Charles University in Prague

Faculty of Mathematics and Physics

MASTER THESIS

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Choquet theory and functional calculus

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Study branch: Mathematical Analysis

Chcel by som poďakovať prof. RNDr. Jaroslav Lukešovi, DrSc. za navhrnutú tému diplomovej práce, za obetavé prečítanie mnohých priebežných verzií textu a predovšetkým za veľa pripomienok, ktoré túto prácu pomohli výrazne zprehľadniť.

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V Prahe dňa 10.4.2006 Milan Kolkus

Túto prácu by som uviedol slovami G.Choqueta:

"Dá se říci, že v matematice stejně jako ve válce, jsou strategové a taktici. Vojenský stratég ma určitou intuitivní představu, jak vést tažení, ponětí o velkých masách a jejich vzájemných vztazích. Taktik se drží terénu, ma technické znalosti a zřetelnou zálibu v organizační činnosti. Spíš bych byl stratég v tom smyslu, že mám přehled o velkých masách a nemám rád a nepraktikuji hromadění znalostí o známych postupech..."[1]

[1]prevzaté z publikácie *Professor Gustave Choquet, Doctor Universitatis Carolinae Honoris Causa Creatus*, MATFYZPRESS, 2002

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Abstract

Názov práce: Choquetová teória a funkcionálny kalkulus
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Abstrakt: Táto práca sa zaoberá možnosťami prenesenia známych výsledkov z konvexnej analýzy do Choquetovej teórie funkčných priestorov, najmä čo sa týka zvazových viet. Odvodili sme, že funkčný priestor $\mathcal{H}$ je simpliciálny práve vtedy, keď istá špeciálna trieda $\mathcal{A}(\mathcal{H})$ $\mathcal{H}$-afinných funkcí tvorí zvaz. Ďalším výsledkom je spektrálna veta pre tento systém $\mathcal{A}(\mathcal{H})$, ktorý stavia i na vyššie zmienenej vete. Pre hlbšie pochopenie súvislostí je dôležitá dobrá znalosť Rieszových priestorov a Banachových zvazov. Základne pojmy sú stručne zhrnuté v úvode. Ako ďalší užitočný nástroj sa ukazuje abstraktná teória integrácie v dvoch rôznych štruktúrach - prístup cez zvazové integrály a tiež pristup cez miery s hodnotami v Banachovom priestore. Pretože sa jedná o pomerne novú problematiku, veľa otvorených problémov a prirodzene vzniknutých otázok s náznakmi možných riešení je zhrnutých v závere práce.

Kľúčové slová: Choquetová teória, funkcionálny kalkulus, abstraktná integácia

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Abstrakc: This thesis concerns with possibilities of known results transfer from convex analysis to Choquet theory of function spaces, mainly as for lattice type theorems. We have proved that function space $\mathcal{H}$ is a simplicial if and only if some family $\mathcal{A}(\mathcal{H})$ of special $\mathcal{H}$-affine functions is a lattice. Next main result is a spectral theorem for this system $\mathcal{A}(\mathcal{H})$. For deeper understanding of connections it is necessary to be familiar with Riesz spaces and Banach lattices. Basic notions are summarized in the Introduction. As a further useful tool it appears to be abstract integration - lattice integrals and Banach space valued measures. Open problems and naturally arised questions with possible ideas for solutions are collected at the end of thesis.

Keywords: Choquet theory of function spaces, functional calculus, abstract integration

Chapter 1

Introduction

It appears a close relationship between Choquet theory of function spaces and function calculus in view of an abstract integration through integrals on lattices and Banach space valued measures. Comparison of both quite different theories we stated to the section Open Problems - Abstract integration.

In many, not only physical motivated tasks, occurs a necessity to put together a real or complex valued function and an element from a seemingly different mathematical structures - Banach algebras, respective C^*-algebras and σ-complete Riesz spaces with unit (for the definition see paragraph Riesz spaces in the section Special spaces). Further conspicious questions linked to this theme are collected in the present paper in the Section Open Problems - Spectral theory. In the following we compare calculuses mentioned above. A spectral theory for function spaces is developed in the Chapter 3.

Let sketch a framework for spectral theories for special algebras with unit e. Namely, on Banach algebras one can establish Dunford holomorphic calculus with the aid of complex analysis and its comfortable properties such that each holomorphic function can be expressed in Taylor series locally at each point. The counter value is a relatively small system of functions from function calculus. One can improve it, but only for Hermitian elements of C^*-algebras. Thanks to complex version of the Stone–Weierstrass theorem we obtain calculus also for continuous complex functions on the spectrum.

In both cases, it is one of key points a possibility to define a resolvent function $(\lambda e - x)^{-1}$. This resolvent yields useful formulas as the Cauchy one in the context of special algebras. Remind that for Hilbert spaces one can establish also the Borel measurable calculus. In this connection let us point out the notions norm completness and orthogonality. A natural reflection of the preceding notions in σ-complete Riesz spaces with unit u is a σ-completness and a lattice orthogonality, for definitions see the section Special spaces. By this way defined orthogonality produce very similar spectral theory to the spectral theory in Hilbert spaces. Orthogonal projections, orthogonal subspaces are meanigful and function calculus is with a resolvent $[(\lambda u - x)^+]u$ and integration over a real spectrum, cf. THEOREM 1.1. For better understanding of those mathematically uncertain considerations, but very intuitive, we refer to [4].

One can conceptualize a notion of $\mathcal{H}$-affinity as a generalization of affinity from the convex analysis to the theory of function spaces, where $\mathcal{H}$ is a subspace of $\mathcal{C}(K)$. One of an imporant argument why we should deal with function spaces instead of quite simpler convex analysis is its higher flexibility which is caused by that we need not go over to state space. An introduction to this nice theory take over from [10]. Overview of known facts about special families of affine functions and also some new generalizations to the function space setting are in the Chapter 2. For the first view to this new theory we refer a reader to [12, Phelps] and to [3, Choquet]. Finally, we refer to the section Open problems - Function spaces for open problems and new possible ideas how to solve them.

1.1 Special spaces

Ordered sets. An ordered set E is called *upper directed* if for any pair of elements $x, y \in E$ there exists an element $z \in E$ such that $x \leq z$ and $y \leq z$. Let denote by $x_1 \vee \ldots \vee x_n$ the *least upper bound* of elements $x_1, \ldots, x_n \in E$ (if it exists) and by $\vee_{n=1}^{\infty} x_n$ the least upper bound (supremum) of sequence $x_n, n = 1, 2, \ldots$ if it exists. If for any $x, y \in E$ the supremum $x \vee y$ exists, then we say that E is an *upper semilattice*. If E is both upper and lower semilattice, then we say that E is a *lattice*. The meaning of dual notions a lower directed, a lower semilattice and the notation $x \wedge y$ should be clear.

Order convergence. Let E be an ordered set. A sequence $\{x_n\}$ of elements of E is said to be *increasing* if $m < n$ implies $x_m \leq x_n$. In this case one writes $x_n \nearrow$. If moreover, the element $\vee_{n=1}^{\infty} x_n$ exists, one writes $x_n \searrow x$ and analogous definitions for $x_n \nearrow$. The sequence $\{x_n\}$ of elements of E is said to *converge with respect to the order relation* to x (abbreviated, (o)-converges to x) if there exist the sequences $\{a_n\}, \{b_n\}$ of E, such that

(a) $a_n \leq x_n \leq b_n$, $n \in \mathbb{N}$,

(b) $a_n \searrow x$ and $b_n \nearrow x$.

In this case one writes $x = (\mathrm{o})\text{-}\lim x_n$.

Dedekind completness. A lattice L is said to be *Dedekind complete* if any subset of L admit a greates lower bound and a least upper bound. The lattice L is said to be *relatively complete* if any bounded subset of L admits a greates lower bound and a least upper bound. If in the previous definitions the subsets of L are assumed to be *countable*, then we get the definition of the *Dedekind σ-complete lattice* (*relatively σ-complete*, respectively).

Riesz spaces. A real vector space V is said to be an *ordered vector space* if an order relation has been given in V, such that the following conditions are satisfied:

(O1) if $x_1, x_2 \in V$ and $x_1 \leq x_2$, then $x_1 + x \leq x_2 + x$ for any $x \in V$,

(O2) if $x_1, x_2 \in V$ and $x_1 \leq x_2$, then $\alpha x_1 \leq \alpha x_2$ for any $\alpha \in \mathbb{R}^+$.

One calls a *Riesz space* (or *vector lattice*) any ordered vector space which is lattice.

One calls a *σ-complete Riesz space* any ordered vector space which is a relatively σ-complete lattice.

Order sums. The notion of *(o) - convergent series* can be introduced in a natural manner:

$$(\mathrm{o})\text{-}\sum_{n=1}^{\infty} x_n = (\mathrm{o})\text{-}\lim_{m} \sum_{n=1}^{m} x_n,$$

and

$$(\mathrm{o})\text{-}\sum_{-\infty}^{+\infty} x_n = (\mathrm{o})\text{-}\sum_{n=1}^{\infty} x_n \ + \ (\mathrm{o})\text{-}\sum_{n=0}^{\infty} x_{-n},$$

if the right side is meaningful.

Example. Let X be an arbitrary nonempty set. Denote by $R(X)$ the set of all real functions on X. $R(X)$ is a complete Riesz space in its natural pointwise ordering. The lattice operations are also defined pointwise. Note that the (o) - convergence and pointwise convergence in $R(X)$ coincides.

Orthogonality. Let Q be a Riesz space. If $x \in Q$, then the *positive part* of x is, by definition, the element $x^+ := x \vee 0$; the *negative part* of x is the element $x^- := x \wedge 0$; the *absolute value* of x is the element $|x| := x^+ + x^-$. Two elements $x_1, x_2 \in Q$ are said to be *orthogonal* if $|x_1| \wedge |x_2| = 0$. One writes then $x_1 \perp x_2$. The *orthogonal complement* of an arbitrary A subset of Q we denote by $A^\perp := \{x \in Q : x \perp A\}$. A set A of elements of Q is said to be *total* if $A^\perp = \{0\}$.

Componets and projectors. A subset P of a Riesz space Q is called a *component* of Q if any element $x \in Q$ can be written as $x = x' + x''$, where $x' \in P$ and $x'' \in P^\perp$. The element x' is called the *projection* of x onto P and it is denoted by $x' = [P]x$. The mapping $x \mapsto [P]x$ of Q into P is called the *projector* (it is denoted by $[P]$). In a σ-complete Riesz space the set $v^{\perp\perp}$ is the smallest component which contains v. It is called the *component generated by* v. The projector determined by $v^{\perp\perp}$ is called a *principial projector* and it is denoted by $[v]$.

Riesz space with unit. In a according with the definition of a total set, an element $c \in Q$ is said to be *total*, if $x \perp c$ implies $x = 0$. A Riesz space is said to be *Riesz space with unit* u if it has total elements and if a positive total element u is choosen in it. The element u is called a *unit element*. If Q is a Riesz space with unit, any element $e \in Q$ for which $e \wedge (u - e) = 0$ is called a *unitary element*.

In what follows let Q be a σ-complete Riesz space with unit u.

Integration to Q. Let φ be a real valued function defined on $\mathbb{R}$, and g be an Q-valued function defined on $\mathbb{R}$. Let us consider a partition $\triangle$ of the real axis, given by points $\lambda_i (i = 0, \pm 1, \pm 2, \ldots)$, such that $\lambda_i < \lambda_{i+1}$ and $\lambda_{i+1} - \lambda_i \leq \varepsilon$ (for a given ε). Let us assume that for any such partition and any choice of the intermediate points γ_i (i.e., $\lambda_i \leq \gamma_i \leq \lambda_{i+1}$) the following sum

$$s_\triangle = (\mathrm{o})\text{-}\sum_{-\infty}^{+\infty} \varphi(\gamma_i)(g(\lambda_{i+1}) - g(\lambda_i)) \tag{1.1}$$

exists. Let $\nu(\triangle) := \sup(\lambda_{i+1} - \lambda_i)$ be the norm of the partition $\triangle$. For any sequence $\{\triangle_n\}_{n\in\mathbb{N}}$ of the partitions, such that $\nu(\triangle_n) \to 0$, we shall consider a sequence of elements of the form (1.1), where the intermediate points are chosen arbitrirarily for each partition $\triangle_n$. If for any sequence $\{\triangle_n\}$, such that $\nu(\triangle_n) \to 0$, the sequence $\{s_{\triangle_n}\}_{n\in\mathbb{N}}$ is (o)-convergent to a given element $x \in Q$, which is indenpendent of the chosen sequence of partitions, one writes

$$x = \int_{-\infty}^{+\infty} \varphi(\lambda) dg(\lambda).$$

Spectral function. For any element $x \in Q$, the function $g^x : \mathbb{R} \to Q$, defined by the formula

$$g^x(\lambda) := [(\lambda u - x)^+]u,$$

is called the *spectral function* of x.

Theorem 1.1. *Any element x of a σ-complete Riesz space with unit can be represented in the form*

$$x = \int_{-\infty}^{+\infty} \lambda dg^x(\lambda).$$

Banach lattices. Let $(B, \vee)$ be a Riesz space. A seminorm ρ on B satisfying $\rho(x) \leq \rho(y)$ whenever $|x| \leq |y|$ is called a *lattice seminorm* and a lattice norm if, in addition, ρ is a norm. In the latter case, $(B, ||\,.\,||)$ is called a *normed Riesz space.* An normed Riesz space which is complete with respect to the norm is called a *Banach lattice.*

M-spaces. A lattice norm $||\,.\,||$ on Riesz space $(M, \vee)$ is called a *M-norm*, if $||x \vee y|| = \max\{||x||, ||y||\}$ for all positive elements $x, y \in M$. A M-normed Banach lattice M is called an *M-space.* The meaning of notions *Banach sublattice, sub-M-space* should be clear.

1.2 Choquet theory of function spaces

Function spaces. By a *function space* $\mathcal{H}$ on a compact Hausdorff topological space K we mean (not necessarily closed) linear subspace of $\mathcal{C}(K)$ containing the constant functions and separating the points of K.

Examples. (a) *Continuous functions.* The whole space $\mathcal{C}(K)$ of all continuous functions on a Hausdorff compact space K represents a simple example of a function space. Clearly, the space $\mathcal{C}(K)$ separates the points of K.

(c) *Convex case – affine functions.* Let X be a convex compact subset of a locally convex space E and $\mathcal{H}$ the linear space $\mathfrak{A}^c(X)$ of all continuous affine functions on X.

(d) *Harmonic case – harmonic functions.* Let U be a bounded open subset of the Euclidean space $\mathbb{R}^d$. The function space $\mathbf{H}(U)$ consists of all continuous functions on $\overline{U}$ which are harmonic on U.

More generally, we can consider a relatively compact open subset U of an abstract harmonic space and the function space $\mathbf{H}(U)$, the linear subspace of $\mathcal{C}(\overline{U})$ of functions which are harmonic on U. We tacitly assume that constant functions are harmonic and $\mathbf{H}(U)$ separates the points of $\overline{U}$.

Representating measures. Let $\mathcal{M}^1(K)$ denote the set of all probability Radon measures on K. We denote by $\mathcal{M}_x(\mathcal{H})$ the set of all $\mathcal{H}$-*representating measures* for $x \in K$, that is,

$$\mathcal{M}_x(\mathcal{H}) := \{\mu \in \mathcal{M}^1(K) : f(x) = \int_K f d\mu \text{ for any } f \in \mathcal{H}\}.$$

$\mathcal{H}$-affine functions. We define the space $\mathcal{A}^b(\mathcal{H})$ of all $\mathcal{H}$-*affine functions* as the family of all bounded Borel functions l on K satisfying the following *barycentric formula*:

$$l(x) = \int_K l \, d\mu \quad \text{for each } x \in K \quad \text{and} \quad \mu \in \mathcal{M}_x(\mathcal{H}).$$

Sometimes we will write shorter $\mu(f)$ instead of $\int_K f d\mu$.

Upper and lower envelopes. Let f be an upper bounded function on K. For $x \in K$, put

$$f^*(x) = \inf\{h(x) : h \in \mathcal{H}\,,\ h \geq f \text{ on } K\ \}. \tag{1.2}$$

Obviously, the *upper envelope* f^* is an upper semicontinuous function on K. Similiary, for a lower bounded function f on K, we define the *lower envelope* f_* so that $f_*(x) = -(-f)^*(x), x \in K$.

Proposition 1.2. *Let $x \in K$. Then the mapping $f \mapsto f^*(x)$ is sublinear functional on $\mathcal{C}(K)$.*

Proof. It is easy to verify that

$$(f+g)^* \leq f^* + g^* \quad \text{and} \quad (\lambda f)^* = \lambda f^*$$

for any $f, g \in \mathcal{C}(K)$ and $\lambda > 0$. □

Lemma 1.3. *Let $f \in \mathcal{C}(K)$ and $x \in K$. Then*

$$[f_*(x), f^*(x)] = \{\mu(f) : \mu \in \mathcal{M}_x(\mathcal{H})\}\,.$$

Proof. Fix an x in K and $f \in \mathcal{C}(K)$. If $\mu \in \mathcal{M}_x(\mathcal{H})$ and $g, h \in \mathcal{H}$, $g \leq f \leq h$, then $g(x) = \mu(g) \leq \mu(f) \leq \mu(h) = h(x)$, so that $f_*(x) \leq \mu(f) \leq f^*(x)$. Pick now $\alpha \in [f_*(x), f^*(x)]$. From LEMMA 1.2 we know that the mapping $p : g \mapsto g^*(x)$ is a sublinear functional on $\mathcal{C}(K)$. The Hahn–Banach theorem provides a linear functional μ_f on $\mathcal{C}(K)$ such that $\mu_f(f) = \alpha$ and $\mu_f \leq p$ on $\mathcal{C}(K)$. Since $\mu_f(g) \leq p(g) = g^*(x) \leq 0$ whenever $g \in \mathcal{C}(K)$ and $g \leq 0$, we see that μ_f is, according to the Riesz representation theorem, a positive Radon measure on K. Let $h \in \mathcal{H}$. Then $h_* = h = h^*$, which yields

$$\mu_f(h) \leq p(h) = h^*(x) = h(x)$$

and simultaneously

$$-\mu_f(h) = \mu_f(-h) \le p(-h) = (-h)^*(x) = -h_*(x) = -h(x)\,.$$

Hence $\mu_f(h) = h(x)$. If $h = 1$ on K, then $\mu_f(h) = h(x) = 1$. Thus $||\mu_f|| = 1$, and we see that $\mu_f \in \mathcal{M}_x(\mathcal{H})$. □

$\mathcal{H}$-concave and $\mathcal{H}$-convex functions

$\mathcal{H}$-concave and $\mathcal{H}$-convex functions. A bounded Borel function f on K is called $\mathcal{H}$-convex, if

$$f(x) \le \mu(f) \quad \text{for any } x \in K \quad \text{and} \quad \mu \in \mathcal{M}_x(\mathcal{H}).$$

In a similar way we define $\mathcal{H}$-*concave* functions. Let $\mathcal{K}(\mathcal{H})$ denote the family of all $\mathcal{H}$-*convex* functions on K and $\mathcal{K}^c(\mathcal{H})$ the family of all continuous $\mathcal{H}$-convex ones. Similarly, we define the family of continuous $\mathcal{H}$-concave functions as

$$\mathcal{S}^c(\mathcal{H}) = \{f \in \mathcal{C}(K) : f(x) \ge \mu(f) \text{ for any } x \in K \text{ and } \mu \in \mathcal{M}_x(\mathcal{H})\}.$$

Of course, $\mathcal{A}^c(\mathcal{H}) = \mathcal{K}^c(\mathcal{H}) \cap \mathcal{S}^c(\mathcal{H})$. Further, let $\mathcal{K}^{usc}(\mathcal{H})$ denote the set of all upper semicontinuous $\mathcal{H}$-convex functions on K, $\mathcal{S}^{lsc}(\mathcal{H})$ the set of all lower semicontinuous $\mathcal{H}$-concave function on K. The meaning of notations $\mathcal{K}^{lsc}(\mathcal{H})$ and $\mathcal{S}^{usc}(\mathcal{H})$ should be clear.

Let denote by $f \vee g$ the pointwise supremum of bounded real functions f and g. We define analogously the pointwise infimum as $f \wedge g$. We denote by f^+, resp. f^- positive, resp. negative part, more precisely $f^+ = f \vee 0$ and $f^- = f \wedge 0$.

Proposition 1.4. *The family $\mathcal{S}^c(\mathcal{H})$ forms a convex cone of functions which is min–stable.*

Proof. Let us just check that $\mathcal{S}^c(\mathcal{H})$ is min–stable: If $k_1, k_2 \in \mathcal{S}^c(\mathcal{H})$, then $k_1 \wedge k_2 \in \mathcal{S}^c(\mathcal{H})$. Indeed, let $x \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. Then

$$\mu(k_1 \wedge k_2) \le \min(\mu(k_1), \mu(k_2)) \le (k_1 \wedge k_2)(x)\,.$$

□

Lemma 1.5. *We have $f = f^*$ on K for any $f \in \mathcal{S}^c(\mathcal{H})$.*

Proof. Pick $x \in K$. With the aid of Lemma 1.3, find $\mu \in \mathcal{M}_x(\mathcal{H})$ so that $f^*(x) = \mu(f)$. Then

$$f^*(x) = \mu(f) \le f(x) \le f^*(x)\,.$$

□

Proposition 1.6. *Let f be an upper bounded function on K. Then*

$$f^* = \inf\{g : g \in \mathcal{A}^c(\mathcal{H})\,,\ g \ge f \text{ on } K\} = \inf\{k : k \in \mathcal{S}^c(\mathcal{H})\,,\ k \ge f \text{ on } K\}\,.$$

Proof. We have

$$f^* \geq \inf\{g : g \in \mathcal{A}^c(\mathcal{H})\,,\ g \geq f \text{ on } K\} \geq \inf\{k : k \in \mathcal{S}^c(\mathcal{H})\,,\ k \geq f \text{ on } K\}\ .$$

Given $k \in \mathcal{S}^c(\mathcal{H})$, $k \geq f$ on K, in view of LEMMA 1.5 we get $k = k^* \geq f^*$. It follows that

$$\inf\{k : k \in \mathcal{S}^c(\mathcal{H}),\ k \geq f \text{ on } K\} \geq f^*\,.$$

□

Lemma 1.7. *Let $\mathcal{H}$ be a function space on K, f be an upper semicontinuous function on K and $x \in K$. Then there exists $\mu \in \mathcal{M}_x(\mathcal{H})$ such that $f^*(x) = \mu(f)$.*

Proof. Denote by $\mathcal{G}$ the lower directed set $\{g \in \mathcal{C}(K) : g \geq f \text{ on } K\}$. By LEMMA 1.3, for any $g \in \mathcal{G}$ there is a measure $\mu_g \in \mathcal{M}_x(\mathcal{H})$ such that $\mu_g(g) = g^*(x)$. Given $\varphi \in \mathcal{G}$, let

$$M_\varphi = \{\mu_g :\ g \in \mathcal{G},\ g \leq \varphi\}.$$

By a compactness argument, there is $\mu \in \bigcap\limits_{\varphi\in\mathcal{G}} \overline{M}_\varphi^{w^*}$. A moment's reflection shows that $\mu \in \mathcal{M}_x(\mathcal{H})$. We observe that

$$\inf\big\{\nu(\varphi) :\ \nu \in M_\varphi\big\} = \inf\big\{\nu(\varphi) :\ \nu \in \overline{M}_\varphi^{w^*}\big\} \leq \mu(\varphi)$$

for each $\varphi \in \mathcal{G}$. Hence

$$\begin{aligned}
f^*(x) \leq& \inf\{g^*(x) : g \in \mathcal{G}\} = \inf\{\mu_g(g) : g \in \mathcal{G}\}\\
\leq& \inf\{\inf\{\mu_g(\varphi) : g \in \mathcal{G}, g \leq \varphi\} :\ \varphi \in \mathcal{G}\} \leq \inf\{\mu(\varphi) :\ \varphi \in \mathcal{G}\}\\
=& \mu(f) \leq \inf\{\mu(h) : h \geq f,\ h \in \mathcal{H}\} = \inf\{h(x) : h \geq f,\ h \in \mathcal{H}\}\\
=& f^*(x)\,,
\end{aligned}$$

which are the inequalities needed to finish the proof. □

Proposition 1.8. *If f is an upper bounded function on K, then f^* is upper semicontinuous and $\mathcal{H}$–concave.*

Proof. Pick $x \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. Then

$$\begin{aligned}
\mu(f^*) &= \mu\big(\inf\{h : h \in \mathcal{H}\ ,\ h \geq f\}\big) \leq \inf\{\mu(h) : h \in \mathcal{H}\ ,\ h \geq f\}\\
&= \inf\{h(x) : h \in \mathcal{H}\ ,\ h \geq f\} = f^*(x)\,.
\end{aligned}$$

This shows that f^* is $\mathcal{H}$–concave. It is plain that f^* is upper semicontinuous. □

Proposition 1.9. *Let f be an upper bounded function on K. Then f is an upper semicontinuous $\mathcal{H}$–concave if and only if $f = f^*$ on K.*

Proof. Let $f \in \mathcal{S}^{usc}(\mathcal{H})$ and $x \in K$. By LEMMA 1.7, there is a measure $\mu \in \mathcal{M}_x(\mathcal{H})$ such that $f^*(x) = \mu(f)$. Then

$$f^*(x) = \mu(f) \leq f(x) \leq f^*(x)\,.$$

Conversely, suppose that $f = f^*$. By PROPOSITION 1.8, the function f^* is upper semicontinuous and $\mathcal{H}$–concave. □

Corollary 1.10. *Let f be an upper bounded function on K. Then*

$$f^* = \inf \{l : l \in \mathcal{A}^{usc}(\mathcal{H})\,,\ l \geq f \text{ on } K\} = \inf \{k : k \in \mathcal{S}^{usc}(\mathcal{H})\,,\ k \geq f \text{ on } K\}\ .$$

Proof. Recall that

$$f^* := \inf \{h : h \in \mathcal{H}\,,\ h \geq f \text{ on } K\}\ .$$

Obviously,

$$f^* \geq \inf \{l : l \in \mathcal{A}^{usc}(\mathcal{H})\,,\ l \geq f \text{ on } K\} \geq \inf \{k : k \in \mathcal{S}^{usc}(\mathcal{H})\,,\ k \geq f \text{ on } K\}\ .$$

Given $l \in \mathcal{A}^{usc}(\mathcal{H})$, $l \geq f$, in view of the preceding PROPOSITION 1.9 we get

$$l = l^* \geq f^* \geq \inf \left\{\tilde{l}(x) : \tilde{l} \in \mathcal{A}^{usc}(\mathcal{H}),\ \tilde{l} \geq f\right\}\ .$$

Taking the infimum over all $l \geq f$ in $\mathcal{A}^{usc}(\mathcal{H})$ finishes the reasoning. □

Corollary 1.11. *Let g be an upper semicontinuous function on K. Then*

$$g^* = \inf \left\{l : l \in \mathcal{A}^{lsc}(\mathcal{H})\,,\ l \geq g \text{ on } K\right\} = \inf \left\{k : k \in \mathcal{S}^{lsc}(\mathcal{H})\,,\ k \geq g \text{ on } K\right\}\ .$$

Proof. Pick $x \in K$ and using LEMMA 1.7 find again $\mu \in \mathcal{M}_x(\mathcal{H})$ such that $\mu(g) = g^*(x)$. Let $k \in \mathcal{S}^{lsc}(\mathcal{H})$, $k \geq g$. Then

$$g^*(x) = \mu(g) \leq \mu(k) \leq k(x)\,.$$

Hence

$$g^* \leq \inf \left\{k : k \in \mathcal{S}^{lsc}(\mathcal{H})\,,\ k \geq g \text{ on } K\right\} \leq \inf \left\{l : l \in \mathcal{A}^{lsc}(\mathcal{H})\,,\ l \geq g \text{ on } K\right\}\ .$$

The reverse inequality is obvious, thus the proof is complete. □

If $\mathcal{H}$ is a function space, we denote $\mathcal{W}(\mathcal{H}) := \{h_1 \vee \cdots \vee h_n : h_i \in \mathcal{H}, i = 1, \ldots, n\}$.

Lemma 1.12. *Let $\mathcal{H}$ be a function space on a compact K. If g is an lower semicontinuous function on K, $f \in \mathcal{K}^{usc}(\mathcal{H})$, $g > f$ on K, then there is a function $k \in \mathcal{W}(\mathcal{H})$ such that $g > k > f$ on K.*

Proof. Fix $x \in K$. By LEMMA 1.7, there is a measure $\mu \in \mathcal{M}_x(\mathcal{H})$ such that $g_* = \mu(g)$. Then

$$g_*(x) = \mu(g) > \mu(f) \geq f(x).$$

Therefore, there exists $h_x \in \mathcal{H}$ such that

$$h_x \leq g \text{ on } K \text{ and } h_x(x) > f(x).$$

Adding a small constant function to h_x, we may assume that $h_x < g$ everywhere on K and still $h_x(x) > f(x)$. We infer from the upper semicontinuity of $f - h_x$ and a compactness argument that there exists $x_1, \ldots, x_n \in K$ such that $k := h_{x_1} \vee \ldots \vee h_{x_n} > f$ on K. The function K has all properties required. □

Corollary 1.13. *Let k be a upper semicontinuous $\mathcal{H}$-convex function on K. If*

$$W := \{w \in \mathcal{W}(\mathcal{H}) : w > k \text{ on } K\},$$

then the set W is lower directed and $k = \inf W$.

Proof. It suffices to establish that $k = \inf W$. Since

$$k = \inf\{g \in \mathcal{C}(K) : g > k \text{ on } K\},$$

using PROPOSITION 1.12 we conclude that $k = \inf W$.

Now we are given $w_1, w_2 \in W$, and we wish fo find $w' \in W$ so that $w' < w_1 \wedge w_2$. Since $(w_1 \wedge w_2)_* > k$, a new application of PROPOSITION 1.12 asserts the existence of $w' \in W$ so that $w_1 \wedge w_2 \geq (w_1 \wedge w_2)_* > w > k$ and the proof is finished. ☐

Theorem 1.14. *Let f be an upper bounded function on K and $\mu \in \mathcal{M}^1(K)$. Then*

$$\mu(f^*) = \inf\{\mu(k) : k \in \mathcal{S}^c(\mathcal{H}),\, k \geq f\}.$$

Proof. We know from PROPOSITION 1.4 that the family $\mathcal{S}^c(\mathcal{H})$ is min–stable, and therefore the set $\inf\{k : k \in \mathcal{S}^c(\mathcal{H}),\, k \geq f\}$ is lower directed and its infimum equals f^* by PROPOSITION 1.6. The assertion follows now from more or less familiar the Lebesgue monotone convergence theorem for lower directed sets of upper semicontinuous functions. ☐

Choquet boundary

Choquet boundary. Define the *Choquet boundary* $Ch_{\mathcal{H}}(K)$ of a function space $\mathcal{H}$ as the set of those points $x \in K$ for which the Dirac measure ε_x is the only $\mathcal{H}$-representating measure for x, that is,

$$Ch_{\mathcal{H}}(K) = \{x \in K : \mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}\}.$$

Theorem 1.15. *A point $x \in K$ belongs to the Choquet boundary of $\mathcal{H}$ if and only if*

$$f(x) = f^*(x) \quad \textit{for every} \quad f \in \mathcal{C}(K).$$

Proof. The assertion is an immediate consequence of LEMMA 1.3. If $x \in \mathrm{Ch}\,\mathcal{H}(K)$ and $f \in \mathcal{C}(K)$, then $\mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}$, and therefore $f(x) = f^*(x)$. Conversely, assume that $f(x) = f^*(x)$ for any $f \in \mathcal{C}(K)$. If $\mu \in \mathcal{M}_x(\mathcal{H})$, then $\mu(f) = f(x)$ for any $f \in \mathcal{C}(K)$. Hence $\mu = \varepsilon_x$. ☐

Lemma 1.16. *A point x belongs to the Choquet boundary of $\mathcal{H}$ if and only if*

$$h^+(x) = (h^+)^*(x) \quad \textit{for every} \quad h \in \mathcal{H},$$

Proof. Suppose $x \in Ch_{\mathcal{H}}(K)$ and $h \in \mathcal{H}$. It is clear that $h^+ \in \mathcal{C}(K)$ and the preceding THEOREM 1.15 gives $h^+(x) = (h^+)^*$. On the other hand, if $x \notin Ch_{\mathcal{H}}(K)$, then by the definition exists $\mu \in \mathcal{M}_x(\mathcal{H})$, such that $\mu \neq \varepsilon_x$, that is, there is $z \in \operatorname{supt}\mu$, $z \neq x$. Since $\mathcal{H}$ separates points of K, we obtain $h' \in \mathcal{H}, h'(x) < h'(z)$. Put $h = h' - h'(x)$, so $h \in \mathcal{H}$ and $h(x) = 0, h(z) > 0$. Therefore

$$h^+(x) = 0 < \mu(h^+) \leq \mu((h^+)^*) \leq (h^+)^*(x),$$

since $(h^+)^*$ is $\mathcal{H}$-concave function according to LEMMA 1.8. ☐

Maximal measures

Choquet's ordering and maximal measures. The convex cone $\mathcal{K}^c(\mathcal{H})$ of all $\mathcal{H}$-convex functions on K determines the partial *Choquet ordering* on the space $\mathcal{M}^+(K)$ of all positive Radon measures on K:

$$\mu \preceq \nu \quad \text{if} \quad \mu(f) \leq \nu(f) \text{ for each } f \in \mathcal{K}^c(\mathcal{H}).$$

Maximal elements of $\mathcal{M}^+(K)$ with respect to this Choquet ordering are called *maximal measures*(or, more precisely, $\mathcal{H}$-*maximal measures*).

We start with trivial observations.

Observation 1.17. *For any $\mu \in \mathcal{M}_x(\mathcal{H})$, we have $\varepsilon_x \prec \mu$.*

Proof. The assertion is just the definition of $\mathcal{K}^c(\mathcal{H})$: whenever $f \in \mathcal{K}^c(\mathcal{H})$ and $\mu \in \mathcal{M}_x(\mathcal{H})$, then $\varepsilon_x(f) = f(x) \leq \mu(f)$. □

Proposition 1.18. *Let $x \in K$, $\mu \in \mathcal{M}_x(\mathcal{H})$, $\nu \in \mathcal{M}^+(K)$ and $\mu \prec \nu$. Then $\nu \in \mathcal{M}_x(\mathcal{H})$.*

Proof. Let $h \in \mathcal{H}$. Then $h(x) = \mu(h) = \nu(h)$ since $\mathcal{H} \subset \mathcal{K}^c(\mathcal{H}) \cap -\mathcal{K}^c(\mathcal{H})$. A particular choice $h = 1$ yields $||\nu|| = 1$. □

Corollary 1.19. *Let $x \in K$ and $\mu \in \mathcal{M}^1(K)$. Then*

$$\varepsilon_x \prec \mu \quad \textit{if and only if} \quad \mu \in \mathcal{M}_x(\mathcal{H})\,.$$

In what follows, we need a strengthened form of Lemma 1.3.

Lemma 1.20. *Let $f \in \mathcal{C}(K)$ and $\lambda \in \mathcal{M}^1(K)$. Then*

$$[\lambda(f_*), \lambda(f^*)] = \left\{\mu(f) : \mu \in \mathcal{M}^1(K)\,,\, \lambda \prec \mu\right\}\,.$$

Proof. The proof is almost the same as that given in Lemma 1.3. Pick $\alpha \in [\lambda(f_*), \lambda(f^*)]$ and imitate it setting

$$p : g \mapsto \lambda(g^*)\,, \quad g \in \mathcal{C}(K)\,.$$

Then p is a sublinear functional on $\mathcal{C}(K)$. The Hahn–Banach theorem with the Riesz representation theorem yields a Radon measure $\mu \in \mathcal{M}^1(K)$ such that

$$\mu(f) = \alpha \quad \text{and} \quad \mu(g) \leq p(g) \quad \text{for} \quad g \in \mathcal{C}(K)\,.$$

It remains to show that $\lambda \prec \mu$. To this end pick $k \in \mathcal{K}^c(\mathcal{H})$. Then $-k \in \mathcal{S}^c(\mathcal{H})$ and an appeal to Lemma 1.5 reveals that $(-k)^* = k$. Therefore

$$\mu(-k) \leq p(-k) = \lambda((-k)^*) = \lambda(-k)\,.$$

Hence $\lambda(k) \leq \mu(k)$, which gives the required inclusion.

For the reverse, let $\lambda \prec \mu$, $v \in \mathcal{K}^c(\mathcal{H})$, $k \in \mathcal{S}^c(\mathcal{H})$, $v \leq f \leq k$. Then

$$\lambda(v) \leq \mu(v) \leq \mu(f) \leq \mu(k) \leq \lambda(k)\,.$$

Using Levi's theorem 1.14 we have

$$\lambda(f_*) \leq \mu(f) \leq \lambda(f^*)\,,$$

which finishes the proof. □

The following result due to G.Mokobodzki characterizes maximal measures.

Theorem 1.21. *Let μ be a positive Radon measure on K. The following assertions are equivalent:*

(i) μ *is maximal,*

(ii) $\mu(f) = \mu(f^*)$ *for any* $f \in \mathcal{C}(K)$,

(iii) $\mu(k) = \mu(k^*)$ *for any* $f \in \mathcal{K}^c(\mathcal{H})$.

Proof. Let $\lambda \in \mathcal{M}^1(K)$ be maximal and let $f \in \mathcal{C}(K)$. By PROPOSITION 1.20 there is a measure $\mu \in \mathcal{M}^1(K)$ such that $\lambda \prec \mu$ and $\mu(f) = \lambda(f^*)$. Since λ is maximal, we have $\mu = \lambda$, and therefore $\mu(f) = \mu(f^*)$. It is obvious that (ii) $\implies$ (iii). To see that (iii) $\implies$ (i), assume that a measure $\lambda \in \mathcal{M}^1(K)$ satisfies $\lambda(v) = \lambda(v^*)$ for each $v \in \mathcal{K}^c(\mathcal{H})$. Let $\mu \in \mathcal{M}^1(K)$, $\lambda \prec \mu$ and fix $v \in \mathcal{K}^c(\mathcal{H})$. Then, using Levi's THEOREM 1.14 we get

$$\begin{aligned}\lambda(v) = \lambda(v^*) = \lambda(\inf\{k : k \in \mathcal{S}^c(\mathcal{H})\,,\, k \geq v\}) &= \inf\{\lambda(k) : k \in \mathcal{S}^c(\mathcal{H})\,,\, k \geq v\}\\ &\geq \inf\{\mu(k) : k \in \mathcal{S}^c(\mathcal{H})\,,\, k \geq v\} \geq \mu(v^*) \geq \mu(v)\,.\end{aligned}$$

Hence $\lambda(v) = \mu(v)$. Since the space $\mathcal{K}^c(\mathcal{H}) - \mathcal{K}^c(\mathcal{H})$ is uniformly dense in $\mathcal{C}(K)$, we conclude that $\lambda = \mu$. ☐

The proof of the following PROPOSITION 1.22 use a fact that in a simplicial function space $\mathcal{H}$ on a metrizable compact there exists continuous strictly $\mathcal{H}$-convex function. Note that in this case $\mathrm{Ch}_{\mathcal{H}}(K)$ is G_δ set. For the proofs of PROPOSTION 1.22 and THEOREM 1.23 see [10].

Proposition 1.22. *Let $\mathcal{H}$ be a function space on a compact K. If K is a metrizable, then measure μ is maximal, if and only if $\mu(K \setminus \mathrm{Ch}_{\mathcal{H}}(K)) = 0$.*

Theorem 1.23 (Choquet representation theorem). *Let $\mathcal{H}$ be a function space on a compact space K admitting a continuous strictly $\mathcal{H}$–convex function h. Then for each $x \in K$ there exists a Radon measure μ on K such that*

$$\mu(K \setminus \mathrm{Ch}_{\mathcal{H}}(K)) = 0 \quad \textit{and} \quad h(x) = \int_K h\,d\mu \quad \textit{for any} \quad h \in \mathcal{H}\,.$$

Theorem 1.24. *Let $\mathcal{H}$ be function space on compact K. If K is metrizable, then measure μ is maximal, if and only if*

$$\mu(h^+) = \mu((h^+)^*) \textit{ for every } h \in \mathcal{H}.$$

Proof. Suppose μ is maximal. If $h \in \mathcal{A}^c(\mathcal{H})$, then $h^+ \in \mathcal{C}(K)$. Using Mokobodzki's THEOREM 1.21 we have that the equality $\mu(h^+) = \mu((h^+)^*)$ holds for every $h \in \mathcal{A}^c(\mathcal{H})$.

On the other hand, assume that μ is not maximal. Denote by x the barycenter of the measure μ. If μ is Dirac measure at point x, then the conclusion is trivial according to LEMMA 1.16. If μ is not the Dirac measure at point x, then there exists $z \in \operatorname{supt}\mu$ such that $h(z) > 0$ and $h(x) = 0$, since $\mathcal{H}$ separates points of K. So $0 < \mu(h^+) \leq \mu((h^+)^*)$. Therefore

$$h^+(x) = 0 < \mu(h^+) \leq \mu((h^+)^*) \leq (h^+)^*(x).$$

Put

$$\lambda := \frac{\mu((h^+)^*)}{(h^+)^*(x)},$$

then

$$\mu(h^+) = \lambda\, h^+(x) + (\mu - \lambda\, \varepsilon_x)(h^+) < \lambda\, (h^+)^*(x) + (\mu - \lambda\, \varepsilon_x)((h^+)^*) = \mu((h^+)^*),$$

which contradicts our assumption that $\mu(h^+) = \mu((h^+)^*)$, for each $h \in \mathcal{H}$. $\square$

The following THEOREM 1.25 is based on an application of well-known Zorn's lemma. For the complete proof, see [10].

Theorem 1.25. *Let μ be a positive Radon measure on K. Then there is a maximal measure λ such that $\mu \prec \lambda$.*

Simplicial function spaces.

Simplicial function spaces. A function space $\mathcal{H}$ on a compact space K is called *simplicial* if for each $x \in K$ there exists a unique maximal measure $\delta_x \in \mathcal{M}_x(\mathcal{H})$.

Abstract Dirichlet problem. For any bounded Borel function f on K we define

$$\mathrm{H}_f : x \mapsto \int_K f \; d\delta_x, \quad x \in K$$

The function H_f is an *(abstract) solution of the Dirichlet problem* for the function f. Let us denote by H mapping

$$\mathrm{H} : f \mapsto \mathrm{H}_f, \text{ for } f \in \mathcal{B}^b(K).$$

Proposition 1.26. *Let* H *be a simplicial function space on a compact space K and let $f \in \mathcal{K}^{usc}(\mathcal{H})$. Then function H_f is an upper semicontinuous $\mathcal{H}$-affine function on K. Moreover, $f^* = \mathrm{H}_f$ on K.*

Proof. Fix $x \in K$ and choose $\mu \in \mathcal{M}_x(\mathcal{H})$. Since $f^* = \inf\{h : h \in \mathcal{H},\ h \geq f\}$, we have $\mu(f) \leq \mu(f^*) \leq \mu(h) = h(x)$ for any $h \in \mathcal{H},\ h \geq f$. Hence $\mu(f) \leq \mu(f^*) \leq f^*(x)$. In particular, $\delta_x(f) \leq f^*(x)$.

Now appeal to LEMMA 1.7 to find a measure $\lambda \in \mathcal{M}_x(\mathcal{H})$ such that $f^*(x) = \lambda(f)$. Thanks to THEOREM 1.25, there is a maximal measure $\nu \in \mathcal{M}^+(K)$ such that $\lambda \prec \nu$. PROPOSITION 1.18 yields $\nu \in \mathcal{M}_x(\mathcal{H})$. The simpliciality of $\mathcal{H}$ implies that $\nu = \delta_x$, and therefore

$$\begin{aligned} f^*(x) \geq \nu(f^*) = \inf\{\nu(k) : k \in \mathcal{S}^c(\mathcal{H}),\ k \geq f\} \geq \\ \geq \inf\{\delta_x(k) : k \in \mathcal{S}^c(\mathcal{H}),\ k \geq f\} = \delta_x(f^*) \geq \delta_x(f) = f^*(x). \end{aligned}$$

$\square$

Proposition 1.27. *The following are equivalent assertions:*

(i) $\mathcal{H}$ *is simplicial,*

(ii) *For every* $f \in \mathcal{K}^c(\mathcal{H})$*, the function* $f^* \in \mathcal{A}^{usc}(\mathcal{H})$*.*

Proof. The implication (i)$\Longrightarrow$(ii) is exactly the preceding PROPOSITION 1.26.

Now suppose (ii). Let $x \in K$ and let $\mu, \nu \in \mathcal{M}_x(\mathcal{H})$ be maximal measures. Our aim is to show that $\mu = \nu$. Since the space $\mathcal{K}^c(\mathcal{H}) - \mathcal{K}^c(\mathcal{H})$ is dense in $\mathcal{C}(K)$, it is sufficient to show that $\mu = \nu$ on $\mathcal{K}^c(\mathcal{H})$. Making use of Mokobodzki's maximality THEOREM 1.21 and the definition of the $\mathcal{H}$–affinity, we see that

$$\mu(s) = \mu(s^*) = s^*(x) = \nu(s^*) = \nu(s)$$

for any $s \in \mathcal{K}^c(\mathcal{H})$. This shows that (ii) implies (i). □

The proof of the next proposition uses in the convex case the Hahn–Banach separation theorem. The general case of a function space $\mathcal{H}$ is solved in J.Spurný's paper [15] using the transfer of $\mathcal{H}$ in to the so-called state space of $\mathcal{H}$. In order to apply THEOREM 4.5 of [15], let us note that a $\mathcal{H}$-affine function on a simplicial space is completely $\mathcal{A}^c(\mathcal{H})$-affine.

Proposition 1.28. *Let* $\mathcal{H}$ *be a simplicial space on a compact space* K*. If* $l \in \mathcal{A}^{usc}(\mathcal{H})$*, then the set*

$$A_l := \{h \in \mathcal{A}^c(\mathcal{H}) : h > l \text{ on } K\}$$

is lower directed and $l = \inf A_l$*.*

If, moreover, K *is metrizable, then there exists increasing sequence* $\{l_n\} \subset A_l$*, such that*

$$l = \inf l_n.$$

Lemma 1.29. *Assume that* $\mathcal{H}$ *is a simplicial space. Then a measure* μ *is maximal if and only if,*

$$\mu(l^+) = \mu((l^+)^*) \quad \text{for every} \quad l \in \mathcal{A}^c(\mathcal{H}).$$

Proof. Suppose μ is maximal. If $l \in \mathcal{A}^c(\mathcal{H})$, then $l^+ \in \mathcal{C}(K)$. Using Mokobodzki's THEOREM 1.21 we have that the equality $\mu(l^+) = \mu((l^+)^*)$ holds for every $l \in \mathcal{A}^c(\mathcal{H})$.

Conversely, we show that $\mu(l^+) = \mu((l^+)^*)$ moreover for every $l \in \mathcal{A}^{usc}(\mathcal{H})$. According to LEMMA 1.28, the set A_l is lower directed and $l = \inf A_l$. Observe that the set

$$A := \{g^+ : g \in \mathcal{A}^c(\mathcal{H}), g > l \text{ on } K\}$$

is also lower directed and $l^+ = \inf A$. Using the Lebesgue monotonne convergence theorem for directed sets we get:

$$\mu(l^+) = \inf_{g \in A_l} \mu(g^+) = \inf_{g \in A_l} \mu((g^+)^*) \geq \mu((l^+)^*) \geq \mu(l^+).$$

Now we show that

$$\mu(l_1 \vee \ldots \vee l_n) = \mu((l_1 \vee \ldots \vee l_n)^*) \tag{1.3}$$

for any $l_1, \dots, l_n$ from $\mathcal{A}^c(\mathcal{H})$. The case $n = 1$ is clear. Assume that the equality holds for some $n \geq 1$. Given continuous $\mathcal{H}$–affine functions $l_1, \dots, l_{n+1}$, put

$$f = (l_1 - l_{n+1}) \vee \dots \vee (l_n - l_{n+1})$$

Then

$$l_1 \vee \dots \vee l_n = f^+ + l_{n+1}.$$

Simpliciality of $\mathcal{H}$ ensures that $f^* \in \mathcal{A}^{usc}(\mathcal{H})$, therefore $\mu((f^*)^+) = \mu(((f^*)^+)^*)$. The following inequalities show that (1.3) holds.

$$\begin{aligned}\mu(l_1 \vee \dots \vee l_{n+1}) = \mu(f^+) + \mu(l_{n+1}) = \mu((f^*)^+) + \mu(l_{n+1}) = \mu(((f^*)^+)^*) + \mu(l_{n+1}) \geq \\ \geq \mu((f^+)^*) + \mu(l_{n+1}) \geq \mu((l_1 \vee \dots \vee l_{n+1})^*) \geq \mu(l_1 \vee \dots \vee l_{n+1})\end{aligned}$$

Finally, the set $\{l_1 \vee \dots \vee l_n : l_1, \dots, l_n \in \mathcal{H}\}$ is dense in $\mathcal{K}^c(\mathcal{H})$ and thanks to THEOREM 1.21 measure μ is maximal. $\square$

Chapter 2

Families of $\mathcal{H}$-affine functions

2.1 The convex case

There are several types of theorems concerning a structure of simplices. We will deal with two of them: lattice type theorems and extension type theorems.

H.Bauer in [2] showed that the set $\mathcal{A}^c(X)$ of all continuous affine functions on a compact convex set X is a Riesz space in its natural ordering if and only if X is a simplex with closed set of extreme points $\operatorname{ext} X$ (so-called Bauer simplex). These assertions are equivalent to the extension theorem that for any continuous bounded function f on $\operatorname{ext} X$ there exists continuous affine function h_f on X such that $f = h_f$ on $\operatorname{ext} X$, see THEOREM 2.1.

Metrizable simplices were studied by E.M.Alfsen in [1] where was inferred extension theorem for bounded Borel functions. In particular, every bounded Borel function on $\operatorname{ext} X$ can be extended uniquely to a function from a set $\mathcal{A}'(X)$, where $\mathcal{A}'(X)$ is the smallest set of real valued functions which contains both l.s.c. and u.s.c. affine functions on X and is closed under bounded pointwise monotonne limits. Moreover, the set $\mathcal{A}'(X)$ consists exactly of those bounded Borel functions on $\operatorname{ext} X$ for which the "barycenter formula" is valid. The above result lean heavily on the fact that the set $\mathcal{A}'(X)$ over a metrizable simplex is a σ-complete Riesz space. This result is not entirely obvious, since $\operatorname{ext} X$ can be a G_δ set, see THEOREM 2.2.

U.Krause in [9] introduced the definition of $\mathcal{A}(X)$ as the smallest set of all real valued functions which contains sums of l.s.c and u.s.c functions and is closed under bounded pointwise monotonne limits. He used result due to H.Bauer that the set $\mathcal{A}^{usc}(X)$ of all u.s.c affine functions is an upper semilattice if and only if X is a simplex. This yields that $\mathcal{A}^{lsc}(X) + \mathcal{A}^{usc}(X)$ is Riesz space. So Krause only replace in Alfsen's definition of the $\mathcal{A}'(X)$ the word "and" by "sums". It is reasonable, because this gives simpler proof without assumption of a metrizability. Furthermore $\mathcal{A}(X)$ coincides with $\mathcal{A}'(X)$ for an arbitrary metrizable simplex X, see THEOREM 2.3 for more details.

Another lattice type and extension type theorem was obtained by S.Teleman in [16] in terms of an extended boundary measure $\widetilde{\mu}$, Choquet topology on $\operatorname{ext} X$, a family $\mathcal{M}^b(\operatorname{ext} X)$ of all bounded universally measurable functions on $\operatorname{ext} X$ and a family $\mathcal{U}(X)$ of all strongly universally measurable functions on X. For the completness we state here Teleman's theorem, see THEOREM 2.4.

In what follows, X is a compact convex subset of a locally convex space.

Theorem 2.1 (Bauer, 1964). *The following assertions are equivalent:*

(i) X *is a simplex with closed* $\operatorname{ext} X$,

(ii) $\mathcal{A}^c(X)$ *is a Riesz space,*

(iii) for any $f \in \mathcal{C}(\operatorname{ext} X)$ *there exists* $h_f \in \mathcal{A}^c(X)$ *such that* $f = h_f$ *on* $\operatorname{ext} X$.

Theorem 2.2 (Alfsen, 1966). *In the following (i) implies (ii) and (ii) implies (iii).*

(i) X *is a metrizable simplex,*

(ii) $\mathcal{A}'(X)$ *is a Riesz space,*

(iii) for any $f \in \mathcal{B}^b(\operatorname{ext} X)$ *there exists* $g_f \in \mathcal{A}'(X)$ *such that* $f = g_f$ *on* $\operatorname{ext} X$.

Theorem 2.3 (Krause, 1970). *The following assertions are equivalent:*

(i) X *is a simplex,*

(ii) $\mathcal{A}^{lsc}(X) + \mathcal{A}^{usc}(X)$ *is a Riesz space,*

(iii) $\mathcal{A}(X)$ *is a Riesz space.*

Theorem 2.4 (Teleman, 1985). *The following assertions are equivalent:*

(i) X *is a simplex,*

(ii) $\mathcal{U}(X)$ *is a Riesz space,*

(iii) for any $f \in \mathcal{M}^b(\operatorname{ext} X)$ *there exists* $u_f \in \mathcal{U}(X)$ *such that* $f = u_f$ *on* $\operatorname{ext} X$.

2.2 The function space generalization

Now, our aim is how to show we can transfer preceding results to a more general setting of function spaces.

Families $\mathcal{A}(\mathcal{H}), \overline{\mathcal{A}}(\mathcal{H})$ and $\mathcal{A}^{'}(\mathcal{H})$. We denote by $\mathcal{A}(\mathcal{H})$ the smallest family of $\mathcal{H}$–affine functions satisfying the following conditions:

(A1) $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H}) \subset \mathcal{A}(\mathcal{H})$,

(A2) if $l_n \in \mathcal{A}(\mathcal{H}), l$ real valued function on K, $l_n \nearrow l$ or $l_n \searrow l$, then $l \in \mathcal{A}(\mathcal{H})$.

If we replace in the previous definition of $\mathcal{A}(\mathcal{H})$ condition (A1) by (A1') we obtain another interesting family denoted $\mathcal{A}^{'}(\mathcal{H})$,

(A1') $\mathcal{A}^{lsc}(\mathcal{H}), \mathcal{A}^{usc}(\mathcal{H}) \subset \mathcal{A}(\mathcal{H})$.

Denote by $\overline{\mathcal{A}}(\mathcal{H})$ the sup-norm closure of the set $\mathcal{A}(\mathcal{H})$ in the set of all bounded real functions on K.

Strong envelopes. Let f be an upper bounded function on K. We define *upper strong envelope* as

$$f^{\circ}(x) = \inf \{h(x) : h \in \mathcal{A}^{lsc}(\mathcal{H}) \,,\; h \geq f \text{ on } K \}.$$

Similiary, for a lower bounded function f on K, we define the *lower strong envelope* $f_{\circ}$ so that $f_{\circ}(x) = -(-f)^{\circ}(x), x \in K$.

Family $\mathcal{U}(\mathcal{H})$. A bounded function f on K is called $\mathcal{H}$-strongly universaly measurable function, if $f_{\circ} = f^{\circ}$ on K. Let denote by $\mathcal{U}(\mathcal{H})$ the set of all $\mathcal{H}$-strongly universaly measurable functions f on K.

Observation 2.5. *If f is a bounded function on K, then $f \leq f^{\circ} \leq f^{*}$. Moreover, if f is an upper semicontinuous function, then $f^{\circ} = f^{*}$.*

Proof. The first part of observation is obvious and the second one is just COROLLARY 1.11. $\square$

Proposition 2.6. *If $\mathcal{H}$ is a function space on compact K, then*

(a) $\mathcal{A}'(\mathcal{H}) \subset \mathcal{A}(\mathcal{H}) \subset \mathcal{A}^{b}(\mathcal{H})$,

(b) $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H}) \subset \mathcal{U}(\mathcal{H})$.

Moreover, if K is metrizable, then

(c) $\mathcal{A}'(\mathcal{H}) = \mathcal{A}(\mathcal{H})$.

Proof. (a) It is obvious that $\mathcal{A}^{lsc}(\mathcal{H}), \mathcal{A}^{usc}(\mathcal{H}) \subset \mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$, so inclusion $\mathcal{A}'(\mathcal{H}) \subset \mathcal{A}(\mathcal{H})$ is trivial. The second inclusion is just Levi's theorem.

(b) Pick $f \in \mathcal{A}^{usc}(\mathcal{H})$. OBSERVATION 2.5 implies that $f = f^{\circ}$. No doubt that $f_{\circ} = f$ and PROPOSITION 1.9 yields that $f^{*} = f$. So $\mathcal{A}^{usc}(\mathcal{H}) \subset \mathcal{U}(\mathcal{H})$ and since $\mathcal{U}(\mathcal{H})$ is a vector space also $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H}) \subset \mathcal{U}(\mathcal{H})$.

(c) Assume that K is a metrizable compact. Pick $f \in \mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$, such that $f = g + h$, $g \in \mathcal{A}^{lsc}(\mathcal{H})$, $h \in \mathcal{A}^{usc}(\mathcal{H})$. According to second part of LEMMA 1.28 there exists an decreasing sequence $h_n \in \mathcal{A}^{c}(\mathcal{H}), h_n \geq h$, $h = \lim h_n$. So $g + h_n \in \mathcal{A}^{lsc}(\mathcal{H})$ and $g + h_n \searrow g + h$, therefore $g + h \in \mathcal{A}'(\mathcal{H})$. Since $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H}) \subset \mathcal{A}'(\mathcal{H})$, we see that the equality $\mathcal{A}(\mathcal{H}) = \mathcal{A}'(\mathcal{H})$ holds. $\square$

Theorem 2.7 (Lattice Theorem). *The following propositions are equivalent:*

(i) *$\mathcal{H}$ is a simplicial space,*

(ii) *$\mathcal{A}^{usc}(\mathcal{H})$ is an upper semilattice,*

(iii) *$\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$ is a Riesz space,*

(iv) *$\mathcal{A}(\mathcal{H})$ is a Riesz space.*

Proof. (i)$\Longrightarrow$(ii). Given $g_1, g_2 \in \mathcal{A}^{usc}(\mathcal{H})$, put $f := (g_1 \vee g_2)^*$. By LEMMA 1.10 $f = \inf\{g \in \mathcal{A}^{usc}(\mathcal{H}) : g \geq g_1 \vee g_2 \text{ on } K\}$ and according to PROPOSITION 1.26 $f \in \mathcal{A}^{usc}(\mathcal{H})$, further $f = \mathrm{H}(g_1 \vee g_2)$. So we can define the supremum operation $\curlyvee$ in $\mathcal{A}^{usc}(\mathcal{H})$ by

$$g_1 \curlyvee g_2 := \mathrm{H}(g_1 \vee g_2) \text{ for every } g_1, g_2 \in \mathcal{A}^{usc}(\mathcal{H}).$$

Clearly, $\curlyvee$ is the supremum operation in $\mathcal{A}^{usc}(\mathcal{H})$ endowed by natural pointwise ordering of functions.

(i)$\Longrightarrow$(iii). The implication (i)$\Longrightarrow$(ii) has just been proved. Let $\curlyvee$ be the supremum operation in $\mathcal{A}^{usc}(\mathcal{H})$ such that $g_1 \curlyvee g_2 = \mathrm{H}(g_1 \vee g_2)$ for every $g_1, g_2 \in \mathcal{A}^{usc}(\mathcal{H})$. Pick $f \in \mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$, $f_1 \in \mathcal{A}^{lsc}(\mathcal{H})$, $f_2 \in \mathcal{A}^{usc}(\mathcal{H})$, $f = f_1 + f_2$. Define

$$f^{\oplus} := \mathrm{H}(f^+).$$

The equalities

$$\mathrm{H}((f_1 + f_2)^+) = \mathrm{H}((-f_1) \vee f_2 + f_1) = (-f_1) \curlyvee f_2 + f_1$$

follow that $f^{\oplus} \in \mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$. We see that

$$f^{\oplus} = \mathrm{H}(f^+) = (f^+)^* \geq f^+. \tag{2.1}$$

On the other hand, if $f' \in \mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$, $f' = f_1' + f_2'$, $f_1' \in \mathcal{A}^{lsc}(\mathcal{H})$, $f_2' \in \mathcal{A}^{usc}(\mathcal{H})$, $f' \geq f^+$, then

$$f' = f_1' + f_2' = (f_1')_* + (f_2')^* = \mathrm{H}(f_1') + \mathrm{H}(f_2') = \mathrm{H}(f').$$

So

$$f' = \mathrm{H}(f'). \tag{2.2}$$

Where we have taken into the account LEMMA 1.26. $\mathrm{H}(f') \geq \mathrm{H}(f^+) = f^{\oplus}$ follow $f' \geq f^{\oplus}$.

Since $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$ is a vector space, we can define the supremum operation $\curlyvee$ in $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$ (which coincides with the supremum operation in $\mathcal{A}^{usc}(\mathcal{H})$, so it is reasonable to denote both by the same symbol $\curlyvee$) by

$$f \curlyvee g := (f - g)^{\oplus} + g \text{ for every } f, g \in \mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$$

and the infimum operation by

$$f \curlywedge g := -(-f) \curlyvee (-g) \text{ for every } f, g \in \mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H}).$$

(i)$\Longrightarrow$(iv). We have proved (i)$\Longrightarrow$(iii). Let $\curlyvee$ be the supremum operation in $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$ such that $f^{\oplus} = \mathrm{H}(f^+)$ for any $f \in \mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$. Denote $\mathcal{Z} := \{f \in \mathcal{A}(\mathcal{H}) : \mathrm{H}(f^+) \in \mathcal{A}(\mathcal{H})\}$. We see that $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H}) \subset \mathcal{Z}$. Using Levi's theorem we obtain that $\mathcal{A}(\mathcal{H}) \subset \mathcal{Z}$, so $\mathcal{A}(\mathcal{H}) = \mathcal{Z}$. Therefore, we can define

$$f^{\oplus} := \mathrm{H}(f^+) \text{ for every } f \in \mathcal{A}(\mathcal{H})$$

(the definition of $f^{\oplus}$ is correct, since it coincides with positive part in the lattice $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$). Using Levi's theorem follows from the (2.1) that $f^{\oplus} \geq f^+$ for every $f \in \mathcal{A}(\mathcal{H})$. For the reverse inequality, pick $f' \in \mathcal{A}(\mathcal{H})$, $f' \geq f^+$ and again using Levi's theorem to (2.2) we obtain that

$$f' = \mathrm{H}(f') \text{ and } \mathrm{H}(f') \geq \mathrm{H}(f^+) = f^{\oplus}.$$

Since $\mathcal{A}(\mathcal{H})$ is a vector space, we can define the lattice operations in it again by $f \curlyvee g := (f-g)^{\oplus} + g$ for every $f, g \in \mathcal{A}(\mathcal{H})$ and the infimum operation by $f \curlywedge g := -(-f) \curlyvee (-g)$ for every $f, g \in \mathcal{A}(\mathcal{H})$.

(ii)$\Longrightarrow$(i). Let $\mathcal{A}^{usc}(\mathcal{H})$ be an upper semilattice with the supremum operation $\curlyvee$. According to PROPOSITION 1.27 it suffices to show that for any $k \in \mathcal{K}^c(\mathcal{H})$ is $k^* \in \mathcal{A}^{usc}(\mathcal{H})$. Let $k \in \mathcal{K}^c(\mathcal{H})$. The COROLLARY 1.13 implies that there exists a decreasing net w_α in $\mathcal{W}(\mathcal{H})$, $w_\alpha = h_1^{(\alpha)} \vee \cdots \vee h_{n(\alpha)}^{(\alpha)}$, $h_1^{(\alpha)}, \dots, h_{n(\alpha)}^{(\alpha)} \in \mathcal{H}$ so that $w_\alpha \geq k$ and $k = \inf w_\alpha$. Put $v_\alpha = h_1^{(\alpha)} \curlyvee \cdots \curlyvee h_{n(\alpha)}^{(\alpha)}$, $v_\alpha \in \mathcal{A}^{usc}(\mathcal{H})$.

We want to show that $k^* = \inf v_\alpha$. Inded, the inequality $k^* \leq \inf v_\alpha$ is obvious. For the converse inequality, pick an arbitrary $h \in \mathcal{H}$, $h > f$ on K and $x \in K$. Since $k = \inf w_\alpha$, there exists w_{α_0} so that $k(x) \leq w_{\alpha_0} < h(x)$. Put $h' = h_1^{(\alpha)} \curlyvee \cdots \curlyvee h_{n(\alpha_0)}^{(\alpha)}$. So $k \leq h'$ on K and $h'(x) \leq h(x)$. Taking the infimum over all $h > k$ on K, $h \in \mathcal{H}$ we obtain that $f^* \geq \inf v_\alpha$.

(iii)$\Longrightarrow$(i). Let $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$ be a lattice with the supremum operation $\curlyvee$. For an arbitrary $l_1, \dots, l_n$ from $\mathcal{H}$ put $l = l_1 \curlyvee \dots \curlyvee l_n$. Observe that $h \geq l_1 \vee \cdots \vee l_n$ if and only if $h \geq l$ for any $h \in \mathcal{H}$. Taking the infimum we obtain $(l_1 \vee \cdots \vee l_n)^* = l^*$. Now we appeal to the Mokobodzki's THEOREM 1.21, so the following inequalities hold for an arbitrary maximal measure μ with barycenter $x \in K$:

$$l(x) = \mu(l) \leq \mu(l^*) = \mu((l_1 \vee \cdots \vee l_n)^*) = \mu(l_1 \vee \cdots \vee l_n) \leq \mu(l) = l(x) \qquad (2.3)$$

Therefore

$$\mu(l_1 \vee \cdots \vee l_n) = l_1 \curlyvee \cdots \curlyvee l_n.$$

If ν is another maximal measure with barycenter $x \in K$, then $\mu(l_1 \vee \cdots \vee l_n) = \nu(l_1 \vee \cdots \vee l_n)$. Similarly for $\curlywedge$, or using the identities $(-f) \curlyvee (-g) = -(f \curlywedge g)$, respectively $(-f) \vee (-g) = -(f \wedge g)$. According to the Stone–Weierstrass theorem the set $\mathcal{W}(\mathcal{H}) - \mathcal{W}(\mathcal{H})$ is dense in $\mathcal{C}(K)$, so $\mu = \nu$ and, by definition, $\mathcal{H}$ is a simplicial space.

(iv)$\Longrightarrow$(i). The proof of this implication is almost the same as the proof of the preceding implication (iii)$\Longrightarrow$(i). Just realize that the barycenter formula also holds for functions from $\mathcal{A}(\mathcal{H})$, see PROPOSITION 2.6 (cf. (2.3)).

□

The proof of the preceding THEOREM 2.7 shows the following observation:

Observation 2.8. *If $\mathcal{H}$ is a simplicial function space, then $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$ is a sublattice of the Riesz space $\mathcal{A}(\mathcal{H})$.*

In what follows, if $\mathcal{H}$ is a simplicial space, then we denote by $\curlyvee$ the supremum operation in $\mathcal{A}(\mathcal{H})$, by $f^{\oplus}$ the positive part of function $f \in \mathcal{A}(\mathcal{H})$, more precisely $f^{\oplus} := f \curlyvee 0$ for any $f \in \mathcal{A}(\mathcal{H})$. The meaning of notation $f^{\ominus}$ should be clear.

Theorem 2.9. *If $\mathcal{H}$ is a simplicial space, then $\mathcal{A}(\mathcal{H})$ is a M-space and the norm closure $\overline{\mathcal{A}}(\mathcal{H})$ of $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$ is an sub-M-space of $\mathcal{A}(\mathcal{H})$.*

Proof. By THEOREM 2.7, $\mathcal{A}(\mathcal{H})$ is a Riesz space and the definition of $\mathcal{A}(\mathcal{H})$ implies that $\mathcal{A}(\mathcal{H})$ is a σ-complete Riesz space.Moreover $\mathcal{A}(\mathcal{H})$ is a Banach space in supremum norm. Indeed, if l_n is a Cauchy sequence in $\mathcal{A}(\mathcal{H})$ with uniform limit l in $\mathcal{B}^b(K)$, then uniform convergence implies that l is a $\mathcal{H}$-affine. For every $n \geq 1$ there exists $n_0(n) \geq 1$, such that $||f - l_i|| \leq 1/n$ for all $i \geq n_0(n)$. Since $f = \inf_n \sup_{i \geq n_0(n)} l_i$, then

$$f(x) = \delta_x(f) = \inf_n \delta_x \left(\sup_{i \geq n_0(n)} l_i \right).$$

In $\mathcal{A}(\mathcal{H})$, σ-distributivity laws hold, so using LEMMA 1.29 we obtain the equality

$$\delta_x \left(\sup_{i \geq n_0(n)} l_i \right) = \curlyvee_{i \geq n_0(n)} l_i(x).$$

The argument of σ-completness gives that $\curlyvee_{i \geq n_0(n)} l_i \in \mathcal{A}(\mathcal{H})$ and, of course, $f \in \mathcal{A}(\mathcal{H})$.

Now we verify that $\mathcal{A}(\mathcal{H})$ is a Banach lattice. Let $l, l' \in \mathcal{A}(\mathcal{H}), |l| < |l'|$. Using the equality $|l| = l \curlyvee -l$ we obtain that

$$l, -l \leq l \curlyvee -l \leq l' \curlyvee -l' \leq ||l'||,$$

thus $||l|| \leq ||l'||$.

We check that $||\,.\,||$ is an M-norm. Indeed, fix $l_1, l_2 \in \mathcal{A}(\mathcal{H}), l_1, l_2$ nonnegative. The inequality $0 \leq l_1, l_2 \leq l_1 \curlyvee l_2$ implies that $||l_1|| \vee ||l_2|| \leq ||l_1 \curlyvee l_2||$. Therefore $l_1, l_2 \leq ||l_1|| \vee ||l_2|| \in \mathcal{A}(\mathcal{H})$, and thus $l_1 \curlyvee l_2 \leq ||l_1|| \vee ||l_2||$. We see that $||l_1 \curlyvee l_2|| \leq ||l_1|| \vee ||l_2||$.

It remains to prove the last part of the proposition. $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H}) \subset \mathcal{A}(\mathcal{H})$ implies that $\overline{\mathcal{A}}(\mathcal{H})$ is a Banach subspace of $\mathcal{A}(\mathcal{H})$. Since $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$ is a sublattice of $\mathcal{A}(\mathcal{H})$, then $\overline{\mathcal{A}}(\mathcal{H})$ is a Banach sublattice of $\mathcal{A}(\mathcal{H})$. Finally, $\mathcal{A}(\mathcal{H})$ is an M-space, then of course $\overline{\mathcal{A}}(\mathcal{H})$ is an M-space. $\square$

Chapter 3

Spectral measures

Let me start with roughly speaking to give an intuition for a better understanding of this chapter. In what follows, assume for a reasons of simplicity, that $\mathcal{H}$ is a Bauer simplicial space on K. We have an Dirichlet operator H which assigns to each continuous function f on K the abstract solution of the Dirichlet problem - a uniquely determined continuous $\mathcal{H}$-affine function on K. It is known that the abstract solution depends only on the values of f on the Choquet boundary of $\mathcal{H}$.

Now concentrate to a deformation of a fixed boundary condition f by an arbitrary continuous function φ on $\mathbb{R}$. Quite precisely, consider a mapping $\mathrm{I}_f(\varphi) = \mathrm{H}_{\varphi\circ f}$ to the set $\mathcal{A}^c(\mathcal{H})$ of all continuous $\mathcal{H}$-affine functions on K. Natural task is to extend I_f to a larger system of real functions which contains at least characteristic functions of an left unbounded intervals. This is motivated by spectral theory, in particular by spectral partitions of $\mathcal{H}$-affine functions.

One of them is the system $\mathcal{B}^b(\mathbb{R})$ of all bounded Borel functions on $\mathbb{R}$. We can obtain this system enclosing the set of continuous bounded (or with compact support) to bounded pointwise monotonne limits. So we can guess that the range of I_f extended to $\mathcal{B}^b(\mathbb{R})$ should be at least some vector space E closed under bounded pointwise monotonne limits which contains $\mathcal{A}^c(\mathcal{H})$. A chance of extending I_f strongly depends on an additional structure of E. U. Krause in [9] extend I_f with range $\mathcal{A}(X)$. It is a smallest possible range in the sense above. We generalize it to the function spaces. The following theorems are based on Krause idea.

Note that another spectral theory was studied by Rogalski [14].

In this chapter $\mathcal{H}$ is a simplical space. For an arbitrary $f \in \mathcal{A}(\mathcal{H})$ define the mapping $\mathrm{I}_f(\varphi) = \mathrm{H}(\varphi \circ f)$, $\varphi \in \mathcal{B}^b(\mathbb{R})$.

Theorem 3.1. *The following assertions hold:*

(a) If $f' \in \mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$, then $\mathrm{I}_{f'}(\mathcal{C}(\mathbb{R})) \subset \overline{\mathcal{A}}(\mathcal{H})$.

(b) If $f \in \mathcal{A}(\mathcal{H})$, then $\mathrm{I}_f(\mathcal{C}(\mathbb{R})) \subset \mathcal{A}(\mathcal{H})$ and $\mathrm{I}_f(\varphi_1 \vee \varphi_2) = \mathrm{I}_f(\varphi_1) \curlyvee \mathrm{I}_f(\varphi_2)$ for any $\varphi_1, \varphi_2 \in \mathcal{C}(\mathbb{R})$.

Proof. (a) Denote $B = \{r \in \mathbb{R} : |r| \leq ||f'||\}$ and by $\mathcal{A}^c(B)$ the set of all continuous affine functions on B and by $\mathcal{W}(B) = \{\varphi_1 \vee \cdots \vee \varphi_2 : \varphi_i \in \mathcal{A}^c(B)\}$. If $\varphi \in \mathcal{A}^c(B)$, then clearly

$$\delta_x((\varphi_1 \vee \cdots \vee \varphi_n) \circ f') = \delta_x((\varphi_1 \circ f') \vee \cdots \vee (\varphi_n \circ f')) = (\varphi_1 \circ f')(x) \curlyvee \cdots \curlyvee (\varphi_n \circ f')(x)$$

for any maximal measure δ_x with barycenter $x \in K$ or, equivalently, $\mathrm{I}_{f'}(\mathcal{W}(B)) \in \mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H})$. According to the Stone–Weierstrass theorem $\mathcal{W}(B) - \mathcal{W}(B)$ is dense in $\mathcal{C}(B)$, so $\mathrm{I}'_f(\varphi) \in \overline{\mathcal{A}}(\mathcal{H})$ for any $\varphi \in \mathcal{C}(B)$.

(b) Levi's theorem and the definition of $\mathcal{A}(\mathcal{H})$ show that $\mathrm{I}_f(\mathcal{C}(\mathbb{R})) \subset \mathcal{A}(\mathcal{H})$. From (a) we know that the equality

$$\mathrm{I}_f(\varphi_1 \vee \varphi_2) = \mathrm{I}_f(\varphi_1) \curlyvee \mathrm{I}_f(\varphi_2)$$

holds whenever $\varphi_1, \varphi_2 \in \mathcal{W}(B)$. Furthermore,

$$\mathrm{I}_f(\varphi_d^+) = \mathrm{I}_f(\varphi_1 \vee \varphi_2) - \mathrm{I}_f(\varphi_2) = \mathrm{I}_f(\varphi_1) \curlyvee \mathrm{I}_f(\varphi_2) - \mathrm{I}_f(\varphi_2) = \mathrm{I}_f(\varphi_d)^{\oplus}$$

where φ_d is a difference of two functions $\varphi_1, \varphi_2 \in \mathcal{W}(B)$. If now $\varphi \in \mathcal{C}(B)$, then for any $\varepsilon > 0$ there exists $\varphi_d \in \mathcal{W}(B) - \mathcal{W}(B)$ such that $\varphi_d - \varepsilon \leq f \leq \varphi_d + \varepsilon$ on B. Therefore

$$\mathrm{I}_f(\varphi^+) \leq \mathrm{I}_f(\varphi_d^+) = \mathrm{I}_f(\varphi_d)^{\oplus} + \varepsilon \leq \mathrm{I}_f(\varphi + \varepsilon)^{\oplus} + \varepsilon \leq \mathrm{I}_f(\varphi)^{\oplus} + 2\varepsilon.$$

So $\mathrm{I}_f(\varphi^+) \leq \mathrm{I}_f(\varphi)^{\oplus}$. The reverse inequality is obvious and the proof is finished. □

Corollary 3.2. *If $\mathcal{H}$ is a Bauer simplicial space, then we have:*

(a') If $f' \in \mathcal{A}^c(\mathcal{H})$, then $\mathrm{I}_{f'}(\mathcal{C}(\mathbb{R})) \subset \mathcal{A}^c(\mathcal{H})$.

Remark 3.3. In the part (b) of the preceding Theorem 3.1 we could replace uniform convergence by a bounded pointwise monotonne convergence and obtain without using a Banach space structure of $\mathcal{A}(\mathcal{H})$ the following:

(b') If $f \in \mathcal{A}(\mathcal{H})$, then $\mathrm{I}_f(\mathcal{B}^b(\mathbb{R})) \subset \mathcal{A}(\mathcal{H})$ and $\mathrm{I}_f(\varphi_1 \vee \varphi_2) = \mathrm{I}_f(\varphi_1) \curlyvee \mathrm{I}_f(\varphi_2)$ for any $\varphi_1, \varphi_2 \in \mathcal{B}^b(\mathbb{R})$.

Natural question arises if any mapping $\mathrm{I} : \mathcal{C}(\mathbb{R}) \to \mathcal{A}(\mathcal{H})$ can be represented by I_f for some $f \in \mathcal{A}(\mathcal{H})$. The following definition specifies assumptions on I.

Let χ_X denotes the characteristic function of an arbitrary set X.

Spectral $\mathcal{A}(\mathcal{H})$-integral. A mapping $\mathrm{I} : \mathcal{C}(\mathbb{R}) \to \mathcal{A}(\mathcal{H})$ is said to be

(N) *nonnegative* if $\mathrm{I}(\varphi) \geq 0$ for any $\varphi \geq 0$,

(P) *probability* if $\mathrm{I}(\chi_{\mathbb{R}}) = \chi_K$,

(PI) *probability $\mathcal{A}(\mathcal{H})$-integral* if it is nonnegative, probability and linear on $\mathcal{C}(\mathbb{R})$.

(SI) *spectral $\mathcal{A}(\mathcal{H})$-integral* if it is probability $\mathcal{A}(\mathcal{H})$-integral and further:

$$\mathrm{I}(\varphi_1 \vee \varphi_2) = \mathrm{I}(\varphi_1) \curlyvee \mathrm{I}(\varphi_2), \text{ for any } \varphi_1, \varphi_2 \in \mathcal{C}(\mathbb{R}).$$

In the sequel we will consider only a probability integrals, so we will write shortly an integral instead of a probability integral.

Remark 3.4. The first part of the assertion (b) from THEOREM 3.1 says that the mapping m_f defined by $\mathrm{m}_f(B) := \mathrm{H}(\chi_B \circ f)$ for an arbitrary Borel subset B of K is a Banach space valued measure, since $\mathcal{A}(\mathcal{H})$ is a Banach space. One can prove Lebesgue's type theorem for an integral with respect to the measure m_f and since $\mathcal{A}(\mathcal{H})$ is also σ-complete Riesz space it follows Levi's type theorem for such Banach space valued measure; with bounded pointwise monotonne limit of a sequence as integrable majorant of this sequence. For more details see [5]. According this it is not entirely obvious fact applied to an $\mathcal{A}(\mathcal{H})$-integral I we can extend it to an extended $\mathcal{A}(\mathcal{H})$-integral $\widetilde{\mathrm{I}} : \mathcal{B}^b(\mathbb{R}) \to \mathcal{A}(\mathcal{H})$ which fulfils Levi (Lebesgue) type theorem.

This not obvious facts are used in the proof of the EXTENSION LEMMA 3.5 applied to the $\mathcal{A}(\mathcal{H})$-integral I, respectively spectral $\mathcal{A}(\mathcal{H})$-integral.

Lemma 3.5 (Extension Lemma). *If $\widetilde{\mathrm{I}}$ is an extended spectral $\mathcal{A}(\mathcal{H})$-integral, then*

$$\widetilde{\mathrm{I}}(\varphi_1 \vee \varphi_2) = \widetilde{\mathrm{I}}(\varphi_1) \curlyvee \widetilde{\mathrm{I}}(\varphi_2), \ \textit{for any } \varphi_1, \varphi_2 \in \mathcal{B}^b(\mathbb{R}).$$

Proof. Denote by $\mathcal{Z} = \{\varphi \in \mathcal{B}^b(\mathbb{R}) : \widetilde{\mathrm{I}}(\varphi^+) = \widetilde{\mathrm{I}}(\varphi)^{\oplus}\}$. By the definition $\mathcal{C}_c(\mathbb{R}) \subset \mathcal{Z}$. We show that $\mathcal{Z}$ is closed under bounded monotonne pointwise limits. Let $\{\varphi_n\}$ be an upper bounded increasing sequence, $\varphi = \sup \varphi_n$. Observe that $\varphi^+ = \sup \varphi_n^+$ and using Levi's theorem for spectral $\mathcal{A}(\mathcal{H})$-integral, (see REMMARK 3.4) we obtain equalities:

$$\begin{aligned}\widetilde{\mathrm{I}}(\varphi^+) &= \sup \widetilde{\mathrm{I}}(\varphi_n^+) = \sup(\widetilde{\mathrm{I}}(\varphi_n)^{\oplus}) = \sup \mathrm{H}(\widetilde{\mathrm{I}}(\varphi_n)^+) = \\ &= \mathrm{H}(\sup(\widetilde{\mathrm{I}}(\varphi_n))^+) = \mathrm{H}((\sup \widetilde{\mathrm{I}}(\varphi_n))^+) = \mathrm{H}(\widetilde{\mathrm{I}}(\varphi)^+) = \widetilde{\mathrm{I}}(\varphi)^{\oplus},\end{aligned}$$

thus $\varphi \in \mathcal{Z}$ and, similarily, for lower bounded decreasing sequences. We see that $\mathcal{Z} = \mathcal{B}^b(\mathbb{R})$ and the proof is finished. ☐

Partition and spectral class. The family $\{l_\lambda\}_{\lambda \in \mathbb{R}} \subset \mathcal{A}(\mathcal{H})$ is called *a partition of* $\mathcal{A}(\mathcal{H})$, if the following conditions hold:

(P1) $0 \le l_\lambda \le 1$,

(P2) $l_\lambda \le l_{\lambda'}$, for $\lambda \le \lambda'$,

(P3) $l_{\lambda'} = \sup_{\lambda < \lambda'} l_\lambda$ (pointwise),

(P4) $\lim_{\lambda \to +\infty} l_\lambda = 1$, $\lim_{\lambda \to -\infty} l_\lambda = 0$ (pointwise).

A partition $\{l_\lambda\}_{\lambda \in \mathbb{R}}$ of $\mathcal{A}(\mathcal{H})$ is called *a spectral partition* of $\mathcal{A}(\mathcal{H})$ if, moreover, the following condition holds:

(SP) $l_\lambda \curlyvee (1 - l_\lambda) = 1$, for every $\lambda \in \mathbb{R}$.

Let I be an $\mathcal{A}(\mathcal{H})$-integral. For $\lambda \in \mathbb{R}$ define the function $l_\lambda^{\mathrm{I}} := \mathrm{I}(\chi(-\infty, \lambda))$ from $\mathcal{A}(\mathcal{H})$.

Theorem 3.6. *The following assertions hold:*

(a) If I *is an* $\mathcal{A}(\mathcal{H})$*-integral, then the family of functions* $\{l_\lambda^{\mathrm{I}}\}_{\lambda\in\mathbb{R}}$ *is the partition of* $\mathcal{A}(\mathcal{H})$ *corresponding to* I*.*

(b) If $\{l_\lambda\}_{\lambda\in\mathbb{R}}$ *is a partition of* $\mathcal{A}(\mathcal{H})$*, then there exists a unique* $\mathcal{A}(\mathcal{H})$*-integral* I*, such that* $l_\lambda = l_\lambda^{\mathrm{I}}$ *for every* $\lambda \in \mathbb{R}$ *and*

$$\mathrm{I}(\varphi)(x) = \int_{\mathbb{R}} \varphi(\lambda) l_\lambda(x) d\lambda, \varphi \in \mathcal{B}^b(\mathbb{R}).$$

Proof. (a) Probability condition (P) and nonnegativity condition (N) implies (P1) and (P2). Levi's theorem shows that (P3) and (P4) are fulfilled.

(b) For $x \in K$, define the function $g_x(\lambda) := l_\lambda(x)$. (P3) implies that g_x is monotonne and continous from the left, thus the integral $\int_{\mathbb{R}} \varphi(\lambda) g_x(\lambda) d\lambda$ exists for every $\varphi \in \mathcal{C}_c(\mathbb{R})$. So we are able to define the function $l_\varphi : x \mapsto \int \varphi\, g_x, x \in K$, for every $\varphi \in \mathcal{C}_c(\mathbb{R})$. One can prove similiarly as in Lemma 3.5 that $l_\varphi \in \mathcal{A}(\mathcal{H})$. Now define the mapping $\mathrm{I}(\varphi)(x) = \int \varphi\, g_x$. Observe that I is an $\mathcal{A}(\mathcal{H})$-integral and $l_\lambda^{\mathrm{I}} = l_\lambda, \lambda \in \mathbb{R}$. □

Denote by $B(T)$ the family of all Borel subsets of an arbitrary topological space T.

Proposition 3.7. *If* I *is an* $\mathcal{A}(\mathcal{H})$*-integral, then* I *is a spectral* $\mathcal{A}(\mathcal{H})$*-integral if and only the partition of* $\mathcal{A}(\mathcal{H})$ *corresponding to* I *is a spectral partion.*

Proof. If I is a spectral $\mathcal{A}(\mathcal{H})$-integral, then $\{\mathrm{I}_\lambda^{\mathrm{I}}\}$ is a spectral partition according to the Theorem 3.5.

On the other hand, let $\{l_\lambda\}$ be a spectral partition. According to the Proposition 3.6 there exists a unique $\mathcal{A}(\mathcal{H})$-integral I, such that $l_\lambda = l_\lambda^{\mathrm{I}}$, for every $\lambda \in \mathbb{R}$. Denote by $l_D = \mathrm{I}(\chi_D)$ for an arbitrary $D \in B(\mathbb{R})$. Further denote

$$\mathcal{D} = \{D \in B(\mathbb{R}) : l_D \curlyvee (1 - l_D) = 1\}.$$

$\mathcal{D}$ is a Dynkin system, that is, a family of subsets of $\mathbb{R}$ such that: $\mathbb{R} \in \mathcal{D}$; if $D_1, D_2 \in \mathcal{D}$ and $D_1 \subset D_2$, then $D_2 \setminus D_1 \in \mathcal{D}$ and if $D_i \in \mathcal{D}, i = 1, 2, \ldots$ is a sequence of mutually disjoint sets from $\mathcal{D}$, then also $\cup_{i=1}^{\infty} D_i \in \mathcal{D}$.

Indeed, given $D_1, D_2 \in \mathcal{D}$, $D_1 \subset D_2$ denote $l_1 = l_{D_1}$ and $l_2 = l_{D_2}$. We see that $l_1 \curlywedge (1 - l_1) = 0$ and $l_2 \curlywedge (1 - l_2) = 0$, therefore $(1 - l_2) \curlywedge (l_2 - l_1) = 0$, so $1 - l_1 = (1 - l_2) \curlyvee (l_2 - l_1)$ and $(1 - l_2) \curlywedge l_1 = 0$ also $1 - (l_2 - l_1) = (1 - l_2) \curlyvee l_1$. This implies that

$$(1 - (l_2 - l_1)) \curlyvee (l_2 - l_1) = (1 - l_2) \curlyvee l_1 \curlyvee (l_2 - l_1) = l_1 \curlyvee (1 - l_1) = 1.$$

We see that the set theoretic difference of sets D_1 and D_2 is from $\mathcal{D}$. Now pick $D_i \in \mathcal{D}, i = 1, 2, \ldots$ a sequence of mutually disjoint sets and denote $D = \cup_{i=1}^{\infty} D_i$. The inequalities $(1 - l_D) \curlywedge l_{D_i} \le (1 - l_{D_i}) \curlywedge l_{D_i}, i = 1, \ldots, k$ yield that

$$(1 - l_D) \curlywedge \left(\sum_{i=1}^{n} l_{D_i} \right) \le \sum_{i=1}^{n} (1 - l_D) \curlywedge l_{D_i} = 0.$$

Since D_i are pairwise disjoint we have that $\sum_{i=1}^n l_{D_i} = l_{\cup_{i=1}^n D}$ and then $l_D = \sup_n \sum_{i=1}^n l_{D_i}$. This implies that $l_D \curlywedge (1 - l_D) = 0$, so $D \in \mathcal{D}$. We have just verified that $\mathcal{D}$ is a Dynkin system, hence $\mathcal{D} = B(\mathbb{R})$.

If D_1 and D_2 are two disjoint Borel sets, then

$$0 \leq l_{D_1} \curlyvee l_{D_2} \leq l_{\mathbb{R}\setminus D_2} \curlyvee l_{D_2} = (1 - l_{D_2}) \curlyvee l_{D_2} = 0.$$

If D_1 and D_2 are arbitrary Borel sets, then for disjoint sets $D_1 \setminus (D_1 \cap D_2)$ and $D_2 \setminus (D_1 \cap D_2)$ the following equalities hold

$$(l_{D_1} - l_{D_1 \cap D_2}) \curlyvee (l_{D_2} - l_{D_1 \cap D_2}) = 0,$$
$$l_{D_1 \cap D_2} = l_{D_1} \curlywedge l_{D_2},$$
$$l_{D_1 \cup D_2} = l_{D_1} + l_{D_2} - l_{D_1 \cap D_2} = l_{D_1} \curlyvee l_{D_2}.$$

Now let $\varphi \in B(\mathbb{R})$ be a simple function(thus is a function with finitely many values $a_i \in R, i = 1, 2, \ldots, n$) or, equivalently, $\varphi = \sum_i a_i \chi_{D_i}$, where $D_i = r \in \mathbb{R} : \varphi(r) = a_i)$. We see that the equality $\varphi^+ = \sum_i a_i^+ \chi_{D_i}$ holds and then the equality $\mathrm{I}(\varphi^+) = \sum_i a_i^+ l_{D_i}$ follows. Since D_i are pairwise disjoint sets, then according to the previous part of the proof we obtain that

$$\left(\sum_{i=1}^k a_i l_{D_i}\right)^{\oplus} = \sum_{i=1}^k (a_i l_{D_i})^{\oplus} = \sum_{i=1}^k a_i^+ l_{D_i}.$$

Therefore $\mathrm{I}(\varphi^+) = \mathrm{I}(\varphi)^{\oplus}$. Now, appeal to the know fact that the set of all simple functions is dense in the set $\mathcal{B}^b(\mathbb{R})$. This shows that $\mathrm{I}(\varphi^+) = \mathrm{I}(\varphi)^{\oplus}$ for an arbitrary function $\varphi \in \mathcal{B}^b(\mathbb{R})$ which concludes the proof. □

Denote by Id the identity function on $\mathbb{R}$.

Expectation of $\mathcal{A}(\mathcal{H})$-integral. Let I be an $\mathcal{A}(\mathcal{H})$-integral. According to Lemma 3.5, we can define *an expectation* e_I *of* $\mathcal{A}(\mathcal{H})$*-integral* I by $\mathrm{e}_\mathrm{I}(x) = \mathrm{I}(\mathrm{Id})(x)$ if it exists. We see that then in this case $\mathrm{e}_\mathrm{I} \in \mathcal{A}(\mathcal{H})$.

Lemma 3.8. *If* I *is a spectral* $\mathcal{A}(\mathcal{H})$*-integral for which the expectation* e_I *exists, then* $\mathrm{I}(\chi_{(-\infty,\lambda)}) = \sup_n (n(\lambda - \mathrm{e}_\mathrm{I})^{\oplus}) \curlywedge 1)$ *for all* $\lambda \in \mathbb{R}$.

Proof. Pick $\lambda_0 \in \mathbb{R}$ and put $\varphi(\lambda) = \lambda_0 - \lambda$. Lemma 3.5 implies that the expecation e_I of I exists and $\mathrm{e}_\mathrm{I} = \mathrm{I}(\mathrm{Id}) \in \mathcal{A}(\mathcal{H})$, also $\mathrm{I}(\varphi^+) = \mathrm{I}(\varphi)^{\oplus} = (\lambda_0 - \mathrm{e}_\mathrm{I})^{\oplus}$. Further from the same Lemma we obtain that

$$\mathrm{I}((n\,\varphi^+) \vee 1) = (n(\lambda_0 - \mathrm{Id})^{\oplus}) \curlywedge 1,$$
$$\mathrm{I}(\sup_n((n\,\varphi^+) \vee 1)) = \sup_n (n(\lambda_0 - \mathrm{e}_\mathrm{I})^{\oplus}) \curlywedge 1.$$

Observe that the following equality $\sup_n (n(\lambda_0 - \mathrm{Id})^+) \wedge 1 = \chi_{(-\infty,\lambda)}$ finishes the proof. □

Theorem 3.9. *The following assertions hold:*

(a) If I *is a spectral* $\mathcal{A}(\mathcal{H})$*-integral, then the family of functions* $\{l_\lambda^{\mathrm{I}}\}_{\lambda\in\mathbb{R}}$ *is the spectral partition of* $\mathcal{A}(\mathcal{H})$ *corresponding to* I.

(b) If $\{l_\lambda\}_{\lambda\in\mathbb{R}}$ *is a spectral partition of* $\mathcal{A}(\mathcal{H})$*, then there exists a unique spectral* $\mathcal{A}(\mathcal{H})$*-integral* I *such that* $l_\lambda = l_\lambda^{\mathrm{I}}$*, for every* $\lambda \in \mathbb{R}$*. and*

$$\mathrm{I}(\varphi)(x) = \int_{\mathbb{R}} \varphi(\lambda) l_\lambda(x) d\lambda, \varphi \in \mathcal{B}^b(\mathbb{R}).$$

Proof. (a) It is just an application of the condition (SI) from the definition of a spectral integral used to the charecteristic function of an arbitrary left unbounded interval.

(b) If $\{l_\lambda\}_{\lambda\in\mathbb{R}}$ is a spectral partition of $\mathcal{A}(\mathcal{H})$, then according to the THEOREM 3.6 there exists a unique $\mathcal{A}(\mathcal{H})$-integral I, such that $l_\lambda = l_\lambda^{\mathrm{I}}$ for every $\lambda \in \mathbb{R}$. Now the conlusion immediatly follows from the PROPOSITION 3.7

□

Theorem 3.10. *The following assertions hold:*

(a) If $f \in \mathcal{A}(\mathcal{H})$*, then the mapping* I_f *is the spectral* $\mathcal{A}(\mathcal{H})$*-integral with expectation* f.

(b) If I *is a spectral* $\mathcal{A}(\mathcal{H})$*-integral for which expectation exists, then there exists unique* $f \in \mathcal{A}(\mathcal{H})$ *such that* $\mathrm{I} = \mathrm{I}_f$.

Proof. (a) It is just the part (b) of THEOREM 3.1.

(b) Since expectation exists, we can put $f = \mathrm{I}(\mathrm{Id})$ and by the definition we obtain that $\mathrm{I} = \mathrm{I}_f$. Uniqueness follows from LEMMA 3.8. □

Theorem 3.11 (Spectral theorem). *For any* $f \in \mathcal{A}(\mathcal{H})$ *there exists a uniquely determined spectral partition* $\{l_\lambda\}_{\lambda\in\mathbb{R}}$ *such that* $f = \int_{\mathbb{R}} \lambda\, l_\lambda(x) d\lambda$.

Proof. According to THEOREMS 3.9 and 3.10 for any $f \in \mathcal{A}(\mathcal{H})$ there exists a unique spectral $\mathcal{A}(\mathcal{H})$-integral I_f with expectation f such that

$$f = \mathrm{I}(\mathrm{Id}) = \int_{\mathbb{R}} \lambda\, l_\lambda^{\mathrm{I}_f} d\lambda.$$

□

Chapter 4

Open problems

Notice that this chapter is more intuitive than formal from the mathematical point of view. It contains collection of open problems and rough strategy how try to overcome them.

4.1 Abstract integration

One can interpret a measure as (in some sense) an additive set function or by another name as an integral. In term of applications it seems demand σ-additivity instead of additivity and request for fulfilment of so-called Daniell condition for integral. In what follows we compare two relatively different approaches: a norm one and a lattice one. For reasons of simplicity, $(X, \mathcal{S})$ is a measurable space.

Order measures. Let $(Q, \curlyvee)$ be a σ-complete Riesz space with unit e. Set function $\mu : \mathcal{S} \to Q$ is said to be

(a) *nonnegative* if $\mu E \geq 0$ for any $E \in \mathcal{S}$,

(b) *order σ-additive* if $\mu(\sum_{i=1}^{\infty} E_i) = (\mathrm{o})\text{-} \sum_{i=1}^{\infty} \mu\, E_i$ for an arbitrary sequence of mutually disjoint sets E_i from $\mathcal{S}$,

(c) *order measure* if it is order σ-additive and $\mu\emptyset = 0$,

(d) *probability measure* if it is a order measure and $\mu X = e$,

(e) *spectral measure* if it is probability measure and $\mu(E_1 \cup E_2) = \mu E_1 \curlyvee E_2$, for any $E_1, E_2 \in \mathcal{S}$.

Banach space valued measures. Let $(B, ||\,.\,||)$ be a Banach space. A set function $\mu : \mathcal{S} \to Q$ is said to be

(g) *norm σ-additive* if $\mu(\sum_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu E_i$, for an arbitrary sequence of mutually disjoint sets E_i from $\mathcal{S}$,

(h) *Banach space valued measure* or shortly Banach measure, if it is norm σ-additive and $\mu\emptyset = 0$.

A notion of an integral is usually associated with addition respect to some measure. For order measures as well as for Banach space valued measures one can thanks to a linearity define an integral for functions with finitely many values(so-called step function). A next step of extending to larger family of functions is markedly different.

Note that an extension of integrals depends also on a range of functions which we would like to integrate. In general tasks it can happens that both measure and function are valued in a different vector spaces. This case is developed in Dinculeanu [5, 1966] or a little bit brieflier in newer [6, 2000] or see also [7, 2002] both by the same author. For our purposes it suffices to deal with integration of real functions. Now we roughly outline constructions and main differences between lattice extending and norm extending.

Integrals on lattices. In the case of order measure the set of all step functions is Riesz space in a natural pointwise ordering. We would like to apply the Daniell extension method, but some technical problem occurs since we do not have an ε-technique as in the real case where we use well-known fact: if s is a supremum of an arbitrary bounded set M of real numbers, then for any $\varepsilon > 0$ there exists an element s' in M such that $s \geq s' > s - \varepsilon$. In general we are forced to assume that Q fulfils additional conditions. In Cristescu [4, 1976] it is (o)-countabilty and σ-regularity. In Riečan [13, 1997] it is condition of weak σ-distributivity. Significant is that for an integral defined in above mentioned papers Levi's theorem holds.

Banach space valued integration. If Banach measure m has a finite variation μ, then one can complete the set of all step functions in an integtral norm $\int ||\,.\,||d\mu$ and obtain a larger class of m-integrable functions. Give a notice that we have no Levi's theorem even if we consider real functions ordered in a natural pointwise ordering. Instead of Levi's theorem one can prove Lebesgue dominated convergence theorem.

Note that from the standpoint of theory one can interpret spectral measure as a lattice homomorphism. In this connection cite the Kantorovich extension theorem based on Hahn–Banach type theorem in the context of Riesz spaces, see Meyer-Nieberg [11, 1991].

Situation is more complicated when a topology comes into effect, that is, if we put $(X, \mathcal{S}) := (T, B(T))$, where T is a locally compact space. An natural question arises if extending the set of all bounded continous functions we obtain the set of all bounded Borel functions $\mathcal{B}^b(T)$. According to Cristescu [4, 1976] answer should be yes, but only for T metrizable and his way of extending fails for continous functions with compact support $\mathcal{C}_c(T)$, since $\mathcal{C}_c(T)$is not a majorizing subspace of $\mathcal{B}^b(T)$. Another situation is discussed in the paper [8, 1976] by Khurana, where is inferred a different extension theorem from $C^c(T)$ to the set of all Borel functions with compact support. He used nets, duals, biduals, weak topology. For T compact, this result is proved in [17, 1972] by an entirely different method.

This section was only introduction to the abstract theory of integration. Applications of preceding metods to solving open problems in function space theory will be studied in the following section Function spaces.

4.2 Function spaces

In what follows, we deal with three type of problems: problems about bilateral relationship of special families of $\mathcal{H}$-affine functions, problems about structure of special families of $\mathcal{H}$-affine functions and problems about extending special $\mathcal{H}$-affine functions.

Further families of $\mathcal{H}$-affine functions. Let X be an arbitrary Hausdorff topological space. Denote by $\mathcal{B}_\alpha(X)$ the set of all Baire functions of the class α and by $\mathcal{B}^m_\alpha(X)$ a similiarly defined set of functions as $\mathcal{B}_\alpha(X)$, which is generated only by bounded pointwise monotonne limits. $\mathcal{B}(X)$ denotes the set of all Baire functions on X. The notation of $\mathcal{B}^m(X)$ should be clear. Further, put $\mathcal{AB}(\mathcal{H}) = \mathcal{A}^b(\mathcal{H}) \cap \mathcal{B}(K)$, respectively $\mathcal{AB}_\alpha(\mathcal{H}) = \mathcal{A}^b(\mathcal{H}) \cap \mathcal{B}_\alpha(K)$. Denote by $\mathcal{A}_\alpha(\mathcal{H})$ the α-th class generated by bounded pointwise limits of sequence from $\mathcal{A}^c(\mathcal{H})$. The meaning of notation $\mathcal{A}^m_\alpha(\mathcal{H})$ should be also obvious.

$\mathcal{A}(\mathcal{H})$ **and** $\mathcal{U}(\mathcal{H})$**.** We have showed that $\mathcal{A}^{lsc}(\mathcal{H}) + \mathcal{A}^{usc}(\mathcal{H}) \subset \mathcal{U}(\mathcal{H})$ (Lemma 2.6). It is not clear if $\mathcal{A}(\mathcal{H}) \subset \mathcal{U}(\mathcal{H})$. Of course, it suffices to show that the set $\mathcal{U}(\mathcal{H})$ is closed under bounded monotonne pointwise limits and this is strongly connected with a limit behavior of the equality of strong envelopes $f_\circ = f^\circ$. Is it possible interpret strong envelopes as upper and lower lattice integrals? Closedness of $\mathcal{U}(\mathcal{H})$ is just Levi's theorem for lattice integrals. In this connection point out that $\mathcal{A}^{lsc}(\mathcal{H}), \mathcal{A}^{usc}(\mathcal{H})$ are only semilattices and $\mathcal{A}^c(\mathcal{H})$ need not to be lattice even for $\mathcal{H}$ simplicial. Without assumption of simpliciality we have only some possibilities make use of upward or downward filterability. It will be ideal to prove that for any $x \in K$ and f universally measurable function there exists $\mathcal{H}$-representing measure μ_x such that $f^\circ(x) = \mu_x(f)$.

The next open problem is whether as matter of the fact $\mathcal{U}(\mathcal{H})$ is a lattice or not. This question is answered only in the convex case by Teleman in [16] with the help of Choquet topologies. We think it is possible to show that $\mathcal{U}(\mathcal{H})$ is a lattice without facial topologies. Why? We have proved that $\mathcal{A}(\mathcal{H})$ is lattice, respective $\mathcal{A}'(\mathcal{H})$ (cf. Proposition 2.6 (c) and Theorem 2.7 (iv)) according Alfsen's ideas from the convex case. Alfsen has subsequently proved an extension type theorem for the family $\mathcal{A}'(X)$. A Teleman's procedure was reversed to Alfsen's one. Teleman inferred at first an extension type theorem for the fxamily $\mathcal{U}(X)$ and then he showed that $\mathcal{U}(X)$ is a lattice. The difficulty is that in a nonmetrizable case the set $\operatorname{ext} X$ of extreme points of X need not to be Borel measurable and maximal measures need not be carried by $\operatorname{ext} X$. The primary topology on K is too insensitive to the set $\operatorname{ext} X$. So this is the reason why we should consider another topology, in Teleman's extension theorem from $\operatorname{ext} X$ is natural Choquet topology. On the other hand, it suffices to show that for $\mathcal{H}$ simplicial the following equality $f^\circ = \mathrm{H}_f$ holds for any universally measurable function f. Naturally, we can define a supremum operation in $\mathcal{U}(\mathcal{H})$ by formula $f \curlyvee g = \mathrm{H}(f \vee g)$ similiary as for $\mathcal{A}(\mathcal{H})$, moreover $\mathcal{A}(\mathcal{H})$ would by sublattice of $\mathcal{U}(\mathcal{H})$, cf. Theorem 2.7.

$\mathcal{A}(\mathcal{H})$ **and** $\mathcal{AB}(\mathcal{H})$**.** The next open problem is if the family $\mathcal{A}(\mathcal{H})$ coincides with the family $\mathcal{AB}(\mathcal{H})$. It is obvious that $\mathcal{A}(\mathcal{H}) \subset \mathcal{AB}(\mathcal{H})$. In a metrizable simplicial case should also the reverse inclusion hold with the aid of simplicial verision of the Alfsen's

THEOREM 2.2. It should be any problem with a generalization, since all assertions used in Alfsen's proof hold in the function space setting. Therefore, for any $f \in \mathcal{AB}(\mathcal{H})$ is $f_{\restriction \mathrm{Ch}_{\mathcal{H}} K} \in \mathcal{B}^b(\mathrm{Ch}_{\mathcal{H}} K)$ and thanks to Alfsen's theorem mentioned above we obtain $\mathrm{H}(f_{\restriction \mathrm{Ch}_{\mathcal{H}} K}) \in \mathcal{A}'(\mathcal{H})$. As open problem remains also extension type theorems for $\mathcal{A}(\mathcal{H})$.

A more delicate problem if $\mathcal{A}_\alpha(\mathcal{H}) = \mathcal{AB}_\alpha(\mathcal{H})$ requires more finer approach. In what follows we aim at the simpler metrizable simplicial case. One can interpert a mapping $\mathrm{H} : \mathcal{C}(K) \to \mathcal{A}(\mathcal{H})$ as a lattice integral. It is known that $\mathrm{H}(\mathcal{B}_1(K)) = \mathcal{A}_1(\mathcal{H})$. According the Khuran THEOREM, see [8] we should obtain (o)-continuity of H. It suffices to think better of differences between monotonne σ-continuity and (o)-continuity. So we conclude that $\mathrm{H}(\mathcal{B}_\alpha(K)) = \mathcal{AB}_\alpha(\mathcal{H})$ and furthemore $\mathrm{H}(\mathcal{B}_\alpha^m(K)) = \mathcal{A}_\alpha^m(\mathcal{H})$.

A problem if $\mathrm{H}(\mathcal{B}_\alpha(\mathrm{Ch}_{\mathcal{H}} K)) = \mathcal{A}_\alpha(\mathcal{H})$, respectively $\mathrm{H}(\mathcal{B}_\alpha^m(\mathrm{Ch}_{\mathcal{H}} K)) = \mathcal{A}_\alpha^m(\mathcal{H})$ is widely complicated, since a Choquet boundary is in general a G_δ set and we can not apply Khuran's theorem. Nevertheless an extension theorem for lattice integrals should be sufficient, but we have to check if $\mathcal{A}(\mathcal{H})$ is a weakly σ-distributive lattice. It should be not so surprising, since $\mathcal{B}^b(\mathrm{Ch}_{\mathcal{H}} K)$ is a weakly σ-distributive lattice in natural pointwise ordering. An idea how to verify that $\mathcal{A}(\mathcal{H})$ is also weakly σ-distributive lattice is try to transfer this condition from $\mathcal{B}^b(\mathrm{Ch}_{\mathcal{H}} K)$ to $\mathcal{A}(\mathcal{H})$ with the lattice integral H. Recall important dependence $\mathrm{H}(f) \curlyvee \mathrm{H}(g) = \mathrm{H}(f \vee g)$, $f, g \in \mathcal{A}(\mathcal{H})$.

4.3 Spectral theory

In this section we sketch another aspect to the interplay between Choquet theory of function spaces and general spectral theory as introduced in the Chapter 3. In what follows, we assume that $\mathcal{H}$ is a simplicial function space. Krause strategy of inference of spectral theory for $\mathcal{A}(\mathcal{H})$ is very similiar to the spectral theory for lattices mixed to the context of function spaces. It would seem that function calculus for lattice $\mathcal{A}(\mathcal{H})$ for $\mathcal{H}$ simplicial is just the special case of a spectral theory for a σ-complete Riesz space with unit, but Krause do not use more general theorems. Even, in Chapter 7 of [9] he stated a function calculus for lattices and C^*- algebras as an application by him developed function calculus for $\mathcal{A}(\mathcal{H})$.

Turn for a while to a connection between the special algebras and Riesz spaces. On Riesz spaces we have function calculus, so we can define the product by formula $a.b = \frac{1}{4}[(a+b)^2 - (a-b)^2]$. On the other hand, on special algebras we have also function calculus, so we can define the supremum operation by $a \curlyvee b = \frac{1}{2}[a + b + \sqrt{(a-b)^2}]$. It appears that both function calculuses are in some sense isomorphic. Since $\mathcal{A}(\mathcal{H})$ is a Riesz space, $\mathcal{A}(\mathcal{H})$ should be Banach algebra. The lattice structure on $\mathcal{A}(\mathcal{H})$ is more natural then algebra one. Conspicious question arises, if for above defined product on $\mathcal{A}(\mathcal{H})$ the equality $f.g = \mathrm{H}(f.g)$ holds for any f, g in $\mathcal{A}(\mathcal{H})$.

Draw to a close, let us mention a connection with representation theorems. Each Banach algebra can be represented using Gelfand's transformation as a Banach algebra of all continous functions on a some compact space. Further, each M-space can be represented according to Kakutani's theorem as an M-space of all continous functions on another compact. It appears as though that the function calculus for $\mathcal{A}(\mathcal{H})$ is a connecting link between all mentioned calculuses.

As the last note we refer to Krause's observation in [9, p. 285] where he outlines a new possible proof of the Choquet representation theorem, cf. 1.23. He asserts that the Choquet representation theorem is just a reformulation of the spectral theorem for $\mathcal{A}(\mathcal{H})$, cf.3.10, (a). A roughly idea is at first apply a general spectral theory for lattices to the lattice $\mathcal{A}(\mathcal{H})$ and obtain spectral theorem for $\mathcal{A}(\mathcal{H})$. As was mentioned, spectral theorem for $\mathcal{A}(\mathcal{H})$ is just a reformulation of the Choquet representation theorem.

Notation	Denotes	See page
$x \vee y, x_1 \vee \ldots \vee x_n$	supremum of elements x, y; $x_1, \ldots, x_n$	6, 10[1]
$x \wedge y, x_1 \wedge \ldots \wedge x_n$	infimum of elements x, y; $x_1, \ldots, x_n$	6
$\vee_{n=1}^{\infty} x_n$	supremum of sequence x_n, if it exists x	6
(o) - $\lim x_n$	order limit of sequence $\{x_n\}, n = 1, 2, ..$	6
(o) - $\sum_1^{\infty} x_n$	order sum of sequence $\{x_n\}, n = 1, 2, ..$	6
(o) - $\sum_{-\infty}^{\infty} x_n$	order sum of $\{x_n\}, n \in \mathcal{Z}$	6
$\lvert x \rvert$	absolute value of element x	7
x^+, x^-	positive (negative) part of element x	7
$x \perp y, A^{\perp}$	orthogonal elements x,y; orthogonal complement of set A	7
$[P]$	projector defined by P	7
(o) - $\int_{-\infty}^{\infty} \varphi(\lambda) dg(\lambda)$	order integral	7
$\mathcal{C}(K)$	continuous functions on K	8
$\mathcal{H}$	function space on Hausdorff compact space K	8
$\mathbf{H}(U)$	functions, harmonic on U and continuous on $\overline{U}$	8
$\mathfrak{U}^c(X)$	continuous affine functions on X	8
$\mathcal{M}^1(K)$	probability Radon measures on K	9
$\mathcal{M}^+(K)$	nonnegative Radon measures on K	14
$\mathcal{M}_x(\mathcal{H})$	$\mathcal{H}$-representating measures with barycenter x	9
f_*, f^*	lower envelope of f, upper envelope of f	9
$\mathcal{A}^b(\mathcal{H})$	Borel bounded $\mathcal{H}$-affine functions	9
$\mathcal{A}^c(\mathcal{H})$	continuous $\mathcal{H}$-affine functions	10
$\mathcal{A}^{lsc}(\mathcal{H}), \mathcal{A}^{usc}(\mathcal{H})$	l.s.c (u.s.c) $\mathcal{H}$-affine functions	10
$\mathcal{K}^c(\mathcal{H}), \mathcal{S}^c(\mathcal{H})$	$\mathcal{H}$-convex ($\mathcal{H}$-concave) functions	10
$\mathcal{K}^{lsc}(\mathcal{H}), \mathcal{K}^{usc}(\mathcal{H})$	l.s.c (u.s.c) $\mathcal{H}$-convex functions	10
$\mathcal{S}^{lsc}(\mathcal{H}), \mathcal{S}^{usc}(\mathcal{H})$	l.s.c (u.s.c) $\mathcal{H}$-concave functions	10
$\mathcal{W}(\mathcal{H})$	'wedge' functions from $\mathcal{H}$	12
$\mathrm{Ch}_{\mathcal{H}}(K)$	Choquet boundary of function space $\mathcal{H}$	13
ε_x	Dirac measure in point x	13
δ_x	maximal measure with barycenter x	14
$\mu \preccurlyeq \nu$	Choquet's ordering of measures μ,ν	14
H	Dirichlet operator	16
$\mathrm{ext}\, X$	set of extreme points of X	19
$\mathcal{A}(X), \mathcal{A}'(X)$	Krause's and Alfsen's special families	19
$\mathcal{U}(X)$	Teleman's family of special affine functions	19
$\mathcal{A}(\mathcal{H}), \mathcal{A}'(\mathcal{H})$	special families of $\mathcal{H}$-affine functions	20
$\overline{\mathcal{A}}(\mathcal{H})$	supremum norm closure of $\mathcal{A}(\mathcal{H})$	20
$f_{\circ}, f^{\circ}$	strong lower envelope of f, strong upper envelope of f	20
$\mathcal{U}(\mathcal{H})$	$\mathcal{H}$-strongly universally measurable functions	21
$\curlyvee, \curlywedge$	supremum (infimum) operation in $\mathcal{A}(\mathcal{H})$	23
$\curlyvee_{n=1}^{\infty} f_n$, $\curlyvee_{n \geq n_0} f_n$	supremum of sequence f_n from $\mathcal{A}(\mathcal{H})$	23
$f^{\oplus}, f^{\ominus}$	positive (negative) part of function f in $\mathcal{A}(\mathcal{H})$	23
χ_X	characteristic function of set X	26

[1]From the Section Choquet theory of function spaces to the end of this paper $\vee$ denotes the supremum operation in the set of real functions on K in its natural pointwise ordering.

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