check{T}$ and $\dot{B} \notin \check{V}$. Consider the set of candidates for members of the new branch, $S = \left\{ t \in T \mid (\exists q \leq p)(q \Vdash \check{t} \in \dot{B}) \right\}$. Note that S is obviously a subtree of T.

In $V[G]$, B is a cofinal branch; therefore, it has exactly one node on each level. For each level α denote that node t_α. For each such node, there is a condition $q_\alpha \in G$ such that $q_\alpha \Vdash \check{t}_\alpha \in \dot{B} \cap \check{T}_\alpha$. Since both p and q_α are in the filter G, they have a common extension p_α. Consider the set X of all such p_α for $\alpha < \kappa$. If $|X| < \kappa$ then there is $C \subseteq \kappa$, such that $|C| = \kappa$ and $\forall \alpha, \beta \in C$ $(p_\alpha = p_\beta)$. Otherwise X has size κ and since $\mathbb{P}$ is κ-Knaster, there is $C \subseteq \kappa$ with $|C| = \kappa$ and all conditions in $\{p_\alpha \mid \alpha \in C\}$ are pairwise compatible.

Using C we can build $D = \{t_\alpha \mid \alpha \in C\}$. We show D is linearly ordered unbounded subset of S and thus determines a cofinal branch B_S through S: Since each node in D is from different level, they are all different from each other, and since C has size κ, D is unbounded in S. To show that they are pairwise comparable, notice that for each β, γ from C, p_β and p_γ are compatible. It follows they have a common extension r and

$$r \Vdash \check{t}_\beta \in \dot{B},\ \check{t}_\gamma \in \dot{B},\ \dot{B} \text{ is a branch.}$$

Obviously t_β and t_γ have to be comparable and $B_S = \{s \in S \mid \exists t \in D\ (s \leq t)\}$ is a cofinal branch through S.

Our next step is to prove that "S splits above each of its nodes", i.e. $(\forall t \in S)$ $(\exists s, u \in S)(s, u \geq t\ \&\ s$ and u are incomparable). Suppose that is not true. Then the following holds: $(\exists t \in S)(\forall s, u \in S)((s, u \geq t) \to (s$ and u are comparable$))$. As $t \in S$, there is some condition q_t such that $q_t \leq p$ and $q_t \Vdash \check{t} \in \dot{B}$. Take a $\mathbb{P}$-generic filter H such that $q_t \in H$. Then in $V[H]$ it holds that $\dot{B}^H$ is a cofinal branch through T and $t \in \dot{B}^H$. In this situation $\dot{B}^H$ equals $\{s \in S \mid s \geq t \vee s \leq t\}$, since above t, S is linearly ordered. This means that $\dot{B}^H$ has already been in V and that is a contradiction, because $p \in H$ and $p \Vdash \dot{B} \notin \check{V}$.

This allows us to construct an antichain of size κ in $\mathbb{P}$. Recall we have found B_S, a cofinal branch through S. If we define $\mathit{pred}(t)$ to be the immediate predecessor of t in T (in the case such node exists), we can define $A = \{s \in S \mid s \notin B_S\ \&\ \exists t \in B_s$ $(t = \mathit{pred}(s))\}$. A contains uncomparable elements by definition and has size κ because B_S is a cofinal branch through S and S splits above each node. Since $A \subseteq S$, for each $\alpha < \kappa$ and a_α from A, there is a condition q_α which forces a_α to be in B. These elements are uncomparable; therefore, these conditions are pairwise incompatible. Otherwise, there would be a condition which forces two uncomparable elements to be in a branch and that is impossible.

From A we have constructed an antichain in $\mathbb{P}$ of size κ, but this contradicts the original assumption that $\mathbb{P}$ is κ-Knaster. □

Corollary 4.13. *If $\mathbb{P}$ is κ-Knaster and $\mathbb{P}$ forces κ to be weakly compact, then κ is weakly compact.*

Proof Clearly κ has to be inaccessible in the ground model if it is inaccessible in the extension. In addition, it has the tree property in the extension. From Lemma 4.12 it follows that it had to have the tree property already in the ground model. $\square$

In fact, the κ-Knaster condition can be replaced by condition "$\mathbb{P} \times \mathbb{P}$ is κ-cc", which is implied by "$\mathbb{P}$ is κ-Knaster". This was also successfully applied to measurable cardinal and we will show that later.

Lemma 4.14. *If $\mathbb{P}$ is a forcing notion such that $\mathbb{P} \times \mathbb{P}$ is κ-cc and T is a κ-tree, then forcing with $\mathbb{P}$ does not add any new cofinal branches to T.*

Proof We recycle some ideas from Lemma 4.12. Same as before, denote B the new cofinal branch of T in $V[G]$, p the condition which forces that, and S the set of all potential candidates for nodes in B. Notice that again it holds that S is a subtree of T with the same height as T and it splits above each of its nodes. This is done exactly as in the proof of Lemma 4.12, as we did not use the κ-Knaster property in these parts of the proof. It was only used to find a branch through S in V and we shall not need that here.

Now work in $V[G]$. B is a cofinal branch not only through T, but also through S. Similarly as before we produce a big antichain, but now in $V[G]$. Define $A_S = \{t \in S \mid t \notin B \;\&\; \exists s \in B\; (s = pred(t))\}$. It is easy to see that A_S contains pairwise uncomparable elements and it has size κ because S splits above each of its nodes. The conditions that forces members of A_S to be in S are therefore incompatible and make an antichain of size κ in $\mathbb{P}$.

We have constructed an antichain of $\mathbb{P}$ of size κ in $V[G]$, contradicting the original assumption $\mathbb{P} \times \mathbb{P}$ is κ-cc (Fact 1.5) $\square$

In the same way as in Corollary 4.13 we obtain:

Corollary 4.15. *If $\mathbb{P}$ is a forcing notion such that $\mathbb{P} \times \mathbb{P}$ is κ-cc and $\mathbb{P}$ forces κ to be weakly compact, then κ is weakly compact.*

It is possible to derive the same result for measurable cardinal, but the proof is little more complicated and directly uses some features of measurability. Notice than for weakly compact cardinal, we obtained the result from more general lemma where no mention of weak compactness occurred.

Lemma 4.16. *If $\mathbb{P}$ is a forcing notion such that $\mathbb{P} \times \mathbb{P}$ is κ-cc and $\mathbb{P}$ forces that κ is measurable, then κ is measurable.*

This proof is taken from [Cum10](proof of Theorem 21.1).

Proof Same as before, it holds that κ is inaccessible in the ground model since it is inaccessible in the extension. Now assume for contradiction that there is a $\mathbb{P}$-generic filter G such that in $V[G]$ there exists U, a normal ultrafilter on κ, but there is no normal ultrafilter on κ in V. Take the condition $p \in \mathbb{P}$ such that $p \Vdash \dot{U}$ is a normal ultrafilter on $\check{\kappa}$. We now work below p. It is possible to build a binary tree T of height κ such that levels of T form increasingly finer partitions of κ into fewer than κ-many pieces. In addition, if a node t is a potential member of $\dot{U}$, then both its successors are also potential members of $\dot{U}$. By potential member we mean, that there is condition q (bellow p), such that $q \Vdash \check{t} \in \dot{U}$. The construction of T goes like this:

1. As a root take κ.
2. At successor step, divide the node into two disjunct pieces such that both are again potential members of $\dot{U}$.
3. At limit steps, take all branches in the tree constructed so far and above each branch put its intersection.

The proof that this construction is correct is in order. First, consider the successor steps. Suppose for some $A \subseteq \kappa$, a potential member of $\dot{U}$, the split could not be done. This means that for every $B \subseteq A$ either all conditions bellow p forces B to be in $\dot{U}$ or all forces $A \setminus B$ to be in $\dot{U}$. However, this allows us to define a measure on A and consequently on κ, since $|A| = \kappa$ as it is a potential member of $\dot{U}$ ($\{X \subseteq \kappa \mid X \cap A \in U_A\}$ would be a measure on κ for U_A a measure on A). We would thus obtain that κ is measurable in the ground model.

Now consider the limit steps. Although it may happen some branches will have empty intersection, we prove the limit steps will nevertheless still be partitions of κ into fewer that κ many pieces. For α limit, consider T', the tree constructed so far, and T_α, the newly constructed αth level of T. Each two nodes t,s from T_α are disjunct, because they came from two different branches b_t and b_s and this means there exists $C, D \subseteq \kappa$ such that $C \in b_t$, $D \in b_s$ and $C \cap D = \emptyset$. Since t and s are intersections of these branches, it follows that they are disjunct. Furthermore, for each $\beta \in \kappa$ the set $\{t \in T' \mid \beta \in t\}$ is a branch through T', its intersection is in T_α and β is in this intersection. It follows that $\bigcup T_\alpha = \kappa$ and T_α is a partition of κ.

To conclude that its size is less than κ, notice that $|T'| = \theta < \kappa$ and so it has only 2^θ many branches. Since κ is inaccessible, $2^\theta < \kappa$ and we can extend every branch and still obtain a level of size less than κ.

This concludes the construction of T in V. We now move to $V[G]$ and look at the situation there. We claim in $V[G]$, the ultrafilter $U = \dot{U}^G$ determines a cofinal branch B through T.

Denote $B = \{t \in T \mid t \in U\}$. U chooses exactly one node from each level. It cannot choose more than one because uncomparable nodes in T are disjunct, and it has to choose at least one because it is κ-complete ultrafilter and each level is a partition of κ into fewer than κ pieces (so if none of these nodes is in U, then neither is their union, but that is κ). In addition, all nodes in B are comparable because they are all in U and incomparable nodes of T are disjunct. It follows B is a cofinal branch through T.

We proceed to define a big antichain of $\mathbb{P}$ in $V[G]$. This is done the same way as in the proof of Lemma 4.14. From the construction of T, it is clear that each member of B has a successor in T which is not in B. Define $A_T = \{t \in T \mid t \notin B \;\&\; \exists s \in B\; (s = pred(t))\}$. Obviously A_T has size κ and contains uncomparable nodes. However, every member of A_T was a potential member of U and so had in $\mathbb{P}$ a witness for his potential membership. Since members of A_T are uncomparable, these witnesses had to be incompatible. It follows they make an antichain of size κ of $\mathbb{P}$ in $V[G]$.

We have built an antichain of size κ in $\mathbb{P}$ in $V[G]$, contradicting the original assumption that $\mathbb{P} \times \mathbb{P}$ is κ-cc (Fact 1.5). $\square$

Next we look at different types of large cardinals and examine their relation to the continuum function and also to Easton's Theorem. We are mainly interested in any direct restrictions such large cardinal would lay on the continuum function and the consistency strength of violating GCH at various types of large cardinals. We also examine some forcing notions that enable us to jump the continuum function at these cardinals. As a warm-up, we examine the small types of large cardinals, proceeding later to the well-known but still interesting weakly compact and measurable cardinals. We then finish with indescribable cardinals, comparing their properties to those found at weakly compact and measurable cardinals.

4.3 The continuum function and small large cardinals

We begin with small types of large cardinals. Here we demonstrate, on the examples of inaccessible and Mahlo cardinals, that it is possible to force the continuum function to jump at small large cardinals. Moreover, with just a small additional constraint on the Easton function, it is possible to generalize Easton's theorem so that the Easton forcing does not destroy these cardinals.

Definition 4.17. Assume F is a function from $CARD$ to $CARD$. We say κ is a *closure point* of F if $\forall\alpha < \kappa\ F(\alpha) < \kappa$.

Obviously inaccessible cardinals are closure points of the continuum function. It is also possible to consider closure points of Easton index function and it is sufficient for inaccessible cardinal to be preserved by Easton forcing if we add an additional constraint that the cardinal is to be a closure point of the index function.

Lemma 4.18. *Suppose κ is inaccessible in M, E is an Easton index function such that κ is a closure point of E and $\mathbb{P}$ is the corresponding Easton forcing. Then κ is inaccesible in $M[\mathbb{P}]$.*

Proof We showed that Easton forcing preserves all cofinalities and hence all cardinals. As a consequence, κ stays regular limit cardinal in $M[\mathbb{P}]$. The additional requirement on E ensures that κ stays not only weakly inaccessible but also (strongly) inaccessible. □

It is interesting that the same constraint on Easton index function is already sufficient for preserving Mahlo cardinals. This follows from the properties of Easton forcing.

Lemma 4.19. *Suppose κ is Mahlo in M, E is an Easton index function such that κ is a closure point of E and $\mathbb{P}$ is the corresponding Easton forcing. Then κ is Mahlo in $M[\mathbb{P}]$.*

Proof We know from Lemma 4.18 that κ stays inaccessible. Moreover, forcing with $\mathbb{P}$ preserves cardinals, so the set of regular cardinals under κ is the same in M and $M[\mathbb{P}]$. It remains to verify that the set of regular cardinals is still stationary below κ. This holds because $\mathbb{P}$ is, in fact, stationarity preserving forcing. This is proved by now known technique, viewing $\mathbb{P}$ as a product $\mathbb{P}(E^{\leq\kappa}) \times \mathbb{P}(E^{>\kappa})$. We aim to show that stationary subsets of κ are preserved after both steps of the product. By Fact 1.3, it is enough to check closure or chain condition.

It is easy to verify that $\mathbb{P}(E^{>\kappa})$ is κ-closed. On the other hand, $\mathbb{P}(E^{\leq\kappa})$ is in general only κ^+-cc. Fortunately, in the case when κ is Mahlo, it is κ-cc (see Theorem 2.2 and Corollary 2.4 of [Bau83]). It follows that the set of regular cardinals bellow κ remains stationary after each step of the product forcing $\mathbb{P}(E^{\leq\kappa}) \times \mathbb{P}(E^{>\kappa})$ and κ remains Mahlo in the extension. □

4.4 The continuum function and weakly compact cardinal

The situation around weakly compact cardinals is more complicated than inaccessible or Mahlo cardinals. Fact 4.10 allows us to manipulate the continuum function below weakly compact cardinal κ as we like and preserve it, so long as we do it on a bounded subset of κ and add only $<\kappa$ many subsets. If we want to manipulate the continuum function at κ, we have to be more careful. As we have shown in Lemma 4.11, weakly compact cardinal can be easily destroyed. The combination of Lemma 4.11 and Corollary 4.13 tells us we cannot hope to add subsets to weakly compact cardinal and below it using just Easton *product* forcing with some additional constraints on Easton function.

Nevertheless, there is a way of adding subsets to weakly compact cardinal and that is using iterated forcing, namely iteration with Easton support. The first idea is apparently due to Silver and appeared as an exercise in [Kun80]. The core idea is to iteratively add subsets to all inaccessible cardinals below the weakly compact cardinal κ and thus prepare the scene for adding subsets to κ. This was later generalized by Hauser ([Hau91]) to the whole hierarchy of indescribable cardinals. He was able to do this by using characterization of indescribable cardinals in terms of elementary embedding. We shall return to indescribable cardinals later. Now we consider ourselves with the case of weakly compact cardinal and show how to violate GCH there. Nice proof can be found in [Cum10], §16. It uses both the characterization of weakly compact cardinal via elementary embedding as well as the characterization using tree property. We shall present it here and give more details on some steps of the proof.

We do this in two steps. First we show how to add at least some subsets to a weakly compact cardinal κ and preserve it. Then we extend it to add more than κ^+ many subsets to κ and still preserve weak compactness.

Lemma 4.20. *Suppose GCH holds in V and κ is weakly compact. Then there is $\mathbb{P}$, a notion of forcing adding new subsets of κ, such that if G is $\mathbb{P}$-generic over V, then κ is weakly compact in $V[G]$.*

Proof Consider $\mathbb{P}_{\kappa+1}$, an iteration of length $\kappa+1$ such that $\forall\lambda \leq \kappa$ at stage λ it forces with $Add(\lambda, \lambda)$ if λ is inaccessible, otherwise it forces with the trivial forcing. The support of the iteration is Easton support, i.e. on limit stages it takes direct limit for inaccessible cardinals, otherwise it takes inverse limit. It is known that this kind of iteration, called reverse Easton iteration, preserves cofinalities. As a result, κ stays inaccessible in the extension.

We would like to have $\mathbb{P}_{\kappa+1} \subseteq V_\kappa$. This is, strictly speaking, not true since the conditions of $\mathbb{P}_{\kappa+1}$ are sequences of lenght $\kappa+1$. Fortunately, we can overcome this with a little trick. For each $p \in \mathbb{P}_{\kappa+1}$ we can define p' such that $dom(p') = supp(p)$ and $\forall\alpha \in dom(p')\ p'(\alpha) = p(\alpha)$. Since direct limit is taken at stage κ, $p' \in V_\kappa$. Then for $\mathbb{P}'_{\kappa+1} = \{p' \mid p \in \mathbb{P}_{\kappa+1}\}$, it holds that $\mathbb{P}'_{\kappa+1} \subseteq V_\kappa$ and $\mathbb{P}_{\kappa+1}$ is isomorphic to $\mathbb{P}'_{\kappa+1}$. We are thus justified to claim $\mathbb{P}_{\kappa+1} \subseteq V_\kappa$.

We shall look at $\mathbb{P}_{\kappa+1}$ as a two-step iteration. The first step is the iteration up to κ, which we denote as $\mathbb{P}_\kappa$ and the second step is $\mathbb{Q} = Add(\kappa, \kappa)^{V[\mathbb{P}_\kappa]}$. Suppose G_κ is $\mathbb{P}_\kappa$-generic over V and g_κ is $\mathbb{Q}$-generic over $V[G_\kappa]$. We proceed to show that κ has the tree property in $V[G]$, where $G = G_\kappa * g_\kappa$.

Suppose T is a κ-tree in $V[G]$. As a κ-tree, it has size κ, so it is in fact isomorphic to some ordering of κ, i.e. structure $\langle\kappa, <_\kappa\rangle$. However, $<_\kappa \subseteq \kappa\times\kappa$, so we can actually code this structure into a subset of κ via a bijection between κ and $\kappa\times\kappa$. This justifies us to treat T as a subset of κ from now on. As such it has a nice name $\dot{T}$ in V.

At this point, we want to use the characterization of weakly compact cardinal via elementary embedding. For that we need to build suitable set M that satisfies all the conditions of Fact 3.5(e). We can use Lemma 4.1, taking X as $V_\kappa \cup \left\{\dot{T}\right\} \cup \{\mathbb{P}_{\kappa+1}\}$ (note V_κ has size κ, since κ is inaccessible), to obtain a model M^* of ZFC^- that satisfies almost everything: It has size κ and contains κ. The only problem is it need not be transitive. To overcome this obstacle we simply collapse M^* using Mostowski collapse lemma. Denote M the result of Mostowski collapse of M^* and π the collapsing isomorphism. We show that M has all the desired properties. Since it is isomorphic to M^* we already know it has size κ and is also a model of ZFC^-. Since it is the result of Mostowski collapse, it is transitive. We only need to check it contains everything we need. Since we have $V_\kappa = H(\kappa) \subseteq M^*$, we obtain $\pi(p) = p$ for every $p \in \mathbb{P}_{\kappa+1}$ since $\mathbb{P}_{\kappa+1} \subseteq V_\kappa$. As a consequence, $\pi(\mathbb{P}_{\kappa+1}) = \mathbb{P}_{\kappa+1}$ and $\pi(\dot{T}) = \dot{T}$. And of course $\pi(\kappa) = \kappa$. Thus M is our desired set and we can apply Fact 3.5(e) to obtain transitive model N of size κ which is closed under $<\kappa$-sequences and an elementary embedding $j : M \longrightarrow N$ with $cp(j) = \kappa$. Notice we made sure that $\mathbb{P}^M_{\kappa+1} = \mathbb{P}_{\kappa+1}$. Since $\dot{T} \in M$, we have $i_G(\dot{T}) = T \in M[G]$. Our next step is to lift j to $M[G]$ in order to study the object $j(T)$.

We now look at the first part of the iteration $\mathbb{P}_{\kappa+1}$, and that is $\mathbb{P}_\kappa$. We are interested in what j does with the conditions of $\mathbb{P}_\kappa$. By elementarity of j we have that $j(\mathbb{P}_\kappa)$ is an iteration of length $j(\kappa)$ with Easton support that adds λ subsets to each inaccessible λ below $j(\kappa)$. Since $V^N_\kappa = V^M_\kappa = V_\kappa$, we have that $j(\mathbb{P}_\kappa)_\kappa = \mathbb{P}_\kappa$, where $j(\mathbb{P}_\kappa)_\kappa$ denotes the iteration consisting of first κ stages of $j(\mathbb{P}_\kappa)$. As a result, G_κ is also $j(\mathbb{P}_\kappa)_\kappa$-generic over N. Since $\mathbb{P}_\kappa$ is a direct limit of $\langle \mathbb{P}_\alpha \mid \alpha < \kappa \rangle$, we get that for each $p \in \mathbb{P}_\kappa$ there is $\beta < \kappa$ such that $(\forall \gamma)(\beta < \gamma < \kappa \rightarrow p(\gamma) = \emptyset)$ holds. Then from elementarity of j we obtain that for $j(p)$ and $j(\beta) = \beta$ it holds that $(\forall \gamma)(\beta < \gamma < j(\kappa) \rightarrow j(p)(\gamma) = \emptyset)$. We thus have a pretty good idea what the elementary embedding does with the conditions from $\mathbb{P}_\kappa$: $j(p) \restriction \kappa = p$ and $j(\gamma) = \emptyset$ for $\kappa \leq \gamma < j(\kappa)$. As a consequence, we are allowed to take any $\mathbb{P}_{\kappa, j(\kappa)}$-generic filter H over $N[G_\kappa]$ that exists in $V[G_\kappa]$ and we know we can lift j to get elementary embedding from $M[G_\kappa]$ to $N[G_\kappa * H]$. For a reason that will be clear later, we take special care of the first stage of $\mathbb{P}_{\kappa, j(\kappa)}$ in $N[G_\kappa]$, i.e. $Add(\kappa, \kappa)$. In $V[G] = V[G_\kappa * g_\kappa]$ we already have a generic filter for this stage, namely g_κ, so we make use of it. We then build arbitrary $\mathbb{P}_{\kappa+1, j(\kappa)}$-generic filter H' over $N[G_\kappa * g_\kappa]$ in $V[G_\kappa * g_\kappa]$. We showed how to construct such generic in Lemma 4.6, we only need to check that the closure condition ${}^{<\kappa}N[G_\kappa * g_\kappa] \subseteq N[G_\kappa * g_\kappa]$ is satisfied (we obviously have that $N[G_\kappa * g_\kappa] \models \mathbb{P}_{\kappa+1, j(\kappa)}$ is κ-closed).

Note we have $V \models {}^{<\kappa}N \subseteq N$ from the way we obtained N. Lemma 4.2 then gives us $V[G_\kappa] \models {}^{<\kappa}N[G_\kappa] \subseteq N[G_\kappa]$ as $\mathbb{P}_\kappa$ is κ-cc (Fact 1.6). Finally $V[G_\kappa] \models Add(\kappa, \kappa)$ is κ-closed and we can use Lemma 4.4 to conclude that $V[G_\kappa * g_\kappa] \models {}^{<\kappa}N[G_\kappa * g_\kappa] \subseteq N[G_\kappa * g_\kappa]$.

This completes everything we need to lift j from M to $M[G_\kappa]$. We now have elementary embedding $j \colon M[G_\kappa] \longrightarrow N[G_\kappa * g_\kappa * H']$. To simplify notation, we shall use $j(G_\kappa)$ instead of $G_\kappa * g_\kappa * H'$ from now on. All it remains is to take care of $\mathbb{Q} = Add(\kappa, \kappa)$ in $M[G_\kappa]$ and $j(\mathbb{Q}) = Add(j(\kappa), j(\kappa))$ in $N[j(G_\kappa)]$. In $V[G_\kappa * g_\kappa]$ we already have g_κ, a $\mathbb{Q}$-generic filter over $M[G_\kappa]$. We need to build h, a $j(\mathbb{Q})$-generic filter over $N[j(G_\kappa)]$ such that $j'' g_\kappa \subseteq h$.

Recall that $\forall p \in \mathbb{Q}\ j(p) = p$ as $j \restriction \kappa = id$ and also $g_\kappa \in N[j(G_\kappa)]$ as we made sure earlier. Note that $r = \bigcup g_\kappa$ is actually a function from κ into 2 and as such is a condition of $j(\mathbb{Q})$. In addition, $\forall p \in g_\kappa\ r \leq j(p) = p$ in $j(\mathbb{Q})$. It follows that r is a strong master condition for Q. Furthermore, $|N[j(G_\kappa)]| = \kappa$ and $j(Q)$ is κ-closed, so we can repeat the construction from Lemma 4.6 to obtain h, a $j(\mathbb{Q})$-generic filter over $N[j(G_\kappa)]$ such that $r \in h$. Because $r \in h$, we also get that $j''g_\kappa \subseteq h$ and we can lift to get elementary embedding $j \colon M[G_\kappa * g_\kappa] \longrightarrow N[j(G_\kappa) * h]$. Since $G = G_\kappa * g_\kappa$, $j(G) = j(G_\kappa) * h$.

At the last step, we return to our tree T. Since T is a κ-tree in $M[G]$, from elementarity of j we get $j(T)$ is a $j(\kappa)$-tree in $N[j(G)]$. This means it has height $j(\kappa)$, hence more than κ. But we can say even more. We show that $j(T)\restriction\kappa$ actually equals T. Since $j\restriction\kappa = id$, it follows that $V_\kappa^{M[G]} = V_\kappa^{N[j(G)]}$ and j is identity on $V_\kappa^{M[G]}$. At the beginning we justified treating T as a subset of κ, so $T \subseteq V_\kappa^{M[G]}$. Then $\forall \alpha < \kappa\ \exists \beta < \kappa\ M[G] \models \forall x \in T_\alpha\ rank(x) < \beta$. From elementarity we get $N[j(G)] \models \forall x \in j(T)_\alpha\ (rank(x) < \beta)$, so all elements of $j(T)\restriction\kappa$ are in $V_\kappa^{N[j(G)]}$. This means that all elements of $j(T)\restriction\kappa$ have their preimage in T and thus $j(T)\restriction\kappa = T$.

To finish the proof, note that any node from level κ of $j(T)$ generates a branch of length κ in $j(T)\restriction\kappa = T$. Formally for $t \in j(T)_\kappa$ the set $\{s \in j(T) \mid s < t\}$ is a cofinal branch through T in $V[G]$. Since T was arbitrary κ-tree, κ has the tree property in $V[G]$. □

We showed how to add some subsets to a weakly compact cardinal without destroying it. Yet with this forcing we did not actually violate the GCH. Fortunately, this is possible with just a little additional work.

Theorem 4.21. *Suppose GCH holds in V, κ is weakly compact, $\lambda > \kappa$ and $\mathbb{P}$ is as $\mathbb{P}_{\kappa+1}$ in Lemma 4.20 except at the last stage it forces with $Add(\kappa,\lambda)$ instead of $Add(\kappa,\kappa)$. If G is $\mathbb{P}_{\kappa+1}$-generic than $V[G] \models \kappa$ is weakly compact and $2^\kappa = \lambda$.*

Proof We show how to reduce this case to the one in Lemma 4.20. As before we view the forcing as a two step iteration $\mathbb{P}_\kappa * \mathbb{Q}$ where $\mathbb{Q} = Add(\kappa,\lambda)$. Suppose H is $\mathbb{P}_\kappa$-generic over V and h is Q-generic over $V[H]$. We work in $V[H]$ and concentrate on the tree property.

Let T be a κ-tree in $V[H * h]$. Without loss of generality $T \subseteq \kappa$. We argue that T has actually been added by a subforcing of $\mathbb{Q}$: Take σ to be a nice $\mathbb{Q}$-name for $T \subseteq \kappa$. Then σ is of the form $\{\{\check{\alpha}\} \times A_\alpha \mid \alpha < \kappa\ \&\ A_\alpha$ is an antichain of $\mathbb{Q}\}$. Set

$X = \bigcup\{dom(p) \mid \exists\alpha < \kappa\; p \in A_\alpha\}$. Note that $|X| = \kappa$ because $Add(\kappa, \lambda)$ is κ^+-cc, so X is defined from conditions of κ antichains each of size at most κ. It follows that there are at most κ many such conditions and all have domain of size less than κ.

Take $Y = \{\gamma \in \lambda \mid \exists\delta \in \kappa\; \langle\delta, \gamma\rangle \in X\}$ and note that $|Y| = \kappa$. Consider a subforcing $\mathbb{R} = \{p \in \mathbb{Q} \mid dom(p) \subseteq \kappa \times Y\}$ of our forcing. It holds that σ is actually a nice $\mathbb{R}$-name and so T is already in $V[H * h']$ where $h' = h \cap \mathbb{R}$. In addition, $\mathbb{R}$ is isomorphic to $Add(\kappa, \kappa)$ via the bijection between κ and Y. It follows that T has been added by a forcing that is the same as we analyzed in Lemma 4.20, so T has a cofinal branch in $V[H * h']$ and therefore also in $V[H * h]$. □

Corollary 4.22. *Con(ZFC + $\exists$ weakly compact cardinal) $\implies$ Con(ZFC + $\exists\kappa$(κ is weakly compact & $2^\kappa > \kappa^+$ & $\forall\lambda < \kappa\; 2^\lambda = \lambda^+$)).*

We showed that if it is consistent for a weakly compact cardinal to exist, then it is consistent that it is the first point where GCH fails. We also gave a construction of a model where this is true. As a consequence, the violation of GCH at weakly compact cardinal does not have more consistency strength then mere existence of weakly compact cardinal. We shall compare this result later to the situation for measurable cardinal and also for indescribable cardinals.

4.5 The continuum function and measurable cardinal

We already mentioned some results about measurable cardinal restricting the behaviour of the continuum function. We know the measurable cardinal cannot be the first where the GCH is violated. Moreover, violation of GCH at measurable cardinal implies violation at many cardinals below it (such that the set of those cardinals is in the ultrafilter).

It is interesting that the violation of GCH at a measurable cardinal is harder than at a weakly compact cardinal. In fact, unlike the case of weakly compact, the consistency strength of a measurable cardinal κ with $2^\kappa > \kappa^+$ is strictly greater than mere measurability of κ. The first forcing extension with measurable cardinal κ with $2^\kappa > \kappa^+$ is due to Silver, but it required κ^{++}-supercompact cardinal κ. This assumption can be relaxed to κ^+-supercompactness, see §13 in [Cum10]. The optimal hypothesis was established in the works of Mitchell [Mit84], Gitik [Git89] and Woodin(unpublished). Woodin showed the violation of GCH can be forced if ${\kappa^{++}}^M = \kappa^{++}$ for M the target of the elementary embedding. Gitik showed this can be forced from a model where Mitchell order $o(\kappa) = \kappa^{++}$ (see e.g. [Cum10], §4 for the definition of Mitchel order of κ) and Mitchell established this is optimal.

Recall that for any Easton function E, there is an extension where the values of the continuum function on regular cardinals is given by E. But this can easily destroy large cardinals. Recently, Honzík and Friedman studied the generalization of Easton construction with the goal to preserve measurable cardinals. In [FH08] they proved this is indeed possible and can be guaranteed if the Easton function satisfies an additional condition regarding measurable cardinals. In [FH12] they showed the condition regarding measurable cardinals can be improved and they indeed found the optimal hypothesis similar to Woodin's way of improving the original Silver's assumption.

In Section 3.3, we briefly discussed the basic properties of a measurable cardinal in relation to the continuum function. We showed that if GCH fails at measurable cardinal then it had to fail at many points below it. We also mentioned that this is not symmetric and promised to prove it, which we do now.

Let κ be a measurable cardinal. Levinski ([Lev95]) proved that if κ is a measurable cardinal then there is a model where GCH fails at every regular cardinal below κ, yet holds at κ. The key is to violate GCH at every inaccessible below κ. We present the proof which is based on the proof of violating GCH at measurable κ assuming κ^+-supercompactness of κ (see [Cum10], §12 and §13). The chosen proof is based on the same technique we have already seen in the proof of Lemma 4.20, the lifting of the elementary embedding. Moreover, it nicely demonstrates the use of master-condition technique in the situation where no single master condition is available. The trick is to build a sequence of "increasingly masterful" conditions, a technique due to Magidor [Mag79].

Theorem 4.23. *Suppose GCH holds in V and κ is a measurable cardinal with $j: V \longrightarrow M$ the elementary embedding generated by a normal measure on κ. Then there is a forcing $\mathbb{P}$ such that κ is measurable in $V[\mathbb{P}]$, $V[\mathbb{P}] \models 2^\kappa = \kappa^+$ and $\forall \lambda < \kappa$ (λ inaccessible $\rightarrow 2^\lambda = \lambda^{++}$).*

Proof Define $\mathbb{P}$ to be the iteration of length $\kappa + 1$ with Easton support such that below κ it forces with $Add(\lambda, \lambda^{++})$ at step λ if λ is inaccessible, otherwise it forces with the trivial forcing. At step κ it forces with $Add(\kappa, \kappa^+)$. We shall denote it $\mathbb{P}_{\kappa+1}$, as is usual with the iteration notation.

If we denote G_κ to be P_κ-generic over V and g_κ to be $Add(\kappa, \kappa^+)$-generic over

$V[G_\kappa]$ then we claim $V[G]$, for $G = G_\kappa * g_\kappa$, is the desired model. It is common knowledge that $P_{\kappa+1}$ preserves cofinalities and hence cardinals. It also adds exactly the number of subsets we want where we want them. The hard part is to show that κ stays measurable in $V[G]$. This can be done by lifting j.

By elementarity, in M $j(\mathbb{P}_{\kappa+1})$ is an iteration of length $j(\kappa)+1$ with Easton support that adds λ^{++} subsets to every inaccessible λ below $j(\kappa)$ (including κ) and then adds $j(\kappa^+)$ subsets to $j(\kappa)$.

Up to stage κ, the forcings in V and M are the same. $\mathbb{P}_\kappa = j(\mathbb{P})_\kappa$ since $\forall\alpha < \kappa$ $P_\alpha \in V_\kappa$ and the direct limit is taken at step κ in $j(\mathbb{P})$. So G_κ is also $j(\mathbb{P})_\kappa$-generic over M.

At stage κ the situation is slightly different. $V[G_\kappa]$ wants to do forcing $Add(\kappa,\kappa^+)$ while $M[G_\kappa]$ wants to do $Add(\kappa,\kappa^{++})^{M[G_\kappa]}$. So we cannot use g_κ directly. Fortunately, we know that $\kappa^+ = (\kappa^+)^M < (\kappa^{++})^M < j(\kappa) < \kappa^{++}$ (Lemma 3.12). It follows that $|(\kappa^{++})^M| = \kappa^+$. Thus, in $V[G_\kappa]$ we can easily define isomorphism i between $Add(\kappa,\kappa^+)$ and $Add(\kappa,\kappa^{++})^{M[G_\kappa]}$ using the bijection between $(\kappa^{++})^M$ and κ^+. Since isomorphic forcings yield the same extensions, we are allowed to force with $Add(\kappa,\kappa^{++})^{M[G_\kappa]}$ instead of $Add(\kappa,\kappa^+)$ and still obtain the same extension. If we take g'_κ to be $Add(\kappa,\kappa^{++})^{M[G_\kappa]}$-generic over $V[G_\kappa]$ then it is also $Add(\kappa,\kappa^{++})^{M[G_\kappa]}$-generic over $M[G_\kappa]$. Moreover, $g_\kappa = i^{-1\prime\prime} g'_\kappa$ is $Add(\kappa,\kappa^+)$-generic over $V[G_\kappa]$. It follows that $V[G] = V[G']$ where $G' = G_\kappa * g'_\kappa$ and we can work in $V[G']$ from now on.

We now proceed to lift j from V to $V[G_\kappa]$. For the sake of notation simplicity, denote $\mathbb{R} = j(\mathbb{P})_{\kappa+1,j(\kappa)}$. We need to build an $\mathbb{R}$-generic over $M[G']$. As we already showed in the proof of Lemma 4.20, any such generic is suitable and will enable us to lift j as $\forall p \in \mathbb{P}_\kappa$ $supp(j(p))$ is bounded in κ. By Lemma 4.7, it is enough to show that $\mathbb{R}$ does not have many antichains and has enough closure. We claim $V[G'] \vDash \mathbb{R}$ is κ^+-closed and has at most κ^+ many maximal antichains in $M[G']$.

From the view of $M[G']$, $\mathbb{R}$ has the standard properties, namely $\mathbb{R}$ is λ-closed where λ is the next inaccessible greater than κ and $\mathbb{R}$ has size $j(\kappa)$ and is $j(\kappa)$-cc (see [Cum10] for more details). Since $M[G'] \vDash \mathbb{R}$ is λ-closed, $M[G'] \vDash \mathbb{R}$ is κ^+-closed. Also $\mathbb{P}$ is κ^+-cc, so by Lemma 4.2, ${}^\kappa M[G'] \subseteq M[G']$. Thus by Lemma 4.5, $V[G'] \vDash \mathbb{R}$ is κ^+-closed. If we denote $\mathcal{A}$ the set of all maximal antichains of $\mathbb{R}$ that are in $M[G']$, then from the size and the chain condition, we have that $M[G'] \vDash |\mathcal{A}| = j(\kappa)^{<j(\kappa)} = j(\kappa)$, since $j(\kappa)$ is inaccessible in $M[G']$. Because $V[G'] \vDash |j(\kappa)| = \kappa^+$, we obtain that $V[G'] \vDash |\mathcal{A}| = \kappa^+$.

The assumptions of Lemma 4.7 are satisfied and we can build a filter H, an $\mathbb{R}$-generic over $M[G']$, and lift $j : V \longrightarrow M$ to get $j : V[G_\kappa] \longrightarrow M[j(G_\kappa)]$, where $j(G_\kappa) = G' * H = G_\kappa * g'_\kappa * H$. Also, the closure of $M[j(G_\kappa)]$ is still preserved, because from $V[G'] \vDash {}^\kappa On \subseteq M[G']$ and $M[G'] \subseteq M[j(G_\kappa)]$ we obtain $V[G'] \vDash {}^\kappa On \subseteq M[j(G_\kappa)]$ and the closure follows from Lemma 4.3.

It remains to take care of the last stage of the iteration. For the sake of simplicity denote $Q = Add(\kappa, \kappa^+)$. Then $j(Q) = Add(j(\kappa), j(\kappa^+))$. We have g_κ, a Q-generic filter over $V[G_\kappa]$. We need to find h, a $j(Q)$-generic over $M[j(G_\kappa)]$ with $j''g_\kappa \subseteq h$. The simplest idea for this kind of proof is to use $j''g_\kappa$ to build a master condition in $M[j(G_\kappa)]$. Unfortunately, here the situation is more difficult since $dom(\cup j''g_\kappa) = \kappa \times j''\kappa^+$ and we do not have enough closure to conclude that $j''g_\kappa$ is in $M[j(G_\kappa)]$. Therefore, a different approach is necessary. We follow [Cum10] who used a technique of Magidor to build the generic filter for lifting even without the single master condition.

We carry out an analysis of the maximal antichains of $j(Q)$. Let A be a maximal antichain of $j(Q)$. By Lemma 3.17 we know $A = j(F)(\kappa)$ for some $F \in V[G_\kappa]$ with $dom(F) = \kappa$. By Łoś's theorem, $\{\alpha < \kappa \mid F(\alpha) \text{ is a maximal antichain of } Q\} \in U$ and without loss of generality we can assume $F(\alpha)$ is a maximal antichain of Q for all $\alpha < \kappa$. This enables us to count the maximal antichains of $j(Q)$. Since Q is κ^+-cc and has size κ^+ (classic argument works because κ is inaccessible in $V[G_\kappa]$), $|\mathcal{A}| = {\kappa^+}^\kappa = \kappa^+$ where $\mathcal{A}$ is the set of all maximal antichains of Q. It follows there are only ${\kappa^+}^\kappa = \kappa^+$ many functions F such that $F \colon \kappa \longrightarrow \mathcal{A}$. It follows that in $V[G_\kappa]$ there are only κ^+ many antichains of $j(Q)$ that are in $M[j(G_\kappa)]$.

Returning to the analysis of A, a maximal antichain of $j(Q)$, consider subforcings of Q of type $Add(\kappa, \xi)$ for $\xi < \kappa^+$ and denote them Q_ξ accordingly. We aim to prove that we can find X, an unbounded subset of κ^+ such that $A \cap j(Q_\xi)$ is maximal in $j(Q_\xi)$ for every $\xi \in X$. Since A can be represented by F that picks up κ many maximal antichains of Q, it is sufficient to show that for some unbouded subset $X \subseteq \kappa^+$ ($\xi \in X \rightarrow \forall \alpha < \kappa \; F(\alpha) \cap Q_\xi$ is maximal in Q_ξ) holds. But this is not hard. Every antichain of Q is relatively small and will therefore appear already in some subforcing Q_ξ. Formally, for A_Q, a maximal antichain of Q, denote β_{A_Q} the supremum of $\{\beta < \kappa^+ \mid \exists p \in A_Q \; \exists \gamma < \kappa \; (\gamma, \beta) \in dom(p)\}$. Then $\beta_{A_Q} < \kappa^+$ because $|A_Q| = \kappa$ and $\forall p \in A_Q \; |p| < \kappa$. For $A = [F]$ we can denote β_A the supremum of $\left\{\beta_{F(\alpha)} \mid \alpha < \kappa\right\}$. $\beta_A < \kappa^+$ as $\beta_{F(\alpha)} < \kappa^+$ for all $\alpha < \kappa$. It follows that for each γ such that $\beta_A < \gamma < \kappa^+$, it holds that $\forall \alpha < \kappa \; F(\alpha) \subseteq Q_\gamma$ and thus $F(\alpha) \cap Q_\gamma = F(\alpha)$ is maximal in Q_γ. As a result, $A \cap j(Q_\gamma)$ is maximal in $j(Q_\gamma)$ for each $\beta_A < \gamma < \kappa^+$.

Now take in $V[G']$ an enumeration $\langle A_i \mid i < \kappa^+ \rangle$ of all the antichains of $j(Q)$ that are in $M[j(G_\kappa)]$. The result of the previous paragraph allows us to take an increasing sequence $\langle \alpha_i \mid i < \kappa^+ \rangle$ of ordinals below κ^+ such that $A_i \cap j(Q_{\alpha_i})$ is maximal in $j(Q_{\alpha_i})$. Furthermore, consider $S = \bigcup \{j(p) \mid p \in g_\kappa\}$. S is a partial function from $j(\kappa) \times j''\kappa^+$ with domain $\kappa \times j''\kappa^+$. We cannot prove $S \in M[j(G_\kappa)]$ since we only have ${}^\kappa M[j(G_\kappa)] \subseteq M[j(G_\kappa)]$. Fortunately, this closure is enough to prove we have in $M[j(G_\kappa)]$ a strong master condition for every subforcing Q_ξ of Q, where $Q_\xi = Add(\kappa, \xi)$ for $\xi < \kappa^+$. Consider $s_\xi = \bigcup \{j(p) \mid p \in g_\kappa \cap Q_\xi\}$. Since $\xi < \kappa^+$, $|Q_\xi| \leq \kappa$ and also $|j(p)| < \kappa$ for $p \in g_\kappa \cap Q_\xi$ because $|p| < \kappa$. It follows $|s_\xi| \leq \kappa$ and we have $s_\xi \in M[j(G_\kappa)]$ from the closure of $M[j(G_\kappa)]$. Moreover, it is a condition of $j(Q_\xi)$ and obviously $\forall p \in g_\kappa \cap Q_\xi$ $(s_\xi \leq j(p))$ from the way we defined s_ξ.

Now we have everything we need to define a decreasing sequence of conditions in $j(Q)$ that will give rise to the desired generic filter. We proceed by induction to build a decreasing sequence of $\langle r_i \mid i < \kappa^+ \rangle$ such that the following conditions are satisfied:

1. $r_i \in j(Q_{\alpha_i})$,
2. $r_i \leq s_{\alpha_i}$,
3. r_i extends some member of A_i.

Suppose we have built the sequence up to i. Now define $r = \bigcup_{j<i} r_j$. Obviously $r \in j(Q_{\alpha_i})$. We claim r and s_{α_i} are compatible.

Consider $\gamma < \kappa$ and $\delta < \alpha_i$ such that $(\gamma, j(\delta)) \in dom(s_{\alpha_i})$. If $(\gamma, j(\delta)) \notin dom(r)$ then there can be no conflict. So assume $(\gamma, j(\delta)) \in dom(r)$. From the definition of r, it follows that $(\gamma, j(\delta)) \in dom(r_j)$ for some $j < i$. Then $r(\gamma, j(\delta)) = r_j(\gamma, j(\delta)) = s_{\alpha_j}(\gamma, j(\delta)) = S(\gamma, j(\delta)) = s_{\alpha_i}(\gamma, j(\delta))$. Thus r and s_{α_i} are compatible. In fact, $r \cup s_{\alpha_i}$ is a condition of $j(Q_{\alpha_i})$. Now recall we took α_i in order to $A_i \cap j(Q_{\alpha_i})$ be maximal antichain in $j(Q_{\alpha_i})$. So there is $a_i \in A_i \cap j(Q_{\alpha_i})$ such that a_i is compatible with $r \cup s_{\alpha_i}$. Take r_i to be the extension of a_i and $r \cup s_{\alpha_i}$.

It is easy to see $h = \{p \in j(Q) \mid \exists i \; r_i \leq p\}$ is a $j(Q)$-generic filter over $M[j(G_\kappa)]$. We defined the sequence $\langle r_i \mid i < \kappa^+ \rangle$ to meet every maximal antichain of $j(Q)$. To verify $j''g_\kappa \subseteq h$, consider $p \in g_\kappa$. Since $|p| < \kappa$, $\exists i \; p \in Q_{\alpha_i}$. Then $s_{\alpha_i} \leq j(p)$ and we obtain $j(p) \in h$ from the fact that $r_i \leq s_{\alpha_i}$.

We are therefore allowed to lift j to $V[G']$ and we obtain elementary embedding $j\colon V[G'] \longrightarrow M[j(G')]$ where $j(G') = G_\kappa * g'_\kappa * H * h$. Since $j(G') \in V[G']$, j is definable in $V[G']$. We have obtained a model $V[G']$ with $V[G'] \models \kappa$ is measurable, $2^\kappa = \kappa^+$ and $\forall\lambda < \kappa$ (λ inaccessible $\to 2^\lambda = \lambda^{++}$). □

4.6 The continuum function and indescribable cardinals

In Section 3.4 we promised to show some results obtained by Hauser concerning the failure of GCH at indescribable cardinals. He came up with a characterization of indescribable cardinals in terms of elementary embeddings (Theorem 3.24) and we have seen how useful such characterization can be since it allows the use of lifting technique in forcing.

In Section 4.4 we examined a special case of indescribable cardinal - weakly compact cardinal. We showed how to force a violation of GCH at weakly compact cardinal. As a consequence, we obtained statements that weakly compact cardinal can be the first where GCH fails and this has no more consistency power that mere existence of weakly compact cardinal.

In [Hau91] Hauser addressed this issue for all Π^m_n indescribable cardinals. He obtained two different results for two categories of indescribable cardinals. The first category contains all Π^1_n indescribable cardinal (including weakly compact) and for these, the situation is the same as in the case of weakly compact cardinal. For the indescribable cardinals of higher order (Π^m_n, $m \geq 2$) however, the situation is different and resembles more the case of measurable cardinal.

Theorem 4.24 (Hauser)**.**

($n \geq 1$). CON(ZFC + $\exists\Pi^1_n$ indescribable cardinal) $\Leftrightarrow$ CON(ZFC + $\exists\kappa$(κ is Π^1_n indescribable, $2^\kappa > \kappa^+$ and $\forall\alpha < \kappa\ 2^\alpha = \alpha^+$)).

($l, n \geq 1, m \geq 2$). CON(ZFC + $\exists\Pi^{m+l-1}_n$ indescribable cardinal) $\Leftrightarrow$ CON(ZFC + $\exists\kappa$(κ is Π^m_n indescribable and $2^\kappa = \kappa^{+l}$)).

The proofs can be found in [Hau91] and are another nice examples of using lifting of elementary embedding. The proofs of implications from left to right are of the type we presented here in proofs of Lemma 4.20 and Theorem 4.23. They use iterated forcing to add subsets to every inaccessible cardinal below the indescribable and then lift the corresponding elementary embedding to show that it stays indescribable in the extension. The key difference is that in the first case it is possible to add only few subsets to cardinals below the indescribable and then add a lot to it, as we demonstrated in the case of weakly compact cardinal. In the second case, Hauser adds λ^{+l} subsets to each inaccessible λ below the indescribable and shows it stays indescribable, although only of lesser order. The implication from right to left is trivial in the first case. In the second case, Hauser shows that an Π^m_n indescribable cardinal κ with $2^\kappa = \kappa^{+l}$ is Π^{m+l-1}_n indescribable in L.

The Π^1_n indescribable cardinals are thus different than the higher order indescribables. If such cardinal exists than it can be the first point where GCH fails and this is not consistency-wise stronger. On the other hand, the failure of GCH at the higher order indescribable is stronger in consistency than mere existence. Also in the proof of the second claim, Hauser violates GCH not only at the indescribable but also at every inaccessible below it. Although Hauser does not state explicitly that this is necessary, it is true that this kind of indescribable cardinal cannot be the first to violate GCH.

Theorem 4.25. *Suppose κ is Π^2_1 indescribable and $\forall\lambda < \kappa\ 2^\lambda = \lambda^+$. Then $2^\kappa = \kappa^+$.*

Proof Recall Hauser's definition of indescribable cardinals in terms of elementary embedding. We can start with a κ-model M such that $V_\kappa \subseteq M$. Then there exists transitive set N with $|N| = |V_{\kappa+1}|$ and ${}^\kappa N \subseteq N$, and an elementary embedding $j : M \longrightarrow N$ with $cp(j) = \kappa$. The important thing is to realize the closure of N under κ sequences is sufficient to obtain $\mathcal{P}^N(\kappa) = \mathcal{P}(\kappa)$ and $(\kappa^+)^N = \kappa^+$ (This should be familiar from the case of measurable cardinal). We first show that GCH holds at κ in N:

$$V \vDash \forall\lambda < \kappa\ 2^\lambda = \lambda^+ \overset{V^M_\kappa = V_\kappa}{\Longrightarrow} M \vDash \forall\lambda < \kappa\ 2^\lambda = \lambda^+ \Longrightarrow$$
$$\overset{\text{elem. of j}}{\Longrightarrow} N \vDash \forall\lambda < j(\kappa)\ 2^\lambda = \lambda^+ \overset{\kappa < j(\kappa)}{\Longrightarrow} N \vDash 2^\kappa = \kappa^+.$$

This implies the existence (in N) of a bijection between $\mathcal{P}^N(\kappa)$ and $(\kappa^+)^N$. But certainly $N \subseteq V$ and since $\mathcal{P}^N(\kappa) = \mathcal{P}(\kappa)$ and $(\kappa^+)^N = \kappa^+$, this bijection is also a witness for $V \vDash 2^\kappa = \kappa^+$. □

Although indescribable cardinals are a result of generalization of a weakly compact cardinal's property, only Π^1_n indescribables behaves exactly like weakly compact cardinal with respect to the failure of GCH. The case of higher order indescribables resembles measurable cardinal much more than weakly compact cardinal.

5 Conclusion

In this thesis we studied the behaviour of the continuum function on regular cardinals in the set theory ZFC and in its extensions by axioms assuming existence of some large cardinal. We mentioned that apart the obvious monotonicity, only one additional constraint on the continuum function, König's inequality, was proved in ZFC. This happened for a good reason. We saw that a consistency result of Easton proves that these constraints are indeed the only ones ZFC lays on the behaviour of the continuum function on regular cardinals. On the other hand, this is not true for singular cardinals. There are more results, some of them very interesting, provable in ZFC concerning the continuum function on singular cardinals.

The case of regular cardinals gets more complicated if we want to manipulate the values of the continuum function and preserve large cardinals. We showed that Easton's theorem easily generalizes to small large cardinals like inaccessible and Mahlo cardinal. For these cardinals the obvious additional constraint of large cardinal being a closure point of the Easton function is sufficient. However, this is not enough for the weakly compact cardinal. It is not possible to use Easton product forcing for adding subsets to a weakly compact cardinal and hope to preserve weak compactness. Fortunately, using iteration with Easton support instead of the product seems to work. We demonstrated the use of iterated forcing in the example where we showed how to violate GCH for the first time at a weakly compact cardinal (Section 4.4) and also in the example where GCH was violated below measurable cardinal on a set of measure one, yet preserved on the measurable (Section 4.5). In these examples we also demonstrated a powerful tool frequently used in the study of large cardinals, the lifting of elementary embedding.

With the iterated forcing it is also possible to generalize Easton's theorem for larger large cardinals. This was first done by Menas [Men76] for supercompact cardinals. Recently, Honzík and Friedman [FH08, FH12] were able to generalize Easton's theorem for measurable and in fact all strong cardinals by combining the classic iteration with the Sacks forcing. Other types of large cardinals were also examined and as it turned out, for many cardinals smaller than measurable the classic iteration is sufficient. The case of Woodin cardinals is studied in [Cod13], the case of weakly compact, Ramsey and strongly Ramsey cardinals in [CG12].

Besides the generalizations of Easton's theorem we were mainly interested in partial results like reflection properties of large cardinals concerning the continuum function and the consistency strength of a large cardinal κ with $2^\kappa > \kappa^+$.

Table 1 summarizes the properties of large cardinals mentioned in this thesis with respect to the continuum function. For comparison we added the property of being consistent with L and included also Ramsey cardinals.

Table 1: Comparing large cardinal properties

large cardinal type	I.	II.	III.
Π^1_n indescribable (including w.c.)	$\times$	$\times$	$\checkmark$
Π^m_n indescribable ($m \geq 2$)	$\checkmark$	$\checkmark$	$\checkmark$
measurable	$\checkmark$	$\checkmark$	$\times$
Ramsey	$\times$	$\times$	$\times$

I. cardinal reflects failure of GCH

II. failure of GCH has more consistency strength

III. cardinal is consistent with L

We showed there is a big difference between weakly compact and measurable cardinal concerning the continuum function. Weakly compact cardinal can be the first point where GCH fails and this is equiconsistent with the existence of a weakly compact cardinal. On the other hand, measurable cardinal reflects down the failure of GCH. In addition, the failure of GCH at a measurable cardinal is consistency-wise stronger than mere existence of a measurable cardinal. Without further knowledge, one might hypothesize that the difference occurs because weakly compact cardinal is consistent with L while measurable cardinal is not. Indescribable cardinals of higher order refute this hypothesis. Regarding the continuum function, they resemble measurable cardinal more than weakly compact, but they are consistent with L. And it does not even work the other way around. Ramsey cardinal is not consistent with L, yet it can be the first point of failure of GCH. Also, the failure of GCH at Ramsey cardinal is equiconsistent with the existence of Ramsey cardinal, so Ramsey cardinals behave more like weakly compact cardinals in this situation.

In order to fully understand the interactions between large cardinals and the continuum function, we would have to study more types of large cardinals. For example, unfoldable and strongly unfoldable cardinals present another interesting intermediate stage between weakly compact and measurable cardinal (see [Ham01] for the basic results about these cardinals and GCH). On the other hand, if we go up the large cardinal hierarchy, the reflection properties grow even stronger.

The supercompact cardinals reflect down the failure of GCH even if it occurred somewhere above them. To be precise, if κ is supercompact and GCH holds below κ, then it holds everywhere (consult §22, Chapter 5 of [Kan08]).

Although we have omitted those topics from this text, we hope that we have succeeded in presenting basic approaches to the study of the behaviour of the continuum function in the presence of large cardinals, and that this work can serve as a stepping stone for those interested in this topic.

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