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DIPLOMOVÁ PRÁCE

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Vychylující moduly konečného typu

Katedra algebry

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Abstrakt: Diplomová práce se zabývá studiem vlastností kotorzních párů v kategorii modulů; zejména nás zajímají podmínky, za nichž je daný kotorzní pár úplný či dokonce konečného typu. Metody dekonstrukce kotorzních párů, které při našem zkoumání rozvineme, posléze využijeme k důkazu tvrzení, že každý vychylující modul je konečného typu. Ukážeme také souvislost prezentované problematiky s tzv. teleskopickou hypotézou.
Klíčová slova: kotorzní pár, axiom konstruovatelnosti, vychylující modul, konečný typ, teleskopická hypotéza

Title: Tilting modules of finite type and methods of deconstruction of cotorsion pairs
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Abstract: The thesis studies properties of cotorsion pairs in the category of modules; we are mostly interested in conditions under which the given cotorsion pair is complete or, actually, of finite type. Methods of deconstruction of cotorsion pairs developed during our inquiry are eventually used to prove that every tilting module is of finite type. We show also a connection of presented problems with so-called telescope conjecture.
Keywords: cotorsion pair, Axiom of Constructibility, tilting module, finite type, telescope conjecture

Prologue

The spine of this work is made by relevant parts of about six recent papers dealing with cotorsion pairs and their application to the approximation theory of modules and tilting theory.

The first section starts with some notation and classical results concerning cotorsion pairs, pure embeddings, and set-theoretic tools involved. These preliminaries are followed by a slightly improved material from [26]. The improvement comes from the most recent paper [29].

The second section consists of the fundamental parts of three papers, [28], [11] and [12], devoted to tilting modules and tilting cotorsion pairs. The central result says that every tilting module is of finite type. As a byproduct, the characterization of tilting and cotilting cotorsion pairs is presented.

In the third section, we give, using in former sections developed methods, a partial solution to the telescope conjecture for module categories formulated, for artin algebras, by Krause and Solberg in [23] and generalized in [6]. The majority of material in this section originates from the latter paper.

1 Completeness of cotorsion pairs

Given an arbitrary ring R, it is generally not possible to classify all R-modules. A way to overcome this obstacle consists in selecting appropriate classes of R-modules, $\mathcal{C}$, and studying $\mathcal{C}$-approximations (envelopes and covers) of R-modules. This approach has successfully been used in module theory starting from classical works on injective envelopes and projective covers by Matlis, Bass et al. in the 1960's, over applications in commutative algebra and representation theory of artin algebras by Auslander's school, to constructions of flat covers by Enochs, Xu et al., and recent applications to tilting theory and finitistic dimension conjectures.

Many approximation classes of modules come from complete cotorsion pairs. However, not all cotorsion pairs are complete, so it is essential to have criteria of completeness available. In the following, we investigate closure properties of the classes $\mathcal{A}$ and $\mathcal{B}$ that imply completeness of the cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ both in ZFC, and in the extension of ZFC with the Axiom of Constructibility (V = L).

We first work under the assumption of V = L. We prove that $\mathfrak{C}$ is complete whenever $\mathfrak{C}$ is generated by a set and $\mathcal{A}$ is closed under pure submodules (Theorem 7). We also show that $\mathfrak{C}$ is complete whenever $\mathfrak{C}$ is hereditary, generated by a set, and $\mathcal{B}$ consists of modules of finite injective dimension (Theorem 11). These results generalize [17, Theorem 14] which says that (under V = L) any cotorsion pair generated by a set is complete in the particular setting of right hereditary rings. However, by [16], these results are independent of ZFC + GCH.

In the rest of the first section, we work in ZFC and prove analogous results, but replacing V = L by further closure properties of the classes $\mathcal{A}$ and $\mathcal{B}$. We show that $\mathfrak{C}$ is complete whenever $\mathcal{B}$ is closed under arbitrary direct sums and either (i) $\mathcal{A}$ is closed under pure submodules, or (ii) $\mathfrak{C}$ is hereditary and $\mathcal{B}$ consists of modules of finite injective dimension (Theorem 12). Moreover, if R is right $\aleph_0$-noetherian and $\mathfrak{C}$ is a hereditary cotorsion pair such that $\mathcal{B}$ is closed under arbitrary direct sums and $\mathcal{B}$ consists of modules of finite injective dimension, then $\mathfrak{C}$ is of countable type (Corollary 13).

1.1 Preliminaries

1.1.1 Algebraic preliminaries

For a ring R, denote by Mod-R the category of all (unitary right R-) modules. For a module M, $\mathrm{gen}(M)$ denotes the minimal cardinality of an R-generating subset in M, and $E(M)$ the injective envelope of M. For $n \geq 0$, the class of all modules of injective (projective, resp.) dimension $\leq n$ is denoted by $\mathcal{I}_n$ ($\mathcal{P}_n$, resp.). For an infinite cardinal κ, by a *κ-bounded product* of modules $(M_i)_{i\in I}$, we mean the submodule $\prod_{i\in I}^{<\kappa} M_i$ of the direct product $\prod_{i\in I} M_i$ formed by all elements with support of cardinality $< \kappa$. So $\prod_{i\in I}^{<\aleph_0} M_i = \bigoplus_{i\in I} M_i$, for example.

For a class of R-modules $\mathcal{C}$, let $\mathcal{C}^{\perp} = \bigcap_{i>0} \mathcal{C}^{\perp_i}$ where $\mathcal{C}^{\perp_i} = \{M \in \text{Mod-}R \mid \mathrm{Ext}^i_R(C, M) = 0 \text{ for all } C \in \mathcal{C}\}$. Similarly, ${}^{\perp}\mathcal{C} = \bigcap_{i>0} {}^{\perp_i}\mathcal{C}$ where ${}^{\perp_i}\mathcal{C} = \{M \in \text{Mod-}R \mid \mathrm{Ext}^i_R(M, C) = 0 \text{ for all } C \in \mathcal{C}\}$. If $\mathcal{C}$ is a singleton $\{N\}$, we usually omit the braces, and write simply $N^{\perp_i}$ or ${}^{\perp_i}N$.

Note that for every class $\mathcal{C}$, ${}^{\perp}\mathcal{C}$ is a *resolving* class, that is, it is closed under extensions, kernels of epimorphisms and contains the projective modules. In particular, it is syzygy-closed. Dually $\mathcal{C}^\perp$ is *coresolving*: it is closed under extensions, cokernels of monomorphisms and contains the injective modules. In particular, it is cosyzygy-closed.

Given an arbitrary ring R, a pair of classes of right R-modules, $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$, is a *cotorsion pair* provided $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$.

Note that $\mathfrak{D} = ({}^{\perp_1}\mathcal{C}, ({}^{\perp_1}\mathcal{C})^{\perp_1})$ and $\mathfrak{E} = ({}^{\perp_1}(\mathcal{C}^{\perp_1}), \mathcal{C}^{\perp_1})$ are cotorsion pairs called the cotorsion pairs *generated* and *cogenerated* by $\mathcal{C}$ respectively. If $\mathcal{C}$ has a representative set of elements, we say that $\mathfrak{D}$ is *generated by a set*, and $\mathfrak{E}$ is *cogenerated by a set*.

We say that a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is of *countable type* provided there is a set of modules, $\mathcal{C}$, possessing a projective resolution consisting of countably generated projective modules such that $\mathcal{B} = \mathcal{C}^{\perp_1}$. Similarly, a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is of *finite type* provided there is a set of modules, $\mathcal{F}$, possessing a projective resolution consisting of finitely generated projective modules such that $\mathcal{B} = \mathcal{F}^{\perp_1}$.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *hereditary* provided $\mathcal{A} = {}^\perp\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^\perp$ (that is, $\operatorname{Ext}^i_R(A, B) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $i \geq 1$). It is easy to see that $(\mathcal{A}, \mathcal{B})$ is hereditary cotorsion pair if and only if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair and $\mathcal{A}$ is resolving if and only if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair and $\mathcal{B}$ is coresolving.

Cotorsion pairs are analogs of the classical torsion pairs (or torsion theories) where Hom_R is replaced by Ext^1_R. Similarly, one can define F-torsion pairs for any additive bifunctor F on Mod-R. We shall present some examples of cotorsion pairs after Lemma 4.

A class $\mathcal{C}$ of modules is *special preenveloping* provided for each module M there are $C \in \mathcal{C}$, $D \in {}^{\perp_1}\mathcal{C}$, and an exact sequence $0 \longrightarrow M \longrightarrow C \longrightarrow D \longrightarrow 0$ (the monomorphism $M \to C$ is called a *special $\mathcal{C}$-preenvelope* of the module M). A special preenveloping class $\mathcal{C}$ containing all injective modules is called *enveloping* provided that for every module M there exists a special $\mathcal{C}$-preenvelope $f : M \to C$ such that $g \in \operatorname{End}_R(C)$ and $gf = f$ imply that g is an automorphism of C. Such special $\mathcal{C}$-preenvelope is called *$\mathcal{C}$-envelope* of the module M. For example, the embeddings $M \hookrightarrow E(M)$ are in fact $\mathcal{I}_0$-envelopes, so $\mathcal{I}_0$ is an enveloping class.

Dually, $\mathcal{C}$ is *special precovering* if for each module M there are $C \in \mathcal{C}$, $D \in \mathcal{C}^{\perp_1}$, and an exact sequence $0 \longrightarrow D \longrightarrow C \longrightarrow M \longrightarrow 0$ (the epimorphism $C \to M$ is a *special $\mathcal{C}$-precover* of M). A special precovering class $\mathcal{C}$ containing all projective modules is called *covering* provided that for every module M there exists a special $\mathcal{C}$-precover $f : C \to M$ such that $g \in \operatorname{End}_R(C)$ and $fg = f$ imply that g is an automorphism of C. Again, such f is called *$\mathcal{C}$-cover* of the module M. The notion of $\mathcal{P}_0$-cover coincides with the classical Bass' notion of projective cover. If $\mathcal{C}$-covers ($\mathcal{C}$-envelopes, resp.) exist, then they are unique up to isomorphism.

A cotorsion pair $\mathfrak{C}$ is *complete* provided that the class $\mathcal{A}$ is special precovering. Salce observed that this is equivalent to $\mathcal{B}$ being a special preenveloping class. So special $\mathcal{A}$-precovers of all modules exist iff special $\mathcal{B}$-preenvelopes do. In this way, complete cotorsion pairs are also helpful to proving dual results in the category Mod-R where no categorial duality is available.

An ascending chain, $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$, of submodules of M is called *continuous* provided that $M_0 = 0$ and $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ for all limit ordinals $\alpha \leq \sigma$. If moreover $M_\sigma = M$, then $\mathcal{M}$ is called a *filtration* of M (of length σ). Similarly (componentwise), we define continuous chains and filtrations consisting of short exact sequences of modules. A filtration is called a *$\mathcal{C}$-filtration* for a class of modules $\mathcal{C}$ if in addition $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$ for each $\alpha < \sigma$. A module M is *$\mathcal{C}$-filtered* if there is a

$\mathcal{C}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$ of M. For κ an infinite cardinal, a filtration $(M_\alpha \mid \alpha \leq \kappa)$ of M is a *κ-filtration* provided $\mathrm{gen}(M_\alpha) < \kappa$ for all $\alpha < \kappa$.

Assume κ is a regular uncountable cardinal. A strictly ascending function $f : \kappa \to \kappa$ is called *continuous* provided that $f(0) = 0$, and $f(\alpha) = \sup_{\beta<\alpha} f(\beta)$ for all limit ordinals $\alpha < \kappa$. If $\mathcal{M} = (M_\alpha \mid \alpha \leq \kappa)$ is a filtration of a module M, and $f : \kappa \to \kappa$ a continuous function, then $\mathcal{M}' = (M_{f(\alpha)} \mid \alpha \leq \kappa)$ (where we put $f(\kappa) = \kappa$) is again a filtration of M, called the *subfiltration* of $\mathcal{M}$ induced by f. Any two κ-filtrations of M coincide on a closed and unbounded subset of κ, cf. [15], thus they possess a common subfiltration.

Let us now state two fundamental results which are standard by now.

Lemma 1. [15, Proposition XII.1.5] *Let M, N be R-modules. Suppose that M has a filtration $(M_\nu \mid \nu \leq \mu)$ such that $M_{\nu+1}/M_\nu \in {}^{\perp_1}N$ for all $\nu < \mu$. Then $M \in {}^{\perp_1}N$.*

Proof. The proof is by induction on μ. Consider a short exact sequence

$$0 \longrightarrow N \xrightarrow{\iota} B \xrightarrow{\pi} M \longrightarrow 0.$$

We must show that this sequence splits. To do this, we define by transfinite induction a continuous increasing chain of homomorphisms $\rho_\nu : M_\nu \to B$ such that $\pi\rho_\nu = \mathrm{id}_{M_\nu}$. Suppose that ρ_ν has been defined for all $\nu < \beta$. If β is a limit ordinal, we let ρ_β to be the union of the ρ_ν. If $\beta = \gamma + 1$, let $\sigma : M_\beta \to B$ be some splitting of $\pi \restriction \pi^{-1}(M_\beta)$, which exists since $\mathrm{Ext}^1_R(M_\beta, N) = 0$ because $\mathrm{Ext}^1_R(M_\beta/M_\gamma, N) = 0$ by assumption and $\mathrm{Ext}^1_R(M_\gamma, N) = 0$ by induction. Since ρ_γ and $\sigma \restriction M_\gamma$ are both splittings of $\pi \restriction \pi^{-1}(M_\gamma)$, there is a homomorphism $\theta : M_\gamma \to N$ such that $\iota\theta = \rho_\gamma - (\sigma \restriction M_\gamma)$. Since $\mathrm{Ext}^1_R(M_\beta/M_\gamma, N) = 0$, θ extends to a homomorphism $\theta' : M_\beta \to N$. If we define $\rho_\beta = \sigma + (\iota\theta')$, then ρ_β extends ρ_γ and $\pi\rho_\beta = \mathrm{id}_{M_\beta}$. □

The following theorem is a crucial one. It provides us with a rich supply of complete cotorsion pairs.

Theorem 2. [18, Theorem 10], [33, Theorem 2.2] *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair.*

(i) *$\mathfrak{C}$ is complete provided $\mathfrak{C}$ is cogenerated by a set of modules (which is equivalent to $\mathfrak{C}$ being cogenerated by the singleton containing the direct sum of all modules from this set).*

(ii) *Let $\mathcal{S}$ be a set. Then $\mathfrak{C}$ is cogenerated by $\mathcal{S}$ if and only if $\mathcal{A}$ consists of all modules isomorphic to direct summands of $\mathcal{S} \cup \{R\}$-filtered modules.*

Proof. (i). Let $\mathcal{B} = S^{\perp_1}$ for a right R-module S. We fix a presentation of S,

$$0 \longrightarrow K \xrightarrow{\mu} F \longrightarrow S \longrightarrow 0,$$

with F a free module. Let λ be an infinite regular cardinal such that K is $< \lambda$-generated, and let $M \in \text{Mod-}R$ be arbitrary. We will construct a $\mathcal{B}$-preenvelope of M.

By induction, we define a chain $(P_\alpha \mid \alpha \leq \lambda)$ of right R-modules as follows:

First $P_0 = M$. For $\alpha < \lambda$, define μ_α as the direct sum of $\mathrm{Hom}_R(K, P_\alpha)$ many copies of μ, so

$$\mu_\alpha \in \mathrm{Hom}_R\big(K^{(\mathrm{Hom}_R(K,P_\alpha))}, F^{(\mathrm{Hom}_R(K,P_\alpha))}\big).$$

Then μ_α is a monomorphism and $\mathrm{Coker}\,\mu_\alpha$ is isomorphic to a direct sum of copies of S. Let $\varphi_\alpha \in \mathrm{Hom}_R(K^{(\mathrm{Hom}_R(K,P_\alpha))}, P_\alpha)$ be the canonical morphism. Note that for each

$\eta \in \mathrm{Hom}_R(K, P_\alpha)$, there exist canonical embeddings $\nu_\eta \in \mathrm{Hom}_R(K, K^{(\mathrm{Hom}_R(K,P_\alpha))})$ and $\nu'_\eta \in \mathrm{Hom}_R(F, F^{(\mathrm{Hom}_R(K,P_\alpha))})$ such that $\eta = \varphi_\alpha \nu_\eta$ and $\nu'_\eta \mu = \mu_\alpha \nu_\eta$.

Now, $P_{\alpha+1}$ is defined via the pushout of μ_α and φ_α:

$$\begin{array}{ccc} K^{(\mathrm{Hom}_R(K,P_\alpha))} & \xrightarrow{\mu_\alpha} & F^{(\mathrm{Hom}_R(K,P_\alpha))} \\ {\scriptstyle\varphi_\alpha}\big\downarrow & & {\scriptstyle\psi_\alpha}\big\downarrow \\ P_\alpha & \xrightarrow{\subseteq} & P_{\alpha+1} \end{array}$$

If $\alpha \leq \lambda$ is a limit ordinal, we put $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$. Set $P = P_\lambda$.

We will prove that $\nu : M \hookrightarrow P$ is a special $\mathcal{B}$-preenvelope of M.

First, we check that $P \in \mathcal{B}$. Since F is projective, we are left to show that any $\varphi \in \mathrm{Hom}_R(K, P)$ factors through μ. Since K is $< \lambda$-generated, there are an index $\alpha < \lambda$ and $\eta \in \mathrm{Hom}_R(K, P_\alpha)$ such that $\varphi(k) = \eta(k)$ for all $k \in K$. The pushout square gives $\psi_\alpha \mu_\alpha = \sigma_\alpha \varphi_\alpha$, where σ_α denotes the inclusion of P_α into $P_{\alpha+1}$. Altogether, we have $\psi_\alpha \nu'_\eta \mu = \psi_\alpha \mu_\alpha \nu_\eta = \sigma_\alpha \varphi_\alpha \nu_\eta = \sigma_\alpha \nu$. It follows that $\varphi = \psi' \mu$ where $\psi' = \psi_\alpha \nu'_\eta \in \mathrm{Hom}_R(F, P_{\alpha+1})$. This proves that $P \in \mathcal{B}$.

It remains to prove that $N = P/M \in \mathcal{A}$. By the construction, N is the union of the continuous chain $(N_\alpha \mid \alpha \leq \lambda)$ where $N_\alpha = P_\alpha/M$. Since $P_{\alpha+1}/P_\alpha$ is isomorphic to a direct sum of copies of S by the pushout construction, so is $N_{\alpha+1}/N_\alpha \cong P_{\alpha+1}/P_\alpha$. Since $S \in \mathcal{A}$, Lemma 1 shows that $N \in \mathcal{A}$.

(ii). The if part is clear using Lemma 1. Let us prove the other part.

We have $\mathcal{B} = S^{\perp_1}$ for some $S \in$ Mod-R. Take $A \in \mathcal{A}$ and let $0 \longrightarrow N \xrightarrow{\mu} F \longrightarrow A \longrightarrow 0$ be a short exact sequence with F free. By the first part of the theorem, there is a special $\mathcal{B}$-preenvelope, $0 \longrightarrow N \xrightarrow{\nu} P \longrightarrow P/N \longrightarrow 0$, of N such that P/N is a union of a continuous chain, $(P_\alpha/N \mid \alpha < \lambda)$, with successive factors isomorphic to S. Consider the pushout of μ and ν:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \big\downarrow & & \big\downarrow & & & & \\ 0 & \longrightarrow & N & \xrightarrow{\mu} & F & \longrightarrow & A & \longrightarrow & 0 \\ & & {\scriptstyle\nu}\big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & P & \longrightarrow & Z & \longrightarrow & A & \longrightarrow & 0 \\ & & \big\downarrow & & {\scriptstyle\pi}\big\downarrow & & & & \\ & & P/N & = & P/N & & & & \\ & & \big\downarrow & & \big\downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Then $Z = \bigcup_{\alpha<\lambda} Z_\alpha$ where Z_α are the preimages of P_α/N in π. So $Z_0 = F$ and the successive factors $Z_{\alpha+1}/Z_\alpha$ are isomorphic to S. Finally, the second row splits since $P \in \mathcal{B}$ and $A \in \mathcal{A}$, so $A \oplus P \cong Z$. □

Recall that a submodule A of an R-module B is called a *pure submodule* if for every finitely presented module F and every $f \in \mathrm{Hom}_R(F, B/A)$ there is a factorization of f

through the canonical projection $\pi : B \to B/A$; in this case, we write $A \subseteq_* B$. Similarly, we say that a short exact sequence $0 \longrightarrow A \xrightarrow{g} B \xrightarrow{h} C \longrightarrow 0$ is *pure* provided that $g(A)$ is a pure submodule of B. In this case, we refer to g as to *pure monomorphism* or *pure embedding*; h is called *pure epimorphism*.

The following lemma, however, shows that there are other ways to approach purity.

Lemma 3. *Let A be an R-submodule of B. Denote by $\mathcal{E}$ the exact sequence $0 \longrightarrow A \xrightarrow{\subseteq} B \xrightarrow{\pi} B/A \longrightarrow 0$. The following are equivalent:*

(i) $A \subseteq_* B$.

(ii) *The following holds for all $0 < m < \omega, 0 < n < \omega$, and all systems of R-linear equations, $\mathcal{S}$, in the variables $x_0, \dots, x_{m-1}$ with $a_j \in A, r_{ij} \in R$ $(i < m, j < n)$*

$$\sum_{i<m} x_i r_{ij} = a_j \quad (j < n):$$

$\mathcal{S}$ has a solution in A whenever $\mathcal{S}$ has a solution in B.

(iii) *$\mathcal{E}$ is a direct limit of a direct system of split short exact sequences.*

(iv) *The sequence $0 \longrightarrow A \otimes_R F \longrightarrow B \otimes_R F \longrightarrow B/A \otimes_R F \longrightarrow 0$ is exact for any (finitely presented) left R-module F.*

(v) *The sequence $0 \longrightarrow (B/A)^c \longrightarrow B^c \longrightarrow A^c \longrightarrow 0$ splits (where M^c denotes the character module of a module M).*

Proof. This result is a classic one. We prove only the equivalence of (i) and (ii). Notice that each finitely presented module F is isomorphic to R^m/G for some $m < \omega$ and some $G \subseteq R^m$ generated by the elements $r_j = \sum_{i<m} 1_i r_{ij}$ $(j < n, r_{ij} \in R)$ where $(1_i \mid i < m)$ is the canonical basis of R^m. Denote by ρ the canonical projection of R^m onto R^m/G. Then for each R-homomorphism $f \in \mathrm{Hom}_R(R^m/G, B/A)$, we have $f(\rho(1_i)) = \pi(b_i')$ for some $b_i' \in B$ $(i < m)$ with $\pi(\sum_{i<m} b_i' r_{ij}) = 0$ for each $j < n$. The exactness of $\mathcal{E}$ then gives

$$\sum_{i<m} b_i' r_{ij} = a_j' \text{ for some } a_j' \in A. \qquad (*)$$

Assume (i). Consider a system $\mathcal{S}$ as in (ii), and define $F = R^m/G$ as above. If $(b_i \mid i < m)$ is solution of $\mathcal{S}$ in B, we can define $f \in \mathrm{Hom}_R(F, B/A)$ by $f(\rho(1_i)) = \pi(b_i)$ (this is possible because $\pi \restriction A = 0$). Then (i) yields $g \in \mathrm{Hom}_R(F, B)$ such that $\pi g = f$. Define $a_i' = b_i - g\rho(1_i)$ $(i < m)$. Then $\pi(a_i') = 0$, so $a_i' \in A$, and also $\sum_{i<m} a_i' r_{ij} = a_j$ for all $j < n$.

Assume (ii). Let $F = R^m/G$ be a finitely presented module and consider $f \in \mathrm{Hom}_R(F, B/A)$. The equality $(*)$ above and (ii) yield existence of $c_i \in A$ $(i < m)$ with $\sum_{i<m} c_i r_{ij} = a_j'$ $(j < n)$. So we can define $g \in \mathrm{Hom}_R(F, B)$ by $g(\rho(1_i)) = b_i' - c_i$. Then $\pi g(\rho(1_i)) = \pi(b_i') = f(\rho(1_i))$, so $\pi g = f$, and (i) holds. □

Later on, we will need some basic properties of pure embeddings.

Lemma 4. *Let $\kappa \geq \operatorname{card} R + \aleph_0$.*

(i) *Let M be a module and X be a subset of M with $\operatorname{card} X \leq \kappa$. Then there is a pure submodule $N \subseteq_* M$ such that $X \subseteq N$ and $\operatorname{card} N \leq \kappa$.*

(ii) *Assume* $C \subseteq B \subseteq A, C \subseteq_* A$ *and* $B/C \subseteq_* A/C$*. Then* $B \subseteq_* A$*.*

(iii) *If* $A \subseteq_* B$ *and* $B \subseteq_* C$ *then* $A \subseteq_* C$*.*

(iv) *Assume* $A_0 \subseteq \cdots \subseteq A_\alpha \subseteq A_{\alpha+1} \subseteq \cdots$ *is a chain of pure submodules of* M*. Then* $\bigcup_\alpha A_\alpha$ *is a pure submodule of* M*.*

Proof. (i) We apply characterization (ii) from Lemma 3 to define $N = \bigcup_{i<\omega} N_i$ where N_0 is the submodule generated by X, and N_{i+1} is the submodule generated by solutions in M of all the R-linear equations with right-hand side in N_i. Since $\kappa \geq \operatorname{card} R + \aleph_0$ and $\operatorname{card} X \leq \kappa$, we can assume that $\operatorname{card} N_{i+1} \leq \kappa$, and (i) easily follows.

(ii) is a direct consequence of the definition of purity.

(iii) and (iv) follow by part (ii) of Lemma 3. □

Examples. Let us state here few examples of (complete) cotorsion pairs.

- The cotorsion pairs $(\mathcal{P}_0, \text{Mod-}R)$, $(\text{Mod-}R, \mathcal{I}_0)$ are called *trivial cotorsion pairs.* They are complete by Theorem 2 (i) because the first one is cogenerated by the single module R and the second one by the set of all cyclic modules.

- For every $n < \omega$, the cotorsion pair $({}^{\perp_1}\mathcal{I}_n, \mathcal{I}_n)$ is cogenerated by a representative set of $\leq \kappa$-generated modules from ${}^{\perp_1}\mathcal{I}_n$ where κ is such that $\operatorname{gen}(I) \leq \kappa$ holds for each right ideal I of R.

- For every $n < \omega$, $(\mathcal{P}_n, \mathcal{P}_n^{\perp_1})$ is a cotorsion pair cogenerated by a representative set of those modules from $\mathcal{P}_n$ which have cardinality $\leq \operatorname{card} R + \aleph_0$. So $\mathcal{P}_n$ is a special precovering class for every $n \in \omega$. Moreover, by the result of Bass, $\mathcal{P}_0$ is covering if and only if R is right perfect.

- Solution of the Flat Cover Conjecture: Let $\mathcal{F}_0$ be the class of all flat right R-modules. Since pure submodules and pure epimorphic images of flat modules are flat, we can use Lemma 4 to construct, for every $F \in \mathcal{F}_0$, an $\mathcal{S}$-filtration where $\mathcal{S}$ is a representative set of those modules from $\mathcal{F}_0$ having cardinality $\leq \operatorname{card} R + \aleph_0$. Using Lemma 1, we deduce that the cotorsion pair $\mathfrak{F} = (\mathcal{F}_0, \mathcal{F}_0^{\perp_1})$ is cogenerated by the set $\mathcal{S}$. So $\mathfrak{F}$ is complete by Theorem 2 (i), and there is a result by Enochs saying that $\mathcal{A}$ is a covering class provided that $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair with $\mathcal{A}$ closed under direct limits. But it is well-known that $\mathcal{F}_0$ is closed under direct limits, thus $\mathcal{F}_0$ is actually a covering class.

Remark. The notion of a pure submodule is generally weaker than the model-theoretic notion of an elementary substructure. Having $\mathcal{A}$ and $\mathcal{B}$ two structures for a language L, we say that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$ if $\mathcal{A}$ is a substructure of $\mathcal{B}$, and for every $0 < n < \omega$, each L-formula $\varphi(x_0, \ldots, x_{n-1})$ with all free variables in the list $x_0, \ldots, x_{n-1}$ and every $(a_i)_{i<n} \in A^n$ we have

$$\mathcal{A} \models \varphi[a_0, \ldots, a_{n-1}] \Longleftrightarrow \mathcal{B} \models \varphi[a_0, \ldots, a_{n-1}].$$

As one can observe from Lemma 3 (ii), in the case of pure submodules (for L a language of R-modules), we are testing only specific formulas called *positive primitive formulas*; they can be written as a sequence of existence quantifiers followed by a conjunction of atomic fomulas.

1.1.2 Set-theoretic preliminaries

In the following, we will sometimes work in the extension of ZF with the Axiom of Constructibility (V = L) which says that every set is constructible. Constructible universe, **L**, is certain, well-understood and well-defined transitive subclass of the universe of all sets **V**. Moreover, **L** is so-called inner model of the set theory; it means that each axiom of the classical set theory (ZF) relativized to **L** holds in **L**. It can be proved that even V = L holds in **L**, so **L** is an inner model of ZF + V = L (ZFL for short); thus ZFL is one of many relatively consistent extensions of ZF. ZFL, however, has a lot of pleasant properties: one can prove Axiom of Choice and Generalized Continuum Hypothesis (GCH) in this theory. Moreover, at the beginning of 1970's, Jensen showed that various combinatorial principles, unprovable in ZFC + GCH, hold in **L**, so ZFL is even stronger than ZFC + GCH. We are going to use one of such principles, Jensen's diamond, to prove some results concerning deconstruction of cotorsion pairs.

In fact, weaker principle, called sometimes generalized weak diamond, will be sufficient for us. Recall that a subset S of an ordinal δ with uncountable cofinality is called *stationary* in δ if it has a non-empty intersection with each closed (in the interval topology) and unbounded subset of δ.

For a family of sets $(S_i \mid i \in I)$, we use the symbol $\bigtimes_{i\in I} S_i$ to denote the (possibly infinite) cartesian product of the sets S_i, that is the set $\{f : I \to \bigcup_{i\in I} S_i \mid f(i) \in S_i \text{ for all } i \in I\}$.

Let κ be a regular uncountable cardinal and E be a stationary subset of κ. Denote by $\Psi_\kappa(E)$ the assertion: "Let A be any set of cardinality κ and $(A_\alpha \mid \alpha < \kappa)$ be a continuous well-ordered chain of subsets of A with $A_0 = \emptyset$, $\bigcup_{\alpha<\kappa} A_\alpha = A$ and $\operatorname{card} A_\alpha < \kappa$, for all $\alpha < \kappa$. For each $\alpha \in E$, let $2 \le p_\alpha < \omega$ and let $P_\alpha : \mathcal{P}(A_\alpha) \to p_\alpha$ be given (here, $\mathcal{P}(S)$ denotes the set of all subsets of S). Then there is $\psi \in \bigtimes_{\alpha\in E} p_\alpha$ such that the set $\{\alpha \in E \mid P_\alpha(X \cap A_\alpha) = \psi(\alpha)\}$ is stationary in κ for every $X \subseteq A$."

The *generalized weak diamond* is the assertion Ψ: "$\Psi_\kappa(E)$ holds true for each regular uncountable cardinal and each stationary subset $E \subseteq \kappa$." As we were pointed out above, Ψ holds under the assumption V = L.

1.2 Completeness under V = L (part I)

Following [17], given a cardinal κ, we define a *κ-refinement* of M (of length σ) as a filtration of M (of length σ) such that M_α is a pure submodule of M and $\operatorname{card} M_{\alpha+1}/M_\alpha \le \kappa$, for all $\alpha < \sigma$.

We start with a slight generalization of [30, Lemma 3.7].

Lemma 5. (V = L) *Let κ be a regular uncountable cardinal, and E a stationary subset of κ. Let R be a ring such that* $\operatorname{card} R \le \kappa$*, and N a module with* $\operatorname{card} E(N) \le \kappa$*. Let M be a module with* $\operatorname{gen}(M) = \kappa$ *and a κ-filtration $(C_\alpha \mid \alpha \le \kappa)$ such that* $\operatorname{Ext}^1_R(C_\alpha, N) = 0$ *for all $\alpha < \kappa$, and $E = \{\alpha < \kappa \mid \operatorname{Ext}^1_R(C_{\alpha+1}/C_\alpha, N) \neq 0\}$. Then* $\operatorname{Ext}^1_R(M, N) \neq 0$.

Proof. Consider a continuous well-ordered chain, $(D_\alpha \mid \alpha < \kappa)$, of subsets of M with $\operatorname{card} D_\alpha < \kappa$ and such that $C_\alpha = \sum_{m\in D_\alpha} mR$, for all $\alpha < \kappa$. Put $D = \bigcup_{\alpha<\kappa} D_\alpha$. Note that $\operatorname{card} D = \kappa$. Let $(B_\alpha \mid \alpha \le \kappa)$ be a κ-filtration of the $\mathbb{Z}$-module $I = E(N)$. Denote by ν the inclusion of N into I, by π the projection of I onto I/N, and by ν_α the inclusion of C_α into $C_{\alpha+1}$, for all $\alpha < \kappa$.

Take $\alpha \in E$. Let $X_\alpha = \mathrm{Hom}_R(C_\alpha, N)$ and $Y_\alpha = \mathrm{Im}(\mathrm{Hom}_R(\nu_\alpha, N))$. By the premise, there is some $f_\alpha \in X_\alpha \setminus Y_\alpha$. Denote by o_α the order of $f_\alpha + Y_\alpha$ in the group $X_\alpha / Y_\alpha = \mathrm{Ext}^1_R(C_{\alpha+1}/C_\alpha, N)$.

We are going to use the principle $\Psi_\kappa(E)$ in the following setting: $A = D \times I$ and $A_\alpha = D_\alpha \times B_\alpha$, $\alpha < \kappa$. Let $\alpha \in E$. If $o_\alpha = \omega$, we put $p_\alpha = 2$. If $o_\alpha < \omega$, we define $p_\alpha = o_\alpha$. In order to define the colourings P_α, $\alpha \in E$, we equip the set of all mappings from D_α to B_α with an equivalence relation $\sim_\alpha$: we put $u \sim_\alpha v$ if and only if there are $n \in \mathbb{Z}$ and $y \in Y_\alpha$ such that $v = u + nf_\alpha \restriction D_\alpha + y \restriction D_\alpha$. Note that the number n is unique (unique modulo p_α) provided $o_\alpha = \omega$ ($o_\alpha < \omega$). Now, for each $\alpha \in E$, we take a colouring $P_\alpha : \mathcal{P}(A_\alpha) \to p_\alpha$ such that $P_\alpha(u) = P_\alpha(v)$ if and only if the number n given by the pair (u, v) is divisible by p_α.

Let $\psi \in \bigtimes_{\alpha \in E} p_\alpha$ be the mapping corresponding to this setting by $\Psi_\kappa(E)$. In order to prove that $\mathrm{Ext}^1_R(M, N) \neq 0$, we shall construct

$$g \in \mathrm{Hom}_R(M, I/N) \setminus \mathrm{Im}(\mathrm{Hom}_R(M, \pi)).$$

By induction on $\alpha < \kappa$, we define $g_\alpha \in \mathrm{Hom}_R(C_\alpha, I/N)$ so that $g_{\alpha+1} \restriction C_\alpha = g_\alpha$ for each $\alpha < \kappa$, and $g_\alpha = \bigcup_{\beta < \alpha} g_\beta$ for all limit $\alpha < \kappa$.

Put $g_0 = 0$. Assume g_α is defined for an ordinal $\alpha < \kappa$. We distinguish the following two cases:

(I) $\alpha \in E$ and there exists $f \in \mathrm{Hom}_R(C_{\alpha+1}, I)$ such that $g_\alpha = \pi f \nu_\alpha$, $\mathrm{Im}(f\nu_\alpha \restriction D_\alpha) \subseteq B_\alpha$, and $P_\alpha(f\nu_\alpha \restriction D_\alpha) = \psi(\alpha)$.

(II) = not (I).

In the case (I), take an f satisfying the conditions of (I). The injectivity of I yields the existence of $h_\alpha \in \mathrm{Hom}_R(C_{\alpha+1}, I)$ such that $h_\alpha \nu_\alpha = f\nu_\alpha - f_\alpha$. Put $g_{\alpha+1} = \pi h_\alpha$. Then $g_{\alpha+1}\nu_\alpha = \pi f \nu_\alpha - \pi f_\alpha = g_\alpha$.

In the case (II), $\mathrm{Ext}^1_R(C_\alpha, N) = 0$ yields the existence of $h_\alpha \in \mathrm{Hom}_R(C_\alpha, I)$ with $g_\alpha = \pi h_\alpha$. The injectivity of I gives some $h_{\alpha+1} \in \mathrm{Hom}_R(C_{\alpha+1}, I)$ such that $h_\alpha = h_{\alpha+1}\nu_\alpha$. Put $g_{\alpha+1} = \pi h_{\alpha+1}$. Then $g_{\alpha+1} \restriction C_\alpha = g_\alpha$.

Finally, put $g = \bigcup_{\alpha < \kappa} g_\alpha$. Then $g \in \mathrm{Hom}_R(M, I/N)$. Proving indirectly, suppose there is $h' \in \mathrm{Hom}_R(M, I)$ such that $g = \pi h'$. Note that the set $\{\alpha < \kappa \mid \mathrm{Im}(h' \restriction D_\alpha) \subseteq B_\alpha\}$ is closed and cofinal in κ. Put $X = h' \restriction D$. By the premise, there is an $\alpha \in E$ such that $g \restriction C_\alpha = \pi h \nu_\alpha$, $P_\alpha(h\nu_\alpha \restriction D_\alpha) = P_\alpha(X \cap A_\alpha) = \psi(\alpha)$, and $\mathrm{Im}(h\nu_\alpha \restriction D_\alpha) \subseteq B_\alpha$, where $h = h' \restriction C_{\alpha+1}$. Hence, the case (I) occurs, and $\pi(h_\alpha - h) = 0$. Then $y_\alpha = (h_\alpha - h)\nu_\alpha \in Y_\alpha$. Moreover, $f\nu_\alpha = h\nu_\alpha + f_\alpha + y_\alpha$, whence $\psi(\alpha) = P_\alpha(f\nu_\alpha \restriction D_\alpha) = P_\alpha(h\nu_\alpha \restriction D_\alpha + f_\alpha \restriction D_\alpha + y_\alpha \restriction D_\alpha) \neq P_\alpha(h\nu_\alpha \restriction D_\alpha)$, a contradiction. Thus $g \notin \mathrm{Im}(\mathrm{Hom}_R(M, \pi))$. □

Lemma 6. (V = L) *Let N be a module such that ${}^{\perp_1}N$ is closed under pure submodules, and κ be a cardinal with $\kappa \geq \mathrm{card}\, R + \mathrm{card}\, E(N) + \aleph_0$. Then for each module $M \in {}^{\perp_1}N$ there are an ordinal σ and a κ-refinement of M of length σ, $(M_\alpha \mid \alpha \leq \sigma)$, such that $M_{\alpha+1}/M_\alpha \in {}^{\perp_1}N$ for all $\alpha < \sigma$.*

Proof. The existence of the κ-refinement of M is proved by induction on the cardinality λ of M. It is clear for $\lambda \leq \kappa$.

Let λ be a regular cardinal $> \kappa$. By induction, we construct a κ-refinement $\mathcal{R}$ of M. First, we enumerate the elements of M, $M = \{m_\alpha \mid \alpha < \lambda\}$, and let $M_0 = 0$.

Let $\alpha < \lambda$. Since $\kappa \geq \operatorname{card} R + \aleph_0$, there is a pure submodule P/M_α of M/M_α containing $m_\alpha + M_\alpha$ such that $\operatorname{card} P/M_\alpha \leq \kappa$ (see Lemma 4 (i)). Since M_α is pure in M by inductive assumption, also P is pure in M by Lemma 4 (ii), and we let $M_{\alpha+1} = P$. If α is a limit ordinal, we let $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ which is again a pure submodule in M by Lemma 4 (iv). Since λ is regular cardinal, $\mathcal{R}$ is a λ-filtration.

Possibly taking a λ-subfiltration, we can without lost of generality assume that $\mathcal{R}$ is a λ-filtration with the following property: if $\alpha < \beta < \lambda$ are such that $\operatorname{Ext}^1_R(M_\beta/M_\alpha, N) \neq 0$, then also $\operatorname{Ext}^1_R(M_{\alpha+1}/M_\alpha, N) \neq 0$.

Since $M \in {}^{\perp_1}N$ and ${}^{\perp_1}N$ is closed under pure submodules, $M_\alpha \in {}^{\perp_1}N$ for every $\alpha < \lambda$, and Lemma 5 yields that the set

$$E = \{\alpha < \lambda \mid \operatorname{Ext}^1_R(M_{\alpha+1}/M_\alpha, N) \neq 0\}$$

is not stationary in λ. So there is a closed and unbounded subset U of λ such that $U \cap E = \emptyset$. Taking the λ-subfiltration of $\mathcal{R}$ indexed by the elements of U, we obtain a λ-filtration, $(F_\alpha \mid \alpha \leq \lambda)$, of M such that $F_{\alpha+1}/F_\alpha \in {}^{\perp_1}N$ for all $\alpha < \lambda$. By inductive assumption, we can refine this λ-filtration into a filtration of length σ (for some ordinal σ) which is a κ-refinement of M.

If λ is singular $> \kappa$, we use the version of Shelah's singular compactness theorem from [15, Theorem IV.3.7]. We call a module M "free" if M has a κ-refinement as in the claim of the Lemma. In order to prove that M is "free", it suffices to show that M is ρ-"free" for any regular cardinal $\kappa < \rho < \lambda$, and apply [15, Lemma XII.1.14] (with $\mu = \kappa$). For the system witnessing the ρ-"freeness" of M, we take the set, $\mathcal{W}$, of all pure submodules of M of cardinality $< \rho$. Since ${}^{\perp_1}N$ is closed under pure submodules, each element of $\mathcal{W}$ is "free" by inductive assumption. Moreover, any subset X of M of cardinality $< \rho$ is contained in an element of $\mathcal{W}$ by Lemma 4 (i). Finally, pure submodules of M are closed under unions of arbitrary well-ordered chains by Lemma 4 (iv). So $\mathcal{W}$ witnesses ρ-"freeness" of M in the sense of [15, Definition IV.1.1]. $\square$

Now, we are in a position to prove our first main result, generalizing [17, Theorem 14] to arbitrary rings.

Theorem 7. (V = L) *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathfrak{C}$ is generated by a set, and $\mathcal{A}$ is closed under pure submodules. Then $\mathfrak{C}$ is complete.*

Proof. By assumption, there is a module N such that $\mathcal{A} = {}^{\perp_1}N$. By Lemma 6 and Lemma 1, $\mathcal{B} = \mathcal{S}^{\perp_1}$ where $\mathcal{S}$ is a representative set of those modules from $\mathcal{A}$ that have cardinality $\leq \operatorname{card} R + \operatorname{card} E(N) + \aleph_0$. So $\mathfrak{C}$ is cogenerated by a set, and hence $\mathfrak{C}$ is complete by Theorem 2 (i). $\square$

1.3 Auxiliary general results

For the other generalization of [17, Theorem 14], we will need the following refinement of a result of Fuchs and Lee [19, Theorem 2.1] (which in turn is based on a construction of Hill [21]) and another two lemmas. These results will be proved in ZFC.

Lemma 8. (Generalized Hill Lemma) *Let R be a ring, κ an infinite regular cardinal and $\mathcal{C}$ a set of $< \kappa$-presented modules Let M be a union of a $\mathcal{C}$-filtration*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots \subseteq M_\sigma = M$$

for some ordinal σ. Then there is a family $\mathcal{F}$ of submodules of M such that:

(H1) *$M_\alpha \in \mathcal{F}$ for all $\alpha \leq \sigma$.*

(H2) *$\mathcal{F}$ is closed under arbitrary sums.*

(H3) *Let $N, P \in \mathcal{F}$ be such that $N \subseteq P$. Then there exists a $\mathcal{C}$-filtration $\bar{\mathcal{P}} = (\bar{P}_\gamma \mid \gamma \leq \tau)$ of the module $\bar{P} = P/N$ such that $\tau \leq \sigma$, and for each $\gamma < \tau$ there is a $\beta < \sigma$ with $\bar{P}_{\gamma+1}/\bar{P}_\gamma$ isomorphic to $M_{\beta+1}/M_\beta$.*

(H4) *Let $\lambda \geq \kappa$ be a regular cardinal. Let $N \in \mathcal{F}$ and X be a subset of M of cardinality $< \lambda$. Then there is a $P \in \mathcal{F}$ such that $N \cup X \subseteq P$ and P/N is $< \lambda$-presented.*

Proof. Let $\mathcal{M}$ denote the filtration $(M_\alpha \mid \alpha \leq \sigma)$ together with an arbitrary family of $< \kappa$-generated modules $(A_\alpha \mid \alpha < \sigma)$ such that $M_{\alpha+1} = M_\alpha + A_\alpha$, for each $\alpha < \sigma$.

We call a subset $S \subseteq \sigma$ 'closed' provided that each $\beta \in S$ satisfies $M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S, \alpha < \beta} A_\alpha$. Let $\mathcal{F}$ be the family of all modules of the form $M(S) = \sum_{\alpha \in S} A_\alpha$ where $S \subseteq \sigma$ is 'closed'. We are going to check the conditions (H1)–(H4) for this definition of $\mathcal{F}$.

Property (H1) is clear since each ordinal $\alpha \leq \sigma$ is a 'closed' subset of σ. For the property (H2), it suffices to check that a union of 'closed' sets is again 'closed' in σ. And really, if $\beta \in S = \bigcup_{i \in I} S_i$, then $\beta \in S_i$ for some $i \in I$, and

$$M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S_i, \alpha < \beta} A_\alpha \subseteq \sum_{\alpha \in S, \alpha < \beta} A_\alpha.$$

In order to prove (H3), let $N = M(S)$ and $P = M(T)$ for some 'closed' subsets $S, T \subseteq \sigma$. Since $S \cup T$ is also 'closed', we can w.l.o.g. assume that $S \subseteq T$. For each $\beta \leq \sigma$, put

$$F_\beta = N + \sum_{\alpha \in T \setminus S, \alpha < \beta} A_\alpha = M(S \cup (T \cap \beta)) \quad \text{and} \quad \bar{F}_\beta = F_\beta / N.$$

Clearly, $\bar{\mathcal{F}} = (\bar{F}_\beta \mid \beta < \sigma)$ is a filtration of $\bar{P}$ such that $\bar{F}_{\beta+1} = \bar{F}_\beta$ for $\beta \notin T \setminus S$, and $\bar{F}_{\beta+1} = \bar{F}_\beta + (A_\beta + N)/N$ otherwise.

Let $\beta \in T \setminus S$. Then $\bar{F}_{\beta+1}/\bar{F}_\beta \cong A_\beta/(A_\beta \cap F_\beta)$. However, $F_\beta = C_\beta + \sum_{\alpha \in S, \beta < \alpha} A_\alpha$, where $C_\beta = \sum_{\alpha \in T, \alpha < \beta} A_\alpha$, so $A_\beta \cap F_\beta \supseteq A_\beta \cap C_\beta = A_\beta \cap M_\beta$ (because $\beta \in T$, so $A_\beta \cap M_\beta \subseteq C_\beta$).

Conversely, if $a \in A_\beta \cap F_\beta$, then $a = c + a_{\alpha_0} + \cdots + a_{\alpha_k}$ where $c \in C_\beta$ $(\subseteq M_\beta)$, $\alpha_i \in S$ and $a_{\alpha_i} \in A_{\alpha_i}$ for all $i \leq k$, and $\alpha_i > \alpha_{i+1}$ for all $i < k$. W.l.o.g., we can assume that α_0 is minimal possible. If $\alpha_0 > \beta$, then $a_{\alpha_0} = a - c - a_{\alpha_1} - \cdots - a_{\alpha_k} \in M_{\alpha_0} \cap A_{\alpha_0} \subseteq \sum_{\alpha \in S, \alpha < \alpha_0} A_\alpha$ (since $\alpha_0 \in S$) in contradiction with the minimality of α_0. So $\alpha_0 < \beta$, and hence $a \in M_\beta$. This proves that $A_\beta \cap F_\beta = A_\beta \cap M_\beta$.

So $\beta \in T \setminus S$ implies $\bar{F}_{\beta+1}/\bar{F}_\beta \cong A_\beta/(M_\beta \cap A_\beta) \cong M_{\beta+1}/M_\beta$ which is a consecutive factor in $\mathcal{M}$. Finally, $\bar{\mathcal{P}}$ is obtained from $\bar{\mathcal{F}}$ by removing repetitions.

For property (H4), we first prove that every subset of σ of cardinality $< \lambda$ is contained in a 'closed' subset of cardinality $< \lambda$. Because λ is an infinite regular cardinal, by (H2), it is enough to prove this only for one-element subsets of σ. That is, to prove that every $\beta < \sigma$ is contained in a 'closed' subset of cardinality $< \lambda$. We induct on β. For $\beta < \lambda$, just take $S = \beta + 1$. Otherwise, the short exact sequence

$$0 \longrightarrow M_\beta \cap A_\beta \longrightarrow A_\beta \longrightarrow M_{\beta+1}/M_\beta \longrightarrow 0$$

shows that $M_\beta \cap A_\beta$ is $< \lambda$-generated. Thus, $M_\beta \cap A_\beta \subseteq \sum_{\alpha \in S_0} A_\alpha$ for a subset $S_0 \subseteq \beta$ of cardinality $< \lambda$. Moreover, we can assume that S_0 is 'closed' in σ by inductive premise, and put $S = S_0 \cup \{\beta\}$. To show that S is 'closed', it suffices to check the definition for β. But $M_\beta \cap A_\beta \subseteq M(S_0) = \sum_{\alpha \in S, \alpha < \beta} A_\alpha$.

Finally, let $N = M(S)$ where S 'closed' in σ, and let X be a subset of M of cardinality $< \lambda$. Then $X \subseteq \sum_{\alpha \in T} A_\alpha$ for a subset T of σ of cardinality $< \lambda$. By the preceding paragraph, we can assume that T is 'closed' in σ. Let $P = M(S \cup T)$. Then P/N is $\mathcal{C}$-filtered by property (H3), and the filtration can be chosen indexed by 1 + the ordinal type of $T \setminus S$, which is less than λ. In particular, P/N is $< \lambda$-presented. $\square$

The following important lemma is a partial converse of Lemma 1. It generalizes [15, Theorem XII.3.3] (which has the additional assumption of $\operatorname{proj.dim}(M_{\alpha+1}/M_\alpha) \leq 1$ for all $\alpha < \kappa$). Lemma 9 essentially says that if $\mathcal{C}$ is closed under arbitrary direct sums or κ-bounded products, then each module $M \in {}^{\perp_1}\mathcal{C}$ with a κ-filtration in ${}^{\perp_1}\mathcal{C}$ is actually ${}^{\perp_1}\mathcal{C}$-filtered by a subfiltration.

Lemma 9. *Let R be a ring, κ a regular uncountable cardinal, and $\mathcal{C}$ a class of modules closed under*

(i) *arbitrary direct sums, or*

(ii) *arbitrary κ-bounded products.*

Let M be a module possessing a κ-filtration $(M_\alpha \mid \alpha \leq \kappa)$ such that $M_\alpha \in {}^{\perp_1}\mathcal{C}$ for all $\alpha \leq \kappa$. Then there is a continuous function $f : \kappa \to \kappa$ such that $M_{f(\beta)}/M_{f(\alpha)} \in {}^{\perp_1}\mathcal{C}$ for all $\alpha < \beta < \kappa$.

Proof. Assume the claim is false. Then the set

$$E = \{\alpha < \kappa \mid \exists \beta : \alpha < \beta < \kappa \;\&\; M_\beta/M_\alpha \notin {}^{\perp_1}\mathcal{C}\}$$

has a non-empty intersection with each closed and unbounded subset of κ. Possibly passing to a subfiltration, we can w.l.o.g. assume that $E = \{\alpha < \kappa \mid \operatorname{Ext}^1_R(M_{\alpha+1}/M_\alpha, \mathcal{C}) \neq 0\}$. Then for each $\alpha \in E$ there are a $C_\alpha \in \mathcal{C}$ and a homomorphism $\delta_\alpha : M_{\alpha+1}/M_\alpha \to E(C_\alpha)/C_\alpha$ that cannot be factorized through the projection $\tau_\alpha : E(C_\alpha) \to E(C_\alpha)/C_\alpha$. For $\alpha < \kappa$, $\alpha \notin E$, we put $C_\alpha = 0$ and $\delta_\alpha = 0$.

Let $I = \prod_{\alpha<\kappa} E(C_\alpha)$, $D = \bigoplus_{\alpha<\kappa} C_\alpha$ in case (i), or $D = \prod^{<\kappa}_{\alpha<\kappa} C_\alpha$ when (ii) holds. Put $F = I/D$. For each subset $A \subseteq \kappa$, define $I_A = \{x \in I \mid x_\beta = 0 \text{ for all } \beta < \kappa, \beta \notin A\}$. In particular, $I_\kappa = I$, and $I_\alpha \cong \prod_{\beta<\alpha} E(C_\beta)$ is injective for each $\alpha \leq \kappa$.

For each $\alpha < \kappa$, we let $F_\alpha = (I_\alpha + D)/D \,(\subseteq F)$ and π_α be the epimorphism $I_\alpha \to F_\alpha$ defined by $\pi_\alpha(x) = x + D$. Then $\operatorname{Ker}(\pi_\alpha) \cong \bigoplus_{\beta<\alpha} C_\beta$ if (i) holds, or $\operatorname{Ker}(\pi_\alpha) \cong \prod_{\beta<\alpha} C_\beta$ in the second case. Whichever case occurs, we have $\operatorname{Ker}(\pi_\alpha) \in \mathcal{C}$.

Let $U = \bigcup_{\alpha<\kappa} I_\alpha$. Then $D \subseteq U \subseteq I$, and we let $G = U/D \,(\subseteq F)$ and $\pi : U \to G$ be the projection modulo D.

For each $\alpha < \kappa$, define $E_\alpha = (I_{\{\alpha\}} + D)/D$. Then there is an isomorphism $\iota_\alpha : E(C_\alpha)/C_\alpha \cong E_\alpha$, and $F_{\alpha+1} = E_\alpha \oplus F_\alpha \,(\subseteq G)$. Moreover, taking $B_\alpha = (I_{(\alpha,\kappa)} + D)/D$, we have $F = F_{\alpha+1} \oplus B_\alpha$, so $G = E_\alpha \oplus F_\alpha \oplus (B_\alpha \cap G)$. Denote by ξ_α the projection onto the first component, E_α, in the latter decomposition of G. Then ξ_α maps $x + D \in G$ to $y + D \in E_\alpha$ where $y_\alpha = x_\alpha$ and $y_\beta = 0$ for all $\alpha \neq \beta < \kappa$.

In order to prove that $\mathrm{Ext}^1_R(M, \mathcal{C}) \neq 0$, it suffices to construct a homomorphism $\varphi : M \to G$ that cannot be factorized through π (because then the map $\mathrm{Hom}_R(M, \pi)$ is not surjective, so $\mathrm{Ext}^1_R(M, D) \neq 0$).

φ will be constructed by induction on $\alpha < \kappa$ as a union of a continuous chain of homomorphisms $(\varphi_\alpha \mid \alpha < \kappa)$ where $\varphi_\alpha : M_\alpha \to F_\alpha$ for all $\alpha < \kappa$, and $\varphi \restriction M_0 = 0$.

For $\alpha < \kappa$, we use the assumption of $\mathrm{Ext}^1_R(M_\alpha, \mathrm{Ker}(\pi_\alpha)) = 0$ to find a homomorphism $\eta_\alpha : M_\alpha \to I_\alpha$ such that $\varphi_\alpha = \pi_\alpha \eta_\alpha$. The injectivity of the module I_α yields a homomorphism $\psi_\alpha : M_{\alpha+1} \to I_\alpha$ such that $\psi_\alpha \restriction M_\alpha = \eta_\alpha$.

Denote by ρ_α the projection $M_{\alpha+1} \to M_{\alpha+1}/M_\alpha$. Define $\varphi_{\alpha+1} = \iota_\alpha \delta_\alpha \rho_\alpha + \pi_\alpha \psi_\alpha$. Then $\varphi_{\alpha+1} \restriction M_\alpha = \pi_\alpha \psi_\alpha \restriction M_\alpha = \pi_\alpha \eta_\alpha = \varphi_\alpha$.

Finally, assume there is $\phi : M \to U$ such that $\varphi = \pi\phi$. Since $U = \bigcup_{\alpha<\kappa} I_\alpha$, the set $C = \{\alpha < \kappa \mid \phi(M_\alpha) \subseteq I_\alpha\}$ is closed and unbounded in κ. So there exists $\alpha \in C \cap E$. Denote by σ the projection $I \to E(C_\alpha)$. Then ϕ induces a homomorphism $\bar{\phi} : M_{\alpha+1}/M_\alpha \to E(C_\alpha)$ defined by $\bar{\phi}\rho_\alpha(m) = \sigma(\phi(m))$ for all $m \in M_{\alpha+1}$.

By the definition of ξ_α, we have $\iota_\alpha \tau_\alpha \sigma(x) = \xi_\alpha \pi(x)$ for each $x \in U$, $\xi_\alpha \restriction F_\alpha = 0$, and $\xi_\alpha \restriction E_\alpha = \mathrm{id}$. So, for each $m \in M_{\alpha+1}$, we get

$$\tau_\alpha \bar{\phi} \rho_\alpha(m) = \iota_\alpha^{-1} \xi_\alpha \pi \phi(m) = \iota_\alpha^{-1} \xi_\alpha \varphi_{\alpha+1}(m) = \iota_\alpha^{-1} \xi_\alpha \iota_\alpha \delta_\alpha \rho_\alpha(m) = \delta_\alpha \rho_\alpha(m).$$

Since ρ_α is surjective, this proves that $\tau_\alpha \bar{\phi} = \delta_\alpha$, in contradiction with the definition of δ_α. □

Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair cogenerated by a set. Then there is an uncountable regular cardinal κ such that $\mathfrak{C}$ is cogenerated by a representative set, $\mathcal{A}_\kappa$, of $< \kappa$-presented modules from $\mathcal{A}$. W.l.o.g., $R \in \mathcal{A}_\kappa$. By Theorem 2 (ii), $\mathcal{A}$ consists of all modules isomorphic to direct summands of $\mathcal{A}_\kappa$-filtered modules. The following lemma from [29] says that we can omit the term 'direct summands'.

Lemma 10. *Let κ be an uncountable regular cardinal, and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair cogenerated by a set, $\mathcal{C}$, of $< \kappa$-presented modules. Then every module in $\mathcal{A}$ is $\mathcal{A}_\kappa$-filtered where $\mathcal{A}_\kappa$ denotes a representative set of $< \kappa$-presented modules from $\mathcal{A}$.*

Proof. Let $K \in \mathcal{A}$, so there is a $\mathcal{C}$-filtered module M such that $M = K \oplus L$ for some $L \subseteq M$. Denote by $\pi_K : M \to K$ and $\pi_L : M \to L$ the corresponding projections. Let $\mathcal{F}$ be the family of submodules of M as in Lemma 8. We proceed in two steps.

Step I: By induction, we construct a filtration $(N_\alpha \mid \alpha \leq \tau)$ of M such that

(1) $N_\alpha \in \mathcal{F}$,

(2) $N_\alpha = \pi_K(N_\alpha) + \pi_L(N_\alpha)$, and

(3) $N_{\alpha+1}/N_\alpha$ is $< \kappa$-presented,

for all $\alpha < \tau$.

First, $N_0 = 0$ and $N_\beta = \bigcup_{\alpha<\beta} N_\alpha$ for all limit ordinals $\beta \leq \tau$. Suppose we have $N_\alpha \subsetneq M$ and we wish to construct $N_{\alpha+1}$. Take $x \in M \setminus N_\alpha$; by property (H4), there is $Q_0 \in \mathcal{F}$ such that $N_\alpha \cup \{x\} \subseteq Q_0$ and Q_0/N_α is $< \kappa$-presented. Let X_0 be a subset of Q_0 of cardinality $< \kappa$ such that the set $\{x + N_\alpha \mid x \in X_0\}$ generates Q_0/N_α. Put

$Z_0 = \pi_K(Q_0) \oplus \pi_L(Q_0)$. Clearly $Q_0/N_\alpha \subseteq Z_0/N_\alpha$. Since $\pi_K(N_\alpha), \pi_L(N_\alpha) \subseteq N_\alpha$, the module Z_0/N_α is generated by the set

$$\{x + N_\alpha \mid x \in \pi_K(X_0) \cup \pi_L(X_0)\}.$$

Thus, we can find $Q_1 \in \mathcal{F}$ such that $Z_0 \subseteq Q_1$ and Q_1/N_α is $< \kappa$-presented.

Similarly, we infer that Z_1/N_α is $< \kappa$-generated for $Z_1 = \pi_K(Q_1) \oplus \pi_L(Q_1)$, and find $Q_2 \in \mathcal{F}$ with $Z_1 \subseteq Q_2$ and Q_2/N_α a $< \kappa$-presented module. In this way, we obtain a chain $Q_0 \subseteq Q_1 \subseteq \cdots$ such that for all $i < \omega$: $Q_i \in \mathcal{F}$, Q_i/N_α is $< \kappa$-presented, and $\pi_K(Q_i) + \pi_L(Q_i) \subseteq Q_{i+1}$. It is easy to see that $N_{\alpha+1} = \bigcup_{i<\omega} Q_i$ satisfies the properties (1)–(3).

Step II: By condition (2), we have $\pi_K(N_{\alpha+1}) + N_\alpha = \pi_K(N_{\alpha+1}) \oplus \pi_L(N_\alpha)$ and similarly for L. Hence

$$(\pi_K(N_{\alpha+1}) + N_\alpha) \cap (\pi_L(N_{\alpha+1}) + N_\alpha) =$$
$$= (\pi_K(N_{\alpha+1}) \oplus \pi_L(N_\alpha)) \cap (\pi_L(N_{\alpha+1}) \oplus \pi_K(N_\alpha)) = \pi_K(N_\alpha) \oplus \pi_L(N_\alpha) = N_\alpha$$

and

$$N_{\alpha+1}/N_\alpha = (\pi_K(N_{\alpha+1}) + N_\alpha)/N_\alpha \oplus (\pi_L(N_{\alpha+1}) + N_\alpha)/N_\alpha.$$

By condition (1), $N_{\alpha+1}/N_\alpha$ is $\mathcal{C}$-filtered. Since

$$(\pi_K(N_{\alpha+1}) + N_\alpha)/N_\alpha \cong \pi_K(N_{\alpha+1})/\pi_K(N_\alpha),$$

$\pi_K(N_{\alpha+1})/\pi_K(N_\alpha)$ is isomorphic to a direct summand of a $\mathcal{C}$-filtered module, and so $\pi_K(N_{\alpha+1})/\pi_K(N_\alpha) \in \mathcal{A}$. By condition (3), $\pi_K(N_{\alpha+1})/\pi_K(N_\alpha)$ is $< \kappa$-presented. We conclude that $(\pi_K(N_\alpha) \mid \alpha \leq \tau)$ is the desired $\mathcal{A}_\kappa$-filtration of $K = \pi_K(N_\tau)$. □

1.4 Completeness under V = L (part II)

Recall that given an infinite cardinal κ, a ring R is right *κ-noetherian* provided that each right ideal of R is $\leq \kappa$-generated. If R is right κ-noetherian and $\lambda > \kappa$, then the class of all injective modules, $\mathcal{I}_0$, is closed under λ-bounded products by Baer's criterion for injectivity. Hence all the classes $\mathcal{I}_n$ $(n < \omega)$ are closed under λ-bounded products.

Theorem 11. (V = L) *Let R be a ring, $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathfrak{C}$ is hereditary and generated by a set, and $\mathcal{B}$ consists of modules of finite injective dimension. Then $\mathfrak{C}$ is complete.*

Proof. By the assumptions, there are an $n < \omega$ and a module $N \in \mathcal{I}_n$ such that $\mathcal{A} = {}^{\perp_1}N$. Let $\mu = \operatorname{card} R + \operatorname{card} E(N) + \aleph_0$ (for the use of Lemma 5 later; see also Theorem 12 (ii)). For $0 \leq i \leq n$, let $\mathfrak{C}_i = (\mathcal{A}_i, \mathcal{B}_i)$ be (hereditary) cotorsion pairs with $\mathcal{A}_i = {}^{\perp_1}(\mathcal{B} \cup \mathcal{I}_i) = {}^{\perp_1}\mathcal{B} \cap {}^{\perp_1}\mathcal{I}_i$. Let $\mathcal{Q}_i$ be a representative set of all $\leq \mu$-generated modules in $\mathcal{A}_i$. By a downward induction on i, we will prove that every module $M \in \mathcal{A}_i$ has a $\mathcal{Q}_i$-filtration $(M_\alpha \mid \alpha \leq \sigma)$.

Let $i = n$. The ring R is clearly μ-noetherian. Since $\mathcal{B} \subseteq \mathcal{I}_n$, we have $\mathcal{A}_n = {}^{\perp_1}\mathcal{I}_n$, and it is well-known that $\mathfrak{C}_n$ is cogenerated by all n-th syzygies of cyclic modules, that is by a set of $< \mu^+$-presented modules. The claim for $i = n$ then follows from Lemma 10.

Let $0 \leq i < n$. We will proceed by induction on $\lambda = \operatorname{gen}(M)$. There is nothing to prove for $\lambda \leq \mu$. Let $\lambda > \mu$ and $M \in \mathcal{A}_i$. Consider the exact sequence

$$0 \longrightarrow K \xrightarrow{\subseteq} R^{(\lambda)} \xrightarrow{\pi} M \longrightarrow 0.$$

We may suppose that $\operatorname{gen}(K) = \lambda$. Obviously $K \in \mathcal{A}_i$ since $\mathfrak{C}_i$ is hereditary. Moreover, K is a syzygy of a module from ${}^{\perp_1}\mathcal{I}_i$, hence $K \in {}^{\perp_1}\mathcal{I}_{i+1}$. So we actually have $K \in \mathcal{A}_{i+1}$. By inductive premis, there is a $\mathcal{Q}_{i+1}$-filtration $\mathcal{K}$ of K. Using Lemma 8 (for $\kappa = \mu^+$), we obtain the family $\mathcal{F}$ for $\mathcal{K}$. Let us define

$$\mathcal{G} = \bigl\{L \subseteq M \mid \bigl(\exists A_L \subseteq \lambda\bigr)\bigl(\pi\bigl(R^{(A_L)}\bigr) = L \ \&\ K \cap R^{(A_L)} \in \mathcal{F}\bigr)\bigr\}.$$

We claim that $\mathcal{G} \subseteq \mathcal{A}_i$. Indeed, let L be a module from $\mathcal{G}$. Then for $B \in \mathcal{B}_i$, we have

$$0 = \operatorname{Ext}^1_R(M, B) \longrightarrow \operatorname{Ext}^1_R(L, B) \longrightarrow \operatorname{Ext}^2_R(M/L, B) \cong \operatorname{Ext}^1_R(K/K \cap R^{(A_L)}, B) = 0.$$

The last equality follows from $\mathcal{B}_i \subseteq \mathcal{B}_{i+1}$ and the fact that $K/K \cap R^{(A_L)} \in \mathcal{A}_{i+1}$ (use (H3) from Lemma 8 and $K \cap R^{(A_L)} \in \mathcal{F}$).

It is obvious that $0 \in \mathcal{G}$, and that $\mathcal{G}$ is closed under well-ordered unions of chains. As the next step, we show that for every regular $\kappa \le \lambda$ with $\kappa > \mu$, and a subset $X \subseteq M$ of cardinality $< \kappa$, there is a $< \kappa$-generated module $L \in \mathcal{G}$ containing X.

Choose a subset $A_0 \subseteq \lambda$ of cardinality $< \kappa$ such that $X \subseteq \pi(R^{(A_0)})$. By (H4) from Lemma 8, there is a $< \kappa$-generated module $K_0 \in \mathcal{F}$ such that $K \cap R^{(A_0)} \subseteq K_0$. Take $A_1 \supseteq A_0$ with $K_0 \subseteq R^{(A_1)}$ and $\operatorname{card} A_1 < \kappa$. Iterating the process, we obtain the chain $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ of $< \kappa$-generated modules from $\mathcal{F}$ and the chain $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ of subsets of λ of cardinality $< \kappa$. Let us define $L = \pi(R^{(\bigcup_{k<\omega} A_k)})$. Then L is a module from $\mathcal{G}$ we have been looking for.

Let λ be regular. By the previous step, we can select from $\mathcal{G}$ a λ-filtration $\mathcal{M}$ of M. Applying Lemma 9 (ii) to $\mathcal{C} = \mathcal{I}_i$ and then Lemma 5 for $M \in {}^{\perp_1}N$, we obtain a subfiltration $\mathcal{M}'$ of $\mathcal{M}$ with consecutive factors from ${}^{\perp_1}\mathcal{I}_i$, and then a subfiltration $\widehat{\mathcal{M}}$ of $\mathcal{M}'$ whose consecutive factors are in $\mathcal{A}$. However, by Lemma 1, these factors even belong to $\mathcal{A}_i$, and they are clearly $< \lambda$-generated. Hence, by inductive premise, they possess $\mathcal{Q}_i$-filtrations. It follows that M has the same property.

If λ is singular, the properties of $\mathcal{G}$ proved above make it possible to apply the singular compactness theorem and conclude that M has a $\mathcal{Q}_i$-filtration.

Finally, using Lemma 1, we obtain that $\mathcal{B} = \mathcal{Q}_0^{\perp_1}$, and Theorem 2 (i) finishes our proof of completeness of $\mathfrak{C}$. $\square$

Remark. (i) Neither Theorem 7 nor Theorem 11 can be proved in ZFC or ZFC + GCH. Eklof and Shelah constructed in [16] a model of ZFC + GCH such that the class of all Whitehead groups (that is, ${}^{\perp_1}\mathbb{Z}$) is not a (special) precovering class of abelian groups. In particular, Lemma 6, and Theorems 7 and 11 fail in that model, so they are independent of ZFC + GCH.

(ii) The proof of Lemma 6 relies on three basic properties of pure extensions: the existence of purifications, the fact that if C is pure in A and B/C is pure in A/C then B is pure in A, and on the union of a chain of pure submodules being a pure submodule, see Lemma 4. We could alternatively assume that ${}^{\perp_1}N$ is closed under elementary substructures and use the corresponding three basic properties of elementary embeddings: the downward Löwenheim-Skolem Theorem, [24, Proposition 2.25], and the fact that a union of a chain of elementary substructures is again an elementary substructure, respectively, to conclude that ${}^{\perp_1}N$ is a special precovering class.

1.5 Completeness in ZFC

Assuming $\mathcal{B}$ closed under arbitrary direct sums, we can prove analogs of Theorems 7 and 11 in ZFC.

Theorem 12. *Let R be a ring, μ an infinite cardinal, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a cotorsion pair such that $\mathcal{B}$ is closed under arbitrary direct sums. Let $\mathcal{Q}$ be a representative set of all $\leq \mu$-generated modules in $\mathcal{A}$. Assume that either*

(i) *$\mathcal{A}$ is closed under pure submodules and $\mu \geq \operatorname{card} R$, or*

(ii) *R is right μ-noetherian, $\mathfrak{C}$ is hereditary, and $\mathcal{B}$ consists of modules of finite injective dimension.*

Then $\mathcal{B} = \mathcal{Q}^{\perp_1}$. In particular, $\mathfrak{C}$ is complete.

Proof. This is proved as in Theorems 7 and 11, with Lemma 9 (i) replacing Lemma 5 in the regular case. (For (ii), we do not define μ in the beginning of the proof of Theorem 11; we just use that R is right μ-noetherian. Also, the definition of N in the proof of Theorem 11 is omitted, and ${}^{\perp_1}N$ is replaced by ${}^{\perp_1}\mathcal{B}$.) $\square$

Recall that a class of modules $\mathcal{C} \subseteq \text{Mod-}R$ is called *definable* if it is closed under direct products, pure submodules and direct limits.

Corollary 13. *Let R be a right $\aleph_0$-noetherian ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair such that $\mathcal{B}$ is closed under arbitrary direct sums and consists of modules of finite injective dimension. Then $\mathfrak{C}$ is of countable type and $\mathcal{B}$ is definable.*

Proof. $\mathfrak{C}$ is of countable type by Theorem 12 (ii) and $\mathcal{B}$ is always closed under direct products. The closure under pure submodules follows from Theorem 45, and $\mathfrak{C}$ hereditary implies $\mathcal{B}$ closed under pure epimorphic images. In particular, $\mathcal{B}$ is closed under direct limits. $\square$

Remark. There are many cotorsion pairs $(\mathcal{A}, \mathcal{B})$ without $\mathcal{B}$ closed under direct sums. In general, it is difficult to verify this closure property, and the property may depend on the set theory we are working in. For example, assuming V = L, every Whitehead group is free, hence $({}^{\perp_1}\mathbb{Z})^{\perp_1} = \text{Mod-}\mathbb{Z}$ is closed under direct sums. By Theorem 12, this is not the case in the model constructed in [16], see previous remark.

2 Tilting modules

The notion of tilting goes back to the beginning of 1980's. Trying to describe the category of finitely generated modules over a finite dimensional algebra Λ, Brenner, Buttler, Happel and Ringel discovered that (what is now called) a finite dimensional 1-tilting Λ-module T induces the pair of categorial equivalences

$$\operatorname{Ker}\operatorname{Ext}^1_\Lambda(T,-) \underset{-\otimes_\Gamma T}{\overset{\operatorname{Hom}_\Lambda(T,-)}{\rightleftarrows}} \operatorname{Ker}\operatorname{Tor}^\Gamma_1(-,T),$$

$$\operatorname{Ker}\operatorname{Hom}_\Lambda(T,-) \underset{\operatorname{Tor}^\Gamma_1(-,T)}{\overset{\operatorname{Ext}^1_\Lambda(T,-)}{\rightleftarrows}} \operatorname{Ker}(-\otimes_\Gamma T),$$

where $\Gamma = \operatorname{End}_\Lambda(T)$. The result above can be viewed as a generalization of the classical Morita duality. In this "equivalence setting", it is important that T is finitely generated, however it seems that infinitely generated tilting modules deserve no less attention.

Namely, infinitely generated tilting modules occur naturally in various areas of contemporary module theory. For example, finiteness of the little finitistic dimension of a right noetherian ring R is equivalent to the existence of a particular tilting R-module T_f, [8]. Explicit computation of T_f then yields a proof of the equality of the little and the big finitistic dimensions for all (non-commutative) Iwanaga-Gorenstein rings, [5]. Similarly, if R is a commutative ring, S is some multiplicative set of regular elements in R, and Q denotes the localization of R in S, then the existence of a decomposition of Q/R into a direct sum of countably presented R-submodules is equivalent to $T_S = Q \oplus Q/R$ being a tilting module of projective dimension ≤ 1, [4].

Though the examples of tilting modules above are typically infinitely generated, there is an implicit finiteness condition connected to tilting. Namely, all examples of tilting modules T are of *finite type*, that is, there is a set, $\mathcal{S}$, of modules possessing a projective resolution consisting of finitely generated projective modules such that $T^\perp = \mathcal{S}^\perp$. Then the tilting class $T^\perp$ is definable, and one can characterize modules in $T^\perp$ by formulas of the first-order language of module theory.

However, to prove that all tilting modules are of finite type requires some preparation. We shall do this in two steps: first, we show that all tilting modules are of *countable type*, that is, there is a set, $\mathcal{C}$, of modules possessing a projective resolution consisting of countably generated projective modules such that $T^\perp = \mathcal{C}^\perp$; in the second step, we combine results from [11] and [12] to pass from the countable type to the finite type.

2.1 Preliminaries

For a ring R, $\dim(R)$ denotes the minimal infinite cardinal κ such that $\operatorname{gen}(I) \leq \kappa$ for all right ideals I of R. For example, $\dim(R) = \aleph_0$ if and only if R is right $\aleph_0$-noetherian.

We denote by $\mathcal{E}$ the class of all modules of the form $\prod^{<\nu}_{i\in I} E_i$ where ν is a regular infinite cardinal and $(E_i \mid i \in I)$ a family of injective modules. The subclass of $\mathcal{E}$ consisting of arbitrary direct sums of injective modules (using only $\nu = \aleph_0$) is denoted by $\mathcal{E}_0$.

The class of n-th cosyzygies of all R-modules is denoted by Cos^n-R; that is, $M \in \operatorname{Cos}^n$-$R$ if and only if there is an exact sequence $E_{n-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow M \longrightarrow 0$ with injective modules E_j, $0 \leq j < n$. Note that $\mathcal{P}_n = {}^{\perp_1}(\operatorname{Cos}^n\text{-}R) = {}^{\perp}(\operatorname{Cos}^n\text{-}R)$ for each $n < \omega$.

For a class of modules $\mathcal{C}$ and an infinite cardinal κ, denote by $\mathcal{C}^{<\kappa}$ and $\mathcal{C}^{\leq\kappa}$ the subclass of all modules in $\mathcal{C}$ possessing a projective resolution consisting of $<\kappa$-generated and $\leq\kappa$-generated, respectively, projective modules. For example, if R is right $\aleph_0$-noetherian or $\mathcal{C} \subseteq \mathcal{P}_1$, then $\mathcal{C}^{\leq\aleph_0}$ consists of all countably presented modules in $\mathcal{C}$. We put mod-$R = (\text{Mod-}R)^{<\aleph_0}$.

A module T is a *tilting module* provided that T has finite projective dimension, $\operatorname{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all $1 \leq i < \omega$ and all cardinals κ, and there are $m < \omega$ and an exact sequence $0 \longrightarrow R \longrightarrow T_0 \longrightarrow \cdots \longrightarrow T_m \longrightarrow 0$ such that T_i is a direct summand in a (possibly infinite) direct sum of copies of T for each $i \leq m$. The class $\mathcal{T}_T = T^\perp$ is called the *tilting class* induced by T, and the (complete, hereditary) cotorsion pair $\mathfrak{C}_T = ({}^\perp(T^\perp), T^\perp)$ the *tilting cotorsion pair* induced by T. A tilting module T and the tilting class $\mathcal{T}_T$ induced by T are of *finite* (*countable*, resp.) *type* if $\mathfrak{C}_T$ is of finite (countable, resp.) type.

If T has projective dimension $\leq n$, then T, $\mathcal{T}_T$, and $\mathfrak{C}_T$, are called the *n-tilting module*, *n-tilting class*, and *n-tilting cotorsion pair*, respectively. In this case, ${}^\perp(T^\perp) \subseteq \mathcal{P}_n$. Note that 0-tilting modules are just the projective generators.

Dually, a module C is a *cotilting module* provided that T has finite injective dimension, $\operatorname{Ext}^i_R(C^\kappa, C) = 0$ for all $1 \leq i < \omega$ and all cardinals κ, and there are $m < \omega$ and an exact sequence $0 \longrightarrow C_m \longrightarrow \cdots \longrightarrow C_0 \longrightarrow W \longrightarrow 0$ such that W is an injective cogenerator, and C_i is a direct summand in a direct product of copies of C for each $i \leq m$. The class $\mathcal{C}_C = {}^\perp C$ is called the *cotilting class* induced by C, and the (hereditary) cotorsion pair $\mathfrak{C}_C = ({}^\perp C, ({}^\perp C)^\perp)$ the *cotilting cotorsion pair* induced by C.

If C has injective dimension $\leq n$, then C, $\mathcal{C}_C$, and $\mathfrak{C}_C$, are called the *n-cotilting module*, *n-cotilting class*, and *n-cotilting cotorsion pair*, respectively. In this case, $({}^\perp C)^\perp \subseteq \mathcal{I}_n$.

Examples. Let us state some examples of tilting and cotilting modules over various rings. For more, see [32].

- Let R be a Dedekind domain. Then $T_P = R_P \oplus \bigoplus_{p \in P} E(R/p)$, where $P \subseteq \operatorname{mSpec}(R)$ and R_P is the preimage of $\bigoplus_{p \in P} E(R/p)$ in the canonical projection $E(R) \to E(R)/R$, is a tilting module. Moreover, every tilting class $\mathcal{T}$ is induced by the tilting module T_P for suitable $P \subseteq \operatorname{mSpec}(R)$. Similarly, every cotilting class $\mathcal{C}$ is induced by the cotilting module $C_P = \prod_{p \in P} J_p \oplus \bigoplus_{q \in \operatorname{Spec}(R) \setminus P} E(R/q)$ for suitable $P \subseteq \operatorname{mSpec}(R)$. Here, J_p denotes the completion of the localization of R at p.

- Let R be a connected wild hereditary algebra over a field k. Denote by τ the Auslander-Reiten translation, and by $\mathcal{R}$ the class of all *Ringel divisible modules*, that is, of all modules D such that $\operatorname{Ext}^1_R(M, D) = 0$ for each regular module M.

  Let M be any regular module. Then for each finite dimensional module N, Lukas constructed an exact sequence $0 \longrightarrow N \longrightarrow A_M \longrightarrow B_M \longrightarrow 0$ where $A_M \in M^\perp$ and B_M is a finite direct sum of copies of $\tau^n M$ for some $n < \omega$. Letting $\mathcal{C}_M = \{\tau^m M \mid m < \omega\}$, we can iterate this construction (for $N = R$, $N = A_M$, etc.) and get an exact sequence $0 \longrightarrow R \longrightarrow C_M \longrightarrow D_M \longrightarrow 0$ where D_M has a countable $\mathcal{C}_M$-filtration. Then $T_M = C_M \oplus D_M$ is a 1-tilting module, called the *Lukas tilting module*. The corresponding 1-tilting class is $\mathcal{R}$, independently on the choice of M.

- Let $n < \omega$ and R be an n-Gorenstein ring; that is, left and right noetherian ring with $\operatorname{inj.dim}(R_R) = \operatorname{inj.dim}({}_RR) = n$. Let

$$0 \longrightarrow R \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_n \longrightarrow 0$$

be the minimal injective coresolution of R. Then $\bigoplus_{i \leq n} E_i$ is an n-tilting module.

Recall that a module M is called *pure-injective* if it is injective relative to all pure embeddings; that is, for every $A, B \in \text{Mod-}R$ with $A \subseteq_* B$ and a homomorphism $f : A \to M$ there is $g \in \operatorname{Hom}_R(B, M)$ such that $g \restriction A = f$. Similarly as in the case of injective modules, every module M has its pure-injective envelope. That is, unique (up to isomorphism) module $PE(M)$ such that $M \subseteq_* PE(M)$, and every pure embedding $\nu : M \to N$ for N pure-injective extends to a split embedding of $PE(M)$ into N. It is known that M is even an elementary substructure of $PE(M)$.

Later on, we will need the following well-known lemma.

Lemma 14. *Let R be a ring, and κ be a cardinal such that $\kappa \geq \dim(R)$. Then any submodule of a $\leq \kappa$-generated module is $\leq \kappa$-generated.*

Proof. First, all submodules of cyclic modules are $\leq \dim(R)$-generated, since they are homomorphic images of right ideals. Further, any $\leq \kappa$-generated module M has a filtration $(M_\alpha \mid \alpha \leq \kappa)$ such that all factors $M_{\alpha+1}/M_\alpha$ are cyclic. If $K \subseteq M$, then $K \cap M_{\alpha+1}/K \cap M_\alpha$ embeds into $M_{\alpha+1}/M_\alpha$ for each $\alpha < \kappa$, and the assertion follows. $\square$

2.2 All tilting modules are of countable type

The central result of this subsection is the following one.

Theorem 15. *Let R be a ring and T a tilting module. Then T is of countable type.*

We also obtain a new characterization of tilting and cotilting cotorsion pairs.

Theorem 16. *Let R be a ring, $n < \omega$, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then:*

(i) *$\mathfrak{C}$ is an n-tilting cotorsion pair if and only if $\mathfrak{C}$ is hereditary, $\mathcal{A}$ consists of modules of projective dimension $\leq n$, and $\mathcal{B}$ is closed under arbitrary direct sums.*

(ii) *$\mathfrak{C}$ is an n-cotilting cotorsion pair if and only if $\mathfrak{C}$ is hereditary, $\mathcal{B}$ consists of modules of injective dimension $\leq n$, and $\mathcal{A}$ is closed under arbitrary direct products.*

2.2.1 Closure properties of cotorsion classes

Proposition 17. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair cogenerated by a class $\mathcal{C}$ of modules of finite projective dimension. Assume that $\mathcal{B}$ is closed under arbitrary direct sums and X is a $\mathcal{B}$-filtered module. Then $X \in \mathcal{B}$.*

Proof. Let $(X_\alpha \mid \alpha \leq \kappa)$ be a $\mathcal{B}$-filtration of X. By induction on κ, we will prove that $X \in \mathcal{B}$ and there is a continuous chain of short exact sequences

$$\delta_\alpha \quad : \quad 0 \longrightarrow K_\alpha \longrightarrow \bigoplus_{\lambda < \alpha} B_\lambda \longrightarrow X_\alpha \longrightarrow 0 \quad (\alpha < \kappa)$$

such that

(i) $K_0 = 0$ and $K_{\alpha+1}/K_\alpha \in \mathcal{B}$ for any $\alpha < \kappa$,

(ii) $B_\lambda \in \mathcal{A} \cap \mathcal{B}$ for all $\lambda < \kappa$,

(iii) The embedding of the middle term of δ_α into the middle term of δ_β is the canonical inclusion, for all $\alpha < \beta < \kappa$.

For $\kappa = 0$, clearly $0 \in \mathcal{B}$, and we just take the short exact sequence of zeros. Let $\kappa = \beta + 1$. Then $X_\kappa \in \mathcal{B}$ immediately by the inductive assumption and the fact that $\mathcal{B}$ is closed under extensions.

For the construction of δ_κ, we use an idea from [31, 2.3]: since $\mathfrak{C}$ is complete, there is a short exact sequence $0 \longrightarrow B' \longrightarrow B_\beta \longrightarrow X_\kappa/X_\beta \longrightarrow 0$ with $B' \in \mathcal{B}$ and $B_\beta \in \mathcal{A}$. We form a pullback:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & X_\beta \arrow[r, equal] \arrow[d] & X_\beta \arrow[d] & \\
0 \arrow[r] & B' \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, "p"] & X_\kappa \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B' \arrow[r] & B_\beta \arrow[r] \arrow[d] & X_\kappa/X_\beta \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $X_\kappa/X_\beta \in \mathcal{B}$, we have $B_\beta \in \mathcal{A} \cap \mathcal{B}$. Thus, the middle column of the diagram splits, and we can use the exact sequence δ_β to form the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_\beta \arrow[r] \arrow[d] & \bigoplus_{\lambda<\beta} B_\lambda \arrow[r] \arrow[d, "i"] & X_\beta \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_\kappa \arrow[r] \arrow[d] & \bigoplus_{\lambda<\kappa} B_\lambda \arrow[r] \arrow[d] & X_\kappa \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B' \arrow[r] \arrow[d] & B_\beta \arrow[r] \arrow[d] & X_\kappa/X_\beta \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The diagram is commutative and has exact rows and columns by the 3×3 Lemma, and the map i can be w.l.o.g. taken as the canonical inclusion. We define δ_κ as the middle row of this diagram.

Finally, assume κ is a limit ordinal. Then we define $\delta_\kappa = \varinjlim_{\alpha<\kappa} \delta_\alpha : 0 \longrightarrow K_\kappa \longrightarrow C_1 \longrightarrow X_\kappa \longrightarrow 0$. To complete the proof, we use an idea from [9, 3.2]: we replace X_κ

with K_κ (and $(X_\beta \mid \beta \leq \kappa)$ with $(K_\beta \mid \beta \leq \kappa)$), and, step by step, construct a long exact sequence:

$$\cdots \longrightarrow C_3 \xrightarrow{f_3} C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} X_\kappa \longrightarrow 0, \qquad C_i \in \mathcal{B} \text{ for } i \geq 1.$$

If $A \in \mathcal{C}$ has projective dimension n, then $0 = \operatorname{Ext}_R^{n+1}(A, \operatorname{Ker}(f_n)) \cong \operatorname{Ext}_R^1(A, X_\kappa)$. This proves that $X_\kappa \in \mathcal{B}$. □

Theorem 18. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair cogenerated by a class of modules of finite projective dimension, and let $\mathcal{B}$ be closed under arbitrary direct sums. Then $\mathcal{B}$ is closed under arbitrary direct limits.*

Proof. First, we prove that $\mathcal{B}$ is closed under unions of arbitrary chains. Let $\mathcal{C} \subseteq \mathcal{B}$ be a chain of modules with respect to inclusion. We construct a $\mathcal{B}$-filtration of $X = \bigcup \mathcal{C}$ by transfinite induction: we set $X_0 = 0$, and $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ for α limit. If $\alpha = \beta + 1$ and $X_\beta \subsetneq X$, we consider $x \in X \setminus X_\beta$ and take X_α as an element of $\mathcal{C}$ containing x. Since $X_\alpha \not\subseteq X_\beta$ and X_β is a union of elements of $\mathcal{C}$, we have $X_\beta \subseteq X_\alpha$. Since the cotorsion pair $\mathfrak{C}$ is hereditary, we have $X_\alpha / X_\beta \in \mathcal{B}$, and Proposition 17 applies.

Now, the result follows from the well-known fact that closure under unions of well-ordered chains implies closure under arbitrary direct limits (see e.g. [1, 1.7]). □

Later on, we will need the following corollary.

Corollary 19. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be as in Theorem 18. Then $\prod_{i\in I}^{<\nu} M_i \in \mathcal{B}$ whenever $(M_i \mid i \in I)$ is a family of modules in $\mathcal{B}$ and ν is a regular infinite cardinal. In particular, $\mathcal{E} \subseteq \mathcal{B}$.*

Proof. The statement follows from the fact that $\prod_{i\in I}^{<\nu} M_i$ is a directed union of the products $\prod_{i\in J} M_i$ for subsets $J \subseteq I$ of cardinality $< \nu$. □

2.2.2 Classes of countable type

We start with a technical lemma that will be applied later on to estimate the number of generators of submodules in our particular setting.

Lemma 20. *Let R be a ring, κ a regular infinite cardinal, and M a module. Let $(M_\alpha \mid \alpha \leq \kappa)$ be a strictly ascending filtration of M. Then there is a family of non-zero injective modules $(E_\alpha \mid \alpha < \kappa)$ and an embedding $e : M \to \prod_{\alpha<\kappa}^{<\kappa} E_\alpha$ such that, for each submodule $N \subseteq M$ with $N \cap (M_{\alpha+1} \setminus M_\alpha) \neq \emptyset$ for all $\alpha < \kappa$, the union of supports of all elements of $e(N)$ equals κ.*

Proof. Let $i_\alpha : M_{\alpha+1}/M_\alpha \to E_\alpha$ be an injective envelope of $M_{\alpha+1}/M_\alpha$ for each $\alpha < \kappa$. We will construct a continuous chain of injective maps $e_\alpha : M_\alpha \to \prod_{\beta<\alpha} E_\beta$ as follows:

$e_0 = 0$; if e_α is already constructed, we can extend it to $f_\alpha : M_{\alpha+1} \to \prod_{\beta<\alpha} E_\beta$ since all E_β's are injective, and put $e_{\alpha+1} = f_\alpha + i_\alpha p_\alpha$ where $p_\alpha : M_{\alpha+1} \to M_{\alpha+1}/M_\alpha$ is the projection.

Consider $e = \bigcup_{\alpha<\kappa} e_\alpha : M \to \prod_{\beta<\kappa}^{<\kappa} E_\beta$. If $N \subseteq M$ and $x \in N \cap (M_{\alpha+1} \setminus M_\alpha)$, then the α-th component of $e(x)$ $(= e_{\alpha+1}(x))$ is $i_\alpha p_\alpha(x)$ $(\neq 0)$, and the claim follows. □

While Lemma 9 (i) will take care of filtrations of regular length, our arguments for singular cardinals will be based on the following two lemmas essentially going back to [14], see also [15, Chap. XII].

Definition 21. Let M be a module, $\mathcal{Q}$ a set of modules, and κ a regular infinite cardinal. Then M is called *κ-$\mathcal{Q}$-free* provided there is a set $\mathcal{S}_\kappa$ consisting of $<\kappa$-generated $\mathcal{Q}$-filtered submodules of M such that:

(a) $0 \in \mathcal{S}_\kappa$,

(b) $\mathcal{S}_\kappa$ is closed under well-ordered chains of length $<\kappa$, and

(c) each subset of M of cardinality $<\kappa$ is contained in an element of $\mathcal{S}_\kappa$.

The set $\mathcal{S}_\kappa$ is said to witness the κ-$\mathcal{Q}$-freeness of M. If $\mathcal{S}_\kappa$ also satisfies

(d) M/N is $\mathcal{Q}$-filtered for each $N \in \mathcal{S}_\kappa$,

then we call M *κ-$\mathcal{Q}$-separable*, and the set $\mathcal{S}_\kappa$ is said to witness the κ-$\mathcal{Q}$-separability of M.

Clearly, every κ-$\mathcal{Q}$-separable module is $\mathcal{Q}$-filtered. The following lemma says that the converse is also true under rather weak assumptions.

Lemma 22. *Let R be a ring, μ an infinite cardinal and $\mathcal{Q}$ a set of $\leq \mu$-presented modules. Then M is λ-$\mathcal{Q}$-separable whenever M is $\mathcal{Q}$-filtered and λ is a regular cardinal $> \mu$. Moreover, it is possible to choose the witnessing sets so that $\mathcal{S}_\lambda \subseteq \mathcal{S}_{\lambda'}$ for all regular cardinals λ, λ' such that $\mu < \lambda < \lambda'$.*

Proof. We use Lemma 8 for $\kappa = \mu^+$ to obtain a family $\mathcal{F}$ for the $\mathcal{Q}$-filtration of M, and set $\mathcal{S}_\lambda = \{F \in \mathcal{F} \mid F \text{ is } <\lambda\text{-generated module}\}$. □

The following is a particular case of the celebrated Shelah's singular compactness theorem.

Lemma 23. [15, XII.1.14 and IV.3.7] *Let R be a ring, λ a singular cardinal, and $\aleph_0 \leq \mu < \lambda$. Let $\mathcal{Q}$ be a set of $\leq \mu$-presented modules. Let M be a module with* $\mathrm{gen}(M) = \lambda$. *Assume that M is κ-$\mathcal{Q}$-free for each regular cardinal $\mu < \kappa < \lambda$. Then M is $\mathcal{Q}$-filtered.*

In our particular setting, we will apply Lemma 9 (i) under more general conditions. For this purpose, we need

Lemma 24. *Let R be a ring, $n < \omega$, and $\mathcal{B}$ a class of modules closed under arbitrary direct sums such that ${}^\perp\mathcal{B} \subseteq {}^\perp\mathcal{E}_0$. Then ${}^\perp\mathcal{B} \cap \mathcal{P}_n = {}^\perp\mathcal{B}_n$ where $\mathcal{B}_n$ is the closure of $\mathcal{B} \cup Cos^n$-R under arbitrary direct sums.*

Proof. Clearly, ${}^\perp\mathcal{B}_n \subseteq {}^\perp\mathcal{B} \cap \mathcal{P}_n$. Conversely, let $M \in {}^\perp\mathcal{B} \cap \mathcal{P}_n$ and $X \in \mathcal{B}_n$. Then X is of the form $B \oplus \bigoplus_{i\in I} C_i$ where $B \in \mathcal{B}$ and $(C_i \mid i \in I)$ is a family of modules from Cos^n-R. That is, we have exact sequences $E_{i,n-1} \longrightarrow \cdots \longrightarrow E_{i,0} \longrightarrow C_i \longrightarrow 0$ with $E_{i,j}$'s injective for all $i \in I$ and $0 \leq j < n$. Since $M \in {}^\perp\mathcal{E}_0$, the exact sequence

$$\bigoplus_{i\in I} E_{i,n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} \bigoplus_{i\in I} E_{i,0} \xrightarrow{f_0} \bigoplus_{i\in I} C_i \longrightarrow 0$$

implies $\mathrm{Ext}_R^k(M, \bigoplus_{i\in I} C_i) \cong \mathrm{Ext}_R^{k+n}(M, \mathrm{Ker}(f_{n-1})) = 0$ for each $0 < k < \omega$. Thus $\mathrm{Ext}_R^k(M, X) = 0$, and we deduce $M \in {}^\perp\mathcal{B}_n$. □

The following lemma will serve as the induction step in the proof of Theorem 15.

For a class of modules $\mathcal{C}$, denote by $\mathfrak{A}_{\aleph_0}(\mathcal{C})$ the assertion: "All modules in $\mathcal{C}$ are $\mathcal{C}^{\leq\aleph_0}$-filtered."

Lemma 25. *Let R be a ring and $\mathcal{B}$ a class of modules closed under arbitrary direct sums such that ${}^\perp\mathcal{B} \subseteq {}^\perp\mathcal{E}$. Then $\mathfrak{A}_{\aleph_0}({}^\perp\mathcal{B} \cap \mathcal{P}_n)$ implies $\mathfrak{A}_{\aleph_0}({}^\perp\mathcal{B} \cap \mathcal{P}_{n+1})$ for each $n < \omega$.*

Proof. Assume $\mathfrak{A}_{\aleph_0}({}^\perp\mathcal{B} \cap \mathcal{P}_n)$ holds. Let κ be a regular uncountable cardinal, $M \in {}^\perp\mathcal{B} \cap \mathcal{P}_{n+1}$ be a module, and $\lambda = \mathrm{gen}(M)$. W.l.o.g., there is a short exact sequence $0 \longrightarrow K \longrightarrow F \stackrel{\pi}{\longrightarrow} M \longrightarrow 0$ where $F = R^{(\lambda)}$, and K is a submodule of F.

Since $M \in {}^\perp\mathcal{B}\cap\mathcal{P}_{n+1}$, we have $K \in {}^\perp\mathcal{B}\cap\mathcal{P}_n$. Let $\mathcal{Q} = {}^\perp\mathcal{B}\cap\mathcal{P}_n^{\leq\aleph_0}$. By assumption and Lemma 22 (for $\mu = \aleph_0$), there are sets $\mathcal{S}_\kappa \subseteq \mathcal{S}_{\kappa^+}$ witnessing the κ-$\mathcal{Q}$-separability and κ^+-$\mathcal{Q}$-separability of K, respectively. Denote by $\mathcal{S}'_\kappa$ the set of all submodules $N \subseteq M$ such that there is a subset $A \subseteq \lambda$ of cardinality $< \kappa$ with $\pi(R^{(A)}) = N$ and $K \cap R^{(A)} \in \mathcal{S}_\kappa$.

Consider $L \in \mathcal{S}_\kappa$. Then L is $\mathcal{Q}$-filtered, so $L \in {}^\perp\mathcal{B}$ by Lemma 1. Moreover, L is $< \kappa$-generated and $\mathcal{P}_n^{\leq\aleph_0}$-filtered, so $L \in \mathcal{P}_n^{<\kappa}$ by the Horseshoe Lemma. This shows that $\mathcal{S}_\kappa \subseteq {}^\perp\mathcal{B} \cap \mathcal{P}_n^{<\kappa}$, and hence $\mathcal{S}'_\kappa \subseteq \mathcal{P}_{n+1}^{<\kappa}$.

We claim that $\mathcal{S}'_\kappa$ witnesses the κ-$\mathcal{Q}'_\kappa$-freeness of M where $\mathcal{Q}'_\kappa = {}^\perp\mathcal{B} \cap \mathcal{P}_{n+1}^{<\kappa}$. Clearly, $0 \in \mathcal{S}'_\kappa$, and $\mathcal{S}'_\kappa$ is closed under well-ordered unions of chains of length $< \kappa$. Moreover, we have the exact sequence $0 = \mathrm{Ext}^i_R(M, B) \longrightarrow \mathrm{Ext}^i_R(N, B) \longrightarrow \mathrm{Ext}^{i+1}_R(M/N, B) \cong \mathrm{Ext}^i_R(K/K \cap R^{(A)}, B) = 0$ for all $B \in \mathcal{B}$, $\pi(R^{(A)}) = N \in \mathcal{S}'_\kappa$ and $i \geq 1$. Thus $\mathcal{S}'_\kappa \subseteq \mathcal{Q}'_\kappa$.

It remains to prove condition (c) of Definition 21. Let X be a subset of M of cardinality $< \kappa$. There is a subset $A_0 \subseteq \lambda$ of cardinality $< \kappa$ such that $X \subseteq \pi(R^{(A_0)})$. Let $L_0 = K \cap R^{(A_0)}$. We will prove that there is a module $K_0 \in \mathcal{S}_\kappa$ containing L_0.

If not, we can inductively construct a strictly ascending κ-filtration $(\tilde{K}_\alpha \mid \alpha \leq \kappa)$ such that $\tilde{K}_\alpha \in \mathcal{S}_\kappa$ and $L_0 \cap (\tilde{K}_{\alpha+1} \setminus \tilde{K}_\alpha) \neq \emptyset$ for all $\alpha < \kappa$. Indeed, take $\tilde{K}_0 = 0$, and for each $\alpha < \kappa$, $L_0 \nsubseteq \tilde{K}_\alpha$ by assumption, so we can find $\tilde{K}_{\alpha+1} \in \mathcal{S}_\kappa$ containing both $\tilde{K}_\alpha$ and an element $x \in L_0 \setminus \tilde{K}_\alpha$. Put $U = \tilde{K}_\kappa$ and consider the map $e : U \to \prod^{<\kappa}_{\alpha<\kappa} E_\alpha$ from Lemma 20. Then the union of supports of all elements of $e(U \cap L_0)$ equals κ. On the other hand, $U \in \mathcal{S}_{\kappa^+}$, so $K/U \in {}^\perp\mathcal{B}$. Since $F/K \cong M \in {}^\perp\mathcal{B}$ and ${}^\perp\mathcal{B} \subseteq {}^{\perp_1}\mathcal{E}$, we can extend e to K, then to F, to get a homomorphism $g : F \to \prod^{<\kappa}_{\alpha<\kappa} E_\alpha$ with $g \restriction U = e$. However, since $\mathrm{card}\, A_0 < \kappa$, the union of supports of all elements of $g(R^{(A_0)})$ has cardinality $< \kappa$, a contradiction.

This proves that there exists $K_0 \in \mathcal{S}_\kappa$ such that $L_0 \subseteq K_0$. Take $A_1 \supseteq A_0$ such that $K_0 \subseteq R^{(A_1)}$ and $\mathrm{card}\, A_1 < \kappa$. Put $L_1 = K \cap R^{(A_1)}$. Continuing in this way, we define a sequence $K_0 \subseteq K_1 \subseteq \cdots$ of elements of $\mathcal{S}_\kappa$, and a sequence $A_0 \subseteq A_1 \subseteq \cdots$ of subsets of λ of cardinality $< \kappa$ such that $K \cap R^{(A_i)} \subseteq K_i$ and $K_i \subseteq R^{(A_{i+1})}$ for all $i < \omega$. Then $K' = \bigcup_{i<\omega} K_i \in \mathcal{S}_\kappa$ and $K' = K \cap R^{(A')}$ where $A' = \bigcup_{i<\omega} A_i$. So $\pi(R^{(A')})$ is an element of $\mathcal{S}'_\kappa$ containing X, and $\mathcal{S}'_\kappa$ witnesses the κ-$\mathcal{Q}'_\kappa$-freeness of M. This completes the proof of the claim.

Now, we will prove $\mathfrak{A}_{\aleph_0}({}^\perp\mathcal{B} \cap \mathcal{P}_{n+1})$ by induction on $\lambda = \mathrm{gen}(M)$ for all $M \in {}^\perp\mathcal{B}\cap\mathcal{P}_{n+1}$. Define $\mathcal{R} = {}^\perp\mathcal{B}\cap\mathcal{P}_{n+1}^{\leq\aleph_0}$. If $\lambda \leq \aleph_0$, then we use Lemma 20 similarly as above to prove that the first syzygy, K, of M is countably generated. Since $K \in {}^\perp\mathcal{B} \cap \mathcal{P}_n$, by induction, we get that K has a projective resolution consisting of countably generated projective modules, so $M \in \mathcal{R}$.

If λ is regular, then we select from $\mathcal{S}'_\lambda$ a λ-filtration, $\mathcal{F}$, of M. Denote by $\mathcal{B}_{n+1}$ the closure of $\mathcal{B}\cup\mathrm{Cos}^{n+1}\text{-}R$ under direct sums as in Lemma 24; we have ${}^\perp\mathcal{B} \cap \mathcal{P}_{n+1} = {}^\perp\mathcal{B}_{n+1}$. Since $0 = \mathrm{Ext}^i_R(N', B) \longrightarrow \mathrm{Ext}^{i+1}_R(N/N', B) \longrightarrow \mathrm{Ext}^{i+1}_R(N, B) = 0$ for all modules

$N' \subseteq N \in \mathcal{F}$, $B \in \mathcal{B}_{n+1}$ and $i \geq 1$, we have $\operatorname{Ext}^i_R(N/N', \mathcal{B}_{n+1}) = 0$ for all $i \geq 2$. Then Lemma 9 (i) yields a λ-subfiltration of $\mathcal{F}$ which is a ${}^\perp\mathcal{B}_{n+1}$-filtration of M. Using inductive hypothesis, we refine this filtration to the desired $\mathcal{R}$-filtration of M.

If λ is singular, then $\mathcal{S}'_\kappa$ witnesses κ-$\mathcal{R}$-freeness of M, for each regular uncountable cardinal $\kappa < \lambda$. So the existence of an $\mathcal{R}$-filtration of M follows by Lemma 23 for $\mu = \aleph_0$. □

A classical result of Kaplansky says that any projective module over any ring is a direct sum of countably generated modules. So $\mathfrak{A}_{\aleph_0}({}^\perp\mathcal{B} \cap \mathcal{P}_0)$ holds for any class of modules $\mathcal{B}$. Lemma 25 thus gives:

Theorem 26. *Let R be a ring and $\mathcal{B}$ a class of modules closed under arbitrary direct sums such that ${}^\perp\mathcal{B} \subseteq {}^\perp\mathcal{E}$. Then for any $n < \omega$, all modules in ${}^\perp\mathcal{B} \cap \mathcal{P}_n$ are ${}^\perp\mathcal{B} \cap \mathcal{P}_n^{\leq \aleph_0}$-filtered.*

Now, it is easy to prove our main theorem.

Proof of Theorem 15. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be the tilting cotorsion pair induced by T, and n be the projective dimension of T. Then $\mathcal{A} = {}^\perp\mathcal{B} \cap \mathcal{P}_n$, so Corollary 19 applies and yields $\mathcal{E} \subseteq \mathcal{B}$. Let $\mathcal{C} = \mathcal{A}^{\leq \aleph_0}$. Then $\mathcal{C}^\perp = \mathcal{B}\,(= T^\perp)$ by Lemma 1 and Theorem 26, so T is of countable type. □

In general, there is a proper class of cotorsion pairs over a fixed ring R, cf. [20]. Since there is always a representative set of isomorphism classes of $\aleph_0$-presented modules, we get:

Corollary 27. *Let R be a ring. Then the cotorsion pairs induced by all tilting modules form a set.*

2.2.3 Tilting and cotilting cotorsion pairs

If we omit the assumption of ${}^\perp\mathcal{B} \subseteq {}^\perp\mathcal{E}$ in Lemma 25 we can still obtain a similar result, with $\aleph_0$ replaced by $\dim(R)$.

For a class of modules $\mathcal{C}$, denote by $\mathfrak{A}_\mu(\mathcal{C})$ the assertion: "All modules in $\mathcal{C}$ are $\mathcal{C}^{\leq \mu}$-filtered."

Lemma 28. *Let R be a ring and $\mathcal{B}$ a class of modules closed under arbitrary direct sums. Let $\mu = \dim(R)$. Then $\mathfrak{A}_\mu({}^\perp\mathcal{B} \cap \mathcal{P}_n)$ implies $\mathfrak{A}_\mu({}^\perp\mathcal{B} \cap \mathcal{P}_{n+1})$ for each $n < \omega$.*

Proof. The proof is the same as for Lemma 25 except that the induction on $\operatorname{gen}(M)$ starts at μ rather than $\aleph_0$, and when proving condition (c) of Definition 21 for $\mathcal{S}'_\kappa$, we find $K_i \in \mathcal{S}_\kappa$ containing L_i using Lemma 14 rather than Lemma 20. Finally, Lemma 24 is not needed when λ is a regular cardinal $\geq \dim(R)$, since each module $M \in \mathcal{P}_n$ with $\operatorname{gen}(M) = \lambda$ has a λ-filtration with successive factors in $\mathcal{P}_n^{<\lambda}$ by [2]. □

As in the case of Theorem 26, Lemma 28 implies:

Theorem 29. *Let R be a ring, $n < \omega$, and $\mathcal{B}$ be a class of modules closed under arbitrary direct sums. Then all modules in ${}^\perp\mathcal{B} \cap \mathcal{P}_n$ are ${}^\perp\mathcal{B} \cap \mathcal{P}_n^{\leq \dim(R)}$-filtered.*

Prior to proving our second main theorem, let us state a characterization of tilting and cotilting cotorsion pairs coming from [3] and [9].

Lemma 30. *Let R be a ring, $n \geq 0$, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then:*

(i) *$\mathfrak{C}$ is an n-tilting cotorsion pair if and only if $\mathfrak{C}$ is complete, hereditary, $\mathcal{A}$ consists of modules of projective dimension $\leq n$, and $\mathcal{B}$ is closed under arbitrary direct sums.*

(ii) *$\mathfrak{C}$ is an n-cotilting cotorsion pair if and only if $\mathfrak{C}$ is complete, hereditary, $\mathcal{B}$ consists of modules of injective dimension $\leq n$, and $\mathcal{A}$ is closed under arbitrary direct products.*

Thus to prove Theorem 16, it is enough to show that completeness of $\mathfrak{C}$ is implied by the other three properties on the right-hand side of the latter characterization. Almost all the work for the tilting case is done. However, cotilting case needs two more lemmas. The crucial one is the fundamental result from [27]:

Lemma 31. *Let R be a ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathcal{A}$ is closed under pure submodules and arbitrary direct products. Then $\mathcal{A}$ is closed under pure epimorphic images, and $\mathfrak{C}$ is complete.*

Lemma 31 allows us to make the induction step in the proof of

Lemma 32. *Let R be a ring, and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathcal{A}$ is closed under arbitrary direct products and ${}^{\perp}\mathcal{B}$ contains all direct products of projective modules. Then ${}^{\perp_n}\mathcal{B}$ is closed under arbitrary direct products for each $n \geq 1$.*

If moreover $\mathcal{B} \subseteq \mathcal{I}_n$ for some $n \geq 0$, then $\mathcal{A}_k = \bigcap_{i \geq k} {}^{\perp_i}\mathcal{B}$ is an $(n-k+1)$-cotilting class for each $1 \leq k \leq n+1$.

Proof. The first claim is proved by induction on n. The case of $n = 1$ is our assumption on $\mathcal{A}$. Let $(M_\alpha \mid \alpha < \kappa)$ be a family of modules in ${}^{\perp_{n+1}}\mathcal{B}$. Consider the short exact sequences $0 \to K_\alpha \to P_\alpha \to M_\alpha \to 0$ with P_α projective for each $\alpha < \kappa$. Since $\mathrm{Ext}_R^{n+1}(M_\alpha, B) \cong \mathrm{Ext}_R^n(K_\alpha, B) = 0$ for all $B \in \mathcal{B}$, the inductive premise gives $\prod_{\alpha<\kappa} K_\alpha \in {}^{\perp_n}\mathcal{B}$, so our assumption on ${}^{\perp}\mathcal{B}$ yields $\prod_{\alpha<\kappa} M_\alpha \in {}^{\perp_{n+1}}\mathcal{B}$.

For the second claim, we first prove by reverse induction on $1 \leq k \leq n+1$ that $\mathcal{A}_k$ is closed under pure submodules. The case of $k = n+1$ is clear since $\mathcal{A}_{n+1} = \text{Mod-}R$ by the assumption on $\mathcal{B}$. Let $M \in \mathcal{A}_{k-1}$, P be a pure submodule in M, and $B \in \mathcal{B}$. Since $\mathcal{A}_{k-1} \subseteq \mathcal{A}_k$, we have $\mathrm{Ext}_R^{k-1}(P, B) \cong \mathrm{Ext}_R^k(M/P, B)$, so it suffices to prove that $\mathcal{A}_k$ is closed under pure epimorphic images. By Lemma 31, it is enough to show that $\mathcal{A}_k$ is closed under pure submodules and arbitrary direct products. However, this is the case by the inductive premise and by the first claim.

So Lemma 31 applies, and for each $1 \leq k \leq n+1$, $\mathfrak{C}_k = (\mathcal{A}_k, \mathcal{A}_k^{\perp})$ is a complete hereditary cotorsion pair such that $\mathcal{A}_k$ is closed under arbitrary direct products. Moreover, $\mathcal{A}_k^{\perp} \subseteq \mathcal{I}_{n-k+1}$ since $\mathcal{B} \subseteq \mathcal{I}_n$. So $\mathfrak{C}_k$ is $(n-k+1)$-cotilting by Lemma 30. $\square$

Now, our second main result follows easily.

Proof of Theorem 16. (i) The cotorsion pair $\mathfrak{C}$ is cogenerated by a set by Theorem 29 and Lemma 1, so $\mathfrak{C}$ is complete by Theorem 2 (i).

(ii) $\mathcal{A} = \mathcal{A}_1$ in the second claim of Lemma 32. $\square$

2.3 All tilting modules are of finite type

2.3.1 Countably generated modules

The aim of this subsection is to investigate conditions under which, for a given class $\mathcal{C}$, a module $M \in \mathcal{C}^{<\aleph_1}$ is a countable direct limit of objects in $\mathcal{C}^{<\aleph_0}$. The key idea is to look at conditions which imply that the first syzygy module of M is $\mathcal{C}^{<\aleph_0}$-filtered.

Let us state here explicitly a corollary of Lemma 8 for the setting of countably generated modules.

Lemma 33. *Let M be countably generated $\mathcal{Q}$-filtered module where $\mathcal{Q}$ is a family of finitely presented modules. Then there is a filtration $(M_n \mid n < \omega)$ of M consisting of finitely presented submodules of M such that M_n and M/M_n are $\mathcal{Q}$-filtered for every $n < \omega$.*

The following technical lemma will be of use later.

Lemma 34. *Let $\mathcal{Q}$ be a family of finitely presented modules containing the regular module R. Let M be a countably presented module and let*

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

be a free presentation of M with F and K countably generated. Assume that K is a direct summand of a $\mathcal{Q}$-filtered module. Then, there exists an exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow M \longrightarrow 0$$

where H and G are countably generated $\mathcal{Q}$-filtered modules.

Proof. Let K be a summand of a $\mathcal{Q}$-filtered module P. Since K is countably generated, Lemma 8 (H4) implies that K is contained in a countably generated $\mathcal{Q}$-filtered submodule of P; thus we may assume that P is countably generated. By Eilenberg's trick, $K \oplus P^{(\omega)} \cong P^{(\omega)}$. Consider the exact sequence

$$0 \longrightarrow K \oplus P^{(\omega)} \longrightarrow F \oplus P^{(\omega)} \longrightarrow M \longrightarrow 0$$

and let $H = K \oplus P^{(\omega)} \cong P^{(\omega)}$, $G = F \oplus P^{(\omega)}$. Then G and H are countably generated $\mathcal{Q}$-filtered modules. $\square$

Lemma 35. *Let $(\mathcal{A}, \mathcal{B})$ be an n-tilting cotorsion pair. If $A \in \mathcal{A}$ is a countably or finitely generated module, then $A \in \mathcal{A}^{<\aleph_1}$ or $A \in \mathcal{A}^{<\aleph_0}$, respectively.*

Proof. Since $\mathcal{A}$ is a resolving class, it is enough to show that every countably or finitely generated module in $\mathcal{A}$ is countably or finitely presented, respectively.

By Theorem 26 and Lemma 1, $(\mathcal{A}, \mathcal{B})$ is of countable type and every module $M \in \mathcal{A}$ is $\mathcal{A}^{<\aleph_1}$-filtered. By Lemma 8 (H4), A is countably presented.

Assume now that $A \in \mathcal{A}$ is finitely generated and let $0 \longrightarrow K \longrightarrow R^m \longrightarrow A \longrightarrow 0$ be a presentation of A. By the first part of the proof, K is countably generated. Write $K = \bigcup_{i<\omega} K_i$, where K_i are finitely generated submodules of K. Consider the map $\varphi : K \to \prod_{i<\omega} E_i$ defined by $\varphi(x) = (x + K_i)_{i<\omega}$, where E_i is an injective envelope of K/K_i for every $i < \omega$. The image of φ is contained in $\bigoplus_{i<\omega} E_i$ which is an object of $\mathcal{B}$. Thus, φ extends to some $\psi : R^m \to \bigoplus_{i<\omega} E_i$. As a consequence, the image of φ is contained in $\bigoplus_{i\leq k} E_i$, for some $k < \omega$. Hence K is finitely generated. $\square$

In order to use an inductive argument we show now that for an n-tilting module T, $n \geq 1$, the cotorsion pair $({}^{\perp}(\Omega(T)^{\perp}), \Omega(T)^{\perp})$, where $\Omega(T)$ stands for a first syzygy of T, is an $(n-1)$-tilting cotorsion pair.

Lemma 36. *Let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair induced by an n-tilting module T with $n \geq 1$. Let $(\mathcal{A}_1, \mathcal{B}_1)$ be the cotorsion pair with $\mathcal{B}_1 = \Omega(T)^{\perp}$ where $\Omega(T)$ is a first syzygy module of T. Then:*

(1) *$(\mathcal{A}_1, \mathcal{B}_1)$ is an $(n-1)$-tilting cotorsion pair,*

(2) *$X \in \mathcal{B}_1$ if and only if (any) first cosyzygy of X belongs to $\mathcal{B}$,*

(3) *$M \in \mathcal{A}$ implies that (any) first syzygy of M belongs to $\mathcal{A}_1$.*

Proof. Let $0 \longrightarrow \Omega(T) \longrightarrow F \longrightarrow T \longrightarrow 0$ be a presentation of T with F free and $\Omega(T)$ a first syzygy module of T. Then $\Omega(T)$ has projective dimension at most $(n-1)$ and, by [9, Lemma 3.4], $\Omega(T)^{\perp}$ is closed under direct sums. By Theorem 16 (i), $\Omega(T)^{\perp}$ is an $(n-1)$-tilting class. Let $(\mathcal{A}_1, \mathcal{B}_1)$ be the associated cotorsion pair, namely $\mathcal{B}_1 = \Omega(T)^{\perp}$ and $\mathcal{A}_1 = {}^{\perp}\mathcal{B}_1 = {}^{\perp_1}\mathcal{B}_1$. For modules X and M consider exact sequences $0 \longrightarrow X \longrightarrow I \longrightarrow \Omega^{-}(X) \longrightarrow 0$ and $0 \longrightarrow \Omega(M) \longrightarrow F \longrightarrow M \longrightarrow 0$, where I is an injective module, F is a free module and $\Omega^{-}(X)$, $\Omega(M)$ are first cosyzygy and syzygy module of X and M, respectively. Then $\operatorname{Ext}^i_R(\Omega(M), X) \cong \operatorname{Ext}^{i+1}_R(M, X)$ and $\operatorname{Ext}^{i+1}_R(M, X) \cong \operatorname{Ext}^i_R(M, \Omega^{-}(X))$ for all $i \geq 1$. The last two statements follow immediately by these formulas. □

The following provides one of the key ingredients for proving our main result.

Lemma 37. *Let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair induced by an n-tilting module T. Let $\Omega(T)$ be a first syzygy of T. Assume that the cotorsion pair $(\mathcal{A}_1, \mathcal{B}_1)$ with $\mathcal{B}_1 = \Omega(T)^{\perp}$ is of finite type. Then any countably generated module $A \in \mathcal{A}$ is isomorphic to a direct limit of a countable direct system of the form:*

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \longrightarrow \cdots \longrightarrow C_k \xrightarrow{f_k} C_{k+1} \longrightarrow \cdots$$

where the modules C_k are in $\mathcal{A}^{<\aleph_0}$.

Proof. Since the cotorsion pair $(\mathcal{A}_1, \mathcal{B}_1)$ is of finite type, it is cogenerated by (a representative set of) $\mathcal{A}_1^{<\aleph_0}$. By Theorem 2 (ii), $\mathcal{A}_1$ coincides with the class of all direct summands of the $\mathcal{A}_1^{<\aleph_0}$-filtered modules.

Fix a countably generated module $A \in \mathcal{A}$ and let

$$0 \longrightarrow K \longrightarrow R^{(\omega)} \longrightarrow A \longrightarrow 0$$

be a presentation of A. Then $K \in \mathcal{A}_1$, thus K is a summand in an $\mathcal{A}_1^{<\aleph_0}$-filtered module. By Lemma 35, A is countably presented; so by the hypotheses and Lemma 34, there exists an exact sequence $0 \longrightarrow H \longrightarrow G \longrightarrow A \longrightarrow 0$ where G and H are countably generated $\mathcal{A}_1^{<\aleph_0}$-filtered modules. By Lemma 33, we can write $H = \bigcup_{k<\omega} H_k$ and $G = \bigcup_{k<\omega} G_k$ where, for every $k < \omega$, H_k and G_k are finitely presented $\mathcal{A}_1^{<\aleph_0}$-filtered modules, and H/H_k, G/G_k are $\mathcal{A}_1^{<\aleph_0}$-filtered. W.l.o.g., we can assume that H is a submodule of G. Given $k < \omega$, there is a j_k such that $H_k \subseteq G_{j_k}$; and we can choose the sequence $(j_k \mid k < \omega)$ to be strictly increasing.

We claim that $G_{j_k}/H_k \in \mathcal{A}^{<\aleph_0}$. By Lemma 35, it is enough to show that $G_{j_k}/H_k \in \mathcal{A}$. Let $B \in \mathcal{B}$; we have to show that $\operatorname{Ext}^1_R(G_{j_k}/H_k, B) = 0$. But H/H_k is $\mathcal{A}_1^{<\aleph_0}$-filtered, thus in $\mathcal{A}_1$, and it is immediate to check that $\mathcal{A}_1 \subseteq \mathcal{A}$. Moreover, $G/H \cong A \in \mathcal{A}$. Hence, every homomorphism $f : H_k \to B$ can be extended to a homomorphism $g : G \to B$, and the restriction of g to G_{j_k} obviously induces an extension of f to G_{j_k}. Thus $\operatorname{Ext}^1_R(G_{j_k}/H_k, B) = 0$, since $G_{j_k} \in \mathcal{A}_1 \subseteq \mathcal{A}$.

Set $C_k = G_{j_k}/H_k$. Since $(j_k \mid k < \omega)$ is increasing and unbounded in ω, the inclusions $G_{j_k} \subseteq G_{j_{k+1}}$ induce maps $f_k : C_k \to C_{k+1}$, and A is a direct limit of the direct system $((C_k, f_k) \mid k < \omega)$. □

2.3.2 Mittag-Leffler condition and pure submodules

Given a countable inverse system of abelian groups

$$\cdots \longrightarrow A_3 \xrightarrow{h_2} A_2 \xrightarrow{h_1} A_1 \xrightarrow{h_0} A_0 \tag{$*$}$$

(we call such a system a *tower* of abelian groups), we can define the homomorphism

$$\Delta : \prod_{i<\omega} A_i \to \prod_{i<\omega} A_i$$

by the formula $\Delta(\dots, a_i, \dots, a_0) = (\dots, a_i - h_i(a_{i+1}), \dots, a_1 - h_1(a_2), a_0 - h_0(a_1))$. Then $\operatorname{Ker}(\Delta) = \varprojlim A_i$. However, the homomorphism Δ is not onto, in general, and we define $\varprojlim^1 A_i$ as the cokernel of Δ. Now, if we have a short exact sequence

$$0 \longrightarrow (A_i)_{i<\omega} \longrightarrow (B_i)_{i<\omega} \longrightarrow (C_i)_{i<\omega} \longrightarrow 0$$

of towers of abelian groups, applying Snake Lemma, we obtain the exact sequence

$$0 \longrightarrow \varprojlim A_i \longrightarrow \varprojlim B_i \longrightarrow \varprojlim C_i \longrightarrow \varprojlim{}^1 A_i \longrightarrow \varprojlim{}^1 B_i \longrightarrow \varprojlim{}^1 C_i \longrightarrow 0.$$

We are interested in conditions under which $\varprojlim^1 A_i = 0$.

Lemma 38. *Let a tower* $(*)$ *be given. If all the maps* h_i *are onto, then* $\varprojlim^1 A_i = 0$. *Moreover* $\varprojlim A_i \neq 0$ *(unless every* $A_i = 0$*), because each of the natural projections* $\varprojlim A_i \to A_j$ *are onto.*

Proof. Given elements $b_i \in A_i$, $i < \omega$, and any $a_0 \in A_0$, inductively choose $a_{i+1} \in A_{i+1}$ to be a lift of $a_i - b_i \in A_i$. The map Δ sends $(\dots, a_1, a_0)$ to $(\dots, b_1, b_0)$, so Δ is onto and $\varprojlim^1 A_i = 0$. If all the $b_i = 0$, then $(\dots, a_1, a_0) \in \varprojlim A_i$. □

Definition 39. A tower $(A_i)_{i<\omega}$ of abelian groups satisfies the *Mittag-Leffler condition* if for each k there exists a $j \geq k$ such that the image of $A_i \to A_k$ equals the image of $A_j \to A_k$ for all $i \geq j$. (The descending chain of images of A_i in A_k is *stationary*.) For example, the Mittag-Leffler condition is satisfied if all the maps $A_{i+1} \to A_i$ in the tower $(A_i)_{i<\omega}$ are onto. We say that $(A_i)_{i<\omega}$ satisfies the *trivial* Mittag-Leffler condition if for each k there exists a $j > k$ such that the map $A_j \to A_k$ is zero.

Proposition 40. [34, Proposition 3.5.7] *If a tower* $(A_i)_{i<\omega}$ *satisfies the Mittag-Leffler condition, then* $\varprojlim^1 A_i = 0$.

Proof. If $(A_i)_{i<\omega}$ satisfies the trivial Mittag-Leffler condition, and $b_i \in A_i$ are given, set $a_k = b_k + \bar{b}_{k+1} + \cdots + \bar{b}_{j-1}$, where $\bar{b}_i$ denotes the image of b_i in A_k. (Note that $\bar{b}_i = 0$ for $i \geq j$.) Then Δ maps $(\ldots, a_1, a_0)$ to $(\ldots, b_1, b_0)$. Thus Δ is onto and $\varprojlim^1 A_i = 0$.

In the general case, let $B_k \subseteq A_k$ be the image of $A_i \to A_k$ for large i. The maps $B_{k+1} \to B_k$ are all onto, so $\varprojlim^1 B_k = 0$ by Lemma 38. The tower $(A_k/B_k)_{k<\omega}$ satisfies the trivial Mittag-Leffler condition, so $\varprojlim^1 A_k/B_k = 0$. From the short exact sequence

$$0 \longrightarrow (B_i)_{i<\omega} \longrightarrow (A_i)_{i<\omega} \longrightarrow (A_i/B_i)_{i<\omega} \longrightarrow 0$$

of towers, we see that $\varprojlim^1 A_i = 0$ as claimed. □

We pass now from countable inverse systems to countable direct systems. It is well-known that every $A \in \text{Mod-}R$ is a direct limit of finitely presented modules. In the case of countably presented A, one can choose the direct system countable and of the form

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} C_2 \longrightarrow \cdots \longrightarrow C_n \xrightarrow{f_n} C_{n+1} \longrightarrow \cdots$$

for C_n finitely presented modules. There exists an analog of the map Δ from preceding paragraphs: if we define $\phi : \bigoplus_{n<\omega} C_n \to \bigoplus_{n<\omega} C_n$ with the formula $\phi\varepsilon_n = \varepsilon_n - \varepsilon_{n+1} f_n$ where $\varepsilon_n : C_n \to \bigoplus_{n<\omega} C_n$ denotes the canonical inclusion, then ϕ fits into the (pure) exact sequence

$$0 \longrightarrow \bigoplus_{n<\omega} C_n \xrightarrow{\phi} \bigoplus_{n<\omega} C_n \longrightarrow \varinjlim C_n \cong A \longrightarrow 0. \qquad (**)$$

Given a homomorphism $\gamma : \bigoplus_{n<\omega} K_n \to \bigoplus_{n<\omega} L_n$ where K_n, L_n $(n < \omega)$ are arbitrary modules, we write γ_{ij} instead of $\pi_i \gamma \varepsilon_j$ (where π_i, ε_j are canonical projection onto L_i, canonical inclusion of K_j, respectively). We say that γ is a *diagonal* map provided that $\gamma_{ij} = 0$ for $i \neq j$.

The next lemma gives a necessary condition to factor a diagonal map γ through ϕ.

Lemma 41. *In the situation $(**)$, let $(M_n)_{n<\omega}$ be right R-modules, and $\gamma : \bigoplus_{n<\omega} C_n \to \bigoplus_{n<\omega} M_n$ be a diagonal map. Assume there is $\psi : \bigoplus_{n<\omega} C_n \to \bigoplus_{n<\omega} M_n$ such that $\psi\phi = \gamma$. Then,*

($\exists$) *there exists a sequence of natural numbers $(l(m))_{m<\omega}$, with $l(m) > m$ for every $m < \omega$, satisfying the following property:*

$$\gamma_{kk} f_{k-1} f_{k-2} \cdots f_m = -\psi_{k\,k+1} f_k f_{k-1} f_{k-2} \cdots f_m,$$

for all $k \geq l(m)$.

Proof. Fix $m < \omega$. Since γ is diagonal and $\psi\phi = \gamma$, we have $(\psi\phi)_{kj} = 0$ whenever $k \neq j$. Thus, if $k > m$, $\psi_{kj} = \psi_{k\,j+1} f_j$ for every $m \leq j < k$. Hence,

$$\psi_{km} = \psi_{kk} f_{k-1} f_{k-2} \cdots f_m$$

and $(\psi\phi)_{kk} = \gamma_{kk}$ yields

$$\psi_{kk} = \psi_{k\,k+1} f_k + \gamma_{kk}.$$

Since C_m is finitely generated, there exists an index $l(m) > m$ such that $\psi_{km} = 0$ for every $k \geq l(m)$. Thus, from the formulas above, we obtain $-\psi_{k\,k+1} f_k f_{k-1} f_{k-2} \cdots f_m = \gamma_{kk} f_{k-1} f_{k-2} \cdots f_m$, for every $k \geq l(m)$. So the sequence $(l(m))_{m<\omega}$ satisfies condition ($\exists$). □

The following proposition puts together factoring of diagonal maps through ϕ with the Mittag-Leffler condition.

Proposition 42. *In the situation* $(**)$*, let* M *be a right* R*-module. For each* $n < \omega$*, we set* $M_n \cong M$*. Assume that for every diagonal homomorphism* $\gamma : \bigoplus_{n<\omega} C_n \to \bigoplus_{n<\omega} M_n$ *there exists a homomorphism* $\psi : \bigoplus_{n<\omega} C_n \to \bigoplus_{n<\omega} M_n$ *such that* $\psi\phi = \gamma$*. Then for every* $m < \omega$*, the chain of subgroups of* $\mathrm{Hom}_R(C_m, M)$

$$\mathrm{Hom}_R(C_{m+1}, M)f_m \supseteq \mathrm{Hom}_R(C_{m+2}, M)f_{m+1}f_m \supseteq \cdots$$
$$\cdots \supseteq \mathrm{Hom}_R(C_{m+n}, M)f_{m+n-1}f_{m+n-2}\cdots f_m \supseteq \cdots$$

is stationary.

The latter is equivalent to the assertion: the inverse system of abelian groups

$$\cdots \longrightarrow \mathrm{Hom}_R(C_2, M) \xrightarrow{\mathrm{Hom}_R(f_1,M)} \mathrm{Hom}_R(C_1, M) \xrightarrow{\mathrm{Hom}_R(f_0,M)} \mathrm{Hom}_R(C_0, M)$$

satisfies the Mittag-Leffler condition.

Proof. As for the equivalence, observe that for every $m < \omega$ and for any $k > m$

$$\mathrm{Hom}_R(f_m, M)\cdots \mathrm{Hom}_R(f_{k-1}, M)\,\mathrm{Hom}_R(C_k, M) = \mathrm{Hom}_R(C_k, M)f_{k-1}\cdots f_m.$$

Now, assume by way of contradiction that there exists an $m < \omega$ for which the chain is not stationary. Then, there exists an infinite set $N \subseteq \omega$ such that, for any $n \in N$, there is a map $\alpha_n \in \mathrm{Hom}_R(C_n, M)$ such that $\alpha_n f_{n-1}f_{n-2}\cdots f_m \notin \mathrm{Hom}_R(C_{n+1}, M)f_nf_{n-1}\cdots f_m$. Consider the diagonal homomorphism $\alpha : \bigoplus_{n<\omega} C_n \to \bigoplus_{n<\omega} M_n$ defined by $\alpha_{nn} = \alpha_n$ for $n \in N$ and $\alpha_{nn} = 0$ otherwise. By hypothesis, α factors through ϕ, hence, by Lemma 41, there exists a positive integer $l(m) > m$ such that for all $k \geq l(m)$, $\alpha_{kk}f_{k-1}f_{k-2}\cdots f_m \in \mathrm{Hom}_R(C_{k+1}, M)f_kf_{k-1}f_{k-2}\cdots f_m$ contradicting the choice of the infinite family $(\alpha_n)_{n<\omega}$. □

Now, we are going to prove a technical lemma which brings pure submodules on the scene.

Lemma 43. *Let* C_1 *and* C_2 *be finitely generated right* R*-modules such that* C_2 *is finitely presented. Let* Y, Z *be right* R*-modules such that* Y *is a pure submodule of* Z*, and let* $\varepsilon : Y \to Z$ *denote the inclusion. Let* $f : C_1 \to C_2$ *and* $h : C_1 \to Y$ *be homomorphisms of right* R*-modules. If* $\hat{g} : C_2 \to Z$ *is such that* $\varepsilon h = \hat{g}f$*, then there exists* $g : C_2 \to Y$ *with* $h = gf$*.*

Proof. We use the description of pure submodules from Lemma 3 (ii). Let us fix a presentation of C_2,

$$0 \longrightarrow K \xrightarrow{\subseteq} R^m \xrightarrow{\pi} C_2 \longrightarrow 0,$$

with $m < \omega$ and K finitely generated. Choose generators $k_i = \sum_{j<m} 1_j r_{ij}$, $i < n$, of K (where 1_j is the j-th canonical generator of R^m, $r_{ij} \in R$ and $n < \omega$). Let $d_i = \sum_{j<m} 1_j s_{ij}$, for $i < l$, $s_{ij} \in R$ and $l < \omega$, be elements of R^m such that $\pi(d_i) = f(c_i)$ where $(c_i \mid i < l)$ generates C_1. The existence of $\hat{g}$ is equivalent to saying that the finite system of R-linear equations in variables $x_0, x_1, \ldots, x_{m-1}$,

$$\sum_{j<m} x_j r_{ij} = 0 \qquad (i < n) \tag{1}$$

$$\sum_{j<m} x_j s_{ij} = h(c_i) \quad (i < l), \tag{2}$$

has a solution in Z. Let $(a_0, a_1, \dots, a_{m-1})$ be a solution in Y guaranteed by Lemma 3 (ii). Then the map $e : R^m \to Y$ defined by the formula $e(1_j) = a_j$ (for each $j < m$) is zero on K by (1); thus there exists $g : C_2 \to Y$ such that $g\pi = e$. Finally, $h = gf$ holds by (2). $\square$

Although it might not look like, what follows is one of the most important short steps on our way to proving that all tilting modules are of finite type.

Proposition 44. *Let Y be a pure submodule of a right R-module Z. In the situation $(**)$, assume that for every diagonal map $\gamma : \bigoplus_{n<\omega} C_n \to Z^{(\omega)}$ there is $\psi : \bigoplus_{n<\omega} C_n \to Z^{(\omega)}$ such that $\psi\phi = \gamma$. Then the inverse system of abelian groups*

$$\cdots \longrightarrow \mathrm{Hom}_R(C_2, Y) \xrightarrow{\mathrm{Hom}_R(f_1, Y)} \mathrm{Hom}_R(C_1, Y) \xrightarrow{\mathrm{Hom}_R(f_0, Y)} \mathrm{Hom}_R(C_0, Y)$$

satisfies the Mittag-Leffler condition.

Proof. By Proposition 42, for every $m < \omega$, the chain of subgroups of $\mathrm{Hom}_R(C_m, Z)$

$$\mathrm{Hom}_R(C_{m+1}, Z)f_m \supseteq \mathrm{Hom}_R(C_{m+2}, Z)f_{m+1}f_m \supseteq \cdots$$

$$\cdots \supseteq \mathrm{Hom}_R(C_{m+n}, Z)f_{m+n-1}f_{m+n-2} \cdots f_m \supseteq \cdots$$

is stationary. Let $\varepsilon : Y \to Z$ be the inclusion. By Lemma 43,

$$\mathrm{Hom}_R(C_{m+n}, Z)f_{m+n-1}f_{m+n-2} \cdots f_m \cap \varepsilon \,\mathrm{Hom}_R(C_m, Y) =$$

$$= \varepsilon \,\mathrm{Hom}_R(C_{m+n}, Y)f_{m+n-1}f_{m+n-2} \cdots f_m,$$

and we see that the corresponding chain of subgroups of $\mathrm{Hom}_R(C_m, Y)$ is stationary too. $\square$

We finish this part by formulating the theorem that could be of independent interest.

Theorem 45. *Let R be a ring. Let $\mathcal{C}$ be a class of right R-modules satisfying that $M^{(\omega)} \in \mathcal{C}$ whenever $M \in \mathcal{C}$. If A is a countably presented right R-module such that $\mathrm{Ext}^1_R(A, M) = 0$ for any $M \in \mathcal{C}$, then $\mathrm{Ext}^1_R(A, N) = 0$ for any right R-module N isomorphic to a pure submodule of a module in $\mathcal{C}$.*

Proof. Consider the presentation $(**)$ of A. Let $M \in \mathcal{C}$, and let N be a pure submodule of M. Since $M^{(\omega)} \in \mathcal{C}$, $\mathrm{Ext}^1_R(A, M^{(\omega)}) = 0$ by hypothesis. Thus, for every homomorphism $\gamma : \bigoplus_{n<\omega} C_n \to M^{(\omega)}$ there exists $\psi : \bigoplus_{n<\omega} C_n \to M^{(\omega)}$ such that $\psi\phi = \gamma$. By Proposition 44, the inverse system of abelian groups $(\mathrm{Hom}_R(C_n, N), \mathrm{Hom}_R(f_n, N))_{n<\omega}$ is Mittag-Leffler.

As modules C_n are finitely presented, when we apply the functor $\mathrm{Hom}_R(C_n, -)$ to the pure exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0,$$

we obtain an inverse system of exact sequences of the form

$$0 \longrightarrow \mathrm{Hom}_R(C_n, N) \longrightarrow \mathrm{Hom}_R(C_n, M) \longrightarrow \mathrm{Hom}_R(C_n, M/N) \longrightarrow 0.$$

As $(\mathrm{Hom}_R(C_n, N), \mathrm{Hom}_R(f_n, N))_{n<\omega}$ is Mittag-Leffler, we can apply Proposition 40 to conclude, that there is an exact sequence

$$0 \longrightarrow \varprojlim \mathrm{Hom}_R(C_n, N) \longrightarrow \varprojlim \mathrm{Hom}_R(C_n, M) \longrightarrow \varprojlim \mathrm{Hom}_R(C_n, M/N) \longrightarrow 0,$$

which in turn gives the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(A, N) \longrightarrow \mathrm{Hom}_R(A, M) \longrightarrow \mathrm{Hom}_R(A, M/N) \longrightarrow 0.$$

Therefore, we also have the exact sequence $0 \longrightarrow \mathrm{Ext}^1_R(A, N) \longrightarrow \mathrm{Ext}^1_R(A, M) = 0$, from which we deduce that $\mathrm{Ext}^1_R(A, N) = 0$ as desired. □

2.3.3 Finite type

The forthcoming proposition reveals the key role pure submodules play in the quest for the main result of the second section.

Proposition 46. *Let $(\mathcal{A}, \mathcal{B})$ be an n-tilting cotorsion pair. Assume that every countably generated module $A \in \mathcal{A}$ is isomorphic to a direct limit of some modules in $\mathcal{A}^{<\aleph_0}$. Then the following are equivalent:*

(1) *the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is of finite type;*

(2) *$\mathcal{B}$ is closed under pure submodules;*

(3) *$\mathcal{B}$ is a definable class.*

Proof. The implication (3) ⇒ (2) is trivial. For the converse, recall that $\mathcal{B}$ is a coresolving class closed under direct sums. So if $\mathcal{B}$ is closed under pure submodules, then it is also closed under direct limits. $\mathcal{B}$ is always closed under products, thus (2) implies (3).

(1) ⇒ (2) is clear.

(3) ⇒ (1). First of all recall that, by Theorem 15, every n-tilting cotorsion pair is of countable type. Hence $\mathcal{B} = (\mathcal{A}^{<\aleph_1})^\perp$. Let $\mathcal{B}' = (\mathcal{A}^{<\aleph_0})^\perp$; then $\mathcal{B}'$ is a definable class containing $\mathcal{B}$. Every definable class is closed under ultrapowers. Since M is an elementary substructure of $PE(M)$ for every module M, $PE(M)$ is an elementary substructure of some ultrapower of M by Frayen's Theorem. Thus every definable class is fully determined by the pure-injective modules it contains.

Let M be a pure-injective module in $\mathcal{B}'$ and let $A \in \mathcal{A}^{<\aleph_1}$. By hypothesis, $A \cong \varinjlim C_n$, $C_n \in \mathcal{A}^{<\aleph_0}$. Then, from a well-known result by Auslander, $\mathrm{Ext}^i_R(A, M) \cong \varprojlim \mathrm{Ext}^i_R(C_n, M) = 0$. Hence $M \in \mathcal{B}$, and we conclude that $\mathcal{B} = \mathcal{B}'$. □

We are now in a position to prove our main result.

Theorem 47. *Let R be any ring and T be an n-tilting R-module, $n \geq 0$. Then T is of finite type.*

Proof. The proof is by induction on the projective dimension n of T.

If $n = 0$, the conclusion is obvious. Next, assume that all m-tilting modules are of finite type for every $m < n$. Let T be a tilting module of projective dimension n and let $(\mathcal{A}, \mathcal{B})$ be the n-tilting cotorsion pair induced by T. By Theorem 15, $(\mathcal{A}, \mathcal{B})$ is of countable type, hence $\mathcal{B} = (\mathcal{A}^{<\aleph_1})^\perp$.

Consider a free presentation of T,

$$0 \longrightarrow \Omega(T) \longrightarrow F \longrightarrow T \longrightarrow 0,$$

and let $(\mathcal{A}_1, \mathcal{B}_1)$ be the cotorsion pair with $\mathcal{B}_1 = \Omega(T)^{\perp}$. By Lemma 36, $(\mathcal{A}_1, \mathcal{B}_1)$ is an $(n-1)$-tilting cotorsion pair; so it is of finite type by inductive hypothesis. In particular $\mathcal{B}_1 = (\mathcal{A}_1^{<\aleph_0})^{\perp}$.

Let $A \in \mathcal{A}$ be a countably generated module. Then, we can apply Lemma 37 to conclude that A is a direct limit of modules from $\mathcal{A}^{<\aleph_0}$. By Proposition 46, for $(\mathcal{A}, \mathcal{B})$ to be of finite type, it suffices to show that $\mathcal{B}$ is closed under pure submodules. But this closure property is assured by Theorem 45. □

3 Telescope conjecture for module categories

In the late 1970's, Bousfield formulated the telescope conjecture for compactly generated triangulated categories $\mathcal{T}$. The conjecture said that any smashing localizing subcategory $\mathcal{L}$ of $\mathcal{T}$ is of finite type, cf. [13], [25]. Keller [22] gave an example disproving the conjecture in the case when $\mathcal{T}$ is the (unbounded) derived category of the module category over a particular (non-noetherian) commutative ring.

However, it appears open whether the conjecture holds true when $\mathcal{T}$ is the stable module category of a self-injective artin algebra R. In this case, the conjecture was shown to be equivalent to a certain property of cotorsion pairs of R-modules, cf. [23, §7]. This lead Krause and Solberg to the following version of the telescope conjecture for module categories of arbitrary artin algebras:

[23, 7.9] "Let R be an artin algebra, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair with $\mathcal{A}$ and $\mathcal{B}$ closed under direct limits. Then $\mathcal{A} = \varinjlim(\mathcal{A} \cap \text{mod-}R)$."

We will deal with the following general version of the Krause-Solberg conjecture, formulated for arbitrary rings:

Conjecture 48. *Let R be a ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ a complete hereditary cotorsion pair with $\mathcal{A}$ and $\mathcal{B}$ closed under direct limits. Then $\mathcal{A} = \varinjlim \mathcal{A}^{<\aleph_0}$.*

We are going to prove that Conjecture 48 holds if and only if $\mathfrak{C}$ is of finite type. The if part has been known for some two or three years; thus the consequence of our second section dealing with tilting modules is that Conjecture 48 holds for every n-tilting cotorsion pair with $\mathcal{A}$ closed under direct limits. However, the only-if part is rather recent result.

We will also show that Conjecture 48 holds provided R is right noetherian ring and $\mathcal{B}$ consists of modules with bounded injective dimension. This will, in particular, solve the noetherian case of n-cotilting cotorsion pairs.

3.1 Preliminaries

Let $\mathcal{M}$ be a subcategory of Mod-R. Unless explicitly said otherwise, we always assume that $\mathcal{M}$ is *full and that it is closed under direct summands, finite direct sums and isomorphic images.*

We denote by $\operatorname{Add} \mathcal{M}$ (respectively $\operatorname{add} \mathcal{M}$) the subcategory of all modules isomorphic to a direct summand of a (finite) direct sum of modules of $\mathcal{M}$, and by $\operatorname{Prod} \mathcal{M}$ the subcategory of all modules isomorphic to a direct summand of a product of modules of $\mathcal{M}$. If $\mathcal{M} = \{M\}$, we write $\operatorname{Add} M$, $\operatorname{add} M$, $\operatorname{Prod} M$.

Let us return to pure-injective modules for a while. An infinite direct sum of copies of a pure-injective module needs not be a pure-injective module in general. For example, $PE(\mathbb{Z}_{(p)})^{(\omega)}$, where $\mathbb{Z}_{(p)}$ is a localization of $\mathbb{Z}$ at the prime p, is not pure-injective abelian group. However, we say that a module M is Σ-*pure-injective* provided that $M^{(\kappa)}$ is pure-injective for all cardinals κ. There is a characterization of Σ-pure-injective modules by Zimmermann and Gruson-Jensen:

Lemma 49. *Let R be a ring and M be a module. The following are equivalent:*

(i) *M is Σ-pure-injective.*

(ii) *There is a cardinal κ such that any direct product of copies of M is a direct sum of modules having cardinality $\leq \kappa$.*

(iii) *There is a cardinal λ such that any direct product of copies of M is a pure submodule in a direct sum of modules having cardinality $\leq \lambda$.*

We recall that a module M with $\operatorname{Add} M$ being closed under products is said to be *product-complete*. Note that M is product-complete iff $\operatorname{Add} M = \operatorname{Prod} M$. Moreover, every product-complete module is Σ-pure-injective by Lemma 49.

Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair. We have already observed that the class $\mathcal{A}$ is always closed under direct sums, and $\mathcal{B}$ is always closed under products. We will say that $\mathfrak{C}$ is *smashing* if $\mathcal{B}$ is also closed under direct sums, and *cosmashing* if $\mathcal{A}$ is also closed under products.

3.2 Telescope conjecture and cotorsion pairs of finite type

This subsection is devoted to proving the beforehand stated equivalence:

Theorem 50. *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ satisfy hypotheses of Conjecture 48. Then $\mathcal{A} = \varinjlim \mathcal{A}^{<\aleph_0}$ if and only if $\mathfrak{C}$ is of finite type.*

3.2.1 Lenzing's result and sufficiency of finite type

Let us begin with the converse implication from Theorem 50. A well-known result of Lenzing, characterizing modules isomorphic to a direct limit of modules from mod-R, appears to be useful in our setting.

Lemma 51. *Assume that $\mathcal{M}$ is a subcategory of* mod-R. *Then the following statements are equivalent for a module A.*

(1) *$A \in \varinjlim \mathcal{M}$.*

(2) *There is a pure epimorphism $\bigoplus_{k\in K} X_k \to A$ for some modules $X_k \in \mathcal{M}$.*

(3) *Every homomorphism $h : F \to A$, where F is finitely presented, factors through a module in $\mathcal{M}$.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear. Let us show (3) $\Rightarrow$ (1).

Take finitely presented modules $(M_i)_{i\in I}$ such that $A = \varinjlim M_i$ with canonical maps $\varphi_{ji} : M_i \to M_j$, $i \leq j$, and $\varphi_i : M_i \to A$.
Claim 1: For all $i \in I$ there is $k \geq i$ such that φ_{ki} factors through a module from $\mathcal{M}$.
Proof: Let $i \in I$. By (3) there are $C \in \mathcal{M}$ and homomorphisms $s : M_i \to C$, $t : C \to A$ such that $\varphi_i = ts$. Note that C is finitely presented and therefore $\varinjlim \operatorname{Hom}_R(C, M_i) \cong \operatorname{Hom}_R(C, A)$. Since I is directed, there are $j \geq i$ and $u \in \operatorname{Hom}_R(C, M_j)$ such that $t = \varphi_j u$. Then $\varphi_j(us - \varphi_{ji}) = 0$, and since M_i is finitely generated, $\varphi_{kj}(us - \varphi_{ji}) = 0$ for some $k \geq j$. Thus $\varphi_{ki} = \varphi_{kj} us$ factors through C.

Our statement now follows from
Claim 2: Assume $A = \varinjlim((M_i)_{i\in I}, (\varphi_{ji})_{i\leq j})$ with canonical maps $\varphi_{ji} : M_i \to M_j$, $i \leq j$, and $\varphi_i : M_i \to A$. Assume further that there is a map $\sigma : I \to I$ such that, for all

$i \in I$, we have $\sigma(i) \geq i$ and factorizations $\varphi_{\sigma(i)i} = \beta_i\alpha_i$ where $\alpha_i : M_i \to N_i$ and $\beta_i : N_i \to M_{\sigma(i)}$. Then $A \cong \varinjlim N_i$.
Proof: Denote by J the set I ordered by putting $i \preceq j$ whenever $i = j$ or $\sigma(i) \leq j$. This is well-defined because for all $i \in I$ we have $\sigma(i) \geq i$. Now, for $i \preceq j$, define $\psi_{ji} : N_i \to N_j$ by setting $\psi_{ji} = \mathrm{id}_{N_i}$ if $i = j$, and $\psi_{ji} = \alpha_j\varphi_{j\sigma(i)}\beta_i$ if $i < j$. Then we obtain a direct system $((N_i)_{i\in J}, (\psi_{ji})_{i\preceq j})$, and the homomorphisms $\rho_i = \varphi_{\sigma(i)}\beta_i : N_i \to A$, $i \in J$, induce the desired isomorphism. □

Now, the promised implication in Theorem 50 will result from a particular case of [7, Theorem 2.3]:

Proposition 52. *Let R be a ring. Let $\mathcal{C}$ be a subcategory of* mod-R *such that $R \in \mathcal{C}$, and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair cogenerated by $\mathcal{C}$. Then $\mathcal{A} \subseteq \varinjlim \mathcal{C}$.*

Proof. First, the isomorphism classes of $\mathcal{C}$ form a set, so $\mathcal{A}$ consists of all direct summands of $\mathcal{C}$-filtered modules (see Theorem 2 (ii)). By Lemma 51, $\varinjlim \mathcal{C}$ is closed under pure epimorphic images, hence under direct summands. So it suffices to prove that $\varinjlim \mathcal{C}$ contains all $\mathcal{C}$-filtered modules.

We proceed by induction on the length, δ, of the filtration. The cases $\delta = 0$ and δ is a limit ordinal are clear (the latter by Lemma 51). Let δ be non-limit, so we have an exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ with $A \in \varinjlim \mathcal{C}$ and $C \in \mathcal{C}$. We will apply Lemma 51 to prove that $B \in \varinjlim \mathcal{C}$.

Let $h : F \to B$ be a homomorphism with F finitely presented. Since $C \in$ mod-R, there is a presentation $0 \longrightarrow G \longrightarrow P \xrightarrow{p} C \longrightarrow 0$ with P finitely generated projective and G finitely presented. There is also $q : P \to B$ such that $p = gq$. We have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F' & \xrightarrow{f'} & F \oplus P & \xrightarrow{(gh)\oplus p} & C & \longrightarrow & 0 \\
 & & {\scriptstyle h'}\downarrow & & {\scriptstyle h\oplus q}\downarrow & & \| & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0
\end{array}$$

Considering the pullback of p and $(gh) \oplus p$, we see that the pullback module U is an extension of G by $F \oplus P$, and F' is isomorphic to a direct summand in U. So U and F' are finitely presented. Since $A \in \varinjlim \mathcal{C}$, Lemma 51 provides for a module $C' \in \mathcal{C}$ and maps $\sigma' : F' \to C'$, $\tau' : C' \to A$ such that $h' = \tau'\sigma'$. Consider the pushout of f' and σ':

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F' & \xrightarrow{f'} & F \oplus P & \xrightarrow{(gh)\oplus p} & C & \longrightarrow & 0 \\
 & & {\scriptstyle \sigma'}\downarrow & & {\scriptstyle \sigma}\downarrow & & \| & & \\
0 & \longrightarrow & C' & \xrightarrow{\rho} & D & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

By assumption, $D \in \mathcal{C}$. By the pushout property, there is $\tau : D \to B$ such that $\tau\sigma = h \oplus q$, hence $\tau\sigma \restriction F = h$. So h factors through D, and $B \in \varinjlim \mathcal{C}$. □

3.2.2 Necessity of finite type

Prior to dealing with the direct implication from Theorem 50, we bring together some preliminary results.

Proposition 53. *Let R be a ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion pair cogenerated by a class $\mathcal{C}$ of countably presented modules. Assume that $B^{(\omega)} \in \mathcal{B}$ whenever $B \in \mathcal{B}$. Then:*

(1) *$\mathfrak{C}$ is smashing, and $\mathcal{B}$ is closed under pure submodules.*

(2) *If $\mathfrak{C}$ is hereditary then $\mathcal{B}$ is definable.*

(3) *If $\mathcal{C} \subseteq \varinjlim \mathcal{A}^{<\aleph_0}$ then $\mathfrak{C}$ is of finite type.*

Proof. (1) First, by Theorem 45, $\mathcal{B}$ is closed under pure submodules. Since $\mathcal{B}$ is closed under arbitrary direct products, and direct sums are pure submodules in direct products, we infer that $\mathfrak{C}$ is smashing.

(2) Since $\mathcal{B}$ is coresolving, (1) also implies that $\mathcal{B}$ is closed under pure-epimorphic images, thus in particular under direct limits. This shows that $\mathcal{B}$ is definable.

(3) It suffices to verify that $\mathcal{B} = (\mathcal{A}^{<\aleph_0})^{\perp_1}$. Clearly $\mathcal{B} \subseteq (\mathcal{A}^{<\aleph_0})^{\perp_1}$. For the reverse inclusion, we first show that the classes $\mathcal{B}$ and $(\mathcal{A}^{<\aleph_0})^{\perp_1}$ contain the same pure-injective modules. Indeed, for any pure-injective module I, the functor $\operatorname{Ext}^1_R(-, I)$ takes direct limits to inverse ones. So, the assumption $\mathcal{C} \subseteq \varinjlim \mathcal{A}^{<\aleph_0}$ implies that any pure-injective module $I \in (\mathcal{A}^{<\aleph_0})^{\perp_1}$ belongs to $\mathcal{B}$. Now, let $M \in (\mathcal{A}^{<\aleph_0})^{\perp_1}$, and let P be the pure-injective envelope of M. Since the class $(\mathcal{A}^{<\aleph_0})^{\perp_1}$ is definable, $P \in (\mathcal{A}^{<\aleph_0})^{\perp_1}$. But then $P \in \mathcal{B}$, and thus $M \in \mathcal{B}$ since M is a pure submodule of P. This proves that $\mathcal{B} = (\mathcal{A}^{<\aleph_0})^{\perp_1}$. □

In view of Proposition 53, our strategy will consists in proving that every cotorsion pair $(\mathcal{A}, \mathcal{B})$ satisfying Conjecture 48 is cogenerated by the class of countably presented modules from $\mathcal{A}$. To this end, we need results which enable us to filter modules from $\mathcal{A}$ by "smaller" modules which still belong to $\mathcal{A}$. The following two lemmas are the first step in this direction.

Lemma 54. *Let C be an injective cogenerator in* Mod-R. *Define $F(X) = C^{\operatorname{Hom}_R(X,C)}$ and $F(\varphi)(f) = f(- \circ \varphi)$, for all $X, Y \in$* Mod-R, *every $\varphi \in \operatorname{Hom}_R(X, Y)$ and $f \in F(X)$. Then F is an endofunctor of* Mod-R *preserving monomorphisms. Moreover, the family $\iota = (\iota_X \mid X \in \text{Mod-}R)$ consisting of canonical embeddings $\iota_X : X \to F(X)$ is a natural transformation from the identity functor to F.*

Proof. It is straightforward to check that F is a functor and ι is a natural transformation. ι_X is an embedding since C is a cogenerator, and the injectivity of C implies that F preserves monomorphisms. □

Lemma 55. *Let R be an arbitrary ring, and $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathcal{B}$ is closed under direct limits. Let λ be a regular uncountable cardinal, $\kappa \geq \lambda$, $A \in \mathcal{A}$ a κ-presented module, and X a subset of A with* $\operatorname{card} X < \lambda$. *Then there exists a $< \lambda$-presented module $\bar{X}$ such that $X \subseteq \bar{X} \subseteq A$. Moreover, $\bar{X}$ can be taken of the form $\pi(R^{(I)})$ where $\pi : R^{(\kappa)} \to A$ is an epimorphism and I is a subset of κ of cardinality $< \lambda$.*

Proof. By assumption, A has a presentation

$$0 \longrightarrow K \xrightarrow{\subseteq} R^{(\kappa)} \xrightarrow{\pi} A \longrightarrow 0$$

with $\operatorname{gen}(K) \leq \kappa$, and there is $I_0 \subseteq \kappa$ of cardinality $< \lambda$ such that $X \subseteq \pi(R^{(I_0)})$. Let $\mathcal{L}$ be the set consisting of all $< \lambda$-generated submodules of K. We claim that $K \cap R^{(I_0)} \subseteq L_0$ for some $L_0 \in \mathcal{L}$.

Let $\mathcal{D} = \{\langle L', L\rangle \in \mathcal{L} \times \mathcal{L} \mid L \nsubseteq L'\}$. Using the notation from Lemma 54, for each $\langle L', L\rangle \in \mathcal{D}$, we define $\tau_{\langle L',L\rangle} : L \to F((L+L')/L')$ as the composition of the canonical projection $L \to (L+L')/L'$ with the embedding $\iota_{(L+L')/L'}$. For $L \in \mathcal{L}$, put $\mathcal{L}_L = \{L' \in \mathcal{L} \mid \langle L', L\rangle \in \mathcal{D}\}$. Note that for every $L, \tilde{L} \in \mathcal{L}$, $L \subseteq \tilde{L}$ implies $\mathcal{L}_L \subseteq \mathcal{L}_{\tilde{L}}$.

Now, for each $L \in \mathcal{L}$, we put

$$G(L) = \prod_{L' \in \mathcal{L}_L} F((L+L')/L'),$$

notice that $G(L) \in \mathcal{I}_0$, and for every $\varepsilon : L \subseteq \tilde{L}$ ($\in \mathcal{L}$), we define

$$G(\varepsilon) = \prod_{L' \in \mathcal{L}_L} F(\varepsilon_{L'})$$

where $\varepsilon_{L'}$ is the inclusion $(L+L')/L' \subseteq (\tilde{L}+L')/L'$. Then G is a functor from the small category $\mathcal{L}$, morphisms of which are just inclusions, to Mod-R. Moreover, G preserves monomorphisms (since F does), and there is the natural transformation $\tau = (\tau_L \mid L \in \mathcal{L})$ from the canonical emdedding $\mathcal{L} \hookrightarrow$ Mod-R to G where τ_L is a fibred product of $(\tau_{\langle L',L\rangle} \mid L' \in \mathcal{L}_L)$: it is routine to check that the square

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\tau_{\tilde{L}}} & G(\tilde{L}) \\ {\scriptstyle\varepsilon}\big\uparrow & & \big\uparrow{\scriptstyle G(\varepsilon)} \\ L & \xrightarrow{\tau_L} & G(L) \end{array}$$

commutes for each $L, \tilde{L} \in \mathcal{L}$ and $\varepsilon : L \subseteq \tilde{L}$ (one needs the fact that ι is a natural transformation).

Let E be a direct limit of the directed system $G(\mathcal{L})$. For every $L \in \mathcal{L}$, denote by ν_L the colimit injection $G(L) \hookrightarrow E$. Since K is a directed union of its $< \lambda$-generated submodules, it follows from the preceding paragraph that there exists the unique homomorphism $f : K \to E$ such that $f \restriction L = \nu_L \tau_L$ for all $L \in \mathcal{L}$. Notice that $\mathcal{L}$ is λ-directed since λ is a regular cardinal, so $G(\mathcal{L})$ has the same property.

Using the assumption put on $\mathcal{B}$, we have $E \in \mathcal{B}$, which allows us to extend f to some $g : R^{(\kappa)} \to E$. Since $\operatorname{card} I_0 < \lambda$ and $G(\mathcal{L})$ is λ-directed, there exists $L_0 \in \mathcal{L}$ such that $g \restriction R^{(I_0)}$ factorizes through ν_{L_0}. We deduce then that $K \cap R^{(I_0)} \subseteq L_0$; if not, there exist $x \in K \cap R^{(I_0)}$ and $L \in \mathcal{L}$ such that $x \in L \setminus L_0$, whence $\tau_{\langle L_0, L+L_0\rangle}(x) \neq 0 \neq \tau_{L+L_0}(x)$ contradicting $f \restriction (K \cap R^{(I_0)})$ being factorized through ν_{L_0}. Our claim is proved.

Since L_0 is a $< \lambda$-generated module, $L_0 \subseteq R^{(I_1)}$ for some $I_0 \subseteq I_1 \subseteq \kappa$ with $\operatorname{card} I_1 < \lambda$. Iterating this construction, we obtain a set $I = \bigcup_{n<\omega} I_n$ such that $K \cap R^{(I)} = L$ for some $L \in \mathcal{L}$, and $\bar{X} = \pi(R^{(I)}) \cong R^{(I)}/L$ has the desired properties. □

Lemma 56. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair such that $\mathcal{A} = \varinjlim \mathcal{A}^{<\aleph_0}$. Let λ be a regular uncountable cardinal, $\kappa \geq \lambda$, $A \in \mathcal{A}$ a κ-presented module, and X be a subset of A of cardinality $< \lambda$. Assume that either*

(i) *R is a right $\aleph_0$-noetherian ring, or*

(ii) *$\mathcal{B}$ is closed under direct limits.*

Then there is a $< \lambda$-presented module $A' \in \mathcal{A}$ such that $X \subseteq A' \subseteq A$.

Proof. *Step 1:* For any $<\lambda$-presented submodule B of A, we construct a $<\lambda$-generated submodule B' of A containing B with the property that any homomorphism of the form $D \xrightarrow{h} B \subseteq B'$ with D finitely presented factors through a module in $\mathcal{A}^{<\aleph_0}$.

To this end, we fix a pure-exact sequence $0 \longrightarrow \operatorname{Ker}(\pi) \longrightarrow \bigoplus_{i\in I} D_i \xrightarrow{\pi} B \longrightarrow 0$ with D_i finitely presented for all $i \in I$. Since B is $<\lambda$-presented, we will w.l.o.g. assume that $\operatorname{card} I < \lambda$. For F a non-empty finite subset of I, let $D_F = \bigoplus_{i\in F} D_i$, and $\pi_F = \pi \restriction D_F$. By induction on $\operatorname{card} F$, we define finitely generated modules $A_F \in \mathcal{A}^{<\aleph_0}$ and $C_F \subseteq A$ such that there is a commutative diagram

$$\begin{array}{ccc} D_F & \xrightarrow{\pi_F} & B \\ {\scriptstyle f_F}\big\downarrow & & \big\downarrow{\scriptstyle \subseteq} \\ A_F & \xrightarrow{g_F} & A \end{array}$$

and $\pi(D_F) \subseteq C_F = \operatorname{Im}(g_F)$. Hereby we proceed as follows:

If $\operatorname{card} F = 1$, then the existence of A_F and C_F follows immediately from Lemma 51 since $A \in \varinjlim \mathcal{A}^{<\aleph_0}$.

If $\operatorname{card} F > 1$, we take $M = D_F \oplus \bigoplus_{\emptyset\neq G\subsetneq F} A_G$ and let $g = \pi_F \oplus \bigoplus_{\emptyset\neq G\subsetneq F} g_G$. By Lemma 51, there exist $A_F \in \mathcal{A}^{<\omega}$, $h_F : M \to A_F$ and $g_F : A_F \to A$ such that $g = g_F h_F$, and we put $C_F = \operatorname{Im}(g_F)$ and $f_F = h_F \restriction D_F$. Note that C_F contains C_G for each $\emptyset \neq G \subsetneq F$.

Now let B' be the union of all C_F where F runs through all non-empty finite subsets of I. This is a directed union of $<\lambda$-many finitely generated submodules of A, so B' is a $<\lambda$-generated submodule of A containing B. Moreover, if $h : D \to B$ is a homomorphism with D finitely presented, then there is a factorization f of h through the pure epimorphism π. But then $\operatorname{Im}(f) \subseteq D_F$ for a non-empty finite subset $F \subseteq I$, and $D \xrightarrow{h} B \subseteq B'$, which equals $g_F f_F f$, factors through $A_F \in \mathcal{A}^{<\aleph_0}$ as required.

Step 2: Consider now the presentation of A from Lemma 55. We will define A' as the union of an increasing chain $(B_n \mid n < \omega)$ of $<\lambda$-presented submodules in A of the form $\pi(R^{(J_n)})$ for some J_n of cardinality $<\lambda$ (where $J_0 \subseteq J_1 \subseteq \cdots$). The chain will be defined by induction on n.

Take $B_0 = \pi(R^{(J_0)})$ $<\lambda$-presented and such that $X \subseteq B_0$ (this is clearly possible in case (i), and it is possible by Lemma 55 in case (ii)). If B_n is defined, there is a $<\lambda$-generated submodule B'_n of A containing B_n constructed as in Step 1. Let $B_{n+1} = \pi(R^{(J_{n+1})})$ be a $<\lambda$-presented submodule of A containing B'_n (again, obtained using the $\aleph_0$-noetherian property of R in case (i), and Lemma 55 in case (ii)).

It remains to prove that $A' \in \mathcal{A}$. By Lemma 51, it suffices to show that every R-homomorphism $h : D \to A'$ with D finitely presented has a factorization through a module in $\mathcal{A}^{<\aleph_0}$. However, $\operatorname{Im}(h) \subseteq B_n$ for some $n < \omega$, and the claim then follows by construction of B'_n in Step 1. $\square$

Theorem 57. *Let R be a ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a smashing cotorsion pair such that $\mathcal{A} = \varinjlim \mathcal{A}^{<\aleph_0}$. Assume that either*

(i) *R is a right $\aleph_0$-noetherian ring, or*

(ii) *$\mathcal{B}$ is closed under direct limits.*

Then $\mathfrak{C}$ is of finite type.

Proof. We denote by $\mathcal{A}_0$ the class of all countably presented modules in $\mathcal{A}$. Let $A \in \mathcal{A}$, and let $\kappa \geq \aleph_0$ be such that A is a κ-presented module. By induction on κ, we will prove that A is $\mathcal{A}_0$-filtered. There is nothing to prove for $\kappa = \aleph_0$.

If κ is a regular uncountable cardinal then Lemma 56 yields a κ-filtration, $\mathcal{F} = (A_\alpha \mid \alpha \leq \kappa)$, of A such that $A_\alpha \in \mathcal{A}$ is $< \kappa$-presented for each $\alpha < \kappa$. By Lemma 9 (i), there is a subfiltration, $\mathcal{G}$, of $\mathcal{F}$ such that all successive factors in $\mathcal{G}$ are $< \kappa$-presented modules from $\mathcal{A}$, so they are $\mathcal{A}_0$-filtered by inductive premise. Hence A is $\mathcal{A}_0$-filtered.

If κ is singular, we use Lemma 23. For each regular uncountable cardinal $\lambda < \kappa$, we let S_λ denote the set of all $< \lambda$-presented submodules $A' \subseteq A$ with $A' \in \mathcal{A}$. Clearly, $0 \in S_\lambda$, and S_λ is closed under unions of well-ordered chains of length $< \lambda$ since $\mathcal{A}$ is closed under arbitrary direct limits. By Lemma 56, each subset of A of cardinality $< \lambda$ is contained in an element of S_λ. By inductive premise, S_λ consists of $\mathcal{A}_0$-filtered modules for all regular $\aleph_0 < \lambda < \kappa$, so S_λ witnesses the λ-$\mathcal{A}_0$-freeness of A in the sense of Definition 21. By Lemma 23, A is $\mathcal{A}_0$-filtered. This proves that each $A \in \mathcal{A}$ is $\mathcal{A}_0$-filtered.

So, we infer from Lemma 1 that $\mathcal{B} = (\mathcal{A}_0)^{\perp_1}$. Finally, Proposition 53 shows that $\mathfrak{C}$ is of finite type. □

The proof of our first main result is on hand now.

Proof of Theorem 50. Combine Proposition 52 and Theorem 57. Notice that the assumption of $\mathcal{B}$ being closed under direct limits is not needed for the proof of the if part. □

3.3 Noetherian case and cotilting cotorsion pairs

Now, we are going to prove a particular case of Conjecture 48 for arbitrary right noetherian rings. This will imply validity of Conjecture 48 in the case when $\mathfrak{C}$ is a cotilting cotorsion pair over a right noetherian ring.

By Theorem 50, the proof of Conjecture 48 amounts to showing that $\mathfrak{C}$ is of finite type. First, we need a lemma, the dual version of which is contained in Lemma 32.

Lemma 58. *Let R be a ring, and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a smashing cotorsion pair cogenerated by a class $\mathcal{C}$ such that $\mathcal{C}^\perp$ contains all direct sums of injective modules. Then $\mathcal{C}^{\perp_n}$ is closed under arbitrary direct sums for each $n \geq 1$.*

Proof. By induction on n. The case of $n = 1$ is clear since $\mathfrak{C}$ is smashing. Let $(M_\alpha \mid \alpha < \kappa)$ be a family of modules in $\mathcal{C}^{\perp_{n+1}}$. Consider short exact sequences

$$0 \longrightarrow M_\alpha \longrightarrow I_\alpha \longrightarrow C_\alpha \longrightarrow 0$$

with I_α injective for each $\alpha < \kappa$. Since $0 = \operatorname{Ext}_R^{n+1}(A, M_\alpha) \cong \operatorname{Ext}_R^n(A, C_\alpha)$ for all $A \in \mathcal{C}$, the inductive premise gives $\bigoplus_{\alpha<\kappa} C_\alpha \in \mathcal{C}^{\perp_n}$, so our assumption on $\mathcal{C}^\perp$ yields $\bigoplus_{\alpha<\kappa} M_\alpha \in \mathcal{C}^{\perp_{n+1}}$. □

Proposition 59. *Let R be a right coherent ring. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a smashing hereditary cotorsion pair cogenerated by a class $\mathcal{C} \subseteq (\text{Mod-}R)^{\leq \aleph_0}$ and such that $\mathcal{B} \subseteq \mathcal{I}_n$ for some $n \in \omega$. Then $\mathfrak{C}$ is of finite type.*

Proof. We will construct cotorsion pairs $\mathfrak{C}_i = (\mathcal{A}_i, \mathcal{B}_i)$, $1 \leq i \leq n+1$, such that $\mathcal{B} = \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \cdots \subseteq \mathcal{B}_n \subseteq \mathcal{B}_{n+1}$ and by reverse induction on i, we will show that $\mathfrak{C}_i$ is of finite type for each $1 \leq i \leq n+1$.

Let us start with the cotorsion pair $\mathfrak{C}_{n+1} = (\mathcal{A}_{n+1}, \mathcal{B}_{n+1})$ cogenerated by the class $\mathcal{S}_{n+1}$ of all modules that are k-th syzygies of modules from mod-R for some $k \geq n$. Then $\mathcal{B}_{n+1} = \bigcap_{k \geq n+1} (\text{mod-}R)^{\perp_k}$, and we claim that $\mathcal{B}_{n+1} \subseteq \bigcap_{k \geq n+1} \mathcal{A}^{\perp_k}$. In fact, $(\text{mod-}R)^{\perp_1}$ coincides with the class of all pure submodules of injective modules since R is right coherent. Moreover, mod-R is resolving, so $(\text{mod-}R)^{\perp_1} = (\text{mod-}R)^{\perp}$. Since $\mathcal{B}$ is definable by Proposition 53 (2), we deduce that $(\text{mod-}R)^{\perp} \subseteq \mathcal{B}$, and our claim follows by dimension shifting.

We now set $\mathcal{B}_i = \mathcal{B}_{n+1} \cap \bigcap_{k \geq i} \mathcal{A}^{\perp_k}$ for $1 \leq i \leq n$. Then, as $\mathcal{B} \subseteq \mathcal{I}_n \subseteq \mathcal{B}_{n+1}$, we have $\mathcal{B} = \mathcal{B}_1$. Moreover, all $\mathcal{B}_i$ are obviously coresolving. Further, applying Lemma 58 to $\mathfrak{C}$ (which is possible because $\mathfrak{C}$ is cogenerated by $\mathcal{A}$ and $\mathcal{A}^{\perp} = \mathcal{B}$ contains all direct sums of injective modules), we infer that all $\mathcal{B}_i$ are closed under direct sums.

For each $1 \leq i \leq n$, we thus obtain a hereditary smashing cotorsion pair $\mathfrak{C}_i = (\mathcal{A}_i, \mathcal{B}_i)$ which is cogenerated by a class of countably presented modules, namely by $\mathcal{S}_i = \mathcal{S}_{n+1} \cup \mathcal{C}_i$, where $\mathcal{C}_i$ denotes the class of all modules that are k-th syzygies of modules from $\mathcal{C}$ for some $k \geq i-1$.

Of course, $\mathfrak{C}_{n+1}$ is of finite type. Let $1 \leq i \leq n$, and let $M \in \mathcal{S}_i$. We have a short exact sequence

$$0 \longrightarrow K \longrightarrow R^{(\omega)} \longrightarrow M \longrightarrow 0.$$

We claim that $K \in \mathcal{A}_{i+1}$. Indeed, if $N \in \mathcal{B}_{i+1} = \mathcal{B}_{n+1} \cap \bigcap_{k \geq i+1} \mathcal{A}^{\perp_k}$ then its first cosyzygy C belongs to $\mathcal{B}_i$, so $\operatorname{Ext}^2_R(\mathcal{A}_i, N) = 0$, and in particular, $\operatorname{Ext}^2_R(M, N) = 0$, hence $\operatorname{Ext}^1_R(K, N) = 0$. This proves the claim.

By inductive premise, $\mathfrak{C}_{i+1}$ is of finite type, hence cogenerated by $\mathcal{A}_{i+1}^{<\aleph_0}$. By Theorem 2 (ii), it follows that K is a direct summand in a $\mathcal{A}_{i+1}^{<\aleph_0}$-filtered module. Using Lemma 34, we obtain the exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow M \longrightarrow 0$$

with H and G countably generated $\mathcal{A}_{i+1}^{<\aleph_0}$-filtered modules. W.l.o.g., we can assume that H is a submodule of G. As in the proof of Lemma 37, we show that $M \in \varinjlim \mathcal{A}_i^{<\aleph_0}$. We state here the argument for the reader's convenience.

By Lemma 33, we can write $H = \bigcup_{k<\omega} H_k$ and $G = \bigcup_{k<\omega} G_k$ where, for every $k < \omega$, H_k and G_k are finitely presented $\mathcal{A}_{i+1}^{<\aleph_0}$-filtered modules, and H/H_k, G/G_k are $\mathcal{A}_{i+1}^{<\aleph_0}$-filtered. Given $k < \omega$, there is j_k such that $H_k \subseteq G_{j_k}$. Moreover, we can choose the sequence $(j_k \mid k < \omega)$ to be strictly increasing.

We claim that $G_{j_k}/H_k \in \mathcal{A}_i^{<\aleph_0}$. Clearly, G_{j_k}/H_k is finitely presented, and R right coherent implies $G_{j_k}/H_k \in \text{mod-}R$, thus we have to show that $\operatorname{Ext}^1_R(G_{j_k}/H_k, B) = 0$ for each $B \in \mathcal{B}_i$. Since $G_{j_k} \in \mathcal{A}_{i+1} \subseteq \mathcal{A}_i$, we need only to check that every $f \in \operatorname{Hom}_R(H_k, B)$ can be extended to a homomorphism from G_{j_k} to B. We have $\operatorname{Ext}^1_R(H/H_k, B) = 0$ because $H/H_k \in \mathcal{A}_{i+1}$, thus we may extend f to a homomorphism f' from H to B, and then, since $G/H \cong M \in \mathcal{A}_i$, to a homomorphism g from G to B. The restriction of g to G_{j_k} obviously induces an extension of f to G_{j_k}. Our claim is proved.

Set $C_k = G_{j_k}/H_k$. Since $(j_k \mid k < \omega)$ is increasing and unbounded in ω, the inclusions $G_{j_k} \subseteq G_{j_{k+1}}$ induce maps $f_k : C_k \to C_{k+1}$, and M is a direct limit of the direct system $((C_k, f_k) \mid k < \omega)$.

But then, since $M \in \mathcal{S}_i$ was arbitrary, it follows that $\mathcal{S}_i \subseteq \varinjlim \mathcal{A}_i^{<\aleph_0}$, and so $\mathfrak{C}_i$ is of finite type by Proposition 53 (3). $\square$

Theorem 60. *Let R be a right noetherian ring and $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a hereditary smashing cotorsion pair. If either*

(1) *$\mathcal{A}$ consists of modules of bounded projective dimension, or*

(2) *$\mathcal{B}$ consists of modules of bounded injective dimension,*

then $\mathfrak{C}$ is of finite type.

Proof. By Theorem 16 (i), (1) implies that $\mathfrak{C}$ is a tilting cotorsion pair, hence $\mathfrak{C}$ is of finite type by Theorem 47. (Indeed, this holds for an arbitrary ring R.)

Assume (2). Then it follows from Corollary 13 that $\mathfrak{C}$ is cogenerated by a class of countably presented modules, so it is of finite type by Proposition 59. $\square$

Before stating the final "cotilting corollary", we need to know a little more about smashing cotilting cotorsion pairs.

Proposition 61. *Let C be a cotilting module with corresponding cotilting cotorsion pair $(\mathcal{A}, \mathcal{B})$. Then the following statements are equivalent.*

(1) *$(\mathcal{A}, \mathcal{B})$ is smashing.*

(2) *C is Σ-pure-injective.*

(3) *There is a product-complete cotilting module C' such that $\mathcal{A} = {}^{\perp}C'$.*

Proof. We know from [3, Lemma 2.4] that $\operatorname{Prod} C = \mathcal{A} \cap \mathcal{B}$, and that $\mathcal{B}$ consists of the modules B having a long exact sequence

$$0 \longrightarrow C_n \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow B \longrightarrow 0$$

with $C_0, C_1, \ldots, C_n \in \operatorname{Prod} C$. We then deduce that $\mathcal{B}$ is closed under direct sums if and only if so is $\operatorname{Prod} C$. Since C is pure-injective by [27], the latter implies that the module C is Σ-pure-injective. So, we have (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

It remains to prove (2) $\Rightarrow$ (3). Assume that C is Σ-pure-injective. By Lemma 49, there is a cardinal κ such that every product of copies of C is a direct sum of modules of cardinality at most κ. Of course, the isomorphism classes of all κ-generated modules lying in $\operatorname{Prod} C$ form a set $\mathcal{K}$. Let C' be the direct sum of all modules in $\mathcal{K}$ and P the direct product of all modules in $\mathcal{K}$. We then have $\operatorname{Prod} C \subseteq \operatorname{Add} C'$. Moreover, $P \in \operatorname{Prod} C$ is Σ-pure-injective. Hence the pure submodule C' of P is a direct summand of P. This proves $\operatorname{Prod} C' \subseteq \operatorname{Prod} C$, and further, by Σ-pure-injectivity, $\operatorname{Add} C' \subseteq \operatorname{Prod} C'$. We then conclude that $\operatorname{Add} C' = \operatorname{Prod} C' = \operatorname{Prod} C$, so C' is a product-complete cotilting module such that $\mathcal{A} = {}^{\perp}C'$. $\square$

Corollary 62. *Let R be a right noetherian ring, and $(\mathcal{A}, \mathcal{B})$ an n-cotilting cotorsion pair. Then the following statements are equivalent.*

(i) *$(\mathcal{A}, \mathcal{B})$ is of finite type.*

(ii) *$\mathcal{B}$ is definable.*

(iii) *There is a Σ-pure-injective cotilting module C such that $\mathcal{A} = {}^{\perp}C$.*

Proof. By Proposition 61, condition (iii) means that $(\mathcal{A}, \mathcal{B})$ is smashing. So, we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). (iii) $\Rightarrow$ (i) is an immediate consequence of Theorem 60 (2). $\square$

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