Charles University in Prague

Faculty of Mathematics and Physics

DOCTORAL THESIS

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Combinatorial problems in geometry

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Study program:	Computer Science
Specialization:	Discrete Models and Algorithms

Prague 2012

Acknowledgements

I would like to thank my supervisor, Pavel Valtr, for his continuous inspiration with nice geometric problems since my early undergraduate years, for his suggestions of interesting international stays and conferences, and for his patience and help with finalizing this thesis. I am grateful to Géza Tóth for his hospitality during my stay in Budapest and for sharing several very interesting problems, which inspired a significant part of my research. I would also like to thank Josef Cibulka for many long and productive discussions about enumerating various combinatorial objects. Our joint result is an essential ingredient for one of the main theorems in the first chapter.

My research was supported by the project 1M0545 of the Ministry of Education of the Czech Republic, by the project CE-ITI (GACR P2020/12/G061) of the Czech Science Foundation, by the by the GraDR EUROGIGA GACR project No. GIG/11/E023, and by the grants SVV-2010-261313 (Discrete Methods and Algorithms) and SVV-2012-265313 (Discrete Models and Algorithms).

The results presented in the second and the fourth chapter were obtained during my stay at Alfréd Rényi Institute of Mathematics in Budapest from November 2006 to February 2007, which was supported by the Phenomena in High Dimensions project, in the framework of the European Community's "Structuring the European Research Area" program.

The results presented in the third chapter were obtained during the DIMACS/DIMATIA REU program at Rutgers University in June and July 2007, supported by the project ME 886 of the Ministry of Education of the Czech Republic.

Part of the research was also conducted during the Special Semester on Discrete and Computational Geometry at École Polytechnique Féderale de Lausanne from September to December 2010, organized and supported by the CIB (Centre Interfacultaire Bernoulli) and the SNSF (Swiss National Science Foundation).

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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In Prague, December 10, 2012 Jan Kynčl

Název práce: Kombinatorické otázky v geometrii

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Vedoucí disertační práce: Doc. RNDr. Pavel Valtr, Dr., Katedra aplikované matematiky

Abstrakt: Dokážeme, že pro každý graf G s n vrcholy a m hranami, který nemá izolované vrcholy, počet tříd slabého izomorfismu jednoduchých topologických grafů, které realizují G, je nejvýše $2^{O(n^2 \log(m/n))}$, nejvýše $2^{O(mn^{1/2} \log n)}$ pro $m < n^{3/2}$ a nejvýše $2^{n^2 \cdot \alpha(n)^{O(1)}}$, je-li G úplný graf. Jako důsledek obdržíme nový horní odhad $2^{O(n^{3/2} \log n)}$ na počet průsečíkových grafů pseudoúseček. Ukážeme, že počet tříd izomorfismu jednoduchých topologických grafů, které realizují G, je nejvýše $2^{m^2+O(mn)}$. Zlepšíme výsledek Károlyiho, Pacha a Tótha konstrukcí arrangementu n úseček v rovině s nejvýše $n^{\log 8/\log 169}$ vzájemně se křížících nebo vzájemně disjunktních úseček. Také ukážeme, že dosažitelnost v orientovaných grafech nakreslených na ploše libovolného rodu je redukovatelná v logspace na dosažitelnost v orientovaných grafech nakreslených v rovině. Nakonec zobecníme výsledek J. Foxe a Cs. Tótha tím, že ukážeme, že průsečíkové číslo grafu klesá spojitě v závislosti na relativním počtu vhodně odebraných hran.

Klíčová slova: jednoduchý topologický graf, věta ramseyovského typu, orientovaná grafová dosažitelnost, pokles průsečíkového čísla

Title: Combinatorial problems in geometry

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Abstract: We prove that for every graph G with n vertices, m edges and no isolated vertices the number of weak isomorphism classes of simple topological graphs that realize G is at most $2^{O(n^2 \log(m/n))}$, at most $2^{O(mn^{1/2} \log n)}$ if $m < n^{3/2}$, and at most $2^{n^2 \cdot \alpha(n)^{O(1)}}$ if G is a complete graph. As a consequence we obtain a new upper bound $2^{O(n^{3/2} \log n)}$ on the number of intersection graphs of n pseudosegments. We show that the number of isomorphism classes of simple topological graphs that realize G is at most $2^{m^2+O(mn)}$. Improving a result of Károlyi, Pach and Tóth, we construct an arrangement of n segments in the plane with at most $n^{\log 8/\log 169}$ pairwise crossing or pairwise disjoint segments. We also show that reachability in directed graphs embedded on a fixed surface of arbitrary genus is logspace-reducible to reachability in directed graphs embedded in the plane. Finally, we generalize a result of J. Fox and Cs. Tóth by proving that the crossing number of a graph decays continuously with respect to the fraction of suitably removed edges.

Keywords: simple topological graph, Ramsey-type theorem, directed graph reachability, decay of crossing number

Contents

Introduction

General overview and motivation

Computers and the Internet, social networks, biological species and their ancestor relations, molecules and chemical reactions, neurons and synapses in the brain, transistors and their connections in the integrated circuits—these are examples of phenomena of extreme interest in the beginning of the 21st century. They all possess an inherent structure, whose common mathematical abstraction is a *graph*.

A graph consists of a set of objects called *vertices* and a set of pairs of vertices, called *edges*. Edges represent pairwise relations between the objects. In the real world, entities with a graph structure are often called networks.

Some networks have additional structure: consider, for example, airports and flight connections, cities and roads, or integrated circuits. In these examples, we have an information about the positions of the vertices and the edges in the space or on a certain surface (the surface of the Earth or a silicon chip).

Other networks have no inherent information about the positions of the vertices, but we may want to visualize them as a diagram on the paper or on the computer screen, so that the structure is comprehensible for a human. This is the aim of the area of research called *graph drawing*.

A *drawing* of a graph is a representation of vertices as points in the plane and edges as simple curves or lines connecting the corresponding pairs of vertices. In general, some crossings between edges are unavoidable. A folklore puzzle asks to connect each of three houses by a path to each of three wells, so that the paths do not cross each other. After several unsuccessful attempts, one usually comes to the conclusion that this problem has no solution. A rigorous proof of this fact relies on the famous Jordan curve theorem. A generalization of this puzzle is known as the Turán's Brick Factory problem [BW10], whose objective is determining the minimum number of crossings needed to connect each of m kilns with each of n storage yards. The problem has been solved for small values of m or n, but general exact solution is not known.

In the example of flight connections between airports, the drawing is already given by the positions of the flight corridors and may involve many crossings. In the case of general networks, a usual goal is to find a drawing with as few crossings as possible. In some applications, however, the drawing is required to have no crossings at all—this is the case of the current technology of integrated circuits.

Drawings of graphs allowing crossings are called *topological graphs*. Topological graphs whose edges are drawn as straight-line segments are called *geometric graphs*. Graphs allowing drawing without crossings are called *planar graphs*. Drawings without crossings are then called *embeddings* or *plane graphs*.

Plane graphs can be generalized to graphs embedded on surfaces of higher genus, like the torus or the projective plane. Surfaces have been traditionally studied in differential geometry and general or algebraic topology. Graph embeddings help in studying the properties of the surfaces by reducing some continuous questions to a discrete setting. On the other hand, embeddings of graphs on surfaces can be used to solve strictly combinatorial problems. For example, they are important tool in Robertson-Seymour's project on graph minors. Perhaps the most famous result of this project is that every proper minor-closed class of graphs can be characterized by a finite set of forbidden minors. An introduction to the theory of graph embeddings on surfaces is provided by the monographs by Gross and Tucker [GT87] and Mohar and Thomassen [MT01]. Many interesting relations of graph embeddings with various objects in other fields of mathematics and physics are explored by Lando and Zvonkin [LZ04].

Topological graphs and theoretical aspects of graph drawing form one of the research areas in discrete geometry. Discrete geometry, also known as combinatorial geometry, studies combinatorial properties of finite sets of points, lines, circles, or convex sets in the plane, or more generally, hyperplanes, spheres, or polytopes in the d-dimensional Euclidean space. Discrete geometry includes classical topics such as tilings, packings and coverings. A famous problem is to determine the most efficient packing of unit balls in the three-dimensional space. Everyone who ever tried to arrange oranges or similar round-shaped objects into a box intuitively knows what the best packing is: simply put the objects in horizontal layers, so that each layer forms a triangular lattice and every two adjacent layers are shifted by the same vector so that the upper layer fits in the holes created by the layer just below. The assertion that this particular packing has the largest possible density is known as the Kepler conjecture. A rigorous proof of this conjecture has been given only recently [Ha05, HF11], with a significant portion of the proof relying on computer assistance and a special software developed for this purpose. As this example illustrates, many problems in discrete geometry are motivated by real-world problems. Discrete geometry has direct applications in computational geometry, robotics or computer graphics, but also often surprising and elegant applications in other branches of mathematics, such as number theory.

Like in many other mathematical disciplines, most of the problems studied in discrete geometry are interesting for their own sake. However, unlike in other disciplines, the problems in discrete geometry are often very simple to state and can be understood by non-experts, even high-school students. Still, many of these problems are open or notoriously difficult to solve, of which the Kepler conjecture is a perfect example. Determining the chromatic number of the plane is another famous problem, which looks rather innocent. The *chromatic number* of the plane is the minimum number of colors needed to color all the points of the plane so that no two points at unit distance receive the same color. It is easy to show that at least 4 and at most 7 colors are needed. But for more than 50 years, neither of these bounds have been improved. Discrete geometry was popularized by Paul Erdős, who enriched the field by posing numerous interesting questions of this type. A representative set of hundreds of open problems has been collected by Brass, Moser and Pach [BMP05]. A handbook edited by Goodman and O'Rourke [GO04] contains a comprehensive overview of the results and open problems in the area in an encyclopedic form. The classical and modern topics of discrete geometry are covered in the excellent textbooks by Pach and Agarwal [PA95], Matoušek [Ma02], Pak [Pa10], or Devadoss and O'Rourke [DO11]. Topics related to convex polytopes are covered in monographs by Grünbaum [Gr03] and Ziegler [Zi95].

The problems and the results

In this thesis, we study several problems from discrete geometry related to topological graphs and intersection representations of graphs. The thesis is based on the following four papers.

[1] J. Kynčl, Improved enumeration of simple topological graphs, manuscript.

[2] J. Kynčl, Ramsey-type constructions for arrangements of segments, *European Journal of Combinatorics* **33**(3) (2012), 336–339.

[3] J. Kynčl and T. Vyskočil, Logspace reduction of directed reachability for bounded genus graphs to the planar case, *ACM Transactions on Computation Theory* **1**(3) (2010), 1–11.

[4] J. Černý, J. Kynčl and G. Tóth, Improvement on the decay of crossing numbers, to appear in *Graphs and Combinatorics*.

Chapters 1, 3 and 4 are slightly modified versions of papers [1], [3] and [4], respectively. Chapter 2 is an extended version of paper [2].

Now we briefly overview the particular topics studied in each chapter and the main results obtained.

Chapter 1: Enumeration of simple topological graphs

How many ways can one draw a graph? This sounds like a natural and easy to state question. It is also the title of a joint paper by Géza Tóth and János Pach [PT06], where this question was seriously studied for the first time (for drawings with crossings, that is). In Chapter 1 we restrict our attention to *simple topological graphs*, that is, drawings of graphs in the plane where every two edges have at most one common point (an endpoint or a crossing) and no three edges pass through a single crossing. Before we attempt to count different drawings, we need to specify which drawings are *different* and which are the *same*.

We say that topological graphs G and H are *isomorphic* if H can be obtained from G by a homeomorphism of the sphere, and *weakly isomorphic* if G and H have the same set of pairs of crossing edges. Clearly, the isomorphism implies the weak isomorphism.

We generalize results of Pach and Tóth [PT06] and the author's previous results [Ky09] on counting different drawings of a graph under both notions of isomorphism. We prove that for every graph G with n vertices, m edges and no isolated vertices, the number of weak isomorphism classes of simple topological graphs that realize G is at most $2^{O(n^2\log(m/n))}$, and at most $2^{O(mn^{1/2}\log n)}$ if $m \le n^{3/2}$. As a consequence we obtain a new upper bound $2^{O(n^{3/2}\log n)}$ on the number of intersection graphs of n pseudosegments. We improve the upper bound on the number of weak isomorphism classes of simple complete topological graphs with n vertices to $2^{n^2\cdot\alpha(n)^{O(1)}}$, using an upper bound on the size of a set of permutations with bounded VC-dimension recently proved by Josef Cibulka and the author [CK12]. We show that the number of isomorphism classes of simple topological graphs that realize G is at most $2^{m^2+O(mn)}$ and at least $2^{\Omega(m^2)}$ for graphs with $m > (6+\varepsilon)n$.

Chapter 2: Ramsey properties of intersection graphs of segments

"Complete disorder is impossible". This motto characterizes the area of combinatorics called Ramsey theory. The basic Ramsey's theorem says that for any given n, in a sufficiently large graph one can find either n vertices that are pairwise joined by an edge (a *clique* of size n) or n vertices such that no two of them are joined by an edge (an *independent set* of size n). In general, the number of vertices in the large graph must be exponential in n. In some classes of graphs, however, a polynomial number of vertices is sufficient. Such classes are said to satisfy the *Erdős-Hajnal property*. These include, for example, the class of intersection graphs of convex sets in the plane. That is, graphs whose vertices can be represented by convex sets in the plane so that two vertices are joined by an edge if and only if the corresponding sets intersect. Larman et al. [LMPT94] showed that among n convex sets in the plane, there are either $n^{1/5}$ pairwise intersecting or $n^{1/5}$ pairwise disjoint sets.

In Chapter 2, we study arrangements of segments in the plane and their intersection graphs. Károlyi, Pach and Tóth [KPT97] constructed an arrangement of n segments with at most $n^{\log 4/\log 27}$ pairwise crossing or pairwise disjoint segments. We improve their result by constructing an arrangement of n segments with at most $n^{\log 8/\log 169}$ pairwise crossing or pairwise disjoint segments. We use the recursive method based on flattenable arrangements which was established by Larman et al. [LMPT94]. We also show that not every arrangement of segments can be flattened, by constructing an intersection graph of segments that cannot be realized by an arrangement of segments crossing a common line. Moreover, we also construct an intersection graph of segments crossing a common line that cannot be realized by a flattenable arrangement.

Chapter 3: Reachability in graphs on surfaces

Directed reachability (or briefly *reachability*) is the following decision problem: given a directed graph G and two of its vertices s, t, determine whether there is a directed path from s to t in G. At first sight, the problem may seem easy or even trivial—using a standard method of breadth-first search one can find the solution in linear time. However, the situation gets much more complex when one has to use a space-efficient algorithm. In this case, the complexity classes L and NL are of particular interest. The classes L (logspace) and NL (non-deterministic logspace) consist of deterministic and non-deterministic algorithms, respectively, that use only a logarithmic amount of additional memory. For instance, a logspace algorithm may store only a constant number of vertex labels when searching the input graph.

Directed reachability is a standard complete problem for the complexity class NL. *Planar reachability* is an important restricted version of the reachability problem, where the input graph is planar. Planar reachability is hard for L and is contained in NL but is not known to be NL-complete or contained in L. Allender et al. [ABC+09] showed that reachability for graphs embedded on the torus is logspace-reducible to the planar case. In Chapter 3, we generalize this result to graphs embedded on a fixed surface of arbitrary genus, both orientable and non-orientable.

Chapter 4: Continuous decay of the crossing number

The *crossing number* $\text{CR}(G)$ of a graph G is the minimum possible number of crossings in a drawing of G in the plane where no three edges cross at the same point. In Chapter 4, we investigate the following question: how does the crossing number of G decrease if we delete some edges? In particular, we are interested in finding those edges that are the least "responsible" for the crossing number. Richter and Thomassen [RT93] proved that every graph G has an edge with $\text{CR}(G-e) \geq \frac{2}{5}\text{CR}(G) - O(1)$. Pach and Tóth [PT00] proved that for every graph G and *any* edge e, we have $\text{CR}(G-e) \geq \text{CR}(G) - m + 1$. Fox and Cs. Tóth [FT08] proved that for every graph that is not too sparse, one can delete a constant fraction of the edges so that the remaining graph G' has crossing number at least $\left(\frac{1}{28} - o(1)\right)\text{CR}(G)$.

We generalize the result of Fox and Cs. Tóth [FT08] and prove that the crossing number of a graph decays in a "continuous fashion" in the following sense. For any $\varepsilon > 0$ there is a $\delta > 0$ such that for a sufficiently large n, every graph G with n vertices and $m \geq n^{1+\varepsilon}$ edges has a subgraph G' of at most $(1-\delta)m$ edges and crossing number at least $(1-\varepsilon)\text{CR}(G)$.

1. Enumeration of simple topological graphs

1.1 Introduction

A *topological graph* $T = (V(T), E(T))$ is a drawing of a graph G in the plane with the following properties. The vertices of G are represented by a set $V(T)$ of distinct points in the plane and the edges of G are represented by a set $E(T)$ of simple curves connecting the corresponding pairs of points. We call the elements of $V(T)$ and $E(T)$ the *vertices* and the *edges* of T. The drawing has to satisfy the following general position conditions: (1) the edges pass through no vertices except their endpoints, (2) every two edges have only a finite number of intersection points, (3) every intersection point of two edges is either a common endpoint or a proper crossing ("touching" of the edges is not allowed), and (4) no three edges pass through the same crossing. A topological graph is *simple* if every two edges have at most one common point, which is either a common endpoint or a crossing. A topological graph is *complete* if it is a drawing of a complete graph.

We use two different notions of isomorphism to enumerate topological graphs.

Topological graphs G and H are *weakly isomorphic* if there exists an incidence preserving one-to-one correspondence between $V(G), E(G)$ and $V(H), E(H)$ such that two edges of G cross if and only if the corresponding two edges of H do.

Note that every topological graph G drawn in the plane induces a drawing G_{S^2} on the sphere, which is obtained by a standard one-point compactification of the plane. Topological graphs G and H are *isomorphic* if there exists a homeomorphism of the sphere which transforms G_{S^2} into H_{S^2}. In Section 1.5 we give an equivalent combinatorial definition.

Unlike the isomorphism, the weak isomorphism can change the faces of the involved topological graphs, the order of crossings along the edges and also the cyclic orders of edges around vertices.

For counting the (weak) isomorphism classes, we consider all the graphs labeled. That is, each vertex is assigned a unique label from the set $\{1, 2, \ldots, n\}$, and we require the (weak) isomorphism to preserve the labels. Mostly it makes no significant difference in the results as we operate with quantities asymptotically larger than $n!$.

For a graph G, let $T_{\mathrm{w}}(G)$ be the number of weak isomorphism classes of simple topological graphs that realize G. Pach and Tóth [PT06] and the author [Ky06] proved the following lower and upper bounds on $T_{\mathrm{w}}(K_n)$.

Theorem 1.1. [Ky06, PT06] *For the number of weak isomorphism classes of simple drawings of K_n, we have*

$$2^{\Omega(n^2)} \leq T_{\mathrm{w}}(K_n) \leq ((n-2)!)^n = 2^{O(n^2 \log n)}.$$

We prove generalized upper and lower bounds on $T_{\mathrm{w}}(G)$ for all graphs G.

Theorem 1.2. *Let G be a graph with n vertices and m edges. Then*

$$T_{\mathrm{w}}(G) \leq 2^{O(n^2 \log(m/n))}.$$

If $m < n^{3/2}$, then

$$T_{\mathrm{w}}(G) \leq 2^{O(mn^{1/2}\log n)}.$$

Let $\varepsilon > 0$. If G is a graph with no isolated vertices and at least one of the conditions $m > (1+\varepsilon)n$ or $\Delta(G) < (1-\varepsilon)n$ is satisfied, then

$$T_{\mathrm{w}}(G) \geq 2^{\Omega(\max(m, n\log n))}.$$

We also improve the upper bound from Theorem 1.1.

Theorem 1.3. *We have*

$$T_{\mathrm{w}}(K_n) \leq 2^{n^2 \cdot \alpha(n)^{O(1)}}.$$

Here $\alpha(n)$ is the inverse of the Ackermann function. It is an extremely slowly growing function, which can be defined in the following way [Ni10]. $\alpha(m) := \min\{k : \alpha_k(m) \leq 3\}$ where $\alpha_d(m)$ is the dth function in the *inverse Ackermann hierarchy*. That is, $\alpha_1(m) = \lceil m/2 \rceil$, $\alpha_d(1) = 0$ for $d \geq 2$ and $\alpha_d(m) = 1 + \alpha_d(\alpha_{d-1}(m))$ for $m, d \geq 2$. The constant in the $O(1)$ notation in the exponent is huge (roughly 4^{30^4}), due to a Ramsey-type argument used in the proof.

Theorem 1.3 is proved in Section 1.3. In the proof of Theorem 1.3 we use the fact that for simple complete topological graphs, the weak isomorphism class is determined by the rotation system. This is combined with a recent combinatorial result, an upper bound on the size of a set of permutations with bounded VC-dimension [CK12]. The method in the proof of Theorem 1.2 is more topological, gives a slightly weaker upper bound, but can be generalized to all graphs.

In Subsection 1.3.5, we generalize Theorem 1.3 by removing almost all topological aspects of the proof. The resulting Theorem 1.15 is a purely combinatorial statement.

In Subsection 1.3.6, we consider the class of simple complete topological graphs with maximum number of crossings and suggest an alternative method for obtaining an upper bound on the number of weak isomorphism classes of such graphs.

An arrangement of *pseudosegments* (or also 1-*strings*) is a set of simple curves in the plane such that any two of the curves cross at most once. An *intersection graph of pseudosegments* (also called a *string graph of rank* 1) is a graph G such that there exists an arrangement of pseudosegments with one pseudosegment for each vertex of G and a pair of pseudosegments crossing if and only if the corresponding pair of vertices forms an edge in G. Using tools from extremal graph theory, Pach and Tóth [PT06] proved that the number of intersection graphs of n pseudosegments is $2^{o(n^2)}$. As a special case of Theorem 1.2 we obtain the following upper bound.

Theorem 1.4. *There are at most $2^{O(n^{3/2}\log n)}$ intersection graphs of n pseudosegments.*

The best known lower bound for the number of (unlabeled) intersection graphs of n pseudosegments is $2^{\Omega(n\log n)}$ and follows, for example, from the the fact that there are $2^{\Theta(n\log n)}$ nonisomorphic permutation graphs with n vertices.

Let $T(G)$ be the number of isomorphism classes of simple topological graphs that realize G. The following theorem generalizes the result $T(K_n) = 2^{\Theta(n^4)}$ from [Ky09].

Theorem 1.5. *Let G be a graph with n vertices, m edges and no isolated vertices. Then*

$$T(G) \leq \binom{6mn}{2mn}\binom{m^2+6mn}{\frac{m^2}{2}+2mn} \cdot 2^{O(n\log n)} \leq 2^{m^2+2mn(1+3\log_2 3)+O(n\log n)}, \text{ and}$$

$$T(G) \leq 2^{m^2+4mn} \cdot \binom{2mn+\frac{m^2}{2}}{2mn} \cdot 2^{O(n\log n)} \leq 2^{m^2+2mn(\log(1+\frac{m}{4n})+2+\log_2 e)+O(n\log n)}.$$

Let $\varepsilon > 0$. For graphs G with $m > (6+\varepsilon)n$ we have

$$T(G) \geq 2^{\Omega(m^2)}.$$

For graphs G with $m > \omega(n)$ we have

$$T(G) \geq 2^{m^2/60} - o(1).$$

The two upper bounds on $T(G)$ come from two essentially different approaches. The first one gives better asymptotic results for dense graphs, whereas the second one is better for sparse graphs (roughly, with at most $35n$ edges). For graphs with $m = O(n)$ the second term in the exponent becomes more significant. Since $m \geq n/2$, the exponent in the first upper bound can be bounded by

$$m^2 \cdot \left(4\log_2 3 + 8\log_2\frac{3}{2} + \frac{17}{2}\cdot\log_2\frac{26}{17} + \frac{9}{2}\cdot\log_2\frac{26}{9}\right) + o(m^2) \leq 23.118m^2 + o(1),$$

using the entropy bound for the binomial coefficient. Similarly, the exponent in the second upper bound can be bounded by

$$m^2 \cdot (1 + 8 + 4\log_2(9/8) + 1/2\cdot\log_2 9) + o(m^2) \leq 11.265m^2 + o(1).$$

For such very sparse graphs (for example, matchings), however, better upper bounds can be deduced more directly from other known results.

The upper bound $T(G) \leq 2^{O(m^2)}$ is trivially obtained from the upper bound on the number of unlabeled plane graphs (or planar maps). Indeed, every drawing $\mathcal{G}$ of G can be transformed into a plane graph H by subdividing the edges of $\mathcal{G}$ by its crossings and regarding the crossings of $\mathcal{G}$ as new 4-valent vertices in H. The graph H has thus at most $n + \binom{m}{2}$ vertices, at most $m + 2\binom{m}{2} = m^2$ edges, no loops and no multiple edges. Tutte [Tu63] showed that there are

$$\frac{2(2M)!3^M}{M!(M+2)!} = 2^{(\log_2(12)+o(1))M}$$

rooted connected planar maps with M edges (see also [BR86, BW85, DM11]). Walsh and Lehman [WL75] showed that the number of rooted connected planar loopless maps with M edges is

$$\frac{6(4M+1)!}{M!(3M+3)!} = 2^{(\log_2(256/27)+o(1))M}.$$

This implies the upper bound $T(G) \leq 2^{(\log_2(256/27)+o(1))m^2}$. Somewhat better estimates could be obtained by reducing the problem to counting 4-regular planar maps [RL01, RLL02], since typically almost all vertices in H are the 4-valent vertices obtained from the crossings of $\mathcal{G}$. But such a reduction would be less straightforward and the resulting upper bound $2^{(\frac{1}{2}\log_2(196/27)+o(1))m^2}$ still relatively high for dense graphs (for graphs with more than $27n$ edges the two upper bounds from Theorem 1.5 are better).

The proof in [Ky09] implies the upper bound $T(K_n) \leq 2^{(1/12+o(1))(n^4)}$, although it is not explicitly stated in the paper. However, the key Proposition 7 in [Ky09] is incorrect. We prove a correct version in Section 1.5.

Note that by the reduction to counting planar maps, for every fixed constant k, we also obtain the upper bound $2^{O(km^2)}$ on the number of isomorphism classes of connected topological graphs with m edges where all pairs of edges are allowed to cross k times.

All the logarithms used in this chapter are binary, unless indicated otherwise.

1.2 Preliminaries

The weak isomorphism classes of topological graphs can be represented in a combinatorial way by abstract topological graphs. An *abstract topological graph* (or briefly an *AT-graph*) is a pair (G, R) where G is a graph and $R \subseteq \binom{E(G)}{2}$ is a set of pairs of its edges. For a topological graph T that is a drawing of G we define the AT-graph of T as (G, R_T) where R_T is the set of pairs of edges having at least one common crossing. A (simple) topological graph T is called a *(simple) realization* of (G, R) if $R_T = R$. Clearly, two topological graphs are weakly isomorphic if and only if they are realizations of the same AT-graph.

The *rotation* of a vertex v in a topological graph T is the clockwise cyclic order of the edges incident with v. The rotation $\rho(v)$ of a vertex v is represented by a cyclic sequence of the vertices adjacent to v. The *rotation system* of T is the set of rotations of all its vertices.

We use the following property of simple complete topological graphs, which directly implies the upper bound on $T_{\mathrm{w}}(K_n)$ in Theorem 1.1.

Proposition 1.6. [Ky09, PT06] *The rotation system of a simple complete topological graph G uniquely determines which pairs of edges of G cross. That is, two simple complete topological graphs with the same rotation system are weakly isomorphic.*

This property can be shown to be satisfied by a broader class of "sufficiently dense" graphs. For example, this property is satisfied by the wheel graph $W_4 = K_5 - 2K_2 = K_{1,2,2}$, and consequently by all graphs G such that every pair of non-adjacent edges belongs to a subgraph of G isomorphic to W_4. This includes, for example, the complete 3-partite graph $K_{1,n,n}$ with $n \geq 2$. But already for complete bipartite graphs, many weakly nonisomorphic drawings can share the same rotation system. For example, there are at least $2^{n/2}$ weakly nonisomorphic simple drawings of $K_{2,n}$ with the same rotation system. To see this, let n be an even positive integer and let $v, w, u_1, u_2, \ldots, u_n$ be the vertices of $K_{2,n}$ with v, w forming the 2-element independent set of the bipartition. Let $(u_1, u_2, \ldots, u_n)$ be the rotation of v and $(u_{n-1}, u_n, \ldots, u_3, u_4, u_1, u_2)$ the rotation of w. For every $i = 1, 2, \ldots, n/2$, there are two ways of drawing the four edges $vu_{2i}, vu_{2i-1}, wu_{2i}, wu_{2i-1}$ (either

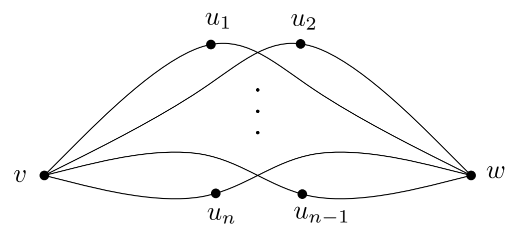

Figure 1.1: Every 4-cycle $vu_{2i-1}wu_{2i}$ in $K_{2,n}$ can be drawn in one of two ways, while keeping the rotation system fixed.

vu_{2i-1} crosses wu_{2i} or wu_{2i-1} crosses vu_{2i}), and these choices can be done independently. See Figure 1.1. Note that by cloning the vertex v into $n-1$ copies we obtain $2^{n/2}$ weakly nonisomorphic drawings of $K_{n,n}$ with the same rotation system.

We note that the converse of Proposition 1.6 is also true: the rotation systems of two weakly isomorphic simple complete topological graphs are either the same or inverse [Gi05, Ky09].

1.3 Simple complete topological graphs

In this section we prove Theorem 1.3.

The upper bound $T_{\mathrm{w}}(K_n) \leq 2^{O(n^2 \log n)}$ in Theorem 1.1 follows directly from Proposition 1.6, since there are at most $(n-2)!$ possible rotations for each vertex, thus at most $((n-2)!)^n = 2^{O(n^2 \log n)}$ possible rotation systems of K_n. However, not every set of rotations is realizable as a rotation system of a simple complete topological graph. For example, the rotation of each vertex in a simple complete topological graph is uniquely determined by the set of rotations of the other $n-1$ vertices. This is easily seen by investigating the drawings of K_4 [Ky06, PT06] (see Observation 1.14) and using the fact that a cyclic permutation of n elements is determined by cyclic subpermutations of all triples.

The smallest forbidden patterns in the rotation system are the 4-tuples of cyclic subpermutations of 3 elements that cannot be realized as rotation systems of a simple drawing of K_4. In fact, in Section 1.3.5 we show that it is possible to prove Theorem 1.3 by combinatorial arguments, using only these simple forbidden patterns.

However, we first show a proof relying more on the topological structure of the drawings, which gives a better upper bound on $T_{\mathrm{w}}(K_n)$, and also provides an intuition for the purely combinatorial proof.

The core idea in both versions of the proof is to reduce the problem of bounding $T_{\mathrm{w}}(K_n)$ to counting single permutations with forbidden subpermutations.

1.3.1 Permutations with bounded VC-dimension

Let S_n be the set of all *n-permutations*, that is, permutations of the set $\{1, 2, \ldots, n\}$. The *restriction of* $\pi \in S_n$ *to the k-tuple* $(a_1, a_2, \ldots, a_k)$ *of positions*, where $1 \leq a_1 < a_2 < \cdots < a_k \leq n$, is the k-permutation π' satisfying $\forall i, j : \pi'(i) < \pi'(j) \Leftrightarrow \pi(a_i) < \pi(a_j)$. Let $\mathcal{P} \subseteq S_n$. The k-tuple of positions $(a_1, \ldots, a_k)$ is *shattered by* $\mathcal{P}$ if each k-permutation appears as a restriction of some $\pi \in \mathcal{P}$ to $(a_1, \ldots, a_k)$. The *VC-dimension of* $\mathcal{P}$ is the size of the largest set of positions shattered by $\mathcal{P}$. In other words, the VC-dimension of $\mathcal{P}$ is at most k if for every $k+1$ positions $a_1, \ldots, a_{k+1}$ there is some forbidden $(k+1)$-permutation that does not appear as a restriction of any $\pi \in \mathcal{P}$ to $(a_1, \ldots, a_{k+1})$. Raz [Ra00] proved that a set of n-permutations of VC-dimension 2 has size at most $2^{O(n)}$. The following result proved by Cibulka and the author [CK12] is the key ingredient in the proof of Theorem 1.3.

Theorem 1.7. [CK12] *For every* $t \geq 2$*, the size of a set of* n*-permutations with VC-dimension* $2t+2$ *is at most*

$$2^{n \cdot ((2/t!)\alpha(n)^t + O(\alpha(n)^{t-1}))}.$$

The upper bound in Theorem 1.7 is asymptotically almost tight, since there are sets of permutations with VC-dimension $2t+2$ of size $2^{n \cdot ((1/t!)\alpha(n)^t - O(\alpha(n)^{t-1})}$; see [CK12].

If the forbidden $(k+1)$-permutation is the same for all $(k+1)$-tuples of positions, we get a better, exponential upper bound on the size of $\mathcal{P}$. This was conjectured by Stanley and Wilf and proved by Marcus and Tardos [MT04], using Klazar's earlier result [Kl00]. Later Cibulka [Ci09] improved Klazar's reduction and obtained the upper bound $2^{O(k \log k)n}$ on the size of $\mathcal{P}$.

1.3.2 Unavoidable topological subgraphs

A *complete convex geometric graph* (shortly a *convex graph*) is a topological graph whose vertices are in convex position and the edges are drawn as straight-line segments; see Figure 1.2, left. We denote by C_m any complete convex geometric graph with m vertices, as all such graphs belong to the same weak isomorphism class.

A simple complete topological graph with m vertices is called *twisted* and denoted by T_m if there exists a *canonical* ordering of its vertices $v_1, v_2, \ldots, v_m$ such that for every $i < j$ and $k < l$ two edges $v_i v_j, v_k v_l$ cross if and only if $i < k < l < j$ or $k < i < j < l$; see Figure 1.2, right. Figure 1.3 shows an equivalent drawing of T_m on the cylindrical surface.

Let G and H be topological graphs. We say that G *contains* H if G has a topological subgraph weakly isomorphic to H.

We use the following asymmetric form of the Ramsey-type result by Pach, Solymosi and Tóth [PST03], which generalizes the Erdős–Szekeres theorem for planar point sets.

Theorem 1.8. [PST03] *For all positive integers* n, m_1, m_2 *satisfying*

$$m_1 m_2 \leq \log_4^{1/4}(n+1),$$

every simple complete topological graph with n *vertices contains* C_{m_1} *or* T_{m_2}.

The graphs C_m and T_m are special cases of simple complete topological graphs with m vertices and $\binom{m}{4}$ crossings, which is the maximum number of crossings possible [HM92]. The existence of a complete subgraph with m vertices and

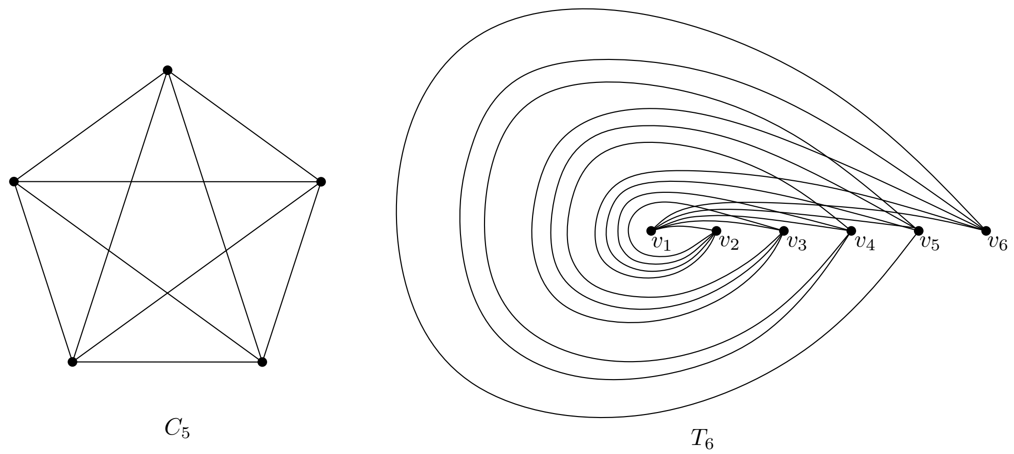

Figure 1.2: The convex graph C_5 and the twisted graph T_6.

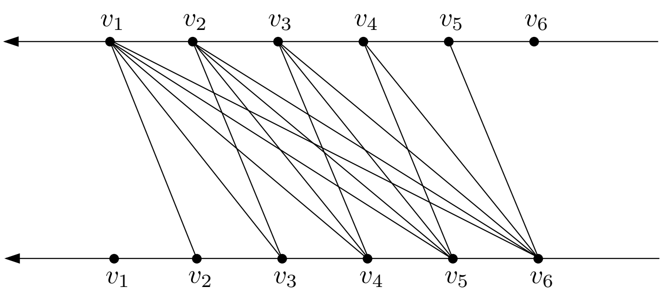

Figure 1.3: A drawing of the twisted graph T_6 on the cylindrical surface.

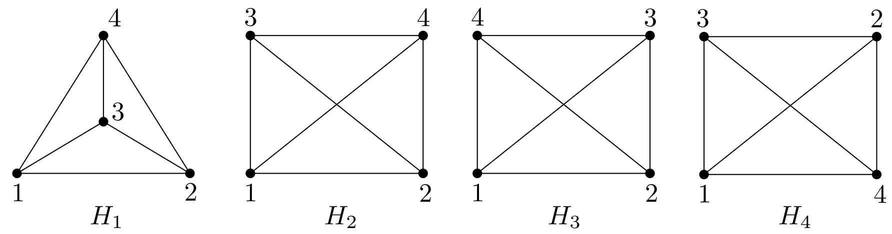

Figure 1.4: Four nonisomorphic simple drawings of K_4.

$\binom{m}{4}$ crossings in a sufficiently large simple complete topological graph G follows directly from Ramsey's theorem and the nonplanarity of K_5 [HMS95], but the bound on the size of G obtained is much larger than that from Theorem 1.8. For the special case $m_1 = m_2 = 5$, Harborth, Mengersen, and Schelp [HMS95] showed a much better upper bound than that following from Theorem 1.8.

Theorem 1.9. [HMS95] *Every simple complete topological graph with* 113 *vertices contains* C_5 *or* T_5.

1.3.3 Forbidden patterns in the rotation system

Let G be a simple complete topological graph and let v be a vertex of G. Our goal is to obtain an upper bound on the number of possible rotations of v in G when the complete subgraph $G - v$ is fixed. To this end, we need to identify some forbidden permutations in the rotation of v.

Lemma 1.10. *Let G be a simple complete topological graph with vertices* $1, 2, 3, 4$. *Suppose that the counter-clockwise order of the vertices of the topological triangle* 123 *is* $1, 2, 3$. *If*

(a) the vertex 4 *is outside the triangle* 123 *and its rotation is* $(1, 2, 3)$, *or*

(b) the vertex 4 *is inside the triangle* 123 *and its rotation is* $(1, 3, 2)$,

then G has no crossings. Otherwise G has one crossing.

Proof. Figure 1.4 shows representatives of all four isomorphism classes of simple complete topological graphs with vertices $1, 2, 3, 4$. The notions of isomorphism and weak isomorphism for these graphs coincide, since in each of the four drawings different pairs of edges cross. Each of the drawings is chosen so that the vertices $1, 2, 3$ appear in counter-clockwise order in the triangle 123 and the vertex 4 is outside the triangle 123. This still leaves some freedom in choosing the outer face of the drawing: we may always choose any of the three faces adjacent to the vertex 4, but the rotation system of the drawing stays the same. Since the rotation of the vertex 4 is $(1, 2, 3)$ in H_1, which is without crossings, and $(1, 3, 2)$ in H_2, H_3 and H_4, which have one crossing, the case when the vertex 4 is in the outer face of 123 follows. The other case follows by the symmetry exchanging the outer and the inner face of the triangle 123. ☐

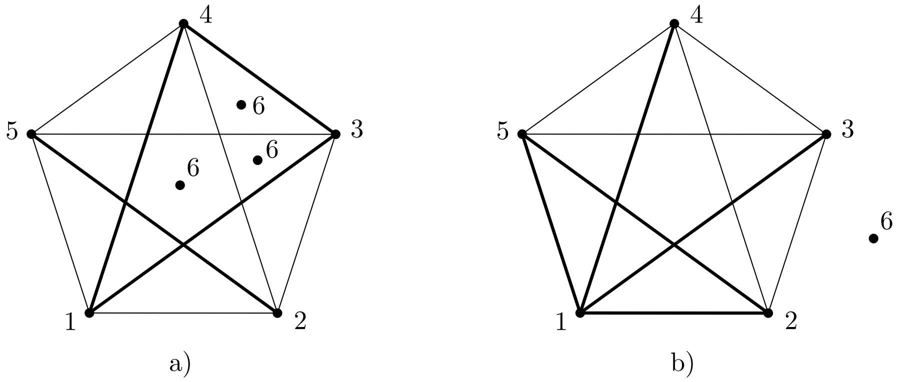

Figure 1.5: Impossibility of adding a vertex with rotation $(1,4,2,5,3)$. The thick edges cannot be crossed by the edge 61.

Lemma 1.11. *Let G be a simple complete topological graph with vertices $1, 2, \ldots,$ 6. Suppose that G contains a convex graph C_5 induced by the vertices $1, 2, \ldots, 5$, which appear in this counter-clockwise order on its outer face. Then the rotation of the vertex 6 is not $(1,4,2,5,3)$.*

Proof. Let H be the induced convex graph $G[\{1,2,3,4,5\}]$. Suppose for contradiction that the rotation of the vertex 6 in G is $(1,4,2,5,3)$. We distinguish two cases according to the face of H in which the vertex 6 is contained. See Figure 1.5.

a) The vertex 6 is in one of the inner faces of H. By symmetry, we may assume that it is either in the inner pentagonal face or in the intersection of the triangles 234 and 134. The rotation of the vertex 6 in $G[\{1,3,4,6\}]$ is $(1,4,3)$. By Lemma 1.10 applied to the triangle 134, the edge 61 lies completely inside the triangle 134. The vertex 6 is also outside the triangle 125 and the rotation of 6 in $G[\{1,2,5,6\}]$ is $(1,2,5)$. By Lemma 1.10, the edges 61 and 25 do not cross. But this is a contradiction as the vertices 6 and 1 are separated by a closed curve formed by portions of the edges $25, 14, 43, 31$, which the edge 16 cannot cross.

b) The vertex 6 is in the outer face of H. By Lemma 1.10 applied to the triangle 125, the edge 61 cannot cross the edge 25. Consequently, the edge 61 crosses no edge of H. Similarly, no other edge adjacent to 6 can cross an edge of H. This contradicts the conclusion of Lemma 1.10 applied to the triangle 134. □

Lemma 1.12. *Let G be a simple complete topological graph with vertices $1, 2, 3, 4,$ 5. Suppose that G contains a convex graph H induced by the vertices $1, 2, 3, 4$, which appear in this counter-clockwise order on its outer face. If the vertex 5 is inside the triangular face of H adjacent to vertices 2 and 3, then its rotation is not $(1,3,2,4)$.*

Proof. See Figure 1.6. Suppose for contradiction that the vertex 5 is inside the triangular face of H adjacent to vertices 2 and 3 and its rotation in G is $(1,3,2,4)$. By Lemma 1.10 applied to the triangles 234 and 134, the edge 54 does not cross

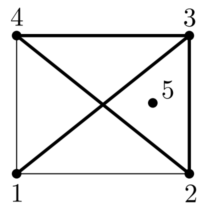

Figure 1.6: Impossibility of adding a vertex with rotation $(1,3,2,4)$ to the triangular face adjacent to vertices 2 and 3. The thick edges cannot be crossed by the edge 54.

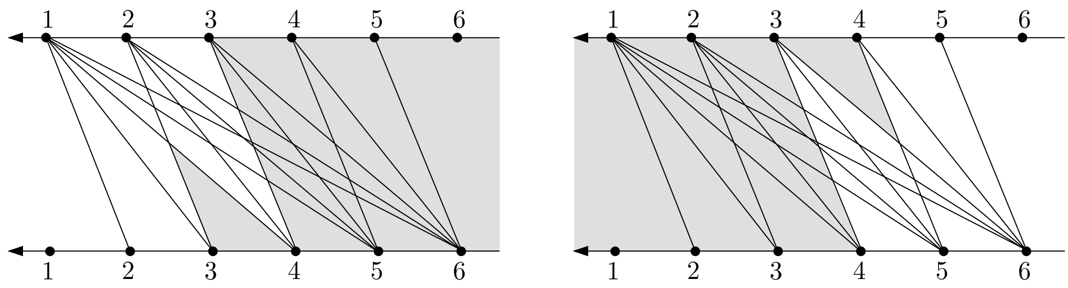

Figure 1.7: Impossibility of adding a vertex with rotation $(1,2,3,6,5,4)$ to the twisted graph T_6. The grey area represents the triangular face adjacent to the vertices 3 and 4 in the subgraphs induced by the vertices $1,2,3,4$ (left) and $3,4,5,6$ (right).

the edges $13, 23, 34$ and 24. But portions of these edges form a closed curve separating the vertices 4 and 5, a contradiction. □

Lemma 1.13. *Let G be a simple complete topological graph with vertices $1,2,\ldots,$ 7. Suppose that G contains a twisted graph T_6 induced by the vertices $1,2,\ldots,6$, in this canonical order. Then the rotation of the vertex 7 is not $(1,2,3,6,5,4)$.*

Proof. Suppose for contradiction that the rotation of the vertex 7 is $(1,2,3,6,5,4)$. The subgraphs $G_1 = G[\{1,2,3,4\}]$ and $G_2 = G[\{3,4,5,6\}]$ are both isomorphic to the convex graph C_4. The 4-cycles corresponding to the outer face of C_4 are 1243 and 3465, respectively. The two triangular faces adjacent to the vertices 3 and 4 in G_1 and G_2 cover the whole plane; see Figure 1.7. It follows that at least one of these two faces contains the vertex 7. The rotation of the vertex 7 is $(1,2,3,4)$ in G_1 and $(6,5,4,3)$ in G_2, which contradicts Lemma 1.12. □

1.3.4 Proof of Theorem 1.3

Now we finish the proof of Theorem 1.3 by combining previous results from this section. Let $g(n)$ be the number of different rotation systems of simple complete topological graphs with n vertices. By Proposition 1.6, we have $T_w(n) \leq g(n)$. We show an upper bound on $g(n)$ by induction.

Let $N = 4^{30^4}$. Assume that $n \geq 2N$, otherwise we have $g(n) \leq g(2N)$, which is a constant. We may also assume for simplicity that $n = 2^k$ where k is a positive integer.

Let G be a simple complete topological graph with vertices $v_1, v_2, \ldots, v_n$. Let G_1 be the subgraph of G induced by the vertices $v_1, \ldots, v_{n/2}$ and let G_2 be the subgraph of G induced by the vertices $v_{n/2+1}, \ldots, v_n$. Fix a rotation system $\mathcal{R}_1$ for G_1 and $\mathcal{R}_2$ for G_2. Choose an arbitrary drawing of G_1 with the rotation system R_1. Let v_i be a vertex of G_2. We show an upper bound on the number of possible rotations of v_i in the subgraph G_1^i of G induced by $V(G_1) \cup \{v_i\}$.

By Theorem 1.8, every simple complete topological graph with N vertices contains C_5 or T_6. Therefore, every induced subgraph of G_1 with N vertices contains a subgraph H weakly isomorphic to C_5 or T_6. By Lemma 1.11 or 1.13, one of the cyclic permutations of the vertices of H is forbidden in the rotation of v_i. Note that Lemmas 1.11 and 1.13 can be applied regardless of the particular way how H is drawn. Consequently, for each N-tuple of vertices in G_1, a non-empty subset of their cyclic permutations is forbidden in the rotation of v_i.

Let $\mathcal{R}_1^i$ denote the set of all possible rotations of v_i in G_1^i. To pass from cyclic permutations to linear permutations, we arbitrarily select a first element in each cyclic permutation from $\mathcal{R}_1^i$ and denote the resulting set of permutations as $\mathcal{P}_1^i$. For each forbidden cyclic permutation ρ of N elements, the permutations from $\mathcal{P}_1^i$ avoid all N linear permutations obtained from ρ. In particular, the VC-dimension of the set $\{\pi^{-1}; \pi \in \mathcal{P}_1^i\}$ is at most $N-1$. Let $f(m)$ be the maximum possible size of a set of m-permutations with VC-dimension $N-1$. By Theorem 1.7,

$$|\mathcal{R}_1^i| = |\mathcal{P}_1^i| \leq f(n/2) \leq 2^{(n/2)\cdot((2/t!)\alpha(n/2)^t + O(\alpha(n/2)^{t-1}))},$$

where $t = (N-2)/2$. For every $i > n/2$, the rotation of v_i in G is uniquely determined by the rotation π_i of v_i in G_1^i, the rotation π_i' of v_i in G_2 and by one of the $\frac{(n/2)(n/2-1)}{n-1}\binom{n-1}{n/2} \leq n2^n$ ways of merging π_i and π_i' together. For $i \leq n/2$, the situation is symmetric.

It follows that the number of all possible rotation systems of G with $\mathcal{R}_1$ and $\mathcal{R}_2$ fixed is at most

$$\begin{aligned}(f(n/2)\cdot n2^n)^n &\leq n^n \cdot 2^{n^2} \cdot 2^{(n^2/2)\cdot((2/t!)\alpha(n/2)^t + O(\alpha(n/2)^{t-1}))} \\ &\leq 2^{c(n^2/2)\cdot\alpha(n)^t},\end{aligned}$$

where c is an absolute constant. Since there are $g(n/2)$ possibilities for each of the rotation systems $\mathcal{R}_1$ and $\mathcal{R}_2$, we have

$$\begin{aligned}g(n) &\leq (g(n/2))^2 \cdot 2^{c(n^2/2)\cdot\alpha(n)^t} \\ &\leq g(2N)^n \cdot 2^{c(n^2/2+2(n/2)^2/2+4(n/4)^2/2+\cdots)\cdot\alpha(n)^t} \\ &\leq g(2N)^n \cdot 2^{c(n^2)\cdot\alpha(n)^t} = 2^{O(n^2\cdot\alpha(n)^t)}.\end{aligned}$$

1.3.5 Combinatorial generalization of Theorem 1.3

Here we generalize Theorem 1.3 to a purely combinatorial statement involving n-tuples of cyclic permutations. The aim is to estimate the number of possible rotation systems of a simple complete topological graph using as little topological information as possible. In particular, the only condition we need comes from drawings of complete graphs with 4 vertices.

graph	rotation system
H_1	$((2,4,3),(1,3,4),(1,4,2),(1,2,3))$
H_2^R	$((2,4,3),(1,4,3),(1,2,4),(1,2,3))$
H_3^R	$((2,3,4),(1,3,4),(1,2,4),(1,2,3))$
H_4^R	$((2,3,4),(1,4,3),(1,4,2),(1,2,3))$
H_1^R	$((2,3,4),(1,4,3),(1,2,4),(1,3,2))$
H_2	$((2,3,4),(1,3,4),(1,4,2),(1,3,2))$
H_3	$((2,4,3),(1,4,3),(1,4,2),(1,3,2))$
H_4	$((2,4,3),(1,3,4),(1,2,4),(1,3,2))$

Table 1.1: The eight possible rotation systems of a simple complete topological graph with 4 vertices. The labels refer to the drawings in Figure 1.4, where H_i^R denotes the mirror image of H_i.

Observation 1.14. [Ky09, PT06] *The eight rotation systems listed in Table 1.1 are the only possible rotation systems of a simple complete topological graph with vertices* $1, 2, 3, 4$.

The eight rotation systems from Observation 1.14 can be characterized by the following *parity condition*. Let $l \in \{1,2,3,4\}$ and $\{i,j,k\} = \{1,2,3,4\} \setminus \{l\}$, with $i = \min\{i,j,k\}$. We call the rotation (i,j,k) at l *positive* if $j < k$ and *negative* if $k < j$. A 4-tuple of rotations at vertices $1,2,3,4$ forms a rotation system of a simple complete topological graph with vertices $1,2,3,4$ if and only if the number of negative rotations is even. Note that this characterization does not depend on the particular linear ordering of the vertices.

An *abstract rotation system* $\mathcal{R}$ on a set $V = \{v_1, \dots, v_n\}$ is an n-tuple of cyclic $(n-1)$-permutations $\pi_{v_1}, \dots, \pi_{v_n}$ where the set of elements of π_{v_i} is $V \setminus \{v_i\}$. A *subsystem of* $\mathcal{R}$ *induced* by a subset $W = \{w_1, \dots, w_k\} \subset V$, denoted by $\mathcal{R}[W]$, is a $|W|$-tuple of cyclic permutations $\rho_{w_1}, \dots, \rho_{w_k}$ where ρ_{w_i} is a restriction of π_{w_i} to the subset $W \setminus \{w_i\}$.

An abstract rotation system is *realizable* if it is a rotation system of a simple complete topological graph. Realizable rotation systems on a set W of size 4 are precisely those satisfying the parity condition for some linear ordering of W. An abstract rotation system $\mathcal{R}$ is *good* if every subsystem of $\mathcal{R}$ induced by a 4-element subset is realizable.

We prove the following theorem, generalizing Theorem 1.3.

Theorem 1.15. *The number of good abstract rotation systems on an n-element set is at most*

$$2^{n^2 \cdot \alpha(n)^{O(1)}}.$$

We do not know whether the upper bound in Theorem 1.15 is asymptotically tight. The best lower bound $2^{\Omega(n^2)}$ on the number of good abstract rotation systems comes from Theorem 1.1.

Problem 1. *Is it true that the number of good abstract rotation systems on an n-element set is bounded by $2^{O(n^2)}$?*

We note that the asymptotic number of abstract rotation systems may vary significantly if a different pattern of the same size is forbidden. There are 16

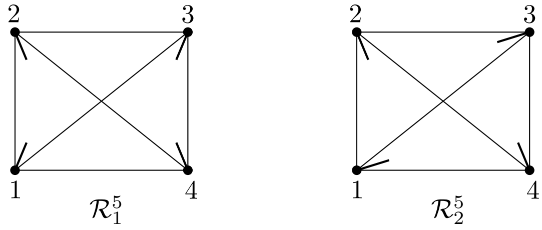

Figure 1.8: Partial realizations of the good abstract rotation systems $\mathcal{R}_1^5$ and $\mathcal{R}_2^5$. Thick segments represent the portions of the edges incident with the vertex 5.

possible abstract rotation systems on every 4-element set. If the forbidden pattern consists of a different set of eight abstract rotation systems, we may obtain $2^{\Omega(n^2 \log n)}$ abstract rotation systems on n elements satisfying this restriction. For example, consider the set $\mathcal{A}$ of all abstract rotation systems on the set $\{1, 2, \ldots, n\}$ where in every induced subsystem on four elements $i < j < k < l$, we forbid the eight abstract rotation systems with rotation (j, l, k) at i. The following construction shows that the size of $\mathcal{A}$ is $2^{\Omega(n^2 \log n)}$. Consider an abstract rotation system $\mathcal{R} = (\pi_1, \pi_2, \ldots, \pi_n)$ where $\pi_i(j) \in \{1, \ldots, i-1\}$ for $j \leq i-1$ and $\pi_i(j) = j+1$ for $j \geq i$. Clearly, the rotation at i in every subsystem of $\mathcal{R}$ induced by four elements $i < j < k < l$ is (j, k, l). The number of such abstract rotation systems is $\prod_{i=1}^{n}(i-1)! = 2^{\Omega(n^2 \log n)}$.

Good abstract rotation systems do not characterize realizable abstract rotation systems completely. For example, the following two good abstract rotation systems on five elements are not realizable:

$$\mathcal{R}_1^5 = ((2,5,3,4),(1,3,4,5),(1,2,5,4),(1,2,5,3),(1,3,4,2)),$$
$$\mathcal{R}_2^5 = ((2,3,5,4),(1,3,4,5),(1,5,2,4),(1,2,5,3),(1,4,3,2)).$$

It is straightforward to check that both $\mathcal{R}_1^5$ and $\mathcal{R}_2^5$ are good. Suppose that these systems are realizable. In both cases, in the subgraph H induced by the vertices $1, 2, 3, 4$, the edges 13 and 24 cross. Fix a drawing of H as a convex graph with vertices $1, 2, 3, 4$ on the outer face in clockwise order; see Figure 1.8. In both cases, the orientations of triangles and the rotations of vertices imply, by Lemma 1.10, that the vertex 5 must lie inside the triangles 132 and 143. But this is impossible as the two triangles have disjoint interiors.

While it is likely that there is no finite characterization of realizable abstract rotation systems by a finite list of forbidden subsystems, it is known that realizable abstract rotation systems can be recognized in polynomial time [Ky11].

To prove Theorem 1.15, we proceed in the same way as in the proof of Theorem 1.3, but we need to replace Theorem 1.8, Lemma 1.11 and Lemma 1.13 by combinatorial analogues.

An abstract rotation system on n elements is called *convex* and denoted by $\mathcal{C}_n$ if the elements can be ordered in a sequence $v_1, v_2, \ldots, v_n$ so that the rotation at v_i is $(v_1, v_2 \ldots, v_{i-1}, v_{1+1}, v_{i+2}, \ldots, v_n)$. An abstract rotation system on n elements is called *twisted* and denoted by $\mathcal{T}_n$ if the elements can be ordered in a sequence $v_1, v_2, \ldots, v_n$ so that the rotation at v_i is $(v_{i-1}, \ldots, v_2, v_1, v_{1+1}, v_{i+2} \ldots, v_n)$. Note that $\mathcal{C}_n$ is a rotation system of the convex graph C_n and $\mathcal{T}_n$ is a rotation system of the twisted graph T_n.

Two abstract rotation systems are *isomorphic* if they differ only by relabeling of their ground set. An abstract rotation system $\mathcal{R}$ *contains* an abstract rotation system $\mathcal{S}$ if $\mathcal{R}$ has an induced subsystem isomorphic to $\mathcal{S}$.

The following theorem generalizes Theorem 1.8.

Theorem 1.16. *For all positive integers m_1, m_2 there is an M such that every good abstract rotation system on M elements contains $\mathcal{C}_{m_1}$ or $\mathcal{T}_{m_2}$.*

To keep the proof simple, we do not try to optimize the value of M, as a function of the parameters m_1 and m_2. However, it is likely that the same bound as in Theorem 1.8 can be proved even in this generalized setting, by adapting the original topological proof [PST03]. We also note that the assumption of being good is not necessary: Theorem 1.16 holds in general for all abstract rotation systems, only with larger values of M.

Proof. Let $(\pi_1, \pi_2, \ldots, \pi_M)$ be a good abstract rotation system on the set $\{1, 2, \ldots, M\}$. Assume without loss of generality that $\pi_1 = (2, 3, \ldots, M)$ and that $\pi_i(1) = 1$ for $i > 1$. For every three elements i, j, k with $1 < i < j < k$, consider the induced abstract rotation system $\mathcal{R}[\{1, i, j, k\}]$. For $l \in \{i, j, k\}$, let $t_{i,j,k}(l) = 1$ if the rotation at l in $\mathcal{R}[\{1, i, j, k\}]$ is positive and $t_{i,j,k}(l) = 0$ if the rotation at l in $\mathcal{R}[\{1, i, j, k\}]$ is negative. The *type* of the triple (i, j, k) is the sequence $t_{i,j,k}(i)t_{i,j,k}(j)t_{i,j,k}(k)$. By the parity condition, we have the following four types of triples: $111, 100, 010$ and 001. By Ramsey's theorem, if M is sufficiently large, there is a subset $W \subseteq \{2, 3, \ldots, M\}$ of size $m = \max(m_1, m_2)$ such that all triples from W have the same type. Without loss of generality, assume that $W = \{2, 3, \ldots, m+1\}$. Let abc, with $a, b, c \in \{0, 1\}$, be the type shared by all the triples from W. If $a = 1$, then for each $l \in W$, the entries $l + 1, l + 2, \ldots, m + 1$ form an increasing sequence in π_l. If $a = 0$, the entries $l+1, l+2, \ldots, m+1$ form a decreasing sequence in π_l. Similarly, the entries $2, 3, \ldots, l-1$ form an increasing sequence in π_l if $c = 1$ and a decreasing sequence if $c = 0$. If $b = 1$, then in π_l, all entries smaller than l appear before all entries larger than l. If $b = 0$, then in π_l, all entries smaller than l appear after all entries larger than l. Therefore, if $abc = 111$ or 010, then W induces an abstract rotation system isomorphic to $\mathcal{C}_m$, and if $abc = 100$ or 001, then W induces an abstract rotation system isomorphic to $\mathcal{T}_m$. □

The following two lemmas generalize Lemma 1.11 and Lemma 1.13. Again, we do not try to optimize the sizes of the two abstract rotation systems $\mathcal{C}_{m_1}$ and $\mathcal{T}_{m_2}$.

Lemma 1.17. *Let $\mathcal{R}$ be a good abstract rotation system on the set $\{1, 2, \ldots, 8\}$. Suppose that the subsystem of $\mathcal{R}$ induced by the vertices $1, 2, \ldots, 7$ is $\mathcal{C}_7$, with $(v_1, \ldots, v_7) = (1, \ldots, 7)$. Then the rotation at 8 is not $(1, 3, 5, 7, 2, 4, 6)$.*

Proof. Let $\mathcal{R} = (\pi_1, \pi_2, \ldots, \pi_8)$ and suppose for contradiction that $\pi_8 = (1, 3, 5, 7, 2, 4, 6)$. Let $i, i+1, i+2$ be three consecutive numbers in the cyclic sequence $(1, 2, \ldots, 7)$. The subsystem $\mathcal{R}[\{i, i+1, i+2, 8\}] = (\rho^i_i, \rho^i_{i+1}, \rho^i_{i+2}, \rho^i_8)$ has at least one negative triple among $\rho^i_i, \rho^i_{i+1}, \rho^i_{i+2}$. If ρ^j_j is negative, that is, $\rho^j_j = (j+1, 8, j+2)$, we have $\pi_j = (1, 2, \ldots, j-1, j+1, 8, j+2, \ldots, 7)$. Similarly, if ρ_j^{j-1} is negative, then $\pi_j = (1, 2, \dots, j-1, 8, j+1, j+2, \dots, 7)$. Finally, if ρ_j^{j-2} is negative, then $\pi_j = (1, 2, \dots, j-2, 8, j-1, j+1, j+2, \dots, 7)$. Therefore, a negative triple ρ_j^i precisely determines the position of the element 8 in the rotation π_j, and each such rotation can arise from at most one negative triple ρ_j^i. It follows that in each of the rotations $\pi_j, j \in \{1, 2, \dots, 7\}$, the element 8 appears in one of the three possible positions between the elements $j-2$ and $j+2$. But then the subsystem $\mathcal{R}[\{1, 3, 5, 10\}] = ((10, 3, 5), (1, 10, 5), (1, 3, 10), (1, 3, 5))$ has exactly one negative triple, a contradiction. □

Let $\mathcal{R} = (\rho_1, \rho_2, \rho_3, \rho_4)$ be an abstract rotation system on a 4-element set. The *signature* of $\mathcal{R}$ is a sequence $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ of four symbols, where ε_i is '+' if ρ_i is positive and '−' if ρ_i is negative.

Lemma 1.18. *Let $m = 816$. Let $\mathcal{R}$ be a good abstract rotation system on the set $\{1, 2, \dots, m\}$. Suppose that the subsystem of $\mathcal{R}$ induced by the vertices $1, 2, \dots, m-1$ is $\mathcal{T}_{m-1}$, with $(v_1, \dots, v_{m-1}) = (1, \dots, m-1)$. Then the rotation at m is not $(1, 3, \dots, m-1, 2, 4, \dots, m-2)$.*

Proof. Let $\mathcal{R} = (\pi_1, \pi_2, \dots, \pi_m)$ and suppose for contradiction that $\pi_m = (1, 3, \dots, m-1, 2, 4, \dots, m-2)$. Let $k = 8$, $W = \{2k+1, 2k+2, \dots, m-4\}$ and $m' = |W| = m - 2k - 4$.

For $i \in W \cup \{m-3, m-2\}$, we say that a rotation π_i is of the *first type* if the element m appears in π_i within the subinterval $(i-2k, \dots, 1, i+1)$, of the *second type* if m appears in π_i within the subinterval $(i+2, \dots, m-1, i-1)$, of the *third type* if $\pi_i = (i-1, \dots, 1, i+1, m, i+2, \dots, m-1)$, and of the *fourth type* if m appears in π_i within the subinterval $(i-1, \dots, i-2k)$. Let W_1 (W_2, W_3) be the set of those elements $i \in W$ such that π_i is of the first (second, third) type, respectively. Let W_4' be the set of those elements $i \in W \cup \{m-3, m-2\}$ such that π_i is of the fourth type.

First we show that $|W_4'| \le 8k$. If $|W_4'| \ge 8k+1$, then at least $4k+1$ elements $i_1 < i_2 < \dots < i_{4k+1}$ of W_4' are all odd or all even. In particular, the rotation ρ_m in the subsystem $\mathcal{R}[\{i_1, i_{2k+1}, i_{4k+1}, m\}] = (\rho_{i_1}, \rho_{i_{2k+1}}, \rho_{i_{4k+1}}, \rho_m)$ is positive. Since the rotations $\pi_{i_1}, \pi_{i_{2k+1}}$ and $\pi_{i_{4k+1}}$ are of the fourth type, we observe that the signature of $\mathcal{R}[\{i_1, i_{2k+1}, i_{4k+1}, m\}]$ is $(+, +, -, +)$, which is a contradiction with the parity condition.

Next we show that $|W_3| \le m'/2 + 3$. Suppose for contradiction that $|W_3| \ge m'/2 + 4$. Let W_3^E be the set of even elements of W_3 and let I be the smallest interval containing W_3^E. Let $W_3^O = W_3 \setminus W_3^E$ be the set of odd elements of W_3. Since $|W_3^E \cup (W_3^O \setminus I)| \le m'/2 + 1$, the interval I contains at least 3 odd elements $o_1 < o_2 < o_3$ of W_3. In particular, for $e_1 = \min I$ and $e_3 = \max I$, we have $e_1, e_3 \in W_3^E$, $o_2 \ge e_1 + 3$ and $e_3 \ge o_2 + 3$. It follows that $\mathcal{R}[\{e_1, o_2, e_3, m\}] = ((m, o_2, e_3), (e_1, m, e_3), (o_2, e_1, m), (o_2, e_1, e_3))$. But this subsystem has signature $(+, -, -, -)$, a contradiction.

For each $i \in W_1$ and $j \in \{1, 2, \dots, k\}$, we consider the subsystem $\mathcal{R}[\{i - 2j + 1, i, i+1, m\}] = (\rho_{i-2j+1}^{i,j}, \rho_i^{i,j}, \rho_{i+1}^{i,j}, \rho_m^{i,j})$. Since the parity of i is opposite to the parity of $i - 2j + 1$ and $i + 1$, the rotation $\rho_m^{i,j}$ is negative. Since the rotation π_i is of the first type, we have $\rho_i^{i,j} = (i - 2j + 1, m, i+1)$. It follows that the signature of $\mathcal{R}[\{i-2j+1, i, i+1, m\}]$ is either $(+, -, +, -)$ or $(-, -, -, -)$. Moreover, there is a $j(i) \in \{0, 1, \dots, k\}$ such that for $j \le j(i)$ the signature of $\mathcal{R}[\{i-2j+1,i,i+1,m\}]$ is $(-,-,-,-)$ and for $j > j(i)$ the signature of $\mathcal{R}[\{i-2j+1,i,i+1,m\}]$ is $(+,-,+,-)$.

Similarly for each $i \in W_2$ and $j \in \{1,2,\dots,k\}$, we consider the subsystem $\mathcal{R}[\{i-2j,i,i+2,m\}] = (\sigma^{i,j}_{i-2j}, \sigma^{i,j}_{i}, \sigma^{i,j}_{i+1}, \sigma^{i,j}_{m})$. We have $\sigma^{i,j}_m = (i-2j,i,i+2)$ and $\sigma^{i,j}_i = (i-2j,i+2,m)$, thus $\mathcal{R}[\{i-2j,i,i+2,m\}]$ has signature either $(+,+,+,+)$ or $(-,+,-,+)$. Again, there is a $j(i) \in \{0,1,\dots,k\}$ such that the signature is $(-,+,-,+)$ for $j \le j(i)$ and $(+,+,+,+)$ for $j > j(i)$.

Let $W_1^+ = \{i \in W_1; j(i) < k\}$. For every $i \in W_1^+$, the signature of $\mathcal{R}[\{i-2k+1,i,i+1,m\}]$ is $(+,-,+,-)$. In particular, the rotation π_{i+1} is of the fourth type. Therefore, $|W_1^+| \le |W_4'| \le 8k$.

Similarly, let $W_2^+ = \{i \in W_2; j(i) < k\}$. For every $i \in W_1^+$, the signature of $\mathcal{R}[\{i-2k,i,i+2,m\}]$ is $(+,+,+,+)$. In particular, the rotation π_{i+2} is of the fourth type. Therefore, $|W_2^+| \le |W_4'| \le 8k$.

Let $W_1^- = W_1 \setminus W_1^+ = \{i \in W_1; j(i) = k\}$. For every $i \in W_2^+$ and every $j \in \{1,2,\dots,k\}$, the signature of $\mathcal{R}[\{i-2j+1,i,i+1,m\}]$ is $(-,-,-,-)$. In particular, $\pi_{i-2j+1} = (i-2j,\dots,1,i-2j+2,\dots,i,m,i+1,\dots,m-1)$. Observe that for every $l \in \{2,\dots,m-5\}$, there is at most one pair i,j such that $i \in W_1^-, j, \in \{1,2,\dots,k\}$ and $l = i-2j+1$. Thus we have $|W_1^-| \le \frac{m-6}{k}$.

Let $W_2^- = W_2 \setminus W_2^+ = \{i \in W_2; j(i) = k\}$. For every $i \in W_2^+$ and every $j \in \{1,2,\dots,k\}$, the signature of $\mathcal{R}[\{i-2j,i,i+2,m\}]$ is $(-,+,-,+)$. In particular, the element m appears in π_{i-2j} in one of the two positions in the subinterval $(i,i+1,i+2)$. This implies that for every $l \in \{1,2,\dots,m-6\}$, there is at most one pair i,j such that $i \in W_2^-, j, \in \{1,2,\dots,k\}$ and $l = i-2j$. Thus we have $|W_2^-| \le \frac{m-6}{k}$.

Putting all the estimates together, we have

$$\begin{aligned} m' = |W| &\le |W_1^+| + |W_1^-| + |W_2^+| + |W_2^-| + |W_3| + |W_4'| \\ &\le \frac{m'}{2} + 3 + \frac{2(m-6)}{k} + 24k \end{aligned}$$

and thus

$$\begin{aligned} k(m-2k-4) &\le 6k + 4(m-6) + 48k^2, \\ (k-4)m &\le 50k^2 + 10k - 24. \end{aligned}$$

By our choice $m = 816$ and $k = 8$, this gives $4 \cdot 816 \le 3256$ and we have a contradiction. □

1.3.6 Graphs with maximum number of crossings

Harborth and Mengersen [HM92] investigated simple complete topological graphs on n vertices with maximum number of crossings, which is $\binom{n}{4}$. They showed the lower bound $e^{\Omega(\sqrt{n})}$ on the number $T_{\mathrm{w}}^{\max}(n)$ of different weak isomorphism classes of such (unlabeled) graphs. Their construction actually gives a better lower bound $T_{\mathrm{w}}^{\max}(n) \ge 2^{n(\log n - O(1))}$ [Ky09].

We do not have any better upper bound on $T_{\mathrm{w}}^{\max}(n)$ than that from Theorem 1.3, thus the problem of determining $T_{\mathrm{w}}^{\max}(n)$ asymptotically seems to be wide open. However, the following observation could help with improving the upper bound to $2^{O(n^2)}$.

Let G be a simple complete topological graph with vertex set V and with $\binom{|V|}{4}$ crossings. Let $v \in V$ and let G' be a subgraph of G induced by $V \setminus \{v\}$. A *face* of G' is a connected region of the set obtained from the plane by removing all the edges of G'. Each bounded face of G' is an intersection of the interiors of a particular subset of triangles of G'. Two faces in two different weakly isomorphic drawings of G' are considered equivalent if they share the same subset of triangles they are contained in. By a *combinatorial face* we mean an equivalence class of faces, but also any particular face from the class. Lemma 1.10 implies that the combinatorial face of G' that contains v uniquely determines the rotation of v in G. Therefore, the number of possible rotations of v, with the weak isomorphism class of G' fixed, is bounded from above by the number $f(G')$ of possible combinatorial faces in a simple topological graph weakly isomorphic to G'. The number $f(G')$ may be exponential, for example when G' is the convex graph C_n. This graph has $n/2$ pairwise crossing edges (main diagonals), which may be drawn through a common point x. Then each of the edges can be redrawn to go around x from the left or from the right. Each of these choices produces a different combinatorial face containing x. On the other hand, it can be shown that $f(C_n) = 2^{O(n)}$, since each of the bounded combinatorial faces of C_n can be assigned to a unique subset of pairwise crossing diagonals, in the following way. Let C be the Hamiltonian cycle of C_n bounding the outer face. To each diagonal e of C we assign the region $r(e)$ bounded by e and by the shorter arc of C determined by the endpoints of e. (For the main diagonals, we choose the "shorter" arc arbitrarily.) Each face f is assigned to a set $R(f)$ of minimal regions $r(e)$ containing f. The set $R(f)$ determines all triangles containing f, and all diagonals e such that $r(e) \in R(f)$ are pairwise crossing. A set of pairwise crossing diagonals in C_n is uniquely determined by the set of their endpoints. Therefore, there are at most 2^{n-1} possible sets $R(f)$ and so $f(C_n) \leq 2^{n-1} + 1$.

We do not know whether similar upper bound holds for all simple complete topological graphs.

Problem 2. *Is it true that for every simple complete topological graph G with n vertices, the number of possible combinatorial faces in simple complete topological graphs weakly isomorphic to G satisfies $f(G) \leq 2^{O(n)}$?*

A positive answer to Problem 2 would imply that $T_{\mathrm{w}}^{\max}(n) = 2^{O(n^2)}$, by the proof in Subsection 1.3.4.

A similar question can be asked in the combinatorial setting. In a simple complete topological graph with n vertices and $\binom{n}{4}$ crossings, every 4-tuple of vertices induces a crossing. Therefore, for every complete subgraph with 4 vertices there are 6 possible rotation systems, corresponding to the rotation systems of the graphs $H_2, H_3, H_4, H_2^R, H_3^R, H_4^R$ in Table 1.1. In addition to the parity condition, these rotation systems satisfy the following condition. There exists a pair $i, j \in \{1, 2, 3, 4\}$ such that for $\{k, l\} = \{1, 2, 3, 4\} \setminus \{i, j\}$, the rotation at k is (i, j, l) and the rotation at l is (i, j, k). In fact, there are always four such pairs i, j, corresponding to the four edges without crossing in the drawing.

Problem 3. *What is the number of abstract rotation systems on n elements, where every subsystem induced by 4 elements is realizable as a rotation system of*

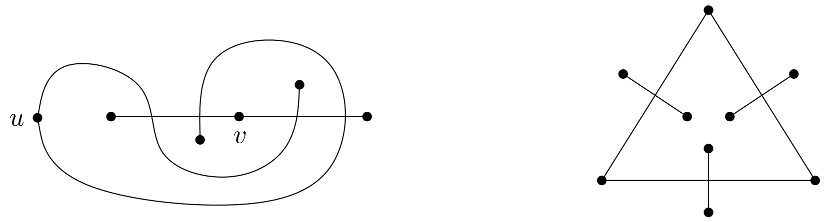

Figure 1.9: Left: A simple drawing of $P_3 + P_3$ which can not be extended by an edge uv. Right: A topologically connected drawing of a graph with four components, with every spanning forest topologically disconnected.

a simple drawing of K_4 with one crossing?

We do not know better lower bound than that implied by the topological construction by Harborth and Mengersen [HM92, Ky09]. The best upper bound comes from Theorem 1.15.

1.4 The upper bound in Theorem 1.2

Let $G = (V, E)$ be a graph with n vertices and m edges. If v is an isolated vertex in G, then $T_w(G) = T_w(G - v)$. Thus, we may assume that G has no isolated vertices. The upper bound on $T_w(G)$ for other graphs G then directly follows.

Let $\mathcal{G}$ be a simple topological graph realizing G. A *topological component* of $\mathcal{G}$ is a maximal connected subset of the plane that is a union of vertices and edges of $\mathcal{G}$. Note that a topological component of $\mathcal{G}$ is a union of components of G. A topological graph $\mathcal{G}$ is *topologically connected* if it has only one topological component.

First we extend G by adding edges connecting the topological components of $\mathcal{G}$ as follows. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two topological components of $\mathcal{G}$. We redraw $\mathcal{C}_2$ so that it has a vertex v_2 on the boundary of its outer face, and place this drawing inside a face of $\mathcal{C}_1$ containing a vertex v_1 on its boundary. Then we may add the edge v_1v_2 as a curve without crossings. We repeat this process until there is only one topological component. Since the graph G had no isolated vertices, we added at most m new edges, so the new graph has n vertices and $\Theta(m)$ edges. In this way, we might have created at most $n^{2m} \le 2^{O(m \log n)}$ different graphs. Thus, for proving the upper bound on $T_w(G)$, we may assume that $\mathcal{G}$ is topologically connected.

Ideally, we would like to extend the graph G to a connected graph, but it is not clear that it is always possible to connect two components of G that form a single topological component in the drawing by an edge so that the resulting drawing is still a simple topological graph. For example, there are simple topological graphs where some pairs of vertices from different components cannot be connected by an edge, so that the resulting drawing is still simple; see Figure 1.9, left.

1.4.1 A construction of a topological spanning tree

Next we construct a *topological spanning tree* $\mathcal{T}$ of $\mathcal{G}$; see Figure 1.10, left. A topological spanning tree $\mathcal{T}$ of $\mathcal{G}$ is a simply connected subset of the single topological component of $\mathcal{G}$ containing all vertices of $\mathcal{G}$ and satisfying the property that the only nonseparating points of $\mathcal{T}$ are the vertices of $\mathcal{G}$. Our goal is to find such a tree consisting of $O(n)$ connected portions of edges of $\mathcal{G}$. If G is a complete graph, we may take as $\mathcal{T}$ the star consisting of all edges incident with one vertex of $\mathcal{G}$ [Ky09], since such edges are internally disjoint. If G is connected, we may start with a drawing of an arbitrary spanning tree of G, but as some edges of the tree may cross, we may need to remove portions of some edges to break cycles. If G has multiple components, the construction is a bit more involved. For example, it is not enough to take a union of spanning trees of the individual components, as some of the spanning trees may be topologically disjoint, even if $\mathcal{G}$ is topologically connected; see Figure 1.9, right. Also we may need to include in $\mathcal{T}$ multiple disjoint portions of the same edge.

Let $C_1, \dots, C_k$ be the connected components of G. We choose their order in such a way that for every $i \in \{1, 2, \dots, k\}$, the drawing of $C_1 \cup \cdots \cup C_i$ is topologically connected. Then for every $i \in \{2, 3, \dots, k\}$, there is an edge e_i in C_i that crosses some edge $f_i \in C_1 \cup \cdots \cup C_{i-1}$. Let T_1 be a spanning tree of C_1 and let e_1 be an edge of T_1. For every $i \in \{2, 3, \dots, k\}$, let T_i be a spanning tree of C_i containing e_i. For every $i \in \{1, 2, \dots, k\}$, let $e_{i,1} = e_i$ and let $e_{i,2}, \dots, e_{i,m_i}$ be the remaining edges of T_i ordered in such a way that for every $j \in \{1, 2, \dots, m_i\}$, the subgraph of T_i formed by the edges $e_{i,1}, e_{i,2}, \dots, e_{i,j}$ is connected.

In the rest of this section we often identify the vertices, edges and subgraphs of G with the corresponding vertices, edges and subgraphs of $\mathcal{G}$.

The construction of $\mathcal{T}$ proceeds in k phases. In the first phase, we construct a topological spanning tree $\mathcal{T}_1$ of C_1, in the following way. We start with the tree $\mathcal{T}_{1,1}$ consisting of the single edge e_1. Let $j \in \{2, 3, \dots, m_1\}$ and suppose that the tree $\mathcal{T}_{1,j-1}$ has been defined. Let $v_{1,j}$ be the vertex of $e_{i,j}$ that is not contained in the edges $e_1, \dots, e_{j-1}$. If $e_{i,j}$ crosses none of the edges $e_{1,1}, \dots, e_{1,j-1}$, then let $\mathcal{T}_{1,j} = \mathcal{T}_{1,j-1} \cup e_{1,j}$. Otherwise, among the crossings of $e_{1,j}$ with the edges $e_{1,1}, \dots, e_{1,j-1}$, let $x_{i,j}$ be the crossing closest to $v_{1,j}$. The tree $\mathcal{T}_{1,j}$ is now obtained from $\mathcal{T}_{1,j-1}$ by attaching the portion of $e_{1,j}$ between $x_{1,j}$ and $v_{1,j}$. Finally, we put $\mathcal{T}_1 = \mathcal{T}_{1,m_1}$.

Let $i \in \{2, 3, \dots, k\}$ and suppose that the tree $\mathcal{T}_{i-1}$ has been defined. In the ith phase, we construct the tree $\mathcal{T}_i$ in the following way. Let $e_i = w_i w_i'$ and let x_i be the crossing of e_i with f_{i-1}. If e_i crosses $\mathcal{T}_{i-1}$ in at least one point, then let $x_{i,1}$ and $x_{i,1}'$ be the crossings of e_i with $\mathcal{T}_{i-1}$ closest to w_i and w_i', respectively. The tree $\mathcal{T}_{i,1}$ is then obtained from $\mathcal{T}_{i-1}$ by attaching the portion of e_i between w_i and $x_{i,1}$ and the portion of e_i between w_i' and $x_{i,1}'$. If e_i is disjoint with $\mathcal{T}_{i-1}$, then we construct $\mathcal{T}_{i,1}$ from $\mathcal{T}_{i-1}$ by adding the whole edge e_i and joining e_i with $\mathcal{T}_{i-1}$ by the shortest portion of f_{i-1} connecting x_i with a point of $\mathcal{T}_{i-1}$, which may be an endpoint of f_{i-1} or a crossing. The rest of the i-th phase is similar to the construction of $\mathcal{T}_1$. In j-th step, we construct $\mathcal{T}_{i,j}$ from $\mathcal{T}_{i,j-1}$ by attaching the portion of $e_{i,j}$ connecting the vertex of $e_{i,j}$ not contained in $\mathcal{T}_{i,j-1}$ with the closest point of $\mathcal{T}_{i,j-1}$ along $e_{i,j}$. Finally, we put $\mathcal{T}_i = \mathcal{T}_{i,m_i}$ and $\mathcal{T} = \mathcal{T}_k$.

It follows from the construction that the tree $\mathcal{T}$ has $n' \leq 2n$ vertices, which are either vertices or crossings of $\mathcal{G}$, and hence at most $2n$ edges, which are portions of edges of $\mathcal{G}$.

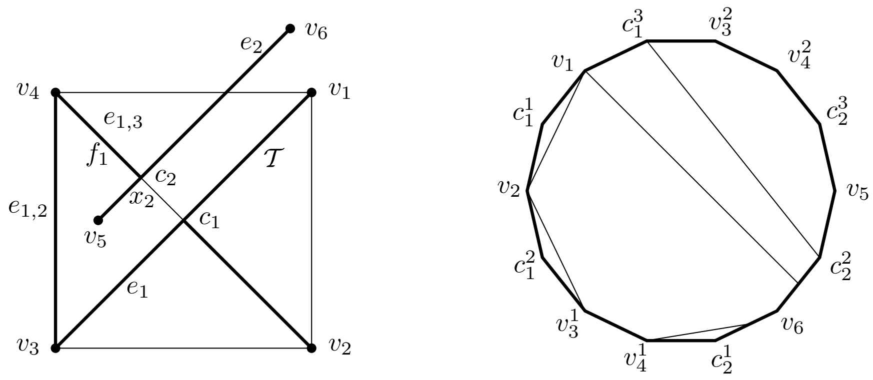

Figure 1.10: A topological spanning tree $\mathcal{T}$ of a simple topological graph with two components (left) and the corresponding $\mathcal{T}$-representation (right).

1.4.2 A construction of a $\mathcal{T}$-representation

Now we construct the *$\mathcal{T}$-representation of $\mathcal{G}$*, which generalizes the star-cut representation defined in [Ky09]. Consider $\mathcal{G}$ drawn on the sphere S^2 and cut the sphere along the edges of $\mathcal{T}$. The resulting open set $S^2 \setminus \mathcal{T}$ can be mapped by an orientation preserving homeomorphism Φ to an open regular $(2n'-2)$-gon D, in such a way that the inverse map Φ^{-1} can be continuously extended to the closure of D so that the vertices and edges of D are mapped to the vertices and edges of $\mathcal{T}$. Note that every edge of $\mathcal{T}$ corresponds to two edges of D, and a vertex of degree d in $\mathcal{T}$ corresponds to d vertices of D. See Figure 1.10, right. During the cutting operation, every edge e of G can be cut into at most n' pieces by the edges of $\mathcal{T}$. Each such piece becomes a *pseudochord* of D. That is, a simple curve in D with endpoints on the boundary of D, and with the property that every two such curves cross in at most one point. Moreover, two pseudochords sharing an endpoint are internally disjoint, as they correspond to portions of edges with a common vertex. To separate the endpoints of the pseudochords, we cut a small disc around each vertex w of D, draw a part of its boundary inside D as an arc g_w and shorten the pseudochords incident with w so that their endpoints are on g_w. For an edge e of D, let O_e be the counter-clockwise order of the endpoints of the pseudochords along e. Similarly, for each vertex w of D, let O_w be the counter-clockwise order of the endpoints of the pseudochords along g_w. The orders O_e and O_w are given as sequences of labels of the pseudochords. The collection of the orders O_e and O_w, which together form a cyclic sequence of endpoints of the pseudochords along the boundary of D, is called the *perimetric order*.

The *$\mathcal{T}$-representation of $\mathcal{G}$* is given by (1) the topological spanning tree $\mathcal{T}$ and (2) the perimetric order O_D. The tree $\mathcal{T}$ is given as an abstract graph with a rotation system, which determines its combinatorial planar embedding.

Note that the perimetric order determines which pairs of pseudochords cross and how the pseudochords connect to the edges. Thus the $\mathcal{T}$-representation of $\mathcal{G}$ determines the weak isomorphism class of $\mathcal{G}$. However, topological graphs weakly isomorphic to $\mathcal{G}$ may have several different $\mathcal{T}$-representations, which differ by the orders of crossings along the edges of $\mathcal{T}$. We say that two $\mathcal{T}$-representations are *weakly isomorphic* if they are representations of weakly isomorphic topological graphs.

1.4.3 Counting topological spanning trees

The upper bound on $T_w(G)$ will follow from an upper bound on the number of weak isomorphism classes of $\mathcal{T}$-representations of simple drawings of G. First we estimate the number of different topological spanning trees.

Lemma 1.19. *Let G be a graph with n vertices, m edges and no isolated vertices. Topologically connected simple realizations of G have at most $2^{O(n \log n)}$ different topological spanning trees, up to a homeomorphism of the plane.*

Proof. Let k be the number of connected components of G. A component with n_i vertices has at most $n_i^{n_i-2}$ spanning trees, hence G has at most $2^{O(n \log n)}$ spanning forests. Let $T_1 \cup T_2 \cup \cdots \cup T_k$ be a fixed spanning forest of G. The inductive construction of the topological spanning tree $\mathcal{T}$ consists of $n-k$ steps. In each step, an edge of some spanning tree T_i is added to the construction. Consider the step where a portion of the edge $e_{i,j}$ is added to the tree $\mathcal{T}_{i,j-1}$. The new edge is attached either to a vertex of $\mathcal{T}_{i,j-1}$ or to an interior point of some edge of $\mathcal{T}_{i,j-1}$. There are two ways how to attach a new edge to an edge of $\mathcal{T}_{i,j-1}$, and d ways how to attach a new edge to a vertex of degree d in $\mathcal{T}_{i,j-1}$. Together, there are $4(n_{i,j}-1) \le 4n' - 4 \le 8n$ different ways how to attach a new edge, where $n_{i,j}$ is the number of vertices of $\mathcal{T}_{i,j-1}$, and there are at most m choices for the edge $e_{i,j}$.

Now consider the step where portions of the edge e_i are added to the tree $\mathcal{T}_{i-1}$. If e_i crosses $\mathcal{T}_{i-1}$, then two portions of e_i are added and this step is equivalent to two previous steps. If e_i does not cross $\mathcal{T}_{i-1}$, then the whole edge e_i and a portion of f_{i-1} are added. There are at most m choices for e_i, m choices for f_{i-1}, two ways how to attach the portion of f_{i-1} to e_i and at most $8n$ different ways how to attach the portion of f_{i-1} to $\mathcal{T}_{i-1}$. Altogether, we have at most $(8nm)^{n-1} \le 2^{O(n \log n)}$ ways how to construct $\mathcal{T}$. □

1.4.4 Counting $\mathcal{T}$-representations

It remains to estimate for each topological spanning tree $\mathcal{T}$, the maximum number of weak isomorphism classes of $\mathcal{T}$-representations. This will be the dominant term in the estimate of $T_w(G)$. Every edge of G corresponds to at most $2n$ pseudochords in the $\mathcal{T}$-representation. Hence the $\mathcal{T}$-representation has at most $2mn$ pseudochords, with at most $\binom{4mn}{8n}(4mn)! \le 2^{O(mn \log n)}$ different perimetric orders. This gives a trivial $2^{O(mn \log n)}$ upper bound on the number of weak isomorphism classes of $\mathcal{T}$-representations.

To determine the weak isomorphism class, we do not need the whole information given by the perimetric order. In fact, we only need to know the number of pseudochords corresponding to each edge of G and the *type* of each pseudochord [Ky11], which we define in the next paragraph. There are at most $(2n)^m \le 2^{O(m \log n)}$ choices of the numbers of pseudochords corresponding to the edges of G in the $\mathcal{T}$-representation. This upper bound is asymptotically dominated by the upper bounds in Theorem 1.2, hence we consider these numbers fixed in the rest of this section.

The *type* $t(p)$ of a pseudochord p is the pair (X, Y) where each of X, Y is either an edge of the polygon D containing the endpoint of p or an endpoint of p on the arc g_w for some vertex w of D. For each vertex w of D representing a vertex v of G, we consider $\deg(v)$ points on g_w as possible values of X and

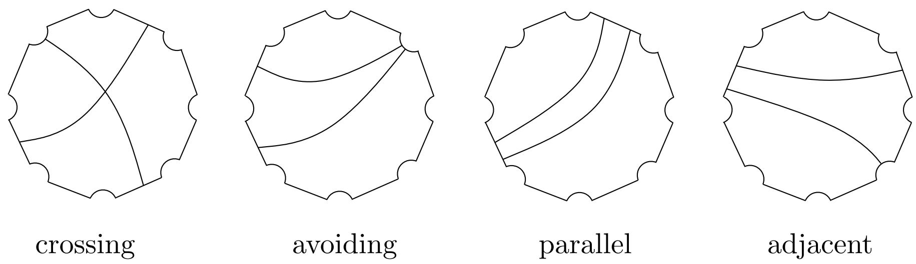

Figure 1.11: Four categories of pairs of types of pseudochords.

Y. For each triple of vertices w_1, w_2, w_3 of D representing a crossing x of $\mathcal{G}$, we have exactly one possible endpoint as a possible value of X and Y, on exactly one of the arcs $g_{w_1}, g_{w_2}, g_{w_3}$. This follows from the fact that $\mathcal{T}$ contains exactly three portions of edges incident with x and only the fourth portion becomes a pseudochord.

Let p and p' be pseudochords with types (X, Y) and (X', Y'), respectively. We say that the types (X, Y) and (X', Y') are

crossing if the elements X, X', Y, Y' are pairwise distinct and their cyclic order around the boundary of D is (X, X', Y, Y') or (X, Y', Y, X'),

avoiding if they are not crossing and all the elements X, X', Y, Y' are pairwise distinct,

parallel if $(X, Y) = (X', Y')$ or $(X, Y) = (Y', X')$, and

adjacent otherwise, that is, if exactly one of the following four equalities holds: $X = X'$, $X = Y'$, $Y = X'$ or $Y = Y'$.

See Figure 1.11 for examples.

If the elements X, Y, X', Y' are pairwise distinct, we can directly determine whether p and p' cross: crossing types imply crossing pseudochords and avoiding types imply disjoint pseudochords. If, for example, $X = X'$ (in which case X is an edge of D), we cannot determine whether p and p' cross, since this depends on the relative position of the endpoints of p and p' on X. The pairs of pseudochords with parallel and adjacent types can be arranged into maximal sequences, called *ladders*, formed by portions of two edges of G, for which we can determine whether they cross or not. See [Ky11] for details.

A pseudochord is called *external* if it represents the initial or the terminal portion of an edge of G. Thus, at least one of the endpoints of an external pseudochord lies on one of the arcs g_w where w is a vertex of D representing a vertex of G. All the other pseudochords are called *internal*. Every external pseudochord can have one of $O((n+m)^2)$ possible types. Every internal pseudochord, representing an internal portion of an edge of G, can have only $O(n^2)$ different types, since for the variables X, Y, we are considering only edges of D and points on the arcs g_w, where w is a vertex of D representing a crossing of $\mathcal{G}$. Altogether, there are at most $(O(n+m)^{4m}) \leq 2^{O(m \log n)}$ combinations of types of the external pseudochords. This is again asymptotically dominated by the upper bounds in Theorem 1.2. In the rest of this section, we consider only internal pseudochords.

For a subset $F \subseteq E$ of edges of G, let $f(F)$ be the number of possible combinations of types of the internal pseudochords corresponding to the edges from F. Similarly, for a set S of internal pseudochords, let $f(S)$ be the number of possible combinations of types of pseudochords from S. Our goal is to obtain a good upper bound on $f(E)$.

A trivial estimate gives the upper bound $f(E) \leq O(n^2)^{mn} = 2^{O(mn \log n)}$. This can be improved by considering the fact that the pseudochords representing a common edge of G do not cross. Also note that for two pseudochords p, p' representing a common edge e, their types are always avoiding. It follows that the set of types of the pseudochords representing e can be represented as a noncrossing matching of size at most $2n$ on a set of at most $8n$ points in convex position, where each point corresponds to an edge or a vertex of D. Observe that the order of the pseudochords along e can be reconstructed from this matching, thus this representation is injective. The number of such matchings is bounded from above by $2^{O(n)}$. Together, this gives the upper bound $f(E) \leq 2^{O(nm)}$.

This estimate can be improved even further. In a simple topological graph, edges incident to a common vertex v do not cross. Therefore, all the internal pseudochords representing edges incident with v are pairwise disjoint. Let $P(v)$ be the set of these pseudochords. Note that two pseudochords from $P(v)$ representing different edges may have avoiding, parallel or adjacent types. Let d be the degree of v. Similarly as before, we can represent the set of types of the pseudochords from $P(v)$ as a noncrossing matching M on a set of at most $8dn$ points in convex position, where each vertex of D is represented by a point and each edge of D is represented by d consecutive points. Again, from the matching M and from the types of the external pseudochords representing the edges incident with v we can uniquely determine which edge each pseudochords represents and how the pseudochords connect together to form the (portions of) edges incident with v. A straightforward upper bound $f(P(v)) \leq 2^{O(dn)}$ follows. To get a better upper bound, we observe that many of these pseudochords share the same type. More precisely, we have up to $2dn$ pseudochords in $P(v)$, but only $O(n)$ different types, since no two of the types are crossing. There are $2^{O(n)}$ ways of choosing the set of pairwise noncrossing types for internal pseudochords. For a fixed set S of $O(n)$ types, we assign to each type $t \in S$ its *weight*, that is, a positive integer $n(t)$ denoting the number of pseudochords from $P(v)$ with type t. The set $\{n(t), t \in S\}$ satisfying the property $\sum_{t \in S} n(t) = |P(v)|$ is called the *weight vector* of S. From the set S and its weight vector, we can reconstruct the matching M and determine the type of each pseudochord and how the pseudochords connect to edges. This idea is similar to encoding curves on a surface using normal coordinates [SSS02, SSS03]. For a fixed S, there are $\binom{O(dn)}{O(n)} = d^{O(n)} = 2^{O(n \log d)}$ different weight vectors. This gives the upper bound $f(P(v)) \leq 2^{O(n \log d)}$. By Jensen's inequality, $f(E) \leq 2^{O(n^2 \log(m/n))}$. Together with Lemma 1.19, this gives the first upper bound in Theorem 1.2.

The previous method gives a good upper bound on $T_w(G)$ for dense graphs. For graphs with $o(n^2)$ edges, the method is useful if the graph has very irregular degree sequence; more precisely, if it has a small number of vertices covering almost all the edges. For graphs with $o(n^{3/2})$ edges and with most of the vertices of degree $\Theta(m/n)$, we get better results by considering larger subsets of edges. We just need to balance the number of edges in the subset to keep the number

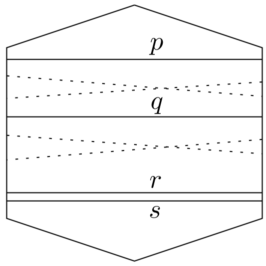

Figure 1.12: Illustration for the proof of Lemma 1.20. The dotted lines represent pseudochords from $P_1(F)$. The pseudochords p, q, r, s all have the same type, r and s also have the same subtype, but p, q and r have pairwise different subtypes.

of their crossings small enough.

Lemma 1.20. *Let $F \subseteq E$ be a set of k edges. Then*

$$f(F) \le \binom{O(m+k^2)}{O(k^2)} \cdot 2^{O(k^2 \log k)} \cdot 2^{O(n+k^2)} \cdot \binom{kn}{O(n+k^2)}.$$

In particular, for $k = \lfloor \sqrt{n} \rfloor$ we have

$$f(F) \le 2^{O(n \log n)}.$$

Proof. Let $P(F)$ be the set of (both external and internal) pseudochords representing the edges of F. Since every two edges cross at most once, there are at most $\binom{k}{2}$ crossings among the pseudochords from $P(F)$. In particular, at most k^2 pseudochords from $P(F)$ cross other pseudochord from $P(F)$. Let $P_1(F) \subseteq P(F)$ be the set of pseudochords crossing at least one pseudochord from $P(F)$. Let $P_0(F)$ be the set of internal pseudochords from $P(F) \setminus P_1(F)$. We estimate the number of perimetric orders of $|P_0(F) \cup P_1(F)|$ pseudochords in D inducing at most $\binom{k}{2}$ crossings. Each such perimetric order, together with the set of types of the external pseudochords from $P(F)$, determine the types of all pseudochords from $P(F)$, since no member of $P(F) \setminus P_1(F)$ crosses a member of $P_0(F) \cup P_1(F)$.

For the pseudochords from $P_1(F)$, we have at most $\binom{O(m+k^2)}{2k^2}$ ways of choosing the set of their endpoints on the boundary of D, and at most $(k^2)! \le 2^{O(k^2 \log k)}$ ways of matching them together. Here we do not need to optimize for matchings inducing $O(k^2)$ crossings. However, Proposition 1.22 in the next section implies the upper bound $2^{O(k^2)}$.

The pseudochords from $P_0(F)$ form a noncrossing matching in the regions of $D \setminus (\bigcup P_1(F))$. To determine the positions of the pseudochords from $P_0(F)$, we need to refine their types into *subtypes* by splitting the edges of D by the endpoints of the pseudochords from $P_1(F)$. See Figure 1.12. There are at most $O(n + k^2)$ portions of edges of D after this splitting, hence at most $2^{O(n+k^2)}$ choices for the set of pairwise noncrossing subtypes of the pseudochords from $P_0(F)$. Finally, there are at most $\binom{kn}{O(n+k^2)}$ ways of assigning a vector of positive integers with total sum at most kn to the chosen set of subtypes. This is sufficient to determine the perimetric order of the pseudochords from $P(F)$ and the lemma follows. $\square$

The second upper bound in Theorem 1.2 is proved as follows. By Lemma 1.19, we may fix a topological spanning tree. Then we partition the edge set of G into

$O(m/\sqrt{n})$ subsets of size at most $\sqrt{n}$ and apply Lemma 1.20 to each of the subsets. Theorem 1.4 is a special case of Theorem 1.2, where the graph G is a matching.

1.5 The upper bound in Theorem 1.5

We start with some additional definitions and a combinatorial definition of the isomorphism of topological graphs. Then we show that we need to consider only topologically connected topological graphs. Finally, we reduce the problem to counting isomorphism classes of arrangements of pseudochords and present two different solutions to this problem. In the first solution we split the problem into two parts: enumerating chord diagrams and enumerating arrangements with fixed boundary, using encoding by binary vectors. The second approach is based on enumerating the dual graphs of the arrangements, which form a subclass of quadrangulations of a disc.

1.5.1 A combinatorial definition of the isomorphism

A *rotation* of a crossing c in a topological graph is the clockwise cyclic order in which the four portions of the two edges crossing at c leave the point c. Note that each crossing has exactly two possible rotations. An *extended rotation system* of a simple topological graph is the set of rotations of all its vertices and crossings. Assuming that T and T' are drawings of the same abstract graph, we say that their (extended) rotation systems are *inverse* if for each vertex $v \in V(T)$ (and each crossing c in T) the rotation of v and the rotation of the corresponding vertex $v' \in V(T')$ are inverse cyclic permutations (and so are the rotation of c and the rotation of the corresponding crossing c' in T'). For example, if T' is a mirror image of T, then T and T' have inverse (extended) rotation systems.

Topologically connected topological graphs G and H are *isomorphic* if (1) G and H are weakly isomorphic, (2) for each edge e of G the order of crossings with the other edges of G is the same as the order of crossings on the corresponding edge e' in H, and (3) the extended rotation systems of G and H are either the same or inverse. This induces a one-to-one correspondence between the faces of G and H such that the crossings and the vertices incident with a face f of G appear along the boundary of f in the same (or inverse) cyclic order as the corresponding crossings and vertices in H appear along the boundary of the face f' corresponding to f. It follows from Jordan-Schönflies theorem that this definition is equivalent to the previous one in Section 1.1.

Let G be a topological graph with more than one topological component. The *face structure* of G is a collection of face boundaries, represented as oriented facial walks in the underlying abstract graph, of all noncontractible faces of G, that is, faces with more than one boundary component. The orientations are chosen in such a way that either for each noncontractible face the facial walk of the outer boundary component is oriented clockwise and the facial walks of all inner boundary components are oriented counter-clockwise, or vice versa. Both choices are regarded as giving the same face structure. By this condition, the orientations of the facial walks in the face structure encode relative orientations of the topological components. Note that the rotation system of G is not sufficient to determine the orientation of topological components that are simple cycles.

Topological graphs G and H with more than one topological component are *isomorphic* if there is a one-to-one mapping between the vertices and edges of G and H satisfying the conditions (1)–(3) and, in addition, (4) the face structures of G and H are the same.

1.5.2 Reduction to topologically connected graphs

Let G be a graph with no isolated vertices. Let $\mathcal{G}$ be a topological graph realizing G. If $\mathcal{G}$ has more than one topological component, we want to extend it to a topologically connected graph by adding edges connecting the topological components, in the same way as in the previous section. However, for this extension to be possible we may need to rearrange the topological components of $\mathcal{G}$, which changes the face structure of $\mathcal{G}$. While preserving the isomorphism classes of the k topological components of $\mathcal{G}$, there are 2^k ways of choosing their orientation and at most $O(n^4)^{2k}$ possible face structures of topological graphs built from these components. Thus there are at most $2^{O(n \log n)}$ rearrangements of topological components of $\mathcal{G}$. Hence, by the same argument as in the previous section, we may further assume that $\mathcal{G}$ is topologically connected.

1.5.3 Arrangements of pseudochords

An essential part of the structure of a particular isomorphism class of simple topological graphs is captured by the following combinatorial object, which slightly generalizes arrangements of pseudolines.

An *arrangement of pseudochords* is a finite set M of simple curves in the plane with endpoints on a common simple closed curve C_M, such that all the curves from M lie in the region bounded by C_M and every two curves in M have at most one common point, which is a proper crossing. The elements of M are called *pseudochords*. The arrangement M is *simple* if no three pseudochords from M share a common crossing. The *perimetric order* of M is the counter-clockwise cyclic order of the endpoints of the pseudochords of M on C_M. The perimetric order of M determines which pairs of pseudochords cross and which do not, but it does not determine the orders of crossings on the pseudochords. Two (labeled) arrangements of pseudochords are *isomorphic* if they have the same perimetric order and the same orders of crossings on the corresponding pseudochords. Equivalently, one arrangement can be obtained from the other one by an orientation preserving homeomorphism. Note that a $\mathcal{T}$-representation of a simple topological graph can be regarded as a simple arrangement of pseudochords.

The following proposition is inspired by Felsner's [Fe97] enumeration of simple wiring diagrams. Originally it appeared in [Ky09] as Proposition 7, but in an incorrect, stronger form.

Proposition 1.21. [Ky09, a correct form of Proposition 7] *The number of isomorphism classes of simple arrangements of n pseudochords with fixed perimetric order inducing k crossings is at most 2^{2k}.*

Proof. Let $M = \{p_1, p_2, \ldots, p_n\}$ be a simple arrangement of pseudochords with endpoints on a circle C_M and with a given perimetric order. Cut the circle at an

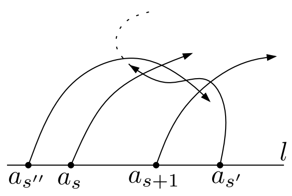

Figure 1.13: $p_{s'}$ cannot be the first pseudochord crossing p_s.

arbitrary point and unfold it by a homeomorphism to a horizontal line l, while keeping all the pseudochords above l. Orient each pseudochord p_i from its left endpoint a_i to its right endpoint b_i. Let k_i be the number of crossings on p_i and let $c^i_1, c^i_2, \dots, c^i_{k_i}$ be the crossings of p_i ordered from a_i to b_i. Let $p_{r(i,j)}$ be the pseudochord that crosses p_i at c^i_j.

For two crossing pseudochords p_i and p_j we say that p_i *is to the left of* p_j if a_i is to the left of a_j. This is equivalent with the rotation of their common crossing being (a_i, b_j, a_j, b_i).

To each p_i we assign a vector $\alpha^i = (\alpha^i_1, \alpha^i_2, \dots, \alpha^i_{k_i}) \in \{0,1\}^{k_i}$ where $\alpha^i_j = 0$ if $p_{r(i,j)}$ is to the left of p_i and $\alpha^i_j = 1$ if p_i is to the left of $p_{r(i,j)}$.

The sum of the lengths of the vectors α^i is equal to $\sum_{i=1}^{n} k_i = 2k$. Hence, there are at most 2^{2k} different sequences $(\alpha^1, \alpha^2, \dots, \alpha^n)$ encoding an arrangement with the given perimetric order and the chosen orientation of pseudochords.

It remains to show that we can uniquely reconstruct the isomorphism class of M from the vectors $\alpha^1, \alpha^2, \dots, \alpha^n$ by identifying the pseudochords $p_{r(i,j)}$. We proceed by induction on k and n. For arrangements without crossings there is only one isomorphism class with a fixed perimetric order. Now, suppose that we can reconstruct the isomorphism class for arrangements with at most $k-1$ crossings and take a sequence $\alpha = (\alpha^1, \alpha^2, \dots, \alpha^n)$ encoding an arrangement M with k crossings.

If some of the vectors α^i is empty, the corresponding pseudochord p_i is empty (has no crossing). We may then draw p_i as an arbitrary curve γ_i from a_i to b_i in the upper half-plane of l. Then we split the arrangement into two parts: the *inner* part consisting of pseudochords with endpoints between a_i and b_i, and the *outer* part with endpoints to the left of a_i or to the right of b_i. We draw both parts separately by induction. Finally, by applying a suitable homeomorphism we place the inner part inside the region bounded by γ_i and l and the outer part outside that region.

Further we assume that M has no empty pseudochords.

Without loss of generality we may assume that the left endpoints are ordered along l as $a_1, a_2, \dots, a_n$ from left to right. Clearly, $\alpha^1 = (1,1,\dots,1)$ and $\alpha^n = (0,0,\dots,0)$. It follows that there exists $s \in \{1,\dots,n-1\}$ such that $\alpha^s_1 = 1$ and $\alpha^{s+1}_1 = 0$.

Claim. *The first crossing on the pseudochords p_s and p_{s+1} is their common crossing. That is, $r(s,1) = s+1$ and $r(s+1,1) = s$.*

Proof of claim. Refer to Figure 1.13. For contradiction, suppose that $r(s,1) = s' \geq s+2$ (the case when $r(s+1,1) \leq s-1$ is symmetric). Then $r(s+1,1) \notin \{s, s'\}$. Hence, $r(s+1,1) = s''$ for some $s'' < s$ and the crossing of p_{s+1} with $p_{s''}$ occurs within the triangle $a_s a_{s+1} c_1^s$. This forces the pseudochords $p_{s'}$ and $p_{s''}$ to cross twice, a contradiction. □

Let $c = c_1^s = c_1^{s+1}$ be the first crossing on p_s and p_{s+1}. Since the two arcs $a_s c$ and $a_{s+1} c$ are free of crossings, there is no endpoint between a_s and a_{s+1} on l. For the induction step, we swap the endpoints a_s and a_{s+1} in the perimetric order of M and delete the first value from the vectors α^s and α^{s+1}. In this way we obtain an encoding α' of an arrangement M' with $k-1$ crossings, which is obtained from M by deleting the arcs $a_s c$ and $a_{s+1} c$, including a small open neighborhood of c. By the induction hypothesis, the isomorphism class of M' can be uniquely reconstructed from α'. By attaching to M' two crossing arcs starting at a_s and a_{s+1} and thus extending the two pseudochords p_s and p_{s+1}, we obtain an arrangement isomorphic to M. □

1.5.4 Counting isomorphism classes of topologically connected topological graphs

Let G be a graph with n vertices, m edges and no isolated vertices. Let $\mathcal{G}$ be a topologically connected simple topological graph that realizes G. The isomorphism class of $\mathcal{G}$ is determined by the isomorphism class of a $\mathcal{T}$-representation of $\mathcal{G}$. To determine the isomorphism class of a $\mathcal{T}$-representation, we need to determine (1) the topological spanning tree $\mathcal{T}$, (2) the perimetric order of the $\mathcal{T}$-representation, and (3) the isomorphism type of the induced arrangement of pseudochords.

(1) By Lemma 1.19, there are at most $2^{O(n \log n)}$ choices for the topological spanning tree $\mathcal{T}$ of $\mathcal{G}$, up to a homeomorphism of the plane. For the rest of the section, we fix one topological spanning tree $\mathcal{T}$ of $\mathcal{G}$.

(2) With $\mathcal{T}$ fixed, a $\mathcal{T}$-representation can have at most $2^{O(mn \log n)}$ diferent perimetric orders, as we have seen in Subsection 1.4.4.

This estimate is good enough when G has $m = \omega(n \log n)$ edges, but we need a better upper bound for sparser graphs. This can be achieved by counting only perimetric orders that induce at most $\binom{m}{2}$ crossings.

There are at most $\binom{4mn}{8n} \leq 2^{O(n \log n)}$ ways of choosing the set of endpoints of the pseudochords along the boundary of the disc D in the $\mathcal{T}$-representation. To determine the perimetric order, we need, in addition, to determine a perfect matching of the endpoints inducing at most $\binom{m}{2}$ crossings.

Such matchings can be also regarded as representations of *circle graphs* with a given number of vertices and edges. In the literature, these structures are called *chord diagrams* [Kh00, Re79]. See Figure 1.14, left. Following the notation in [Re79], let $C(n,k)$ denote the number of diagrams of n chords with k crossings. It is well known that $C(n,0)$, which is the number of noncrossing perfect matchings of $2n$ points on the circle, is equal to the nth Catalan number. Precise enumeration results for $C(n,k)$ in the form of generating functions were obtained by Touchard [To50] and Riordan [Ri75], but explicit formulas for $C(n,k)$ were computed only for $k \leq 6$ [To50]. The following asymptotic upper bound is implicit in Read's paper [Re79].

Proposition 1.22. [Re79] *For the number of diagrams of n chords with at most*

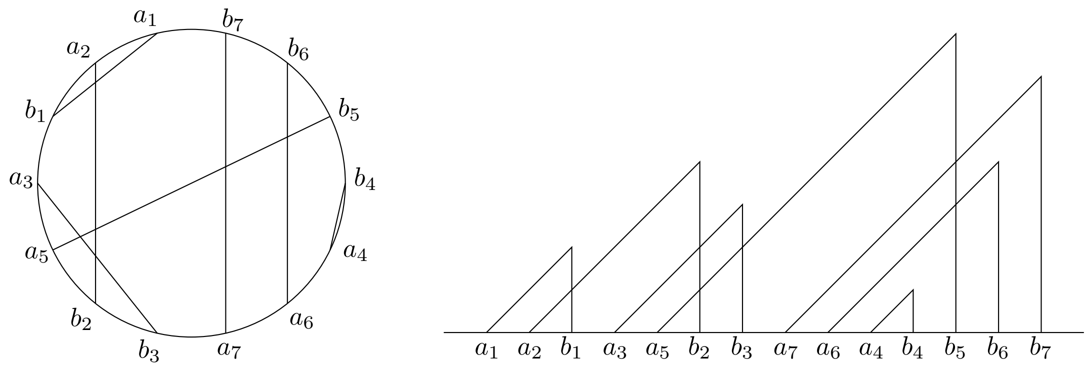

Figure 1.14: A chord diagram with seven chords and six crossings and a corresponding sawtooth diagram with $\kappa = (1, 2, 1, 0, 2, 0, 0)$.

k crossings, we have the upper bound

$$\sum_{i=0}^{k} C(n,i) \leq C(n)\binom{n+k}{n}$$

where $C(n)$ is the nth Catalan number.

Proof. Like in the proof of Proposition 1.21, the key "trick" is breaking the symmetry of the circle by cutting it at one point and unfolding onto a horizontal line l. The chords then become arcs in the upper half-plane with endpoints on l. Each such arc has a distinguished left endpoint and a right endpoint. Instead of arbitrary arcs, Read [Re79] constructs triangular "teeth" consisting of a diagonal segment from the left endpoint followed by a vertical segment to the right endpoint and calls the resulting drawing the *sawtooth diagram* associated to the original chord diagram. See Figure 1.14, right.

Let L be the set of all the left endpoints of the chords on l, and R the set of all the right endpoints. For every point x on l, there are at least as many left endpoints than right endpoints to the left of x. Therefore the sets L and R correspond to the sets of n left and n right parentheses that are correctly matched. There are exactly $C(n)$ such partitions (L, R) of the $2n$ points on l.

One partition (L, R) can be shared by more sawtooth diagrams, if crossings are allowed. To determine the sawtooth diagram (and the corresponding chord diagram) uniquely, we encode the intersection graph of the chords as follows. Let $b_1, b_2, \ldots, b_n$ be the points of R ordered from left to right. For $i = 1, 2, \ldots, n$, let c_i be the chord with right endpoint b_i, let a_i be the left endpoint of c_i and let k_i be the number of chords with left endpoint to the right of a_i that cross c_i. We claim that the vector $\kappa = (k_1, k_2, \ldots, k_n)$, together with the partition (L, R), uniquely determines the sawtooth diagram. This can be seen by drawing the diagram from left to right. All the crossings of the chord c_i with chords with left endpoint to the right of a_i occur on the vertical segment of c_i with endpoint b_i. Therefore, every time we reach the x-coordinate of some b_i, we take the $(k_i + 1)$th diagonal segment from the bottom and connect its right endpoint by a vertical line to b_i. All the other diagonal segments are extended further to the right.

Since $\sum_{i=1}^{n} k_i \leq k$, for every partition (L, R) there are at most $\binom{n+k}{k}$ possible vectors κ and the proposition follows. □

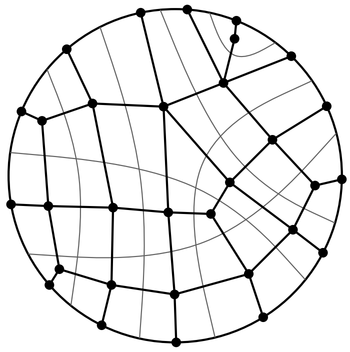

Figure 1.15: A simple arrangement of 7 pseudochords with 9 crossings and its dual quadrangulation.

By Proposition 1.22, by the entropy bound for binomial coefficients and by the inequality $\log_e(1+x) \le x$, the number of possible perimetric orders of the $\mathcal{T}$-representation is at most

$$2^{O(n\log n)} \cdot C(2mn)\binom{2mn+\binom{m}{2}}{2mn} \le 2^{2mn\log(1+\frac{m}{4n})+\frac{m^2}{2}\log(1+\frac{4n}{m})+4mn+O(n\log n)}$$
$$\le 2^{2mn(\log(1+\frac{m}{4n})+2+\log_2 e)+O(n\log n)}.$$

(3) By Proposition 1.21, there are less than 2^{m^2} isomorphism classes of simple arrangements of pseudochords induced by the $\mathcal{T}$-representation with a given perimetric order. Together with Proposition 1.22 and previous discussion, this proves the second upper bound in Theorem 1.5.

Arrangements and quadrangulations

Here we show an alternative approach to enumerating simple arrangements of pseudochords.

A *quadrangulation* of the disc D is a 2-connected plane graph embedded in D such that its outer face coincides with the boundary of D and every inner face is bounded by a 4-cycle. A quadrangulation is called *simple* if it has no separating 4-cycle. The vertices of the quadrangulation lying on the boundary of D are called *external*, all the other vertices are *internal*.

Mullin and Schellenberg [MS68] proved that there are

$$\frac{(3M+3)!(2N+M-1)!}{(M-1)!(2M+3)!N!(N+M+1)!} \le \binom{3M+3}{M}\binom{2N+M-1}{N}$$

isomorphism classes of rooted simple quadrangulations of the disc with N internal and $2M+4$ external vertices.

The *dual graph* of a simple arrangement of pseudochords is constructed as follows. Place one vertex inside each 2-dimensional cell and one vertex in the interior of every boundary edge. Then join all pairs of vertices that correspond to adjacent 2-cells or to a boundary edge and its adjacent 2-cell. See Figure 1.15.

Observe that the dual graph of a simple arrangement of n pseudochords with k crossings is a simple quadrangulation with $2n$ external and $n+k+1$ internal vertices. From the quadrangulation the original arrangement can be uniquely reconstructed up to isomorphism. However, not all simple quadrangulations can be obtained in this way: the graph of the 3-dimensional cube is such an example.

By plugging $M = n - 2$ and $N = n + i + 1$ into Mullin's and Schellenberg's formula and summing over $i = 0, 1, \dots, k$ we obtain the following upper bound.

Proposition 1.23. *There are at most*

$$\binom{3n-3}{n-2}\binom{3n+2k}{n+k+1}$$

isomorphism classes of simple arrangements of n pseudochords with at most k crossings. □

Instead of using Proposition 1.22 and 1.21, we may directly apply Proposition 1.23 with $n := 2mn$ and $k := \binom{m}{2}$. This gives the first upper bound in Theorem 1.5.

1.6 The lower bounds

In this section we present constructions of many pairwise different simple drawings of a given graph G, proving the lower bounds in Theorem 1.5 and 1.2. Since we are dealing with arbitrary graphs, we use the following tool to find large subgraphs with more "regular" structure.

Let G be a graph and let A, B be disjoint subsets of its vertices. By $G[A, B]$ we denote the bipartite graph $(A \cup B, E_G(A, B))$ consisting of all edges with one endpoint in A and the other endpoint in B.

Lemma 1.24. *Let q, r be positive integers with $q \geq 3$ and $1 \leq r \leq \binom{q}{2}$. Let H be a graph with vertex set $\{1, 2, \dots, q\}$ and with r edges. Let $G = (V, E)$ be a graph with n vertices and m edges. There is a partition of the vertex set V into q clusters $V_1, \dots, V_q$ such that for every edge $\{i, j\}$ of H the number of edges in the bipartite graph $G[V_i, V_j]$ is at least*

$$\frac{2m}{q^2}\left(1 - \sqrt{\frac{r(q-2)}{2} \cdot \frac{n}{m}} - O\left(\sqrt{\frac{m}{n^3}}\right)\right).$$

This is a variant of the result by Kühn and Osthus [KO07, Theorem 3], who consider the case of $r = \binom{q}{2}$ and assume that G has maximum degree bounded by a constant fraction of n. The proof of Lemma 1.24 is similar to that of Theorem 3 in [KO07]. The main idea is to use the second order method to analyze the random partition.

During the analysis we need to bound the number of pairs of adjacent edges in a graph G, which we denote by $p(G)$. Let $\mathcal{G}(n, m)$ be the class of all graphs with n vertices and m edges and let $f(n, m)$ be the maximum of $p(G)$ over all $G \in \mathcal{G}(n, m)$. Ahlswede and Katona [AK78] proved that the maximum of $p(G)$ is always attained for at least one of two special graphs in $\mathcal{G}(n, m)$, a *quasi-star* or a *quasi-clique*. Ábrego et al. [AFNW09] completely characterized all graphs $G \in \mathcal{G}(n, m)$ for which $p(G) = f(n, m)$. The problem of computing $f(n, m)$ has been studied and partially solved by many researches; see [AFNW09] or [Ni07] for an overview of previous results. Although all the values of $f(n,m)$ have been computed, the behavior of the function depends on certain nontrivial number-theoretic properties of the parameters m,n [AFNW09]. Nikiforov [Ni07] proved tight asymptotic upper bounds on $f(n,m)$, which may be stated in a simplified form as follows.

Lemma 1.25. [Ni07, Theorem 2] *For all n and m,*

$$f(n,m) \le \sqrt{2}m^{3/2} \qquad \text{if } m \ge n^2/4, \text{ and}$$
$$f(n,m) \le \frac{1}{2}\left((n^2-2m)^{3/2} - n^3\right) + 2nm \qquad \text{if } m < n^2/4.$$

We use a weaker, even more simplified asymptotic upper bound, which is easier to apply. For our purposes, we need the bound to be tight only for small values of m.

Corollary 1.26. *For all n and m,*

$$f(n,m) \le \frac{1}{2}nm + O\left(\frac{m^2}{n}\right).$$

Proof. If $m \ge n^2/4$, then by Lemma 1.25 we have

$$f(n,m) \le \sqrt{2}m^{3/2} \cdot \frac{2m^{1/2}}{n} \le \frac{2\sqrt{2}m^2}{n}.$$

If $m < n^2/4$, then by Lemma 1.25, the desired upper bound is equivalent to the inequality

$$3mn - n^3 + (n^2-2m)^{3/2} \le O(m^2/n).$$

Using the inequality $\sqrt{1-x} \le 1 - x/2$, which holds for $x \le 1$, we have

$$3mn - n^3 + (n^2-2m)^{3/2} = 3mn + n^3((1-2m/n^2)^{3/2} - 1)$$
$$\le 3mn + n^3\left(\left(1-\frac{m}{n^2}\right)^3 - 1\right) = \frac{3m^2}{n} - \frac{m^3}{n^3}.$$

□

Proof of Lemma 1.24. Let $V_1, V_2, \ldots, V_q$ be a random partition of the vertex set V, where each vertex is assigned independently to cluster V_i with probability $1/q$. For $\{i,j\} \in E(H)$, let $X_{i,j}$ be a random variable counting the number of edges in the bipartite graph $G[V_i,V_j]$. Clearly, we have $\mathrm{E}X_{i,j} = 2m/q^2$. Let $\sigma^2 = \sigma^2_{i,j} = \mathrm{VAR}X_{i,j}$.

By Chebyshev's inequality, we have

$$P\left(X_{i,j} < \frac{2m}{q^2} - \sqrt{r}\sigma\right) < \frac{1}{r}.$$

It follows that there is a partition $V_1, V_2, \ldots, V_q$ such that for every edge $\{i,j\}$ of H, the graph $G[V_i,V_j]$ has at least $\frac{2m}{q^2} - \sqrt{r}\sigma$ edges.

To complete the proof, we need to estimate σ from above. Let $X = X_{i,j}$ for some $\{i,j\} \in E(H)$. We have

$$\sigma^2 = \mathrm{E}X^2 - (\mathrm{E}X)^2 = \mathrm{E}X^2 - \frac{4m^2}{q^4}.$$

For every edge e of G, let X_e be the indicator variable of the event that e has one endpoint in V_i and the other endpoint in V_j. Clearly, $X = \sum_{e\in E} X_e$. Recall that $p(G)$ denotes the number of pairs of adjacent edges in G. We have

$$\begin{aligned}
\mathrm{E}X^2 &= \sum_{e\in E} \mathrm{E}X_e^2 + 2\cdot \sum_{e,e'\in E;\, e\neq e'} \mathrm{E}X_e X_{e'}\\
&= \frac{2m}{q^2} + 2\cdot\frac{2}{q^3}\cdot p(G) + 2\cdot\frac{4}{q^4}\left(\binom{m}{2} - p(G)\right)\\
&= \frac{2m}{q^2} + \frac{8}{q^4}\binom{m}{2} + \left(\frac{4}{q^3} - \frac{8}{q^4}\right) p(G).
\end{aligned}$$

By Corollary 1.26, $p(G) \le \frac{1}{2}nm + O\left(\frac{m^2}{n}\right)$. Hence,

$$\begin{aligned}
\sigma^2 &\le \frac{2m}{q^2} + \frac{4m^2}{q^4} + \frac{4q-8}{q^4}\cdot\frac{1}{2}nm + O\left(\frac{m^2}{n}\right) - \frac{4m^2}{q^4}\\
&\le \frac{2q-4}{q^4}\cdot nm + O\left(\frac{m^2}{n}\right)\\
&\le \left(\frac{\sqrt{2q-4}}{q^2}\cdot\sqrt{nm} + O\left(\frac{m^{3/2}}{n^{3/2}}\right)\right)^2
\end{aligned}$$

and the lemma follows.

□

1.6.1 The lower bound in Theorem 1.5

The construction giving the first lower bound in Theorem 1.5 generalizes the construction from [Ky09].

Let $\varepsilon > 0$ and let $G = (V, E)$ be a graph with n vertices and m edges. We apply Lemma 1.24 with $q = 6$, $r = 3$ and $E(H) = \{\{1,4\},\{2,5\},\{3,6\}\}$. If $m > (6+\varepsilon)\cdot n$, then Lemma 1.24 implies that there is a partition of V into six clusters $V_1, V_2, \dots, V_6$ such that each of the three subgraphs $G[V_1,V_4], G[V_2,V_5], G[V_3,V_6]$ has $\Omega(m)$ edges. We may assume that $G[V_3,V_6]$ has the least number of edges of these three graphs.

Like in [Ky09], we construct $2^{\Omega(m^2)}$ drawings of G that are all weakly isomorphic to the same geometric graph with vertices in convex position. For each $k = 1, 2, \dots, 6$, we place the vertices of the set V_k on the unit circle, inside a small neighborhood of the point $(\cos(\frac{k\pi}{3}), \sin(\frac{k\pi}{3}))$; see Figure 1.16, left. For every pair of vertices $u \in V_k$ and $v \in V_l$ such that $|k-l| \neq 3$, we draw the edge uv as a straight-line segment. For $k \in \{1,2,3\}$, the edges between the sets V_k and V_{k+3} are drawn inside a narrow rectangle R_k such that all the crossings among this group of edges occur outside the region $R = R_1 \cap R_2 \cap R_3$, and for $k, l \in \{1,2,3\}$, $k \neq l$, all the crossings between the edges of $\mathrm{G}[V_k, V_{k+3}]$ and $\mathrm{G}[V_l, V_{l+3}]$ lie inside R. In the region R, the edges connecting V_2 with V_5 form $\Omega(m)$ parallel curves.

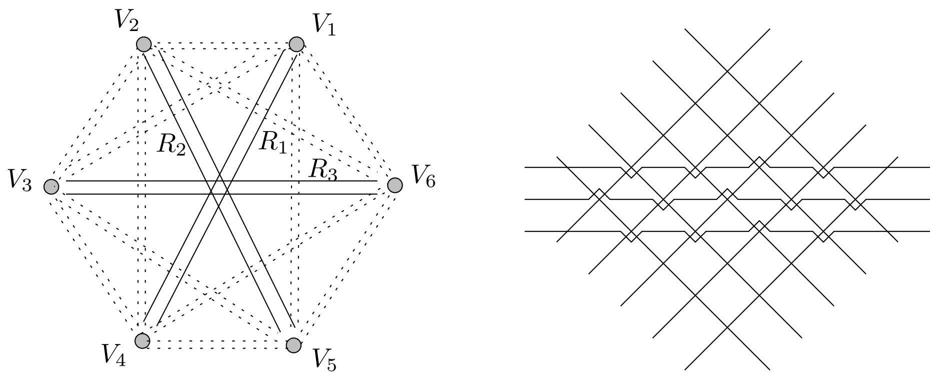

Figure 1.16: A construction of $2^{\Omega(m^2)}$ pairwise nonisomorphic drawings of a given graph.

Together with the edges connecting V_1 with V_4, they form an $\Omega(m) \times \Omega(m)$ grid inside R.

We partition the crossings of this grid into $\Omega(m)$ parallel diagonals forming horizontal rows. Each (horizontal) edge e connecting V_3 with V_6 is drawn along one of the diagonal d_i. Each edge is assigned to a different diagonal. In the neighborhood of each crossing c in d_i we can decide whether the edge e passes above or below c; see Figure 1.16, right. These two possibilities give us two nonisomorphic topological graphs, and the choices can be made independently at each crossing of the grid. Since we make the choice at $\Omega(m^2)$ crossings, we obtain $2^{\Omega(m^2)}$ pairwise nonisomorphic drawings of G.

For graphs with superlinear number of edges, Lemma 1.24 gives a partition where each of the graphs $G[V_i, V_j]$ has $c = 2m/q^2 - o(m)$ edges. In the previous construction, this gives a grid with $c^2 = 4m^2/q^4 - o(m^2)$ crossings and hence $2^{3m^2/q^4 - o(m^2)}$ pairwise nonisomorphic drawings of G, since 3/4 of the crossings can be covered by c parallel diagonals. For $q = 6$, this gives the lower bound $T(G) \geq 2^{m^2/432 - o(m^2)}$.

The constant 1/432 can be easily improved. Previous construction used as a "template" a convex geometric drawing of K_6. This topological graph has one *free triangle*, that is, a triangular face bounded by three pairwise crossing edges. A free triangle may be *switched* by moving a portion of one of the boundary edges over the crossing of the other two edges. This feature is then amplified by replacing the free triangle by the grid construction. A set of k free triangles is *independent* if no two of the triangles share a vertex. Equivalently, every two triangles share at most one boundary edge. This guarantees that each of the 2^k combinations of switched triangles is possible. There are simple drawings of K_6 with two independent free triangles [HM92, HM74]. If we use one of them as a template, we get $2^{m^2/216 - o(m^2)}$ pairwise nonisomorphic drawings of G.

Using larger simple complete topological graphs as templates, much better lower bounds can be obtained. Instead of free triangles, we may consider, in general, *free k-tuples*, which consist of k pairwise crossing edges with all $\binom{k}{2}$ crossings close to each other, forming locally an arrangement of k pseudolines. A system of free k-tuples is *independent* if no two k-tuples share a crossing.

When replacing a free 4-tuple by the grid construction, we use both horizontal and vertical diagonals of the grid. After drawing the horizontal and vertical edges

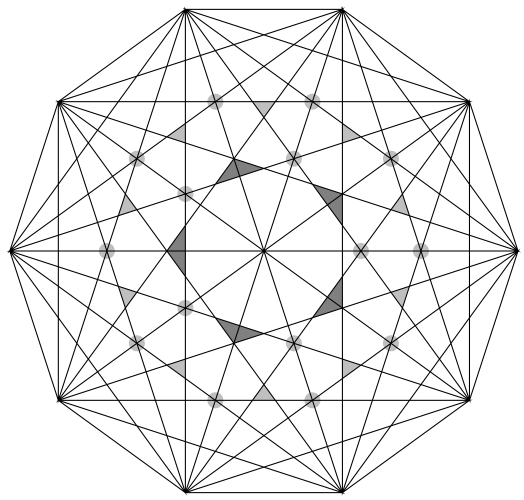

Figure 1.17: A complete convex geometric graph with 10 vertices and an independent system of 25 free triangles (light grey), 5 free 4-tuples (dark grey) and one free 5-tuple.

along the diagonals, half of the crossings in the grid become free 4-tuples and the other half free triangles. Every free 4-tuple can be drawn in 8 different ways. Therefore, by replacing each of the original four edges by c parallel edges, we obtain $2^{(1/2+3\cdot 1/2)c^2} = 2^{2c^2}$ pairwise nonisomorphic drawings. That is, every free 4-tuple in the template with q vertices contributes $8m^2/q^4$ to the exponent in the lower bound on $T(G)$.

For example, the regular convex drawing of K_{10} on Figure 1.17 has, after small perturbation, one free 5-tuple, 5 free 4-tuples and 25 free triangles, all independent. Using this drawing as a template, we obtain the lower bound $T(G) \geq 2^{m^2 \cdot 123/10^4 - o(m^2)} > 2^{m^2/82} - o(1)$ (for simplicity, we estimate the contribution of the free 5-tuple by the contribution of a free 4-tuple).

Further improvement can be obtained using all possible partial arrangements of three pairwise crossing systems of parallel pseudolines, in place of the grid construction, which produces only a subset of all such arrangements. Felsner and Valtr [FV11] proved that there are $2^{(4.5\log_2 3 - 6 - o(1))n^2} > 2^{1.132 \cdot n^2} - o(1)$ partial arrangements of $3n$ pseudolines that form three pairwise crossing subsets of n parallel pseudolines. They observed that these partial arrangements are dual to rhombic tilings of a regular hexagon and used MacMahon's formula enumerating these tilings. This also implies the rough lower bound $2^{2.264 \cdot n^2} - o(1)$ on the number of partial arrangements of $4n$ pseudolines that form four pairwise crossing subsets of n parallel pseudolines. Using these estimates with the template from Figure 1.17, we obtain the lower bound $T(G) \geq 2^{m^2 \cdot 167.585/10^4 - o(m^2)} > 2^{m^2/60} - o(1)$.

This lower bound on $T(G)$ is very likely far from being optimal. However, it is probably hard to close the gap between the lower and upper bound on $T(G)$, given that even for pseudoline arrangements, the best known lower and upper bounds on their number differ significantly [FV11].

1.6.2 The lower bound in Theorem 1.2

Fix $\varepsilon > 0$ and let G be a graph with n vertices and m edges, with no isolated vertices, and satisfying at least one of the conditions $m > (1+\varepsilon)n$ or $\Delta(G) < (1-\varepsilon)n$.

The first construction

First we show that $T_{\mathrm{w}}(G) \geq 2^{\Omega(m)}$ for graphs with $m > (4+\varepsilon)n$, generalizing a construction by Pach and Tóth [PT06].

Without loss of generality assume that n is odd. Let W be a random subset of $(n+1)/2$ vertices of G. The expected number of edges in the induced graph $G[W]$ is

$$\frac{\binom{(n+1)/2}{2}}{\binom{n}{2}} m = \frac{n+1}{4n} m = \left(\frac{1}{4} + \frac{1}{4n}\right) m.$$

Let W_0 be a subset of $(n+1)/2$ vertices inducing at least $(1/4 + 1/(4n))m$ edges. Every graph with m edges has a bipartite subgraph with at least $m/2$ edges. Let $W_0 = W_1 \cup W_2$ be a bipartition such that the bipartite graph $G[W_1, W_2]$ has at least $(1/8 + 1/(8n))m$ edges.

We place the vertices of V on three parallel vertical lines as follows. The vertices of $W' = V \setminus W_0$ are placed on the y-axis to the points $(0, i/2)$, $i = 0, 1, \ldots, (n-3)/2$, the vertices of W_1 to the points $(-1, i)$, $i = 0, 1, \ldots, |W_1| - 1$, and the vertices of W_2 to the points $(1, i), i = 1, 2, \ldots, |W_2| - 1$. Observe that the midpoint of every straight-line segment with one endpoint in W_1 and the other endpoint in W_2 lies in W'.

The idea of obtaining exponentially many pairwise different drawings of G is now similar as in the grid construction in the previous subsection. The edges of $G[W']$ are drawn as arcs close to the y-axis. Every edge $e = w_1 w_2$ of $G[W_1, W_2]$ is drawn as an arc along the straight-line segment $w_1 w_2$, in one of two possible ways: either close above or close below the segment. See Figure 1.18. Let w' be the midpoint of $w_1 w_2$. If w' is adjacent to a vertex $u \in V \setminus \{w_1, w_2\}$, then in one of the two drawings the edges $w_1 w_2$ and $w'u$ cross and in the other one they are disjoint. Since the minimum degree in G is at least 1, for every $w' \in W'$, there is at most one pair of vertices $w_1 \in W_1$ and $w_2 \in W_2$ such that w' is a midpoint of the segment $w_1 w_2$ and w' is not adjacent to $V \setminus \{w_1, w_2\}$. This implies that for at least

$$(1/8 + 1/(8n))m - (n-1)/2 > ((1/8 + 1/(8n) - 1/(8+2\varepsilon))m = \Omega(m)$$

edges of $G[W_1, W_2]$, the two choices produce two weakly nonisomorphic drawings. Since the choices for all the edges are independent, this gives $2^{\Omega(m)}$ pairwise weakly nonisomorphic drawings of G in total.

The second construction

The lower bound on $T_{\mathrm{w}}(G)$ can be improved for sparse graphs with minimum degree at least 1 that have at least $(1+\varepsilon)n$ edges or maximum degree at most $(1-\varepsilon)n$. Such assumptions are needed to guarantee a nontrivial number of pairs of independent edges, to avoid graphs like stars, which can only be drawn without

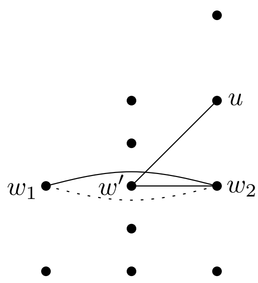

Figure 1.18: Two ways of drawing the edge w_1w_2.

crossings. For every such graph G we show the lower bound $T_{\mathrm{w}}(G) \geq 2^{\Omega(n \log n)}$. Moreover, all the drawings in this construction are *geometric graphs*; that is, the edges are drawn as straight-line segments.

Lemma 1.27. *Let G be a graph with n vertices, m edges, no isolated vertices and satisfying $m > (1+\varepsilon)n$ or $\Delta(G) < (1-\varepsilon)n$. Then the vertex set of G can be partitioned into three parts V_1, V_2, V_3 such that $|V_1| \geq n/4$, every vertex from V_1 has a neighbor in V_2 and the induced graph $G[V_3]$ has at least $\lfloor \varepsilon/2 \cdot n \rfloor$ edges.*

Proof. We distinguish two cases.

Case 1: G has a spanning forest F with no isolated vertices such that its components can be partitioned into two subforests F_1 and F_2, each with at least $\lfloor \varepsilon n \rfloor$ vertices. Assume that $|V(F_1)| \geq |V(F_2)|$. We set $V_3 = V(F_2)$. Now V_1 and V_2 are defined as the color classes of a proper 2-coloring of F_1, with $|V_1| \geq |V_2|$.

Case 2: No spanning forest as in Case 1 exists. Let F be a spanning forest with no isolated vertices and maximum possible number of components. If some of the components has a path of length three as a subgraph, then by removing the middle edge of the path, the tree splits into two smaller nontrivial components, contradicting the choice of F. It follows that every component of F is a star, that is, a graph isomorphic to $K_{1,k}$ for some $k \geq 1$. Let T_0 be the largest component in F. By the assumption, T_0 is a star with more than $\lceil (1-\varepsilon)n \rceil$ vertices. In particular, $\Delta(G) \geq \Delta(T_0) \geq (1-\varepsilon)n$. Hence we have $m > (1+\varepsilon)n$. This means that G has more than εn edges that do not belong to T_0. Let V_3 be the set of vertices spanned by $\lfloor \varepsilon/2 \cdot n \rfloor$ such edges, together with all vertices that do not belong to T_0. Finally, we set V_2 to be the one-element set containing the central vertex of T_0 and $V_1 = V(T_0) \setminus (V_2 \cup V_3)$. □

Let V_1, V_2, V_3 be the partition from Lemma 1.27. Let H be a bipartite subgraph of $G[V_3]$ with at least $\lfloor \varepsilon/4 \cdot n \rfloor$ edges. Split the set V_3 into two parts according to the bipartition of H and place all vertices from one part in a small disc with center $(0,0)$ and radius $r < 1/3$ and all vertices from the second part in a small disc with center $(1,0)$ and radius r, so that the vertices are in general position. Draw all edges of $G[V_3]$ as straight-line segments. See Figure 1.19. There are two lines t_1, t_2 parallel to the y-axis going through points $(x_1, 0)$ and $(x_2, 0)$, respectively, such that $r < x_1 < x_2 < 1-r$ and no two edges of $G[V_3]$ cross between t_1 and t_2. In particular, the edges of $G[V_3]$ split the vertical strip S between t_1 and t_2 into at least $\lfloor \varepsilon/4 \cdot n \rfloor + 1$ regions. Place all vertices of V_2 inside S above the horizontal line with y-coordinate r, and each of the vertices of V_1 in

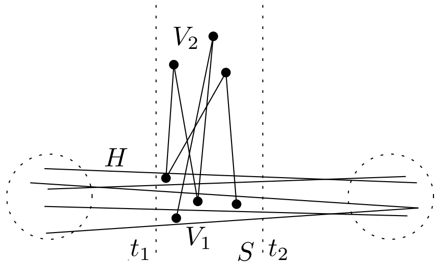

Figure 1.19: An illustration of the second construction for the lower bound in Theorem 1.2.

one of the $\lfloor \varepsilon/4 \cdot n \rfloor + 1$ regions of S, so that all vertices are in general position. Draw all the remaining edges as straight-line segments. The choice of the region for each vertex v of V_1 determines how many edges from H an edge connecting v with V_2 crosses. In total, this gives $(|E(H)| + 1)^{|V_1|} \geq (\varepsilon/4 \cdot n)^{n/4} \geq 2^{\Omega(n \log n)}$ pairwise weakly nonisomorphic geometric drawings of G.

1.7 Concluding remarks and open problems

The problem of counting the asymptotic number of "nonequivalent" simple drawings of a graph in the plane has been answered only partially. Many open questions remain.

The gap between the lower and upper bounds on $T_{\mathrm{w}}(G)$ proved in Theorem 1.2 is wide open, especially for graphs with low density. For graphs with cn^2 edges, the lower and upper bounds on $\log T_{\mathrm{w}}(G)$ differ by a logarithmic factor. We conjecture that the correct answer is closer to the lower bound.

We do not even know whether $T_{\mathrm{w}}(G)$ is a monotone function with respect to the subgraph relation, since there are simple topological graphs that cannot be extended to simple complete topological graphs. See Figure 1.9, left, for an example. Due to somewhat "rigid" properties of simple complete topological graphs, we have a much better upper bound for the complete graph than, say, for the complete bipartite graph on the same number of vertices.

Problem 4. *Does the complete graph K_n maximize the value $T_w(G)$ among the graphs G with n vertices? More generally, is it true that $T_w(H) \leq T_w(G)$ if $H \subseteq G$?*

Our methods for proving upper bounds on the number of weak isomorphism classes of simple topological graphs do not generalize to the case of topological graphs with two crossings per pair of edges allowed.

Problem 5. *What is the number of weak isomorphism classes of drawings of a graph G where every two independent edges are allowed to cross at most twice and every two adjacent edges at most once?*

For the complete graph with n vertices, Pach and Tóth [PT06] proved the lower bound $2^{\Omega(n^2 \log n)}$ and the upper bound $2^{o(n^4)}$.

A nontrivial lower bound can be proved also in the case when G is a matching. Ackerman et al. [APZ12] constructed a system of n x-monotone curves where every pair of curves intersect in at most one point where they either cross or touch, with $\Omega(n^{4/3})$ pairs of touching curves. Eyal Ackerman (personal communication) noted that this also follows from an earlier result by Pach and Sharir [PS91], who constructed an arrangement of n segments with $\Omega(n^{4/3})$ vertically visible pairs of disjoint segments. By changing the drawing in the neighborhood of every touching point, we obtain $2^{\Omega(n^{4/3})}$ different intersection graphs of 2-intersecting curves, also called *string graphs of rank* 2 [PT06]. This improves the trivial lower bound observed by Pach and Tóth [PT06].

In section 1.3, we proved that certain patterns are forbidden in the rotation systems of simple complete topological graphs, or more generally, in good abstract rotation systems. The problem of counting topological graphs was thus reduced to a combinatorial problem of counting permutations with forbidden patterns, by the recursion in Subsection 1.3.4. A general problem of this type can be formulated as follows. Given a constant N and a collection $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$ of sets of N-element permutation patterns, we say that a set $\mathcal{P}$ of permutations on n elements is $\mathcal{F}$*-restricted* if for each N-tuple $X = (x_1, x_2, \dots, x_N)$ of positions, there is an $i \in \{1, 2, \dots, m\}$ such that for every permutation $\pi \in \mathcal{P}$, all permutations from F_i are forbidden as restrictions of π at X. What is the maximum size of an $\mathcal{F}$-restricted set $\mathcal{P}$ of permutations on n elements?

For example, in the special case of the Stanley-Wilf conjecture, the collection $\mathcal{F}$ consists of a single one-element set. A set of permutations with VC-dimension at most k is an $\mathcal{F}$-restricted set where the collection $\mathcal{F}$ consists of $(k+1)!$ one-element sets, each containing a different permutation of $\{1, 2, \dots, k+1\}$.

In Subsection 1.3.4, we reduced the upper bound in Theorem 1.3 to the upper bound on the size of an $\mathcal{F}$-restricted set where $\mathcal{F}$ consists of the following $2\binom{N}{5} + 2\binom{N}{6}$ sets. For every set $A \subset \{1, 2, \dots, N\}$ of five positions, the collection $\mathcal{F}$ contains a set F_A of all permutations of N elements whose restriction to A is $(1, 4, 2, 5, 3)$ or some of its four cyclic shifts, and a set F'_A of all permutations of N elements whose restriction to A is $(1, 3, 5, 2, 4)$ or some of its four cyclic shifts. Similarly, for every set $B \subset \{1, 2, \dots, N\}$ of six positions, the collection $\mathcal{F}$ contains a set F_B of all permutations of N elements whose restriction to B is $(1, 2, 3, 6, 5, 4)$ or some of its five cyclic shifts, and a set F'_B of all permutations of N elements whose restriction to B is $(1, 4, 5, 6, 3, 2)$ or some of its five cyclic shifts. This follows from Lemma 1.11, 1.13 and from the proof of Theorem 1.8, where the canonical linear ordering of the vertices of the unavoidable convex or twisted graphs is consistent with the linear ordering of the vertices of the given simple complete topological graph. Such $\mathcal{F}$-restricted sets of permutations are a special case of sets with VC-dimension smaller than N, which can have superexponential size [CK12], and generalize the sets with a single forbidden permutation pattern, for which a single exponential upper bound on their size is known [Kl00, MT04]. Therefore one might ask for which collections $\mathcal{F}$ it is true that $\mathcal{F}$-restricted sets of permutations have only exponential size.

A positive answer to the following problem would improve the upper bound in Theorem 1.3 to $T_{\mathrm{w}}(K_n) \le 2^{O(n^2)}$, which would be asymptotically optimal.

Problem 6. *Let $N > 6$ be a constant positive integer. Let $\mathcal{P}$ be a set of permutations of n elements such that for every N-tuple X of positions, there is either a 5-tuple $A \subset X$ such that the pattern $(1, 3, 5, 2, 4)$ and all its cyclic shifts are forbidden as restrictions at A, or a 5-tuple $A' \subset X$ such that the pattern $(1, 4, 2, 5, 3)$*

and all its cyclic shifts are forbidden as restrictions at A', or a 6*-tuple* $B \subset X$ *such that* $(1, 2, 3, 6, 5, 4)$ *and all its cyclic shifts are forbidden as restrictions at* B*, or a* 6*-tuple* $B' \subset X$ *such that* $(1, 4, 5, 6, 3, 2)$ *and all its cyclic shifts are forbidden as restrictions at* B'*. Is it true that* $|\mathcal{P}| \leq 2^{O(n)}$*?*

For $N = 4$ and $\mathcal{F} = \{\{(1, 2, 3, 4)\}, \{(3, 4, 1, 2)\}\}$, a construction in [CK12] shows an $\mathcal{F}$-restricted set of permutations of superexponential size. Such a construction does not necessarily satisfy the conditions in Problem 6 since, for example, the pattern $(3, 4, 1, 2)$ is a restriction of just one cyclic shift of $(1, 2, 3, 6, 5, 4)$, one cyclic shift of $(1, 4, 5, 6, 3, 2)$ and of no cyclic shift of either $(1, 3, 5, 2, 4)$ or $(1, 4, 2, 5, 3)$. On the other hand, this construction does give a superexponential lower bound on the size of sets of permutations satisfying the restrictions implied by Lemma 1.17 and Lemma 1.18, which appear in the proof of the combinatorial Theorem 1.15. This follows from the fact that every cyclic shift of the inverse of $(1, 3, \ldots, 815, 2, 4, \ldots, 814)$ contains both patterns $(1, 2, 3, 4)$ and $(3, 4, 1, 2)$. Therefore, a positive solution to Problem 1 will require a different approach.

2. Ramsey properties of intersection graphs of segments

2.1 Introduction

An *arrangement of segments* is a finite set of compact straight-line segments in the plane in general position (that is, no three endpoints are collinear). We study the following Ramsey-type problem [LMPT94]: what is the largest number $r(k)$ such that there exists an arrangement of $r(k)$ segments with at most k pairwise crossing and at most k pairwise disjoint segments?

Larman et al. [LMPT94] proved that $k^5 \geq r(k) \geq k^{\log 5/\log 2} > k^{2.3219}$. The upper bound has remained unchanged since then. Károlyi et al. [KPT97] improved the lower bound to $r(k) \geq k^{\log 27/\log 4} > k^{2.3774}$.

We improve the construction for the lower bound even further and prove the following theorem.

Theorem 2.1. *For infinitely many positive integers k there exists an arrangement of $k^{\log 169/\log 8} > k^{2.4669}$ segments with at most k pairwise crossing and at most k pairwise disjoint segments.*

Similar questions were studied by Fox, Pach and Cs. Tóth [FPT11] for string graphs, a class of graphs generalizing intersection graphs of segments. They proved, as a consequence of a stronger result, that for each positive integer k there is a constant $c(k) > 0$ such that in any system of n curves in the plane where every two curves intersect in at most k points, there is a subset of $n^{c(k)}$ curves that are pairwise disjoint or pairwise crossing.

2.2 Proof of Theorem 1

Both previous constructions for the lower bound [KPT97, LMPT94] use the same approach. The starting configuration is an arrangement M_0 of n_0 segments with at most k_0 pairwise crossing or pairwise disjoint segments. In the i-th step, an arrangement M_i of n_0^{i+1} segments is constructed from the arrangement M_{i-1} by replacing each of its segments by a flattened copy (a precise definition will follow) of M_0, which acts as a "thick segment". Then two segments from different copies of M_0 cross if and only if the two corresponding segments in M_{i-1} cross. Our new arrangement M_i has then at most k_0^{i+1} pairwise crossing or pairwise disjoint segments. This gives a lower bound $r(k) \geq k^{\log n_0/\log k_0}$ for infinitely many values of k.

We improve the construction by making a better starting arrangement. Unlike the previous constructions, our basic pieces will be arrangements with different maximal numbers of pairwise crossing and pairwise disjoint segments. By putting them together, we obtain our starting arrangement M_0.

Let $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$ denote the Cayley graph of the cyclic group $\mathbb{Z}_{13}$ corresponding to the generators 1 and 5. That is, $V(\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)) = \{1, 2, \ldots, 13\}$ and $E(\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)) = \{\{i, j\}; 1 \leq i < j \leq 13, (j - i) \in \{1, 5, 8, 12\}\}$. See Figure 2.1.

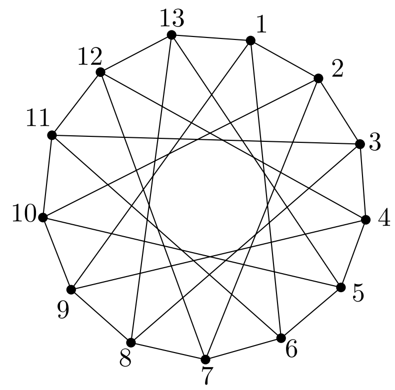

Figure 2.1: A Cayley graph $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$.

Lemma 2.2. *The graph* $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$ *contains no clique of size* 3 *and no independent set of size* 5.

Proof. Suppose that $a < b < c$ are three vertices of $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$ inducing a clique. Then the numbers $k = c - a, l = c - b$ and $m = b - a$ belong to the set $\{1, 5, 8, 12\}$, but this set contains no triple k, l, m satisfying the equation $k = l + m$; a contradiction.

Now suppose that $A = \{a < b < c < d < e\}$ is an independent set of $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$. By the pigeon-hole principle, A contains two vertices with difference 2 (modulo 13). Thus, we can, without loss of generality, assume that $a = 1$ and $b = 3$. It follows that $\{c, d, e\} \subseteq \{5, 7, 10, 12\}$. But A cannot contain both 5 and 10, neither both 7 and 12. Hence $|A \cap \{5, 7, 10, 12\}| \leq 2$; a contradiction. □

A (k, l)-*arrangement* is an arrangement of segments with at most k pairwise crossing and at most l pairwise disjoint segments.

The *intersection graph* $G(M)$ of an arrangement M is the graph whose vertices are the segments of M and two vertices are joined by an edge if and only if the corresponding segments intersect.

An arrangement M of segments is *flattenable* if for every $\varepsilon > 0$ there is an arrangement M_ε with $G(M_\varepsilon) = G(M)$ and two discs D_1, D_2 of radius ε whose centers are at unit distance, such that each segment from M_ε has one endpoint in D_1 and the second endpoint in D_2. A *flattened copy* of M is the arrangement M_ε with sufficiently small ε.

The key result is the following lemma.

Lemma 2.3. 1. *There exists a flattenable* $(2, 4)$-*arrangement of* 13 *segments.*

2. *There exists a flattenable* $(4, 2)$-*arrangement of* 13 *segments.*

Note that 13 is the largest possible number of segments for these two types of arrangements since every graph with more than 13 vertices contains either a clique of size 5 or an independent set of size 3 [Ra94].

Both previous constructions [KPT97, LMPT94] used *convex* starting arrangement, that is, an arrangement of segments with endpoints in convex position. Convex arrangements are flattenable by a relatively simple argument [KPT97]. However, Kostochka [Ko88] proved that any convex (k, k)-arrangement has at

	left x	left y	right x	right y
1	$-\varepsilon$	0	$1-2\varepsilon$	$2\varepsilon^2+2\varepsilon^6$
2	ε^2	$\varepsilon-\varepsilon^3$	$1-\varepsilon^2$	ε^3
3	0	$\varepsilon^4+\varepsilon^6$	1	$\varepsilon^3+3\varepsilon^4$
4	0	$\varepsilon^4-\varepsilon^6$	$1-2\varepsilon$	$2\varepsilon^2-\varepsilon^6$
5	$-\varepsilon+\varepsilon^2$	0	$1-2\varepsilon^2$	$2\varepsilon^3-2\varepsilon^4$
6	$-\varepsilon$	$2\varepsilon^6$	$1-\varepsilon$	$2\varepsilon^6$
7	0	ε^6	1	$\varepsilon^3+2\varepsilon^4$
8	0	ε	$1+\varepsilon^3$	0
9	0	ε	$1-2\varepsilon^2$	$2\varepsilon^3-\varepsilon^4$
10	$-\varepsilon^2+3\varepsilon^3$	$3\varepsilon^6$	$1-2\varepsilon$	$2\varepsilon^2+\varepsilon^6$
11	$-\varepsilon^2$	ε^6	$1-2\varepsilon^2$	$2\varepsilon^3-3\varepsilon^4$
12	0	ε^4	1	0
13	$-\varepsilon$	0	$1+\varepsilon$	0

Table 2.1: Arrangement $M_a(\varepsilon)$.

most $(1+o(1))\cdot k^2\log k$ segments. He also gave a construction of a convex (k,k)-arrangement with $\Omega(k^2\log k)$ segments (see also [Ce08]). Černý [Ce08] investigated convex (k,l)-arrangements for small values of k. He showed, in particular, that any convex $(2,4)$-arrangement has at most 12 segments, and any convex $(4,2)$-arrangement has at most 11 segments.

Our starting arrangements thus cannot be convex. Hence their flattening will require a special approach.

Proof. For each sufficiently small $\varepsilon>0$, we construct an arrangement $M_a(\varepsilon)$ with intersection graph $\mathrm{Cay}(\mathbb{Z}_{13};1,5)$ and an arrangement $M_b(\varepsilon)$ whose intersection graph is the complement of $\mathrm{Cay}(\mathbb{Z}_{13};1,5)$. See Figure 2.2 for an illustration.

In Tables 2.1 and 2.2, we provide precise coordinates of the endpoints of all the 13 segments, as functions of ε. To achieve general position of the segments, which is required by our definition, we can slightly perturb the endpoints while preserving the intersection graph of the arrangement.

Since the coordinates of all the left endpoints converge to $(0,0)$ and the coordinates of all the right endpoints converge to $(1,0)$, it remains to verify that for sufficiently small $\varepsilon>0$, each of these two described arrangements has the desired intersection graph. This is a straightforward calculation, which can be done by the following simple algorithm.

We use the fact that the functions describing the coordinates are polynomials in ε. For $i\in 1,2,\dots,13$, let s_i be the i-th segment of the arrangement and let $l_x(i),l_y(i),r_x(i),r_y(i)$ be the polynomials representing the coordinates of the left and the right endpoint of s_i. For each pair $i<j$, we need to determine whether s_i and s_j cross if ε is small enough.

Let s be a segment with endpoints (l_x,l_y) and (r_x,r_y) and let s' be a segment with endpoints (l'_x,l'_y) and (r'_x,r'_y). Let p be the line containing s, and let p' be the line containing s'. The segments s and s' intersect if and only if $s'\cap p\neq\emptyset$ and $s\cap p'\neq\emptyset$. We have $p=\{(x,y);ax+by+c=0\}$, where $a=r_y-l_y$, $b=r_x-l_x$ and $c=r_xl_y-l_xr_y$. Thus, $s'\cap p\neq\emptyset$ if and only if $(al'_x+bl'_y+c)(ar'_x+br'_y+c)\leq 0$. The relation $s\cap p'\neq\emptyset$ can be expressed similarly.

	left x	left y	right x	right y
1	ε	$\varepsilon^2 - \varepsilon^3 + \varepsilon^4 - 2\varepsilon^5$	$1 + \varepsilon^2$	$-\varepsilon^4 + \varepsilon^6$
2	0	$\varepsilon^2 + 3\varepsilon^5$	$1 - \varepsilon^3$	ε^7
3	0	$\varepsilon^2 + 4\varepsilon^5$	$1 + \varepsilon$	$-\varepsilon^3$
4	0	$2\varepsilon^3$	$1 + 3\varepsilon^4$	$-\varepsilon^8$
5	$\varepsilon - \varepsilon^2 + \varepsilon^3$	$\varepsilon^2 - \varepsilon^3 + \varepsilon^4 - \varepsilon^8$	$1 + \varepsilon$	$-\varepsilon^4$
6	0	$\varepsilon^2 + \varepsilon^5$	$1 + \varepsilon$	$-\varepsilon^3$
7	0	$\varepsilon^2 + 5\varepsilon^5$	$1 + 3\varepsilon^4$	$-3\varepsilon^7$
8	$\varepsilon - \varepsilon^2 + \varepsilon^3 + \varepsilon^4 + 2\varepsilon^5$	$\varepsilon^2 - \varepsilon^3 + \varepsilon^4 + \varepsilon^5 + \varepsilon^6$	$1 + \varepsilon - \varepsilon^4$	$-\varepsilon^3$
9	0	ε^2	$1 + \varepsilon$	$-\varepsilon^4$
10	0	0	$1 + 5\varepsilon^3$	0
11	0	$\varepsilon^2 + 2\varepsilon^5$	$1 + 3\varepsilon^4 - 2\varepsilon^5$	ε^8
12	$\varepsilon - \varepsilon^3$	$\varepsilon^3 - \varepsilon^4$	$1 + \varepsilon$	$-\varepsilon^4$
13	0	0	1	ε

Table 2.2: Arrangement $M_b(\varepsilon)$.

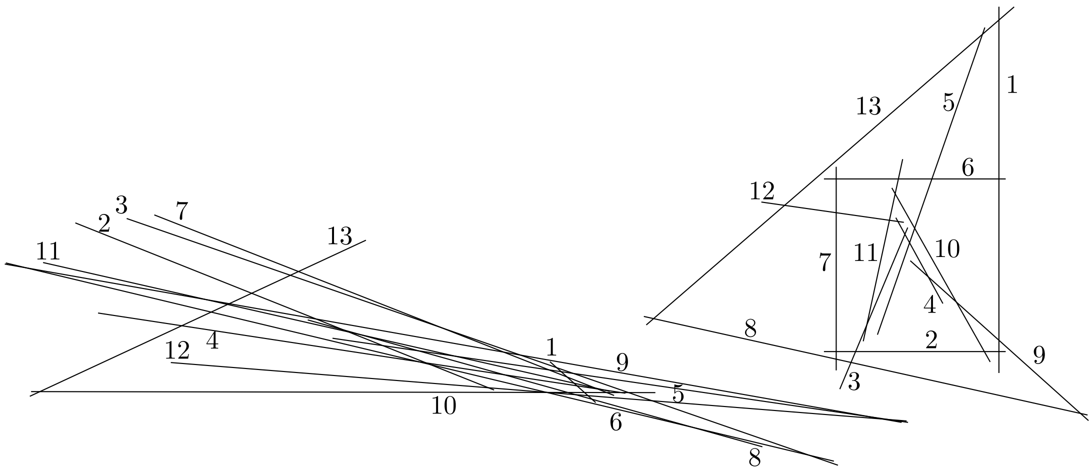

Figure 2.2: A partially flattened $(4,2)$-arrangement of 13 segments (left) and a $(2,4)$-arrangement of 13 segments (right).

The algorithm now follows. For each i, compute the polynomials $a_i = r_y(i) - l_y(i)$, $b_i = r_x(i) - l_x(i)$ and $c_i = r_x(i)l_y(i) - l_x(i)r_y(i)$. Then for each pair $i \neq j$, compute the polynomial $d_{i,j} = (a_i l_x(j) + b_i l_y(j) + c_i)(a_i r_x(j) + b_i r_y(j) + c_i)$. Now s_i and s_j intersect if and only if each $d_{i,j}$ and $d_{j,i}$ is nonpositive in some positive neighborhood of 0. That is, the polynomial is either zero or the coefficient by the non-zero term of the smallest order is negative.

A program verifying both constructions can be downloaded from the following webpage: `http://kam.mff.cuni.cz/~kyncl/programs/segments`. □

Now we are ready to finish the proof of Theorem 2.1. Take a sufficiently flattened arrangement $M_a(\varepsilon)$ and replace each of its segments by a copy of a sufficiently flattened arrangement $M_b(\delta)$. In this way we obtain our starting flattenable $(8,8)$-arrangement M_0 of 169 segments. Then we proceed by the method described at the beginning of this section.

2.3 Non-flattenable arrangements

Since the flattenable arrangements are the main tool in the construction in the previous section, it is natural to ask whether every arrangement of segments can be flattened. A necessary condition for an arrangement to be flattenable is the existence of a line crossing all the segments, in a sufficiently flattened realization. We show the following.

Theorem 2.4. *There exists an intersection graph of segments that cannot be realized by an arrangement of segments crossing a common line.*

Theorem 2.5. *There exists an arrangement of segments crossing a common line that is not flattenable.*

2.3.1 Proof of Theorem 2.4

Let G be an intersection graph of the arrangement in Figure 2.3. The arrangement consists of 7 *horizontal* and 7 *vertical* segments forming a grid, the 56 *frame* segments forming a cycle, 28 *joining* segments connecting a grid segment with a segment of the frame (each grid segment is joined to the frame by two joining segments and every other segment from the frame is used), and finally 8 *short* segments, each crossing one vertical and one horizontal segment from the grid.

We prove Theorem 2.4 in a slightly stronger form.

An *arrangement of pseudosegments* is a set of simple curves in the plane such that every two of the curves have at most one common point and any such point is a proper crossing. If M is an arrangement of pseudosegments, then each curve from M, and also any curve c such that $M \cup \{c\}$ is an arrangement of pseudosegments, is called a *pseudosegment*.

Proposition 2.6. *For any arrangement M of pseudosegments whose intersection graph is G, no pseudosegment can cross all the curves from M.*

Proof. Let M be an arrangement of pseudosegments whose intersection graph is G. We use the terms *frame/grid/horizontal/vertical/joining/short* pseudosegment in a similar meaning as above. The union Γ of the frame pseudosegments

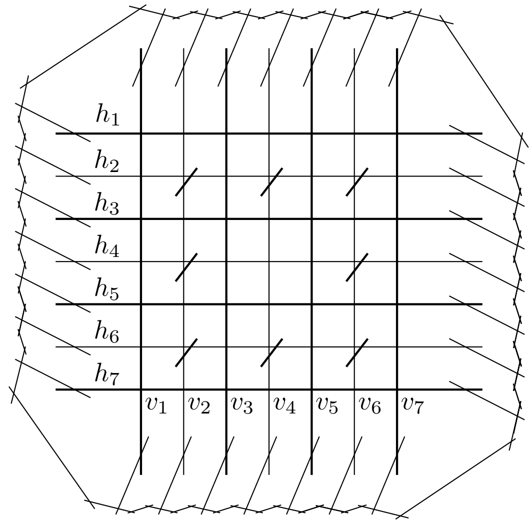

Figure 2.3: A construction for Theorem 2.4.

contains a unique closed curve γ. Each frame pseudosegment intersects γ in a connected arc and the cyclic order of these arcs along γ is uniquely determined (up to inversion). Both Γ and γ cut the plane into two connected regions. Since the subgraph of G induced by the grid and short vertices is connected and separated from the frame cycle by the joining vertices, the union of the grid and short pseudosegments is connected and disjoint from Γ. Thus, we can, without loss of generality, assume that all the grid and short pseudosegments lie in the region Ω bounded by Γ.

The order of the intersections of the joining pseudosegments with Γ along the boundary of Ω is uniquely determined. Each grid pseudosegment together with its two joining pseudosegments divides Ω into two connected components. All the 7 vertical pseudosegments with their joining pseudosegments divide Ω into 8 connected components and the "horizontal" order of the vertical pseudosegments is uniquely determined. Each horizontal pseudosegment has to start in the leftmost region and end in the rightmost region and is forced to cross the vertical pseudosegments in the same order and orientation. Similarly each vertical pseudosegments has to cross all the horizontal pseudosegments in the same order and orientation. It follows that the grid pseudosegments form a "pseudogrid" homeomorphic to the grid in Figure 2.3. Therefore, we can further assume that the grid pseudosegments are straight-line segments forming a regular square grid.

Label the vertical and the horizontal segments of the grid consecutively by $v_1, v_2, \dots v_7$ and $h_1, h_2, \dots h_7$. The odd-numbered segments form a *coarse* grid of 3×3 *big* squares. Each of the eight short pseudosegments is contained in one of the big squares, since for each orthogonal pair v_{2i}, h_{2j} of the even-labeled grid segments, the big square determined by the segments $v_{2i-1}, v_{2i+1}, h_{2j-1}$ and h_{2j+1} is the only face in the arrangement $M \setminus (\{v_{2i}, h_{2j}\} \cup M_s)$ intersected by both v_{2i} and h_{2j} (here M_s denotes the set of short pseudosegments in M). Therefore, each pseudosegment that crosses all pseudosegments in M must intersect at least 8 big squares in the coarse grid. We show that such pseudosegment does not exist.

Let p be a pseudosegment. Suppose that p has both its endpoints outside the grid. Then p can enter and leave the grid at most twice, since in each traversal of

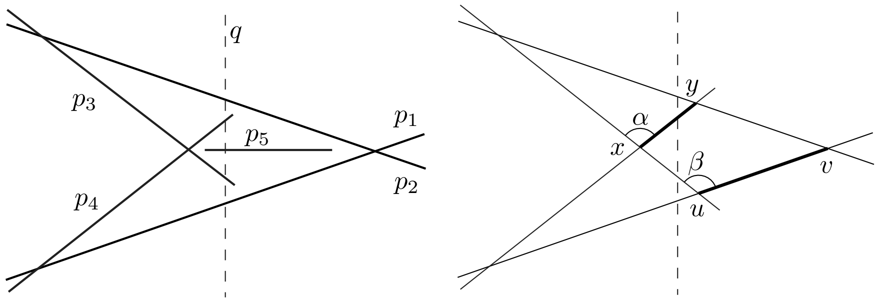

Figure 2.4: Arrangement M_0, a core of the construction for Theorem 2.5.

the grid p crosses two of the four boundary segments v_1, v_7, h_1, h_7. If p intersects k big squares in a traversal, it has to cross at least $k+1$ segments of the coarse grid (including two of the boundary segments). It follows that p can intersect at most 5 big squares in one traversal, and at most 6 big squares in two traversals.

Now suppose that p starts outside and ends inside the grid. Suppose further that p intersects k big squares during the first traversal and then l big squares after entering the grid for the second time. Then p has to cross at least $k+1+l$ coarse grid segments. Since p avoids one of the boundary segments, we have $k + l \leq 6$.

If p starts and ends inside the grid and intersects k big squares before it reaches the boundary of the grid for the first time, l during the following traversal, and m after it enters the grid for the second time, it has to cross at least $k+l+1+m$ coarse grid segments. That gives us $k + l + m \leq 7$.

It follows that any pseudosegment can intersect at most 7 big squares, thus at most 7 short pseudosegments. □

2.3.2 Proof of Theorem 2.5

The core of the construction is the arrangement of five segments in Figure 2.4.

Lemma 2.7. *The arrangement M_0 of segments $p_1, \dots, p_5$ crossing a common vertical line q in Figure 2.4, left, cannot be homeomorphically flattened. More precisely, for a sufficiently small ε, there is no homeomorphism of the plane mapping each segment p_i onto a segment, the line q onto a line, the left endpoint of each segment to an ε-neighborhood of the point $(0, 0)$, and the right endpoint of each segment to an ε-neighborhood of the point $(1, 0)$.*

Proof. Suppose for contradiction that M_0 is already flattened by such a homeomorphism (for sufficiently small ε). Let $x \in p_3 \cap p_4$ and $v \in p_1 \cap p_2$. Let y be an intersection of p_2 with the line extending the segment p_4. Similarly, let u be an intersection of p_1 with the line extending the segment p_3. See Figure 2.4, right. As all the right endpoints are close to $(1, 0)$, the points y, u and v are also close to $(1, 0)$ since they are to the right from the right endpoint of p_3 or p_4, and to the left from the right endpoint of p_1 or p_2. The slopes of all the segments are close to 0, thus $\beta > \alpha > \pi/2$. It follows that $\|x - y\| < \|u - v\|$, hence x is close to $(1, 0)$ as well.

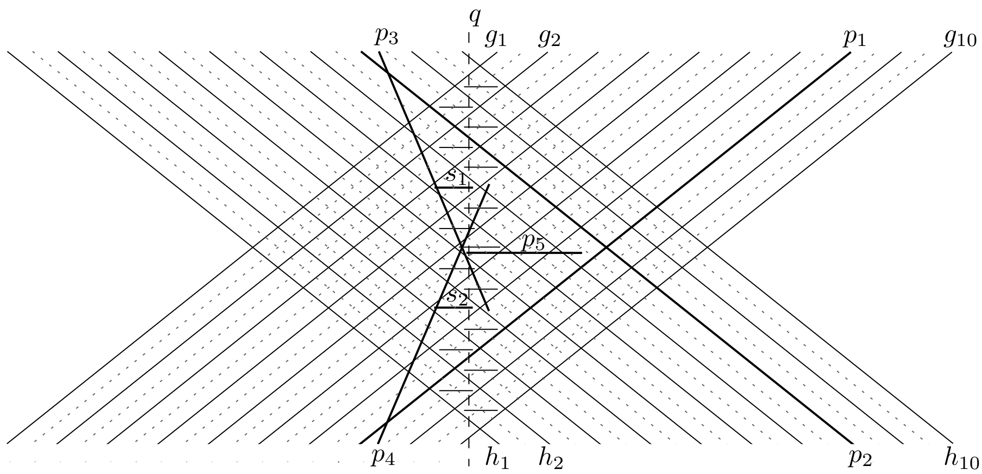

Figure 2.5: Arrangement M_1 consisting of core, grid and auxiliary segments.

The segments p_3 and p_4 and the line q form a triangle T, which contains the left endpoint of p_5. Since all the vertices of T are close to $(1, 0)$, the left endpoint of p_5 is close to $(1, 0)$ as well, a contradiction. □

By Lemma 2.7, we only have to add some other segments to M_0 so that in any realization of the resulting arrangement M in the plane such that all segments cross a common line q, the subarrangement M_0 (together with the line q) is homeomorphic to the arrangement in Figure 2.4, left.

We add 18 segments parallel to p_2 and 18 segments parallel to p_1, so that they form an 18×18 grid as in Figure 2.5. All these 38 segments are called *grid* segments. As in the construction in the previous section, by taking every odd grid segment we get a coarse 9×9 grid. These segments are denoted by $g_1, \ldots, g_{10}$ and $h_1, \ldots, h_{10}$ and drawn by full lines in Figure 2.5. We add 17 *short* segments to 17 cells of the coarse grid along the diagonal, each short segment crossing two (even) grid segments. We obtain an arrangement M_1 where the intersections between segments are defined by the drawing in Figure 2.5.

To get the final arrangement M, we add a frame and some joining segments, as in the construction in the previous section. We add one joining segment for each p_3 and p_4, and two joining segments for each grid segment. In total, we add 78 joining segments connected to every other segment of a cycle of length 156. It is easy to ensure that all the added segments still cross the line q; see Figure 2.6 for an example with smaller grid.

Now we fix an arbitrary (sufficiently flattened) realization M' of M such that there is a line q crossing all segments from M'.

By the same argument as in the previous section, the grid segments form a grid homeomorphic to the grid in Figure 2.5. The line q can pass through at most 17 cells of the coarse grid, since it crosses two of the segments g_1, g_{10}, h_1, h_{10} when entering and leaving the grid, and one other segment g_i or h_i between every two cells in the coarse grid. Each of the 17 short segments has to lie in the same cell as in Figure 2.5. It follows that q passes exactly through these 17 cells and also in the same order as in Figure 2.5. As a consequence we get that the orientation of the segments $g_1, \ldots, g_{10}$ and $h_1, \ldots, h_{10}$ induced by the grid is consistent with

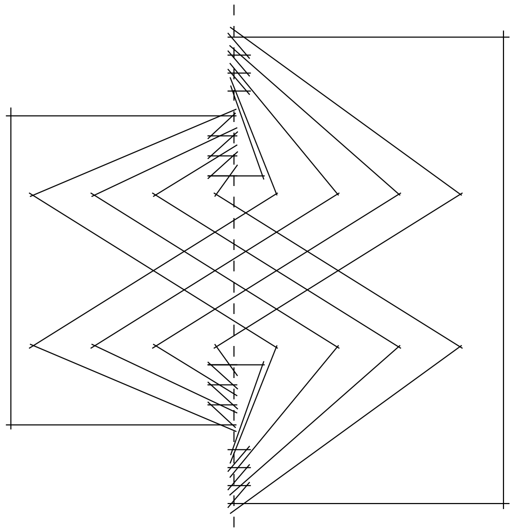

Figure 2.6: An example of the frame and the joining segments added to a small grid arrangement in such a way that all segments cross a common line.

the left-right orientation induced by the line q, as in Figure 2.5.

The segment p_5 has to lie inside the same 3×3 subgrid of the coarse grid as in Figure 2.5. Moreover, since it crosses q, it also has to start and end in the same two cells (but the cells it passes through are not uniquely determined). Since both p_3 and p_4 are connected to the frame between specific pairs of grid segments, one of their endpoints lies outside the grid and the other endpoint lies in the same cell as in Figure 2.5.

We can restrict the position of p_3 and p_4 even further. Since p_3 crosses the short segment s_1, it has to pass through the corresponding cell. As a consequence we get that the intersection of p_3 with q lies "below" the intersection of p_5 with q, otherwise p_3 would cross p_5 or cross h_7 twice. Similarly, as p_4 crosses s_2, it has to cross q "above" the intersection of p_5 with q. Also, by the same reason, starting from the endpoint inside of the grid, both p_3 and p_4 cross q before they cross p_1 or p_2. Therefore, the sub-arrangement of $p_1, \ldots, p_5$ and q in M' is homeomorphic to the arrangement in Figure 2.4 and the proof of Theorem 2.5 is finished.

3. Reachability in graphs on surfaces

3.1 Introduction

Directed reachability (or briefly reachability) is a standard complete problem for the complexity class NL (nondeterministic logspace). The problem is defined as follows: given a directed graph G and two of its vertices s, t, determine whether there is a directed path from s to t in G. Such a path is briefly called a *directed s-t-path*. In our definition of a directed graph we allow two edges in opposite directions between the same pair of vertices, but we do not allow loops. In the literature the reachability problem is also referred to as (directed) s-t-connectivity. Planar reachability is an important restricted version of the reachability problem, where the input graph is planar. It can be assumed that the graph is given with its planar embedding since a planar embedding of a planar graph can be constructed in logspace [AM04]. Planar reachability is hard for L and is contained in NL but is not known to be NL-complete or contained in L. Recently Bourke, Tewari and Vinodchandran [BTV09] improved the complexity upper bound by showing that planar reachability is in UL (unambiguous logspace), which is the class of decision problems that can be solved by a nondeterministic logspace Turing machine that has at most one accepting computation on any input. Allender et al. [ABC+09] showed that reachability for graphs embedded on the torus is logspace-reducible to planar reachability, and hence belongs to UL as well. We generalize this result to graphs embedded on arbitrary fixed surface, orientable or nonorientable.

Theorem 3.1. *For each fixed connected compact surface S, the reachability problem for graphs embedded in S is logspace-reducible to planar reachability.*

Note that we have to assume that the graph is given together with its embedding in S, since it is not known whether such an embedding can be found in logspace if S has positive genus.

Thierauf and Wagner [TW09] generalized the result of Bourke, Tewari and Vinodchandran [BTV09] in another direction by showing that reachability in $K_{3,3}$-free graphs and K_5-free graphs is logspace-reducible to planar reachability.

The reachability problem is known to be in L for undirected graphs [Re08], for directed series parallel graphs [JLR06], and planar directed acyclic graphs (DAGs) with a single source [ABC+09]. The last result has been recently extended to planar DAGs with $O(\log n)$ sources [SBV09] and subsequently to DAGs with $2^{O(\sqrt{\log n})}$ sources embedded on a surface of genus $2^{O(\sqrt{\log n})}$ [SV10].

Recently, Datta et al. [DKTV09] gave an alternative proof of the fact that directed reachability in bounded genus graphs is in UL, using a method of deterministic isolation.

3.2 Preliminaries

A *surface* is, roughly speaking, a topological space locally homeomorphic to the plane. We are interested in surfaces that are connected and compact (that is, without boundary). Every such surface is either *orientable* (has two sides) or *nonorientable* (has only one side). Each orientable (nonorientable) connected compact surface S is obtained from the sphere by attaching $g \geq 0$ handles (crosscaps, respectively). The number g is called the *genus* of S. The simplest orientable surfaces are the sphere (with genus 0) and the torus (with genus 1). The simplest nonorientable surfaces are the projective plane (with genus 1) and the Klein bottle (with genus 2). For more detailed introduction into surfaces and graph embeddings refer to [MT01].

Let S be a connected compact surface of genus $g > 0$, orientable or nonorientable. Let Π be an embedding of a given directed graph G with n vertices in S. If S is orientable, the embedding is given by the rotation system of G. The *rotation system* is a set of rotations of all vertices, where the *rotation* of a vertex v is the (clockwise) cyclic permutation of edges incident with v. If S is nonorientable, the embedding is given by the rotation system of G and an orientation function $\lambda : E(G) \to \{1, -1\}$ which assigns -1 to an edge e if and only if e changes orientation. We may assume that G is connected since the reachability problem for undirected graphs is in L [Re08]. We may also assume that the surface S is minimal for Π, that is, each face of the embedding is homeomorphic to a disc. Such an embedding is called a 2-*cell embedding.*

The *Euler characteristic* of S, denoted by $\chi(S)$, is defined as $\chi(S) = 2 - 2g$ if S is orientable, and $\chi(S) = 2 - g$ if S is nonorientable. Equivalently, if v, e and f denote the number of vertices, edges and faces of a 2-cell embedding of G in S, then $\chi(S) = v - e + f$. In this way we can also define the Euler characteristic $\chi(\Pi)$ of the embedding Π. The Euler characteristic of Π can be computed in logspace in the following way. We enumerate all the faces of Π by traversing along facial walks (in clockwise direction), starting from every ordered pair of adjacent vertices, which we call a *vector*. We label all the vectors as $u_1, u_2, \ldots, u_{2m}$. In the beginning we set $f = 0$. In step i ($i = 1, 2, \ldots, 2m - 1$), we list all facial walks starting with vectors $u_1, u_2, \ldots, u_i$ and check whether the vector u_{i+1} appeared in the list, as a part of some walk. If not, the facial walk starting with u_{i+1} determines a new face and we increase f by one.

3.3 Proof of the main theorem

3.3.1 Main idea

Let s and t be two given vertices of G. We describe a logspace construction of a planar graph G'' containing vertices s'' and t'' such that G'' contains a directed s''-t''-path if and only if G contains a directed s-t-path.

We follow the approach of Allender et al. [ABC+09]. The main idea is to find cycles of G that do not separate S, and cut the surface S along them. This operation reduces the genus, so after finitely many steps we get a planar embedding of some resulting graph G'. The cutting operation, however, can destroy the connectivity properties of the graph. This is fixed by gluing several copies of G' together. If the original surface S has negative Euler characteristic, a naive gluing can produce a graph of exponential size, which is not constructible in logspace. We fix this problem by showing that a graph of polynomial size is sufficient to restore the connectivity properties; this is essentially the main new ingredient in this proof. The core idea is that there are only polynomially many distinct "topological" types of (directed) paths in G, as each path in G is a curve in S that does not cross itself.

3.3.2 Finding a nonseparating cycle

The construction starts with finding a spanning tree T of G [AM04, NT95]. The construction of the spanning tree reduces to undirected reachability which is in L due to Reingold [Re08]. The spanning tree is not stored in the memory, however. Instead we get a function that takes an edge e as an input and answers TRUE if and only if $e \in T$, using only a logarithmic amount of memory. This is, in fact, a common interpretation of any "logspace construction" (of objects of polynomial size).

A cycle in G is called *nonseparating* if the corresponding closed curve in the embedding does not separate S. The following lemma shows that a nonseparating cycle can be found efficiently. It is stated in [ABC+09] for orientable surfaces only, but the proof works for nonorientable surfaces as well.

Lemma 3.2. [ABC+09] *Let G be a connected graph 2-cell embedded in a surface S of positive genus. Let T be a spanning tree of G. Then there exists an edge $e \in E(G) \setminus T$ such that the (fundamental) cycle contained in $T \cup \{e\}$ is nonseparating.*

A cycle C in G is *one-sided* if C has an odd number of orientation-changing edges (those with $\lambda(e) = -1$), and *two-sided* otherwise. Note that on an orientable surface every cycle is two-sided, and any one-sided cycle (on a nonorientable surface) is nonseparating. Given a cycle C in G, we can test in logspace whether C is one-sided, by traversing along the cycle and multiplying the orientations of its edges. In case C is two-sided, the test whether C is nonseparating can be also performed in logspace [ABC+09], by checking for an existence of a vertex v that is connected by a path (internally disjoint with C) to each side of C. It follows by Lemma 3.2 that a nonseparating cycle can be found in logspace.

3.3.3 Cutting operation

Now we describe the cutting operation. When cutting G along a two-sided cycle $C = v_1v_2 \dots v_k$, we replace the cycle C with two new cycles $C' = v'_1v'_2 \dots v'_k$ and $C'' = v''_1v''_2 \dots v''_k$; see Figure 3.1. For each edge v_iw ($w \in G - C$) on the left side of C, we create an edge v'_iw and for each edge v_iw on the right side of C we create an edge v''_iw. The new edges are directed in the same way as the corresponding edges in G. The rotation system of the new graph is determined by the rotation system of G in the obvious way; the main difference is that the copies of adjacent edges of C become adjacent in the rotations of the vertices v'_i and v''_i. The orientation of each new edge is the same as the orientation of the corresponding original edge. Alternatively, we can perform several switching operations to make the orientations of all the cycle edges positive. A *switching operation* at a vertex v reverses the rotation of v and changes the orientations of all its incident edges. The switching operation does not change the embedding as it preserves all facial cycles.

The result of the cutting operation is an embedding into a surface of Euler characteristic $\chi(S)+2$. This surface is obtained by patching the two holes created

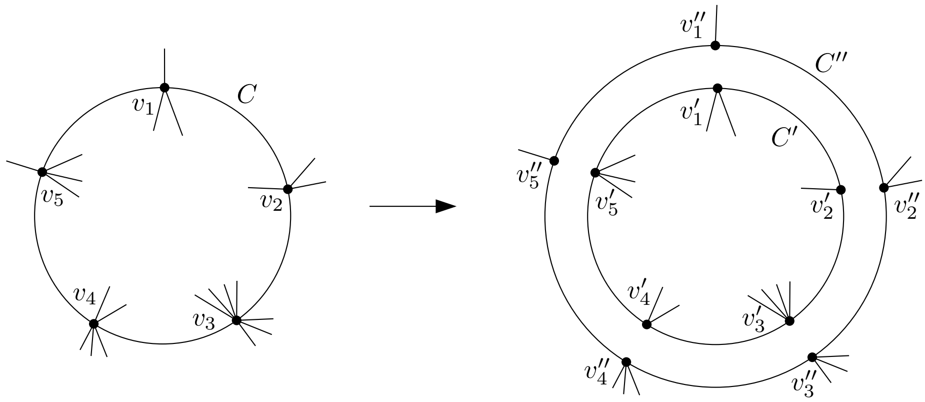

Figure 3.1: Cutting along a two-sided cycle C. For simplicity, the orientations of the edges of C are chosen to be positive and the directions of the edges are not drawn.

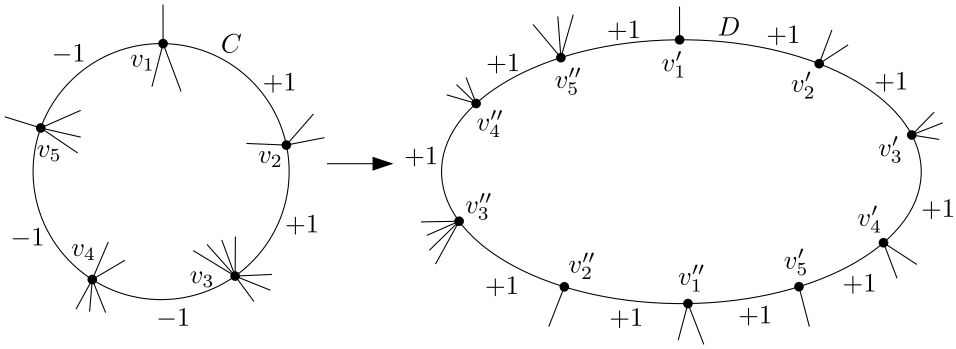

Figure 3.2: Cutting along a one-sided cycle C on a nonorientable surface. The labels $+1$ and -1 denote the orientations of the cycle edges. For simplicity, the directions of the edges are not drawn.

by the cutting with two discs, so the cycles C' and C'' become facial cycles of the new embedding.

When cutting of G along a one-sided cycle $C = v_1v_2 \dots v_k$, we replace C with a new cycle $D = v_1'v_2' \dots v_k'v_1''v_2'' \dots v_k''$; see Figure 3.2. Although D is one-sided, we can still distinguish the left and the right side of the path $P = v_1v_2 \dots v_k$. For each edge v_iw ($w \in G - C$) on the left side of P we create an edge $v_i'w$ and for each edge v_iw on the right side of P we create an edge $v_i''w$. The directions and the orientations of new edges and the rotation system of the new embedding are determined similarly as in the previous case. The result is an embedding into a surface of Euler characteristic $\chi(S)+1$. This surface is obtained by patching the hole created by the cutting with a disc. The cycle D becomes a facial cycle of the new embedding.

3.3.4 Reducing the genus

Starting from the given embedding Π of the graph G, we sequentially cut the graph along nonseparating cycles, as long as the Euler characteristic of the em-

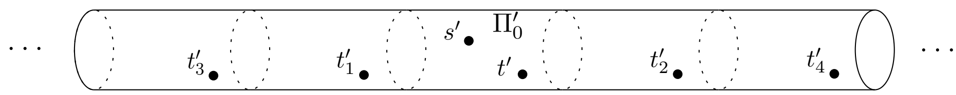

Figure 3.3: The surface obtained after gluing several copies of Π' with two holes.

bedding is smaller than 2. Since each cutting operation increases the Euler characteristic (and decreases the genus), after at most g cuttings we get a planar embedding Π' of a graph G'. Since the cutting operation preserves facial cycles, the embedding Π' contains facial cycles $C'_1, C''_1, C'_2, C''_2, \ldots, C'_{g'}, C''_{g'}$ and $D_1, D_2, \ldots, D_{g''}$, where each pair C'_i, C''_i corresponds to a cutting along a two-sided cycle C_i and each D_i corresponds to a cutting along a one-sided cycle. The faces bounded by the cycles C'_i, C''_i and D_i are called *holes*. Each cutting increases the number of vertices at most twice and can be performed in logspace. Since at most g cuttings are performed, the graph G' has at most $2^g \cdot n$ vertices and can be constructed in logspace as well.

The vertices s and t might be split into more vertices during the cuttings. In such a case we just choose one of the copies of s and one of the copies of t and call them s' and t', respectively. The existence of a directed s'-t'-path in G' implies the existence of a directed s-t-path in G, but not the other way. It might happen that a directed path in G was cut into several pieces during the cuttings.

3.3.5 Restoring connectivity

To restore the connectivity of G', we can glue a certain number of copies of the embedding Π' together. Two copies can be glued by the cycle C'_i in one copy with the cycle C''_i in the other copy so that corresponding vertices and edges are identified. Or, we can glue them by the cycle $D_i = v'_1 v'_2 \ldots v'_k v''_1 v''_2 \ldots v''_k$ in one copy with the same cycle D_i in the other copy, so that the vertices v'_i, v''_i from one copy are identified with v''_i, v'_i from the other copy, respectively.

From the description of the cutting operations it follows that the total number of holes in Π' is equal to $2 - \chi(S) \leq 2g$. In case S is the projective plane, the embedding Π' has a single hole D_1, so we can glue at most two copies of Π' together to get an embedding in the sphere (which is the universal cover of the projective plane). In all other cases $\chi(S) \leq 0$ so Π' has at least two holes and we can glue arbitrary number of copies together. In case S is the torus or the Klein bottle, we have $\chi(S) = 0$ and two holes in Π'. In other words, Π' is an embedding in the cylindrical surface. Thus every gluing of several copies of Π' will again result in an embedding in the cylindrical surface; see Figure 3.3. For all S with $\chi(S) < 0$ we have at least 3 holes in Π' and the gluing will result in an embedding in a surface with a tree-like structure; see Figure 3.4. Note that in every case the resulting embedding will remain planar, since the surface obtained by the gluing is homeomorphic to a subset of the plane.

We start with Π'_0, a "root" copy of Π', and glue a copy of Π' to each hole of Π'_0. Then in $2^g \cdot n - 1$ subsequent steps we glue a copy of Π' to each hole of the embedding constructed in the previous step; see Figure 3.4. We obtain a planar embedding of a graph H that has better connectivity than G'. More precisely, G has a directed s-t-path if and only if s' in Π'_0 is connected by a directed path

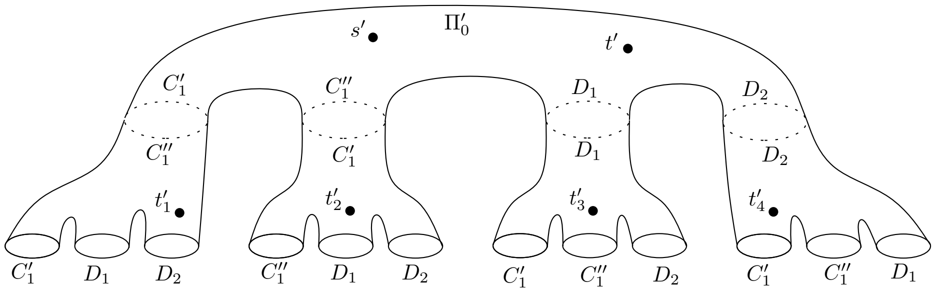

Figure 3.4: The surface obtained after the first step of gluing copies of Π' with more than two holes.

to one of the copies of t' in H. This follows from the observation that each directed s-t-path P in G that was cut into i components (including one-vertex components that arise when more cutting cycles pass through the same vertex) is restored after at most $i-1$ steps, as a *lifted* directed path $\widetilde{P}$ in H.

In each step the number of holes in the underlying surface (and hence the number of new copies of Π' to be added in the next step) increases $(1-\chi(S))$-times. It follows that if S is the projective plane our construction terminates after the first step and the graph H will be only twice as large as G'. In case S is the torus or the Klein bottle, the graph H consists of $1+2\cdot(2^g\cdot n-1)$ copies of Π' and so it has only quadratic size in n and can be constructed in logspace. In all other cases, however, H has exponential size and thus cannot be constructed in logspace.

We show that instead of H we can take a subgraph of H of polynomial size with the same connectivity.

Lemma 3.3. *There is a logspace-constructible directed plane graph K (of polynomial size) containing vertices $s', t'_1, t'_2, \ldots, t'_N$ such that s is connected by a directed path to t in G if and only if s' is connected by a directed path to one of the vertices t'_i in K.*

For the final step of the reduction we use the following result of Allender et al. [ABC+09].

Lemma 3.4. [ABC+09] *Given a directed planar graph H containing vertices $s', t'_1, t'_2, \ldots, t'_N$, there is a logspace-constructible directed planar graph H' containing vertices s'' and t'' such that H' contains a directed s''-t''-path if and only if H contains a directed path from s' to one of the vertices t'_i.*

The proof of Lemma 3.4 uses a reduction of planar reachability to a special case where both vertices s and t are on the outer face. Such a reduction is performed for each pair s', t'_i in H, yielding a planar graph H_i with s' and t'_i on its outer face. Then a new vertex s'' is added and connected by a directed edge to a copy of s' in each H_i. Similarly, a new vertex t'' is added and each vertex t'_i is connected by a directed edge to t''.

3.3.6 Proof of Lemma 3.3

The main idea of the construction is that only polynomially many branches of the tree structure of H are needed to cover the lifts of all s-t-paths in G. We

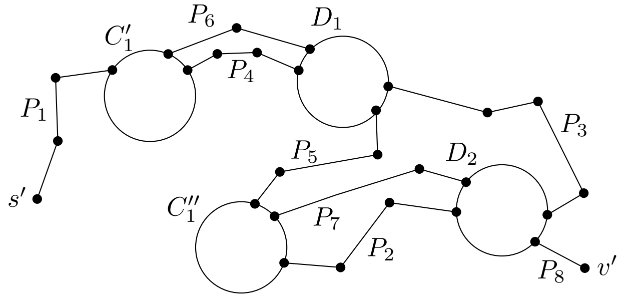

Figure 3.5: A path P of type $(C_1', D_2, D_1, C_1', D_1, C_1', D_2)$ cut into eight pieces.

show how to identify these branches and how to enumerate them in logspace.

Let P be a (not necessarily directed) s-v-path in G, where v is an arbitrary vertex. Let $(P_1, P_2, \ldots, P_k)$, $k \leq 2^g \cdot n$, denote the sequence of paths in G' that arise from P by the cuttings. Each P_i is a curve (possibly with zero length) with endpoints on the boundary of some hole (except P_1 starts in s' and P_k ends in v', where v' is one of the copies of v in G').

The *type* of P is a sequence $(X_1, X_2, \ldots, X_{k-1})$ of cycles C_i', C_i'' or D_i such that X_i contains the terminal vertex of P_i and corresponds to the cutting operation that separated the components P_i and P_{i+1}. See Figure 3.5. If X_i is the cycle C_j' (C_j'', D_j), then let X_i' denote the cycle C_j'' $(C_j', D_j$, respectively). Similarly we define the *type* for every *s-v-curve*, that is, a curve embedded in S starting in s and ending in v.

The type of P determines through which copies of G' in H (we shall call these copies the *regions* of H) the lifted path $\widetilde{P}$ passes. The number of different types of directed paths in G may be exponential, but we show that all the corresponding lifted paths are contained in a union of only polynomial number of regions. To show this it is enough to prove that there are only polynomially many regions containing the terminal vertex of some lifted s-v-path. Every such region is called the *terminal region* of the corresponding path.

If $X_i' = X_{i+1}$ for some $i \leq k-2$, then the path P_{i+1} starts and ends on the boundary of the same hole, which means that the paths P_i and P_{i+2} terminate in the same region. Thus after removing X_i and X_{i+1} from the sequence $X = (X_1, X_2, \ldots, X_{k-1})$ we get a sequence Y, which is a type of a curve that we get from P by replacing P_{i+1} and parts of P_i and P_{i+2} in a neighborhood of X_i with a curve drawn along the cycle X_i; see Figure 3.6. Hence Y determines the same terminal region as X. The resulting s-v-curve does not consist of the edges of G only, but it still passes through each vertex of G at most once. Even if more reductions occur at the same cycle X_i, we can make sure that the new curve does not cross itself, which follows from the planarity of Π'. An s-v-curve P and its type $(X_1, X_2, \ldots, X_{k-1})$ are called *reduced* if there is no i such that $X_i' = X_{i+1}$. From the preceding observation it follows that for each s-v-path P in G there is a reduced s-v-curve with the same terminal region.

Let P be a reduced s-v-curve in S. We define an undirected *characteristic multigraph* of P, which we denote by $M(P)$, together with its planar embedding; see Figure 3.7. The vertex set of $M(P)$ consists of the points s' and v' in the embedding Π' and points c_i', c_i'' and d_i chosen in the interiors of the holes bounded

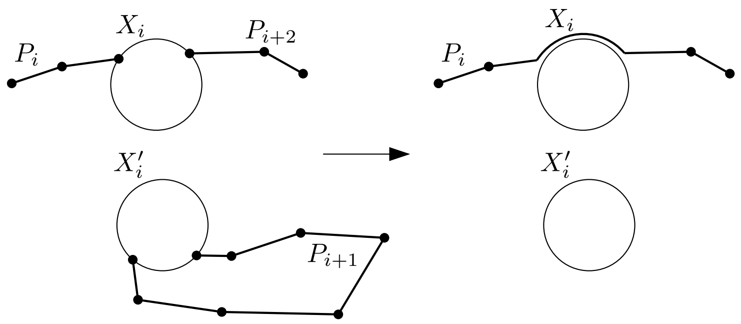

Figure 3.6: Reducing a path.

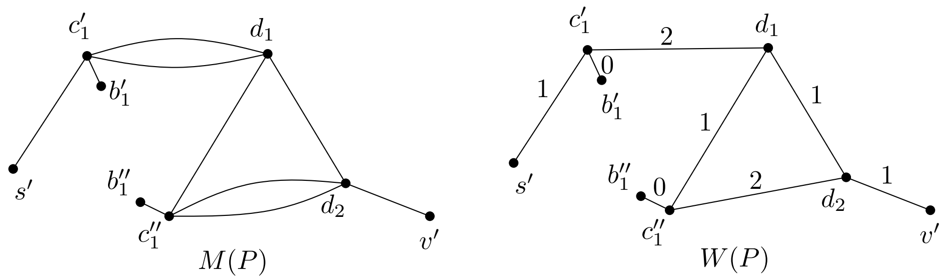

Figure 3.7: The characteristic multigraph of the path P from Figure 3.5 (left) and its corridor multigraph (right).

by the cycles C_i', C_i'' and D_i, respectively. For each part P_i we draw an edge between the vertices x_{i-1}' and x_i inside the cycles X_{i-1}' and X_i (except for P_1 and P_k, where one of the vertices is s' or v', respectively). The edge is drawn along the curve P_i in Π' and extended to the points x_{i-1}' and x_i at each end.

We also add auxiliary leaf vertices b_i' and b_i'' joined by auxiliary edges to c_i' and c_i''. The points b_i' and b_i'' are chosen as points in the interiors of the cycles C_i' and C_i'', between two consecutive edges $x_i'v_j', x_i'v_{j+1}'$ and $x_i''v_j'', x_i''v_{j+1}''$, where v_j' corresponds to v_j'' and v_{j+1}' corresponds to v_{j+1}'' when the cycles C_i' and C_i'' are glued together. Then we can define a linear ordering L_i' (L_i'') of the (nonauxiliary) edges incident with c_i' (c_i''), starting at b_i' (b_i'') and going around the vertex c_i' (c_i'', respectively) in clockwise direction. This allows us to reconstruct the type of the original curve P from the characteristic multigraph $M(P)$, by walking along the edges of $M(P)$ in a deterministic way. The curve starts at s' and ends at v'. When the curve arrives at the vertex c_i' (c_i'') along the j-th smallest edge in L_i' (L_i''), it continues from c_i'' (c_i') along the j-th largest edge in L_i'' (L_i', respectively). When the curve arrives at d_i, it continues from the same vertex along the opposite edge incident with d_i.

If the multigraph $M(P)$ has more than one component, we sequentially add auxiliary edges joining two different components, to get a connected multigraph $M'(P)$.

Now we construct a compact description of the multigraph $M'(P)$. We define a weighted embedded undirected *corridor multigraph* $W(P)$ as follows. Put $W_0(P) = M(P)$ and set the weights of all edges to 1 (except the auxiliary edges that get weight 0). Then repeat the following step: if there is a face f in $W_i(P)$

bounded by only two edges e_1 and e_2 with weights $w(e_1)$ and $w(e_2)$, construct $W_{i+1}(P)$ from $W_i(P)$ by merging the edges e_1 and e_2 into a single edge with weight $w(e_1) + w(e_2)$, thus removing the face f. After at most $2^g \cdot n$ steps we get a multigraph $W(P)$ where all faces are bounded by at least three edges. Indeed, $W(P)$ contains no loops as P is a reduced curve, and all bigons were eliminated in the construction of $W(P)$. By Euler's formula, $W(P)$ has at most $3(2g+2)+6$ edges, while the sum of the weights of the edges is at most $2^g \cdot n$. Clearly, we can reconstruct $M(P)$ (and hence the type and the terminal region of P) from the corridor multigraph $W(P)$ by replacing each edge of weight w by w parallel edges.

The construction of $W(P)$ can be achieved in logspace as follows. Take the edges of $M'(P)$ in clockwise order around each vertex. If the current edge e_i forms a bigon with the previous edge e_{i-1}, increase the weight of the corresponding edge in the corridor multigraph by one. Otherwise form a new edge of weight 1 (or 0 if e_i is auxiliary) if e_i has not been visited already from its other endpoint.

Up to homeomorphism, there are only a constant number of connected plane multigraphs with at most $3(2g + 2) + 6$ edges. Thus all such multigraphs can be generated in constant space. To get all the possible corridor multigraphs, we need to assign weights to the edges. As each of the multigraphs has a constant number of edges and the sum of weights is linear in n, there are only polynomially many different assignments of the weights to the edges. Therefore, there are only polynomially many corridor multigraphs, up to homeomorphism. Clearly, all the weight assignments (decompositions of a number $m \leq 2^g \cdot n$ into a sum of constantly many nonnegative integers) can be generated in logspace.

By the previous discussion, we can easily determine the type of the curve corresponding to a given corridor multigraph in logspace. The graph K is now constructed by taking a copy of G' for each of the generated types, and gluing the appropriate pairs of these copies together.

4. Continuous decay of the crossing number

4.1 Introduction

For any graph G, let $n(G)$ denote the number of its vertices and $m(G)$ the number of its edges. If it is clear from the context, we simply write n and m instead of $n(G)$ and $m(G)$. The crossing number $\text{CR}(G)$ of a graph G is the minimum number of edge crossings over all drawings of G in the plane. In the optimal drawing of G, crossings are not necessarily distributed uniformly among the edges. Some edges could be more "responsible" for the crossing number than some other edges. For any fixed k, it is not hard to construct a graph G whose crossing number is k, but G has an edge e such that $G \setminus e$ is planar. Richter and Thomassen [RT93] started to investigate the following general problem. We have a graph G, and we want to remove a given number of edges. By *at least* how much does the crossing number decrease? They conjectured that there is a constant c such that every graph G with $\text{CR}(G) = k$ has an edge e with $\text{CR}(G - e) \geq k - c\sqrt{k}$. They only proved that G has an edge with $\text{CR}(G - e) \geq \frac{2}{5}k - O(1)$.

Pach, Radoičić, Tardos, and Tóth [PRTT06] proved that for every graph G with $m(G) \geq \frac{103}{16}n(G)$, we have $\text{CR}(G) \geq 0.032\frac{m^3}{n^2}$. It is not hard to see [PT00] that for *any* edge e, we have $\text{CR}(G - e) \geq \text{CR}(G) - m + 1$. These two results imply an improvement of the Richter–Thomassen bound if $m \geq 8.1n$, and also imply the Richter–Thomassen conjecture for graphs with $\Omega(n^2)$ edges.

J. Fox and Cs. Tóth [FT08] investigated the case where we want to delete a *positive fraction* of the edges.

Theorem A. [FT08] *For every $\varepsilon > 0$, there is an n_ε such that every graph G with $n(G) \geq n_\varepsilon$ vertices and $m(G) \geq n(G)^{1+\varepsilon}$ edges has a subgraph G' with*

$$m(G') \leq \left(1 - \frac{\varepsilon}{24}\right) m(G)$$

and

$$\text{CR}(G') \geq \left(\frac{1}{28} - o(1)\right) \text{CR}(G).$$

We generalize Theorem A and show that the decay of the crossing number is continuous in the fraction of deleted edges.

Theorem. *For every $\varepsilon, \gamma > 0$, there is an $n_{\varepsilon,\gamma}$ such that every graph G with $n(G) \geq n_{\varepsilon,\gamma}$ vertices and $m(G) \geq n(G)^{1+\varepsilon}$ edges has a subgraph G' with*

$$m(G') \leq \left(1 - \frac{\varepsilon\gamma}{1224}\right) m(G)$$

and

$$\text{CR}(G') \geq (1 - \gamma)\text{CR}(G).$$

Recently, Balogh, Leanos and Salazar [BLS12] proved a variant of Theorem A for sparse graphs with no "artificially inflated" edges. That is, graphs that contain no nontrivial planar subgraphs attached to the rest of the graph by a pair of vertices.

4.2 Proof of the Theorem

Our proof is based on the argument of Fox and Tóth [FT08], the only new ingredient is Lemma 4.1.

Definition. *Let $r \geq 2, p \geq 1$ be integers. A $2r$-earring of size p is a graph that is a union of an edge uv and p edge-disjoint paths between u and v, each of length at most $2r-1$. The edge uv is called the main edge of the $2r$-earring.*

Lemma 4.1. *Let $r \geq 2, p \geq 1$ be integers. There exists n_0 such that every graph G with $n \geq n_0$ vertices and $m \geq 6prn^{1+1/r}$ edges contains at least $m/3pr$ edge-disjoint $2r$-earrings, each of size p.*

Proof. By the result of Alon, Hoory, and Linial [AHL02], for some n_0, every graph with $n \geq n_0$ vertices and at least $n^{1+1/r}$ edges contains a cycle of length at most $2r$.

Suppose that G has $n \geq n_0$ vertices and $m \geq 6prn^{1+1/r}$ edges. Take a *maximal* edge-disjoint set $\{E_1, E_2, \ldots, E_x\}$ of $2r$-earrings, each of size p. Let $E = E_1 \cup E_2 \cup \cdots \cup E_x$, the set of all edges of the earrings and let $G' = G - E$. Now let E'_1 be a $2r$-earring of G' of maximum size. Note that this size is less than p. Let $G'_1 = G' - E'_1$. Similarly, let E'_2 be a $2r$-earring of G'_1 of maximum size and let $G'_2 = G'_1 - E'_2$. Continue analogously, as long as there is a $2r$-earring in the remaining graph. We obtain the $2r$-earrings $E'_1, E'_2, \ldots, E'_y$, and the remaining graph $G'' = G'_y$ contains no $2r$-earring. Let $E' = E'_1 \cup E'_2 \cup \cdots \cup E'_y$.

We claim that $y < n^{1+1/r}$. Suppose on the contrary that $y \geq n^{1+1/r}$. Take the main edges of $E'_1, E'_2, \ldots, E'_y$. We have at least $n^{1+1/r}$ edges so by the result of Alon, Hoory, and Linial [AHL02] some of them form a cycle C of length at most $2r$. Let i be the smallest index with the property that C contains the main edge of E'_i. Then C, together with E'_i would be a $2r$-earring of G'_{i-1} of greater size than E'_i, contradicting the maximality of E'_i.

Each of the earrings $E'_1, E'_2, \ldots, E'_y$ has at most $(p-1)(2r-1)+1$ edges so we have $|E'| \leq y(p-1)(2r-1) + y < (2pr-1)n^{1+1/r}$. Since G'' contains no $2r$-earring, it contains neither a cycle of length at most $2r$, since this would be a $2r$-earring of size one. Therefore, by [AHL02], for the number of edges of G'' we have the upper bound $e(G'') < n^{1+1/r}$.

It follows that the set $E = \{E_1, E_2, \ldots, E_x\}$ contains at least $m - 2prn^{1+1/r} \geq \frac{2}{3}m$ edges. Each of $E_1, E_2, \ldots, E_x$ has at most $p(2r-1)+1 \leq 2pr$ edges, therefore, $x \geq m/3pr$. □

Lemma 4.2. [FT08] *Let G be a graph with n vertices, m edges, and degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Let ℓ be the integer such that $\sum_{i=1}^{\ell-1} d_i < 4m/3$ but $\sum_{i=1}^{\ell} d_i \geq 4m/3$. If n is large enough and $m = \Omega(n \log^2 n)$ then*

$$\mathrm{CR}(G) \geq \frac{1}{65} \sum_{i=1}^{\ell} d_i^2.$$

Proof of the Theorem. Let $\varepsilon, \gamma \in (0,1)$ be fixed. Choose integers r, p such that $\frac{1}{\varepsilon} < r \leq \frac{2}{\varepsilon}$, and $\frac{67}{\gamma} < p \leq \frac{68}{\gamma}$. It follows that we have $\frac{1}{r} < \varepsilon \leq \frac{2}{r}$, and $\frac{67}{p} < \gamma \leq \frac{68}{p}$. Then there is an $n_{\varepsilon,\gamma}$ with the following properties: (a) $n_{\varepsilon,\gamma} \geq n_0$ from Lemma 4.1, (b) $(n_{\varepsilon,\gamma})^{1+\varepsilon} > 18pr \cdot (n_{\varepsilon,\gamma})^{1+1/r}$.

Let G be a graph with $n \geq n_{\varepsilon,\gamma}$ vertices and $m \geq n^{1+\varepsilon}$ edges. Let $v_1, \ldots, v_n$ be the vertices of G, of degrees $d_1 \leq d_2 \leq \cdots \leq d_n$. Define ℓ as in Lemma 4.2, that is, $\sum_{i=1}^{\ell-1} d_i < 4m/3$ but $\sum_{i=1}^{\ell} d_i \geq 4m/3$. Let G_0 be the subgraph of G induced by $v_1, \ldots, v_\ell$. Observe that G_0 has $m' \geq m/3$ edges. Therefore, by Lemma 4.1 G_0 contains at least $m'/3pr \geq m/9pr$ edge-disjoint $2r$-earrings, each of size p.

Let M be the set of the main edges of these $2r$-earrings. We have $|M| \geq m/9pr \geq \frac{\varepsilon\gamma}{1224}m$. Let $G' = G - M$ and $G'_0 = G_0 - M$.

Take an optimal drawing $D(G')$ of the subgraph $G' \subset G$. We have to draw the missing edges to obtain a drawing of G. Our method is a randomized variation of the embedding method, which has been applied by Leighton [L83], Richter and Thomassen [RT93], Shahrokhi et al. [SSSV97], Székely [S04], and most recently by Fox and Tóth [FT08]. For every missing edge $e_i = u_iv_i \in M \subset G_0$, e_i is the deleted main edge of a $2r$-earring $E_i \subset G_0$. So there are p edge-disjoint paths in G_0 from u_i to v_i. For each of these paths, draw a curve from u_i to v_i infinitesimally close to that path, on either side. Call these p curves *potential u_iv_i-edges* and call the resulting drawing D. Note that a potential u_iv_i-edge crosses itself if the corresponding path does. In such cases, we redraw the potential u_iv_i-edge in the neighborhood of each self-crossing to get a noncrossing curve.

To get a drawing of G, for each $e_i = u_iv_i \in M$, choose one of the p potential u_iv_i-edges at random, independently and uniformly, with probability $1/p$, and draw the edge u_iv_i as that curve.

There are two types of new crossings in the obtained drawing of G. *First category crossings* are infinitesimally close to a crossing in $D(G')$, *second category crossings* are infinitesimally close to a vertex of G_0 in $D(G')$.

The expected number of first category crossings is at most

$$\left(1 + \frac{2}{p} + \frac{1}{p^2}\right) \mathrm{CR}(G') = \left(1 + \frac{1}{p}\right)^2 \mathrm{CR}(G').$$

Indeed, for each edge of G', there can be at most one new edge drawn next to it, and that is drawn with probability at most $1/p$. Therefore, in the close neighborhood of a crossing in $D(G')$, the expected number of crossings is at most $(1 + \frac{2}{p} + \frac{1}{p^2})$. See figure 4.1(a).

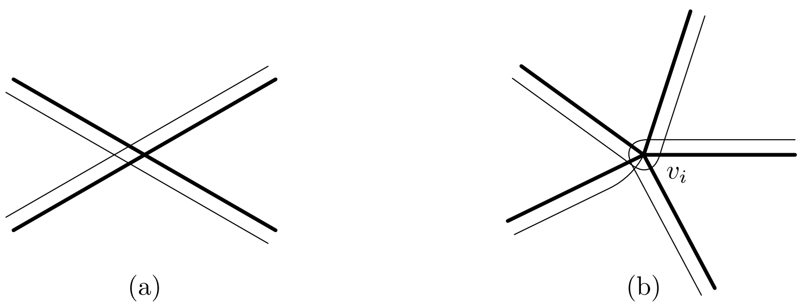

Figure 4.1: The thick edges are edges of G', the thin edges are the potential edges. Figure shows (a) a neighborhood of a crossing in $D(G')$ and (b) a neighborhood of a vertex v_i in G'.

In order to estimate the expected number of second category crossings, consider the drawing D near a vertex v_i of G_0. In the neighborhood of the vertex

v_i we have at most d_i original edges. Since we draw at most one potential edge along each original edge, there can be at most d_i potential edges in the neighborhood. Each potential edge can cross each original edge at most once, and any two potential edges can cross at most twice. See figure 4.1(b). Therefore, the total number of second category crossings in D in the neighborhood of v_i is at most $2d_i^2$. (This bound can be substantially improved with a more careful argument, see e. g. [FT08], but we do not need anything better here.) To obtain the drawing of G, we keep each of the potential edges with probability $1/p$, so the expected number of crossings in the neighborhood of v_i is at most $(\frac{1}{p} + \frac{1}{p^2})d_i^2$, using the fact that the self-crossings of the potential uv-edges have been eliminated.

Therefore, the total expected number of crossings in the random drawing of G is at most $(1 + \frac{2}{p} + \frac{1}{p^2})\text{CR}(G') + (\frac{1}{p} + \frac{1}{p^2})\sum_{i=1}^{\ell} d_i^2$.

There exists an embedding with at most this many crossings, therefore, by Lemma 4.2 we have

$$\begin{aligned}\text{CR}(G) &\leq \left(1 + \frac{1}{p}\right)^2 \text{CR}(G') + \left(\frac{1}{p} + \frac{1}{p^2}\right)\sum_{i=1}^{\ell} d_i^2 \\ &\leq \left(1 + \frac{1}{p}\right)^2 \text{CR}(G') + \left(\frac{65}{p} + \frac{65}{p^2}\right)\text{CR}(G).\end{aligned}$$

It follows that

$$\left(1 - \frac{65}{p} - \frac{65}{p^2}\right)\text{CR}(G) \leq \left(1 + \frac{1}{p}\right)^2 \text{CR}(G'),$$

so

$$\begin{aligned}\left(1 - \frac{65}{p} - \frac{65}{p^2}\right)\left(1 - \frac{1}{p}\right)^2 \text{CR}(G) &\leq \left(1 - \frac{1}{p^2}\right)^2 \text{CR}(G'), \\ \left(1 - \frac{65}{p} - \frac{65}{p^2}\right)\left(1 - \frac{2}{p}\right)\text{CR}(G) &\leq \text{CR}(G'), \\ \left(1 - \frac{67}{p}\right)\text{CR}(G) &\leq \text{CR}(G'),\end{aligned}$$

consequently,

$$(1 - \gamma)\,\text{CR}(G) \leq \text{CR}(G').$$

□

4.3 Concluding remarks

In the statement of our Theorem we cannot require that *every* subgraph G' with $(1 - \delta)m(G)$ edges has crossing number $\text{CR}(G') \geq (1 - \gamma)\text{CR}(G)$, instead of just *one* such subgraph G'. In fact, the following statement holds.

Proposition 4.3. *For every $\varepsilon \in (0, 1)$ there exist graphs G_n with $n(G_n) = \Theta(n)$ vertices and $m(G_n) = \Theta(n^{1+\varepsilon})$ edges with subgraphs $G'_n \subset G_n$ such that*

$$m(G'_n) = (1 - o(1))\,m(G_n)$$

and

$$\text{CR}(G'_n) = o(\text{CR}(G_n)).$$

Proof. Roughly speaking, G_n will be the disjoint union of a large graph G'_n with low crossing number and a small graph H_n with large crossing number. More precisely, let $G = G_n$ be a disjoint union of graphs $G' = G'_n$ and $H = H_n$, where G' is a disjoint union of $\Theta(n^{1-\varepsilon})$ complete graphs, each with $\lfloor n^{\varepsilon} \rfloor$ vertices and H is a complete graph with $\lfloor n^{(3+5\varepsilon)/8} \rfloor$ vertices. We have $m(G) = \Theta(n^{1+\varepsilon})$ and $m(H) = \Theta(n^{(3+5\varepsilon)/4}) = o(m(G))$, since $\frac{3+5\varepsilon}{4} < 1 + \varepsilon$. By the crossing lemma (see e. g. [PRTT06]), $\text{CR}(G) \geq \text{CR}(H) = \Omega(n^{(3+5\varepsilon)/2})$, but $\text{CR}(G') = O(n^{1-\varepsilon} \cdot n^{4\varepsilon}) = O(n^{1+3\varepsilon}) = o(\text{CR}(G))$, because $\frac{3+5\varepsilon}{2} > 1 + 3\varepsilon$. □

In the preliminary paper [CKT08] we conjectured that we can require that a positive fraction of all subgraphs G' of G with $(1-\delta)m(G)$ edges has crossing number $\text{CR}(G') \geq (1-\gamma)\text{CR}(G)$. The following construction shows that the conjecture does not hold in general for graphs with less than $n^{4/3-\Omega(1)}$ edges.

Proposition 4.4. *For every $\varepsilon \in (0, 1/3)$ and $\delta > 0$ there exist graphs G_n with $n(G_n) = \Theta(n)$ vertices and $m(G_n) = \Theta(n^{1+\varepsilon})$ edges with the following property. Let G'_n be a random subgraph of G_n such that we choose each edge of G_n independently with probability $p = 1 - \delta$. Then*

$$\Pr\left[\text{CR}(G'_n) \leq o(\text{CR}(G_n))\right] > 1 - e^{-\delta n^{\Omega(1/3-\varepsilon)}}.$$

Proof. As in Proposition 4.3, the idea is to build the graph $G = G_n$ from two disjoint graphs K and H, where K is a large graph with low crossing number and H is a small graph with large crossing number. In addition, deleting a random constant fraction of edges from H will break all the crossings in H with high probability.

Now we describe the constructions more precisely. Let $\gamma > 0$ be a constant such that $3\varepsilon + 4\gamma < 1$ and $3\varepsilon + 5\gamma > 1$. Let K be a disjoint union of $\Theta(n^{1-\varepsilon})$ complete graphs, each with n^{ε} vertices (we omit the explicit rounding to keep the notation simple). We have $m(K) = \Theta(n^{1+\varepsilon})$ and $\text{CR}(K) = \Theta(n^{1+3\varepsilon})$.

The graph H consists of five *main* vertices $v_1, v_2, \ldots, v_5$ and $n^{1-2\gamma}$ internally vertex disjoint paths of length n^{γ} connecting each pair v_i, v_j. The graph H has $n(H) = \Theta(n^{1-\gamma})$ vertices and $m(H) = \Theta(n^{1-\gamma})$ edges. We claim that $\text{CR}(H) = n^{2-4\gamma}$. The upper bound follows from the fact that the crossing number of K_5 is 1. We take a drawing of K_5 with one crossing and replace each edge e by $n^{1-2\gamma}$ paths drawn close to e. For the lower bound, take a drawing of H minimizing the number of crossings. Let $p_{i,j}$ be a path with the minimum number of crossings among the paths connecting v_i and v_j. By redrawing all the other paths connecting v_i and v_j along $p_{i,j}$ the crossing number of the drawing does not change. The paths $p_{i,j}$ together form a subdivision of K_5, therefore at least one pair $p_{i,j}, p_{k,l}$ of the paths crosses. Due to the redrawing, every path connecting v_i and v_j crosses every path connecting v_k and v_l, which makes $n^{2-4\gamma}$ crossings. By the choice of γ, we have $n^{1+3\varepsilon} = o(n^{2-4\gamma})$, therefore $\text{CR}(G) = \Theta(\text{CR}(H))$ and $\text{CR}(K) = o(\text{CR}(G))$.

Let G' be a random subgraph of G where each edge of G is taken independently with probability $p = 1-\delta$. Let $H' = G' \cap H$. We show that with high probability, H' is a forest, in particular $\text{CR}(H') = 0$. This happens if at least one edge is missing from every path connecting two main vertices of H. The probability of such an event is at least

$$1 - n \cdot (1-\delta)^{n^\gamma} \geq 1 - e^{-\delta n^\gamma + \log n} \geq 1 - e^{-\delta n^{\Omega(1/3-\varepsilon)}}.$$

It follows that with this probability, $\mathrm{CR}(G') \leq \mathrm{CR}(K) \leq o(\mathrm{CR}(G))$. □

Note that in the previous construction the number δ does not have to be constant: it is enough to delete a random $\delta = c \log n / n^\gamma$ fraction of the edges to get the same conclusion with probability almost 1.

Balogh, Leanos and Salazar [BLS12] extended the construction from Proposition 4.4 also to the case $\varepsilon = 1/3$.

The question whether deleting a small random constant fraction of the edges of a graph G decreases the crossing number only by a small constant fraction remains open for graphs with more than $n^{4/3}$ edges. We do not know the answer even to the following weaker version of the question.

Problem 7. *Let $\varepsilon \in (0, 2/3)$ and $p \in (0,1)$ be constants. Does there exist $c(p) > 0$ and n_0 such that for every graph G with $n(G) > n_0$ and $m(G) > n(G)^{4/3+\varepsilon}$, a random subgraph G' of G with each edge taken with probability p has crossing number at least $c(p) \cdot \mathrm{CR}(G)$, with probability at least $1/2$?*

The graphs in Proposition 4.4 have small number of edges responsible for almost all the crossings. Is this the only way how to force a random subgraph of G to have crossing number $o(\mathrm{CR}(G))$?

Problem 8. *Let $\varepsilon > 0$. Does there exist n_0 and δ such that every graph G with $n(G) \geq n_0$ and $m(G) \geq n(G)^{1+\varepsilon}$ has a subset F of $o(m(G))$ edges such that every subgraph G' of G with $m(G') \geq (1-\delta)m(G)$ and $E(G') \subset E(G) \setminus F$ has $\mathrm{CR}(G') \geq (1-\varepsilon)\mathrm{CR}(G)$?*

Bibliography

[AFNW09] B. M. Ábrego, S. Fernández-Merchant, M. G. Neubauer and W. Watkins, Sum of squares of degrees in a graph, *JIPAM. Journal of Inequalities in Pure and Applied Mathematics* **10**(3) (2009), Article 64, 34 pp.

[APZ12] E. Ackerman, R. Pinchasi and S. Zerbib, On touching curves, *Bernoulli Reunion Conference on Discrete and Computational Geometry*, EPFL, Lausanne, 2012.

[AK78] R. Ahlswede and G.O.H. Katona, Graphs with maximal number of adjacent pairs of edges, *Acta Mathematica Academiae Scientiarum Hungaricae* **32**(1-2) (1978), 97–120.

[ABC+09] E. Allender, D. A. M. Barrington, T. Chakraborty, S. Datta and S. Roy, Planar and grid graph reachability problems, *Theory of Computing Systems* **45**(4) (2009), 675–723.

[AM04] E. Allender and M. Mahajan, The complexity of planarity testing, *Information and Computation* **189**(1) (2004), 117–134.

[AHL02] N. Alon, S. Hoory and N. Linial, The Moore bound for irregular graphs, *Graphs and Combinatorics* **18** (2002), 53–57.

[BLS12] J. Balogh, J. Leanos and G. Salazar, On the decay of crossing numbers of sparse graphs, arXiv:1203.0510v1, 2012.

[BW10] L. Beineke and R. Wilson, The early history of the brick factory problem, *The Mathematical Intelligencer* **32**(2) (2010), 41–48.

[BR86] E. A. Bender and L. B. Richmond, A survey of the asymptotic behaviour of maps, *Journal of Combinatorial Theory, Series B* **40**(3) (1986), 297–329.

[BW85] E. A. Bender and N. C. Wormald, The number of loopless planar maps, *Discrete Mathematics* **54**(2) (1985), 235–237.

[BTV09] C. Bourke, R. Tewari and N. V. Vinodchandran, Directed planar reachability is in unambiguous log-space, *ACM Transactions on Computation Theory* **1**(1) (2009), 1–17.

[BMP05] P. Brass, W. Moser, and J. Pach, *Research problems in discrete geometry*, Springer, New York, 2005, ISBN 978-0387-23815-8.

[Ce08] J. Černý, *Combinatorial and computational geometry*, Ph.D. thesis, Charles University, Prague, 2008.

[CKT08] J. Černý, J. Kynčl and G. Tóth, Improvement on the decay of crossing numbers, *Proceedings of the 15th International Symposium on Graph Drawing (Graph Drawing 2007), Lecture Notes in Computer Science* **4875**, 25–30, Springer, Berlin, 2008.

[Ci09] J. Cibulka, On constants in the Füredi–Hajnal and the Stanley–Wilf conjecture, *Journal of Combinatorial Theory, Series A* **116**(2) (2009), 290–302.

[CK12] J. Cibulka and J. Kynčl, Tight bounds on the maximum size of a set of permutations with bounded VC-dimension, *Journal of Combinatorial Theory, Series A* **119**(7) (2012), 1461–1478.

[DKTV09] S. Datta, R. Kulkarni, R. Tewari and N. V. Vinodchandran, Space complexity of perfect matching in bounded genus bipartite graphs, *Journal of Computer and System Sciences* **78** (2012) 765–779.

[DO11] S. L. Devadoss and J. O'Rourke, *Discrete and computational geometry*, Princeton University Press, Princeton, NJ, 2011, ISBN 978-0-691-14553-2.

[DM11] M. Drmota and M. Noy, Universal exponents and tail estimates in the enumeration of planar maps, *Electronic Notes in Discrete Mathematics* **38** (2011), 309–317.

[Fe97] S. Felsner, On the number of arrangements of pseudolines, *ACM Symposium on Computational Geometry (Philadelphia, PA, 1996)*, *Discrete and Computational Geometry* **18** (1997), 257–267.

[FV11] S. Felsner and P. Valtr, Coding and Counting Arrangements of Pseudolines, *Discrete and Computational Geometry* **46**(3) (2011), 405–416.

[FPT11] J. Fox, J. Pach and Cs. D. Tóth, Intersection patterns of curves, *Journal of the London Mathematical Society, Second Series* **83**(2) (2011), 389–406.

[FT08] J. Fox and Cs. D. Tóth, On the decay of crossing numbers, *Journal of Combinatorial Theory, Series B* **98**(1) (2008), 33–42.

[Gi05] E. Gioan, Complete graph drawings up to triangle mutations, *Graph-theoretic concepts in computer science, Lecture Notes in Computer Science* **3787**, 139–150, Springer, Berlin, 2005.

[GO04] J. E. Goodman and J. O'Rourke, *Handbook of Discrete and Computational Geometry, Second edition*, *Discrete Mathematics and its Applications (Boca Raton)*, Chapman & Hall/CRC, Boca Raton, FL, 2004, ISBN 1-58488-301-4.

[GT87] J. L. Gross and T. W. Tucker, *Topological graph theory*, *Wiley-Interscience Series in Discrete Mathematics and Optimization*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1987, ISBN 0-471-04926-3.

[Gr03] B. Grünbaum, *Convex polytopes, Second edition*, Prepared and with a preface by V. Kaibel, V. Klee and G. M. Ziegler, *Graduate Texts in Mathematics* **221**, Springer-Verlag, New York, 2003, ISBN 0-387-00424-6; 0-387-40409-0.

[Ha05] T. C. Hales, A Proof of the Kepler Conjecture, *Annals of Mathematics, Second Series* **162**(3) (2005), 1065–1185.

[HF11] T. C. Hales and S. P. Ferguson, *The Kepler Conjecture: The Hales-Ferguson Proof*, Springer, New York, 2011, ISBN 978-1-4614-1128-4.

[HM74] H. Harborth and I. Mengersen, Edges without crossings in drawings of complete graphs, *Journal of Combinatorial Theory, Series B* **17** (1974), 299–311.

[HM92] H. Harborth and I. Mengersen, Drawings of the complete graph with maximum number of crossings, *Proceedings of the Twenty-third Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1992)*, *Congressus Numerantium* **88** (1992), 225–228.

[HMS95] H. Harborth, I. Mengersen and R. Schelp, The drawing Ramsey number $\mathrm{Dr}(K_n)$, *Australasian Journal of Combinatorics* **11** (1995), 151–156.

[JLR06] A. Jakoby, M. Liśkiewicz and R. Reischuk, Space efficient algorithms for directed series-parallel graphs, *Journal of Algorithms in Cognition, Informatics and Logic* **60**(2) (2006), 85–114.

[KPT97] G. Károlyi, J. Pach and G. Tóth, Ramsey-type results for geometric graphs I, *ACM Symposium on Computational Geometry (Philadelphia, PA, 1996)*, *Discrete and Computational Geometry* **18**(3) (1997), 247–255.

[Kh00] A. Khruzin, Enumeration of chord diagrams, arXiv:math/0008209v1, 2000.

[Kl00] M. Klazar, The Füredi–Hajnal Conjecture Implies the Stanley–Wilf Conjecture, *Formal Power Series and Algebraic Combinatorics, Moscow 2000*, 250–255, Springer, 2000.

[Ko88] A. Kostochka, Upper bounds on the chromatic number of graphs, *Trudy Institute Matematiki (Novosibirsk)* **10**, *Modeli i Metody Optim.* (1988), 204–226.

[KO07] D. Kühn and D. Osthus, Maximizing several cuts simultaneously, *Combinatorics, Probability and Computing* **16** (2007), 277–283.

[Ky06] J. Kynčl, Crossings in topological graphs, master thesis, Charles University, Prague, 2006.

[Ky09] J. Kynčl, Enumeration of simple complete topological graphs, *European Journal of Combinatorics* **30**(7) (2009), 1676–1685.

[Ky11] J. Kynčl, Simple realizability of complete abstract topological graphs in P, *Discrete and Computational Geometry* **45**(3) (2011), 383–399.

[LZ04] S. K. Lando and A. K. Zvonkin, *Graphs on surfaces and their applications*, *Encyclopaedia of Mathematical Sciences* **141**, *Low-Dimensional Topology II*, Springer, Berlin, 2004, ISBN 3-540-00203-0.

[LMPT94] D. Larman, J. Matoušek, J. Pach and J. Törőcsik, A Ramsey-type result for convex sets, *The Bulletin of the London Mathematical Society* **26**(2) (1994), 132–136.

[L83] T. Leighton, *Complexity issues in VLSI*, MIT Press, Cambridge, MA, 1983.

[MT04] A. Marcus and G. Tardos, Excluded permutation matrices and the Stanley–Wilf conjecture, *Journal of Combinatorial Theory, Series A* **107**(1) (2004), 153–160.

[Ma02] J. Matoušek, *Lectures on discrete geometry*, *Graduate Texts in Mathematics* **212**, Springer, New York, 2002, ISBN 0-387-95373-6.

[MT01] B. Mohar and C. Thomassen, *Graphs on surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2001, ISBN 0-8018-6689-8.

[MS68] R.C. Mullin and P.J. Schellenberg, The enumeration of c-nets via quadrangulations, *Journal of Combinatorial Theory* **4**(3) (1968), 259–276.

[Ni07] V. Nikiforov, The sum of the squares of degrees: Sharp asymptotics, *Discrete Mathematics* **307**(24) (2007), 3187–3193.

[NT95] N. Nisan and A. Ta-Shma, Symmetric Logspace is closed under complement, *Chicago Journal of Theoretical Computer Science* (1995), Article 1, 11 pp.

[Ni10] G. Nivasch, Improved bounds and new techniques for Davenport–Schinzel sequences and their generalizations, *Journal of the ACM* **57**(3) (2010), 1–44.

[PA95] J. Pach and P. Agarwal, *Combinatorial Geometry*, J. Wiley, New York, 1995, ISBN 978-0-471-58890-0.

[PRTT06] J. Pach, R. Radoičić, G. Tardos and G. Tóth, Improving the Crossing Lemma by finding more crossings in sparse graphs, *Discrete and Computational Geometry* **36**(4) (2006), 527–552.

[PS91] J. Pach and M. Sharir, On vertical visibility in arrangements of segments and the queue size in the Bentley-Ottmann line sweeping algorithm, *SIAM Journal on Computing* **20**(3) (1991), 460–470.

[PST03] J. Pach, J. Solymosi and G. Tóth, Unavoidable configurations in complete topological graphs, *Discrete and Computational Geometry* **30** (2003), 311–320.

[PT00] J. Pach and G. Tóth, Thirteen problems on crossing numbers, *Geombinatorics* **9** (2000), 194–207.

[PT06] J. Pach and G. Tóth, How many ways can one draw a graph?, *Combinatorica* **26**(5) (2006), 559–576.

[Pa10] I. Pak, *Lectures on Discrete and Polyhedral Geometry*, `http://www.math.ucla.edu/~pak/book.htm`, 2010.

[Ra94] S. Radziszowski, Small Ramsey Numbers, *Electronic Journal of Combinatorics* **1** (1994), Dynamic Survey 1, 30 pp. (electronic).

[Ra00] R. Raz, VC-Dimension of Sets of Permutations, *Combinatorica* **20**(2) (2000), 241–255.

[Re79] R. C. Read, The chord intersection problem, *Annals of the New York Academy of Sciences* **319**(1) (1979), 444–454.

[Re08] O. Reingold, Undirected connectivity in log-space, *Journal of the ACM* **55**(4) (2008), Art. 17, 24 pp.

[RL01] H. Ren and Y. Liu, Enumerating near-4-regular maps on the sphere and the torus, *Discrete Applied Mathematics* **110**(2–3) (2001), 273–288.

[RLL02] H. Ren, Y. Liu and Z. Li, Enumeration of 2-connected Loopless 4-regular maps on the plane, *European Journal of Combinatorics* **23**(1) (2002), 93–111.

[RT93] B. Richter and C. Thomassen, Minimal graphs with crossing number at least k, *Journal of Combinatorial Theory, Series B* **58** (1993), 217–224.

[Ri75] J. Riordan, The distribution of crossings of chords joining pairs of $2n$ points on a circle, *Mathematics of Computation* **29**(129) (1975), 215–222.

[SSS02] M. Schaefer, E. Sedgwick and D. Štefankovič, Algorithms for normal curves and surfaces, *Computing and combinatorics, Lecture Notes in Computer Science* **2387**, 370–380, Springer, Berlin, 2002.

[SSS03] M. Schaefer, E. Sedgwick and D. Štefankovič, Recognizing string graphs in NP, *Journal of Computer and System Sciences* **67** (2003), 365–380.

[SSSV97] F. Shahrokhi, O. Sýkora, L. Székely and I.Vrťo, Crossing numbers: bounds and applications, in: *Intuitive geometry (Budapest, 1995), Bolyai Society Mathematical Studies* **6**, 179–206, Budapest, 1997.

[SBV09] D. Stolee, C. Bourke and N. V. Vinodchandran, A log-space algorithm for reachability in planar DAGs with few sources, *Electronic Colloquium on Computational Complexity*, TR09-049 (2009), technical report. Also in: *Proceedings of the 2010 IEEE 25th Conference on Computational Complexity (CCC)*, 131–138.

[SV10] D. Stolee and N. V. Vinodchandran, Space-efficient algorithms for reachability in surface-embedded graphs, *Electronic Colloquium on Computational Complexity*, TR10-154 (2010), technical report. Also in: *Proceedings of the 2012 IEEE 27th Conference on Computational Complexity (CCC)*, 326–333.

[S04] L. Székely, Short proof for a theorem of Pach, Spencer, and Tóth, in: *Towards a theory of geometric graphs, Contemporary Mathematics* **342**, 281–283, AMS, Providence, RI, 2004.

[TW09] T. Thierauf and F. Wagner, Reachability in $K_{3,3}$-free graphs and K_5-free graphs is in unambiguous log-space, *Electronic Colloquium on Computational Complexity*, TR09-029 (2009), technical report. Also in: *Proceedings of the 17th international conference on Fundamentals of computation theory (FCT 2009)*, *Fundamentals of Computation Theory, Lecture Notes in Computer Science* **5699** (2009), 323–334.

[To50] J. Touchard, Contribution a l'étude du probleme des timbres poste, *Canadian Journal of Mathematics* **2** (1950), 385–398.

[Tu63] W. T. Tutte, A census of planar maps, *Canadian Journal of Mathematics* **15** (1963), 249–271.

[WL75] T. R. S. Walsh and A. B. Lehman, Counting rooted maps by genus III: Nonseparable maps, *Journal of Combinatorial Theory, Series B* **18** (1975), 222–259.

[Zi95] G. Ziegler, *Lectures on polytopes*, *Graduate Texts in Mathematics* **152**, Springer-Verlag, New York, 1995, ISBN 0-387-94365-X.