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DOCTORAL THESIS

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Existence and Qualitative Properties of Solutions to Certain Systems of Fluid Mechanics

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague, 18th October, 2012 Václav Mácha

Název práce: Existence a kvalitativní vlastnosti řešení některých systémů mechaniky tekutin.
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Abstrakt: V předložené práci studujeme existenci a jednoznačnost řešení zobecněné Stokesovy úlohy, dále se pak věnujeme vyšší diferencovatelnosti a hölderovské spojitosti řešení jak zobecněného Stokesova systému tak zobecněného Navier–Stokesova systému. V případě řešení lineární rovnice jsme dosáhli plné regularity v libovolné dimenzi, v případě nelineárního problému pracujeme pouze v dimenzi dvě nebo tři. V dimenzi 2 jsme schopní dokázat plnou regularitu řesení, v dimenzi 3 obdržíme pouze částečnou regularitu řešení. Pro přehlednost jsou všechny hlavní výsledky uvedeny v první kapitole.

Klíčová slova: Stokesův problém, Navier–Stokesův problém, částečná regularita.

Title: Existence and qualitative properties of solutions to certain systems of fluid mechanics.
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Abstract: In the presented work, we study the existence and uniqueness of solutions to the generalized Stokes problem. We, further, focus on the higher differentiability and the Hölder continuity of solutions to the generalized Stokes and generalized Navier-Stokes system. We reach the full regularity in an arbitrary dimension for a linear case, while in a nonlinear case we work only in dimensions $d = 2, 3$. In dimension $d = 2$ we are able to proof the full regularity of solution, in dimension $d = 3$ we obtain only a partial regularity. All main results are introduced in the first section.

Keywords: Stokes problem, Navier–Stokes problem, Partial regularity

Contents

List of notation

Notation	Meaning	Condition
$\mathbb{R}^d$	d–dimensional Euclidean space	
$B_R(x)$	$\{y \in \mathbb{R}^d, \|y-x\| < R\}$	$x \in \mathbb{R}^d,\ R > 0$
$B_R^+(x)$	$\{y \in \mathbb{R}^d, \|y-x\| < R, y_d > 0\}$	$x \in \mathbb{R}^d,\ R > 0$
B_R^{d-1}	$\{x \in \mathbb{R}^d, \|x\| < R, x_d = 0\}$	$R > 0$
$B_R^\Omega(x)$	$B_R(x) \cap \Omega$	$x \in \mathbb{R}^d, \Omega \subset \mathbb{R}^d, R > 0$
$\mathbb{R}^{d^2}_{\text{sym}}$	Space of symmetric $d \times d$ matrices	
I	An identity matrix	
$L^\pi(\Omega, \mathbb{R}^d)$	Lebesgue spaces	$\Omega \subset \mathbb{R}^n,\ 1 \le \pi \le \infty$
	of functions $f : \Omega \mapsto \mathbb{R}^d$	
$\\|.\\|_\pi, \\|.\\|_{\pi,\Omega}$	Norm on $L^\pi(\Omega, \mathbb{R}^d)$	
$W^{k,\pi}(\Omega, \mathbb{R}^d)$	Sobolev spaces	$\Omega \subset \mathbb{R}^n,\ 1 \le \pi \le \infty$
	of functions $f : \Omega \mapsto \mathbb{R}^d$	$k \in \mathbb{N}$
$\\|.\\|_{k,\pi}, \\|.\\|_{k,\pi,\Omega}$	Norm on $W^{k,\pi}(\Omega, \mathbb{R}^d)$	
$W_0^{1,\pi}$	$\overline{C_0^\infty(\Omega, \mathbb{R}^d)}^{\\|.\\|_{1,\pi}}$	$1 \le \pi < \infty$
π'	$\frac{\pi}{\pi-1}$	$1 < \pi < \infty$
$W^{-1,\pi}$	$\left(W_0^{1,\pi'}\right)'$	$\pi \in (1, \infty)$
$L^2_0(\Omega)$	$\left\{g \in L^2(\Omega, \mathbb{R}), \int_\Omega g = 0\right\}$	$\Omega \subset \mathbb{R}^d$
$W^{1,2}_{0,\text{div}}$	$\{u \in W_0^{1,2}(\Omega, \mathbb{R}^d), \operatorname{div} u = 0\}$	$\Omega \subset \mathbb{R}^d$
$[.,.]_X$	Duality between X' and X	
$\langle .,. \rangle_H$	Scalar product on H	H is a Hilbert space
μ	Lebesgue measure	
$(f)_{E,\sigma}$	$\sigma(E)^{-1} \int_E f d\sigma$	σ is a measure on $\mathbb{R}^d$,
		$E \subset \mathbb{R}^d$, $\sigma(E) > 0$,
		f is $\sigma-$measurable
$(f)_{x,R}$	$(f)_{B_R(x),\mu}$	$x \in \mathbb{R}^d,\ R > 0$
$\mathcal{H}^n$	$n-$dimensional Hausdorff measure	
	as stated in [11]	
$(f)_\Gamma$	$(f)_{\Gamma,\mathcal{H}^{d-1}}$	$\Gamma \subset \mathbb{R}^d$
		is a (d-1)-dimensional
		manifold
$\text{Ran}(F)$	Range of an operator F	
$\text{Ker}(F)$	Kernel of an operator F	
VMO	Space of functions with	
	vanishing mean oscillations	
VMO_B	$VMO \cap L^\infty$	

Chapter 1

Introduction

1.1 Motivation

Non-Newtonian fluid is a type of fluid whose flow properties differ from those of Newtonian fluids which are described by the Navier–Stokes system. However, there are many physical phenomena which can not be expressed by the typical Navier–Stokes model, such as shear thinning, shear thickening, die swell, etc. The viscosity of non–Newtonian fluids is not generally constant but depends on shear rate and, as many experimental works show, there are several liquids whose viscosity depends on pressure. On the other hand, changes in the density of these liquids are negligible as the pressure grows (see for example [3, 7]). Thus we can model these liquids as being incompressible and, in this case, the governing equation has a form

$$\begin{aligned} u_t - \operatorname{div} T(\mathcal{D}u, p) + \operatorname{div}(u \otimes u) + \nabla p &= f \text{ in } (0,\tau) \times \Omega, \\ \operatorname{div} u &= 0 \text{ in } (0,\tau) \times \Omega, \end{aligned} \tag{1.1.1}$$

where $\Omega \subset \mathbb{R}^d$ is a body, p stands for pressure, u is a velocity field, $\mathcal{D}u$ denotes a symmetrical gradient of u, i.e. $\mathcal{D}u = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$, and f represents body forces. Further, T stands for the deviatoric stress tensor and $\operatorname{div}(u \otimes u)$ is a convective term.

A plenty of works studying this system under various boundary and growth conditions have been published, see for example [4, 5, 9, 17, 18, 26] and references given there. However, there are still many open questions, mostly regarding regularity of solutions.

In this work, we deal with a steady case, i.e. we study an equation

$$\begin{aligned} -\operatorname{div} T(\mathcal{D}u, p) + \operatorname{div}(u \otimes u) + \nabla p &= f \text{ in } \Omega, \\ \operatorname{div} u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{1.1.2}$$

We assume that there exist positive constants c_1, c_2, c_3 such that[1] the deviatoric stress tensor T obeys the following growth condition for all $\xi \in \mathbb{R}^{d^2}_{\mathrm{sym}}$, all $D \in \mathbb{R}^{d^2}_{\mathrm{sym}}$ and $\pi \in \mathbb{R}$:

$$\begin{aligned} c_1|\xi|^2 < \frac{\partial T(D,\pi)}{\partial D}(\xi \otimes \xi) &< c_2|\xi|^2, \\ \left|\frac{\partial T(D,\pi)}{\partial \pi}\right| &< c_3. \end{aligned} \tag{1.1.3}$$

Partial regularity of solution to (1.1.2) in interior domains has been studied in [24, 25]. N. D. Huy studied partial regularity up to a straight boundary in his dissertation thesis ([15]). Chapter 4 of this work is devoted to the partial Hölder regularity for system (1.1.2) in a bounded C^2 domain Ω. In the remainder of this work, we assume that the tensor T fulfills

$$\begin{gathered} T(0,\pi) = 0, \quad \forall \pi \in \mathbb{R} \\ \exists S : \mathbb{R}^{d^2} \times \mathbb{R} \to \mathbb{R}; \quad T(D,\pi) = \frac{\partial S(D,\pi)}{\partial D}, \quad \forall (D,\pi) \in \mathbb{R}^{d^2} \times \mathbb{R}. \end{gathered} \tag{1.1.4}$$

In order to obtain partial regularity, we use so called indirect approach to regularity. To learn more about this approach we refer reader to [11] where this procedure is used to obtain partial regularity of solution to certain elliptic systems. The blow–up system of (1.1.2) has a form of the generalized Stokes system which can be read as follows

$$\begin{aligned} -\operatorname{div}(A\mathcal{D}u) + B\nabla p &= f \text{ on } \Omega, \\ \operatorname{div} u &= g \text{ on } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.1.5}$$

The coefficients A and B come from identities

$$\begin{aligned} A^{kl}_{ij} &= \frac{1}{2}\left(\frac{\partial T_{ij}}{\partial \xi_{kl}} + \frac{\partial T_{il}}{\partial \xi_{kj}}\right)(a,e), \\ B_{kj} &= \delta_{kj} - \frac{\partial T_{ij}}{\partial \tau}(a,e) \end{aligned}$$

[1] Hereinafter in this text we use a letter c for an arbitrary constant which can vary from line to line. A subscribed letter c (e.g. c_1, c_2) stands for a specific constant.

where $a \in \mathbb{R}^{d^2}$ and $e \in \mathbb{R}$ are defined later.
The existence and uniqueness of solution to (1.1.5) is well known for $B = I$ - in this case it is sufficient to test the equation by selenoidal functions and to use Lax-Milgram lemma and de Rham theorem [30]. Also the Hilbert regularity and the Hölder regularity is known and its proof can be found in [14] and [8]. The case of a constant matrix B, generally not equal to identity, was studied in [14] where existence, uniqueness and higher differentiability of solution was proven. One may ask whether this kind of results can be obtained even for a non-constant matrix B. The existence and uniqueness of solution to such problem was provided in my diploma thesis. However, these results are mentioned here for completeness of this work. Moreover, we provide two regularity results. The first part of this thesis was published in two articles, namely [22] and [23].
All main results are formulated in the next section.

1.2 Main results

In case of a linear system, we present two existence results and two regularity results. In nonlinear case, we full regularity for dimension $d = 2$ and partial regularity for dimension $d = 3$. As a byproduct we obtain higher differentiability in a bounded domain.

1 Theorem. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let a matrix $A \in L^\infty(\Omega, \mathbb{R}^{d^4})$ be elliptic and symmetric. Then there exists a neighborhood $U \subset W^{1,\infty}(\Omega, \mathbb{R}^{d^2})$ of an identity matrix such that for a matrix $B \in U$ and for every $f \in W^{-1,2}(\Omega, \mathbb{R}^d)$ and $g \in L^2_0(\Omega)$ there exists a unique weak solution (u,p) of equation* (1.1.5). *In addition, following inequality holds*

$$\|u\|_{1,2} + \|p\|_2 \quad \leq \quad c(\|f\|_{-1,2} + \|g\|_2)$$

with c independent of u, p, f and g.

2 Theorem. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let a matrix $A \in L^\infty(\Omega, \mathbb{R}^{d^4})$ be elliptic and symmetric. Then there exists a neighborhood $V \subset W^{1,\infty}(\Omega, \mathbb{R}^{d^2})$ of an identity matrix, which is generally bigger then U from the previous theorem, such that for a matrix $B \in V$, $g = 0$ and $f \in W^{-1,2}$ the following is true.*

- *If $[f, (B^{-1})^T \psi]_{W^{1,2}_0} = 0$ for all weak solutions ψ to dual equation* (3.1.6) *then there exists a weak solution to* (1.1.5). *The space of functions f, for which solution does not exist, has a finite dimension.*

- *For every couple of weak solutions* (u_1,p_1) *and* (u_2,p_2) *to* (1.1.5) *it holds that*

$$[\operatorname{div}((B^{-1}A)^T\nabla\psi + (\nabla B^{-1}A)^T\psi),(u_1-u_2)]_{W^{1,2}_{0,div}} = 0$$

for every $\psi \in W^{1,2}_{0,\mathrm{div}}$. *Moreover, the space of weak solutions to* (1.1.5) *has a finite dimension.*

We also show higher differentiability of solutions for the linear system and for the smooth data.

3 Theorem. *Let* $k \in \mathbb{N}\cup\{0\}$. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded* C^{k+2} *domain. Suppose that* $f \in W^{k,2}(\Omega,\mathbb{R}^d)$, $g \in W^{k+1,2}(\Omega,\mathbb{R})$, $A \in W^{k+1,\infty}(\Omega,\mathbb{R}^{d^4})$, $B \in W^{k+1,\infty}(\Omega,\mathbb{R}^{d^2})$, $B \in V$ *and let* $(u,p) \in W^{1,2}_0(\Omega,\mathbb{R}^d)\times L^2(\Omega,\mathbb{R})$ *be a weak solution to* (1.1.5). *Then* $(u,p) \in W^{k+2,2}(\Omega,\mathbb{R}^d)\times W^{k+1,2}(\Omega,\mathbb{R})$ *and*

$$\|u\|_{k+2,2} + \|p\|_{k+1,2} \le c(\|f\|_{k,2} + \|g\|_{k+1,2} + \|u\|_{1,2}).$$

In case $B \in U$, *we get*

$$\|u\|_{k+2,2} + \|p\|_{k+1,2} \le c(\|f\|_{k,2} + \|g\|_{k+1,2}).$$

And the following result deals with Hölder regularity of solutions to the linear system.

4 Theorem. *Let* $\Omega \subset \mathbb{R}^d$ *be a* C^1 *domain and* $\Omega_1 \subset \Omega$ *be a nonempty open subset and let* $A \in VMO_B$ *be elliptic and symmetric. Then there exists a neighborhood* $U' \in W^{1,\infty}(\Omega,\mathbb{R}^{d^2})$ *of an identity matrix such that following holds. Let* $B \in U'$, $f = \operatorname{div} F$, $F \in L^{2,\mu}(\Omega,\mathbb{R}^{d^2})$ *and* $g = 0$. *Moreover, let solution* $(u,p) \in W^{1,2}(\Omega)\times L^2(\Omega)$ *to* (1.1.5) *fulfills* $\int_{\Omega_1} p = 0$. *Then there exists a constant* c *such that*

$$\|\mathcal{D}u\|_{L^{2,\mu}} + \|p\|_{L^{2,\mu}} \le c\|F\|_{L^{2,\mu}}$$

for all $\mu < d$.

The main result for the nonlinear system can be read as follows.

5 Theorem. *Let* $d \le 3$ *and let* $\Omega \subset \mathbb{R}^d$ *be a* C^2 *domain and* $f \in L^{2+\delta}(\Omega,\mathbb{R}^d)\cap L^{2,d-1+\alpha}(\Omega,\mathbb{R}^{d^2})$ *for some* $\delta > 0$ *and* $\alpha \in (0,1)$. *Then there is a positive constant* γ *such that if* $c_3 < \gamma$ *then for any weak solution* (u,p) *to* (1.1.2) *there exists a closed set* $\Omega' \subset \overline{\Omega}$ *such that* $\mathcal{H}^{d-2}(\Omega') = 0$ *and* ∇u *and* p *are Hölder continuous in* $\overline{\Omega}\setminus\Omega'$.

Chapter 2

Preliminaries

2.1 Definitions

Unless stated otherwise, we assume that the domain $\Omega \subset \mathbb{R}^d$ is bounded and Lipschitz. The space $L^\infty(\Omega, \mathbb{R}^d)$ is considered with a norm $\|u\|_\infty = \sqrt{\sum_{i=1}^d \|u_i\|_\infty^2}$. We consider one additional norm on the space $W_0^{1,2}$ except the standard one ($\|\nabla u\|_2$), namely $\|u\|_D := \|\mathcal{D}u\|_2$. We use the same notation for norms in a dual space, thus for $u' \in W^{-1,2}$ the notation $\|u'\|_D$ means $\sup\{|(u', u)_{W_0^{1,2}}|; u \in W_0^{1,2}; \|u\|_D \leq 1\}$. Spaces $W_0^{1,2}$ and $W_{0,\mathrm{div}}^{1,2}$ are Hilbert spaces with scalar product $\langle u, v\rangle_D = \int_\Omega \mathcal{D}u\mathcal{D}v$. For operator T on a Hilbert space, we denote its Hilbert adjoint operator by T'.

We also provide a definition of Morrey and VMO spaces and their basic properties which are used later. For more informations about this spaces we refer to [19] and [6].

6 Definition - Morrey Spaces. *Let $0 \leq \mu < d$. We define a space $L^{2,\mu}(\Omega, \mathbb{R}^n)$ as a space of the functions $u \in L^2(\Omega, \mathbb{R}^n)$ for which $\|u\|_{L^{2,\mu}} < \infty$ where*

$$\|u\|_{L^{2,\mu}} \stackrel{def}{=} \sup_{x \in \Omega, 0 < \rho < \mathrm{diam}(\Omega)} \left(\frac{1}{\rho^\mu} \int_{B_\rho^\Omega(x)} |u(y)|^2 dy \right)^{1/2}$$

Additionally, we define a space $W_{0,\mathrm{div}}^{1,2,\mu}(\Omega)$ as a space of functions belonging to $W_{0,\mathrm{div}}^{1,2}(\Omega)$ with $\nabla u \in L^{2,\mu}(\Omega, \mathbb{R}^{d^2})$.

7 Definition. *For a real valued function $f \in L^1(\Omega, \mathbb{R})$ and $r > 0$, $x \in \Omega$ we define:*

$$n(x, r)(f) \stackrel{def}{=} \sup_{0 < \rho \leq r} \frac{1}{|B_\rho^\Omega(x)|} \int_{B_\rho^\Omega(x)} |f(y) - (f)_{B_\rho^\Omega(x)}| dy$$

and $n(r)(f) \stackrel{def}{=} \sup_{x \in \Omega} n(x, r)(f)$. We define the space $VMO(\Omega, \mathbb{R}^d)$ by the fol-

lowing relation:

$$VMO(\Omega,\mathbb{R}^d) = \{f \in L^1(\Omega,\mathbb{R}^d), n(r)(f) < +\infty \textit{ for all } r \in (0,\operatorname{diam}(\Omega)\rangle \textit{ and } \lim_{r\to 0+} n(r) = 0\}$$

Moreover, we work with a space $VMO_B(\Omega,\mathbb{R}^d) = VMO(\Omega,\mathbb{R}^d) \cap L^\infty(\Omega,\mathbb{R}^d)$.

8 Definition. *A matrix* $A \in L^\infty(\Omega,\mathbb{R}^{d^2\times d^2})$ *is said to be symmetric if* $A_{ij}^{kl} = A_{il}^{kj} = A_{kj}^{il}$ *for all* $i,j,k,l \in \{1,\dots,d\}$ *and for almost all* $x \in \Omega$.
We call a matrix $A \in L^\infty(\Omega,\mathbb{R}^{d^2\times d^2})$ *elliptic if there exists a constant* $\alpha > 0$ *such that* $A(x)(\xi\otimes\xi) \geq \alpha\|\xi\|^2$ *for all* $\xi \in \mathbb{R}^{d^2}_{\text{sym}}$ *and for almost all* $x \in \Omega$.

9 Definition. *For* $A \in L^\infty(\Omega,\mathbb{R}^{d^2\times d^2})$ *symmetric,* $B \in W^{1,\infty}(\Omega,\mathbb{R}^{d^2})$, $f \in W^{-1,2}(\Omega,\mathbb{R}^d)$ *and* $g \in L^2_0(\Omega)$, *a weak solution to* (1.1.5) *is defined as a couple* $(u,p) \in W_0^{1,2}(\Omega,\mathbb{R}^d) \times L^2(\Omega,\mathbb{R})$ *fulfilling*[1]*:*

$$\begin{aligned}\int_\Omega A_{ij}^{kl}(\mathcal{D}u)_{jl}(\mathcal{D}\varphi)_{ik} + \int_\Omega p\frac{\partial\,(B_{kj}\varphi_k)}{\partial x_j} &= [f,\varphi]_{W_0^{1,2}} \qquad \forall\varphi \in W_0^{1,2}(\Omega,\mathbb{R}^d),\\ \operatorname{div} u &= g \textit{ a.e on } \Omega. \end{aligned} \tag{2.1.1}$$

We call the weak solution unique if for any $\Omega_1 \subset \Omega$ *there exists only one weak solution* (u,p) *such that* $\int_{\Omega_1} p = 0$.

10 Definition. *Let* $f \in W^{-1,2}(\Omega,\mathbb{R}^d)$. *We say, that* $(u,p) \in W^{1,2}_{0,\text{div}}(\Omega) \times L^2(\Omega,\mathbb{R})$ *is a weak solution to* (1.1.2), *if, for* $\forall\varphi \in W^{1,2}_{0,\text{div}}$, *it holds that*

$$\int_\Omega T_{ij}(\mathcal{D}u,p)\frac{\partial\varphi_j}{\partial x_i} + \int_\Omega u_j u_i \frac{\partial\varphi_j}{\partial x_i} = [f,\varphi]_{W_0^{1,2}}$$

and, for all $\forall\varphi \in W_0^{1,2}(\Omega,\mathbb{R}^d)$

$$\int_\Omega p\operatorname{div}\varphi = -\int_\Omega T(\mathcal{D}u,p)\nabla\varphi - \int_\Omega (u\otimes u)\nabla\varphi + [f,\varphi]_{W_0^{1,2}}.$$

2.2 Observations

11 Lemma. *There exist constants* c_4 *and* c_5 *such that for every* $u \in W_0^{1,2}(\Omega,\mathbb{R}^d)$ *following inequalities hold:*

$$\frac{1}{c_4}\|u\|_2 \leq \|u\|_D \tag{2.2.1}$$

$$\|u\|_D \leq \|\nabla u\|_2 \leq c_5\|u\|_D \tag{2.2.2}$$

[1]Summation convention is used throughout this paper.

Proof. The proof of the first inequality in (2.2.2) is obvious. The rest comes from Korn's inequality (see cf. [13]). Inequality (2.2.1) immediatelly follows from (2.2.2) and from Poincaré inequality (see c.f. [1], Theorem 6.30). □

12 Assumptions. *Let a matrix $A \in L^\infty$ be symmetric and elliptic with a constant $\alpha > 0$, a matrix $B = I - K$, $K \in W^{1,\infty}(\Omega, \mathbb{R}^{d^2})$, $\|K\|_\infty < 1$.*

- *We say that an assumption A_1 is fulfilled if the inequality*

$$\frac{c_5\sqrt{d}\|K\|_\infty}{(1-\|K\|_\infty)} + \frac{c_4\sqrt{d}\|\nabla K\|_\infty}{(1-\|K\|_\infty)^2} < \frac{\alpha}{\|A\|_\infty} \tag{2.2.3}$$

  holds.

- *If*

$$\|K\|_\infty < \frac{\alpha\sqrt{d}}{c_5\|A\|_\infty\sqrt{d}+\alpha}, \tag{2.2.4}$$

  we say that an assumption A_2 is fulfilled.

13 Lemma. *There exists a bounded linear operator $T : L^2_0(\Omega) \mapsto W^{1,2}_0(\Omega, \mathbb{R}^d)$ fulfilling*

$$\operatorname{div} Tg = g \qquad \forall g \in L^2_0(\Omega). \tag{2.2.5}$$

Proof. For proof see [30], Lemma 2.1.1 in Chapter II. □

14 Corollary. *Let there exist a weak solution to equation* (1.1.5) *for $g = 0$. Then there exists a weak solution to equation* (1.1.5) *for any $g \in L^2_0(\Omega)$.*
Let a weak solution to equation (1.1.5) *with $g = 0$ be unique. Then a weak solution to* (1.1.5) *is unique for any $g \in L^2_0(\Omega)$.*
Let (u, p) be a weak solution to (1.1.5) *with $g = 0$ which satisfies $\|u\|_{1,2} + \|p\|_2 \leq c\|f\|_{-1,2}$. Then a weak solution to* (1.1.5) *with the same data A, B and f but general $g \in L^2_0(\Omega)$ fulfills*

$$\|u\|_{1,2} + \|p\|_2 \leq c(\|f\|_{-1,2} + \|g\|_2). \tag{2.2.6}$$

Proof. Let $g \in L^2_0$. Then, according to Lemma 13, we get the existence of u_1 such that $\operatorname{div} u_1 = g$ with $\|u_1\|_{1,2} \leq c\|g\|_2$. We define a function $u \overset{def}{=} u_0 + u_1$ where $u_0 \in W^{1,2}_{0,\mathrm{div}}$ such that u_0 solve

$$-\operatorname{div} A\mathcal{D}u_0 + B\nabla p = f + \operatorname{div} A\mathcal{D}u_1.$$

The existence of such a solution is granted by the assumptions of this corollary. The function u solves system (1.1.5) due to its linearity. Since $\|u_0\|_{1,2} + \|p\|_2 \leq c\|f\|_{-1,2}$ we immediately obtain (2.2.6).

Now suppose that there exists a unique solution to (1.1.5) such that $\operatorname{div} u = 0$. For contradiction assume that there exist at least two solutions (u_1, p_1) and (u_2, p_2) solving (1.1.5) with the same f, A, B and g and with $\operatorname{div} u_1 = \operatorname{div} u_2 = g$. Their difference solve

$$\begin{aligned} -\operatorname{div} A\mathcal{D}(u_1 - u_2) + B\nabla(p_1 - p_2) &= 0, \\ -\operatorname{div}(u_1 - u_2) &= 0. \end{aligned}$$

Naturally, one solution to this problem is zero and according to the assumptions this solution is unique. Thus we get $(u_1, p_1) = (u_2, p_2)$ and the corollary is proved. □

15 Lemma. *Let $\Omega_0 \subset \Omega$. There exists a constant c such that for each $f \in W^{-1,2}(\Omega, \mathbb{R}^d)$ satisfying*

$$[f, \varphi]_{W^{1,2}} = 0 \qquad \forall \varphi \in W^{1,2}_{0,\operatorname{div}}(\Omega)$$

there exists a uniquely determined $p \in L^2(\Omega, \mathbb{R})$ satisfying

$$\nabla p = f, \qquad \int_{\Omega_0} p = 0, \qquad \|p\|_2 \leq c\|f\|_{-1,2}.$$

Proof. For proof see [30], Lemma 2.1.1 in chapter II. □

16 Lemma. *Let $B = I - K$, where $K \in W^{1,\infty}(\Omega, \mathbb{R}^{d^2})$ and $\|K\|_\infty < 1$. Then there exists an inversion $C \overset{def}{=} B^{-1} \in W^{1,\infty}(\Omega, \mathbb{R}^{d^2})$ of the form $C = I + L$, where $L = \sum_{i=1}^{\infty} K^i$. Moreover, following estimates holds*

$$\|\nabla C\|_\infty = \|\nabla L\|_\infty \leq \left(\frac{\sqrt{d}\|\nabla K\|_\infty}{(1 - \|K\|_\infty)^2} \right),$$

$$\|L\|_\infty \leq \frac{\sqrt{d}\|K\|_\infty}{1 - \|K\|_\infty}.$$

Proof. Space $L^\infty(\Omega, \mathbb{R}^{d^2})$ equipped with a norm

$$\|X\|_a \overset{def}{=} \sup \left\{ \sqrt{\sum_{i,k=1}^{d} \left(\sum_{j=1}^{d} \|X_{ij} Y_{jk}\|_\infty \right)^2} ; Y \in L^\infty(\Omega, \mathbb{R}^{d^2}), \|Y\|_\infty \leq 1 \right\}$$

is Banach algebra hence we can use Neumann Lemma (i.e. Theorem 10.7 in [28]). Moreover, $\frac{\|X\|_\infty}{\sqrt{d}} \leq \|X\|_a \leq \|X\|_\infty$. The assumption $\|K\|_\infty < 1$ implies that $\|K\|_a < 1$ and thus B is invertible and $\|B^{-1}\|_a < \infty$. Because $B^{-1} \in L^\infty(\Omega, \mathbb{R}^{d^2})$ we get $\dfrac{1}{\det B} \in L^\infty(\Omega)$ (it follows immediately from $\dfrac{1}{\det B} = \det B^{-1}$). We denote the cofactor matrix to B by $\overline{B}$. Following identities hold true for inverse matrices

$$\begin{aligned} B^{-1}_{ij} &= \frac{1}{\det B}\overline{B_{ji}}, \\ \frac{\partial B^{-1}_{ij}}{\partial x_k} &= \frac{\frac{\partial \overline{B_{ji}}}{\partial x_k}\det B - \overline{B_{ji}}\frac{\partial(\det B)}{\partial x_k}}{(\det B)^2} \end{aligned}$$

and thus we obtain $B^{-1} \in W^{1,\infty}(\Omega, \mathbb{R}^d)$. Moreover, the precise form of matrix $C = B^{-1}$, which comes from Neumann Lemma, can be written as

$$C = I + L = I + \sum_{i=1}^{\infty} K^i.$$

We use triangle inequality together with property of Banach algebra ($\|x.y\| \leq \|x\|.\|y\|$) to get

$$\|L\|_a = \left\| \sum_{i=1}^{\infty} K^i \right\|_a \leq \sum_{i=1}^{\infty} \|K^i\|_a \leq \sum_{i=1}^{\infty} \|K\|_a^i = \frac{\|K\|_a}{1 - \|K\|_a}.$$

For $\frac{\partial L}{\partial x_j}$ it holds

$$\frac{\partial L}{\partial x_j} = \frac{\partial}{\partial x_j}\sum_{i=1}^{\infty} K^i = \sum_{i=1}^{\infty}\frac{\partial}{\partial x_j}\left(K^i\right) = \sum_{i=1}^{\infty}\sum_{l=1}^{i} K^{i-l}\frac{\partial K}{\partial x_j}K^{l-1}$$

and following estimate can be derived for the norm of $\left\|\frac{\partial L}{\partial x_j}\right\|_a$

$$\left\|\frac{\partial L}{\partial x_j}\right\|_a \leq \left\|\frac{\partial K}{\partial x_j}\right\|_a \sum_{i=1}^{\infty} i\|K\|_a^{i-1} = \frac{\left\|\frac{\partial K}{\partial x_j}\right\|_a}{(1 - \|K\|_a)^2}.$$

Obviously

$$\left\|\frac{\partial L}{\partial x_j}\right\|_\infty \leq \frac{\sqrt{d}\left\|\frac{\partial K}{\partial x_j}\right\|_\infty}{\left(1 - \|K\|_\infty\right)^2}.$$

After summation we get

$$\|\nabla L\|_\infty^2 = \sum_{j=1}^{d}\left\|\frac{\partial L}{\partial x_j}\right\|_\infty^2 \leq \frac{d}{\left(1 - \|K\|_\infty\right)^4}\sum_{j=1}^{d}\left\|\frac{\partial K}{\partial x_j}\right\|_\infty^2 = \left(\frac{\sqrt{d}\|\nabla K\|_\infty}{\left(1 - \|K\|_\infty\right)^2}\right)^2.$$

□

17 Lemma - Fredholm's alternative. *Let H be a Hilbert space equipped with a norm $\|.\|_H$ and let there be three bounded linear operators F, G, $E : H \mapsto H$ such that F is invertible and E is compact. Moreover let $|\lambda| > \|G\|_{H^*}\|F^{-1}\|_{H^*}$. Then following holds:*

1. $\operatorname{Ran}(\lambda F' + G' + E') = \operatorname{Ker}(\lambda F + G + E)^{\perp}$,
2. $\operatorname{Ran}(\lambda F + G + E) = \operatorname{Ker}(\lambda F' + G' + E')^{\perp}$,
3. $\dim(\operatorname{Ker}(\lambda F + G + E)) < \infty$.

Proof. Composition of operators $\lambda F + G$ and F^{-1} is $\lambda I + GF^{-1}$. This operator is obviously invertible since $\lambda > \|G\|_{H^*}\|F^{-1}\|_{H^*}$. Also operator $\lambda F + G$ is invertible because F is one-to-one. So we can apply the operator $(\lambda F + G)^{-1}$ and work with operators $(I + (\lambda F + G)^{-1}E)$ and $(I + E(\lambda F + G)^{-1})$. The operator E is compact and the same holds true for the operators $(\lambda F + G)^{-1}E$ and $E(\lambda F + G)^{-1})$. Hence Fredholm alternative (cf [20]) together with following identities:

$$\operatorname{Ran}(I + E(\lambda F + G)^{-1}) = \operatorname{Ran}(\lambda F + G + E),$$

$$\operatorname{Ker}(I + (\lambda F + G)^{-1}E) = \operatorname{Ker}(\lambda F + G + E)$$

yield

$$\begin{aligned}\operatorname{Ran}(\lambda F' + G' + E') = \operatorname{Ran}(I + E'((\lambda F + G)^{-1})') = \operatorname{Ran}(I + ((\lambda F + G)^{-1}E)') = \\ = \operatorname{Ker}(I + (\lambda F + G)^{-1}E)^{\perp} = \operatorname{Ker}(\lambda F + G + E)^{\perp}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Ran}(\lambda F + G + E) = \operatorname{Ran}(I + E(\lambda F + G)^{-1}) = \operatorname{Ker}(I + (E(\lambda F + G)^{-1})')^{\perp} = \\ = \operatorname{Ker}(I + ((\lambda F + G)^{-1})'E')^{\perp} = \operatorname{Ker}(\lambda F' + G' + E')^{\perp}.\end{aligned}$$

The finite dimension of the null space is a direct result of the Fredholm alternative. □

18 Observations. *The space $L^{2,\mu}$ can be identified with L^2 for $\mu = 0$. The space $L^{2,\alpha}$ is embedded into $L^{2,\mu}$ for $\mu < \alpha < d$ (see for instance [19]).*
Immediately from the definition we see that for $f \in L^{2,\mu}(\Omega)$ and $g \in L^{\infty}(\Omega)$ we get $gf \in L^{2,\mu}(\Omega)$ and $\|gf\|_{L^{2,\mu}} \leq \|g\|_{\infty}\|f\|_{L^{2,\mu}}$.

19 Lemma. *Let Ω be a C^1 domain and $n \in \mathbb{N}$.*

1. *Let $d \geq 3$. For any $\mu < d - 2$ there exists a constant c such that for all $f \in L^{2,\mu}(\Omega, \mathbb{R}^n)$ there is a function $F \in L^{2,\mu+2}(\Omega, \mathbb{R}^{n\times d})$ fulfilling $f = -\operatorname{div} F$ in the weak sense (i.e. $\int_\Omega f\varphi = \int_\Omega F\nabla\varphi$ for all $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^n)$) and $\|F\|_{2,\mu+2} \leq c\|f\|_{2,\mu}$.*
2. *Let $d \leq 2$. Then there exists a constant c such that for all $f \in L^2(\Omega, \mathbb{R}^n)$ there is a function $F \in L^{2,\mu}(\Omega, \mathbb{R}^{n\times d})$, $0 < \mu < d$ fulfilling $f = \operatorname{div} F$ in the weak sense and $\|F\|_{2,\mu} \leq c\|f\|_2$.*

Proof. Let us consider a weak solution w of the following system

$$\begin{aligned} -\Delta w &= f \text{ on } \Omega, \\ w &= 0 \text{ on } \partial\Omega. \end{aligned}$$

In case $d \geq 3$ the Theorem 3.16 in [31] immediately gives the existence of a constant c independent of f such that the estimate $\|\nabla w\|_{2,\mu+2} \leq c\|f\|_{2,\mu}$ is fulfilled. Let $d \leq 2$. Then $\nabla w \in W^{1,2}$ and $W^{1,2}$ is embedded into $L^{2,\mu}$ for $\mu \in (0, d)$ (see Theorems 2.3 and 2.1 in [31]). Now it suffices to set $F = \nabla w$. $\square$

If Ω is a C^2 domain, we can suppose that $\partial\Omega$ can be described in a neighborhood of $x_0 \in \partial\Omega$ as a function $\Gamma_{x_0} : \mathbb{R}^{d-1} \mapsto \mathbb{R}^d$ fulfilling $\Gamma_{x_0}(0) = x_0$ and, since both systems (1.1.2) and (1.1.5) are invariant under rotation and translation, we require that $\frac{\partial \Gamma_i}{\partial x_j}(0) = \delta_{ij}$, $i \in \{1, \ldots, d\}$, $j \in \{1, \ldots, d-1\}$. Furthermore, we can assume that there exist constants $\alpha, \beta > 0$ such that[2]

$$\{(x', x_d) \in \mathbb{R}^d, |x'| < \alpha, \Gamma(x') < x_d < \Gamma(x') + \beta\} \subset \Omega$$

and

$$\{(x', x_d) \in \mathbb{R}^d, |x'| < \alpha, \Gamma(x') - \beta < x_d < \Gamma(x')\} \subset \mathbb{R}^d \setminus \overline{\Omega}.$$

See Definition A.3.29 in [16] for more. We define a new function $F_{x_0} : \mathbb{R}^d \mapsto \mathbb{R}^d$ by $F_{x_0}(x) = \Gamma_{x_0}(x') + (0, x_d)$. We write $F_{x_0,R}(x)$ for $F_{x_0}(Rx)$. The image of $B_1^+(0)$ under mapping $F_{x_0,R}$ is denoted as $\Omega_{x_0,R}$. For simplicity of notation, we omit suffix x_0 if possible.

20 Observations. *Let Ω be a C^2-domain, $x_0 \in \partial\Omega$. Then*

(i) $\nabla F_{x_0,R}(0) = RI$.

(ii) $\nabla F_{x_0,R}(x) = RI + R^2\omega(x)$, *where ω is a function, which is bounded uniformly with respect to x_0 and R.*

(iii) There exist $c > 0$ and $R_0 > 0$ such that, for all $R < R_0$ and $x \in B_1^+(0)$,

$$R^d - cR^{d+1} \leq |\det \nabla F_{x_0,R}(x)| \leq R^d + cR^{d+1}.$$

(iv) Especially, there exist $R_1 \in (0, R_0)$ and $c, c' > 0$ such that, for all $0 < R < R_1$ and for all $x \in B_1^+(0)$, there exists $F_{x_0,R}^{-1}$ and

$$\begin{aligned} cR^d &\leq |\det \nabla F_{x_0,R}(y)| \leq c'R^d, \\ cR^{-d} &\leq |\det \nabla F_{x_0,R}^{-1}(x)| \leq c'R^{-d}, \end{aligned}$$

for all $y \in B_1^+(0)$ and $x \in \Omega_{x_0,R}$.

[2] Here x' is understood as the first $(d-1)$–tuple of coordinates of x, i.e. $x = (x_1, x_2, \ldots, x_{d-1}, x_d) = (x', x_d)$.

(v) There exist R_2 and constants $c, c' > 0$ such that, for all $R \in (0, R_2)$,

$$\Omega_{x_0,cR} \subset (B_R(x_0) \cap \Omega) \subset \Omega_{x_0,c'R}.$$

Proof. (i) It follows immediately from the definition of $F_{x_0,R}$.

(ii) According to the mean value theorem, we have

$$\frac{\partial F_{x_0,R}}{\partial x_i}(x) - \frac{\partial F_{x_0,R}}{\partial x_i}(0) = \frac{\partial^2 F_{x_0,R}}{\partial x_i \partial x_j}(\xi) x_i$$

for some $\xi \in B_1^+(0)$. The definition of $F_{x_0,R}$ implies $\|\nabla^2 F_{x_0,R}(\xi)\|_\infty = cR^2 \|\nabla^2 \Gamma(F_{x_0,R}(\xi))\|_\infty$. Since Ω is a C^2 domain, $\nabla^2 \Gamma$ is bounded and the rest follows immediately.

(iii) It follows immediately from the definition of determinant and *(ii)*.

(iv) According to *(iii)*, for R sufficiently small, we have $|\det \nabla F_{x_0,R}| > 0$ and, due to the inverse function theorem, $F_{x_0,R}$ is invertible. We can also assume that $cR \leq \frac{1}{2}$ and thus

$$\frac{R^d}{2} = R^d - \frac{1}{2} R^d \leq R^d - cRR^d \leq |\det \nabla F_{x_0,R}| \leq R^d + cRR^d \leq R^d \left(1 + \frac{1}{2}\right).$$

The identity

$$1 = |\det I| = \left|\det \left(\nabla F_{x_0,R} \nabla F_{x_0,R}^{-1}\right)\right| = |\det \nabla F_{x_0,R}| \left|\det \nabla F_{x_0,R}^{-1}\right|$$

implies the rest.

(v) Let $x \in \Omega_{x_0,R}$. Then there exists $y \in B_1^+(0)$ such that $x = F_{x_0,R}(y)$. Further, since $\nabla F_{x_0,R}$ is bounded according to *(ii)*, $F_{x_0,R}$ is Lipschitz with a constant $R + cR^2$. Thus, $|x - x_0| \leq (R + cR^2)|y - 0|$ and $x \in B_{R(1+cR)}(x_0)$. Thus, for R sufficiently small, the first inclusion is proven.
Let $x \in B_{x_0,R} \cap \Omega$ for R sufficiently small. Then $|x - x_0| < R$ and since $F_{x_0,R'}^{-1}$ is Lipschitz with constant cR'^{-1} we get $|F_{x_0,R'}^{-1}(x) - 0| < c\frac{R}{R'}$. It is enough to choose $R' = Rc$ and, consequently $x \in F_{x_0,R'}(B_1^+(0))$.

□

21 Lemma. *Let T satisfy* (1.1.3) *and* (1.1.4) *and let $D, D_1, D_2 \in \mathbb{R}^{d^2}$ and $p, p_1, p_2 \in \mathbb{R}$. Then*

(i) $\frac{c_1}{2} |D_1 - D_2|^2 \leq (T(D_1, p_1) - T(D_2, p_2))(D_1 - D_2) + \frac{c_3^2}{2c_1} |p_1 - p_2|^2$,

(ii) $T(D, p) D \geq \frac{c_1}{4} (|D|^2 - 1)$,

(iii) $|T(D,p)| \le c_2\,(1+|D|)$.

Proof. The proof of inequality (i) follows the proof of Lemma 3.3 in [9]. Set

$$D_{1,2}(s) = D_2 + s(D_1 - D_2) \quad \text{and} \quad p_{1,2}(s) = p_2 + s(p_1 - p_2).$$

We have

$$\begin{aligned} T(D_1,p_1) - T(D_2,p_2) &= \int_0^1 \frac{\partial}{\partial s} T(D_{1,2}(s), p_{1,2}(s))ds \\ &= \int_0^1 \frac{\partial T(D_{1,2}(s), p_{1,2}(s))}{\partial D}(D_1 - D_2)ds \\ &\quad + \int_0^1 \frac{\partial T(D_{1,2}(s), p_{1,2}(s))}{\partial p}(p_1 - p_2)ds. \end{aligned}$$

We denote $(T(D_1,p_1)-T(D_2,p_2))(D_1-D_2)$ by $M_{1,2}$. Young and Hölder inequality together with assumption (1.1.3) imply

$$\begin{aligned} c_1|D_1 - D_2|^2 \le& \int_0^1 \frac{\partial T(D_{1,2}(s), p_{1,2}(s))}{\partial D}(D_1 - D_2)(D_1 - D_2)ds \\ \le& M_{1,2} + \left| \int_0^1 \frac{\partial T(D_{1,2}(s), p_{1,2}(s))}{\partial p}(p_1 - p_2)(D_1 - D_2)ds \right| \\ \le& M_{1,2} + c_3|p_1 - p_2||D_1 - D_2| \\ \le& M_{1,2} + \frac{c_3^2}{2c_1}|p_1 - p_2|^2 + \frac{c_1}{2}|D_1 - D_2|^2 \end{aligned}$$

and the desired inequality follows immediately. The inequalities (ii) and (iii) comes from Lemma 1.19, Chapter 5 in [27]. □

22 Lemma - Poincaré inequalities. *Let Ω be a C^2 domain and let $f \in W^{1,p}(\Omega)$, let $\Omega_R \subset \Omega$ be a neighborhood of a point $x_0 \in \partial\Omega$ described as $\Omega_R = F_{x_0,R}(B_1^+(0))$ and let $\Gamma_R = \overline{\Omega_R} \cap \partial\Omega$. Then $\|f\|_{avg} := |(f)_{\Gamma_R}| + \|\nabla f\|_{p,\Omega_R}$ is equivalent to $\|.\|_{1,p,\Omega_R}$.*
Especially, there exists a constant c independent on f such that

$$c\|f\|_{p,\Omega_R} \le R^{\frac{d}{p}}|(f)_{\Gamma_R}| + R\|\nabla f\|_{p,\Omega_R}$$

and

$$\|f - (f)_{\Gamma_R}\|_{p,\Omega_R} \le cR\|\nabla f\|_{p,\Omega_R}$$

hold for all $R < R_0$, where R_0 is sufficiently small.

Proof. The equivalence of norms can be found in [16] as Lemma A.3.80.
For the proof of the inequalities we suppose that R_0 is small such that F_R is

invertible for all $R < R_0$ and $\| \det F_R \|_\infty \le cR$. We use a rescaling argument. A function f fulfills

$$
\begin{aligned}
\|f\|^p_{p,\Omega_R} = \int_{\Omega_R} |f|^p = & \int_{\Omega_{R_0}} |f(F_R(F^{-1}_{R_0}(y)))|^p . |\det \nabla F_R(y)| . |\det \nabla F^{-1}_{R_0}(y)| dy \\
\le & c \| \det \nabla F_R \|_\infty \| \det \nabla F^{-1}_{R_0} \|_\infty \int_{\Omega_{R_0}} |f(F_R(F^{-1}_{R_0}(y)))|^p dy \\
\le & c \left(\frac{R}{R_0} \right)^d \int_{\Omega_{R_0}} |f(F_R(F^{-1}_{R_0}(y)))|^p dy
\end{aligned}
$$

According to the above mentioned equivalence of norms, we get

$$
\begin{aligned}
& \int_{\Omega_{R_0}} |f(F_R(F^{-1}_{R_0}(y)))|^p dy \\
\le & c \left(|(f(F_R(F^{-1}_{R_0}(y))))_{\Gamma_{R_0}}| + \left(\int_{\Omega_{R_0}} |\nabla_y f(F_R(F^{-1}_{R_0}(y)))|^p dy \right)^{\frac{1}{p}} \right)^p \\
\le & c |(f(x))_{\Gamma_R}|^p + c \int_{\Omega_{R_0}} |\nabla_x f(F_R(F^{-1}_{R_0}(y))) \nabla F_R(F^{-1}_{R_0}(y)) \nabla F^{-1}_{R_0}(y)|^p dy \\
\le & c |(f(x))_{\Gamma_R}|^p + \left(\frac{R}{R_0} \right)^p \int_{\Omega_{R_0}} |\nabla_x f(F_R(F^{-1}_{R_0}(y)))|^p dy.
\end{aligned}
$$

The last term can be estimated via change of variables as follows

$$
\begin{aligned}
\int_{\Omega_{R_0}} |\nabla_x f(F_R(F^{-1}_{R_0}(y)))|^p dy \le & \int_{\Omega_R} |\nabla_x f(x)|^p |\det F^{-1}_R \det F_{R_0}| dx \\
\le & c \left(\frac{R_0}{R} \right)^d \|\nabla f\|^p_{p,\Omega_R}.
\end{aligned}
$$

We put these three inequalities together and, since R_0 is fixed, we get

$$
\|f\|^p_{p,\Omega_R} \le cR^d |(f)_{\Gamma_R}| + cR^p \|\nabla f\|^p_{p,\Omega_R}.
$$

This inequality applied to a function $(f - (f)_{\Gamma_R})$ implies

$$
\|f - (f)_{\Gamma_R}\|_{p,\Omega_R} \le c \left(R^{\frac{d}{p}} |(f - (f)_{\Gamma_R})_{\Gamma_R}| + R \|\nabla f - \nabla (f)_{\Gamma_R}\|_{p,\Omega_R} \right) \le cR \|\nabla f\|_{p,\Omega_R}
$$

and the lemma is proven. □

23 Lemma . *Let $G \subset \mathbb{R}^d$ be an open set, $v \in L^1_{\text{loc}}(G, \mathbb{R})$, $0 \le \alpha < d$ and set*

$$
E_\alpha(v) = \left\{ x \in G, \limsup_{\rho \to 0+} \rho^{-\alpha} \int_{B_\rho(x)} |v| > 0 \right\}
$$

. Then $\mathcal{H}^\alpha(E_\alpha(v)) = 0$.

Proof. See Theorem 2.2, Chapter IV in [11]. □

24 Corollary. *Let $G \subset \mathbb{R}^d$ be an open set, $s \in (0,1]$ and*[3] $v \in W^{s,p}_{\text{loc}}(G, \mathbb{R})$. *Set*

$$F = \{x \in G, \lim_{\rho \to 0+} (v)_{x,\rho} \text{ does not exist}\} \cup \{x \in G, \lim_{\rho \to 0+} |(v)_{x,\rho}| = \infty\}.$$

Then for all $\varepsilon > 0$

$$\mathcal{H}^{d-ps+\varepsilon}(F) = 0.$$

Proof. For $s = 1$ we refer to [11]. Let $s \in (0,1)$. From definition of $W^{s,p}$, it may be concluded that $w = \frac{|v(x)-v(y)|^p}{(x-y)^{d+sp}} \in L^1(G \times G, \mathbb{R})$. We consider a set $E \subset G \times G$ defined as $E = E_{d-ps+\varepsilon}(w)$. Set

$$\operatorname{diag} E := \{x \in G, (x,x) \in E\}.$$

It suffices to show that $F \subset \operatorname{diag} E$. So let $x \notin \operatorname{diag} E$. For some r_0 sufficiently small, it holds that $\sup_{0<r<r_0} \left(r^{-d+ps-\varepsilon} \int_{B_r(x,x)} \frac{|v(z)-v(y)|^p}{(z-y)^{d+sp}} dz dy \right) \le M < \infty$. Let $0 < \frac{r}{2} \le t < r < r_0$. Then

$$\begin{aligned}
|(v)_{x,r} - (v)_{x,t}| &= c \left| r^{-d} \int_{B_r(x)} v(y) dy - t^{-d} \int_{B_t(x)} v(z) dz \right| \\
&= c \left| (tr)^{-d} \int_{B_t(x)} \left(\int_{B_r(x)} v(y) dy \right) dz - (rt)^{-d} \int_{B_r(x)} \left(\int_{B_t(x)} v(z) dz \right) dy \right| \\
&\le c (tr)^{-d} \int_{B_t(x) \times B_r(x)} |v(y) - v(z)| dy dz \\
&\le c (tr)^{-d/p} \left(\int_{B_t(x) \times B_r(x)} |v(y) - v(z)|^p dy dz \right)^{1/p} \\
&\le c \left(r^{-d+ps} \int_{B_t(x) \times B_r(x)} \frac{|v(y) - v(z)|^p}{|y-z|^{d+ps}} dy dz \right)^{1/p} \\
&\le c r^{\varepsilon/p} \left(r^{-d+ps-\varepsilon} \int_{B_t(x) \times B_r(x)} \frac{|v(y) - v(z)|^p}{|y-z|^{d+ps}} dy dz \right)^{1/p} \\
&\le c_6 M^{1/p} r^{\varepsilon/p},
\end{aligned}$$

which gives the continuity of $\sigma(r) \stackrel{def}{=} (u)_{x,r}$ as a function of $r \in (0, \infty)$ for fixed x. It remains to prove that $\lim_{r \to 0} \sigma(r)$ exists and is finite. Let $\{r_i\}_{i=1}^\infty$ be non-increasing sequence converging to zero. Then $\sigma(r_i)$ is Cauchy sequence. Indeed, for every $\theta > 0$ there exists $i_0 \in \mathbb{N}$ such that $r_j^{\varepsilon/p} < \frac{\theta(1-\left(\frac{1}{2}\right)^{\varepsilon/p})}{c_6 M^{1/p}}$ whenever $j \ge i_0$. We set $s_0 = r_j$ and $s_k = \frac{s_{k-1}}{2} = \frac{r_j}{2^k}$. For every $i > j$ there exists l such that

[3] i.e. $v \in L^p(G)$ and $\frac{v(x)-v(y)}{|x-y|^{d/p+s}} \in L^p_{\text{loc}}(G \times G)$ for $s \in (0,1)$

$s_{l+1} \le r_i < s_l$. Then

$$\begin{aligned}|\sigma(r_i)-\sigma(r_j)| &\le |\sigma(r_i)-\sigma(s_l)| + \sum_{k=1}^{l} |\sigma(s_k)-\sigma(s_{k-1})| \\ &\le c_6 M^{1/p} s_l^{\varepsilon/p} + \sum_{k=0}^{l-1} c_6 M^{1/p} s_k^{\varepsilon/p} \le c_6 M^{1/p} \sum_{k=0}^{l} \left(\frac{r_j}{2^k}\right)^{\varepsilon/p} \\ &\le c_6 M^{1/p} r_j^{\varepsilon/p} \frac{1}{1-\left(\frac{1}{2}\right)^{\varepsilon/p}} \le \theta.\end{aligned}$$

Hence $\lim_{r\to 0+}(u)_{x,r}$ exists and it is finite, thus $x \notin F$. □

25 Lemma. *Let $(w,q) \in W^{1,2}(B_1^+(0)) \times L^2(B_1^+(0))$ be a weak solution to a system*

$$\begin{aligned}-\operatorname{div} A\mathcal{D}w + (I-B)\nabla q &= 0 \text{ on } B_1^+(0), \\ \operatorname{div} w &= 0 \text{ on } B_1^+(0), \\ w &= 0 \text{ on } B_1^{d-1},\end{aligned}$$

where $A \in \mathbb{R}^{d^4}$, $B \in \mathbb{R}^{d^2}$ are constant matrices and there exist $\lambda > 0$, $\Lambda > 0$ and $\gamma > 0$ such that following inequality holds true for all $\xi \in \mathbb{R}^{d^2}$

$$\begin{aligned}\lambda|\xi|^2 \le A(\xi\otimes\xi) &\le \Lambda|\xi|^2, \\ B &\le \gamma.\end{aligned}$$

If[4] $\gamma \le \frac{\lambda}{(\lambda + c_7\Lambda)c_7}$, *then for all $\tau, \alpha \in (0,1)$, $R \le 1$ there is a positive constant C^* such that*

$$E^{w,q}(0,\tau R) \le C^* \tau^\alpha E^{w,q}(0,R)$$

where C^* depends only on λ, Λ, γ and d.

Proof. See Lemma 2.2 in [14]. □

[4] Here the constant c_7 comes from Bogovskiĭ operator.

Chapter 3

Generalized Stokes System

3.1 Existence and Uniqueness

In this chapter, we assume that $A \in L^\infty(\Omega, \mathbb{R}^{d^4})$ and $B \in W^{1,\infty}(\Omega, \mathbb{R}^{d^2})$ are non–constant matrices. Under assumption A_2, according to Lemma 16, there exists a matrix $B^{-1} \in W^{1,\infty}(\Omega, \mathbb{R}^{d^2})$. Set $C = B^{-1}$ and $L = C - I$. For $\varphi \in W_0^{1,2}(\Omega)$, a function $C^T\varphi$ is in $W_0^{1,2}(\Omega)$. Thus, we apply $C^T\varphi$ as a test function to (2.1.1) and we get

$$\int_\Omega A\mathcal{D}u\mathcal{D}\left(C^T\varphi\right) + \int_\Omega p \operatorname{div} B^T C^T \varphi = [f, C^T\varphi]_{W_0^{1,2}}.$$

Let $h \in W^{-1,2}(\Omega, \mathbb{R}^d)$ be given by $[h,\varphi]_{W_0^{1,2}} = [f, C^T\varphi]_{W_0^{1,2}}$. It follows that

$$\int_\Omega CA\mathcal{D}u\nabla\varphi + \int_\Omega (\nabla C)A\mathcal{D}u\varphi + \int_\Omega p \operatorname{div}\varphi = [h,\varphi], \tag{3.1.1}$$

where $(CA)_{ij}^{ml} = C_{mk}A_{ij}^{kl}$ and $((\nabla C)A)_i^{ml} = \frac{\partial C_{mk}}{\partial x_i}A_{ij}^{kl}$. Hence the problem (1.1.5) is equivalent to

$$\begin{aligned} -\operatorname{div}(CA)\mathcal{D}u + (\nabla C)A\mathcal{D}u + \nabla p &= h, \\ \operatorname{div} u &= g, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{3.1.2}$$

Proof of Theorem 1. Let assumption A_1 hold. By Corollary 14 it is enough to consider the case $g = 0$. Testing (2.1.1) by the function $C^T\varphi$, $\operatorname{div}\varphi = 0$ we get (according to (3.1.1))

$$\int_\Omega A\mathcal{D}u\mathcal{D}\varphi + \int_\Omega (LA)\mathcal{D}u\nabla\varphi + \int_\Omega (\nabla L)A\mathcal{D}u\varphi = [h,\varphi]_{W_0^{1,2}}. \tag{3.1.3}$$

Consider three linear operators $F, G, E : W^{1,2}_{0,\mathrm{div}}(\Omega) \mapsto W^{1,2}_{0,\mathrm{div}}(\Omega)$ defined as follows

$$\begin{array}{lll} F : u \mapsto \mathbb{F}u & \text{such, that } \langle \mathbb{F}u, \varphi\rangle_D = \int_\Omega A\mathcal{D}u\mathcal{D}\varphi, \\ G : u \mapsto \mathbb{G}u & \text{such, that } \langle \mathbb{G}u, \varphi\rangle_D = \int_\Omega (LA)\mathcal{D}u\nabla\varphi, \\ E : u \mapsto \mathbb{E}u & \text{such, that } \langle \mathbb{E}u, \varphi\rangle_D = \int_\Omega (\nabla L)A\mathcal{D}u\varphi. \end{array} \tag{3.1.4}$$

Since

$$\begin{aligned} |\langle \mathbb{F}u, \varphi\rangle| &= \left| \int_\Omega \sum_{n,m=1}^{d} \left(\sum_{j,l=1}^{d} A^{ml}_{nj}(x)(\mathcal{D}u)_{lj}(x) \right) (\mathcal{D}\varphi)_{mn} \right| \\ &\leq \int_\Omega \sum_{m,n=1}^{d} \left(\sum_{j,l=1}^{d} |A^{ml}_{nj}(x)||(\mathcal{D}u)_{lj}(x)| \right) |(\mathcal{D}\varphi)_{mn}| \\ &\leq \|A\|_\infty \left(\int_\Omega \sum_{j,l=1}^{d} |(\mathcal{D}u)_{lj}(x)|^2 \right)^{\frac{1}{2}} \left(\int_\Omega \sum_{m,n=1}^{d} |(\mathcal{D}\varphi)_{mn}|^2 \right)^{\frac{1}{2}} \\ &\leq \|A\|_\infty \|\mathcal{D}u\|_2 \|\mathcal{D}\varphi\|_2 = \|A\|_\infty \|u\|_D \|\varphi\|_D, \end{aligned} \tag{3.1.5}$$

we get

$$\|F\|_D \leq \|A\|_\infty.$$

The operators G and E can be estimated in the same way as follows

$$\begin{aligned} \|G\|_D &\leq c_5 \|L\|_\infty \|A\|_\infty \\ \|E\|_D &\leq c_4 \|\nabla L\|_\infty \|A\|_\infty. \end{aligned}$$

Thus the operators F, G and E are well defined. The matrix A is elliptic with a constant α, whence $\langle Fu, u\rangle \geq \alpha \|u\|^2_D$ and operator F is bijective according to Lax-Milgram lemma (see cf. [29], Corollary 8.2). This gives

$$\|F\|^{-1}_D \leq \|F^{-1}\|_D \leq \frac{1}{\alpha}.$$

The operator $F + G + E$ is bijective if and only if $I + F^{-1}(G + E)$ is bijective. Let us compute

$$\|F^{-1}(G+E)\|_D \leq \|F^{-1}\|_D(\|G\|_D + \|E\|_D) \leq \frac{1}{\alpha}\|A\|(c_5\|L\|_\infty + c_4\|\nabla L\|_\infty).$$

We conclude, using the estimates from Lemma 16, that

$$\|F^{-1}(G+E)\|_D \leq \frac{\|A\|_\infty}{\alpha} \left(\frac{c_5\sqrt{d}\|K\|_\infty}{(1-\|K\|_\infty)} + \frac{c_4\sqrt{d}\|\nabla K\|_\infty}{(1-\|K\|_\infty)^2} \right).$$

Due to A_1 we get that $\|F^{-1}(G+E)\|_D < 1$. Hence, $I + F^{-1}(G+E)$ is bijective and there is only one solution $u \in W^{1,2}_{0,\mathrm{div}}$ fulfilling (3.1.3). Moreover, one gets

$\|u\|_{1,2} \leq c\|f\|_{-1,2}$ according to Lemma 11. Note that there exists a constant c such that $\|f\|_D < c\|f\|_{-1,2}$. Now we can express p from equation (3.1.2) by

$$\nabla p = \operatorname{div}(CA)\mathcal{D}u - \nabla CA\mathcal{D}u + h$$

and, since $[\operatorname{div}(CA)\mathcal{D}u - \nabla CA\mathcal{D}u + h, \varphi]_{-1,2} = 0$ for $\varphi \in W^{1,2}_{0,\mathrm{div}}(\Omega)$ according to (3.1.3), existence of p is proved due to the Lemma 15. Moreover, this lemma leads to an estimate

$$\|u\|_{1,2} + \|p\|_2 \leq c\|f\|_{-1,2}.$$

□

Throughout the rest of this section we assume that A_2 holds. We work with three operators F, E and G defined in (3.1.4).

26 Lemma. $\operatorname{Ker}(F' + G' + E')$ *is a set of all weak solutions* $\psi \in W^{1,2}_{0,\mathrm{div}}(\Omega)$ *to a system*

$$\begin{aligned} \operatorname{div}(CA)^T \nabla\psi + \operatorname{div}((\nabla C)A)^T \psi &= 0, \\ \operatorname{div}\psi &= 0, \\ \psi|_{\partial\Omega} &= 0. \end{aligned} \tag{3.1.6}$$

A set $\operatorname{Ran}(F' + G' + E')$ *can be described as*

$$\begin{aligned} \{\varphi \in W^{1,2}_{0,\mathrm{div}}(\Omega), \exists\psi \in W^{1,2}_{0,div}(\Omega),& \\ \langle\varphi, z\rangle_D = [\operatorname{div}(CA)^T\nabla\psi + \operatorname{div}((\nabla C)A)^T\psi, z]\ \forall z \in W^{1,2}_{0,div}(\Omega)\}.& \end{aligned} \tag{3.1.7}$$

Proof. Let ψ be a weak solution to the equation (3.1.6), which means that ψ satisfies the equation

$$\int_\Omega CA\mathcal{D}\varphi\nabla\psi + \int_\Omega (\nabla C)A\mathcal{D}\varphi\psi = 0 \qquad \forall\varphi \in W^{1,2}_{0,div}(\Omega).$$

The left hand side of this equation coincides with $\langle(F + G + E)\varphi, \psi\rangle_D$ and an identity

$$\langle(F + G + E)\varphi, \psi\rangle_D = \langle(F' + G' + E')\psi, \varphi\rangle_D \tag{3.1.8}$$

completes the proof of the first part.

Now, φ is in $\operatorname{Ran}(F' + G' + E')$ if and only if there exists $\psi \in W^{1,2}_{0,\mathrm{div}}(\Omega)$ such that for all $z \in W^{1,2}_{0,\mathrm{div}}(\Omega)$

$$\begin{aligned} &\langle\varphi, z\rangle = \langle(F' + G' + E')\psi, z\rangle_D = \langle\psi, (F + G + E)z\rangle_D = \\ &\int_\Omega CA\mathcal{D}z\nabla\psi + (\nabla C)A\mathcal{D}z\psi = [\operatorname{div}(CA)^T\nabla\psi + \operatorname{div}((\nabla C)A)^T\psi, z] \end{aligned} \tag{3.1.9}$$

which is the desired conclusion. □

27 Lemma. *The operator* $E : W^{1,2}_{0,\mathrm{div}}(\Omega) \mapsto W^{1,2}_{0,\mathrm{div}}(\Omega)$ *is compact.*

Proof. We may factorize E as follows

$$\begin{array}{ccc} W^{1,2}_{0,\mathrm{div}}(\Omega) & \xrightarrow{E} & W^{1,2}_{0,\mathrm{div}}(\Omega) \\ \mathcal{E}\downarrow & & \uparrow \mathcal{H} \\ (L^2(\Omega,\mathbb{R}^d))' & \xrightarrow{\mathcal{I}} & (W^{1,2}_{0,\mathrm{div}}(\Omega))', \end{array}$$

Here $\mathcal{H}$ is an identification between a Hilbert space and its dual, while $\mathcal{I}$ is dual to the compact embedding between $W^{1,2}_{0,\mathrm{div}}$ and L^2, thus $\mathcal{I}$ is compact (see [28] Theorem 4.19). $\mathcal{E}$ is defined in the same way as E, it means

$$\mathcal{E}(u)\varphi = \int_\Omega (\nabla L)A\mathcal{D}u\varphi$$

for all $\varphi \in L^2(\Omega,\mathbb{R}^d)$. □

Proof of Theorem 2. Let assumption A_2 hold. As in the previous section we focus on the equation

$$(F+G+E)u = h.$$

By A_2 we get $\|F^{-1}\|\|G\| < 1$, thus all assumptions to Lemma 17 are satisfied, since E is compact due to Lemma 27. Applying Lemma 26 we get the claim. □

3.2 Higher differentiability

Before formulating a proof of the main results, we show a proof of the interior regularity via bootstrap argument presented in [22].

28 Lemma. *Let* $\Omega' \subset \Omega$ *be a nonempty open and bounded set which fulfills* $\mathrm{dist}(\Omega',\partial\Omega) \geq \gamma > 0$. *Moreover, let* $A \in W^{1,\infty}(\Omega, R^{d^2\times d^2})$, $B \in W^{2,\infty}(\Omega, R^{d^2})$, $f \in L^2(\Omega, R^d)$, $g \in W^{1,2}$ *satisfying* $\int_\Omega g = 0$, *let condition* A_1 *be fulfilled and* (u,p) *be a weak solution to* (1.1.5). *Then* $\left(\frac{\partial u}{\partial x_1}, \frac{\partial p}{\partial x_1}\right) \in W^{1,2}(\Omega') \times L^2(\Omega')$ *and*

$$\begin{aligned} \left\|\frac{\partial u}{\partial x_1}\right\|_{1,2,\Omega'} &\leq c(\|f\|_2 + \|g\|_{1,2}), \\ \left\|\frac{\partial p}{\partial x_1}\right\|_{2,\Omega'} &\leq c(\|f\|_2 + \|g\|_{1,2}). \end{aligned}$$

Proof. Denote $V = \overline{\Omega'}$. Then V is a compact set and there exists an open set $\Omega_V \subset \Omega$ such that $V \subset \Omega_V$ and $\mathrm{dist}(\Omega_V,\partial\Omega) > \frac{\gamma}{2}$. We choose an arbitrary smooth bounded function ϑ such that $\mathrm{dist}(\mathrm{supp}\,\vartheta,\partial\Omega) > \frac{\gamma}{4}$ and $\vartheta(x) = 1\ \forall x \in \Omega'$. We multiply formally (1.1.5) by a function ϑ (i.e. we apply a test function $\vartheta\varphi$ instead of φ). Thus

$$\begin{aligned} -(\operatorname{div} A\mathcal{D}u)\vartheta + (B\nabla p)\vartheta &= f\vartheta \text{ on } \Omega, \\ \vartheta \operatorname{div} u &= g\vartheta \text{ on } \Omega. \end{aligned} \tag{3.2.1}$$

It holds that

$$\begin{aligned} (-\operatorname{div} A\mathcal{D}(u\vartheta)) &= -\frac{1}{2}\left(\frac{\partial}{\partial x_i} A_{ij}^{kl} \frac{\partial (u\vartheta)_l}{\partial x_j} + \frac{\partial}{\partial x_i} A_{ij}^{kl} \frac{\partial (u\vartheta)_j}{\partial x_l}\right)_{k=1}^{d} \\ &= -(\operatorname{div} A\mathcal{D}u)\vartheta - \operatorname{div}(A((\nabla\vartheta)u)) - A\mathcal{D}u\nabla\vartheta \\ B\nabla(p\vartheta) &= \left(B_{ki}\left(\frac{\partial \vartheta p}{\partial x_i}\right)\right)_{k=1}^{d} = \left(B_{ki}\frac{\partial p}{\partial x_i}\vartheta + B_{ki} p \frac{\partial \vartheta}{\partial x_i}\right)_{k=1}^{d} \\ &= (B\nabla p)\vartheta + (B\nabla\vartheta)p. \end{aligned}$$

Hence the system (3.2.1) is equivalent to

$$\begin{aligned} -\operatorname{div} A\mathcal{D}(u\vartheta) + B\nabla(p\vartheta) &= f\vartheta - F(u,p,A,B,\vartheta) \text{ on } \Omega, \\ \operatorname{div}(u\vartheta) &= g\vartheta + u\nabla\vartheta \text{ on } \Omega, \end{aligned} \tag{3.2.2}$$

where F is defined as

$$F(u,p,A,B,\vartheta) = \operatorname{div}(A(\nabla\vartheta u)) + A\mathcal{D}u\nabla\vartheta - B\nabla\vartheta p$$

and the L^2 norm of F can be estimated by

$$\|R\|_2 \le c(\|A\|_{1,\infty}\|\vartheta\|_{2,\infty}\|u\|_{1,2} + \|B\|_\infty\|\nabla\vartheta\|_\infty\|p\|_2).$$

We set $\overline{u} = u\vartheta$, $\overline{p} = p\vartheta$, $\overline{f} = f\vartheta - F(u,p,A,B,\vartheta)$ and $\overline{g} = g\vartheta + u\nabla\vartheta$. The equation (3.2.2) can be written as

$$\begin{aligned} -\operatorname{div} A\mathcal{D}\overline{u} + B\nabla\overline{p} &= \overline{f} \text{ on } \Omega, \\ \operatorname{div}\overline{u} &= \overline{g} \text{ on } \Omega. \end{aligned}$$

The Green's formula shows that

$$\int_\Omega \overline{g} = \int_\Omega \vartheta g + \int_\Omega u\nabla\vartheta = \int_\Omega \vartheta \operatorname{div} u + \int_\Omega u\nabla\vartheta = \int_{\partial\Omega} u\vartheta\nu = 0.$$

Here ν stands for a unit outer normal. In order to shorten the notation, we write $\Delta_{\delta e_1} u(x)$ instead of $u(x+\delta e_1) - u(x)$. By the linearity of (1.1.5),

$$\begin{aligned} -\operatorname{div} A\mathcal{D}\left(\frac{\Delta_{\delta e_1}\overline{u}}{\delta}\right) + B\nabla\left(\frac{\Delta_{\delta e_1}\overline{p}}{\delta}\right) &= \frac{1}{\delta}\Delta_{\delta e_1}\overline{f} + \frac{1}{\delta}\left(\operatorname{div}(\Delta_{\delta e_1}A)\mathcal{D}\overline{u}(.+\delta e_1)\right) \\ &\quad + \frac{1}{\delta}\left((\Delta_{\delta e_1}B)\nabla\overline{p}(.+\delta e_1)\right), \\ \operatorname{div}\frac{\Delta_{\delta e_1}\overline{u}}{\delta} &= \frac{\Delta_{\delta e_1}\overline{g}}{\delta}, \\ \Delta_{\delta e_1}\overline{u}|_{\partial\Omega} &= 0. \end{aligned} \tag{3.2.3}$$

For δ small enough, there exists a constant c, which is independent of δ, such that

$$\begin{aligned}
\left\|\frac{1}{\delta}\Delta_{\delta e_1}\overline{f}\right\|_{-1,2} &\leq c\|\overline{f}\|_2,\\
\left\|\frac{1}{\delta}\Delta_{\delta e_1}\overline{g}\right\|_{2} &\leq c\|\overline{g}\|_{1,2},\\
\left\|\frac{1}{\delta}\Delta_{\delta e_1}B\right\|_{1,\infty} &\leq c\|B\|_{2,\infty},\\
\left\|\frac{1}{\delta}\Delta_{\delta e_1}A\right\|_{\infty} &\leq c\|A\|_{1,\infty}
\end{aligned}$$

and

$$\begin{aligned}
\|(\Delta_{\delta e_1}B)\nabla\overline{p}(.+\delta e_1)\|_{-1,2} &\leq c\|p\|_2\|\Delta_{\delta e_1}B\|_{1,\infty},\\
\|\operatorname{div}(\Delta_{\delta e_1}A)\mathcal{D}\overline{u}(.+\delta e_1)\|_{-1,2} &\leq c\|u\|_{1,2}\|\Delta_{\delta e_1}A\|_{\infty}.
\end{aligned}$$

Moreover,

$$\|\overline{f}\|_2+\|\overline{g}\|_{1,2}+\|u\|_{1,2}+\|p\|_2\leq c(\|f\|_2+\|g\|_2),$$

where $c=c(\theta,A,B,\Omega)$. The equation (3.2.3) satisfies the assumptions of Theorem 1 thus

$$\begin{aligned}
\left\|\frac{\Delta_{\delta e_1}\overline{u}}{\delta}\right\|_{1,2} &\leq c(\|f\|_2+\|g\|_{1,2})\\
\left\|\frac{\Delta_{\delta e_1}\overline{p}}{\delta}\right\|_{2} &\leq c(\|f\|_2+\|g\|_{1,2})
\end{aligned}$$

We conclude from [10], Lemma 15.5 that $\frac{\partial\overline{u}}{\partial x_1}$ is in $W^{1,2}$, $\frac{\partial\overline{p}}{\partial x_1}$ is in L^2, and that

$$\begin{aligned}
\|\frac{\partial}{\partial x_1}(\vartheta u)\|_{1,2} &\leq c(\|f\|_2+\|g\|_{1,2}),\\
\|\frac{\partial}{\partial x_1}(\vartheta p)\|_{2} &\leq c(\|f\|_2+\|g\|_{1,2}).
\end{aligned}$$

The L^2 norm of $\frac{\partial}{\partial x_1}p$ can be estimated in the same way. □

The derivative with respect to the first canonical vector was chosen just for simplification of the proof. It is obvious, that the previous lemma can be modified for a derivative with respect to any canonical vector.

29 Theorem. *Let Ω' be an arbitrary nonempty open subset of Ω such that* $\operatorname{dist}(\Omega',\Omega)\geq\gamma>0$. *Let $A\in W^{1,\infty}(\Omega,\mathbb{R}^{d^4})$, $B\in W^{2,\infty}(\Omega,\mathbb{R}^{d^2})$, $f\in L^2(\Omega,\mathbb{R}^d)$, $g\in W^{1,2}(\Omega,\mathbb{R})$ and let condition A_1 be fulfilled. Then a weak solution (u,p) is in space $W^{2,2}(\Omega',\mathbb{R}^d)\times W^{1,2}(\Omega',\mathbb{R})$ and following estimates hold*

$$\begin{aligned}
\|u\|_{2,2,\Omega'} &\leq c(\|f\|_2+\|g\|_{1,2}),\\
\|p\|_{1,2,\Omega'} &\leq c(\|f\|_2+\|g\|_{1,2}).
\end{aligned}$$

Regularity of some special cases of the system (1.1.5) can be found in [10], [14]. Here we use the result published in [12].

30 Definition. *We say that a matrix A is weakly coercive if there exists $\lambda > 0$ such that for all $u \in W_0^{1,2}(\Omega, \mathbb{R}^d)$*

$$\int_\Omega A\mathcal{D}u\mathcal{D}u > \lambda \|\nabla u\|_2^2.$$

31 Theorem. *Let $k \in \mathbb{N} \cup \{0\}$, Ω be a bounded domain of class C^{k+2}. We assume that $A \in W^{k+1,\infty}(\Omega, \mathbb{R}^{d^4})$ is weakly coercive, $g \in W^{k+1,2}(\Omega, \mathbb{R})$ and $f \in W^{k,2}(\Omega, \mathbb{R}^d)$. Then any weak solution (u, p) to a system*

$$\begin{aligned} -\operatorname{div} A\mathcal{D}u + \nabla p &= f \text{ in } \Omega, \\ \operatorname{div} u &= g \text{ in } \Omega, \\ u|_{\partial\Omega} &= 0 \end{aligned} \tag{3.2.4}$$

belongs to $W^{k+2,2}(\Omega) \times W^{k+1,2}(\Omega)$, and

$$\|u\|_{k+2,2} + \|p\|_{k+1,2} \le c(\|f\|_{k,2} + \|g\|_{k+1,2} + \|u\|_2).$$

Proof. For details, we refer reader to Theorem 1.2 and Remark 1.5 in Part II in [12]. The proof of the theorem given there can be easily generalized. □

As written before the system (1.1.5) can be arranged as

$$\begin{aligned} -\operatorname{div} CA\mathcal{D}u + \nabla p &= f - \mathcal{D}CA\nabla u, \\ \operatorname{div} u &= g. \end{aligned} \tag{3.2.5}$$

We recall that $C = B^{-1}$.

32 Lemma. *The matrix CA is weakly coercive under A_2.*

Proof. We suppose that $C = I - L$. Let us compute

$$\int_\Omega CA\mathcal{D}u\mathcal{D}u = \int_\Omega A\mathcal{D}u\mathcal{D}u - LA\mathcal{D}u\mathcal{D}u \ge \alpha\|\mathcal{D}u\|_2^2 - \|L\|_\infty\|A\|_\infty\|\mathcal{D}u\|_2^2 \ge$$
$$(\alpha - \|A\|_\infty\|L\|_\infty)|\mathcal{D}u|^2$$

Assumption A_2 grants that $\|A\|\|L\| < \alpha$ and thus the proof is complete. □

As a consequence we obtain a proof of Theorem 3.

Proof of 3. Let assumption A_2 hold. It suffices to show the claim for a weak solution to (3.2.5). By Lemma 32, the matrix CA is weakly coercive, thus Theorem 31 gives

$$\|u\|_{k+2,2} + \|p\|_{k+1,2} \leq c(\|f\|_{k,2} + \|\mathcal{D}C\|_\infty \|A\|_\infty \|u\|_{k+1,2} + \|g\|_{k+1,2} + \|u\|_2).$$

For $k = 0$ we get

$$\|u\|_{2,2} + \|p\|_{1,2} \leq c(\|f\|_2 + \|\mathcal{D}C\|_\infty \|A\|_\infty \|u\|_{1,2} + \|g\|_{1,2} + \|u\|_2),$$

thus

$$\|u\|_{2,2} + \|p\|_{1,2} \leq c(\|f\|_2 + \|g\|_{1,2} + \|u\|_{1,2}).$$

Let the estimate

$$\|u\|_{k+2,2} + \|p\|_{k+1,2} \leq c(\|f\|_{k,2} + \|g\|_{k+1,2} + \|u\|_{1,2}) \tag{3.2.6}$$

hold for some $k \in \mathbb{N}$. Then for $k + 1$ we get, according to Theorem 31,

$$\|u\|_{k+3,2} + \|p\|_{k+2,2} \leq c(\|f\|_{k+1,2} + \|\mathcal{D}C\|_\infty \|A\|_\infty \|u\|_{k+2,2} + \|g\|_{k+2,2} + \|u\|_2).$$

From (3.2.6) we have an estimate on $\|u\|_{k+2,2}$ and we immediately get the first claim of Theorem 3. If A_1 holds, Theorem 1 give us an estimate on $\|u\|_{1,2}$, whence $\|u\|_{k+2,2} + \|p\|_{k+1,2} \leq c(\|f\|_{k,2} + \|g\|_{k+1,2})$. □

3.3 Hölder regularity

In this section, we use results on solutions to the system

$$\begin{aligned} -\operatorname{div} A\mathcal{D}u + \nabla p &= \operatorname{div} F \text{ on } \Omega, \\ \operatorname{div} u &= 0 \text{ on } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{3.3.1}$$

proved in [8]. Results concerning regularity of weak solutions to (3.3.1) are given in the following theorem.

33 Theorem. *Let $A \in VMO_B$ be elliptic and Ω be a C^1 domain. Then there exists a positive constant c_8 such that, for any (u, p) which solves* (3.3.1) *and a right hand side $F \in L^{2,\mu}(\Omega, \mathbb{R}^{d^2})$, $(0 \leq \mu < d)$, we have*

$$\|\mathcal{D}u\|_{L^{2,\mu}} + \|p\|_{L^{2,\mu}} \leq c_8 \|F\|_{L^{2,\mu}}. \tag{3.3.2}$$

Let (u,p) be a weak solution to (1.1.5). Then assumption $p \in L^{2,\mu}$ leads to the claim that p and $\mathcal{D}u$ are in $L^{2,\mu+2}$. This fact is formulated in the following lemma.

34 Lemma. *Let Ω be a C^1 domain, $\Omega_1 \subset \Omega$ be a nonempty open subset, $0 < \mu < d-2$ (resp. $\mu = 0$ for $d \le 2$) and $\nu \in [\mu, \mu+2]$ (resp. $\nu \in [0,d)$ for $d \le 2$). Let $A \in VMO_B$ be symmetric and elliptic, $f = \operatorname{div} F$, $F \in L^{2,\nu}(\Omega, \mathbb{R}^{d^2})$, $B \in W^{1,\infty}(\Omega, \mathbb{R}^{d^2})$, $c_8\|I-B\|_\infty =: l < 1$ and $g = 0$. We suppose, moreover, that a weak solution $(u,p) \in W^{1,2}(\Omega) \times L^{2,\mu}(\Omega)$ to (1.1.5) fulfills $\int_{\Omega_1} p = 0$. Then there exists a constant c such that*

$$\|\mathcal{D}u\|_{L^{2,\nu}} + \|p\|_{L^{2,\nu}} \le c(\|F\|_{L^{2,\nu}} + \|p\|_{L^{2,\mu}}).$$

Proof. From $B = I - K$, the first equation in (1.1.5) can be rewritten as

$$-\operatorname{div} A\mathcal{D}u + (I-K)\nabla p = \operatorname{div} F,$$

which is equivalent to

$$-\operatorname{div} A\mathcal{D}u + \nabla p = \operatorname{div} F + \operatorname{div}(Kp) - (\operatorname{div} K)p. \tag{3.3.3}$$

The first and third terms on the right hand side are in appropriate Morrey spaces. To handle the second term, we use Banach fixed-point theorem. Let us equip the space $W^{1,2,\nu}_{0,\mathrm{div}}(\Omega) \times L^{2,\nu}(\Omega)$ with a norm $\|(u,p)\| \overset{def}{=} \|\mathcal{D}u\|_{2,\nu} + \|p\|_{2,\nu}$. Fix (u,p) and, for a given F, we define an operator $P : W^{1,2,\nu}_{0,\mathrm{div}} \times L^{2,\nu} \mapsto W^{1,2,\nu}_{0,\mathrm{div}} \times L^{2,\nu}$ by

$$P(v,q) = (w,r) \overset{def}{\Leftrightarrow}$$
$$-\operatorname{div} A\mathcal{D}w + \nabla r = \operatorname{div} F + \operatorname{div}(Kq) - (\operatorname{div} K)p \quad \& \int_{\Omega_1} r = 0. \tag{3.3.4}$$

The right hand side of the equation in (3.3.4) can be expressed as $\operatorname{div} G$ where G is in a space $L^{2,\nu}(\Omega, \mathbb{R}^{d^2})$. Indeed, F and Kq are in $L^{2,\nu}$ and $(\operatorname{div} K)p$ is in $L^{2,\mu}$. Thus, according to Lemma 19, $(\operatorname{div} K)p$ can be expressed as a divergence of some function from $L^{2,\nu}(\Omega, \mathbb{R}^{d^2})$. Theorem 1 gives the existence of a unique solution to the equation (3.3.4) and from Theorem 33 it follows that this solution is in $W^{1,2,\nu}_{0,\mathrm{div}}(\Omega) \times L^{2,\nu}(\Omega, \mathbb{R})$. Thus target space of the operator P is $W^{1,2,\nu}_{0,\mathrm{div}}(\Omega) \times L^{2,\nu}(\Omega, \mathbb{R})$ and the operator is well defined.

Let us estimate a norm $\|P(v_1,q_1) - P(v_2,q_2)\| = \|\mathcal{D}w_1 - Dw_2\|_{L^{2,\nu}} + \|r_1 - r_2\|_{L^{2,\nu}}$. Due to the linearity of (1.1.5) we have

$$-\operatorname{div} A\mathcal{D}(w_1 - w_2) + \nabla(r_1 - r_2) = -\operatorname{div}(K(q_1 - q_2)).$$

According to Theorem 33 and Lemma 19

$$\|\mathcal{D}w_1 - \mathcal{D}w_2\|_{L^{2,\nu}} + \|r_1 - r_2\|_{L^{2,\nu}} \le c_8 \|K\|_\infty \|q_1 - q_2\|_{L^{2,\nu}} = l\|q_1 - q_2\|_{L^{2,\nu}}.$$

Hence, due to assumptions, the mapping P is a contraction. Note that the whole procedure can be done even for P extended on $W^{1,2}_{0,\mathrm{div}}(\Omega) \times L^2(\Omega, \mathbb{R})$. That is, $P : W^{1,2}_{0,\mathrm{div}}(\Omega) \times L^2(\Omega, \mathbb{R}) \mapsto W^{1,2}_{0,\mathrm{div}}(\Omega) \times L^2(\Omega, \mathbb{R})$ is also a contraction. Therefore, there exists a fixed point, i.e. a pair $(v_0, q_0) \in W^{1,2,\nu}_{0,\mathrm{div}}(\Omega) \times L^{2,\nu}(\Omega, \mathbb{R})$ such that $P(v_0, q_0) = (v_0, q_0)$. Because P is a contraction on the space $W^{1,2}_{0,\mathrm{div}}(\Omega) \times L^2(\Omega, \mathbb{R})$, this fixed point coincides with the solution (u, p). We get

$$\|\mathcal{D}u\|_{L^{2,\nu}} + \|p\|_{L^{2,\nu}} \le c\|F\|_{L^{2,\nu}} + l\|p\|_{L^{2,\nu}} + c\|p\|_{L^{2,\mu}}.$$

The claim follows immediately due to the assumption $l < 1$. □

As a consequence of the previous lemma we get a proof of Theorem 4.

Proof of Theorem 4. Let $B \in W^{1,\infty}(\Omega, \mathbb{R}^{d^2})$ and let $c_8\|I - B\|_\infty =: l < 1$. For a dimension two or less we get the claim immediately from Lemma 34. We now assume that a dimension is greater than two. Note that, according to Theorem 1, we get the claim for $\mu = 0$. Suppose for a moment that the claim is true for some μ_0. Then Lemma 34 gives the validity of the claim for $\mu < \min\{d, \mu_0 + 2\}$ and the Theorem is proven by induction. □

3.4 Few additional lemmas

35 Lemma. *Let Ω be a bounded Lipschitz domain, $A \in L^\infty(\Omega, \mathbb{R}^{d^4})$ be an elliptic matrix and $(u, p) \in W^{1,2}(\Omega, \mathbb{R}^d) \times L^2(\Omega, \mathbb{R})$, $\int_\Omega p = 0$, be a weak solution to the system*

$$\begin{aligned} -\operatorname{div} A\mathcal{D}u + \nabla p &= \operatorname{div} F, \\ \operatorname{div} u &= g, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{3.4.1}$$

Then there exists $\delta > 0$ such that, for $F \in L^{2+\delta}(\Omega, \mathbb{R}^{d^2})$ and $g \in L^{2+\delta}(\Omega, \mathbb{R})$,

$$\|\mathcal{D}u\|_{2+\delta} + \|p\|_{2+\delta} \le c(\|F\|_{2+\delta} + \|g\|_{2+\delta}). \tag{3.4.2}$$

Proof. According to Bogovskiǐ lemma (see [2] for more) there exists u_1 such that $\operatorname{div} u_1 = g$, $u_1|_{\partial\Omega} = 0$ and $\|\mathcal{D}u_1\|_{2+\delta} \le c\|g\|_{2+\delta}$.

Let (u_0, p) solve the following system

$$\begin{aligned} -\operatorname{div} A\mathcal{D}u_0 + \nabla p &= \operatorname{div} F + \operatorname{div} A\mathcal{D}u_1, \\ \operatorname{div} u_0 &= 0, \\ u_0|_{\partial\Omega} &= 0. \end{aligned}$$

According to Lemma 2.6 in [17], we have

$$\|\mathcal{D}u_0\|_{2+\delta} \le c\|F + A\mathcal{D}u_1\|_{2+\delta} \le c(\|F\|_{2+\delta} + \|g\|_{2+\delta}).$$

Finally, Lemma 2.7 in [2] implies

$$\|p\|_{2+\delta} \le c(\|F + A\mathcal{D}(u_0 + u_1)\|_{2+\delta}) \le c(\|F\|_{2+\delta} + \|g\|_{2+\delta}).$$

As a consequence, the pair $(u = u_0 + u_1, p)$ solves (3.4.1) and (3.4.2) holds □

36 Lemma. *Let $A \in L^\infty(B_R(0), \mathbb{R}^{d^4})$ be an elliptic matrix and let $(u,p) \in W^{1,2}(B_R^+(0), \mathbb{R}^d) \times L^2(B_R^+(0), \mathbb{R})$, $\int_{B_R^+} p = 0$, (resp. $(u,p) \in W^{1,2}(B_R(0), \mathbb{R}^d) \times L^2(B_R(0), \mathbb{R})$, $\int_{B_R} p = 0$) be a weak solution to a system*

$$\begin{aligned} -\operatorname{div} A\mathcal{D}u + \nabla p &= \operatorname{div} F, \\ \operatorname{div} u &= g, \\ u|_{\partial B_R^+(0)} &= 0, \\ (\textit{resp.}\ \ u|_{\partial B_R(0)} &= 0). \end{aligned} \tag{3.4.3}$$

Then there exists $\delta > 0$ such that, for functions $F \in L^{2+\delta}(B_R(0), \mathbb{R}^{d^2})$ and $g \in L^{2+\delta}(B_R(0), \mathbb{R})$, we get $p \in L^{2+\delta}(B_R^+(0), \mathbb{R})$ (resp. $p \in L^{,2+\delta}(B_R(0), \mathbb{R})$). Moreover, there exists a constant c_9 independent of R and right hand side such that

$$\|p\|_{2+\delta} \le c_9(\|F\|_{2+\delta} + \|g\|_{2+\delta}).$$

Proof. For $R = 1$, it follows from Lemma 35. For arbitrary $R > 0$, it suffices to use change of variables. Set $\tilde{u}(x) = u(Rx)$, $\tilde{p}(x) = p(Rx)$, $\tilde{F}(x) = F(Rx)$ and $\tilde{g}(x) = g(Rx)$ for $x \in B_1^+(0)$. Then $(\tilde{u}, \tilde{p})$ solves

$$\begin{aligned} -\operatorname{div} A\mathcal{D}\tilde{u} + \nabla R\tilde{p} &= \operatorname{div} R\tilde{F} \text{ in } B_1^+(0), \\ \operatorname{div} \tilde{u} &= R\tilde{g} \text{ in } B_1^+(0), \\ \tilde{u}|_{\partial B_1^+(0)} &= 0, \\ (\text{resp.}\ \ \tilde{u}|_{\partial B_1(0)} &= 0). \end{aligned}$$

By Lemma 35, we get

$$\|R\tilde{p}\|_{2+\delta} \le c(\|R\tilde{F}\|_{2+\delta} + \|R\tilde{g}\|_{2+\delta}),$$

where c does not depend on R, which implies the result. □

37 Remark. *Let assumptions of the previous lemma hold. It is also true, that*

$$\|p\|_2 \le c_{10}(\|F\|_2 + \|g\|_2).$$

Furthermore, according to Lemma 2.6 in [17], it holds that $c_{10}c_9^{-1} < 1$.

38 Corollary. *Let* $A \in L^\infty(B_R(0), \mathbb{R}^{d^4})$ *be an elliptic matrix and let a matrix* $B \in L^\infty(B_R(0), \mathbb{R}^{d^2})$ *satisfy* $\|B\|_\infty < c_9^{-1}$. *Let* $(u,p) \in W^{1,2}(B_R^+(0), \mathbb{R}^d) \times L^2(B_R^+(0), \mathbb{R})$ *(resp.* $(u,p) \in W^{1,2}(B_R(0), \mathbb{R}^d) \times L^2(B_R(0)\mathbb{R})$*) be a weak solution to a system*

$$\begin{aligned} -\operatorname{div} A\mathcal{D}u + \nabla p &= \operatorname{div} F - \operatorname{div}(Bp), \\ \operatorname{div} u &= g, \\ u|_{\partial B_R^+(0)} &= 0, \\ (\textit{resp.}\ \ u|_{\partial B_R(0)} &= 0). \end{aligned} \tag{3.4.4}$$

Then there exists $\delta > 0$ *and* c_{11} *such that, for* $F, g \in L^{2+\delta}(B_R(0))$, *we get* $u \in W^{1,2+\delta}(B_R^+(0), \mathbb{R}^d)$, $p \in L^{2+\delta}(B_R^+(0), \mathbb{R})$ *(resp.* $u \in W^{1,2+\delta}(B_R^+(0), \mathbb{R}^d)$, $p \in L^{2+\delta}(B_R^+(0), \mathbb{R})$*). Moreover, if* $\int_{B_R^+} p = 0$ *(resp.* $\int_{B_R} p = 0$*), then*

$$\|\mathcal{D}u\|_{2+\delta} + \|p\|_{2+\delta} \le c_{11}(\|F\|_{2+\delta} + \|g\|_{2+\delta}).$$

Proof. We give the proof only for the upper half ball; the other case can be proven in a similar way. For given $q \in L^{2+\delta}(B_R^+(0), \mathbb{R})$ let v, q' be a weak solution to a system

$$\begin{aligned} -\operatorname{div} A\mathcal{D}v + \nabla q' &= \operatorname{div} F - \operatorname{div}(Bq) \text{ in } B_R^+(0), \\ \operatorname{div} v &= 0 \text{ in } B_R^+(0), \\ v|_{\partial B_R^+(0)} &= 0, \\ \int_{B_R^+} q' &= \int_{B_R^+} p \end{aligned}$$

and we define operator $T : L^{2+\delta}(B_R^+(0), \mathbb{R}) \mapsto L^{2+\delta}(B_R^+(0), \mathbb{R})$ as $T(q) = q'$. This operator is well defined according to the previous lemma. Let $q_1, q_2 \in L^{2+\delta}$ be arbitrary and set $q_1' = T(q_1)$ and $q_2' = T(q_2)$. The linearity of the generalized Stokes problem implies

$$\begin{aligned} -\operatorname{div} A\mathcal{D}(v_1 - v_2) + \nabla(q_1' - q_2') &= \operatorname{div}(B(q_1 - q_2)) \text{ in } B_R^+(0), \\ \operatorname{div}(v_1 - v_2) &= 0 \text{ in } B_R^+(0), \\ (v_1 - v_2)|_{\partial B_R^+(0)} &= 0 \end{aligned}$$

and $\int_{B_R}(q_1' - q_2') = 0$. From Lemma 36 we obtain

$$\|q_1' - q_2'\|_{2+\delta} \le c_9\|B\|_\infty\|q_1 - q_2\|_{2+\delta} \le \gamma\|q_1 - q_2\|_{2+\delta}$$

where $\gamma = c_9\|B\|_\infty < 1$. Hence T is a contraction and thus there exists $q \in L^{2+\delta}(B_R^+(0), \mathbb{R})$ such that $T(q) = q$ and

$$\begin{aligned} -\operatorname{div} A\mathcal{D}v + \nabla q &= \operatorname{div} F - \operatorname{div} Bq \text{ in } B_R^+(0), \\ \operatorname{div} v &= 0 \text{ in } B_R^+(0), \\ v|_{\partial B_R^+(0)} &= 0. \end{aligned}$$

It can be derived from Lemma 36 that $v \in W^{1,2+\delta}$. Functions (v, q) coincide with (u, p) since (3.4.4) has a unique solution as proven further. Therefore, for $\int_{B_R^+} p = 0$, we get following estimate by Lemma 35

$$\|\mathcal{D}u\|_{2+\delta} + \|p\|_{2+\delta} \leq c\left(\|F\|_{2+\delta} + \|g\|_{2+\delta} + \|Bp\|_{2+\delta}\right) \leq c_{11}\left(\|f\|_{2+\delta} + \|g\|_{2+\delta}\right).$$

It remains to prove the uniqueness of solution to (3.4.4). Let (u_1, p_1), $(u_2, p_2) \in W^{1,2}(B_R^+(0), \mathbb{R}^d) \times L^2(B_R^+(0), \mathbb{R})$ be weak solutions to (3.4.4) such that $\int_{B_R^+(0)} p_1 = \int_{B_R^+(0)} p_2$. Then

$$\begin{aligned} -\operatorname{div} A\mathcal{D}(u_1 - u_2) + \nabla(p_1 - p_2) &= -\operatorname{div} B(p_1 - p_2), \\ \operatorname{div}(u_1 - u_2) &= 0, \\ u_1 - u_2|_{\partial B_R^+(0)} &= 0 \end{aligned}$$

and $\int_{B_R^+(0)} p_1 - p_2 = 0$. Thus, according to Lemma 36,

$$\|p_1 - p_2\|_2 \leq c_{10}c_9^{-1}\|p_1 - p_2\|_2.$$

Since $c_{10}c_9^{-1} < 1$, we get $p_1 = p_2$ and, consequently, $u_1 = u_2$. □

39 Corollary. *Let $R_1 > 0$ and let $A \in L^\infty(B_{R_1}^+(0), \mathbb{R}^{d^4})$ be an elliptic matrix and let $B \in L^\infty(B_{R_1}^+(0), \mathbb{R}^{d^2})$ satisfy $\|B\|_\infty < c_9^{-1}$. Then there exists R_0 such that for all $R \in (0, R_0)$ the following holds.*
Let $(u, p) \in W^{1,2}(B_R^+(0), \mathbb{R}^d) \times L^2(B_R^+(0)\mathbb{R})$ be a weak solution to a system

$$\begin{aligned} -\operatorname{div} A\mathcal{D}u + \nabla p &= \operatorname{div} F - \operatorname{div}(Bp) + RS(u, p) \text{ on } B_R^+(0) \\ \operatorname{div} u &= g \text{ on } B_R^+(0) \\ u|_{\partial B_R^+} &= 0, \end{aligned} \tag{3.4.5}$$

where $S : W^{1,2+\delta}(B_R^+(0), \mathbb{R}^d) \times L^{2+\delta}(B_R^+(0), \mathbb{R}) \mapsto W^{-1,2+\delta}(B_R^+(0), \mathbb{R}^d)$ is a linear operator which is bounded independently of R.
Then there exists $\delta > 0$ such that for $(F, g) \in L^{2+\delta}(B_R^+(0), \mathbb{R}^{d^2} \times \mathbb{R})$, we get $(u, p) \in W^{1,2+\delta}(B_R^+(0), \mathbb{R}^d) \times L^{2+\delta}(B_R^+(0), \mathbb{R})$

Proof. As in the previous proof, we use Banach fixed–point theorem. We define $T : W_0^{1,2+\delta}(B_R^+(0), \mathbb{R}^d) \times L_0^{2+\delta}(B_R^+(0)) \mapsto W_0^{1,2+\delta}(B_R^+(0), \mathbb{R}) \times L_0^{2+\delta}(B_R^+(0))$ as follows

$$T(v,r) = (u,p) \Leftrightarrow \begin{array}{rcl} -\operatorname{div} A\mathcal{D}u + \nabla p &=& \operatorname{div} F - \operatorname{div}(Bp) + RS(v,r) \\ \operatorname{div} u &=& q \\ u|_{\partial B_R^+} &=& 0. \end{array}$$

Let $(u_i, p_i) = T(v_i, r_i)$, $i \in \{1, 2\}$. Then

$$\begin{array}{rcl} -\operatorname{div} A\mathcal{D}(u_1 - u_2) + \nabla(p_1 - p_2) &=& -\operatorname{div}(B(p_1 - p_2)) \\ && +RS(v_1 - v_2, q_1 - q_2) \text{ in } B_R^+(0) \\ \operatorname{div}(u_1 - u_2) &=& 0 \text{ in } B_R^+(0) \\ (u_1 - u_2)|_{\partial B_R^+} &=& 0. \end{array}$$

According to Lemma 38 it holds, that

$$\|\mathcal{D}(u_1 - u_2)\|_{2+\delta, B_R^+} + \|p_1 - p_2\|_{2+\delta, B_R^+} \leq Rc_{11}c \left(\|\mathcal{D}(v_1 - v_2)\|_{2+\delta, B_R^+} + \|q_1 - q_2\|_{2_\delta, B_R^+} \right).$$

It is enough to choose R_0 such that $R_0 c_{11} c < 1$ and the operator T is a contraction for any $R \in (0, R_0)$. Uniqueness of solution to Stokes problem implies the claim of the corollary. □

Chapter 4

Navier–Stokes System with Pressure–dependent Viscosity

Throughout this chapter, we focus on the equation (1.1.2) in dimension d equal 2 or 3.

4.1 Existence of Solution

40 Lemma. *Let Ω be a Lipschitz domain, $c_3 < \frac{c_1}{(c_1+c_2)c_7}$. Then there exists a constant $c > 0$ such that for all $f \in W^{-1,2}(\Omega, \mathbb{R}^d)$ there exists a weak solution $(u, p) \in W^{1,2}(\Omega, \mathbb{R}^d) \times L^2_0(\Omega)$ to* (1.1.2) *satisfying*

$$\|\nabla u\|_2 + \|p\|_2 \leq c\|f\|_{-1,2}$$

Proof. Since we use the same method as in [9] where an analogous result is proven for the growth $m < 2$, we provide only a sketch of the proof. This sketch is divided into two steps. At first, we introduce an approximative problem

$$\begin{aligned} -\operatorname{div} T(\mathcal{D}u^\varepsilon, p^\varepsilon) + (u^\varepsilon \nabla)u^\varepsilon + \frac{\operatorname{div} u^\varepsilon}{2} u^\varepsilon + \nabla p^\varepsilon &= f \text{ in } \Omega, \\ -\varepsilon \Delta p^\varepsilon + \varepsilon p^\varepsilon + \operatorname{div} u^\varepsilon &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega \\ \frac{\partial p^\varepsilon}{\partial \nu} &= 0 \text{ on } \partial\Omega \end{aligned} \tag{4.1.1}$$

and we show the existence of solution $(u^\varepsilon, p^\varepsilon)$ to (4.1.1). Then we find a sequence $(u^{\varepsilon_n}, p^{\varepsilon_n})$ converging to (u, p) and we show that (u, p) is a solution to (1.1.2).

Existence of solution to the approximative problem

In order to prove the existence of solution to (4.1.1), we use the Galerkin approximations.

Let $\{\alpha^k\}_{k=1}^\infty$ be a basis in $W^{1,2}(\Omega,\mathbb{R})$ and $\{a^k\}_{k=1}^\infty$ be a basis in $W_0^{1,2}(\Omega,\mathbb{R}^d)$. For $n\in\mathbb{N}$ set

$$p^n=\sum_{k=1}^n c_k^n\alpha^k,\qquad u^n=\sum_{k=1}^n d_k^n a^k,$$

where p^n and u^n solve a system

$$\varepsilon\int_\Omega \nabla p^n\nabla\alpha^r+\varepsilon\int_\Omega p^n\alpha^r-\int_\Omega u^n\nabla\alpha^r=0,\qquad r=1,\dots,n, \tag{4.1.2}$$

$$\int_\Omega T(\mathcal{D}u^n,p^n)\mathcal{D}a^s+\int_\Omega(u^n\nabla)u^na^s+\int_\Omega\frac{\operatorname{div}u^n}{2}u^na^s= \\ -\int_\Omega\nabla p^na^s+[f,a^s]_{W_0^{1,2}},\qquad s=1,\dots,n. \tag{4.1.3}$$

We multiply (4.1.2) by c_r^n, (4.1.3) by d_s^n and we sum all together over $r=1,\dots,n$ and $s=1,\dots,n$. Since

$$\int_\Omega(u^n\nabla)u^nu^n+\int_\Omega\operatorname{div}u^n\frac{|u^n|^2}{2}=0, \tag{4.1.4}$$

we get

$$\varepsilon\left(\|\nabla p^n\|_2^2+\|p^n\|_2^2\right)+\int_\Omega T(\mathcal{D}u^n,p^n)\mathcal{D}u^n=[f,u^n]_{W_0^{1,2}}.$$

Lemma 21 implies

$$\varepsilon\left(\|\nabla p^n\|_2^2+\|p^n\|_2^2\right)+\|\nabla u^n\|_2^2\le c_{12}$$

and

$$\|T(\mathcal{D}u^n,p^n)\|_2^2\le c_{12}.$$

Thus, up to a subsequence, $(u^n,p^n)\to(u,p)$ weakly in $W^{1,2}(\Omega,\mathbb{R}^d)\times W^{1,2}(\Omega,\mathbb{R})$ and $(u^n,p^n)\to(u^\varepsilon,p^\varepsilon)$ strongly in $L^4(\Omega,\mathbb{R}^d)\times L^2(\Omega,\mathbb{R})$. Moreover, $T(\mathcal{D}u^n,p^n)\to\chi$ weakly in $L^2(\Omega,\mathbb{R}^{d^2})$. That is enough to assert that, for all $\varphi\in W^{1,2}(\Omega,\mathbb{R})$ and for all $\psi\in W_0^{1,2}(\Omega,\mathbb{R}^d)$,

$$\varepsilon\int_\Omega\nabla p^\varepsilon\nabla\varphi+\varepsilon\int_\Omega p^\varepsilon\varphi+\int_\Omega\operatorname{div}u^\varepsilon\varphi=0, \tag{4.1.5}$$

$$\int_\Omega(u^\varepsilon\nabla)u^\varepsilon\psi+\frac{1}{2}\int_\Omega(\operatorname{div}u^\varepsilon)u^\varepsilon\psi+\int_\Omega\chi\mathcal{D}\psi-\int_\Omega p^\varepsilon\operatorname{div}\psi=[f,\psi]_{W^{-1,2}}, \tag{4.1.6}$$

$$\varepsilon\left(\|\nabla p^\varepsilon\|_2^2+\|p^\varepsilon\|_2^2\right)+\int_\Omega\chi\mathcal{D}u^\varepsilon=[f,u^\varepsilon]_{W^{-1,2}}.$$

In order to conclude the first part of the proof, it is sufficient to show $T(p^\varepsilon,\mathcal{D}u^\varepsilon)=\chi$. We still proceed as in [9]. First we prove strong convergence of $\mathcal{D}u^n$ to $\mathcal{D}u^\varepsilon$

in $L^2(\Omega,\mathbb{R}^{d^2})$. Lemma 21 implies

$$\begin{aligned}
c_1\|\mathcal{D}u^n-\mathcal{D}u^\varepsilon\|_2^2 \le& \int_\Omega \left(T(\mathcal{D}u^n,p^n)-T(\mathcal{D}u^\varepsilon,p^\varepsilon)\right)\left(\mathcal{D}u^n-\mathcal{D}u^\varepsilon\right)+\frac{c_3}{2c_1}\|p^n-p^\varepsilon\|_2^2\\
=&\int_\Omega T(\mathcal{D}u^n,p^n)\mathcal{D}u^n-\int_\Omega T(\mathcal{D}u^\varepsilon,p^\varepsilon)\mathcal{D}(u^n-u^\varepsilon)\\
&-\int_\Omega T(\mathcal{D}u^n,p^n)\mathcal{D}u^\varepsilon+\frac{c_3}{2c_1}\|p^n-p^\varepsilon\|_2^2\\
=&[f,u^n]_{W_0^{1,2}}-\varepsilon(\|\nabla p^n\|_2^2+\|p^n\|_2^2)-\int_\Omega T(\mathcal{D}u^\varepsilon,p^\varepsilon)\mathcal{D}(u^n-u^\varepsilon)\\
&-\int_\Omega T(\mathcal{D}u^n,p^n)\mathcal{D}u^\varepsilon+\frac{c_3}{2c_1}\|p^n-p^\varepsilon\|_2^2,
\end{aligned}$$

And, due to a weak lower semi-continuity of norms, we obtain

$$c_1\lim_{n\to\infty}\|\mathcal{D}u^n-\mathcal{D}u^\varepsilon\|_2^2\le[f,u^\varepsilon]_{W_0^{1,2}}-\varepsilon\left(\|\nabla p^\varepsilon\|_2^2+\|p^\varepsilon\|_2^2\right)-\int_\Omega \chi\mathcal{D}u^\varepsilon\le 0.$$

Thus, $\mathcal{D}u^n\to\mathcal{D}u^\varepsilon$ strongly in L^2, $(\mathcal{D}u^n,p^n)\to(\mathcal{D}u^\varepsilon,p^\varepsilon)$ almost everywhere in Ω. Due to the Vitali theorem,

$$\int_\Omega T(\mathcal{D}u^n,p^n)\mathcal{D}\psi\to\int_\Omega T(\mathcal{D}u^\varepsilon,p^\varepsilon)\mathcal{D}\psi=\int_\Omega\chi\mathcal{D}\psi.$$

Convergence of approximative solutions

We need to estimate p^ε and u^ε independently of ε. We take $\varphi=p^\varepsilon$ in (4.1.5) and $\psi=u^\varepsilon$ in (4.1.6). We get

$$\begin{aligned}
\varepsilon\left(\|\nabla p^\varepsilon\|_2^2+\|p^\varepsilon\|_2^2\right)+\int_\Omega p^\varepsilon\operatorname{div}u^\varepsilon &= 0,\\
\int_\Omega T(\mathcal{D}u^\varepsilon,p^\varepsilon)\mathcal{D}u^\varepsilon-\int_\Omega p^\varepsilon\operatorname{div}u^\varepsilon &= [f,u^\varepsilon]_{W_0^{1,2}}.
\end{aligned}$$

Consequently,

$$\varepsilon\left(\|\nabla p^\varepsilon\|_2^2+\|p^\varepsilon\|_2^2\right)+\|\nabla u^\varepsilon\|_2^2\le c_{13}$$

and, due to Lemma 21,

$$\|T(\mathcal{D}u^\varepsilon,p^\varepsilon)\|_2\le c_{13}.$$

We test equation (4.1.1) by φ^ε defined by

$$\begin{aligned}
\operatorname{div}\varphi^\varepsilon &= p^\varepsilon \text{ in } \Omega,\\
\varphi^\varepsilon &= 0 \text{ on } \partial\Omega.
\end{aligned}$$

We emphasize, that $\int_\Omega p^\varepsilon=0$ due to $(4.1.1)_2$ and $(4.1.1)_4$. Further, due to the Bogovskiĭ lemma, $\|\varphi\|_{1,2}\le c_7\|p^\varepsilon\|_2$. We obtain

$$\|p^\varepsilon\|_2^2=\int_\Omega T(\mathcal{D}u^\varepsilon,p^\varepsilon)\mathcal{D}\varphi^\varepsilon-[f,\varphi^\varepsilon]_{W_0^{1,2}}+\int_\Omega(u^\varepsilon\nabla)u^\varepsilon\varphi^\varepsilon+\frac{1}{2}\int_\Omega(\operatorname{div}u^\varepsilon)u^\varepsilon\varphi^\varepsilon.$$

It can be derived, using Lemma 21, that

$$
\begin{aligned}
\|p^\varepsilon\|_2^2 \leq& \,(c(1+\|\mathcal{D}u^\varepsilon\|_2)+\|f\|_{-1,2})\,\|\varphi^\varepsilon\|_{1,2}+2\|\nabla u^\varepsilon\|_2\|u^\varepsilon\|_4\|\varphi^\varepsilon\|_4\\
\leq& c\|\varphi^\varepsilon\|_{1,2}\leq c\|p^\varepsilon\|_2,
\end{aligned}
$$

and therefore $\|p^\varepsilon\|_2\leq c$. Thus, up to a subsequence, $(u^\varepsilon,p^\varepsilon)\to(u,p)$ weakly in $W_0^{1,2}(\Omega,\mathbb{R}^d)\times L^2(\Omega)$ and $T(\mathcal{D}u^\varepsilon,p^\varepsilon)\to\chi$ weakly in L^2. Above obtained estimate is enough to proceed to a limit in (4.1.1) as follows

$$
\begin{aligned}
\int_\Omega \chi\mathcal{D}\varphi+\int_\Omega(u\nabla)u\varphi-\int_\Omega p\operatorname{div}\varphi &= [f,\varphi]_{W_0^{1,2}},\\
\operatorname{div}u &= 0.
\end{aligned}
$$

As in the first step, it is sufficient to show that $\chi=T(\mathcal{D}u,p)$ which can be done by proving that $(\mathcal{D}u^\varepsilon,p^\varepsilon)\to(\mathcal{D}u,p)$ strongly in L^2. We define φ^ε as

$$
\begin{aligned}
\operatorname{div}\varphi^\varepsilon &= p^\varepsilon-p \text{ in } \Omega,\\
\varphi^\varepsilon &= 0 \text{ on } \partial\Omega.
\end{aligned}
$$

We remind, that $\varphi^\varepsilon\to0$ weakly in $W^{1,2}(\Omega,\mathbb{R}^d)$. Hence, by testing (4.1.1) by φ^ε, we get

$$
\begin{aligned}
\|p^\varepsilon-p\|_2^2=&\int_\Omega p(p^\varepsilon-p)-[f,\varphi^\varepsilon]_{W_0^{1,2}}+\frac{1}{2}\int_\Omega(\operatorname{div}u^\varepsilon)u^\varepsilon\varphi^\varepsilon+\int_\Omega T(\mathcal{D}u,p)\mathcal{D}\varphi^\varepsilon\\
&+\int_\Omega(u^\varepsilon\nabla)u^\varepsilon\varphi^\varepsilon+\int_\Omega\left(T(\mathcal{D}u^\varepsilon,p^\varepsilon)-T(\mathcal{D}u,p)\right)\mathcal{D}\varphi^\varepsilon
\end{aligned}
$$

and consequently,

$$
\lim_{\varepsilon\to0}\|p^\varepsilon-p\|_2^2=\lim_{\varepsilon\to0}\int_\Omega\left(T(\mathcal{D}u^\varepsilon,p^\varepsilon)-T(\mathcal{D}u,p)\right)\mathcal{D}\varphi^\varepsilon. \tag{4.1.7}
$$

It can be easily seen that

$$
\begin{aligned}
\int_\Omega\left(T(\mathcal{D}u^\varepsilon,p^\varepsilon)-T(\mathcal{D}u,p)\right)\mathcal{D}\varphi^\varepsilon\leq& \,c_2\int_\Omega|\mathcal{D}u^\varepsilon-\mathcal{D}u||\mathcal{D}\varphi^\varepsilon|+c_3\int_\Omega|p^\varepsilon-p||\mathcal{D}\varphi^\varepsilon|\\
\leq& c_2\|\mathcal{D}u^\varepsilon-\mathcal{D}u\|_2\|\mathcal{D}\varphi^\varepsilon\|_2+c_3\|p^\varepsilon-p\|_2\|\mathcal{D}\varphi^\varepsilon\|_2\\
=& c_2c_7\|\mathcal{D}u^\varepsilon-\mathcal{D}u\|_2\|p^\varepsilon-p\|_2+c_3c_7\|p^\varepsilon-p\|_2^2
\end{aligned} \tag{4.1.8}
$$

and further,

$$
\frac{c_1}{2}\|\mathcal{D}u^\varepsilon-\mathcal{D}u\|_2^2\leq\int_\Omega\left(T(\mathcal{D}u^\varepsilon,p^\varepsilon)-T(\mathcal{D}u,p)\right)(\mathcal{D}u^\varepsilon-\mathcal{D}u)+\frac{c_3^2}{2c_1}\|p^\varepsilon-p\|_2^2.
$$

We test (4.1.1) by $\varphi^\varepsilon=u^\varepsilon-u$. We obtain

$$
\begin{aligned}
\int_\Omega\left(T(\mathcal{D}u^\varepsilon,p^\varepsilon)-T(\mathcal{D}u,p)\right)(\mathcal{D}u^\varepsilon-\mathcal{D}u)=&-\int_\Omega T(\mathcal{D}u,p)\mathcal{D}(u^\varepsilon-u)\\
&+\int_\Omega p^\varepsilon\operatorname{div}(u^\varepsilon-u)+[f,u^\varepsilon-u]_{W_0^{1,2}}-\int_\Omega(u^\varepsilon\nabla)u^\varepsilon(u^\varepsilon-u)\\
&-\frac{1}{2}\int_\Omega(\operatorname{div}u^\varepsilon)u^\varepsilon(u^\varepsilon-u).
\end{aligned}
$$

Since $\int_\Omega p^\varepsilon \operatorname{div} u^\varepsilon = -\varepsilon(\|\nabla p^\varepsilon\|_2^2 + \|p^\varepsilon\|_2^2)$, we conclude that

$$\begin{aligned}\int_\Omega (T(\mathcal{D}u^\varepsilon, p^\varepsilon) - T(\mathcal{D}u, p)) (\mathcal{D}u^\varepsilon - \mathcal{D}u) + \varepsilon \left(\|\nabla p^\varepsilon\|_2^2 + \|p^\varepsilon\|_2^2\right) \\ = -\int_\Omega T(\mathcal{D}u, p)\mathcal{D}(u^\varepsilon - u) + [f, u^\varepsilon - u]_{W_0^{1,2}} - \int_\Omega (u^\varepsilon \nabla) u^\varepsilon (u^\varepsilon - u) \\ - \frac{1}{2}\int_\Omega (\operatorname{div} u^\varepsilon) u^\varepsilon (u^\varepsilon - u).\end{aligned}$$

Therefore

$$\lim_{\varepsilon\to 0} \frac{c_1}{2}\|\mathcal{D}u^\varepsilon - \mathcal{D}u\|_2^2 \leq \lim_{\varepsilon\to 0} \frac{c_3^2}{2c_1}\|p^\varepsilon - p\|_2^2 \tag{4.1.9}$$

and, consequently,

$$\lim_{\varepsilon\to 0} \|\mathcal{D}u^\varepsilon - \mathcal{D}u\|_2 \leq \lim_{\varepsilon\to 0} \frac{c_3}{c_1}\|p^\varepsilon - p\|_2 \tag{4.1.10}$$

From (4.1.7), (4.1.8) and (4.1.9) it may be concluded that

$$(1 - c_3c_7) \lim_{\varepsilon\to 0} \|p^\varepsilon - p\|_2^2 \leq \frac{c_2c_7c_3}{c_1} \lim_{\varepsilon\to 0} \|p^\varepsilon - p\|_2^2.$$

As $(1 - c_3c_7(1 + \frac{c_2}{c_1})) > 0$, it can be derived that

$$\lim_{\varepsilon\to 0} \|p^\varepsilon - p\|_2 = 0$$

and from (4.1.10) we get

$$\lim_{\varepsilon\to 0} \|\mathcal{D}(u^\varepsilon - u)\|_2 = 0,$$

whence the proof is complete. □

4.2 Higher differentiability

41 Lemma. *Let Ω be a C^2 domain and let $f \in L^2(\Omega, \mathbb{R}^d)$. Let assumption* (1.1.3) *be satisfied with $c_3 < \frac{c_1}{(c_1 + c_7c_2)c_7}$. Then a weak solution to* (1.1.2) *belongs to $W^{2,2}(\Omega, \mathbb{R}^d) \times W^{1,2}(\Omega, \mathbb{R})$.*

Proof. As an interior regularity has been proven already (see e.g. [24]), we focus only on boundary regularity. Unknowns u and p satisfy following integral identity

$$\int_\Omega T(\mathcal{D}u, p)\mathcal{D}\varphi - (u \otimes u)\nabla\varphi - p \operatorname{div} \varphi - f\varphi = 0$$

for all $\varphi \in W_0^{1,2}$. Let $0 \in \partial\Omega$ and suppose that φ is supported in some sufficiently small neighborhood $\Omega_{0,R}$. A precise value of R will be specified later. We define functions

$$\hat{u}(x) = u\left(F_R\left(\frac{x}{R}\right)\right),$$
$$\hat{p}(x) = p\left(F_R\left(\frac{x}{R}\right)\right),$$
$$\hat{f}(x) = f\left(F_R\left(\frac{x}{R}\right)\right),$$
$$\psi(x) = \varphi\left(F_R\left(\frac{x}{R}\right)\right), \tag{4.2.1}$$

where $x \in B_R^+(0)$. We remind that $F_R\left(\frac{x}{R}\right) = F(x)$. We set $y = F(x)$. Following relations hold[1]

$$\nabla \hat{u}(x) = \nabla_y u(F(x)) \nabla F(x) = \nabla_y u(F(x)) I + R \nabla_y u(F(x)) \omega(x),$$
$$\mathcal{D}\hat{u}(x) = \mathcal{D}_y u(F(x)) + R\omega(x) \nabla_y u(F(x))$$

and thus $(\hat{u}, \hat{p})$ satisfy the equation

$$\int_{B_R^+(0)} T(\mathcal{D}_y u(F), p(F)) \mathcal{D}_y \varphi(F)\, |\det \nabla F| + \int_{B_R^+(0)} u(F) \otimes u(F) \nabla_y \varphi(F)\, |\det \nabla F|$$
$$- \int_{B_R^+(0)} p(F) \operatorname{div}_y \varphi(F)\, |\det \nabla F| - \int_{B_R^+(0)} f(F) \varphi(F)\, |\det \nabla F| = 0.$$

Let R be sufficiently small and $x \in B_R^+(0)$. Then we have

$$\nabla F^{-1}(y) = I + R\omega(y),$$
$$\nabla^2 F(x) < \infty.$$

The functions $(\hat{u}, \hat{p})$ fulfill

$$\int_{B_R^+(0)} T(\mathcal{D}\hat{u} + R\omega \nabla \hat{u}, \hat{p}) \mathcal{D}\psi \nabla F^{-1} - \int_{B_R^+(0)} (\hat{u} \otimes \hat{u}) \mathcal{D}\psi \nabla F^{-1}$$
$$- \int_{B_R^+(0)} \hat{p} \operatorname{Tr}(\nabla \psi \nabla F^{-1}) = \int_{B_R^+(0)} f\psi \tag{4.2.2}$$

for all $\psi \in W_0^{1,2}(B_R^+(0))$. In further calculations, we omit the term $|\det \nabla F|$. We provide only a sketch of the proof because we follow step-by-step the proof presented in [24]. Let $i \in \{1, \ldots, d-1\}$. We emphasize, that the operator $\Delta_{\delta e_i}$ is defined as $\Delta_{\delta e_i} f(x) = f(x + \delta e_i) - f(x)$. We apply operator $\frac{1}{\delta}\Delta_{\delta e_i}$ on equation (4.2.2). We denote $\frac{1}{\delta}\Delta_{\delta e_i}$ by Δ and $\frac{1}{\delta}\Delta_{-\delta e_i}$ by Δ_- to shorten the notation. We set

$$\hat{A}(x) = \int_0^1 \frac{\partial T(\mathcal{D}\hat{u}(x) + R\omega \nabla \hat{u}(x) + t\Delta_{\delta e_i}(\mathcal{D}\hat{u}(x) + R\omega \nabla \hat{u}(x)), p(x) + t\Delta_{\delta e_i} p(x))}{\partial \mathcal{D}} dt$$
$$\hat{B}(x) = \int_0^1 \frac{\partial T(\mathcal{D}\hat{u}(x) + R\omega \nabla \hat{u}(x) + t\Delta_{\delta e_i}(\mathcal{D}\hat{u}(x) + R\omega \nabla \hat{u}(x)), p(x) + t\Delta_{\delta e_i} p(x))}{\partial p} dt.$$

[1]The ω denotes, as usual, arbitrary matrix–, vector–, or real–valued function which is bounded independently on R and on the right hand side.

From equation (4.2.2), we conclude, that $(\hat{u}, \hat{p})$ satisfies

$$\int_{B_R^+(0)} \hat{A}\Delta\mathcal{D}\hat{u}\mathcal{D}\psi + \int_{B_R^+(0)} \hat{B}\Delta\hat{p}\mathcal{D}\psi + R\int_{B_R^+(0)} (\hat{A}\Delta\mathcal{D}\hat{u}\mathcal{D}\psi + \hat{B}\Delta\hat{p}\mathcal{D}\psi)\omega$$
$$+\int_{B_R^+(0)} T(\mathcal{D}\hat{u}+R\omega\hat{u},\hat{p})\mathcal{D}\psi\Delta\nabla F^{-1}+\int_{B_R^+(0)} \Delta(u\otimes u)\mathcal{D}\psi\nabla F^{-1}+\int_{B_R^+(0)} (u\otimes u)\mathcal{D}\psi\Delta\nabla F^{-1}$$
$$+\int_{B_R^+(0)} \Delta p\,\mathrm{Tr}(\nabla\psi\nabla F^{-1}) = \int_{B_R^+(0)} f\Delta_-\psi.$$

Choose a test function $\psi(x) = \eta^2(x)\Delta\hat{u}(x)$, where $\eta \in C^\infty(B_R^+)$ is a nonnegative cut-off function. In what follows, norms $\|\omega\|_\infty$, $\|\Delta_-\nabla F^{-1}\|_\infty$, $\|\eta\|_{1,\infty}$, $\|\hat{u}\|_{1,2}$ and $\|\hat{p}\|_2$ will be included in a general constant c. We obtain

$$(c_1 - Rc)\,\|\eta\mathcal{D}\Delta\hat{u}\|_{2,B_R^+} \le \int_{B_R^+} \eta^2(I+R\omega)\hat{A}\mathcal{D}\Delta(\hat{u}\eta)\mathcal{D}\Delta(\hat{u}\eta)$$
$$= -\int_{B_R^+} (I+R\omega)2\eta\hat{A}\mathcal{D}(\Delta\hat{u})\nabla\eta\Delta\hat{u}$$
$$-\int_{B_R^+} (I+R\omega)\hat{B}(\Delta p)\eta\mathcal{D}(\Delta\hat{u})\eta - 2\int_{B_R^+} (I+R\omega)\hat{B}(\Delta p)\eta\Delta\hat{u}\nabla\eta$$
$$+\int_{B_R^+} T(\mathcal{D}\hat{u}+R\omega\nabla\hat{u},\hat{p})\eta^2\mathcal{D}(\nabla\hat{u})\Delta\nabla F^{-1}+\int_{B_R^+} T(\mathcal{D}\hat{u}+R\omega\nabla\hat{u},\hat{p})2\eta\nabla\eta\nabla\hat{u}\Delta\nabla F^{-1}$$
$$+\int_{B_R^+} (I+R\omega)\Delta(\hat{u}\otimes\hat{u})\eta\mathcal{D}(\Delta u)\eta + \int_{B_R^+} (I+R\omega)\Delta(\hat{u}\otimes\hat{u})2\eta\nabla\eta\Delta u$$
$$+\int_{B_R^+} (\hat{u}\otimes\hat{u})2\eta\nabla\eta\Delta\hat{u}\Delta\nabla F^{-1} + \int_{B_R^+} (\hat{u}\otimes\hat{u})\eta^2\mathcal{D}(\nabla\hat{u})\Delta\nabla F^{-1}$$
$$+\int_{B_R^+} \Delta p\,\mathrm{Tr}(2\eta\nabla\eta\Delta\hat{u}\nabla F^{-1})+\int_{B_R^+} \Delta\hat{p}\eta^2\,\mathrm{Tr}(\Delta\nabla\hat{u}(I+R\omega))+\int_{B_R^+} f\Delta_-(\eta^2\Delta\hat{u}(x))$$
$$= -I_1 - I_2 - I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10} + I_{11} + I_{12}.$$

Since $\mathrm{Tr}(\nabla\hat{u}(I+R\omega)) = 0$, we immediately get $I_{11} = 0$. For I_1 and I_3 it is enough to use the Young inequality and boundedness of $\hat{A}$ and $\hat{B}$ to get

$$|I_1| \le c(\varepsilon) + \varepsilon\,\|\eta\mathcal{D}\Delta\hat{u}\|_{2,B_R^+} \tag{4.2.3}$$

and

$$|I_3| \le c(\varepsilon) + \varepsilon\,\|\eta\Delta\hat{p}\|_{2,B_R^+}\,. \tag{4.2.4}$$

The Young inequality also gives

$$|I_{10}| \le c(\varepsilon) + \varepsilon\,\|\eta\Delta\hat{p}\|_{2,B_R^+}\,. \tag{4.2.5}$$

The boundedness of $\hat{B}$ yields

$$|I_2| \le c_3\|\eta\Delta\hat{p}\|_{2,B_R^+}\|\eta\mathcal{D}\Delta\hat{u}\|_{2,B_R^+}. \tag{4.2.6}$$

The term $T(\mathcal{D}\hat{u} + R\omega\nabla\hat{u}, \hat{p})$ is estimated from above according to Lemma 21. Thus we have

$$|I_4| + |I_5| \leq c(\varepsilon) + \varepsilon \left\|\eta\mathcal{D}\Delta\hat{u}\right\|_{2,B_R^+}. \tag{4.2.7}$$

For term I_6 we have

$$\begin{aligned}|I_6| \leq & \left(\int_{B_R^+} (\eta\mathcal{D}\Delta\hat{u})^2\right)^{1/2} \left(\int_{B_R^+} (|\Delta\hat{u}||\hat{u}||\eta|)^2\right)^{1/2} \\ \leq & \|\eta\mathcal{D}\Delta\hat{u}\|_{2,B_R^+}\|\hat{u}\|_{6,B_R^+}\|\eta\Delta\hat{u}\|_{3,B_R^+}.\end{aligned}$$

The interpolation inequality $\|f\|_{3,B_R^+} \leq c\|f\|_{1,2,B_R^+}^{d/6}\|f\|_{2,B_R^+}^{1-d/6}$ (see Theorem 5.8 in [1]) implies

$$\begin{aligned}|I_6| &\leq c\|\eta\Delta\mathcal{D}\hat{u}\|_{2,B_R^+}\|\nabla(\eta\Delta\hat{u})\|_{2,B_R^+}^{d/6}\|\eta\Delta\hat{u}\|_{2,B_R^+}^{1-d/6} \leq c\|\eta\Delta\mathcal{D}\hat{u}\|_{2,B_R^+}\|\mathcal{D}(\eta\Delta\hat{u})\|_{2,B_R^+}^{d/6} \\ &\leq c\|\eta\Delta\mathcal{D}\hat{u}\|_{2,B_R^+}\left(\|\nabla\eta\Delta\hat{u}\|_{2,B_R^+}^{d/6} + \|\eta\Delta\mathcal{D}\hat{u}\|_{2,B_R^+}^{d/6}\right) \\ &\leq c\|\eta\Delta\mathcal{D}\hat{u}\|_{2,B_R^+} + c\|\eta\Delta\mathcal{D}\hat{u}\|_{2,B_R^+}^{1+d/6} \leq c(\varepsilon) + \varepsilon\|\eta\Delta\mathcal{D}u\|_{2B_R^+}^2.\end{aligned}$$

The same procedure may be applied on I_6, I_7 and I_8. Thus

$$|I_6| + |I_7| + |I_8| + |I_9| \leq c(\varepsilon) + \varepsilon\|\eta\Delta\mathcal{D}u\|_{2B_R^+}^2. \tag{4.2.8}$$

Finally,

$$|I_{12}| \leq c(\varepsilon) + \varepsilon \left\|\eta\mathcal{D}\Delta\hat{u}\right\|_{2,B_R^+}^2 \tag{4.2.9}$$

Inequalities (4.2.3), (4.2.4), (4.2.5), (4.2.6), (4.2.7), (4.2.8) and (4.2.9) yield

$$\begin{aligned}(c_1 - cR)\left\|\eta\mathcal{D}\Delta\hat{u}\right\|_{2,B_R^+} \leq \varepsilon &\left(\|\eta\mathcal{D}\Delta\hat{u}\|_{2,B_R^+}^2 + \|\eta\Delta\hat{p}\|_{2,B_R^+}^2\right) + \\ &+ (c_3 + \varepsilon)\left\|\eta\Delta\hat{p}\right\|_{2,B_R^+}\left\|\eta\mathcal{D}\Delta\hat{u}\right\|_{2,B_R^+} + c(\varepsilon) \end{aligned} \tag{4.2.10}$$

In order to get an estimate of pressure, we choose $\Phi \in W_0^{1,2}$ as a solution to the following problem

$$\begin{aligned}\operatorname{div}\Phi &= \eta\Delta\hat{p} - |B_R^+|^{-1}\int \eta(x)\Delta\hat{p}(x)dx \text{ in } \Omega, \\ \Phi &= 0 \text{ on } \partial\Omega.\end{aligned}$$

It holds that $\|\Phi\|_{1,2.B_R^+} \leq c_7\|\eta\Delta\hat{p}\|_{2,B_R^+}$ We use a test function $\psi = \eta\Phi$ to get

$$\begin{aligned}0 = \int_{B_R^+}\left(\hat{A}\mathcal{D}\Delta\hat{u} + \hat{B}\Delta\hat{p}\right)\eta\mathcal{D}\Phi(I+R\omega) + \int_{B_R^+}\Delta T(\mathcal{D}\hat{u}+R\omega\hat{u},\hat{p})\nabla\eta\Phi(I+R\omega) + \\ \int_{B_R^+}\Delta(\hat{u}\otimes\hat{u})\mathcal{D}(\eta\Phi) - \int_{B_R^+}\Delta p\eta\operatorname{div}(\eta\Phi) - \int_{B_R^+} f\Delta_-\Phi \\ - \int_{B_R^+}\Delta p\eta\operatorname{Tr}(\Phi(I-\nabla F^{-1})) + \int_{B_R^+} T(\mathcal{D}\hat{u} + R\omega\hat{u},\hat{p})\mathcal{D}(\eta\Phi)\Delta\nabla F^{-1} \\ + \int_{B_R^+}(\hat{u}\otimes\hat{u})\mathcal{D}(\eta\Phi)\Delta\nabla F^{-1} = J_1 + J_2 + J_3 - J_4 - J_5 - J_6 + J_7 + J_8.\end{aligned}$$

Hölder inequality implies

$$|J_1| \le \left(c_2 \|\eta\mathcal{D}\Delta\hat{u}\|_{2,B_R^+} + c_3 \|\eta\Delta\hat{p}\|_{2,B_R^+}\right)\left(c_7 \|\eta\Delta\hat{p}\|_{2,B_R^+} + c\right). \qquad (4.2.11)$$

Easily, using Young inequality,

$$|J_2| \le |\int_{B_R^+} T(\mathcal{D}\hat{u} + R\omega\hat{u}, \hat{p})\Delta_-(\nabla\eta\Phi)| \le c(\varepsilon) + \varepsilon\|\eta\Delta p\|^2_{2,B_R^+}, \qquad (4.2.12)$$

where ε stands for arbitrary real positive number. Further

$$\begin{aligned} |J_3| \le & \left|\int_{B_R^+} \Delta\hat{u}\hat{u}\eta\nabla\Phi\right| + \left|\int_{B_R^+} \Delta(\hat{u}\otimes\hat{u})\nabla\eta\Phi\right| \\ \le & \|\Delta\hat{p}\eta\|_{2,B_R^+}\|\hat{u}\eta\Delta\hat{u}\|_{2,B_R^+} + |\int_{B_R^+} (\hat{u}\otimes\hat{u})\Delta_-(\nabla\eta\Phi)|. \end{aligned}$$

Because $\|\hat{u}\eta\Delta\hat{u}\|_{2,B_R^+} \le \|\hat{u}\|_{6,B_R^+}\|\eta\Delta\hat{u}\|_{3,B_R^+} \le c\|\hat{u}\|_{6,B_R^+}\|\nabla(\eta\Delta\hat{u})\|^{d/6}_{2,B_R^+}\|\eta\Delta\hat{u}\|^{1-d/6}_{2,B_R^+}$, we get

$$|J_3| \le \epsilon\|\eta\Delta\mathcal{D}\hat{u}\|_{2,B_R^+}\|\eta\Delta p\|_{2,B_R^+} + \varepsilon\|\eta\Delta p\|^2_{2,B_R^+} + c(\varepsilon). \qquad (4.2.13)$$

Further,

$$\begin{aligned} J_4 \ge & \int_{B_R^+} \Delta p(\eta \operatorname{div}\Phi + \nabla\eta\Phi) \ge \|\eta\Delta p\|_2 - \left(\int_{B_R^+}(\Delta p)\eta\right)^2 - \int_{B_R^+} |p\Delta_-(\nabla\eta\Phi)| \\ \ge & \|(\Delta p)\eta\|^2_{2,B_R^+} - c(\varepsilon) - \varepsilon\|(\Delta p)\eta\|^2_{2,B_R^+}. \end{aligned} \qquad (4.2.14)$$

Easily

$$|J_5| \le c(\varepsilon) + \varepsilon \|\eta\Delta\hat{p}\|_{2,B_R^+}. \qquad (4.2.15)$$

Finally

$$|J_6| \le cR\|\Delta p\eta\|^2_{2,B_R^+}, \qquad (4.2.16)$$

$$|J_7| \le c(\varepsilon) + \varepsilon\|\Delta p\eta\|^2_{2,B_R^+} \qquad (4.2.17)$$

and, since $(u\otimes u) \in L^3$, we get

$$|J_8| \le c(\varepsilon) + \varepsilon\|\Delta p\eta\|^2_2. \qquad (4.2.18)$$

Thus we have

$$\|\eta\Delta\hat{p}\|^2_{2,B_R^+} \le (c_2c_7+\varepsilon+cR)\|\eta\Delta\mathcal{D}\hat{u}\|_{2,B_R^+}\|\eta\Delta p\|_{2,B_R^+} + (c_3c_7+\varepsilon+cR)\|\eta\Delta p\|^2_{2,B_R^+} + c(\varepsilon). \qquad (4.2.19)$$

Here we use Young inequality in a form $ab \le \frac{a^2}{2(c_7c_2+c_1)} + \frac{b^2(c_7c_2+c_1)}{2}$. We obtain

$$\left(1 - c_3c_7 - \frac{c_2c_7}{2(c_7c_2+c_1)} + \varepsilon + cR\right)\|\eta\Delta\hat{p}\|^2_{2,B_R^+} \le \left(\frac{c_2c_7(c_2c_7+c_1)}{2} + \varepsilon + Rc\right)\|\eta\Delta\mathcal{D}\hat{u}\|^2_{2,B_R^+} + c(\varepsilon).$$

The assumption $c_3 < \frac{c_1}{(c_1+c_2c_7)c_7}$ implies that there exist $R > 0$ and $\varepsilon > 0$ such that

$$1 - c_3c_7 - \frac{c_1c_7}{2(c_2c_7 + c_1)} - \varepsilon - Rc > \frac{c_2c_7}{2(c_2c_7 + c_1)}.$$

Thus

$$\|\eta\Delta\hat{p}\|^2_{2,B_R^+} \leq \left((c_2c_7 + c_1)^2 + \varepsilon + Rc\right)\|\eta\Delta\mathcal{D}\hat{u}\|^2_{2,B_R^+} + c(\varepsilon). \tag{4.2.20}$$

The same Young inequality applied on (4.2.10) implies

$$\begin{aligned}\|\eta\Delta\mathcal{D}\hat{u}\|^2_{2,B_R^+} \leq \frac{c_3}{c_1}\left(\frac{1}{2(c_2c_7 + c_1)}\|\eta\Delta\hat{p}\|^2_{2,B_R^+} + \frac{c_2c_7 + c_1}{2}\|\eta\Delta\mathcal{D}\hat{u}\|^2_{2,B_R^+}\right)\\ + (\varepsilon + Rc)\|\eta\mathcal{D}\nabla\hat{u}\|^2_{2,B_R^+} + c(\varepsilon)\\ \leq \left(\frac{c_3(c_2c_7 + c_1)}{c_1} + \varepsilon + Rc\right)\|\eta\Delta\mathcal{D}\hat{u}\|^2_{2,B_R^+} + c(\varepsilon)\end{aligned}$$

According to assumptions, we can choose R and ε such that $\frac{c_3(c_7c_2+c_1)}{c_1}+\varepsilon+Rc < 1$ and thus we get

$$\|\eta\Delta\mathcal{D}\hat{u}\|^2_{2,B_R^+} + \|\eta\Delta\hat{p}\|^2_{2,B_R^+} \leq c.$$

Now it is enough to choose η as

$$\eta = \begin{cases} 1 \text{ in } B^+_{R/2} \\ 0 \text{ in } \mathbb{R}^d \setminus B_R \\ \text{smoothly} \end{cases}.$$

Thus

$$\left\|\frac{\partial\nabla\hat{u}}{\partial x_i}\right\|_{2,B_{R/2}} + \left\|\frac{\partial\hat{p}}{\partial x_i}\right\|_{2,B_{R/2}} \leq c(\|u\|_{1,2}, \|p\|_2, \|f\|_2, \omega, R, T)$$

for all $i \in \{1, \ldots, d-1\}$.

It suffices to show that also the derivatives with respect to the normal vector are bounded in proper spaces. The functions $(\hat{u}, \hat{p})$ satisfy equation

$$\begin{aligned} - \operatorname{div} T(\mathcal{D}\hat{u} + (\nabla F^{-1} - I)\nabla\hat{u}, \hat{p})\nabla F^{-1} & \\ -\nabla\hat{p}\nabla F^{-1} &= g, && (4.2.21)\\ \operatorname{div}\hat{u} &= \operatorname{Tr}((\nabla F^{-1} - I)\nabla\hat{u}), && (4.2.22)\end{aligned}$$

where $g \in L^{\frac{3}{2}}$ contains right hand side and the convective term. We rewrite this system in point of view of an unknown vector $s = \left(\frac{\partial^2\hat{u}_1}{\partial x_d^2}, \ldots, \frac{\partial^2\hat{u}_d}{\partial x_d^2}, \frac{\partial\hat{p}}{\partial x_d}\right)$. The equation (4.2.21) can be reformulated as follows

$$\overline{A}^{kl}_{ij}\frac{\partial^2 u_i}{\partial x_l\partial x_j} + \left(\delta_{kl} + R\omega + (I - R\omega)\frac{\partial T_{kl}(\mathcal{D}\hat{u} - R\omega\mathcal{D}\hat{u}, \hat{p})}{\partial\hat{p}}\right)\frac{\partial p}{\partial x_l} = g', \tag{4.2.23}$$

where $\overline{A} = -(I - R\omega)\frac{\partial T(\mathcal{D}\hat{u} - R\omega\mathcal{D}\hat{u},\hat{p})}{\partial\mathcal{D}}$. Therefore $\|\overline{A}\| \le c_2\|I + R\omega\|$. We emphasize that according to the assumptions $|\frac{\partial T}{\partial p}| < c_3 < 1$ and thus, for $R > 0$ sufficiently small, there exists an inverse matrix $C = \left(\delta_{kl} + (I - R\omega)\frac{\partial T_{kl}(\mathcal{D}\hat{u} - R\omega\mathcal{D}\hat{u},\hat{p})}{\partial\hat{p}}\right)^{-1}$. We multiply (4.2.23) by C and we put all the already estimated terms on the right hand side. Hence, we obtain, for $m \in \{1, \dots, d\}$[2]

$$(-\tilde{A}s)_m = (Cg')_m + \frac{\partial p}{\partial x_m}(1 - \delta_{dm}) - \sum_{l,j\in\{1,\dots,d\}^2\setminus\{(d,d)\}, i\in\{1,\dots,d\}} (C\overline{A})^{ml}_{ij}\frac{\partial^2 u_i}{\partial x_l \partial x_j}, \tag{4.2.24}$$

where $\tilde{A}$ is defined as a $d \times (d+1)$ matrix

$$\tilde{A}_{mi} = \begin{pmatrix} & 0 \\ (C\overline{A})^{md}_{id} & \vdots \\ & 0 \\ & 1 \end{pmatrix}. \tag{4.2.25}$$

We denote the right hand side of (4.2.24) by $\tilde{g}$. We add to (4.2.24) the equation (4.2.22) differentiated with respect to x_d. We get

$$(\tilde{A}'s)_m = \tilde{g}_m(1 - \delta_{m(d+1)}) + \delta_{m(d+1)}\frac{\partial u_i}{\partial x_j}\frac{\partial^2 F_j^{-1}}{\partial x_i \partial x_d}. \tag{4.2.26}$$

Here

$$\tilde{A}' = \begin{pmatrix} \tilde{A} \\ 0, \dots, 0, 1, 0 \end{pmatrix} + R\omega.$$

Further, we denote the right hand side of (4.2.26) by $\tilde{g}'$. We compute $\det \tilde{A}'$. We expand the determinant of $\tilde{A}'$ along the last row and along the last column. We get $\det \tilde{A}' = \det \tilde{A}_M + Rc$ where $\tilde{A}_M$ is the $(d-1) \times (d-1)$ matrix that results from $\tilde{A}$ by removing the last two columns and last row. The matrix $C\overline{A}$ is elliptic. Indeed, $\overline{A}$ is elliptic with constant $c_1 - Rc$ because $\frac{\partial T}{\partial D}$ is elliptic. Further, $C\overline{A}$ is elliptic with a constant $c_1 - c_2\frac{c_3}{1-c_3} - Rc$ which is, for R small enough, greater than zero according to the assumptions. Thus also a matrix $\tilde{A}_M$ is elliptic and it has nonzero determinant. We get, that, for R sufficiently small, there exists an inverse matrix $(\tilde{A}')^{-1} \in L^\infty$. From (4.2.26) we have for arbitrary $r \in \mathbb{R}$

$$\|s\|_r \le C\left(\|\tilde{g}'\|_r + \left\|\left(\frac{\partial p}{\partial x_i}\right)_{i=1,\dots,d-1}\right\|_r + \left\|\left(\frac{\partial^2 u_i}{\partial x_l \partial x_j}\right)_{i,j=1,\dots,d-1}\right\|_r\right). \tag{4.2.27}$$

Since $\tilde{g}' \in L^{\frac{3}{2}}$, we have $\nabla^2 u \in L^{\frac{3}{2}}(B_R^+(0), \mathbb{R}^{d^3})$. The Sobolev embedding theorem implies $u \in W^{1,3} \cap L^6$, thus $u\nabla u \in L^2(B_R^+(0), \mathbb{R}^d)$ and the right hand side $\tilde{g}'$ in

[2] Here d is not a summation index.

(4.2.26) is bounded in L^2. By iterating this process, we obtain

$$\|s\|_2 \le C \left(\|g'\|_2 + \left\| \left(\frac{\partial p}{\partial x_i} \right)_{i=1,\dots,d-1} \right\|_2 + \left\| \left(\frac{\partial^2 u_i}{\partial x_l \partial x_j} \right)_{i,j=1,\dots,d-1} \right\|_2 \right), \quad (4.2.28)$$

which concludes the proof. □

4.3 Higher integrability

42 Lemma. *Let* $c_3 < \min\left\{\frac{c_1}{(c_1+c_7c_2)c_7}, c_9^{-1}\right\}$ *and* Ω *be a* C^2 *domain. Then there exists a constant* $\delta > 0$ *such that, for* $f \in L^{2+\delta}(\Omega, \mathbb{R}^d)$*, a weak solution* (u,p) *to* (1.1.2) *belongs to* $W^{2,2+\delta}(\Omega, \mathbb{R}^d) \times W^{1,2+\delta}(\Omega, \mathbb{R})$*.*

Proof. Assume that $0 = x_0 \in \Omega$ and let $R > 0$ be such that $B_{2R} \subset \Omega$. Since all assumptions of the previous lemma holds, we can assume, that $(u,p) \in W^{2,2}(\Omega, \mathbb{R}^d) \times W^{1,2}(\Omega, \mathbb{R})$. We differentiate (1.1.2) with respect to x_i for $i \in \{1, \dots, d\}$ fixed. We get

$$-\operatorname{div} \frac{\partial T}{\partial D} \mathcal{D}\left(\frac{\partial u}{\partial x_i}\right) - \operatorname{div} \frac{\partial T}{\partial p} \frac{\partial p}{\partial x_i} + \nabla \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i}(f - \operatorname{div}(u \otimes u)). \quad (4.3.1)$$

Set $A = \frac{\partial T}{\partial D}(\mathcal{D}u, p)$, $B = \frac{\partial T}{\partial p}(\mathcal{D}u, p)$, $U = \frac{\partial u}{\partial x_i}$ and $P = \frac{\partial p}{\partial x_i}$. The equation (4.3.1) can be rewritten as

$$-\operatorname{div} A\mathcal{D}U + \nabla P = \frac{\partial}{\partial x_i}(f - \operatorname{div}(u \otimes u)) + \operatorname{div} BP.$$

We multiply this equation by a cut–of function $\eta \in C^\infty$ which is defined by

$$\eta(x) = \begin{cases} 1 & x \in B_{R/2} \\ 0 & x \in \mathbb{R}^d \setminus B_R \end{cases}.$$

Thus functions $(\tilde{U}, \tilde{P}) \overset{def}{=} (U\eta, P\eta)$ solve

$$\begin{aligned} -\operatorname{div} A\nabla\tilde{U} + \nabla\tilde{P} &= F + \operatorname{div} B\tilde{P}, \\ \operatorname{div} \tilde{U} &= g, \\ \tilde{U}\Big|_{B_R} &= 0, \end{aligned}$$

where

$$F = \eta \frac{\partial}{\partial x_i}(f - \operatorname{div}(u \otimes u)) + (\nabla\eta)P + (\nabla\eta)BP + \operatorname{div}(A(\nabla\eta)U)$$

and

$$g = \frac{\partial \eta}{\partial x_j} U_j.$$

Since $\nabla U \in L^2(\Omega, \mathbb{R}^{d^2})$, U belongs to $L^6(\Omega, \mathbb{R}^d)$. Thus we have $g \in L^{2+\delta}(\Omega, \mathbb{R})$. Further, F can be written as $F = \operatorname{div} F'$, where $F' \in L^{2+\delta}(\Omega, \mathbb{R}^{d^2})$. Indeed, since $u \nabla u \in L^5(\Omega, \mathbb{R}^d)$, the term $\eta \frac{\partial}{\partial x_i}(f - \operatorname{div}(u \otimes u)) + \operatorname{div} A(\nabla \eta) U$ is in space $W^{-1,2+\delta}(\Omega, \mathbb{R}^d)$. Moreover, $F - \eta \frac{\partial}{\partial x_i}(f - \operatorname{div}(u \otimes u)) \in L^2(\Omega, \mathbb{R}^d)$ because $U \in W^{1,2}(\Omega, \mathbb{R}^d)$, $P \in L^2(\Omega, \mathbb{R})$ and $B \in L^\infty(\Omega, \mathbb{R}^{d^2})$. Thus, according to Corollary 38, we get that $(\nabla U, P)$ are in space $L^{2+\delta}(\Omega, \mathbb{R}^{d^2} \times \mathbb{R})$. Since i can be chosen arbitrarily, we immediately obtain $u \in W^{2,2+\delta}(B_{\frac{R}{2}}, \mathbb{R}^d)$ and $p \in W^{1,2+\delta}(B_{\frac{R}{2}}, \mathbb{R})$.
Let $0 = x_0 \in \partial\Omega$ and $\Omega_{x_0,R}$ be the neighborhood defined earlier. We define quantities $\hat{u}$, $\hat{p}$ and $\hat{f}$ by (4.2.1) and we differentiate equation (1.1.2) with respect to x_i, $i \in \{1, \ldots, d-1\}$. We assume that $\frac{\partial \hat{u}}{\partial x_i}$ is equal to zero on $\partial B_R^+(0)$. We set $A = \frac{\partial T}{\partial \mathcal{D}}((\mathcal{D}\hat{u} + R\omega\nabla\hat{u}, \hat{p}))$ and $B = \frac{\partial T}{\partial p}((\mathcal{D}\hat{u} + R\omega\nabla\hat{u}, \hat{p}))$ and we have

$$\int_{B_R+} A\mathcal{D}\frac{\partial \hat{u}}{\partial x_i}\mathcal{D}\psi + \int_{B_R^+} B\frac{\partial \hat{p}}{\partial x_i}\mathcal{D}\psi - \int_{B_R^+} \frac{\partial \hat{p}}{\partial x_i}\operatorname{div}\psi = RS_1\left(\frac{\partial \hat{u}}{\partial x_i}, \frac{\partial \hat{p}}{\partial x_i}, \psi\right) + S_2(\hat{u}, \hat{p}, \psi), \quad (4.3.2)$$

where

$$S_1\left(\frac{\partial \hat{u}}{\partial x_i}, \frac{\partial \hat{p}}{\partial x_i}, \psi\right) = \int_{B_R^+} \Bigg(T(\mathcal{D}\hat{u} + R\omega\nabla\hat{u}, \hat{p})\omega\nabla\frac{\partial \hat{u}}{\partial x_i}\mathcal{D}\psi\nabla F + \frac{\partial \hat{p}}{\partial x_i}(\operatorname{Tr}\nabla\psi\omega) + A\mathcal{D}\frac{\partial \hat{u}}{\partial x_i}\mathcal{D}\psi\omega + B\frac{\partial \hat{p}}{\partial x_i}\mathcal{D}\psi\omega\Bigg)$$

and

$$S_2(\hat{u}, \hat{p}, \psi) = \int_{B_R^+} \Bigg(-f\frac{\partial \psi}{\partial x_i} - T(\mathcal{D}\hat{u} + R\omega\nabla\hat{u}, \hat{p})\frac{\partial \nabla F}{\partial x_i}\mathcal{D}\psi + \hat{p}\operatorname{Tr}(\nabla\psi\frac{\partial \nabla F}{\partial x_i}) + \frac{\partial}{\partial x_i}(\hat{u}\otimes\hat{u})\mathcal{D}\psi\nabla F + (\hat{u}\otimes\hat{u})\mathcal{D}\psi\frac{\partial \nabla F}{\partial x_i} + T(\mathcal{D}\hat{u} + R\omega\nabla\hat{u}, \hat{p})\mathcal{D}\psi\frac{\partial \nabla F}{\partial x_i}\Bigg).$$

It holds that

$$|S_2(\hat{u}, \hat{p}, \psi)| \leq c\left(\|f\|_{L^{2+\delta}} + \|\nabla\hat{u}\|_{L^{2+\delta}} + \|\hat{p}\|_{L^{2+\delta}} + \left\|\frac{\partial}{\partial x_i}(\hat{u}\otimes\hat{u})\right\|_{L^{2+\delta}} + \|\hat{u}\otimes\hat{u}\|_{L^{2+\delta}}\right)\|\psi\|_{W_0^{1,(2+\delta)'}}.$$

Thus the term $S_2(\hat{u}, \hat{p}, \psi)$ can be represented as $\int G\nabla\psi$ where $G \in L^{2+\delta}(\Omega, \mathbb{R}^{d^2})$.
For S_1 we have, due to Hölder inequality,

$$\left\|S_1\left(\frac{\partial \hat{u}}{\partial x_i}, \frac{\partial \hat{p}}{\partial x_i}, .\right)\right\|_{-1,2+\delta} = \sup_{\psi\in W_0^{1,(2+\delta)'}, \|\psi\|_{1,(2+\delta)'}\leq 1}\left|S_1\left(\frac{\partial \hat{u}}{\partial x_i}, \frac{\partial \hat{p}}{\partial x_i}, \psi\right)\right| \leq c\left(\left\|\nabla\frac{\partial \hat{u}}{\partial x_i}\right\|_{2+\delta} + \left\|\frac{\partial \hat{p}}{\partial x_i}\right\|_{2+\delta}\right).$$

According to Lemma 39 there exists $R_0 > 0$ such that for all $R < R_0$ it holds that $\left(\frac{\partial \hat{u}}{\partial x_i}, \frac{\partial \hat{p}}{\partial x_i}\right) \in W^{1,2+\delta}(B_R^+(0), \mathbb{R}^d) \times L^{2+\delta}(B_R^+(0), \mathbb{R})$.
The same considerations can be done even for a function, which is not supported in B_R^+. It is enough to take $\frac{\partial \hat{u}}{\partial x_i}\eta$ instead of $\frac{\partial \hat{u}}{\partial x_i}$ where η is a nonnegative smooth cut–off function defined as

$$\eta(x) = \begin{cases} 1 & x \in B^+_{R/2} \\ 0 & x \in \mathbb{R} \setminus B_{3R/4} \end{cases}.$$

The regularity of the derivation with respect to the normal vector can be done similarly as in proof of Lemma 41. Hence, since $\overline{\Omega}$ is compact, we get the claim of the lemma. □

43 Corollary. *Let all assumptions of Lemma 42 holds. Then there exists $\delta > 0$ such that $(\mathcal{D}u, p) \in W^{1/2,2+\delta}(\partial\Omega)$.*

Proof. Follows immediately from properties of the trace operator. □

4.4 Key lemma and its consequences

For needs of this section, we define quantity $E^{u,p}(x,R)$ for $\alpha \in (0,1)$ as follows

$$E^{u,p}(x,R) = R^{\frac{2-d}{2}} \|\nabla^2 u\|_{2,\Omega_{x,R}} + R^{\frac{2-d}{2}} \|\nabla p\|_{2,\Omega_{x,R}} + R^\alpha.$$

Throughout this section, we assume that Ω is a bounded C^2 domain.

44 Key lemma. *Let* (1.1.3) *be satisfied with $c_3 < \frac{c_1}{(c_1+c_7c_2)c_7}$, let $\alpha \in (0,1)$ and let $f \in L^{2,\mu}(\Omega, \mathbb{R}^d)$ where $\mu > d - 1 + \alpha$. There exists $R_0 > 0$ such that for all $M > 0$ and $\tau \in (0,1)$ there exists $\varepsilon > 0$ for which the following implication holds: Let $(u,p) \in W^{1,2}(\Omega, \mathbb{R}^d) \times L^2(\Omega, \mathbb{R})$ be a weak solution of system* (1.1.2) *and let for any $x_0 \in \partial\Omega$ and $R \in (0, R_0)$ the inequalities*

$$E^{u,p}(x_0, R) < \varepsilon, \;\; (|\nabla u|)_{\Gamma_h} + |\,(p)_{\Gamma_h}\,| \le M$$

hold. Then

$$E^{u,p}(x_0, \tau R) \le 2C^* \tau^\alpha E^{u,p}(x_0, R).$$

Proof. We prove this lemma via blow up system.
Throughout the proof, we write F_h instead of F_{x_h,R_h} and Ω_h instead of Ω_{x_h,R_h}. We define a set Γ_h as $\Gamma_h = \partial\Omega_h \cap \partial\Omega$. For a contradiction, we suppose that there exist M, τ, $x_h \in \partial\Omega$, $\varepsilon_h \to 0$, $R_h \to 0$, as h tends to zero, and weak solutions (u_h, p_h) to (1.1.2) satisfying

$$E^{u_h,p_h}(x_h, R_h) = \varepsilon_h, \; |(\nabla u_h)_{\Gamma_h}| + |(p_h)_{\Gamma_h}| \le M \tag{4.4.1}$$

and

$$E^{u_h,p_h}(x_h,\tau R_h) > 2C^*\tau^\alpha E^{u_h,p_h}(x_h,R_h).$$

We, moreover, assume that[3]

$$\begin{aligned}(p_h)_{\Gamma_h} &\to a \text{ in } \mathbb{R}\\ (\mathcal{D}^* u_h)_{\Gamma_h} &\to e \text{ in } \mathbb{R}^{d\times d}.\end{aligned}$$

Further, from the assumption (4.4.1), it follows that $\frac{R_h}{\varepsilon_h} = R_h^{1-\alpha}\frac{R_h^\alpha}{\varepsilon_h} \to 0$ as h tends to zero. We set $x = F_h(y)$ and we introduce new rescaled quantities v_h, q_h and f_h, defined by

$$\begin{aligned}v_h(y) &= \frac{u_h(F_h(y)) - (\nabla u)_{\Gamma_h}\cdot(0,\dots,0,y_d)R_h}{R_h\varepsilon_h},\\ q_h(y) &= \frac{p_h(F_h(y)) - (p_h)_{\Gamma_h}}{\varepsilon_h},\\ f_h(y) &= \frac{R_h}{\varepsilon_h} f(F_h(y)).\end{aligned}$$

Their derivatives fulfill

$$\begin{aligned}\nabla_y v_h(y) &= \frac{\nabla_x u_h(F_h(y)) - (\nabla_x u_h)_{\Gamma_h}\cdot(0,\dots,0,1)}{\varepsilon_h} + \frac{R_h}{\varepsilon_h}\omega\nabla_x u_h(F_h(y)),\\ \mathcal{D}_y v_h(y) &= \frac{\mathcal{D}_x u_h(F_h(y)) - (\mathcal{D}_x^* u_h)_{\Gamma_h}}{\varepsilon_h} +\\ &\quad + \frac{1}{2}\left(\frac{R_h}{\varepsilon_h}\omega\nabla_x u_h(F_h(y)) + \left(\frac{R_h}{\varepsilon_h}\omega\nabla_x u_h(F_h(y))\right)^T\right),\\ \nabla_y^2 v_h(y) &= \frac{1}{R_h\varepsilon_h}\left(\nabla_x^2 u_h(F_h(y))\,(\nabla F_h(y))^2\right) +\\ &\quad + \frac{1}{\varepsilon_h R_h}\nabla_x u_h(F_h(y))\nabla^2 F_h(y),\\ \nabla_y q_h(y) &= \frac{R_h\nabla_x p_h(F_h(y))}{\varepsilon_h} + \frac{R_h^2}{\varepsilon_h}\omega\nabla_x p_h(F_h(y)).\end{aligned} \tag{4.4.2}$$

By the change of variables, we have, due to properties of F_h (see Observations 20),

$$\begin{aligned}(|R_h^d| - c|R_h^{d+1}|)\int_{B_1^+(0)} |\nabla_x p_h(F_h(y))|^2 dy &\le \int_{\Omega_h} |\nabla_x p_h|^2 dx\\ &\le (|R_h^d| + c|R_h^{d+1}|)\int_{B_1^+(0)} |\nabla_x p_h(F_h(y))|^2 dy,\\ (|R_h^d| - c|R_h^{d+1}|)\int_{B_1^+(0)} |\nabla_x^2 u_h(F_h(y))|^2 dy &\le \int_{\Omega_h} |\nabla_x^2 u_h|^2 dx\\ &\le (|R_h^d| + c|R_h^{d+1}|)\int_{B_1^+(0)} |\nabla_x^2 u_h(F_h(y))|^2 dy.\end{aligned}$$

---

[3] We use the convention $\nabla = \left(\nabla', \frac{\partial}{\partial_d}\right)$. The operator $\mathcal{D}^*$ is defined as $\mathcal{D}^* u \stackrel{def}{=} \frac{1}{2}\left(\left(0, \frac{\partial}{\partial_d}u\right) + \left(0, \frac{\partial}{\partial_d}u\right)^T\right)$

Thus

$$\begin{aligned}\frac{1}{\sqrt{R_h^d+cR_h^{d+1}}}\|\nabla_x p_h\|_{2,\Omega_h} \leq&\|\nabla_x p_h(F_h(.))\|_{2,B_1^+(0)} \leq \frac{1}{\sqrt{R_h^d-cR_h^{d+1}}}\|\nabla_x p_h\|_2,\\ \frac{1}{\sqrt{R_h^d+cR_h^{d+1}}}\|\nabla_x^2 u_h\|_{2,\Omega_h} \leq&\|\nabla_x^2 u_h(F_h(.))\|_{2,B_1^+(0)} \leq \frac{1}{\sqrt{R_h^d-cR_h^{d+1}}}\|\nabla_x^2 u_h\|_{2,\Omega_h}.\end{aligned} \tag{4.4.3}$$

The identity $(\nabla F_h)^2 = R_h^2 I + R_h^3\omega + R_h^4\omega$ implies that

$$\begin{aligned}\|\nabla^2 v_h\|_{2,B_1^+(0)} + \|\nabla q_h\|_{2,B_1^+(0)} \leq& \left\|\frac{1}{R_h\varepsilon_h}\nabla_x^2 u_h(F_h(.))(\nabla F_h)^2\right\|_{2,B_1^+(0)} +\\ &+\left\|\frac{1}{R_h\varepsilon_h}\nabla_x u_h(F_h(.))\nabla^2 F_h\right\|_{2,B_1^+(0)} +\\ &+\left\|\frac{R_h}{\varepsilon_h}\nabla_x u_h(F_h(.))\right\|_{2,B_1^+(0)} + \left\|\frac{R_h^2}{\varepsilon_h}\omega\nabla_x p_h(F_h(.))\right\|_{2,B_1^+(0)}\\ \leq&\frac{R_h^{\frac{2-d}{2}}}{\varepsilon_h\sqrt{1-cR_h}}\left(\|\nabla^2 u_h\|_{2,\Omega_h}+\|\nabla p_h\|_{2,\Omega_h}\right)+\\ &+cR_h\frac{R_h^{\frac{2-d}{2}}}{\varepsilon_h\sqrt{1-cR_h}}\left(\|\nabla^2 u_h\|_{2,\Omega_h}+\|\nabla p_h\|_{2,\Omega_h}\right)+\\ &+\left\|\frac{1}{R_h\varepsilon_h}\nabla_x u_h(F_h(.))\nabla^2 F_h\right\|_{2,B_1^+(0)}\\ \leq&\frac{1+cR_h}{\varepsilon_h\sqrt{1-cR_h}}E^{u_h,p_h}(x_h,R_h) + \left\|\frac{1}{R_h\varepsilon_h}\nabla_x u_h(F_h(.))\nabla^2 F_h\right\|_{2,B_1^+(0)}\end{aligned}$$

and similarly

$$\begin{aligned}&\|\nabla^2 v_h\|_{2,B_1^+(0)} + \|\nabla q_h\|_{2,B_1^+(0)} \geq\\ &\qquad\geq\frac{1-cR_h}{\varepsilon_h\sqrt{1+cR_h}}E^{u_h,p_h}(x_h,R_h) - \left\|\frac{1}{R_h\varepsilon_h}\nabla_x u_h(F_h(.))\nabla^2 F_h\right\|_{2,B_1^+(0)}.\end{aligned}$$

The term $\left\|\frac{1}{R_h\varepsilon_h}\nabla_x u_h(F_h(.))\nabla^2 F_h\right\|_{2,B_1^+(0)}$ converges to zero as h tends to zero. Indeed, according to the Poincaré inequality (Lemma 22) we get

$$\begin{aligned}\left\|\frac{1}{R_h\varepsilon_h}\nabla_x u_h(F_h(.))\nabla^2 F_h\right\|_{2,B_1^+(0)} \leq&\frac{cR_h}{\varepsilon_h}\|\nabla_x u_h(F_h(.))\|_{2,B_1^+(0)} \leq c\frac{R_h}{\varepsilon_h R^{\frac{d}{2}}}\|\nabla u_h\|_{2,\Omega_h}\\ \leq&c\left(\frac{R_h}{\varepsilon_h}|(\nabla u_h)_{\Gamma_h}| + R_h\frac{R_h^{1-\frac{d}{2}}}{\varepsilon_h}\|\nabla^2 u_h\|_{2,\Omega_h}\right)\\ \leq&c\left(\frac{R_h}{\varepsilon_h}M + R_h\frac{E^{u_h,p_h}(x_h,R_h)}{\varepsilon_h}\right)\\ \leq&c\left(\frac{R_h}{\varepsilon_h}M + R_h\right)\to 0.\end{aligned}$$

It follows that

$$E^{v_h,p_h}(0,1) \to 1 \text{ as } h \to 0,$$
$$E^{v_h,p_h}(0,\tau) > 2C^*\tau^\alpha E^{v_h,p_h}(0,1) \text{ for } h \text{ sufficiently small.} \tag{4.4.4}$$

Boundedness of the second gradient of v_h and the first gradient of p_h in space L^2 implies that, up to a subsequence,

$$(v_h, p_h) \to (v,p) \text{ in } W^{2,2}(B_1^+(0)) \times W^{1,2}(B_1^+(0)) \text{ weakly.}$$

We set $x = F_h(y)$ and $\psi(y) = \varphi(F_h(y)) = \varphi(x)$. Every term in a weak formulation of the equation (1.1.2) can be reformulated as follows

$$\int_{\Omega_{R_h}(x_h)} f(x)\varphi(x)dx = \int_{B_1^+(0)} f(F_h(y))\psi(y)|\det \nabla F_h(y)|dy,$$

$$\begin{aligned}\int_{\Omega_{R_h}(x_h)} & u_h(x)\otimes u_h(x)\mathcal{D}\varphi(x)dx = \\ &= \frac{1}{R_h}\int_{B_1^+(0)} u_h(F_h(y))\otimes u_h(F_h(y))\mathcal{D}\psi(y)|\det\nabla F_h(y)|dy- \\ &\qquad - R_h\int_{B_1^+(0)} u_h(F_h(y))\otimes u_h(F_h(y))\mathcal{D}\psi(y)\omega|\det\nabla F_h(y)|dy,\end{aligned}$$

similarly

$$\begin{aligned}\int_{\Omega_{R_h}(x_h)} & T(\mathcal{D}u_h(x), p_h(x))\mathcal{D}\varphi(x)dx = \\ &= \frac{1}{R_h}\int_{B_1^+(0)} T(\mathcal{D}u_h(F_h(y)), p_h(F_h(y)))\mathcal{D}\psi(y)|\det\nabla F_h(y)|dy- \\ &\qquad - R_h\int_{B_1^+(0)} T(\mathcal{D}u_h(F_h(y)), p_h(F_h(y)))\mathcal{D}\psi(y)\omega|\det\nabla F_h(y)|dy\end{aligned}$$

and, due to $\operatorname{div}\psi(y) = \operatorname{Tr}(\nabla\varphi\nabla F_h) = R_h \operatorname{div}\varphi + R_h^2\nabla\varphi\omega$,

$$\begin{aligned}\int_{\Omega_{R_h}(x_h)} p_h(x)\operatorname{div}\varphi(x)dx &= \frac{1}{R_h}\int_{B_1^+(0)} p_h(F_h(y))\operatorname{div}\psi(y)|\det\nabla F_h(y)|dy+ \\ &\quad + R_h\int_{B_1^+(0)} (p_h(F_h(y)))\operatorname{Tr}(\nabla\varphi(F_h(y))\omega)|\det\nabla F_h(y)|dy.\end{aligned}$$

Hence, for all $\psi \in W_0^{1,2}(B_1^+(0))$, holds

$$\begin{aligned} I_1+I_2+I_3+I_4+I_5+I_6 &= 0, \\ \operatorname{div} v_h &= -\frac{1}{\varepsilon_h}\left(\frac{\partial u_{hd}}{\partial x_d}\right)_{\Gamma_h} + \frac{R_h}{\varepsilon_h}\omega\operatorname{div} u_h, \\ v_h|_{B_1^{d-1}} &= 0 \end{aligned} \tag{4.4.5}$$

where the terms I_i are defined as

$$
\begin{aligned}
I_1 &= \int_{B_1^+(0)} \frac{1}{\varepsilon_h} T\left(\mathcal{D}v_h\varepsilon_h + (\mathcal{D}^* u_h)_{\Gamma_h}, q_h\varepsilon_h + (p_h)_{\Gamma_h}\right) : \mathcal{D}\psi \frac{|\det \nabla F_h(y)|}{R_h^d},\\
I_2 &= \int_{B_1^+(0)} \frac{1}{\varepsilon_h}\Bigg(T\bigg(\mathcal{D}v_h\varepsilon_h + (\mathcal{D}^* u_h)_{\Gamma_h} + R_h \frac{1}{2}\left(\nabla u_h\omega +\right.\\
&\left.(\nabla u_h\omega)^T\right)\bigg), q_h\varepsilon_h + (p_h)_{\Omega_{R_h}(x_h)}\Bigg) -\\
&T\left(\mathcal{D}v_h\varepsilon_h + (\mathcal{D}^* u_h)_{\Gamma_h}, q_h\varepsilon_h + (p_h)_{\Gamma_h})\right) : \mathcal{D}\psi \frac{|\det \nabla F_h(y)|}{R_h^d},\\
I_3 &= \int_{B_1^+(0)} -\frac{1}{\varepsilon_h}\left(q_h\varepsilon_h + (p_h)_{\Gamma_h}\right) \operatorname{div}\psi \frac{|\det \nabla F_h(y)|}{R_h^d},\\
I_4 &= -\int_{B_1^+(0)} f_h\psi \frac{|\det \nabla F_h(y)|}{R_h^d},\\
I_5 &= \int_{B_1^+(0)} \frac{1}{\varepsilon_h}\left(v_h R_h\varepsilon_h + (\nabla u_h)_{\Gamma_h}(0,\dots,0,1)R_h\right) \otimes\\
&\otimes \left(v_h R_h\varepsilon_h + (\nabla u_h)_{\Gamma_h}(0,\dots,0,1)R_h\right) \mathcal{D}\psi \frac{|\det \nabla F_h(y)|}{R_h^d},\\
I_6 &= \frac{R_h^2}{\varepsilon_h}\int_{B_1^+(0)} \left((p_h(F_h(y)))\operatorname{Tr}(\nabla\varphi(F_h(y))\omega) +\right.\\
&+ u_h(F_h(y)) \otimes u_h(F_h(y))\mathcal{D}\psi(y)\omega\\
&\left.+ T(\mathcal{D}u_h(F_h(y)), p_h(F_h(y)))\mathcal{D}\psi(y)\omega\right) \frac{|\det \nabla F_h(y)|}{R_h^d}.
\end{aligned}
$$

Since $\operatorname{div}_y u_h = 0$, we have

$$\operatorname{Tr}\left(\nabla_y u_h(F_h(y))(\nabla F_h(y))^{-1}\right) = 0$$

and identity $F_h(y) = \frac{1}{R_h}(I + R_h\omega)$ implies

$$\operatorname{div}_y u_h(F_h(y)) = -R_h\omega\nabla_y u_h(F_h(y)).$$

By the zero-Dirichlet boundary condition, $\left.\frac{\partial u_{hi}(F_h(y))}{\partial y_j}\right|_{(y',0)} = 0$ for all $y' \in B_1^{d-1}$, $i \in \{1,\dots,d\}$ and $j \in \{1,\dots,d-1\}$. Thus, for every $y \in B_1^{d-1}$,

$$|\nabla_y u_{hd}(F_h(y))| \leq cR_h|\nabla_y u_h(F_h(y))|.$$

Thus, for $\left(\frac{\partial u_{hd}}{\partial x_d}\right)_{\Gamma_h}$, we have

$$\begin{aligned}\int_{\Gamma_h}\left|\frac{\partial u_{hd}}{\partial x_d}(x)\right|dx=&\int_{B_1^{d-1}}\left|\frac{\partial u_{hd}}{\partial x_d}(F_h(y))\right||\det\nabla F_h(y)|dy\\ \le& c\int_{B_1^{d-1}}\left|\nabla_y u_{hd}(F_h(y))(\nabla F_h)^{-1}\right||\det\nabla F_h(y)|dy\\ \le& cR_h\left\|(\nabla F_h)^{-1}\right\|_\infty\int_{B_1^{d-1}}|\nabla_y u_h(F_h(y))||\det\nabla F_h(y)|dy\\ =& cR_h\left\|(\nabla F_h)^{-1}\right\|_\infty\int_{B_1^{d-1}}|\nabla_x u_h(F_h(y))\nabla F_h||\det\nabla F_h(y)|dy\\ \le& cR_h\|(\nabla F_h)^{-1}\|_\infty\|\nabla F_h\|_\infty\int_{\Gamma_h}|\nabla_x u_h|dx\le cR_h|\Gamma_h|M.\end{aligned}$$

Therefore $\frac{1}{\varepsilon_h}\left|\left(\frac{\partial u_{hd}}{\partial x_d}\right)_{\Gamma_h}\right|\le c\frac{R_h}{\varepsilon_h}M\to 0$. Also $\frac{R_h}{\varepsilon_h}\omega\operatorname{div}u_h\to 0$ and thus $\operatorname{div}v_h$ tends to zero.

The term $\frac{|\det\nabla F_h(y)|}{R_h^d}$ tends to 1 in L^∞ as h goes to zero. Thus we omit it in further computations. The term I_6 tends to zero as the integral is bounded and $\frac{R_h}{\varepsilon_h}\to 0$. Similarly, also the terms I_5 and I_2 goes to zero. The term I_4 can be handled as

$$\begin{aligned}|I_4|=&\frac{1}{\varepsilon_h R_h^{d-1}}\left|\int_{\Omega_h}f\psi\right|\le\frac{R_h}{\varepsilon_h}R_h^{-d}\int_{\Omega_h}|f\psi|\le\\ &\le\frac{R_h^{(\mu+1-d)}}{\varepsilon_h}R^{-\mu}\|f\|_{2,\Omega_h}\|\psi\|_{2,\Omega_h}\le R^{\mu+1-d-\alpha}\frac{R^\alpha}{\varepsilon_h}\|f\|_{2,\mu}\|\psi\|_2\to 0.\end{aligned}$$

We rewrite the term I_1 as follows

$$\begin{aligned}I_1=&\frac{1}{\varepsilon_h}\Bigg(\int_{B_1^+(0)}T(\mathcal{D}v_h\varepsilon_h+(\mathcal{D}^*u_h)_{\Gamma_h},q_h\varepsilon_h+(p_h)_{\Gamma_h}):\mathcal{D}\psi\\ &-\underbrace{\int_{B_1^+(0)}T((\mathcal{D}^*u_h)_{\Gamma_h},(p_h)_{\Gamma_h}):\mathcal{D}\psi}_{=0}\Bigg)\\ =&\frac{1}{\varepsilon}\int_{B_1^+(0)}\int_0^1\frac{\partial}{\partial s}T(s\mathcal{D}v_h\varepsilon_h+(\mathcal{D}^*u_h)_{\Gamma_h},sq_h\varepsilon_h+(p_h)_{\Gamma_h})ds\\ =&\int_{B_1^+(0)}\left(\int_0^1\frac{\partial T(s\mathcal{D}v_h\varepsilon_h+(D^*u_h)_{\Gamma_h},sq_h\varepsilon_h+(p_h)_{\Gamma_h})}{\partial D}ds\right)\mathcal{D}v_h:\mathcal{D}\psi(y)dy+\\ &+\int_{B_1^+(0)}\left(\int_0^1\frac{\partial T(s\mathcal{D}v_h\varepsilon_h+(D^*u_h)_{\Gamma_h},sq_h\varepsilon_h+(p_h)_{\Gamma_h})}{\partial p}ds\right)q_h\mathcal{D}\psi(y)dy.\end{aligned}$$

Thus

$$I_1\to\int_{B_1^+(0)}A\mathcal{D}v:\mathcal{D}\psi(y)dy+\int_{B_1^+(0)}Bq\mathcal{D}\psi(y)dy$$

where A and B are defined as

$$A \stackrel{def}{=} \frac{\partial T(a,e)}{\partial D},$$
$$B \stackrel{def}{=} \frac{\partial T(a,e)}{\partial p}.$$

From the fact that $\int_{B_1^+(0)} (p_h)_{\Gamma_h} \operatorname{div} \psi = 0$ for all $\psi \in W_0^{1,2}(B_1^+(0), \mathbb{R}^d)$, we derive that

$$\begin{aligned} I_3 &= \int_{B_1^+(0)} -\frac{1}{\varepsilon_h} \left(q_h \varepsilon_h + (p_h)_{\Gamma_h}\right) \operatorname{div} \psi \\ &= \int_{B_1^+(0)} q_h \operatorname{div} \psi \to \int_{B_1^+(0)} q \operatorname{div} \psi. \end{aligned}$$

We may conclude that v and q solve

$$\begin{aligned} -\operatorname{div} A\mathcal{D}v + (I-B)\nabla q &= 0 \text{ in } B_1^+(0), \\ \operatorname{div} v &= 0 \text{ in } B_1^+(0) \end{aligned} \tag{4.4.6}$$

and by Lemma 25

$$E^{v,q}(x, \tau R) \le C\tau^\alpha E^{v,q}(x,R). \tag{4.4.7}$$

Our goal is to prove that

$$\begin{aligned} 2C^*\tau^\alpha < E^{v_h,q_h}(0,\tau) \to E^{v,q}(0,\tau) \le \\ \le C^*\tau^\alpha E^{v,q}(0,1) \le C^*\tau^\alpha \liminf_{h\to 0} E^{v_h,q_h}(0,1) \le C^*\tau^\alpha \end{aligned} \tag{4.4.8}$$

which is a contradiction. The first inequality comes from (4.4.4). The third inequality is true due to (4.4.7). The weak lower semicontinuity of norm gives the forth inequality and the fifth inequality is trivial. It remains to show that

$$E^{v_h,q_h}(0,\tau) \to E^{v,q}(0,\tau).$$

We do it by proving that $(v_h,q_h) \to (v,q)$ strongly in $W^{2,2}(B_\tau^+(0)) \times W^{1,2}(B_\tau^+(0))$. Throughout the rest of this proof, we neglect the term $\frac{|\det \nabla F_h|}{R_h^d}$ for simplicity. We differentiate (4.4.5) with respect to x_i, $i \in \{1,\dots,d-1\}$. Set

$$\begin{aligned} A_h &= \frac{\partial T}{\partial D}\left(\mathcal{D}v_h\varepsilon_h + (\mathcal{D}^*u_h)_{\Gamma_h}, q_h\varepsilon_h + (p_h)_{\Gamma_h}\right) = \frac{\partial T}{\partial D}(a_h,e_h) \\ B_h &= \frac{\partial T}{\partial p}\left(\mathcal{D}v_h\varepsilon_h + (\mathcal{D}^*u_h)_{\Gamma_h}, q_h\varepsilon_h + (p_h)_{\Gamma_h}\right) = \frac{\partial T}{\partial p}(a_h,e_h). \end{aligned}$$

Further, we set $w_h = \frac{\partial v_h}{\partial x_i}$ and $r_h = \frac{\partial q_h}{\partial x_i}$. The functions w_h and r_h satisfy

$$\begin{aligned} -\operatorname{div} A_h\mathcal{D}w_h + \operatorname{div}\left((I-B_h)\cdot r_h\right) &= S_h, \\ \operatorname{div} w_h &= g_h \\ w_h\big|_{B_1^{d-1}} &= 0 \end{aligned} \tag{4.4.9}$$

where $S_h \in W^{-1,2}$ is defined as

$$
\begin{aligned}
[S_h,\varphi]_{W^{-1,2}} = R_h^2 \int_{B_1^+(0)} 2\left(w_h \otimes (v_h\varepsilon_h + (\nabla u_h)_{\Gamma_h}(0,\dots,0,1))\right)\mathcal{D}\varphi + \int_{B_1^+(0)} f_h \frac{\partial \varphi}{\partial x_i} \\
+ \int_{B_1^+(0)} \left(\frac{\partial T}{\partial \mathcal{D}}(a_h + R_h(\nabla u_h\omega + (\nabla u_h\omega)^T), e_h) - \frac{\partial T}{\partial \mathcal{D}}(a_h, e_h)\right)\mathcal{D}\frac{\partial v_h}{\partial x_i}\mathcal{D}\varphi \\
+ \int_{B_1^+(0)} \left(\frac{\partial T}{\partial p}(a_h + R_h(\nabla u_h\omega + (\nabla u_h\omega)^T), e_h) - \frac{\partial T}{\partial p}(a_h, e_h)\right)\frac{\partial q_h}{\partial x_i}\mathcal{D}\varphi \\
+ \int_{B_1^+(0)} \frac{\partial T}{\partial \mathcal{D}}(a_h + R_h(\nabla u_h\omega + (\nabla u_h\omega)^T), e_h)\frac{\partial}{\partial x_i}\nabla u_h(F(h))R_h\omega\mathcal{D}\varphi \\
+ \int_{B_1^+(0)} \frac{\partial T}{\partial \mathcal{D}}(a_h + R_h(\nabla u_h\omega + (\nabla u_h\omega)^T), e_h)\nabla u_h(F_h)\frac{\partial}{\partial x_i}\nabla F_h\mathcal{D}\varphi \\
+ \frac{R_h}{\varepsilon_h}\int_{B_1^+(0)} \Bigg[\frac{\partial}{\partial x_i}\left(u_h(F_h)\otimes u_h(F_h)\right)\mathcal{D}\varphi R_h\omega + u_h(F_h)\otimes u_h(F_h)\mathcal{D}\varphi\frac{\partial}{\partial x_i}\nabla F_h \\
+\left(\frac{\partial T}{\partial \mathcal{D}}(\mathcal{D}u_h(F_h), p_h(F_h))\mathcal{D}\frac{\partial}{\partial x_i}u_h(F_h) + \frac{\partial T}{\partial p}(\mathcal{D}u_h(F_h), p_h(F_h))\frac{\partial}{\partial x_i}p_h(F_h)\right)\mathcal{D}\varphi R_h\omega \\
+ T(\mathcal{D}u_h(F_h), p_h(F_h))\mathcal{D}\varphi\frac{\partial}{\partial x_i}\nabla F_h\Bigg] \\
= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7. \quad (4.4.10)
\end{aligned}
$$

Further, g_h is defined as follows

$$
g_h = \frac{1}{\varepsilon_h R_h}\operatorname{Tr}\left(\nabla^2 u_h\left(F_h(x)\right)\left(\frac{\partial F_h(x)}{\partial x_i} - R_h I\right)\nabla F_h(x) + \nabla u_h\left(F_h(x)\right)\frac{\partial}{\partial x_i}\nabla F_h(x)\right)
$$

From (4.4.9) and $\frac{\partial}{\partial x_i}$(4.4.6) we deduce

$$
\begin{aligned}
-\operatorname{div} A\mathcal{D}(w_h - w) + (I-B)\nabla(r_h - r) &= S_h + \operatorname{div}(A_h - A)\mathcal{D}w_h \\
&\quad + \operatorname{div}(B_h - B)r_h \\
\operatorname{div}(w_h - w) &= g_h. \qquad (4.4.11)
\end{aligned}
$$

Let there be a real smooth cut–off function $\theta \geq 0$, $\theta = \begin{cases} 1 & \text{for } x \in B_\tau^+(0) \\ 0 & \text{for } x \in \mathbb{R}^d \setminus B_1^+(0) \end{cases}$
Set $\tilde{w}_h = (w_h - w)\theta$ and $\tilde{r}_h = (r_h - r)\theta$. We multiply system (4.4.11) by θ to get

$$
\begin{aligned}
-\operatorname{div} A\mathcal{D}\tilde{w}_h + (I-B)\nabla\tilde{r}_h &= \theta S_h + \theta\operatorname{div}(A_h - A)\mathcal{D}w_h \\
&\quad + \theta\operatorname{div}(B_h - B)r_h + (I-B)(\nabla\theta)(r_h - r) \\
&\quad - \nabla\theta A\mathcal{D}(w_h - w) - \operatorname{div} A(\nabla\theta)(w_h - w) \\
\operatorname{div}\tilde{w}_h &= \theta g_h + \nabla\theta(w_h - w) \\
\tilde{w}_h|_{\partial B_1^+} &= 0. \qquad (4.4.12)
\end{aligned}
$$

We denote the left hand side of (4.4.12) by $\tilde{S}_h$. We test equation (4.4.12) by $\tilde{w}_h$. We get

$$c_1\|\mathcal{D}\tilde{w}_h\|_2^2 \le A\int_{B_1^+} \mathcal{D}\tilde{w}_h\mathcal{D}\tilde{w}_h = \int_{B_1^+} \tilde{r}_h \operatorname{div}\tilde{w}_h + \int_{B_1^+} B\tilde{r}_h\mathcal{D}\tilde{w}_h + [\tilde{S}_h, \tilde{w}_h]_{-1,2}$$
$$\le \|\tilde{r}_h\|_2\|\operatorname{div}\tilde{w}_h\|_2 + c_3\|\tilde{r}_h\|_2\|\mathcal{D}\tilde{w}_h\|_2.$$

Since $\|\operatorname{div}\tilde{w}_h\|_2 = \|\theta g_h + \nabla\theta(w_h - w)\|_2 = o(h) \to 0$, we get, using Young inequality

$$c_1\|\mathcal{D}\tilde{w}_h\|_2^2 \le \varepsilon\left(\|\mathcal{D}\tilde{w}_h\|_2^2 + \|\tilde{r}_h\|_2^2\right) + c_3\|\tilde{r}_h\|_2\|\mathcal{D}\tilde{w}_h\|_2 + c[\tilde{S}_h, w_h]_{-1,2} + o(h). \quad (4.4.13)$$

Further, we test equation (4.4.12) by φ_h which solves

$$\begin{aligned} \operatorname{div}\varphi_h &= \tilde{r}_h - (\tilde{r}_h)_{B_1^+} \\ \varphi_h|_{\partial B_1^+} &= 0. \end{aligned}$$

We get

$$\|\nabla\tilde{r}_h\|_2^2 = \int_{B_1^+} A\mathcal{D}\tilde{w}_h\mathcal{D}\varphi_h + \int_{B_1^+} B\tilde{r}_h\mathcal{D}\varphi_h + [\tilde{S}_h, \varphi_h]_{-1,2}$$
$$\le c_2c_7\|\mathcal{D}\tilde{w}_h\|_2\|\tilde{r}_h\|_2 + (c_3c_7 + \varepsilon)\|\tilde{r}_h\|_2^2 + [\tilde{S}_h, \varphi_h]_{-1,2}.$$

We use Young inequalities in the same way as in (4.2.10) to conlude

$$\|\mathcal{D}\tilde{w}_h\|_2^2 + \|\tilde{r}_h\|_2^2 \le c([\tilde{S}_h, \tilde{w}_h]_{-1,2} + [\tilde{S}_h, \varphi_h]_{-1,2}) + o(h).$$

We show that the terms $[\tilde{S}_h, \tilde{w}_h]_{-1,2}$ and $[\tilde{S}_h, \varphi_h]_{-1,2}$ tend to zero. In what follows, we estimate a term $[\tilde{S}_h, \varphi_h]$ since a method is the same even for the second term. The terms $J_1, \dots, J_7$ come from (4.4.10) with $\varphi = \varphi_h$.

$$[\tilde{S}_h, \varphi]_{-1,2} = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + \int_{B_1^+} \nabla\theta(A_h - A)\mathcal{D}w_h\varphi_h$$
$$+ \int_{B_1^+} \theta(A_h - A)\mathcal{D}w_h\mathcal{D}\varphi_h + \int_{B_1^+} \nabla\theta(B_h - B)r_h\varphi_h + \int_{B_1^+} \theta(B_h - B)r_h\nabla\varphi_h$$
$$- \int_{B_1^+} \nabla\theta A\mathcal{D}(w_h - w)\varphi_h - \int_{B_1^+} A\nabla\theta(w_h - w)\mathcal{D}\varphi_h$$
$$= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9 + J_{10} + J_{11} - J_{12} - J_{13}.$$

Since $w_h \to w$ and $\varphi_h \to 0$ strongly in $L^2(\Omega, \mathbb{R}^d)$, it can be derived that J_{12} and J_{13} tend to zero.

Further, $A_h \to A$ almost everywhere, $B_h \to B$ a.e., $\frac{\partial T}{\partial D}(a_h + R_h(\nabla u_h\omega + (\nabla u_h\omega)^T), e_h) \to \frac{\partial T}{\partial D}(a_h, e_h)$ a.e. and also $\frac{\partial T}{\partial p}(a_h + R_h(\nabla u_h\omega + (\nabla u_h\omega)^T), e_h) \to \frac{\partial T}{\partial p}(a_h, e_h)$ almost everywhere. Thus, terms J_3, J_4, J_8, J_9, J_{10} and J_{11} go to zero.

Term J_2 can be estimated similarly as term I_4.

Because w_h and v_h are both bounded in L^4, we get $J_1 \to 0$ due to $R_h \to 0$.

Further, $\frac{R_h}{\varepsilon_h} \to 0$, thus $J_7 \to 0$.

The fact $R_h \to 0$ also implies $J_5 \to 0$ and, since $\left\|\frac{\partial}{\partial x_i} \nabla F_h\right\|_\infty \leq R^2 c$, we easily get $J_6 \to 0$.

Thus we have $\left(\frac{\partial v_h}{\partial x_i}, \frac{\partial p}{\partial x_i}\right) \to \left(\frac{\partial v}{\partial x_i}, \frac{\partial p}{\partial x_i}\right)$ strongly in $W^{1,2}(B_\tau^+) \times L^2(B_\tau^+)$ for all $i \in \{1, \ldots, d-1\}$. The convergence of derivations with respect to the normal vector can be done similarly as at the end of proof of Lemma 41. □

45 Lemma. *Let assumptions* (1.1.3) *be satisfied with* $c_3 < \frac{c_1}{(c_1 + c_7 c_2) c_7}$ *and let* $f \in L^{2,\mu}(\Omega, \mathbb{R})$ *where* $\mu > d - 1 + \alpha$. *There exists* R_0 *such that for all* $M > 0$ *and* $\gamma \in (0, \alpha)$ *there exists* $\tau \in (0,1)$ *and* $\varepsilon > 0$ *for which the following implication holds.*

Let $(u,p) \in W^{1,2}(\Omega, \mathbb{R}^d) \times L^2(\Omega, \mathbb{R})$ *be a weak solution of the system* (1.1.2) *and let for all* $R \in (0, R_0)$ *and for all* $x_0 \in \partial\Omega$ *the inequalities*

$$E^{u,p}(x_0, R) < \varepsilon, \ (|\nabla u|)_{\Gamma_{x_0,R}} + |\,(p)_{\Gamma_{x_0,R}}\,| \leq \frac{M}{4}$$

hold. Then

$$E^{u,p}(x_0, \tau^k R) \leq \frac{1}{2^k} \tau^{k\gamma} E^{u,p}(x_0, R),$$

for all $k \in \mathbb{N}$

.

Proof. According to Lemma 22, we get for $0 < R < R'$

$$\begin{aligned} |(p)_{\Gamma_{x,R}} - (p)_{\Gamma_{x,R'}}| &= \frac{c}{R^{\frac{d}{2}}} \|(p)_{\Gamma_{x,R}} - (p)_{\Gamma_{x,R'}}\|_{2,\Omega_{x,R}} \\ &\leq c R^{\frac{-d}{2}} \left(\|(p)_{\Gamma_{x,R}} - p\|_{2,\Omega_{x,R}} + \|p - (p)_{\Gamma_{x,R'}}\|_{2,\Omega_{x,R'}} \right) \\ &\leq c_{14} R^{1-\frac{d}{2}} \|\nabla p\|_{2,\Omega_{x,R}} + c_{15} R' R^{-\frac{d}{2}} \|\nabla p\|_{2,\Omega_{x,R'}}. \end{aligned}$$

Fix τ such that $2C^* \tau^{\alpha-\gamma} < \frac{1}{2}$ and $\tau < \frac{1}{2}$. According to Lemma 44 there exists ε_1 such that

$$E^{u,p}(x_0, \tau R) \leq 2C^* \tau^\alpha E^{u,p}(x_0, R)$$

whenever

$$E^{u,p}(x_0, R) < \varepsilon_1.$$

We suppose that $E^{u,p}(x_0, R) < \varepsilon_2$ where ε_2 is such that $(c_{14} + 2c_{15}\tau^{-\frac{d}{2}})\varepsilon_2 < \frac{M}{4}$. According to the Lemma 44, the conclusion is true for $k = 0$.

Let the conclusion be true for some $k \in \mathbb{N}$ and let $\left|(p)_{\Gamma_{x_0,\tau^{i-1}R_0}}\right| < \frac{M}{2}$ for all $i \leq k-1$. We have

$$E^{u,p}(x_0, \tau^k R_0) \leq \frac{1}{2^k}\tau^{k\gamma}E^{u,p}(x_0, R_0).$$

We get $E^{u,p}(x_0, \tau^k R_0) < \frac{1}{2^k}\min\{\varepsilon_1, \varepsilon_2\}$ due to the assumptions . The function p fulfills

$$\begin{aligned}|(p)_{\Gamma_{x_0,\tau^k R_0}}| &\leq |(p)_{\Gamma_{x_0,\tau^k R_0}} - (p)_{\Gamma_{x_0,\tau^{k-1}R_0}}| + |(p)_{\Gamma_{x_0,\tau^{k-1}R_0}}| \\ &\leq c_{14}(\tau^k R_0)^{1-\frac{d}{2}}\|\nabla p\|_{2,\Omega_{x_0,\tau^k R_0}} + c_{15}\tau^{-\frac{d}{2}}(\tau^{k-1}R_0)^{1-\frac{d}{2}}\|\nabla p\|_{2,\Omega_{x_0,\tau^{k-1}R_0}} \\ &\qquad + (|p|)_{\Gamma_{x_0,\tau^{k-1}R_0}}.\end{aligned}$$

The estimate $(\tau^k R_0)^{1-\frac{d}{2}}\|\nabla p\|_{2,\Omega_{x_0,\tau^k R_0}} \leq E^{u,p}(x_0, \tau^k R_0) \leq \frac{1}{2^k}\varepsilon_2$ implies

$$|(p)_{\Gamma_{x_0,\tau^k R_0}}| \leq \frac{1}{2^k}\left(c_{14} + 2c_{15}\tau^{\frac{-d}{2}}\right)\varepsilon_2 + |(p)_{\Gamma_{x_0,\tau^{k-1}R_0}}|.$$

Therefore

$$|(p)_{\Gamma_{x_0,\tau^k R_0}}| \leq \frac{M}{4}\sum_{i=1}^{k}\frac{1}{2^i} + (p)_{\Gamma_{x_0,R_0}} \leq \frac{M}{2}.$$

The same conclusion can be drawn for $(|\nabla u|)_{\Gamma_{x_0,\tau^k R_0}}$. Thus $(|\nabla u|)_{\Gamma_{x_0,\tau^k R_0}} + |(p)_{\Gamma_{x_0,\tau^k R_0}}| \leq M$ and we can use Key Lemma to get

$$E^{u,p}(x, \tau^{k+1}R_0) \leq 2C^*\tau^{\alpha-\gamma}\tau^{\gamma}E^{u,p}(x, \tau^k R_0) \leq \frac{\tau^{\gamma}}{2}\frac{\tau^{k\gamma}}{2^k}E^{u,p}(x, R_0).$$

□

For $(u,p) \in W^{2,2}(\Omega) \times W^{1,2}(\Omega)$, $x \in \overline{\Omega}$ and $0 < R$ we define quantities $\mathcal{E}_0^{u,p}(x,R)$ and $\mathcal{E}^{u,p}(x,R)$ as follows

$$\begin{aligned}\mathcal{E}_0^{u,p}(x,R) &\stackrel{def}{=} R^{\frac{2-d}{2}}\|\nabla^2 u\|_{2,B_R(x)\cap\Omega} + R^{\frac{2-d}{2}}\|\nabla p\|_{2,B_R(x)\cap\Omega}, \\ \mathcal{E}^{u,p}(x,R) &\stackrel{def}{=} \mathcal{E}_0^{u,p}(x,R) + R^{\alpha}\end{aligned}$$

Inclusions $\Omega_{x,\frac{R}{2}} \subset (B_R(x)\cap\Omega) \subset \Omega_{x,2R}$ are valid for R less or equal to certain R_0 hence it can be seen that there exists a constant c, which depends only on Ω, such that

$$\frac{1}{c}E^{u,p}(x,R) \leq \mathcal{E}^{u,p}(x,R) \leq cE^{u,p}(x,R)$$

for all $x \in \Gamma$.

Lemma 3.4 in [24] is a variant of the Key lemma for interior and can be read as follows.

46 Lemma. *Let assumption* (1.1.3) *be satisfied with* $c_3 < \frac{c_1}{(c_1+c_7c_2)c_7}$ *and let* $f \in L^{2,\mu}$ *where* $\mu > d-1+\alpha$. *There exists* $R_0 > 0$ *such that for all* $M > 0$ *and* $\tau \in (0,1)$ *there exists an* $\varepsilon > 0$ *for which the following implication holds.*
Let $(u,p) \in W^{1,2}(\Omega,\mathbb{R}^d) \times L^2(\Omega,\mathbb{R})$ *be a weak solution of the system* (1.1.2) *and let for any* $x_0 \in \Omega$ *and* $R \in (0,R_0)$ *the inequalities*

$$\mathcal{E}^{u,p}(x_0,R) < \varepsilon, \ \left|(u)_{B_R(x_0)}\right| + \left|(\mathcal{D}u)_{B_R(x_0)}\right| + \left|(p)_{B_R(x_0)}\right| \leq M$$

hold. Then

$$\mathcal{E}^{u,p}(x_0,\tau R) \leq 2C^*\tau^\alpha \mathcal{E}^{u,p}(x_0,R).$$

Following lemma can be obtained in similar way as Lemma 45.

47 Lemma. *Let assumptions 1.1.3 be satisfied with* $c_3 < \frac{c_1}{(c_1+c_7c_2)c_7}$ *and let* $f \in L^{2,\mu}$ *where* $\mu > d-1+\alpha$. *There exists* $R_0 > 0$ *such that for all* $M > 0$ *and* $\gamma \in (0,\alpha)$ *there exists* $\tau \in (0,1)$ *and* $\varepsilon > 0$ *for which the following implication hold:*
Let $(u,p) \in W^{1,2}(\Omega,\mathbb{R}^d) \times L^2(\Omega,\mathbb{R})$ *be a weak solution of the system* (1.1.2) *and let for any* $x_0 \in \Omega$ *and* $R \in (0,R_0)$ *the inequalities*

$$\mathcal{E}^{u,p}(x_0,R) < \varepsilon, \ \left|(u)_{B_R(x_0)}\right| + \left|(\mathcal{D}u)_{B_R(x_0)}\right| + \left|(p)_{B_R(x_0)}\right| \leq \frac{M}{4}$$

hold. Then

$$\mathcal{E}^{u,p}(x_0,\tau^k R) \leq \frac{1}{2^k}\tau^{k\gamma}\mathcal{E}^{u,p}(x_0,R)$$

for all $k \in \mathbb{N}$

48 Corollary. *Let* (u,p) *be a weak solution of* (1.1.2) *and let* (1.1.3) *be satisfied with* $c_3 = \frac{c_1}{(c_1+c_7c_2)c_7}$. *If* $x_0 \in \partial\Omega$ *fulfills*

$$\begin{aligned} \liminf_{R\to 0} E^{v,p}(x_0,R) =& 0, \\ \limsup_{R\to 0} |(p)_{\Gamma_{x_0,R}}| + (|\nabla u|)_{\Gamma_{x_0,R}} <& \infty, \\ \limsup_{R\to 0} |(p)_{\Omega_{x_0 r,}}| + |(\mathcal{D}u)_{\Omega_{x_0,R}}| + |(u)_{\Omega_{x_0,R}}| <& \infty, \end{aligned}$$

then $\mathcal{D}u$ *and* p *are Hölder continuous on some neighborhood of* x_0.

Proof. Let $x_0 \in \Gamma$ satisfy the assumptions of the corollary. Our aim is to prove that there exists constant c_{16} and $\gamma > 0$ such that for all $x \in \Omega_{x_0,\frac{R}{2}}$, where $R > 0$ is sufficiently small, and for all $\rho \leq \frac{R}{2}$ it holds that

$$\mathcal{E}^{u,p}(x,\rho) \leq c_{16}\rho^\gamma. \tag{4.4.14}$$

This condition directly implies that $(\nabla^2 u, \nabla p) \in \mathcal{L}^{2,d-2+\gamma}(\Omega, \mathbb{R}^{d^2}) \times \mathcal{L}^{2,d-2+\gamma}(\Omega, \mathbb{R})$ and thus ∇u and p are Hölder continuous.
It holds that $\liminf_{R\to 0} E^{u,p}(x_0, R) = 0 \Rightarrow \forall c > 0; \forall R_0 > 0; \exists R < R_0; E^{u,p}(x_0, R) < c$ and thus, according to the continuity of integral, there exists $R_1 \in \left(0, \frac{R_0}{4}\right)$, a neighborhood Γ_{x_0,R_1} and a constant $c_{17} > 0$ such that for all $x \in \Gamma_{x_0,R_1}$ it holds that $E^{u,p}(x_0, R_1) \le c_{17}$. Further, as $\liminf_{R\to 0} \mathcal{E}^{u,p}(x_0, R) = 0$, we assume, without loss of generality, that $\mathcal{E}^{u,p}(x, R_1) \le c_{17}$ for all $x \in \Omega_{x_0,R_1}$.
Let $\rho < \frac{R_1}{3}$. We suppose that $x \in \Gamma_{x_0, \frac{R_1}{3}}$. We find $k \in \mathbb{N}$ such that $\tau^{k+1} R_1 < \rho \le \tau^k R_1$ where τ comes from Lemma 45. It can be easily seen that

$$E^{u,p}(x, \rho) \le \max\left\{1, \tau^{\frac{2-d}{2}}\right\} E^{u,p}\left(x, \tau^k R_1\right).$$

Thus, according to Lemma 45, there exists constant c_{18} such that

$$\begin{aligned} c\mathcal{E}^{u,p}(x, \rho) \le E^{u,p}(x, \rho) &\le c\tau^{k\gamma} E^{u,p}(x_0, R_1) \\ &\le c\left(R_1 \tau^{k+1}\right)^{\gamma} \frac{E^{u,p}(x, R_1)}{(R_1 \tau)^{\gamma}} \le \rho^{\gamma} c_{18}. \end{aligned} \tag{4.4.15}$$

Let $x \in \Omega_{x_0,R_1/3} \setminus \Gamma_{x_0,R_1/3}$. We distinguish between two situations. If $\rho \le \operatorname{dist}(x, \Gamma_{x_0,R_1/3})$, we can simply repeat previous method using Lemma 47 instead of Lemma 45 and we get that existence of a constant c_{19} such that

$$\mathcal{E}^{u,p}(x, \rho) \le c_{19} \rho^{\gamma}. \tag{4.4.16}$$

In order to complete the proof we need to show, that $\mathcal{E}^{u,p}(x, \rho)\rho^{-\gamma}$ is bounded independently of ρ and x even for $\rho > \operatorname{dist}(x, \partial\Omega)$.
If $\rho > \operatorname{dist}(x, \partial\Omega)$, we can find $x_1 \in \Gamma_{x_0,R_1/3}$ such that $B_\rho(x) \cap \Omega \subset B_{3\rho}(x_1) \cap \Omega$. Thus there exists a constant c_{20} such that

$$\mathcal{E}^{u,p}(x, \rho) \le (3^{(d-2)/2} + 3^{\alpha}) \mathcal{E}^{u,p}(x_1, 3\rho) \le c\mathcal{E}^{u,p}(x_1, 3\rho) \le cc_{17}\rho^{\gamma} \le c_{20}\rho^{\gamma}. \tag{4.4.17}$$

Combining inequalities (4.4.15), (4.4.16) and (4.4.17) we get the validity of (4.4.14) on some neighborhood of x_0. $\square$

4.5 Proof of the main theorem

Let $c_3 < \min\left\{\frac{c_1}{(c_1+c_2c_7)c_7}, c_9^{-1}\right\}$. We call a point $x \in \partial\Omega$ singular if there is no relative neighborhood of x where $\mathcal{D}u$ and p are Hölder continuous. We denote the set of all singular points by Σ. As a consequence of the previous corollary we get $\Sigma \subset \bigcup_{i=1}^{3} \Sigma_i$ where

$$
\begin{aligned}
\Sigma_1 =& \{x \in \partial\Omega, \liminf_{R\to 0} E^{u,p}(x,R) > 0\},\\
\Sigma_2 =& \{x \in \partial\Omega, \limsup_{R\to 0} (|\mathcal{D}u|)_{\Gamma_{x_0,R}} + |(p)_{\Gamma_{x_0,R}}| = \infty\},\\
\Sigma_3 =& \{x \in \partial\Omega, \limsup_{R\to 0} |(p)_{\Omega_{x_0,R}}| + |(\mathcal{D}u)_{\Omega_{x_0,R}}| + |(u)_{\Omega_{x_0,R}}| = \infty\}.
\end{aligned}
$$

We know, according to Lemma 42, that $(\mathcal{D}u, p) \in W^{1/2,2+\delta}(\partial\Omega)$ and, according to Corollary 24, we get

$$\mathcal{H}^{d-2}(\Sigma_2) = 0.$$

Note that $(\mathcal{D}u, p) \in W^{1,6}$ and thus Corollary 24 also implies

$$\mathcal{H}^{d-2}(\Sigma_3) = 0.$$

Due to the Lemma 23

$$\mathcal{H}^{d-2}(\Sigma_1) = 0.$$

Thus

$$\mathcal{H}^{d-2}(\Sigma) \leq \mathcal{H}^{d-2}(\Sigma_1) + \mathcal{H}^{d-2}(\Sigma_2) + \mathcal{H}^{d-2}(\Sigma_2) = 0$$

and the proof of the main theorem is completed.

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