Report on the doctoral thesis

"Combinatorial problems in geometry"

submitted by

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Dear Professor Kratochvíl,

This dissertation contains several very nice and interesting results related to drawings and other geometric representations of graphs. It is a very fast growing field of combinatorial and computational geometry, partly because of its theoretical importance and interest, and partly because of its practical applications. Applications include social sciences, information systems, computational biology, VLSI circuit design, and many others. There is a popular and high level annual conference devoted to this topic, called Graph Drawing.

Let me point out some of the results.

In the first chapter the author gives estimates on the number of different drawings of a graph. To do this, we have to define what "different drawing" means. A drawing is simple if any two edges (curves) have at most one common point. Two drawings are isomorphic, if there is a homeomorphism of the plane that transforms one drawing to the other, and weakly isomorphic, if the list of crossing pairs of edges is the same in the two drawings. Jan gives several estimates on the number of non-isomorphic, and the number of weakly non-isomorphic drawings of graphs. These estimates are much better and much more general than previous bounds of Pach and Tóth, and of Jan, respectively. The proofs combine various techniques and results with beautiful ideas. As an example, it is shown that the number of weakly non-isomorphic simple drawings of a complete graph K_n is at most $2^{n^2\alpha(n)^c}$. This improves the bound $2^{cn^2 \log n}$ of Pach and Tóth. The main ingredient of the proof is to find certain forbidden subconfigurations, and estimate the number of permutations with some forbidden subpermutations (called permutations with bounded VC-dimension).

The chapter is closed with a very interesting discussion and collection of open problems, conjectures, and ideas.

Twenty years ago Larman, Matoušek, Pach, and Törőcsik proved that any set of k^5 segments in the plane contains k pairwise crossing or pairwise disjoint segments. They also proved that the statement does not always hold for $k^{\log 5/\log 2} \approx k^{2.32}$ segments. Károlyi, Pach, and Tóth slightly improved it, they constructed a set of roughly $k^{2.37}$ segments with no k pairwise crossing or pairwise disjoint.

Jan managed to further improve it, to roughly $k^{2.46}$, by a very sophisticated construction. It is made in two steps, just like the previuous constructions. First a particular finite set of segments is constructed, then one has to show how to "iterate" it. Here both steps are much more complicated than previously, and it is checked by a computer program. There are some other, related results in this chapter.

Reachability problem is the following: given a directed graph, decide if there is a directed path from vertex s to t. Planar reachability is the same problem for planar graphs. Recently it was shown that reachability problem for graphs drawn (without crossings) on the torus is logspace-reducible to the planar version. Jan generalized this result to any compact surface.

The crossing number of a graph is the minimum number of edge crossings over all of its drawings in the plane. Jan, together with Jakub Černý and myself proved that the crossing number of a graph decays in a "continuous fashion" in the following sense. For any $\varepsilon > 0$ there is a $\delta > 0$ such that for n sufficiently large, every graph G with n vertices and $m \geq n^{1+\varepsilon}$ edges, has a subgraph G' of at most $(1-\delta)m$ edges and crossing number at least $(1-\varepsilon)\text{CR}(G)$. This generalizes the result of J. Fox and Cs. Tóth.

Summarizing, it is a very nice thesis with many important contributions. It is very well and carefully written. Usually I have to put a long list of typos and errors in such a report, this time there is nothing I have to mention. I really enjoyed reading it. Some results require deep insight, knowledge, and hard work. These results, methods, ideas will surely inspire further research. Jan definitely deserves the doctoral degree.

Sincerely,

Géza Tóth, Budapest, 02.14.2013.