Charles University in Prague

Faculty of Mathematics and Physics

DOCTORAL THESIS

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Ramsey-type results for ordered hypergraphs

Department of Applied Mathematics

Supervisor of the doctoral thesis: Doc. RNDr. Pavel Valtr, Dr.

Study programme: Informatics

Study branch: Discrete Models and Algorithms

Prague 2016

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In Prague, 30th May 2016 Martin Balko

Název práce: Ramseyovské výsledky pro uspořádané hypergrafy

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Vedoucí práce: Doc. RNDr. Pavel Valtr, Dr., Katedra aplikované matematiky

Abstrakt: Představíme *uspořádaná Ramseyova čísla*, která jsou obdobou Ramseyových čísel pro grafy s lineárně uspořádanými vrcholy. Studujeme růst uspořádaných Ramseyových čísel uspořádaných grafů vzhledem k počtu vrcholů. Nalezneme uspořádaná párování se superpolynomiálními uspořádanými Ramseyovými čísly. Ukážeme, že uspořádaná Ramseyova čísla uspořádaných grafů s omezenou degenerovaností a intervalovým chromatickým číslem jsou nanejvýš polynomiální. Dokážeme, že uspořádaná Ramseyova čísla jsou nanejvýš polynomiální pro uspořádané grafy s omezenými délkami hran. Nalezneme 3-regulární grafy se superlineárními uspořádanými Ramseyovými čísly nad všemi uspořádáními. Poslední dva výsledky řeší problémy od autorů Conlon, Fox, Lee a Sudakov. Odvodíme přesnou formuli pro uspořádaná Ramseyova čísla monotónních cyklů a použijeme ji k získání přesné formule pro geometrická Ramseyova čísla cyklů, která byla představena Károlyim a spol. Vyvrátíme domněnku Peterse a Szekerese o zesílení slavné Erdősovy–Szekeresovy domněnky nad uspořádanými hypergrafy. Dokážeme přesnou formuli pro minimální počet průsečíků v jednoduchých x-monotónních nakresleních úplných grafů a ukážeme kombinatorickou charakterizaci těchto nakreslení pomocí obarvení uspořádaných úplných 3-uniformních hypergrafů.

Klíčová slova: uspořádaný graf, uspořádané Ramseyovo číslo, Erdősova–Szekeresova věta, průsečíkové číslo, monotónní nakreslení

Title: Ramsey-type results for ordered hypergraphs

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Abstract: We introduce *ordered Ramsey numbers*, which are an analogue of Ramsey numbers for graphs with a linear ordering on their vertices. We study the growth rate of ordered Ramsey numbers of ordered graphs with respect to the number of vertices. We find ordered matchings whose ordered Ramsey numbers grow superpolynomially. We show that ordered Ramsey numbers of ordered graphs with bounded degeneracy and interval chromatic number are at most polynomial. We prove that ordered Ramsey numbers are at most polynomial for ordered graphs with bounded bandwidth. We find 3-regular graphs that have superlinear ordered Ramsey numbers, regardless of the ordering. The last two results solve problems of Conlon, Fox, Lee, and Sudakov. We derive the exact formula for ordered Ramsey numbers of monotone cycles and use it to obtain the exact formula for geometric Ramsey numbers of cycles that were introduced by Károlyi et al. We refute a conjecture of Peters and Szekeres about a strengthening of the famous Erdős–Szekeres conjecture to ordered hypergraphs. We obtain the exact formula for the minimum number of crossings in simple x-monotone drawings of complete graphs and provide a combinatorial characterization of these drawings in terms of colorings of ordered complete 3-uniform hypergraphs.

Keywords: ordered graph, ordered Ramsey number, Erdős–Szekeres theorem, crossing number, monotone drawing

Acknowledgments

First of all, I would like to thank my supervisor, Pavel Valtr, for all his guidance, advice, and support during my studies. He negotiated several of my stays abroad and also led seminars where my first encounters with research and most of the result from this thesis originate. Many thanks belong to Jan Kynčl for a lot of advice and many productive discussions over the years.

I would like to thank to all my coauthors and collaborators. Thanks to Josef Cibulka, Radoslav Fulek Vít Jelínek, and Karel Král for discussions about problems covered in this thesis. I would also like to thank all colleagues at the Department of Applied Mathematics and the Computer Science Institute of Charles University for a nice and friendly atmosphere.

Some of the results were obtained during my stays at the DIMACS institute of Rutgers University, Alfréd Rényi Institute of Mathematics, and École Polytechnique Fédérale de Lausanne.

My research was supported by the projects CE-ITI (GAČR P202/12/G061) and GAČR 14-14179S of the Czech Science foundation, GAČR GIG/11/E023 GraDR in the framework of ESF EUROGIGA program, and by the grants GAUK 1262213 and GAUK 690214 of the Grant Agency of Charles University. The support of the grants SVV–2013–267313 and SVV–2015–260223 is also highly acknowledged.

Finally, I would like to thank my family, especially my parents, for their support and encouragement throughout all my studies.

Contents

Introduction

Overview and motivation

Ramsey theory refers to a large body of results whose underlying philosophy can be captured by the statement "Every structure of a given kind contains a large well-organized substructure". This part of discrete mathematics has developed spectacularly in the last few decades, emerging into a field whose results are important in many areas, including combinatorics, geometry, logics, and number theory.

Probably the first result in Ramsey theory is *Hilbert's cube lemma* [Hil92] proved by Hilbert in 1892. For positive integers $d, a, n_1, \ldots, n_d$, we call the set $\{a + \sum_{i=1}^{d} \alpha_i n_i \colon \alpha_1, \ldots, \alpha_d \in \{0,1\}\}$ an *affine d-cube* generated by $a, n_1, \ldots, n_d$. Hilbert's cube lemma says that for given positive integers d and c there is a positive integer $\mathrm{H}(c,d)$ such that every partitioning of $[\mathrm{H}(c,d)]$ into c parts contains an affine d-cube in one of the parts. Here we use $[n]$ to denote the set $\{1,\ldots,n\}$ for some positive integer n. We use this abbreviation throughout the thesis. Although Hilbert's cube lemma seems to be the first example of a Ramsey-type result, it did not draw much attention.

About 25 years later, Schur [Sch16] showed that for every positive integer c there is a positive integer $\mathrm{S}(c)$ such that every partition of $[\mathrm{S}(c)]$ into c parts contains three integers x, y, z in one of the parts such that $x + y = z$. This result is now known as *Schur's theorem* and Schur used it to reprove a result of Dickson on a modular version of Fermat's conjecture. In 1927, Van der Waerden [Wae27] proved a conjecture that was independently posed by Schur and Baudet. His result, known as *Van der Waerden's theorem*, says that for positive integers c and m there is a positive integer $\mathrm{W}(c,m)$ such that every partitioning of $[\mathrm{W}(c,m)]$ into c parts contains an m-term arithmetic progression in one of the parts.

Hilbert's cube lemma, Schur's theorem, and Van der Waerden's theorem can be viewed as the earliest roots of Ramsey theory. Although these three results were proved before the 1930s and thus hold precedence in the matter, it is a partition theorem for finite sets proved by Ramsey [Ram30], which is considered to be the most classical result in the field and from which Ramsey theory derives its name. This result, called *Ramsey's theorem*, says that for any positive integers c, k, and n there is a positive integer $\mathrm{R}(c,k,n)$ such that for every partitioning of the collection of all k-element subsets of $[\mathrm{R}(c,k,n)]$ into c parts there is an n-element subset S of $[\mathrm{R}(c,k,n)]$ with all k-element subsets of S in the same part.

Ramsey proved his theorem in 1928 while trying to give a decision procedure for propositional logic. Nowadays, Ramsey's theorem is usually stated in the language of graph theory. A particular case of Ramsey's theorem says that for every positive integer n and every graph G on n vertices there is an integer N such that for every graph H on N vertices either G is a subgraph of H or G is a subgraph of the complement of H. We denote the smallest such N by $\mathrm{R}(G)$ and we call it the *Ramsey number of G*.

The study of Ramsey numbers of graphs is a classical and influential topic, which played a key role in the development of many fields of combinatorics such as the probabilistic method or the theory of quasirandomness. Exact formulas for Ramsey numbers are known for some classes of graphs, for example for stars [BR73], paths [GG67], and cycles [CS71, BE73, Ros73, FS74]. However, determining Ramsey numbers is notoriously difficult in general. For example, despite much effort expended in the last 70 years, the best lower bound for $\mathrm{R}(K_n)$ is $\Omega(n2^{n/2})$ by Erdős [Erd47] and Spencer [Spe75] while the currently best upper bound is due to Conlon [Con09] and gives $\mathrm{R}(K_n) \leq n^{-c\log n/\log\log n}2^{2n}$ for a positive constant c.

In this thesis we study Ramsey-type results with a connection to discrete geometry, a field that studies combinatorial properties of geometric objects such as finite point sets, lines, hyperplanes, circles, or polytopes. The foundations of discrete geometry were laid by Paul Erdős, who popularized this field by posing numerous interesting and natural problems. Many problems in discrete geometry are interesting for their own sake, but some have also led to applications in computational geometry, robotics, and computer graphics as well as in other branches of mathematics, such as number theory.

The connection between Ramsey theory and discrete geometry is almost as old as Ramsey theory itself, since one of the earliest and most popular applications of Ramsey's theorem is the *Erdős–Szekeres theorem*, a foundational result in discrete geometry. It says that for every integer $k \geq 2$ there is a least number $\mathrm{ES}(k)$ such that every set of $\mathrm{ES}(k)+1$ points in the plane in *general position* (no three points lie on a common line) contains k points in convex position. The statement is a generalization of Esther Klein's problem, which was named the *Happy Ending Problem* by Paul Erdős, as it eventually led to the marriage of George Szekeres and Esther Klein.

The Erdős–Szekeres theorem was proved by Erdős and Szekeres [ES35] in 1935 in their classic paper where they actually rediscovered Ramsey's theorem and found an alternative proof with a better estimate on $\mathrm{R}(K_n)$ than the one obtained by Ramsey [Ram30]. Moreover, they also included a proof of another famous result, the *Erdős–Szekeres lemma on monotone subsequences*, which says that, for every positive integer k, every sequence of $(k-1)^2+1$ distinct numbers contains either an increasing sequence of k terms or a decreasing sequence of k terms. With the paper of Erdős and Szekeres, the popularization of Ramsey's theorem among non-logicians began.

There are several great books on Ramsey theory. We refer the interested reader to the excellent book by Graham, Rothschild, and Spencer [GRS90], as well as to the monograph by Nešetřil and Rödl [NR90]. A reference on both classical results and some of the more recent breakthroughs in the subject is the recent book by Prömel [Prö13]. The history of Ramsey theory, recent developments, and some promising future directions are explored in the book by Soifer [Soi11]. We also recommend a recent survey by Conlon, Fox, and Sudakov [CFS15] about the current state-of-art in graph Ramsey theory.

A broad and detailed overview of classical and modern topics in discrete geometry is provided in excellent books by Pach and Agarwal [AJ95] and Matoušek [Mat02]. A collection of hundreds of open problems in discrete geometry can be found in the book by Brass, Moser, and Pach [BMJ05].

Summary of the main results

In this thesis we study several Ramsey-type problems, most of which were motivated by problems from discrete geometry. In particular, we focus on Ramsey-type results for ordered structures such as ordered graphs and hypergraphs. The thesis is based on the following papers.

(1) M. Balko, J. Cibulka, K. Král, and J. Kynčl. Ramsey numbers of ordered graphs. Submitted, 2015, preprint available at `http://arxiv.org/abs/1310.7208`, 2015. Extended abstract in *Electronic Notes in Discrete Mathematics* **49**:419–424, 2015.

(2) M. Balko, V. Jelínek, and P. Valtr. On ordered Ramsey numbers of bounded-degree graphs. In preparation, 2016.

(3) M. Balko and P. Valtr. A SAT attack on the Erdős–Szekeres conjecture. Submitted, 2016. Extended abstract in *Electronic Notes in Discrete Mathematics* **49**:425–431, 2015.

(4) M. Balko, R. Fulek, and J. Kynčl. Crossing numbers and combinatorial characterization of monotone drawings of K_n. *Discrete and Computational Geometry* **53**(1):107–143, 2015.

(5) M. Balko and J. Kynčl. Bounding the pseudolinear crossing number of K_n via simulated annealing. Extended abstract in the (informal) *Proceedings of the XVI Spanish Meeting on Computational Geometry*, pages 37–40, 2015.

We now briefly overview the topics studied in each chapter and we describe the main results. Chapter 1 contains some results from (1). Chapter 2 is based on (2) and also contains some results from (1). Chapter 3 is based on an extended version of (3) and partially on (1). In Chapter 4 we include results from (4) and (5).

Chapter 1: Ordered Ramsey numbers. In the first chapter of this thesis, we introduce Ramsey numbers for *ordered graphs*. That is, graphs with the vertex set ordered by some linear order. A first systematic study of Ramsey numbers for general ordered graphs has been conducted only recently by Balko, Cibulka, Král, and Kynčl in (1) and independently by Conlon, Fox, Lee, and Sudakov [CFLS14].

An ordered graph $\mathcal{G}$ is an *ordered subgraph* of an ordered graph $\mathcal{H}$ if the underlying graph of $\mathcal{G}$ is isomorphic to a subgraph of the underlying graph of $\mathcal{H}$ via isomorphism that preserves the ordering of vertices. The *ordered Ramsey number of* $\mathcal{G}$, denoted by $\overline{\mathrm{R}}(\mathcal{G})$, is the minimum integer N such that every 2-coloring of edges of the ordered complete graph on N vertices contains a monochromatic copy of $\mathcal{G}$ as an ordered subgraph.

There has been much progress in the study of ordered Ramsey numbers of so-called monotone paths [CP02, EM13, FPSS12, MSW15, MS14], in particular with a connection to the Erdős–Szekeres theorem and its generalizations. However, there has been surprisingly little work on more general ordered graphs and hypergraphs.

We study ordered Ramsey numbers for general ordered graphs. In Chapter 1 we start by introducing the necessary notation and initial observations.

Our study then continues by focusing on specific classes of ordered graphs. We show several estimates on ordered Ramsey numbers of ordered stars and paths and we prove the exact formula for ordered Ramsey numbers of so-called monotone cycles (Theorem 1.11).

Chapter 2: Growth rate of ordered Ramsey numbers. In this chapter we investigate how different orderings affect the growth rate of ordered Ramsey numbers.

We observe that ordered Ramsey numbers differ substantially from the usual Ramsey numbers by constructing arbitrarily large ordered matchings $\mathcal{M}_n$ on n vertices whose ordered Ramsey numbers are at least $n^{\log n/(5 \log \log n)}$ (Theorem 2.1). That is, $\overline{\mathrm{R}}(\mathcal{M}_n)$ is superpolynomial in n. This is in sharp contrast with the well-known fact that n-vertex graphs of bounded maximum degree have Ramsey numbers linear in n [CVSTJ83].

The *interval chromatic number* of an ordered graph $\mathcal{G}$ is the least number of intervals the vertex set of $\mathcal{G}$ can be partitioned into such that every interval from the partition induces an independent set in $\mathcal{G}$.

We improve a lower bound of Conlon, Fox, Lee, and Sudakov [CFLS14] on ordered Ramsey numbers of ordered matchings with interval chromatic number 2 (Theorem 2.3).

We then provide two polynomial upper bounds on two classes of sparse ordered graphs. First, we show that $\overline{\mathrm{R}}(\mathcal{G})$ is at most polynomial in n for every ordered graph $\mathcal{G}$ on n vertices with bounded degeneracy and interval chromatic number (Corollary 2.8). Then we prove a polynomial upper bound on $\overline{\mathrm{R}}(\mathcal{G})$ for ordered graphs $\mathcal{G}$ that satisfy a certain recursive decomposition (Theorem 2.11). The latter result implies a polynomial upper bound on ordered Ramsey numbers of ordered graphs with bounded bandwidth (Corollary 2.12), which solves an open problem of Conlon, Fox, Lee, and Sudakov [CFLS14, Problem 6.9]. The *bandwidth* of an ordered graph $\mathcal{G} = (G, \prec)$ with vertices $u_1 \prec \cdots \prec u_n$ is the minimum of $|i - j|$ over all edges $\{u_i, u_j\}$ of $\mathcal{G}$.

We show that there are 3-regular graphs that have superlinear ordered Ramsey numbers, regardless of the ordering (Theorem 2.19), solving another problem of Conlon, Fox, Lee, and Sudakov [CFLS14, Problem 6.7]. On the other hand, we prove that every graph G on n vertices with maximum degree 2 admits an ordering $\mathcal{G}$ of G such that $\overline{\mathrm{R}}(\mathcal{G})$ is linear in n (Theorem 2.25).

Chapter 2 is then concluded with a collection of open problems about ordered Ramsey numbers.

While presenting some of the results from this chapter at the conference Summit 240 in Budapest (2014), we learned about a recent work by Conlon, Fox, Lee, and Sudakov [CFLS14] who independently investigated Ramsey numbers of ordered graphs. There are some overlaps with our results.

Among many other results, Conlon et al. [CFLS14] proved that as n goes to infinity, almost every ordering $\mathcal{M}$ of a matching on n vertices satisfies $\overline{\mathrm{R}}(\mathcal{M}) \geq n^{\log n/(20 \log \log n)}$. This gives a similar bound as Theorem 2.1, where we construct one particular ordered matching on n vertices. Conlon et al. [CFLS14] also showed that every n-vertex ordered graph $\mathcal{G}$ with degeneracy k and interval chromatic number p satisfies $\overline{\mathrm{R}}(\mathcal{G}) \leq n^{32k \log p}$; see Theorem 2.9. This gives a better estimate than Corollary 2.8.

Recently, the study of ordered Ramsey numbers and ordered graphs has become a rather active topic [ARU16, CGK+15, CS15, CS16, MSW15, MR16] that is also included in a survey of Conlon, Fox, and Sudakov [CFS15] about recent developments in graph Ramsey theory.

Chapter 3: Applications. The third chapter of the thesis is devoted to some Ramsey-type problems, in which ordered graphs and hypergraphs naturally appear. This chapter also contains several applications of results from Chapters 1 and 2.

As the first application, we observe that the Erdős–Szekeres lemma on monotone subsequences is a special case of a Ramsey-type result for ordered paths. So, in a certain sense, the origins of ordered Ramsey numbers can be traced back to one of the first results in Ramsey theory.

Similarly, the Erdős–Szekeres theorem can be stated and proved in the framework of ordered hypergraphs. We include a new proof of this theorem by Moshkovitz and Shapira [MS14] that is based on ordered Ramsey numbers of monotone hyperpaths.

In their seminal 1935 paper, Erdős and Szekeres [ES35] conjectured that the maximum size $\mathrm{ES}(k)$ of a set of points in the plane in general position with no k points in convex position satisfies $\mathrm{ES}(k) = 2^{k-2}$. This problem, known as the *Erdős–Szekeres conjecture*, is still open. In 2006, Peters and Szekeres [PS06] conjectured that a certain strengthening of the Erdős–Szekeres conjecture to ordered hypergraphs holds. We refute the conjecture of Peters and Szekeres by providing a counterexample that was found by employing an exhaustive computer search (Theorem 3.3).

At the end of Chapter 3, we show some applications in the theory of geometric Ramsey numbers, which were introduced by Károlyi et al. [KPT97, KPTV98]. We derive the exact formula for geometric Ramsey numbers of cycles (Theorem 3.13), strengthening a bound by Károlyi et al. [KPTV98]. We also show that convex geometric Ramsey numbers of outerplanar graphs are at most quasipolynomial in the number of vertices (Corollary 3.16), improving a bound by Cibulka et al. [CGK+15].

Chapter 4: Crossing numbers of K_n. A typical Ramsey-type result says that once a structure is large enough, there is always a copy of a certain well-organized substructure. It is a natural question to ask *how many* such copies are guaranteed in every sufficiently large structure.

Determining the crossing numbers of complete graphs may be viewed as a particular instance of such question. For a graph G, let $\mathrm{cr}(G)$ be the minimum number of crossings in every drawing of G where no three edges cross at the same point. It is a well-known fact that every drawing of K_5 contains at least one crossing. That is, $\mathrm{cr}(K_5) \geq 1$ (in fact, $\mathrm{cr}(K_5) = 1$). Determining the formula for $\mathrm{cr}(K_n)$ for general n turned out to be very difficult. According to famous *Hill's conjecture* [Guy60, HH63] from the 1950s, we have $\mathrm{cr}(K_n) = Z(n)$, where

$$Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor \left\lfloor\frac{n-1}{2}\right\rfloor \left\lfloor\frac{n-2}{2}\right\rfloor \left\lfloor\frac{n-3}{2}\right\rfloor.$$

It is known that $\mathrm{cr}(K_n) \leq Z(n)$ [BK64, Guy60, HH63, Har02], but, despite several attempts over the last fifty years, Hill's conjecture still remains open.

We say that a drawing of a graph G is *simple* if no two adjacent edges cross and no two edges have more than one common crossing. This is a natural class of drawings to consider, as it is a well-known fact that every drawing with the minimum number of crossings is simple. A drawing of G is *x-monotone* if every edge of G is represented by a curve that intersects every vertical line in at most one point.

In Chapter 4, we restrict the setting of Hill's conjecture to simple x-monotone drawings. We show that Hill's conjecture holds under this restriction by proving that there are at least $Z(n)$ crossings in every simple x-monotone drawing of K_n (Theorem 4.1).

The proof of this result is based on a generalization of a technique used by Ábrego et al. [AAFM$^+$13] for so-called 2-page book drawings. Recently, there has been further progress on this topic and Hill's conjecture is known to hold for a wider class of drawings of K_n. In Chapter 4, we briefly survey this development.

Then we provide a combinatorial characterization of simple x-monotone drawings of K_n that is based on colorings of edges of the ordered complete 3-uniform hypergraph on n vertices by two colors. We show that there is a correspondence between simple x-monotone drawings and a class of such colorings with forbidden subconfigurations on at most five vertices (Theorem 4.8). This puts the previous result into a perspective of Ramsey-type results on ordered hypergraphs. In particular, determining the minimum number of crossings in simple x-monotone drawings of K_n can be regarded as a variant of a Ramsey multiplicity problem for ordered hypergraphs.

A drawing D of a graph is *pseudolinear* if the edges of D can be extended to doubly-infinite curves that form an arrangement of pseudolines. The *pseudolinear crossing number* of K_n is the minimum number of crossings in a pseudolinear drawing of K_n. Felsner and Weil [FW01] proved a characterization of pseudolinear drawings of K_n that is similar to Theorem 4.8. We apply this characterization in a simulated annealing algorithm to prove that the pseudolinear crossing number of K_n is at most $0.380448\binom{n}{4} + O(n^3)$, shrinking the current gap between the lower and upper bound on the pseudolinear crossing number of K_n roughly by five percent (Theorem 4.10).

We enclose this chapter with some open problems, one of which strengthens the original Hill's conjecture.

1. Ordered Ramsey numbers

1.1 Preliminaries

We study the analogue of Ramsey's theorem for graphs with ordered vertex sets. The concept of ordered graphs appeared earlier in the literature [Kla04a, Kla04b, MSW15, PT06], but we are not aware of any Ramsey-type results for such graphs except for the case of monotone paths and hyperpaths [CP02, EM13, FPSS12, MSW15, MS14].

The main goal of Chapters 1 and 2 is to understand the effects of different vertex orderings on the ordered Ramsey number of a given graph, and to compare the ordered and unordered Ramsey numbers.

In the rest of the thesis, we omit the ceiling and floor signs whenever they are not crucial. Unless indicated otherwise, all logarithms in this thesis are base 2.

Hypergraphs. We consider only finite graphs and hypergraphs with no multiple edges and no loops (that is, one-element edges).

A *coloring* of a hypergraph $H = (V, E)$ is a mapping $f \colon E \to C$ where C is a finite set of *colors*. A coloring with c colors is called a *c-coloring*. In a 2-coloring of a graph G with colors red and blue, we call a vertex u of G a *red neighbor* (a *blue neighbor*) of a vertex v of G if the edge $\{u, v\}$ is colored red (blue, respectively).

The *complete* k-uniform hypergraph on n vertices, denoted by K_n^k, is a hypergraph whose edges are all k-element subsets of the n vertices. A general Ramsey's theorem states that for given positive integers c, k, and n, there is an integer N such that every c-coloring of K_N^k contains a monochromatic copy of K_n^k. The minimum such N is called the *Ramsey number* and we denote it by $\mathrm{R}_k(K_n^k; c)$. For graphs we write $\mathrm{R}(K_n; c)$ instead of $\mathrm{R}_2(K_n^2; c)$. Classical results of Erdős [Erd47] and Erdős and Szekeres [ES35] give the exponential bounds

$$2^{n/2} \le \mathrm{R}(K_n; 2) \le 2^{2n}. \tag{1.1}$$

Despite many improvements during the last sixty years (see [Con09] for example), the constant factors in the exponents have remained the same.

Since every k-uniform hypergraph on n vertices is contained in K_n^k, we can consider the following generalization of Ramsey numbers. Let c be a positive integer and let $H_1, \ldots, H_c$ be k-uniform hypergraphs. Ramsey's theorem then implies that there exists a smallest number $\mathrm{R}_k(H_1, \ldots, H_c)$ such that every c-coloring of a complete k-uniform hypergraph with at least $\mathrm{R}_k(H_1, \ldots, H_c)$ vertices contains a monochromatic copy of H_i in color i for some $i \in [c]$. The case when all the hypergraphs $H_1, \ldots, H_c$ are isomorphic to H is called the *diagonal case* and we just write $\mathrm{R}_k(H; c)$ instead of $\mathrm{R}_k(H_1, \ldots, H_c)$. We also abbreviate $\mathrm{R}_k(H; 2)$ as $\mathrm{R}_k(H)$, and $\mathrm{R}_2(H; 2)$ as $\mathrm{R}(H)$.

Ordered hypergraphs. An *ordered hypergraph* $\mathcal{H}$ is a pair $(H, \prec)$ where H is a hypergraph and $\prec$ is a total ordering of its vertex set. The ordering $\prec$ is called a *vertex ordering* of H. Many notions related to hypergraphs, such as vertex degrees or a coloring, can be defined analogously for ordered hypergraphs.

Figure 1.1: Examples of 2-uniform (part (a)) and 3-uniform (part (b)) monotone hyperpaths.

For an ordered hypergraph $\mathcal{H} = (H, \prec)$ and its vertices x, y, we say that x is a *left neighbor* of y and that y is a *right neighbor* of x if x and y belong to a common edge and $x \prec y$. We say that two ordered hypergraphs $(H_1, \prec_1)$ and $(H_2, \prec_2)$ are *isomorphic* if H_1 and H_2 are isomorphic via a one-to-one correspondence $g \colon V(H_1) \to V(H_2)$ that also preserves the orderings; that is, for every $x, y \in V(H_1)$, $x \prec_1 y \Leftrightarrow g(x) \prec_2 g(y)$. An ordered (hyper)graph $\mathcal{H} = (H, \prec_1)$ is an *ordered sub(hyper)graph* of $\mathcal{G} = (G, \prec_2)$, written $\mathcal{H} \subseteq \mathcal{G}$, if H is a sub(hyper)graph of G and $\prec_1$ is a suborder of $\prec_2$. Up to isomorphism, there is only one ordered complete k-uniform hypergraph on n vertices, which we denote as $\mathcal{K}^k_n$, or $\mathcal{K}_n$ if $k = 2$.

We now introduce Ramsey numbers of ordered hypergraphs, called *ordered Ramsey numbers*. For given ordered k-uniform hypergraphs $\mathcal{H}_1, \dots, \mathcal{H}_c$, we denote by $\overline{\mathrm{R}}_k(\mathcal{H}_1, \dots, \mathcal{H}_c)$ the smallest number N such that every c-coloring of $\mathcal{K}^k_N$ contains, for some $i \in [c]$, a monochromatic copy of $\mathcal{H}_i$ in color i as an ordered subhypergraph. If all ordered hypergraphs $\mathcal{H}_i$ are isomorphic to $\mathcal{H}$, we write the ordered Ramsey number as $\overline{\mathrm{R}}_k(\mathcal{H}; c)$. In the case of graphs (that is, if $k = 2$) we write $\overline{\mathrm{R}}(\mathcal{H}_1, \dots, \mathcal{H}_c)$ or $\overline{\mathrm{R}}(\mathcal{H}; c)$, respectively. We abbreviate $\overline{\mathrm{R}}_k(\mathcal{H}; 2)$ as $\overline{\mathrm{R}}_k(\mathcal{H})$, and $\overline{\mathrm{R}}(\mathcal{H}; 2)$ as $\overline{\mathrm{R}}(\mathcal{H})$. If a coloring f of an ordered hypergraph contains no monochromatic copy of $\mathcal{H}$, we say that f *avoids* $\mathcal{H}$.

Since $\mathcal{K}^k_n$ is uniquely determined up to isomorphism, we have $\overline{\mathrm{R}}_k(\mathcal{K}^k_{r_1}, \dots, \mathcal{K}^k_{r_c}) = \mathrm{R}_k(K^k_{r_1}, \dots, K^k_{r_c})$ for arbitrary positive integers $k, c, r_1, \dots, r_c$. Since every ordered k-uniform hypergraph on r vertices is an ordered subhypergraph of $\mathcal{K}^k_r$, we have $\overline{\mathrm{R}}_k(\mathcal{H}_1, \dots, \mathcal{H}_c) \le \overline{\mathrm{R}}_k(\mathcal{K}^k_{r_1}, \dots, \mathcal{K}^k_{r_c})$ where r_i is the number of vertices of $\mathcal{H}_i$. We have thus proved the following fact.

Observation 1.1. *For arbitrary positive integers c and k, let $\mathcal{H}_1 = (H_1, \prec_1), \dots, \mathcal{H}_c = (H_c, \prec_c)$ be an arbitrary collection of ordered k-uniform hypergraphs. Then we have*

$$\mathrm{R}_k(H_1, \dots, H_c) \le \overline{\mathrm{R}}_k(\mathcal{H}_1, \dots, \mathcal{H}_c) \le \mathrm{R}_k\left(K^k_{|V(H_1)|}, \dots, K^k_{|V(H_c)|}\right).$$

To study the asymptotic growth of ordered Ramsey numbers, we introduce *ordering schemes* for some classes of hypergraphs. An ordering scheme is a particular rule for ordering the vertices of the hypergraphs consistently in the whole class. For example, a k-uniform *monotone hyperpath* $(P^k_n, \lhd_{mon})$ is an ordered k-uniform hypergraph with vertices $v_1 \lhd_{mon} \cdots \lhd_{mon} v_n$ and $n - k + 1$ edges, each consisting of k consecutive vertices; see Figure 1.1 for an illustration. Throughout the thesis we use the symbol $\lhd$ instead of $\prec$ to emphasize the fact that the vertex ordering follows some ordering scheme.

For an ordered graph $(G, \prec)$, we say that a vertex v of G is to the *left* (*right*) of a subset U of vertices of G if v precedes (is preceded by, respectively) every vertex of U in $\prec$. More generally, for two subsets U and W of vertices of G, we say that U is *to the left of* W and W is *to the right of* U if every vertex of U

precedes every vertex of W in $\prec$. For an ordered graph $(G, \prec)$, we say that a subset I of vertices of G is an *interval* if for every pair of vertices u and v of I such that $u \prec v$, every vertex w of G satisfying $u \prec w \prec v$ is contained in I.

1.2 Ordered stars

A *star* on n vertices is the complete bipartite graph $K_{1,n-1}$. Ramsey numbers of unordered stars are known exactly [BR73] and they are given by

$$\mathrm{R}(K_{1,n-1}; c) = \begin{cases} c(n-2)+1 & \text{if } c \equiv n-1 \equiv 0 \pmod 2, \\ c(n-2)+2 & \text{otherwise.} \end{cases}$$

The position of the central vertex of an ordered star determines the ordering of the star uniquely up to isomorphism. We use $\mathcal{S}_{r,s}$ to denote the ordered star with $r-1$ vertices to the left and $s-1$ vertices to the right of the central vertex; see Figure 1.2.

For $c, r_1, \ldots, r_c \geq 2$, computing $\overline{\mathrm{R}}(\mathcal{S}_{1,r_1}, \ldots, \mathcal{S}_{1,r_c})$ is straightforward. In the diagonal case, the ordered Ramsey numbers $\overline{\mathrm{R}}(\mathcal{S}_{1,n}; c)$ are equal to the Ramsey numbers $\mathrm{R}(K_{1,n-1}; c)$ for every n and c, if n is even or c is odd.

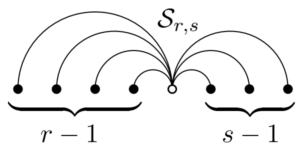


Figure 1.2: The ordered star $\mathcal{S}_{r,s}$.

Observation 1.2. *For all integers $c, r_1, \ldots, r_c \geq 2$, we have*

$$\overline{\mathrm{R}}(\mathcal{S}_{1,r_1}, \ldots, \mathcal{S}_{1,r_c}) = 2 + \sum_{i=1}^{c} (r_i - 2).$$

Proof. Let $\mathcal{K}_N$ be an ordered complete graph with $N \geq 2 + \sum_{i=1}^{c}(r_i - 2)$ vertices and edges colored with c colors. By the pigeonhole principle, for some $i \in [c]$, the leftmost vertex in $\mathcal{K}_N$ has at least $r_i - 1$ incident edges of color i. These edges form a copy of $\mathcal{S}_{1,r_i}$.

The following c-coloring of the edges of $\mathcal{K}_N$ with $N := 1 + \sum_{i=1}^{c}(r_i - 2)$ has no star $\mathcal{S}_{1,r_i}$ in color i. Partition all the vertices of $\mathcal{K}_N$ except for the leftmost vertex into c subsets $V_1, \ldots, V_c$ such that $|V_i| = r_i - 2$. Then color each edge with its right vertex in V_i by color i. $\square$

It is obvious that $\overline{\mathrm{R}}(\mathcal{S}_{1,r_1}, \mathcal{S}_{r_2,1}) = \overline{\mathrm{R}}(\mathcal{S}_{r_1,1}, \mathcal{S}_{1,r_2}) = \overline{\mathrm{R}}(\mathcal{S}_{1,r_2}, \mathcal{S}_{r_1,1})$ for all integers $r_1, r_2 \geq 2$ and that $\overline{\mathrm{R}}(\mathcal{S}_{1,2}, \mathcal{S}_{r,1}) = r$ for every integer $r \geq 2$. Choudum and Ponnusamy [CP02] determined the ordered Ramsey numbers of all pairs of ordered stars by the following recursive formulas.

Theorem 1.3 ([CP02])**.** *For all integers $r_2 \geq r_1 > 2$, we have*

$$\overline{\mathrm{R}}(\mathcal{S}_{1,r_1}, \mathcal{S}_{r_2,1}) = \left\lfloor \frac{-1 + \sqrt{1 + 8(r_1 - 2)(r_2 - 2)}}{2} \right\rfloor + r_1 + r_2 - 2.$$

Moreover, for all integers $r_1, r_2, s_1, s_2 \geq 2$, we have

$$\overline{\mathrm{R}}(\mathcal{S}_{1,r_1}, \mathcal{S}_{r_2,s_2}) = \overline{\mathrm{R}}(\mathcal{S}_{1,r_1}, \mathcal{S}_{r_2,1}) + r_1 + s_2 - 3$$

and

$$\overline{\mathrm{R}}(\mathcal{S}_{r_1,s_1}, \mathcal{S}_{r_2,s_2}) = \overline{\mathrm{R}}(\mathcal{S}_{r_1,1}, \mathcal{S}_{r_2,s_2}) + \overline{\mathrm{R}}(\mathcal{S}_{1,s_1}, \mathcal{S}_{r_2,s_2}) - 1.$$

We show that ordered Ramsey numbers of all ordered stars are linear with respect to the number of vertices and at most exponential with respect to the number of colors.

Theorem 1.4. *For all integers $c \geq 2$, $n_1, \dots, n_c \geq 3$, and for every collection of ordered stars $\mathcal{S}_1, \dots, \mathcal{S}_c$ where n_i is the number of vertices of $\mathcal{S}_i$, we have*

$$\overline{\mathrm{R}}(\mathcal{S}_1, \dots, \mathcal{S}_c) \leq 2^c + 2^{c+1} \cdot \sum_{i=1}^{c} (n_i - 3) < 2^{c+1} \cdot \sum_{i=1}^{c} n_i.$$

In the proof we use Turán's theorem, which we now state for convenience.

Theorem 1.5 ([Tur41]). *Let r be a positive integer and let G be a graph on n vertices that contains no copy of K_{r+1}. Then the number of edges in G is at most $\left(1 - \frac{1}{r}\right) \cdot \frac{n^2}{2}$.*

Proof of Theorem 1.4. For every $i \in [c]$, let r_i and s_i be positive integers such that $\mathcal{S}_i = \mathcal{S}_{r_i,s_i}$. Let N be a positive integer. Assume that the edges of $\mathcal{K}_N$ are colored by colors from $[c]$ so that for every $i \in [c]$, there is no $\mathcal{S}_{r_i,s_i}$ in color i. Thus, in the ordered subgraph $\mathcal{G}_i$ of $\mathcal{K}_N$ formed by the edges of color i, every vertex has at most $r_i - 2$ left neighbors or at most $s_i - 2$ right neighbors. Let $\mathcal{H}_i$ be the ordered subgraph of $\mathcal{G}_i$ obtained by deleting every edge incident from the left to a vertex with at most $r_i - 2$ left neighbors, and every edge incident from the right to a vertex with at most $s_i - 2$ right neighbors. Clearly, $|E(\mathcal{G}_i) \setminus E(\mathcal{H}_i)| \leq N \cdot (r_i + s_i - 4) = N \cdot (n_i - 3)$. It follows that the ordered graph $\mathcal{H} := \bigcup_{i=1}^{c} \mathcal{H}_i$ has at least $\binom{N}{2} - N \cdot \sum_{i=1}^{c} (n_i - 3) = N \cdot (N/2 - 1/2 - \sum_{i=1}^{c} (n_i - 3))$ edges.

By the construction, each of the ordered graphs $\mathcal{H}_i$ is bipartite. Hence, the ordered graph $\mathcal{H}$ is 2^c-partite (in other words, 2^c-colorable). Therefore, by Turán's theorem (Theorem 1.5), $|E(\mathcal{H})| \leq (1 - 1/2^c) \cdot N^2/2 = N \cdot (N/2 - N/2^{c+1})$. Putting the two estimates together, we obtain that $N/2^{c+1} \leq 1/2 + \sum_{i=1}^{c} (n_i - 3)$, from which the theorem follows. □

Additionally, we give a lower bound for ordered Ramsey numbers of ordered stars that have at least one edge incident to the central vertex from each side. For "symmetric" stars $\mathcal{S}_{r_i,r_i}$ with $r_i \geq 2$, the lower bound is within a constant multiplicative factor from the upper bound in Theorem 1.4.

Proposition 1.6. *For all integers $c \geq 2$ and $r_1, \dots, r_c, s_1, \dots, s_c \geq 2$, we have*

$$\overline{\mathrm{R}}(\mathcal{S}_{r_1,s_1}, \dots, \mathcal{S}_{r_c,s_c}) > 2^{c-1} \cdot \max\left(\max_{i \in [c]} \{r_i + s_i - 2\}, 2 + 2 \cdot \sum_{i=1}^{c} (\min(r_i, s_i) - 2) \right).$$

Proof. Let $a := \max_{i\in[c]}\{r_i + s_i - 2\}$, $b := 1 + \sum_{i=1}^{c}(\min(r_i, s_i) - 2)$, and $N_1 := \max(a, 2b)$. Without loss of generality, we assume that $a = r_1+s_1-2$. We start the construction with coloring the edges of $\mathcal{K}_{N_1}$. If $N_1 = a$, we color every edge of $\mathcal{K}_{N_1}$ by color 1. Now suppose that $N_1 = 2b$. For every $i \in [c]$, let $t_i := \min(r_i, s_i) - 2$, and let b_i be the partial sum $\sum_{j=1}^{i} t_j$. In particular, $b_c = b - 1$. Let $b_0 := 0$ and let $v_1, v_2, \ldots, v_{N_1}$ be the vertices of $\mathcal{K}_{N_1}$ from left to right. For every $k, l \in [b]$, $k < l$, color the edge $\{v_k, v_l\}$ by color i if $b_{i-1} < l - k \le b_i$. In this coloring of the subgraph $\mathcal{K}_b = \mathcal{K}_{N_1}[v_1, \ldots, v_b]$, every vertex has at most t_i left neighbors and at most t_i right neighbors joined by an edge of color i. Color the subgraph $\mathcal{K}_{N_1}[v_{b+1}, \ldots, v_{2b}]$ analogously as $\mathcal{K}_b$, and finally, color every edge $\{v_i, v_j\}$ with $i \le b < j$ by color 1.

For $i \in \{2, 3, \ldots, c\}$, let $N_i := 2^{i-1} \cdot N_1$. We color the graphs $\mathcal{K}_{N_i}$ by induction on i. Once $\mathcal{K}_{N_i}$ is colored, we split the vertices $v_1, \ldots, v_{N_{i+1}}$ of $\mathcal{K}_{N_{i+1}}$ into two intervals of length N_i, and color the subgraph induced by each of the two intervals using the coloring of $\mathcal{K}_{N_i}$. Then we color every edge between the two intervals by color i.

It remains to verify that there is no monochromatic copy of $\mathcal{S}_{r_i,s_i}$ in the resulting coloring of $\mathcal{K}_{N_c}$. In the case $N_1 = a$, every vertex has at most $r_1 + s_1 - 3$ neighbors joined by an edge of color 1 and for every other $i \in [c]$, it either has no left or no right neighbors in color i. In the case $N_1 = 2b$, for every $i \in [c]$, every vertex has at most $r_i - 2$ left neighbors or at most $s_i - 2$ right neighbors in color i. $\square$

1.3 Ordered paths

Gerencsér and Gyárfás [GG67] determined the exact values for the Ramsey numbers $\mathrm{R}(P_r, P_s)$ of two paths P_r and P_s.

Theorem 1.7 ([GG67]). *For $2 \le r \le s$, we have $\mathrm{R}(P_r, P_s) = s + \lfloor \frac{r}{2} \rfloor - 1$.*

Perhaps the most natural ordering of a path is the monotone path. We recall that the *monotone path* $(P_n, \lhd_{mon})$ is an ordered graph with vertices $v_1 \lhd_{mon} \cdots \lhd_{mon} v_n$ and $n - 1$ edges, each consisting of a pair of consecutive vertices.

The following result of Choudum and Ponnusamy [CP02] gives the exact formula for the ordered Ramsey numbers of monotone paths and it is closely related to the famous Erdős–Szekeres lemma on monotone subsequences; see Section 3.1. Milans, Stolee, and West [MSW15] gave the following proof in the language of ordered Ramsey theory. Their proof is based on a simple and elegant proof of the Erdős–Szekeres lemma by Seidenberg [Sei59].

Proposition 1.8 ([CP02, MSW15]). *For all positive integers $c, r_1, \ldots, r_c$, we have*

$$\overline{\mathrm{R}}((P_{r_1}, \lhd_{mon}), \ldots, (P_{r_c}, \lhd_{mon})) = 1 + \prod_{i=1}^{c}(r_i - 1).$$

Proof. Let $N := \prod_{i=1}^{c}(r_i - 1)$. For the lower bound, we let the vertices of $\mathcal{K}_N$ be c-tuples $(t_1, \ldots, t_c)$ with $t_i \in [r_i - 1]$ for every $i \in [c]$ and we order the vertices of $\mathcal{K}_N$ lexicographically. We color an edge $\{(t_1, \ldots, t_c), (s_1, \ldots, s_c)\}$ of $\mathcal{K}_N$ by $\min\{i \in [c] : t_i \ne s_i\}$. That is, by the first coordinate where the two vertices differ.

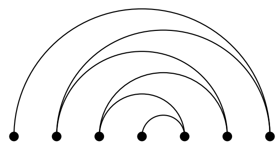


Figure 1.3: The alternating path $(P_7, \triangleleft_{alt})$.

For every $i \in [c]$, there is no monotone path on r_i vertices with edges of color i in this coloring of $\mathcal{K}_N$, as every edge of a monotone path of color i increases the ith coordinate, which is always smaller than r_i.

For the upper bound, we show that every coloring of $\mathcal{K}_{N+1} = (K_{N+1}, \prec)$ contains a monotone path on at least r_i vertices with edges of color i for some $i \in [c]$. For a vertex v of $\mathcal{K}_{N+1}$ and $i \in [c]$, let v_i be the number of vertices of the longest monotone path that has all edges of color i and that ends in v.

If i is the color of an edge $\{u, v\}$ with $u \prec v$, then we extend the longest monotone path of color i that ends in u by $\{u, v\}$ and obtain $u_i < v_i$. Thus there are no two vertices $u \prec v$ of $\mathcal{K}_{N+1}$ such that $(u_1, \dots, u_c) = (v_1, \dots, v_c)$. By the pigeonhole principle, there is a vertex v of $\mathcal{K}_{N+1}$ and $i \in [c]$ such that the ith coordinate of $(v_1, \dots, v_c)$ is at least r_i. This gives a monotone path of color i on at least r_i vertices in $\mathcal{K}_{N+1}$. □

In particular, Proposition 1.8 shows that $\overline{\mathrm{R}}((P_n, \triangleleft_{mon}))$ is quadratic in n. In contrast, we show a family of ordered paths whose ordered Ramsey numbers are linear in the number of vertices.

Let $v_1, \dots, v_n$ be the vertices of P_n in the order as they appear along the path. The *alternating path* $(P_n, \triangleleft_{alt})$ is an ordered path where $v_1 \triangleleft_{alt} v_3 \triangleleft_{alt} v_5 \triangleleft_{alt} \cdots \triangleleft_{alt} v_n \triangleleft_{alt} v_{n-1} \triangleleft_{alt} v_{n-3} \triangleleft_{alt} \cdots \triangleleft_{alt} v_2$ for n odd and $v_1 \triangleleft_{alt} v_3 \triangleleft_{alt} v_5 \triangleleft_{alt} \cdots \triangleleft_{alt} v_{n-1} \triangleleft_{alt} v_n \triangleleft_{alt} v_{n-2} \triangleleft_{alt} \cdots \triangleleft_{alt} v_2$ for n even; see Figure 1.3.

Proposition 1.9. *For every positive integer n, we have*

$$5\lfloor n/2 \rfloor - 4 \le \overline{\mathrm{R}}((P_n, \triangleleft_{alt})) \le 5n.$$

Moreover, the following Turán-type result is true. If $\varepsilon > 0$ is a real constant, then every ordered graph on $N \ge n/\varepsilon$ vertices with at least εN^2 edges contains $(P_n, \triangleleft_{alt})$ as an ordered subgraph.

Proof. For the lower bound on $\overline{\mathrm{R}}((P_n, \triangleleft_{alt}))$, we assume that $n \ge 4$ and we let $r := \lfloor n/2 \rfloor - 1$ and $N := 5r$. We use an upper triangular $\{0, 1\}$-matrix $A = (a_{i,j})_{i,j=1}^N$ to represent a red-blue coloring c of $\mathcal{K}_N$ that avoids $(P_n, \triangleleft_{alt})$. The construction of A is sketched in part (a) of Figure 1.4. For $1 \le i < j \le N$, the edge $\{i, j\}$ of $\mathcal{K}_N$ is blue in c if $A_{i,j} = 0$ and red in c if $A_{i,j} = 1$. For integers k and l with $1 \le k \le l \le 5$, we use $B_{k,l}$ to denote the $r \times r$ blocks that partition A. The block $B_{k,l}$ contains entries $a_{i,j}$ with $(k-1)r + 1 \le i \le kr$ and $(l-1)r + 1 \le j \le lr$. There are two types of blocks. *Red blocks*, containing only 1-entries, and *blue blocks*, containing only 0-entries. The blocks $B_{1,5}, B_{2,2}, B_{2,3}, B_{3,3}, B_{3,4}, B_{4,4}$ are red and all the other blocks are blue.

Suppose for contradiction that c contains a monochromatic copy $\mathcal{P}$ of $(P_n, \triangleleft_{alt})$. Then there are entries $a_{i_1,j_1} = \cdots = a_{i_{n-1},j_{n-1}}$ in A such that, for $t = 1, \dots, n-2$, we have $i_t < i_{t+1}$ and $j_t = j_{t+1}$ if t is odd and $i_t = i_{t+1}$ and $j_t > j_{t+1}$ if t is even.

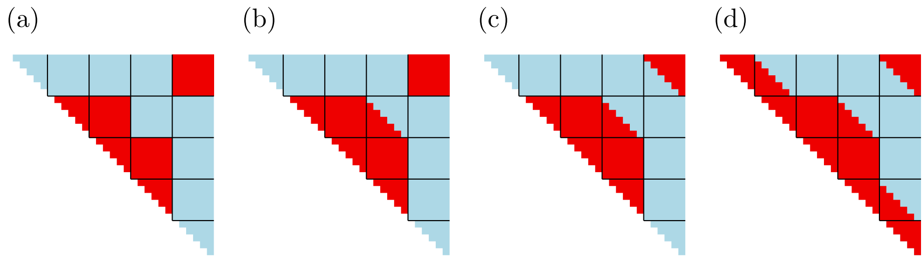


Figure 1.4: (a)–(d) Various constructions for the lower bound on $\overline{\mathrm{R}}((P_n, \lhd_{alt}))$. The red entries represent 1, the blue entries represent 0.

Every block $B_{k,l}$ has at most r rows and at most r columns of A, but $\mathcal{P}$ spans at least $r+1$ rows and $r+1$ columns. Therefore there are three blocks B_1, B_2, B_3 of the same color that satisfy one of the following two conditions. Either the block B_1 is above B_2 and B_3 is to the left of B_2 or the block B_1 is to the right of B_2 and B_3 is below B_2. However, the matrix A contains no such triples of blocks, a contradiction.

Some alternative constructions of A are illustrated in parts (b), (c), and (d) of Figure 1.4.

Now, we prove the Turán-type statement from the second part of the proposition. The upper bound on $\overline{\mathrm{R}}((P_n, \lhd_{alt}))$ then follows, as, for $N := 5n \geq 5$ and $\varepsilon := 1/5$, there are at least $\frac{1}{2}\binom{N}{2} \geq N^2/5 = \varepsilon N^2$ edges of the same color in every 2-coloring of $\mathcal{K}_N$.

For a given $\varepsilon > 0$, let $\mathcal{H} = (H, \prec)$ be an ordered graph on $N \geq n/\varepsilon$ vertices with at least εN^2 edges. Without loss of generality, we assume that the vertex set of $\mathcal{H}$ is $[N]$. We also assume $n \geq 3$, otherwise the statement is trivial.

For a vertex v of $\mathcal{H}$, the *leftmost neighbor of* v in $\mathcal{H}$ is the minimum from $\{u \in V(H) \colon u \prec v, \{u, v\} \in E(H)\}$, if it exists. The *rightmost neighbor of* v in $\mathcal{H}$ is the maximum from $\{u \in V(H) \colon v \prec u, \{u, v\} \in E(H)\}$, if it exists.

We consider the following process of removing edges of $\mathcal{H}$ that proceeds in steps $1, \ldots, n-2$. In every odd step of the process, we remove edges $\{u, v\}$ for every vertex v of $\mathcal{H}$ such that u is the leftmost neighbor of v (if it exists). In every even step, we remove edges $\{u, v\}$ for every vertex v of $\mathcal{H}$ such that u is the rightmost neighbor of v (if it exists). Clearly, we remove at most N edges of $\mathcal{H}$ in every step. In total, we remove at most $(n-2)N$ edges of $\mathcal{H}$ once the process is finished.

From the choice of N, we have $\varepsilon N^2 \geq nN > (n-2)N$ and thus there is at least one edge $\{v_{n-1}, v_n\}$ of $\mathcal{H}$ that we did not remove. Without loss of generality we assume that $v_n \prec v_{n-1}$ for n odd and $v_{n-1} \prec v_n$ for n even.

We now follow the process of removing the edges of $\mathcal{H}$ backwards and we construct the alternating path $(P_n, \lhd_{alt})$ on vertices $v_1, \ldots, v_n$. For $i = n-2, \ldots, 1$, we let v_i be the vertex that was removed in the ith step of the removing process as a neighbor of v_{i+1}. If there is no such vertex v_i, then we would remove the edge $\{v_{i+1}, v_{i+2}\}$ in the ith step of the removing process, which is impossible. Consequently, the construction of $(P_n, \lhd_{alt})$ stops at v_1 and we obtain an alternating path on n vertices as an ordered subgraph of $\mathcal{H}$. $\square$

We do not know the precise multiplicative factor in $\overline{\mathrm{R}}((P_n, \lhd_{alt}))$. A stronger upper bound was obtained by Balko et al. [BCKK13] who showed

$$\overline{\mathrm{R}}((P_n, \lhd_{alt})) \leq 2n - 3 + \sqrt{2n^2 - 8n + 11}$$

by applying a result from extremal theory of $\{0,1\}$-matrices. Our computer experiments [BCKK] indicate that $\overline{\mathrm{R}}(P_n, \lhd_{alt})$ could be of the form $\lfloor (n-2)\frac{1+\sqrt{5}}{2} \rfloor + n$; see Table 1.1. In our experiments we used the Glucose SAT solver [AS13].

n	2	3	4	5	6	7	8	9	10	11	12	13
$\overline{\mathrm{R}}((P_n, \lhd_{alt}))$	2	4	7	9	12	15	17	≥ 20	≥ 22	≥ 25	≥ 28	≥ 30

Table 1.1: Estimates of the ordered Ramsey numbers $\overline{\mathrm{R}}((P_n, \lhd_{alt}))$ for $n \leq 13$.

For general ordered paths, Cibulka et al. [CGK+15] showed that for every ordered path $\mathcal{P}_r$ and the ordered complete graph $\mathcal{K}_s$ we have

$$\overline{\mathrm{R}}(\mathcal{P}_r, \mathcal{K}_s) \leq 2^{\lceil \log s \rceil (\lceil \log r \rceil + 1)}.$$

That is, for every ordered path $\mathcal{P}_n$ we have $\overline{\mathrm{R}}(\mathcal{P}_n) \leq n^{O(\log n)}$. This bound also follows from a result of Conlon et al. [CFLS14] (Theorem 2.9).

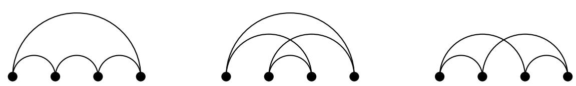
The best known lower bound on $\overline{\mathrm{R}}(\mathcal{P}_n)$ comes from Theorem 2.1, which implies that there are arbitrarily long ordered paths $\mathcal{P}_n$ on n vertices such that $\overline{\mathrm{R}}(\mathcal{P}_n) \geq n^{\log n / (5 \log \log n)}$.

1.4 Ordered cycles

It is a folklore result in Ramsey theory that $\mathrm{R}(C_3) = \mathrm{R}(C_4) = 6$ [CH72]. The first results on Ramsey numbers of cycles were obtained by Chartrand and Chuster [CS71] and by Bondy and Erdős [BE73]. These were later extended by Rosta [Ros73] and by Faudree and Schelp [FS74]. Together, these results give exact formulas for Ramsey numbers of cycles in the two-color case. For all integers $r, s \geq 3$, we have

$$\mathrm{R}(C_r, C_s) = \begin{cases} 2r - 1 & \text{if } (r,s) \neq (3,3), r \geq s \geq 3, \text{ and } s \text{ is odd}, \\ \frac{2r+s}{2} - 1 & \text{if } (r,s) \neq (4,4), r \geq s \geq 4, \text{ and } r, s \text{ are even}, \\ \max\{\frac{2r+s}{2}, 2s\} - 1 & \text{if } r > s \geq 4, s \text{ is even and } r \text{ is odd}. \end{cases}$$

The smallest cycle whose ordered Ramsey numbers are nontrivial to determine is C_4. There are three pairwise nonisomorphic orderings of C_4; see Figure 1.5.



$(C_4, \prec_A)$ $(C_4, \prec_B)$ $(C_4, \prec_C)$

Figure 1.5: Three possible orderings of C_4.

We determine the ordered Ramsey number of each of the three orderings of C_4 from Figure 1.5.

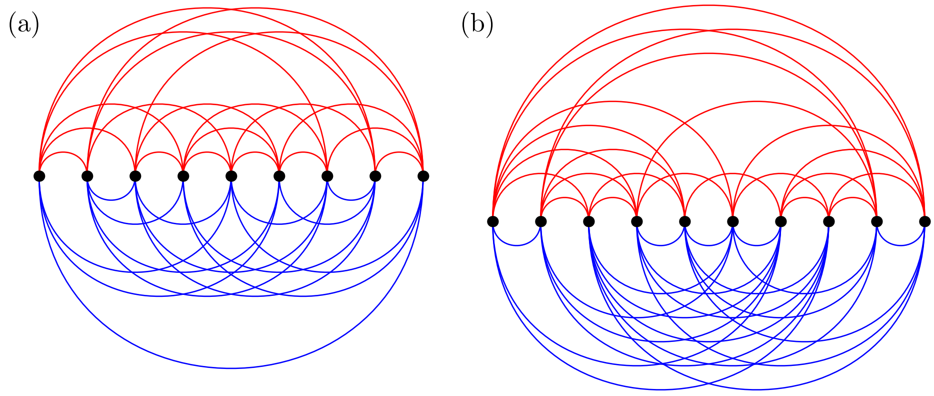


Figure 1.6: Colorings of $\mathcal{K}_9$ and $\mathcal{K}_{10}$ with no monochromatic copy of $(C_4, \prec_B)$ and $(C_4, \prec_C)$, respectively.

Proposition 1.10. *We have*

1) $\overline{\mathrm{R}}((C_4, \prec_A)) = 14$,
2) $\overline{\mathrm{R}}((C_4, \prec_B)) = 10$,
3) $\overline{\mathrm{R}}((C_4, \prec_C)) = 11$.

Proof. The colorings showing the lower bounds can be found in Figures 1.6 and 1.8. These colorings were found using a computer experiment and all supplementary data can be also found on a separate webpage [BCKK]. We now show the upper bounds.

1) This result follows from Theorem 1.11, which is proved later in this section.

2) Consider $(K_{10}, \prec)$ with vertices $v_1 \prec v_2 \prec \cdots \prec v_{10}$ and edges colored red and blue. We put each of the vertices $\{v_4, v_5, \ldots, v_{10}\}$ into one of the following six classes. Class (i, c), where $i \in \{1, 2, 3\}$ and $c \in \{\text{red}, \text{blue}\}$, contains vertices connected by an edge of color c to both vertices in $\{v_1, v_2, v_3\} \setminus \{v_i\}$. Note that each of the seven vertices $\{v_4, v_5, \ldots, v_{10}\}$ is in one or three of these classes. Consequently, one of the six classes contains at least two vertices. Thus there are two vertices that share two left neighbors of the same color, which implies a monochromatic copy of $(C_4, \prec_B)$.

3) Exhaustive computer search showed that $\overline{\mathrm{R}}((C_4, \prec_C)) = 11$ [BCKK]. Here we prove a weaker upper bound $\overline{\mathrm{R}}((C_4, \prec_C)) \leq 13$.

   Consider $(K_{13}, \prec)$ with vertices $v_1 \prec \cdots \prec v_{13}$, edges colored red and blue, and no monochromatic $(C_4, \prec_C)$. Without loss of generality, v_1 has six red neighbors among $\{v_2, v_3, \ldots, v_{12}\}$. If v_1 and v_{13} had two common red neighbors then they would form a red copy of $(C_4, \prec_C)$. Thus there is a set $R \subseteq \{v_2, v_3, \ldots, v_{12}\}$ of at least five vertices such that each of them is adjacent to v_1 by a red edge and to v_{13} by a blue edge. By Theorem 1.3 we have $\overline{\mathrm{R}}(\mathcal{S}_{1,3}, \mathcal{S}_{3,1}) = 5$. Therefore the complete graph formed by the five vertices of R contains either a vertex with at least two red edges incident from the left or a vertex with at least two blue edges incident from the right. In both cases we obtain a monochromatic copy of $(C_4, \prec_C)$. □

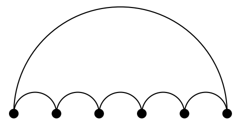


Figure 1.7: The monotone cycle $(C_6, \lhd_{mon})$.

For the rest of the section, we consider a particular family of ordered cycles. A *monotone cycle* $(C_n, \lhd_{mon})$ on n vertices consists of a monotone path with vertices $v_1 \lhd_{mon} \cdots \lhd_{mon} v_n$ and the edge $\{v_1, v_n\}$; see Figure 1.7.

We now show an exact formula for ordered Ramsey numbers of monotone cycles.

Theorem 1.11. *For all integers $r \geq 2$ and $s \geq 2$, we have*

$$\overline{\mathrm{R}}((C_r, \lhd_{mon}), (C_s, \lhd_{mon})) = 2rs - 3r - 3s + 6.$$

In the proof of Theorem 1.11, we use the following simple lemma, which is implicitly proved in [KPT97]. We include its proof for completeness.

Lemma 1.12 ([KPT97])**.** *For positive integers r and s, we have*

$$\overline{\mathrm{R}}((P_r, \lhd_{mon}), \mathcal{K}_s) = \overline{\mathrm{R}}((P_r, \lhd_{mon}), (P_s, \lhd_{mon})) = (r-1)(s-1) + 1.$$

Proof. The lower bound $(r-1)(s-1)+1 \leq \overline{\mathrm{R}}((P_r, \lhd_{mon}), (P_s, \lhd_{mon}))$ follows from Proposition 1.8. For the upper bound $\overline{\mathrm{R}}((P_r, \lhd_{mon}), \mathcal{K}_s) \leq (r-1)(s-1)+1$, we apply induction on r. Let $\mathcal{G}$ be an ordered complete graph with $(r-1)(s-1)+1$ vertices and with edges colored red and blue. The statement is true for $r = 2$, since either $\mathcal{G}$ is a blue copy of $\mathcal{K}_s$ or $\mathcal{G}$ has a red edge. Let $r \geq 3$. By the induction hypothesis, $\mathcal{G}$ has either a blue copy of $\mathcal{K}_s$ or at least $(r-1)(s-1)+1-(r-2)(s-1) = s$ distinct vertices that are the rightmost vertices of a red copy of $(P_{r-1}, \lhd_{mon})$. Either every edge between these vertices is blue, which gives a blue copy of $\mathcal{K}_s$, or a red edge extends one of the red paths $(P_{r-1}, \lhd_{mon})$ to a red copy of $(P_r, \lhd_{mon})$. □

Proof of Theorem 1.11. The upper bound was proved by Károlyi et al. [KPTV98, Theorem 2.1]. We include the proof here for completeness.

Let $\mathcal{G}$ be an ordered complete graph with $N := 2rs - 3r - 3s + 6$ vertices and with edges colored red and blue. The leftmost vertex, v_1, has either at least $(r-2)(s-1)+1$ red neighbors or at least $(r-1)(s-2)+1$ blue neighbors. In the first case, by Lemma 1.12, $\mathcal{G}$ has a red copy of $(P_{r-1}, \lhd_{mon})$ that forms a red copy of $(C_r, \lhd_{mon})$ together with v_1, or a blue copy of $(C_s, \lhd_{mon})$. The second case is symmetric.

Now we prove the lower bound. Let $N := 2rs - 3r - 3s + 5$. We construct a coloring of $\mathcal{K}_N = (K_N, \prec)$ that avoids a red copy of $(C_r, \lhd_{mon})$ and a blue copy of $(C_s, \lhd_{mon})$. See Figure 1.8 for an example of such coloring for $r = s = 4$. We partition the vertices of $\mathcal{K}_N$ into disjoint intervals $I_1, \ldots, I_{2r-3}$, from left to right. For r odd, the $(r-1)/2$ leftmost and $(r-1)/2$ rightmost intervals are of size $s-1$ and the remaining $r-2$ intervals are of size $s-2$. For r even, the $(r-2)/2$ leftmost and $(r-2)/2$ rightmost intervals are of size $s-2$ and the remaining $r-1$ intervals are of size $s-1$. In both cases we have N vertices in total.

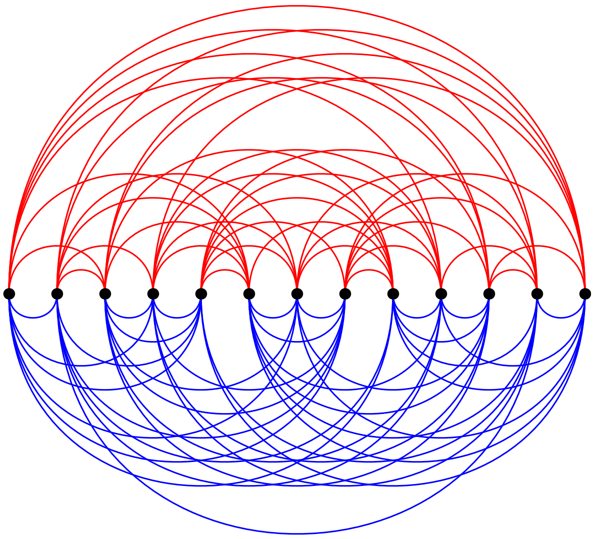


Figure 1.8: A coloring of $\mathcal{K}_{13}$ with no monochromatic monotone cycle of length 4.

We call the intervals of size $s-1$ *long* and the intervals of size $s-2$ *short*. We label the vertices of I_i as $v_1^i, v_2^i, \ldots, v_{|I_i|}^i$ from left to right. We call the index j the *index of the vertex* v_j^i.

The coloring of the edges of $\mathcal{K}_N$ is defined as follows. For every $i \in [2r-3]$, we color all the edges among the vertices of I_i blue. We define four types of edges with vertices in different intervals. The type of an edge is determined by the pair of intervals containing its vertices. The color of an edge is determined by its type and by the relative value of the indices of its vertices. We say that an edge $e = \{v_k^i, v_l^j\}$ between intervals I_i and I_j with $i < j$ is of *type*

- $T_<$ if $j - i \leq r-2$ and $|I_i| \leq |I_j|$. In this case we color e blue if $k < l$ and red otherwise.
- $T_\geq$ if $j - i > r-2$ and $|I_i| < |I_j|$. In this case we color e blue if $k \geq l$ and red otherwise.
- $T_>$ if $j - i > r-2$ and $|I_i| \geq |I_j|$. In this case we color e blue if $k > l$ and red otherwise.
- $T_\leq$ if $j - i \leq r-2$ and $|I_i| > |I_j|$. In this case we color e blue if $k \leq l$ and red otherwise.

The definition of the types and the distribution of the types in $\mathcal{K}_N$ are illustrated in Figures 1.9 and 1.10, respectively.

The distribution of long and short intervals implies the following claim.

Claim 1.13. *Every monochromatic monotone path $\mathcal{P}$ in the constructed coloring of $\mathcal{K}_N$ contains at most one edge $\{u, v\}$, $u \prec v$, with u in a long interval and v in a short interval, and at most one edge $\{u', v'\}$, $u' \prec v'$, with u' in a short interval and v' in a long interval.* □

First we show that our coloring of $\mathcal{K}_N$ contains no red copy of $(C_r, \lhd_{mon})$. Suppose for contradiction that there is such a copy $\mathcal{C}$. Let $\mathcal{P}$ be the monotone path on r vertices contained in $\mathcal{C}$. Let u be the leftmost vertex of $\mathcal{P}$ and v the rightmost

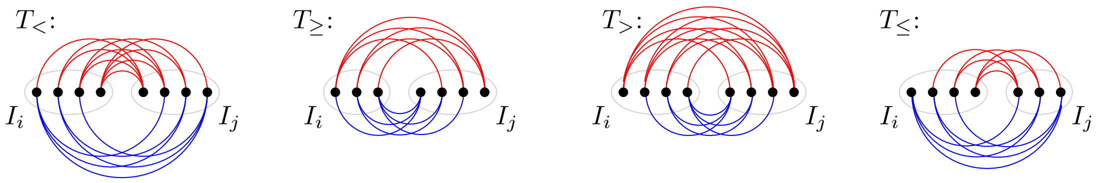


Figure 1.9: The types of pairs (I_i, I_j) for $s = 5$ and colorings of corresponding edges.

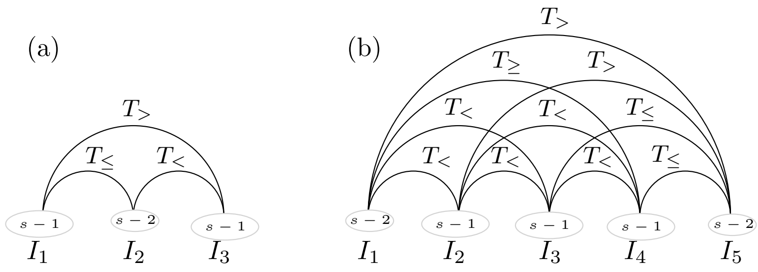


Figure 1.10: Distribution of types of pairs (I_i, I_j) in $\mathcal{K}_N$ for (a) $r = 3$ and (b) $r = 4$.

vertex of $\mathcal{P}$. The edge $\{u, v\}$ is thus the longest edge of $\mathcal{C}$. Note that $\mathcal{C}$ contains at most one vertex from each interval I_i, as every interval contains only blue edges. The path $\mathcal{P}$ contains no edge of type $T_>$ or $T_\geq$, since otherwise $\mathcal{P}$ would skip vertices from at least $r-2$ intervals, leaving at most $2r - 3 - (r-2) = r-1$ intervals. Hence the vertex indices in $\mathcal{P}$ are nonincreasing from left to right, as $\mathcal{P}$ uses red edges of types $T_<$ and $T_\leq$ only.

Since the edge $\{u, v\}$ skips at least $r-2$ intervals, it is of type $T_>$ or $T_\geq$, and thus the index of v is at least as large as the index of u. In combination with the previous observation, this implies that the indices of u and v are equal. Consequently, $\{u, v\}$ is of type $T_>$, and every edge of $\mathcal{P}$ is of type $T_<$. Since there are at most $r-1$ long intervals and at most $r-1$ short intervals, the path $\mathcal{P}$ contains at least one vertex from a long interval and at least one vertex from a short interval. Since every edge of $\mathcal{P}$ is of type $T_<$, this implies that u is in a short interval and v is in a long interval. This is a contradiction since $\{u, v\}$ is of type $T_>$.

Now we show that our coloring of $\mathcal{K}_N$ contains no blue copy of $(C_s, \lhd_{mon})$. Suppose for contradiction that there is such a copy $\mathcal{C}$. Let $\mathcal{P}$ be the monotone path on s vertices contained in $\mathcal{C}$. Let u be the leftmost vertex of $\mathcal{P}$ and v the rightmost vertex of $\mathcal{P}$. This time, $\mathcal{C}$ can contain edges between vertices from the same interval. However, u and v belong to different intervals, as no interval contains s vertices. We distinguish a few cases.

1) First, assume that $\mathcal{P}$ contains only edges with both vertices in the same interval, edges of type $T_<$, and edges of type $T_\leq$. Then the vertex indices along $\mathcal{P}$ are nondecreasing from left to right. By Claim 1.13, at most one edge of $\mathcal{P}$ is of type $T_\leq$. Thus there is at most one edge of $\mathcal{P}$ between vertices with the same vertex index. Since every vertex has index at most $s-1$, we see that $\mathcal{P}$ has exactly one edge of type $T_\leq$ and that the index of v is $s-1$. In particular, v is in a long interval. This implies that from left to right, $\mathcal{P}$ visits a long, a short, and a long interval, in this order. The

distribution of short and long intervals implies that r is odd, u is in a long interval I_i, v is in a long interval I_j, and $j - i > r - 2$. This further implies that $\{u, v\}$ is of type $T_>$, but this contradicts the fact that the index of u is 1 and the index of v is $s - 1$.

2) In the remaining case, $\mathcal{P}$ has an edge f between intervals I_i and I_j with $j - i > r - 2$. There is exactly one such edge since the total number of intervals is $2r - 3$. Every other edge of $\mathcal{P}$ is of type $T_<$, or of type $T_\leq$, or has both vertices in the same interval. Since $e = \{u, v\}$ is longer than f, it is of type $T_>$ or $T_\geq$. Therefore the index of u is larger than or equal to the index of v. Let x be the left vertex of f and y the right vertex of f. Let $\mathcal{P}_1$ be the subpath of $\mathcal{P}$ with endpoints u and x, and let $\mathcal{P}_2$ be the subpath of $\mathcal{P}$ with endpoints y and v.

   (a) Suppose that $\mathcal{P}$ has no edge of type $T_\leq$. Then the indices of vertices in both paths $\mathcal{P}_1$ and $\mathcal{P}_2$ are strictly increasing from left to right. Since $\mathcal{P}_1$ and $\mathcal{P}_2$ have $s - 2$ edges in total, it follows that the index of u is equal to the index of v, the index of y is 1 and the index of x is $s - 1$. In particular, x is in a long interval and e is of type $T_\geq$. This implies that u is in a short interval and v is in a long interval, but this is in contradiction with the distribution of long and short intervals.

   (b) Suppose that $\mathcal{P}$ has an edge g of type $T_\leq$. By Claim 1.13, there is exactly one such edge. Since g goes from a long interval to a short interval, the edge e cannot go from a short interval to a long interval, by the distribution of long and short intervals. Thus e is of type $T_>$. Consequently, the index of u is larger than the index of v. The indices of vertices in both paths $\mathcal{P}_1$ and $\mathcal{P}_2$ are strictly increasing from left to right, with the exception of the edge g, whose vertices can have equal indices. It follows that the index of x is $s - 1$. Consequently, x is in a long interval and so f is of type $T_>$. This is a contradiction, since $\mathcal{P}$ cannot have an edge of type $T_>$ together with an edge of type $T_\leq$, by the distribution of long and short intervals. This finishes the proof that our coloring of $\mathcal{K}_N$ contains no blue copy of $(C_s, \lhd_{mon})$, and the proof of Theorem 1.11. □

Note that we have proved a slightly stronger statement: in our coloring of $\mathcal{K}_N$, there is no red monotone cycle of length at least r and no blue monotone cycle of length at least s.

As a consequence of Theorem 1.11, we obtain tight bounds for geometric and convex geometric Ramsey numbers of cycles introduced by Károlyi et al. [KPT97, KPTV98]; see Section 3.4.

2. Growth rate of ordered Ramsey numbers

We are interested in the effects of vertex orderings on the ordered Ramsey numbers of various classes of graphs. We have already seen that the Ramsey number of a graph and the ordered Ramsey number of its ordering can be asymptotically different: for example, $\overline{\mathrm{R}}((P_n, \lhd_{mon}))$ is quadratic in n (Proposition 1.8) while $\mathrm{R}(P_n)$ is linear (Theorem 1.7).

The gap is much wider for hypergraphs. Let t_h denote the *tower function of height h* defined by $t_1(x) = x$ and $t_h(x) = 2^{t_{h-1}(x)}$ for $h \geq 2$. It is known that for all positive integers Δ and k, there exists a constant $C(\Delta, k)$ such that if H is a k-uniform hypergraph with n vertices and maximum degree Δ, then $\mathrm{R}_k(H) \leq C(\Delta, k) \cdot n$ [CFS09]. On the other hand, Moshkovitz and Shapira [MS14] showed that for every $k \geq 3$ we have $\overline{\mathrm{R}}_k((P_n^k, \lhd_{mon})) = t_{k-1}(2n - o(n))$.

Using a standard probabilistic argument, one can show that there is a constant $c > 0$ such that the Ramsey number $\mathrm{R}(G)$ of every graph G with n vertices and $n^{1+\varepsilon}$ edges is at least $2^{cn^{\varepsilon}}$. On the other hand, it is well-known that if G is an n-vertex graph of bounded maximum degree, then $\mathrm{R}(G)$ is linear in n [CVSTJ83].

In a sharp contrast to the latter fact, we construct ordered matchings whose ordered Ramsey numbers grow superpolynomially (Theorem 2.1). Here, an *ordered matching* is an ordered graph whose underlying graph is 1-regular. We also improve a lower bound by Conlon et al. [CFLS14] for ordered Ramsey numbers of ordered matchings with interval chromatic number 2 (Theorem 2.3). That is, for ordered matchings whose vertex set can be partitioned into two intervals that induce independent sets.

We give polynomial upper bounds on ordered Ramsey numbers for two classes of sparse ordered graphs. Namely, for ordered graphs with bounded degeneracy and interval chromatic number (Corollary 2.8) and for ordered graphs of bounded bandwidth (Corollary 2.12). The latter result solves an open problem of Conlon, Fox, Lee, and Sudakov [CFLS14, Problem 6.9].

We show that there are 3-regular graphs that have superlinear ordered Ramsey numbers, regardless of the ordering (Theorem 2.19), solving a problem of Conlon, Fox, Lee, and Sudakov [CFLS14, Problem 6.7]. On the other hand, we prove that every graph G on n vertices with maximum degree 2 admits an ordering $\mathcal{G}$ of G such that $\overline{\mathrm{R}}(\mathcal{G})$ is linear in n (Theorem 2.25).

We conclude the chapter by mentioning some open problems about ordered Ramsey numbers.

2.1 Lower bounds

The following result shows that ordered Ramsey numbers can grow superpolynomially even if the underlying graph is a matching. As we will see later in Section 2.2, this bound is almost tight for ordered graphs of bounded degeneracy.

We say that pairwise disjoint intervals $I_1, \ldots, I_m$ that partition the vertex set of an ordered graph $(G, \prec)$ are *consecutive*, if $u \prec v$ for all $u \in I_i$, $v \in I_{i+1}$, and $i \in [m-1]$.

Theorem 2.1. *There are arbitrarily large ordered matchings $\mathcal{M}$ on n vertices such that*

$$\overline{\mathrm{R}}(\mathcal{M}) \geq n^{\frac{\log n}{5 \log\log n}}.$$

Proof. Let $r \geq 3$ and let $R_r := \mathrm{R}(K_r) - 1$. We construct a sequence of ordered matchings $\mathcal{M}_{r,k}$, $k \geq 1$, with $n_{r,k}$ vertices and a sequence of 2-colorings $c_{r,k}$ of ordered complete graphs $\mathcal{K}_{N_{r,k}}$ such that $c_{r,k}$ avoids $\mathcal{M}_{r,k}$. Then we choose $k(r)$ so that $n_{r,k(r)}$ is roughly exponential in r. This will imply that $N_{r,k(r)}$ is superpolynomial in $n_{r,k(r)}$ when r grows to infinity.

First we show an inductive construction of the colorings $c_{r,k}$. Let $N_{r,1} := R_r$ and let $c_{r,1}$ be a 2-coloring of $\mathcal{K}_{N_{r,1}}$ avoiding $\mathcal{K}_r$. Let $k \geq 1$ and suppose that a coloring $c_{r,k}$ of $\mathcal{K}_{N_{r,k}}$ has been constructed. Let $N_{r,k+1} := R_r \cdot N_{r,k}$. Partition the vertex set of $\mathcal{K}_{N_{r,k+1}}$ into R_r disjoint consecutive intervals $I_1, I_2, \dots, I_{R_r}$, each of size $N_{r,k}$. Color the complete subgraph induced by each I_i by $c_{r,k}$. The remaining edges of $\mathcal{K}_{N_{r,k+1}}$ form a complete R_r-partite ordered graph $\mathcal{F}_{r,k+1}$, which can be colored to avoid $\mathcal{K}_r$ in the following way. Suppose that $v_1, v_2, \dots, v_{R_r}$ are the vertices of $\mathcal{K}_{N_{r,1}}$. Then for every i, j, $1 \leq i < j \leq R_r$, and for every edge e of $\mathcal{F}_{r,k+1}$ with one vertex in I_i and the other vertex in I_j, let $c_{r,k+1}(e) := c_{r,1}(\{v_i, v_j\})$. Clearly, $N_{r,k} = (R_r)^k$ for every $k \geq 1$.

The matchings $\mathcal{M}_{r,k}$ are also constructed inductively. We start with constructing $\mathcal{M}_{r,1}$, which serves as a basic building block. Roughly speaking, we expand the vertices of $\mathcal{K}_r$ to form a matching and take R_r shifted copies of this matching; see Figure 2.1. More precisely, consider the integers $1, 2, \dots, r^2 R_r$ as vertices, and let $l_i := (i-1)rR_r$, for $1 \leq i \leq r$. For every pair i, j, where $1 \leq i < j \leq r$, we add the R_r edges $\{l_i + j, l_j + i\}, \{l_i + j + r, l_j + i + r\}, \{l_i + j + 2r, l_j + i + 2r\}, \dots,$ $\{l_i + j + (R_r - 1)r, l_j + i + (R_r - 1)r\}$. Note that the vertices $l_i + i + mr$, where $1 \leq i \leq r$ and $0 \leq m < R_r$, are isolated. After removing these vertices we obtain an ordered matching $\mathcal{M}_{r,1}$ with $t_r := r(r-1)R_r$ vertices.

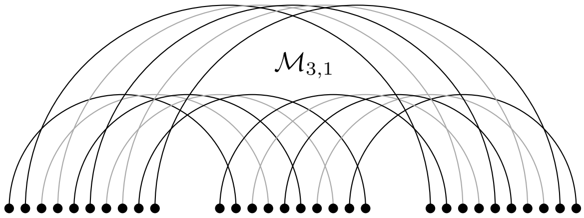


Figure 2.1: The matching $\mathcal{M}_{3,1}$.

Let $n_{r,1} := t_r$. Now let $k \geq 1$ and suppose that $\mathcal{M}_{r,k}$ has been constructed. Let $J_1, L_1, J_2, L_2, \dots, L_{r-1}, J_r$ be consecutive intervals of vertices of size $|L_i| = n_{r,k}$ and $|J_i| = (r-1)R_r$. The matching $\mathcal{M}_{r,k+1}$ is obtained by placing a copy of $\mathcal{M}_{r,k}$ on each of the $r-1$ intervals L_i and a copy of $\mathcal{M}_{r,1}$ on the union of the r intervals J_i; see Figure 2.2. We have $n_{r,k+1} = (r-1)n_{r,k} + t_r$.

Now we show that for every k, the coloring $c_{r,k}$ of $\mathcal{K}_{N_{r,k}}$ avoids $\mathcal{M}_{r,k}$. Trivially, $c_{r,1}$ avoids $\mathcal{M}_{r,1}$ since $n_{r,1} = t_r > R_r = N_{r,1}$. Let $k \geq 1$ and suppose that $c_{r,k}$ avoids $\mathcal{M}_{r,k}$. Let $I_1, \dots, I_{R_r}$ be the intervals of vertices of $\mathcal{K}_{N_{r,k+1}}$ from the construction of $c_{r,k+1}$ and let $J_1, L_1, \dots, L_{r-1}, J_r$ be the intervals of vertices of $\mathcal{M}_{r,k+1}$ from the construction of $\mathcal{M}_{r,k+1}$. Let the edges of $\mathcal{K}_{N_{r,k+1}}$ be colored by $c_{r,k+1}$. Consider a copy of $\mathcal{M}_{r,k+1}$ in $\mathcal{K}_{N_{r,k+1}}$. If two intervals J_j and J_{j+1} intersect some interval I_i, then $L_j \subset I_i$. Since L_j induces $\mathcal{M}_{r,k}$ in $\mathcal{M}_{r,k+1}$ and I_i induces $\mathcal{K}_{N_{r,k}}$ colored with

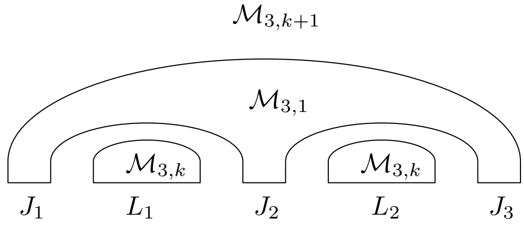


Figure 2.2: The construction of $\mathcal{M}_{3,k+1}$.

$c_{r,k}$ in $\mathcal{K}_{N_{r,k+1}}$, the copy of $\mathcal{M}_{r,k+1}$ is not monochromatic by induction. Thus we may assume that every interval I_i is intersected by at most one interval J_j.

Partition each interval J_j into R_r intervals $J_j^1, J_j^2, \dots, J_j^{R_r}$ of length $r-1$, in this order. At most $R_r - 1$ of the rR_r intervals J_j^l contain vertices from at least two intervals I_i, $1 \le i \le R_r$. Thus there is an l such that for every j, $1 \le j \le r$, the whole interval J_j^l is contained in some interval $I_{i(j)}$. Moreover, all the intervals $I_{i(j)}$ are pairwise distinct by our assumption. By the construction of $\mathcal{M}_{r,k+1}$, there is exactly one edge $e_{j,j'}$ in $\mathcal{M}_{r,k+1}$ between every pair of intervals J_j^l, $J_{j'}^l$. By the coloring of $\mathcal{F}_{r,k+1}$, we have $c_{r,k+1}(e_{j,j'}) = c_{r,1}(\{v_{i(j)}, v_{i(j')}\})$. Since the edges $\{v_{i(j)}, v_{i(j')}\}$ form a complete subgraph with r vertices in $\mathcal{K}_{N_{r,1}}$ and $c_{r,1}$ avoids $\mathcal{K}_r$, the copy of $\mathcal{M}_{r,k+1}$ in $\mathcal{K}_{N_{r,k+1}}$ is not monochromatic. Thus $c_{r,k+1}$ avoids $\mathcal{M}_{r,k+1}$.

Solving the recurrence for $n_{r,k}$, we get

$$n_{r,k} = (1 + (r-1) + \cdots + (r-1)^{k-1}) \cdot t_r < (r-1)^k \cdot t_r < r^{k+2} \cdot R_r.$$

Now we assume that r is sufficiently large and we choose $k(r)$ as follows. Let $c := (\log R_r)/r$. By (1.1), we have $c \in [1/2, 2)$. Let $k(r) := \lfloor (cr/\log r) - 2 \rfloor = (cr/\log r) - 2 - \varepsilon$, where $\varepsilon \in [0,1)$. Let $n := n_{r,k(r)}$, $N := N_{r,k(r)}$ and $\mathcal{M} := \mathcal{M}_{r,k(r)}$. We have

$$\begin{aligned} n = n_{r,k(r)} &< r^{k(r)+2} \cdot R_r \le 2^{cr + \log R_r} = 2^{2cr} \quad \text{and} \\ N = N_{r,k(r)} &= (R_r)^{k(r)} = 2^{crk(r)} > 2^{(c^2 r^2/\log r) - 3cr}. \end{aligned}$$

Using these bounds together with the trivial bound $2^{cr} = R_r < n$, we get

$$\begin{aligned} \log N - \frac{\log^2 n}{5 \log\log n} &> \frac{c^2 r^2}{\log r} - 3cr - \frac{4c^2 r^2}{5(\log r + \log c)} \\ &= c^2 r^2 \left(\frac{1}{\log r} - \frac{3}{cr} - \frac{4}{5(\log r + \log c)} \right) \\ &> 0 \end{aligned}$$

where the last inequality is satisfied for $r > 540$. The theorem follows. □

We remark that our colorings $c_{r,k}$ of $\mathcal{K}_{N_{r,k}}$ are not constructive, since we use the probabilistic lower bound from Ramsey's theorem.

Since any ordered path on n vertices contains an ordered matching on at least $n-1$ vertices, it follows from Theorem 2.1 and Proposition 1.9 that there are arbitrarily large n-vertex graphs G and two orderings $\mathcal{G}$ and $\mathcal{G}'$ of G such that $\overline{\mathrm{R}}(\mathcal{G})$ is linear in n while $\overline{\mathrm{R}}(\mathcal{G}')$ is superpolynomial in n.

We note that, independently, Conlon et al. [CFLS14] proved that as n goes to infinity, almost every ordering $\mathcal{M}_n$ of a matching on n vertices satisfies $\overline{\mathrm{R}}(\mathcal{M}_n) \geq n^{\frac{\log n}{20 \log \log n}}$.

For an ordered graph $\mathcal{G}$, the *interval chromatic number* of $\mathcal{G}$ is the minimum number of intervals the vertex set of $\mathcal{G}$ can be partitioned into such that there is no edge between vertices of the same interval.

For positive integers n_1 and n_2, we use $\mathcal{K}_{n_1,n_2}$ to denote the ordering of K_{n_1,n_2} in which the two parts form consecutive intervals of sizes n_1 and n_2, taken from left to right.

The following simple observation shows that once the interval chromatic number of an n-vertex ordered matching is 2, then the corresponding ordered Ramsey number is only quadratic in n.

Observation 2.2. *For every positive integer n, every ordered matching $\mathcal{M}$ on $2n$ vertices with interval chromatic number 2 satisfies*

$$\overline{\mathrm{R}}(\mathcal{M}) \leq 2n^2.$$

Proof. Let the edges of $\mathcal{K}_{2n^2}$ be colored red and blue. First, we partition the vertex set of $\mathcal{K}_{2n^2}$ into two consecutive intervals A and B, each of length n^2. We assume that the vertex set of $\mathcal{M}$ is $[2n]$. Since the interval chromatic number of $\mathcal{M}$ is 2, the intervals $[n]$ and $J := [2n] \setminus [n]$ induce independent sets in $\mathcal{M}$. Now, we show that there is a monochromatic copy of $\mathcal{M}$ in the coloring of $\mathcal{K}_{2n^2}$.

For every $i \in [n]$, we let $v_i \in J$ be the neighbor of i in $\mathcal{M}$ and we consider partitions $A_1, \ldots, A_n$ and $B_{n+1}, \ldots, B_{2n}$ of A and B, respectively, into consecutive intervals of size n. Now, if there is a red edge of $\mathcal{K}_{2n^2}$ between A_i and B_{v_i} for every $i \in [n]$, then there is a red copy of $\mathcal{M}$ in the coloring of $\mathcal{K}_{2n^2}$. Otherwise there is a pair (A_j, B_{v_j}) for some $j \in [n]$ such that all edges of $\mathcal{K}_{2n^2}$ between A_j and B_{v_j} are blue. That is, there is a blue copy of $\mathcal{K}_{n,n}$ in the coloring of $\mathcal{K}_{2n^2}$ and, since $\mathcal{M} \subseteq \mathcal{K}_{n,n}$, we have a blue copy of $\mathcal{M}$ in the coloring of $\mathcal{K}_{2n^2}$ as well. □

Observation 2.2 gives asymptotically the best known upper bound on $\overline{\mathrm{R}}(\mathcal{M}_{2n})$. Note that we have, in fact, showed $\overline{\mathrm{R}}(\mathcal{M}, \mathcal{K}_{n,n}) \leq 2n^2$. Conlon et al. [CFLS14] provided an almost matching lower bound by showing that there is a constant $C > 0$ such that for all n there is an ordered matching $\mathcal{M}_{2n}$ of interval chromatic number two with $2n$ vertices satisfying

$$\overline{\mathrm{R}}(\mathcal{M}_{2n}) \geq \frac{Cn^2}{\log^2 n \log \log n}.$$

Conlon et al. [CFLS14] proved this result using the well-known *Van der Corput sequence*. Some applications of this sequence appear in the discrepancy theory [Mat99].

For a positive integer n, the *random n-permutation* is a permutation of the set $[n]$ chosen independently uniformly at random from the set of all $n!$ permutations of the set $[n]$.

For a positive integer n and the random n-permutation π, the *random ordered n-matching* $\mathcal{M}(\pi)$ is the ordered matching with the vertex set $[2n]$ and with edges $\{i, n + \pi(i)\}$ for every $i \in [n]$. Note that the interval chromatic number of every random ordered n-matching is 2.

The random ordered n-matching satisfies an event A *asymptotically almost surely* if the probability that A holds tends to 1 as n goes to infinity.

We improve the lower bound of Conlon et al. [CFLS14] by eliminating the $(\log\log n)$-factor in the denominator. Moreover, we show that the new bound is satisfied by almost every ordered matching with interval chromatic number 2.

Theorem 2.3. *There is a constant $C > 0$ such that the random ordered n-matching $\mathcal{M}(\pi)$ asymptotically almost surely satisfies*

$$\overline{\mathrm{R}}(\mathcal{M}(\pi)) \geq C\left(\frac{n}{\log n}\right)^2.$$

If A and B are subsets of the vertex set of a graph G, then we use $e_G(A, B)$ to denote the number of edges that have one vertex in A and one vertex in B. In particular, $e_G(A, A)$ is the number of edges of the subgraph $G[A]$ of G induced by A. A similar notion, $e_{\mathcal{G}}(A, B)$, is used for an ordered graph $\mathcal{G}$ and two subsets A and B of its vertices.

Lemma 2.4. *Let d, n, r, S be positive integers and let $X_1, \dots, X_d \subseteq [n]$ and $Y_1, \dots, Y_d \subseteq [2n] \setminus [n]$ be two collections of pairwise disjoint sets such that $|X_1| \geq \cdots \geq |X_d|$, $|Y_1| \geq \cdots \geq |Y_d|$, and $|X_d||Y_d| \geq S$. Let T be a set of r pairs (X_i, Y_j) with $1 \leq i, j \leq d$. Then the probability that we have $e_{\mathcal{M}(\pi)}(X_i, Y_j) = 0$ for every $(X_i, Y_j) \in T$ is less than*

$$e^{-\frac{S}{n}\left\lfloor(3d-\sqrt{9d^2-8r})/4\right\rfloor^2}.$$

Proof. We may assume without loss of generality that $|X_1| = \cdots = |X_d|$ and $|Y_1| = \cdots = |Y_d|$, since removing elements from the sets X_i and Y_j does not decrease the probability. Let $x := |X_1| = \cdots = |X_d|$ and $y := |Y_1| = \cdots = |Y_d|$. From symmetry, we may assume $x \leq y$.

We estimate the probability P that the random n-permutation π maps no element e from $[n]$ to $\pi(e)$ such that $(e, n + \pi(e)) \in (X_i, Y_j)$ for $(X_i, Y_j) \in T$. Assume that the elements of $[n]$ are in some total order $e_1, \dots, e_n$ in which the values $\pi(e_1), \dots, \pi(e_n) \in [n]$ are assigned. The probability P is then at most $\prod_{k=1}^{n} \min\{\frac{n-|F_{e_k}|}{n-k+1}, 1\}$, where $F_{e_k} \subseteq [2n] \setminus [n]$ is a set of elements f such that $e_k \in X_i$, $f \in Y_j$ for $(X_i, Y_j) \in T$.

There is a set Z of $z := \lfloor(3d - \sqrt{9d^2 - 8r})/4\rfloor$ indices from $[d]$ such that for every $i \in Z$ there are at least $2z$ sets Y_j with $(X_i, Y_j) \in T$. Otherwise $z < \lfloor(3d - \sqrt{9d^2 - 8r})/4\rfloor$ and there are fewer than $zd + (d - z)2z \leq r$ pairs in T. Let $i_1, \dots, i_z$ be the elements of Z.

For every $k = 1, \dots, z$, let N_k be the set of indices j from $[d]$ such that $(X_{i_k}, Y_j) \in T$. Since $\cup_{j\in N_k} Y_j = F_e$ for every $e \in X_{i_k}$, the probability P is at most

$$\prod_{k=1}^{z}\left(\frac{n-\sum_{j\in N_k}|Y_j|}{n-\sum_{l=1}^{k}|X_{i_l}|}\right)^{|X_{i_k}|} = \prod_{k=1}^{z}\left(\frac{n-|N_k|y}{n-kx}\right)^{x}.$$

The denominators are positive, as $x \leq n/d$ and $z < d$.

Since $x \leq y$ and $|N_k| \geq 2z$ for every $k \in [z]$, the last term is at most

$$\left(\frac{n-2zy}{n-zy}\right)^{zx} \leq \left(\frac{n-zy}{n}\right)^{zx} = \left(1-\frac{zy}{n}\right)^{zx} < e^{-xyz^2/n}.$$

Since $xy \geq S$, the probability P is less than

$$e^{-Sz^2/n} = e^{-\frac{S}{n}\left\lfloor (3d-\sqrt{9d^2-8r})/4 \right\rfloor^2}. \qquad \square$$

In the rest of the section, we set $d := 3\log n$, $S := 2 \cdot 10^4 n$, and $r := \log^2 n/4$.

Lemma 2.5. *The random ordered n-matching $\mathcal{M}(\pi)$ satisfies the following statement asymptotically almost surely: for all collections $I_1, \dots, I_d \subseteq [n]$ and $J_1, \dots, J_d \subseteq [2n] \setminus [n]$ of pairwise disjoint intervals satisfying $|I_1| \geq \dots \geq |I_d|$, $|J_1| \geq \dots \geq |J_d|$, and $|I_d||J_d| \geq S$, the number of pairs (I_i, J_j) with $1 \leq i, j \leq d$ and $e_{\mathcal{M}(\pi)}(I_i, J_j) > 0$ is larger than $d^2 - r$.*

Proof. Let $\mathcal{M}(\pi)$ be the random ordered n-matching. Let X be a random variable expressing the number of collections $I_1, \dots, I_d$ and $J_1, \dots, J_d$ of intervals from the statement of the lemma with r pairs (I_i, J_j), $1 \leq i, j \leq d$, that satisfy $e_{\mathcal{M}(\pi)}(I_i, J_j) = 0$. We show that the expected value of X tends to zero as n goes to infinity. The rest then easily follows from Markov's inequality.

The number of collections $I_1, \dots, I_d \subseteq [n]$ and $J_1, \dots, J_d \subseteq [2n] \setminus [n]$ of pairwise disjoint intervals is at most

$$\binom{n+2d}{2d}^2 \leq \left(\frac{e(n+2d)}{2d}\right)^{4d} < 2^{4d\log n}.$$

The number of choices of r elements from a set of d^2 elements is

$$\binom{d^2}{r} \leq \left(\frac{ed^2}{r}\right)^r < 2^{7r},$$

where the last inequality follows from the expression of d and r.

We fix collections $I_1, \dots, I_d$ and $J_1, \dots, J_d$ and r pairs (I_i, J_j) with $1 \leq i, j \leq d$. By Lemma 2.4, the probability that $\mathcal{M}(\pi)$ has no edge between I_i and J_j, where (I_i, J_j) is among r chosen pairs, is less than $e^{-\frac{S}{n}\left\lfloor (3d-\sqrt{9d^2-8r})/4 \right\rfloor^2}$.

Altogether, the expected value of X is bounded from above by

$$2^{4d\log n} \cdot 2^{7r} \cdot e^{-\frac{S}{n}\left\lfloor (3d-\sqrt{9d^2-8r})/4 \right\rfloor^2} < 2^{12\log^2 n + \frac{7}{4}\log^2 n - \frac{60}{4}\log^2 n} = 2^{-\frac{5}{4}\log^2 n}.$$

Thus the expected value of X tends to zero as n goes to infinity, which concludes the proof. $\square$

In the rest of the section, we set $M := \frac{n \log\log n}{8\log n}$, $s := \frac{n}{8\log n}$, and $t := \frac{n}{20\log n}$.

Lemma 2.6. *The random ordered n-matching $\mathcal{M}(\pi)$ satisfies the following statement asymptotically almost surely: for every k with $1 \leq k \leq t$ and for all partitions $I_1, \dots, I_k$ of $[n]$ and $J_k, \dots, J_t$ of $[2n] \setminus [n]$ into consecutive intervals of sizes at most s such that $|I_{i_1}| \geq \dots \geq |I_{i_k}|$, $|J_{j_1}| \geq \dots \geq |J_{j_{t-k+1}}|$, and $|I_{i_{d+1}}||J_{j_{d+1}}| < S$, there are more than M pairs $(I_{i_l}, J_{j_{l'}})$ with $l, l' > d$ and $e_{\mathcal{M}(\pi)}(I_{i_l}, J_{j_{l'}}) > 0$.*

We note that some of the intervals from $I_1, \dots, I_k, J_k, \dots, J_t$ might be empty. Note that it follows from the choice of s that the number of intervals $I_1, \dots, I_k$ is at least $8\log n$ and, similarly, the number of intervals $J_k, \dots, J_t$ is at least $8\log n$.

Proof. We assume that n is sufficiently large. The number of partitions $I_1, \dots, I_k$, $J_k, \dots, J_t$ from the statement is at most

$$\binom{2n+t}{t} \le \left(\frac{e(2n+t)}{t}\right)^t < (120 \log n)^{n/(20 \log n)} < 2^{n \log\log n/(19 \log n)}.$$

For such fixed partition $I_1, \dots, I_k, J_k, \dots, J_t$ of $[2n]$, the number of choices of M pairs $(I_{i_l}, J_{j_{l'}})$ with $l, l' > d$ is at most

$$\binom{t^2}{M} \le \left(\frac{et^2}{M}\right)^M < 2^{n \log\log n/8}.$$

Since every interval of the partition has size at most s, there are at most $2ds$ edges with one endpoint in $I_{i_1} \cup \dots \cup I_{i_d} \cup J_{j_1} \cup \dots \cup J_{j_d}$. Thus the number of edges of $\mathcal{M}(\pi)$ between intervals I_{i_l} and $J_{j_{l'}}$ with $l, l' > d$ is at least $n/4$, as we have $n - 2ds = n - 3n/4 = n/4$.

Let P be the probability that for a fixed partition $I_1, \dots, I_k, J_k, \dots, J_t$ and for a fixed set T of M pairs $(I_{i_l}, J_{j_{l'}})$ with $l, l' > d$ the random n-matching $\mathcal{M}(\pi)$ satisfies $e_{\mathcal{M}(\pi)}(I_{i_l}, J_{j_{l'}}) = 0$ for every pair $(I_{i_l}, J_{j_{l'}})$ with $l, l' > d$ that is not contained in T. We show that the probability P is less than

$$\max_{\alpha \in [1/4,1]} \frac{\binom{MS}{\alpha n}}{(\alpha n)!}.$$

We fix an ordered matching $\mathcal{M}$ that is formed by edges incident to a vertex in $I_{i_1} \cup \dots \cup I_{i_d} \cup J_{j_1} \cup \dots \cup J_{j_d}$. Let $n_{\mathcal{M}}$ be the number of edges in $\mathcal{M}$ and let $P_{\mathcal{M}}$ be the probability that $\mathcal{M}$ is in $\mathcal{M}(\pi)$. Then there are $n - n_{\mathcal{M}} \ge n/4$ edges of $\mathcal{M}(\pi)$ that are not in $\mathcal{M}$. These edges are contained in pairs from T with probability less than $\binom{MS}{n-n_{\mathcal{M}}}/((n-n_{\mathcal{M}})!)$, as $|I_i||J_j| < S$ for every pair $(I_i, J_j) \in T$. Taking the maximum of $\binom{MS}{\alpha n}/(\alpha n)!$ over $\alpha \in [1/4, 1]$, we have

$$P < \sum_{\mathcal{M}} P_{\mathcal{M}} \frac{\binom{MS}{n-n_{\mathcal{M}}}}{(n-n_{\mathcal{M}})!} \le \max_{\alpha \in [1/4,1]} \frac{\binom{MS}{\alpha n}}{(\alpha n)!} \sum_{\mathcal{M}} P_{\mathcal{M}} = \max_{\alpha \in [1/4,1]} \frac{\binom{MS}{\alpha n}}{(\alpha n)!},$$

where the summation goes over all ordered matchings $\mathcal{M}$ that are formed by edges incident to a vertex in $I_{i_1} \cup \dots \cup I_{i_d} \cup J_{j_1} \cup \dots \cup J_{j_d}$.

Using the standard estimates $(a/e)^a \le a!$ and $\binom{a}{b} \le \left(\frac{ea}{b}\right)^b$ for positive integers a and b, we bound $\binom{MS}{\alpha n}/(\alpha n)!$ from above by

$$\left(\frac{eMS/(\alpha n)}{\alpha n/e}\right)^{\alpha n} = \left(\frac{e^2 MS}{(\alpha n)^2}\right)^{\alpha n} = \left(\frac{2 \cdot 10^4 e^2 \log\log n}{8\alpha^2 \log n}\right)^{\alpha n}$$
$$< (\log n)^{-4\alpha n/7} \le 2^{-n \log\log n/7}$$

for a sufficiently large n.

Let X be the random variable expressing the number of partitions $I_1, \dots, I_k$, $J_k, \dots, J_t$ of $[2n]$ from the statement of the lemma such that the number of pairs $(I_{i_l}, J_{j_{l'}})$ satisfying $l, l' > d$ and $e_{\mathcal{M}(\pi)}(I_{i_l}, J_{j_{l'}}) > 0$ is at most M. It follows from our observations that, for a sufficiently large n, the expected value of X is at most

$$2^{n \log\log n/(19 \log n)} \cdot 2^{n \log\log n/8} \cdot 2^{-n \log\log n/7}.$$

Thus we see that the expected value of X tends to zero as n goes to infinity. The rest of the statement then follows from Markov's inequality. □

We now prove Theorem 2.3. The main idea of the proof is similar to the one used by Conlon et al. [CFLS14] in the proof of their lower bound for ordered Ramsey numbers of ordered matchings with interval chromatic number 2.

Proof of Theorem 2.3. Let $\mathcal{M}$ be a $2n$-vertex ordered matching of interval chromatic number 2 that satisfies the statements from Lemma 2.5 and Lemma 2.6. We know that the random n-matching satisfies both statements asymptotically almost surely.

Let $\mathcal{R}$ be the ordered complete graph with loops on the vertex set $[t]$. Let c be the coloring of $\mathcal{R}$ that assigns either a red or a blue color to each edge of $\mathcal{R}$ independently at random with probability 1/2. Let $A_1, \ldots, A_t$ be the partition of the vertex set of $\mathcal{K}_{st}$ into t consecutive intervals of size s. We define a coloring c' of $\mathcal{K}_{st}$ as follows. The color $c'(e)$ of an edge e of $\mathcal{K}_{st}$ is $c(\{i,j\})$ if one endvertex of e is in A_i and the other one is in A_j.

We show that the probability that there is a red copy of $\mathcal{M}$ in c' is less than 1/2. Let $\mathcal{M}_0$ be a red copy of $\mathcal{M}$ in c' and let I and J be the parts of $\mathcal{M}_0$ in this order.

For $i = 1, \ldots, t$, we set $I_i := A_i \cap I$. Let $k \leq t$ be the maximum i such that I_i is nonempty. For $j = k, \ldots, t$, we set $J_j := A_j \cap J$. This gives us partitions $I_1, \ldots, I_k$ and $J_k, \ldots, J_t$ of I and J, respectively, into consecutive intervals, each of size at most s. We consider the orderings $I_{i_1}, \ldots, I_{i_k}$ and $J_{j_1}, \ldots, J_{j_{t-k+1}}$ of $I_1, \ldots, I_k$ and $J_k, \ldots, J_t$, respectively, such that $|I_{i_1}| \geq \cdots \geq |I_{i_k}|$ and $|J_{j_1}| \geq \cdots \geq |J_{j_{t-k+1}}|$.

First, we assume that $|I_{i_d}||J_{j_d}| \geq S$. Since $\mathcal{M}$ satisfies the statement from Lemma 2.5, there are at least $d^2 - r$ pairs (I_i, J_j) with $i \in \{i_1, \ldots, i_d\}$, $j \in \{j_1, \ldots, j_d\}$, and $e_{\mathcal{M}_0}(I_i, J_j) > 0$. From the choice of c', the red copy $\mathcal{M}_0$ thus corresponds to a red ordered subgraph of $\mathcal{R}$ in c with $2d$ or $2d-1$ vertices and with at least $d^2 - r$ edges, one of which might be a loop. By the union bound, the probability that there is such an ordered graph in $\mathcal{R}$ is at most

$$\begin{aligned}\left(\binom{t}{2d} + \binom{t}{2d-1}\right)\binom{d^2}{r}2^{-d^2+r} &\leq 2\left(\frac{et}{2d}\right)^{2d}\left(\frac{ed^2}{r}\right)^{r}2^{-d^2+r} \\ &< 2^{6\log^2 n} \cdot 2^{2\log^2 n} \cdot 2^{-9\log^2 n + \log^2 n/4} \\ &= 2^{-\frac{3}{4}\log^2 n}.\end{aligned}$$

For a sufficiently large n, this expression is less than 1/4.

Now, we assume that $|I_{i_d}||J_{j_d}| < S$. The matching $\mathcal{M}$ satisfies the statement from Lemma 2.6 and thus there are more than M pairs $(I_{i_l}, J_{j_{l'}})$ with $l, l' > d$ and $e_{\mathcal{M}(\pi)}(I_{i_l}, J_{j_{l'}}) > 0$. From the choice of c', the red copy $\mathcal{M}_0$ corresponds to a red ordered subgraph of $\mathcal{R}$ in c with at least M edges that are determined by the partition $I_1, \ldots, I_k, J_k, \ldots, J_t$ of $[2n]$. By the union bound, the probability that there is such ordered graph in c is at most

$$\binom{2n+t}{t}2^{-M} \leq \left(\frac{e(2n+t)}{t}\right)^t 2^{-M} < 2^{\frac{n\log\log n}{19\log n}} \cdot 2^{-\frac{n\log\log n}{8\log n}} = 2^{-\frac{11n\log\log n}{152\log n}}.$$

If n is sufficiently large, then this term is less than 1/4.

In total, the probability that there is a red copy of $\mathcal{M}$ in c' is less than $1/4 + 1/4 = 1/2$. From symmetry, a blue copy of $\mathcal{M}$ appears in c' with probability less than 1/2. Altogether, the probability that there is no monochromatic copy of $\mathcal{M}$ in c' is positive. In other words, we have $\overline{\mathrm{R}}(\mathcal{M}) \geq st = \frac{1}{160}\left(\frac{n}{\log n}\right)^2$. □

2.2 Bounded degeneracy and interval chromatic number

For a positive integer k, a graph G is *k-degenerate* if there is an ordering $v_1, \dots, v_n$ of its vertices such that every vertex v_i has at most k neighbors v_j in G with $j < i$. The *degeneracy* of G is the smallest k such that G is k-degenerate. The degeneracy of an ordered graph $\mathcal{G} = (G, \prec)$ is the degeneracy of the underlying graph G.

We give a polynomial upper bound for ordered Ramsey numbers of ordered graphs with bounded degeneracy and bounded interval chromatic number. We also include a result of Conlon et al. [CFLS14], which shows that $\overline{\mathrm{R}}(\mathcal{G})$ is at most quasipolynomial in n for every n-vertex ordered graph $\mathcal{G}$ of bounded degeneracy.

Lemma 2.7. *Let k, t, n be positive integers and let $\mathcal{G}$ be an ordered k-degenerate graph on n vertices. Then $\overline{\mathrm{R}}(\mathcal{G}, \mathcal{K}_{t,t}) \leq n^2 t^{k+1}$.*

Proof. Assume that $\mathcal{G} = (G, \prec)$. Let $N := n^2 t^{k+1}$ and assume that the edges of $\mathcal{K}_N$ are colored red and blue. We partition the vertices of $\mathcal{K}_N$ into n disjoint consecutive intervals of length nt^{k+1}. The ith such interval is denoted by $I(v)$ where v is the ith vertex of $\mathcal{G}$ in the ordering $\prec$.

We try to construct a blue copy $h(\mathcal{G})$ of $\mathcal{G}$ in $\mathcal{K}_N$ in n steps. In each step of the construction we find an image $h(w) \in I(w)$ of a new vertex w of $\mathcal{G}$ or a red copy of $\mathcal{K}_{t,t}$.

For every vertex v of $\mathcal{G}$ that has no image $h(v)$ yet, we keep a set $U(v) \subseteq I(v)$ of possible candidates for $h(v)$. At the beginning we set $U(v) := I(v)$ for every $v \in V(G)$. Throughout the proof, we will keep the property that the size of $U(v)$ is a multiple of t.

Let $\lessdot$ be an ordering of the vertices of $\mathcal{G}$ such that every vertex v of $\mathcal{G}$ has at most k left neighbors in $\lessdot$. This ordering exists as $\mathcal{G}$ is k-degenerate. Note that the ordering $\lessdot$ might differ from the ordering $\prec$.

Let w be the first vertex of $\mathcal{G}$ in the ordering $\lessdot$ that has no image $h(w)$ yet. Suppose that $u_1, \dots, u_s \in V(G)$ are the right neighbors of w in $\lessdot$. We show how to find the image $h(w)$ or a red copy of $\mathcal{K}_{t,t}$ in $\mathcal{K}_N$.

Let $i \in [s]$. We claim that in $U(w)$ every vertex except for at most $t-1$ vertices has at least $|U(u_i)|/t$ blue neighbors in $U(u_i)$ or there is a red copy of $\mathcal{K}_{t,t}$ with edges between $U(w)$ and $U(u_i)$.

Suppose first that there is a subset $W \subseteq U(w)$ of size t such that each vertex of W has fewer than $|U(u_i)|/t$ blue neighbors in $U(u_i)$. In such a case we delete from $U(u_i)$ every vertex that is a blue neighbor of some vertex of W. Afterwards, there are still at least

$$|U(u_i)| - |W| \cdot \left(\frac{|U(u_i)|}{t} - 1 \right) = |U(u_i)| - t \cdot \left(\frac{|U(u_i)|}{t} - 1 \right) = t$$

vertices left in $U(u_i)$ and every such vertex has only red neighbors in W. Thus we have a red copy of $\mathcal{K}_{t,t}$ in $\mathcal{K}_N$.

By our claim, there is a red copy of $\mathcal{K}_{t,t}$ in $\mathcal{K}_N$ or a set $Z(w) \subseteq U(w)$ of size at least $|U(w)| - s(t-1) > |U(w)| - nt$ such that for every $i \in [s]$, every vertex of $Z(w)$ has at least $|U(u_i)|/t$ blue neighbors in $U(u_i)$. We may assume that the latter case occurs, as otherwise we are done.

We choose an arbitrary vertex $h(w)$ of $Z(w)$ to be the image of w in the constructed blue copy $h(\mathcal{G})$ of $\mathcal{G}$. For this we need to know that $Z(w)$ is nonempty; we show this at the end of the proof. For every $i \in [s]$, we update the set $U(u_i)$ to be a set of $|U(u_i)|/t$ blue neighbors of $h(w)$ in $U(u_i)$.

After these updates, we choose the first vertex in $\lessdot$ that does not have an image yet and proceed with the next step. If every vertex of $\mathcal{G}$ has an image, then we have found a blue copy of $\mathcal{G}$.

It remains to show that the set $Z(w)$ is nonempty in each step. Since w has at most k left neighbors in $\lessdot$, we have updated $U(w)$ at most k times. The size of $U(w)$ is initially nt^{k+1} and it is divided by t in every update. Thus, in the end, $|U(w)| \geq nt$. Consequently, $|Z(w)| > |U(w)| - nt \geq 0$. □

For positive integers n and $p \geq 2$, let $\mathcal{K}_p(n)$ be the ordered complete p-partite graph with parts of size n forming consecutive intervals.

Corollary 2.8. *Let k, n, and $p \geq 2$ be positive integers and let $\mathcal{G}$ be an ordered k-degenerate graph on n vertices. Then*

$$\overline{\mathrm{R}}(\mathcal{G}, \mathcal{K}_p(n)) \leq n^{(1+2/k)(k+1)^{\lceil \log p \rceil} - 2/k}.$$

In particular, every ordered k-degenerate graph $\mathcal{G}$ with n vertices and interval chromatic number p satisfies

$$\overline{\mathrm{R}}(\mathcal{G}) \leq n^{(1+2/k)(k+1)^{\lceil \log p \rceil} - 2/k}.$$

Proof. First, we define a function $f_{k,n}(q) \colon \mathbb{N} \to \mathbb{N}$ as

$$f_{k,n}(q) := n^{(1+2/k)(k+1)^q - 2/k}.$$

This function satisfies the recurrence $f_{k,n}(1) = n^{k+3}$ and $f_{k,n}(q) = n^2 \cdot (f_{k,n}(q-1))^{k+1}$ for every integer $q \geq 2$.

We assume without loss of generality that $p = 2^q$ for some positive integer q. We proceed by induction on q. The case $q = 1$ follows immediately from Lemma 2.7 applied with $t := n$.

Now let $q \geq 2$. Let $\mathcal{K}_N$ be an ordered complete graph with $N := f_{k,n}(q)$ vertices and edges colored red and blue. We show that there is always a blue copy of $\mathcal{G}$ or a red copy of $\mathcal{K}_p(n)$ in $\mathcal{K}_N$.

According to Lemma 2.7, there is a blue copy of $\mathcal{G}$ or a red copy of $\mathcal{K}_{t,t}$ for $t := f_{k,n}(q-1)$. In the first case we are done, thus we assume that the latter case occurs. Let A be the left part of size t and B the right part of size t in the red copy of $\mathcal{K}_{t,t}$.

Since the induced ordered subgraph $\mathcal{K}_N[A]$ has $f_{k,n}(q-1)$ vertices, there is a blue copy of $\mathcal{G}$ or a red copy of $\mathcal{K}_{p/2}(n)$ in $\mathcal{K}_N[A]$ by the inductive assumption. An analogous statement holds for the ordered subgraph $\mathcal{K}_N[B]$.

Thus, if there is no blue copy of $\mathcal{G}$ in $\mathcal{K}_N[A]$ and in $\mathcal{K}_N[B]$, then the two red copies of $\mathcal{K}_{p/2}(n)$ together with the red edges between $\mathcal{K}_N[A]$ and $\mathcal{K}_N[B]$ form a red copy of $\mathcal{K}_p(n)$ in $\mathcal{K}_N$. □

Although Corollary 2.8 gives a polynomial bound, the exponent is rather large with respect to p. Independently, Conlon et al. [CFLS14] proved a stronger bound. We include their result here, as it has some interesting corollaries. For example, it gives a quasipolynomial upper bound on ordered Ramsey numbers of k-degenerate ordered graphs.

Theorem 2.9 ([CFLS14, Theorem 3.1]). *Let $\mathcal{G}$ be an ordered k-degenerate graph on n vertices with maximum degree Δ. Let m and p be positive integers, $q := \lceil \log p \rceil$, and $D := 8p^2m$. Then we have*

$$\overline{\mathrm{R}}(\mathcal{G}, \mathcal{K}_p(m)) \leq 2^{q^2k+q}\Delta^q n^q D^{kq+1}.$$

In particular, every ordered k-degenerate graph $\mathcal{G}$ with n vertices and interval chromatic number $p \geq 2$ satisfies

$$\overline{\mathrm{R}}(\mathcal{G}) \leq n^{32k \log p}.$$

Since the interval chromatic number of every ordered graph on n vertices is at most n, Theorem 2.9 immediately gives a quasipolynomial upper bound on ordered Ramsey number of ordered graphs with bounded degeneracy. For k bounded, this upper bound almost matches the lower bound from Theorem 2.1.

Corollary 2.10 ([CFLS14, Theorem 3.1]). *Every ordered k-degenerate graph $\mathcal{G}$ on n vertices satisfies*

$$\overline{\mathrm{R}}(\mathcal{G}) \leq n^{32k \log n}.$$

2.3 Bounded bandwidth

For given positive integers k and $q \geq 2$, we say that an ordered graph $\mathcal{G} = (G, \prec)$ is (k,q)-*decomposable* if $\mathcal{G}$ has at most k vertices or if it admits the following recursive decomposition: there is a nonempty interval $I \subseteq V(G)$ with at most k vertices such that the interval I_L of vertices of $\mathcal{G}$ that are to the left of I and the interval I_R of vertices of $\mathcal{G}$ that are to the right of I satisfy

1) $|I_L|, |I_R| \leq \frac{q-1}{q} \cdot |V(G)|$,

2) there is no edge between I_L and I_R, and

3) the induced ordered subgraphs $(G[I_L], \prec\restriction_{I_L})$ and $(G[I_R], \prec\restriction_{I_R})$ are (k,q)-decomposable.

Theorem 2.11. *Let k and $q \geq 2$ be fixed positive integers. There is a constant C'_k such that every (k,q)-decomposable ordered graph $\mathcal{G}$ on n vertices satisfies*

$$\overline{\mathrm{R}}(\mathcal{G}) \leq C'_k \cdot n^{128k(q-1)}.$$

The constant C'_k depends on k and the proof of Theorem 2.11 gives a bound $C'_k \leq 2^{O(k \log k)}$.

We say that the *length of an edge* $\{u, v\}$ in an ordered graph $\mathcal{G} = (G, \prec)$ is $|i - j|$ if u is the ith vertex and v is the jth vertex of G in the ordering $\prec$. The *bandwidth* of $\mathcal{G}$ is the length of the longest edge in $\mathcal{G}$. Since every ordered graph with bandwidth k is $(k,2)$-decomposable, Theorem 2.11 implies the following.

Corollary 2.12. *For every fixed positive integer k, there is a constant C'_k such that every n-vertex ordered graph $\mathcal{G}$ of bandwidth k satisfies*

$$\overline{\mathrm{R}}(\mathcal{G}) \leq C'_k \cdot n^{128k}.$$

This result gives a positive answer to a problem of Conlon et al. [CFLS14, Problem 6.9] who asked whether for any natural number k there exists a constant c_k such that $\overline{\mathrm{R}}(\mathcal{H}) \leq n^{c_k}$ for any ordered graph $\mathcal{H}$ on n vertices with bandwidth at most k. By Corollary 2.12, one may take $c_k = O(k)$. It is plausible that the correct value of c_k is significantly smaller than this bound.

In the rest of the section, we prove Theorem 2.11. We prove the following general form of the theorem, which allows us to use double induction.

Theorem 2.13. *For fixed positive integers k, $q \geq 2$ and (k,q)-decomposable ordered graphs $\mathcal{G}$ and $\mathcal{H}$ with r and s vertices, respectively, we have*

$$\overline{\mathrm{R}}(\mathcal{G},\mathcal{H}) \leq C_k \cdot 2^{64k(\lceil \log_{q/(q-1)} r\rceil + \lceil \log_{q/(q-1)} s\rceil)}$$

where C_k is a sufficiently large constant with respect to k.

We start with the following auxiliary result.

Lemma 2.14. *For a positive integer N, let the edges of $\mathcal{K}_N$ be colored red and blue. Then there is a set U with at least $\lfloor N/(16\cdot 10^5)\rfloor$ vertices of $\mathcal{K}_N$ satisfying at least one of the following conditions:*

(a) every vertex of U has at least $N/11$ blue neighbors to the left and $N/11$ blue neighbors to the right of U,

(b) every vertex of U has at least $N/11$ red neighbors to the left and $N/11$ red neighbors to the right of U.

Proof. We assume that $N \geq 16\cdot 10^5$, otherwise the statement is trivial. We define the following two conditions for a vertex v of $\mathcal{K}_N$:

(i) v has at least $\frac{20}{217}N$ blue left and at least $\frac{20}{217}N$ blue right neighbors,

(ii) v has at least $\frac{20}{217}N$ red left and at least $\frac{20}{217}N$ red right neighbors.

First, we show that there is a set W with at least $N/2000$ vertices such that either every vertex of W satisfies (i) or every vertex of W satisfies (ii). Let B be the set of vertices of $\mathcal{K}_N$ that satisfy the condition (i) and let R be the set of vertices of $\mathcal{K}_N$ that satisfy (ii). Suppose that $|B| < N/2000$ and $|R| < N/2000$, otherwise we are done.

Let $\mathcal{K}'$ be the ordered complete graph obtained from $\mathcal{K}_N$ by removing the vertices of $B \cup R$. From the assumptions $\mathcal{K}'$ has more than $(1-\frac{2}{2000})N = \frac{999}{1000}N$ vertices and contains no monochromatic ordered star $\mathcal{S}_{t,t}$ for $t := \lceil \frac{20}{217}N\rceil + 1$. Therefore $\mathcal{K}'$ has fewer than $\overline{\mathrm{R}}(\mathcal{S}_{t,t},\mathcal{S}_{t,t})$ vertices.

Using Theorem 1.3 and the fact that $\overline{\mathrm{R}}(\mathcal{S}_{t,1},\mathcal{S}_{t,t}) = \overline{\mathrm{R}}(\mathcal{S}_{1,t},\mathcal{S}_{t,t})$, we have

$$\begin{aligned}\overline{\mathrm{R}}(\mathcal{S}_{t,t},\mathcal{S}_{t,t}) &= \overline{\mathrm{R}}(\mathcal{S}_{t,1},\mathcal{S}_{t,t}) + \overline{\mathrm{R}}(\mathcal{S}_{1,t},\mathcal{S}_{t,t}) - 1 = 2(\overline{\mathrm{R}}(\mathcal{S}_{1,t},\mathcal{S}_{t,1}) + 2t - 3) - 1\\ &= 2\left(\left\lfloor \frac{-1+\sqrt{1+8(t-2)^2}}{2}\right\rfloor + 2t - 2 + 2t - 3\right) - 1 < (8+2\sqrt{2})t.\end{aligned}$$

Altogether we have $|V(\mathcal{K}')| < (8+2\sqrt{2})(\lceil \frac{20}{217}N\rceil + 1) < \frac{999}{1000}N < |V(\mathcal{K}')|$, a contradiction. Thus there is a set W such that all its vertices satisfy one of the two conditions, say, (i).

Now, we find the set U as a subset of W. To do so, we partition the vertex set of $\mathcal{K}_N$ into $\frac{16 \cdot 10^5}{2000} = 800$ intervals $I_1, \dots, I_{800}$ such that each contains at least $\lfloor N/(16 \cdot 10^5) \rfloor$ vertices of W. This is possible as $|W| \geq N/2000$. Clearly, there is an interval I_i with at most $N/800$ vertices of $\mathcal{K}_N$. We set $U := I_i \cap W$.

Since every vertex of U has at least $\frac{20}{217}N$ blue left neighbors, it also has at least $\frac{20}{217}N - N/800 > N/11$ blue neighbors to the left of I_i and thus to the left of U. Analogously, every vertex of U has at least $N/11$ blue neighbors to the right of U. Therefore, U satisfies condition (a) of the lemma. □

We use the following two classical results further in the proof. The Kővári–Sós–Turán theorem [KST54] gives an upper bound on the maximum number of edges in a bipartite graph that contains no copy of a given complete bipartite graph.

Theorem 2.15 ([Bol04, HC58, KST54]). *Let $Z(m, n; s, t)$ be the maximum number of edges in a bipartite graph $G = (A \cup B, E)$ with $|A| = m$ and $|B| = n$ that does not contain $K_{s,t}$ as a subgraph with s vertices in A and t vertices in B. Assuming $2 \leq s \leq m$ and $2 \leq t \leq n$, we have*

$$Z(m, n; s, t) < (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m < s^{1/t}nm^{1-1/t} + tm.$$

Erdős and Szekeres proved the following upper bound on off-diagonal Ramsey numbers of complete graphs.

Theorem 2.16 ([ES35]). *For every $r, s \geq 2$, we have $\mathrm{R}(K_r, K_s) \leq \binom{r+s-2}{r-1}$.*

By Observation 1.1, we have the same upper bound for the ordered Ramsey numbers $\overline{\mathrm{R}}(\mathcal{K}_r, \mathcal{K}_s)$.

Proof of Theorem 2.13. Let $\mathcal{G}$ and $\mathcal{H}$ be (k, q)-decomposable ordered graphs with r and s vertices, respectively. Let $N = N_{k,q}(r, s) := C_k \cdot 2^{64k(\lceil \log_{q/(q-1)} r \rceil + \lceil \log_{q/(q-1)} s \rceil)}$ where C_k is a constant sufficiently large with respect to k. Assume that the edges of $\mathcal{K}_N$ are colored red and blue. We show that there is a blue copy of $\mathcal{G}$ or a red copy of $\mathcal{H}$ in $\mathcal{K}_N$. We proceed by double induction on $\lceil \log_{q/(q-1)} r \rceil$ and $\lceil \log_{q/(q-1)} s \rceil$.

We assume that either $\lceil \log_{q/(q-1)} r \rceil = 0$ or $\lceil \log_{q/(q-1)} s \rceil = 0$ for the induction basis. In these cases we have $r = 1$ or $s = 1$, respectively, and the statement is trivial.

Now assume that the theorem is true for every pair $\mathcal{G}'$, $\mathcal{H}'$ of (k, q)-decomposable ordered graphs with r' and s' vertices, respectively, such that $\lceil \log_{q/(q-1)} r' \rceil < \lceil \log_{q/(q-1)} r \rceil$ or $\lceil \log_{q/(q-1)} s' \rceil < \lceil \log_{q/(q-1)} s \rceil$.

Let U be the subset of vertices of $\mathcal{K}_N$ from Lemma 2.14. Without loss of generality, we assume that U satisfies part (a) of the lemma. That is, U has at least $\lfloor N/(16 \cdot 10^5) \rfloor$ vertices such that each of them has at least $N/11$ blue neighbors to the left and $N/11$ blue neighbors to the right of U.

By Theorem 2.16, there is a blue copy of $\mathcal{K}_{61k}$ or a red copy of $\mathcal{K}_s$ in $\mathcal{K}_N[U]$ if $|U| \geq \binom{61k+s-2}{61k-1}$. This condition is satisfied if $C_k \geq 16 \cdot 10^5 \cdot (61k)^{61k}$, since $\binom{61k+s-2}{61k-1} \leq 2^{61k \cdot \log(61k+s)} \leq (61k)^{61k} \cdot 2^{61k \cdot \log s} \leq (61k)^{61k} \cdot 2^{61k(\log_{q/(q-1)} s)}$. If $\mathcal{K}_N[U]$ contains a red copy of $\mathcal{K}_s$, we are done. Thus, assume that $\mathcal{K}_N[U]$ contains a blue copy of $\mathcal{K}_{61k}$, and let $U_1 \subset U$ be its vertex set.

Now we apply Theorem 2.15 to obtain a set $U_2 \subset U_1$ of size $6k$ whose vertices have at least $N/2^{64k}$ common blue neighbors to the left of U. Then we apply

Theorem 2.15 again to obtain a set $V \subset U_2$ of size k whose vertices have at least $N/2^{64k}$ common blue neighbors to the right of U.

Let J_L be the interval of vertices of $\mathcal{K}_N$ that are to the left of U and J_R the interval of vertices of $\mathcal{K}_N$ that are to the right of U. By the construction of U, we have $|J_L|, |J_R| \geq N/11$, and thus $|J_L|, |J_R| \leq 10N/11$. Without loss of generality, we assume that $|J_R| \leq N/2$.

The number of blue edges between J_L and U_1 is at least $(N/11) \cdot |U_1| \geq |J_L| \cdot |U_1|/10$. By Theorem 2.15, we have

$$\begin{aligned} Z(|J_L|, |U_1|; |J_L|/2^{60k}, 6k) &< (|J_L|/2^{60k})^{1/(6k)} \cdot 61k \cdot |J_L|^{1-1/(6k)} + 6k \cdot |J_L| \\ &= |J_L| \cdot (61k/2^{10} + 6k) \leq \frac{|J_L| \cdot 61k}{10} = \frac{|J_L| \cdot |U_1|}{10}. \end{aligned}$$

Thus, there is a blue complete bipartite graph between at least $|J_L|/2^{60k}$ vertices in J_L and $6k$ vertices in U_1. These $6k$ vertices form the set U_2.

The number of blue edges between U_2 and J_R is at least $(N/11) \cdot |U_2| \geq |U_2| \cdot |J_R| \cdot 2/11$. By Theorem 2.15, we have

$$\begin{aligned} Z(|J_R|, |U_2|; |J_R|/2^{7k}, k) &< (|J_R|/2^{7k})^{1/k} \cdot 6k \cdot |J_R|^{1-1/k} + k \cdot |J_R| \\ &= |J_R| \cdot (6k/2^7 + k) \leq \frac{|J_R| \cdot 6k \cdot 2}{11} = \frac{2|J_R| \cdot |U_2|}{11}. \end{aligned}$$

Thus, there is a blue complete bipartite graph between at least $|J_R|/2^{7k}$ vertices in J_R and k vertices in U_2. These k vertices form the set V. Since $|J_L|, |J_R| \geq N/11$, the vertices of V have at least $N/(2^{60k} \cdot 11) > N/2^{64k}$ common blue neighbors to the left of V and at least $N/(2^{7k} \cdot 11) > N/2^{64k}$ common blue neighbors to the right of V.

Since $\mathcal{G}$ is (k,q)-decomposable, we can partition the vertices of $\mathcal{G}$ into three intervals I_L, I, and I_R where $0 < |I| \leq k$ and $|I_L|, |I_R| \leq r(q-1)/q$ such that I is to the right of I_L and to the left of I_R, the intervals I_L and I_R induce (k,q)-decomposable ordered graphs $\mathcal{G}_L$ and $\mathcal{G}_R$, respectively, and there is no edge between $\mathcal{G}_L$ and $\mathcal{G}_R$.

From our choice of N, we have

$$\begin{aligned} N/2^{64k} &= C_k \cdot 2^{64k(\lceil \log_{q/(q-1)} r \rceil + \lceil \log_{q/(q-1)} s \rceil - 1)} \\ &= C_k \cdot 2^{64k(\lceil \log_{q/(q-1)} r(q-1)/q \rceil + \lceil \log_{q/(q-1)} s \rceil)} \geq N_{k,q}(\lfloor r(q-1)/q \rfloor, s) \end{aligned}$$

and so $\overline{\mathrm{R}}(\mathcal{G}_L, \mathcal{H}), \overline{\mathrm{R}}(\mathcal{G}_R, \mathcal{H}) \leq N/2^{64k}$. Therefore, using the inductive assumption, we can find either a blue copy of $\mathcal{G}_L$ or a red copy of $\mathcal{H}$ in the common blue left neighborhood of V. Similarly, we can find a blue copy of $\mathcal{G}_R$ or a red copy of $\mathcal{H}$ in the common blue right neighborhood of V. Suppose that we do not obtain a red copy of $\mathcal{H}$ in any of these two cases. Then we find a blue copy of $\mathcal{G}$ by choosing $|I|$ vertices of V as a copy of I and connect them to the blue copies of $\mathcal{G}_L$ and $\mathcal{G}_R$. □

2.4 Fixed graph and variable number of colors

Here we discuss the asymptotics of ordered Ramsey numbers $\overline{\mathrm{R}}(\mathcal{G}; c)$ of a fixed ordered graph $\mathcal{G}$ as a function of the number of colors c. That is, for the rest of

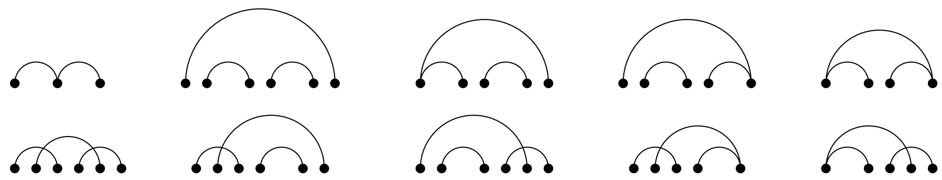


Figure 2.3: Minimal nonseparable ordered graphs with interval chromatic number at least 3.

the section we assume that $\mathcal{G}$ is a fixed ordered graph and that c can be arbitrarily large.

The unordered Ramsey numbers are at most polynomial for bipartite graphs and at least exponential otherwise; this follows from the Kővári–Sós–Turán theorem (Theorem 2.15) and from the existence of a decomposition of K_n into $\lceil \log n \rceil$ bipartite subgraphs, respectively. For ordered Ramsey numbers we observe a similar dichotomy, but the characterization is more subtle.

An ordered graph $\mathcal{G}$ is *separable* if the vertex set of $\mathcal{G}$ can be partitioned into two nonempty intervals I_1, I_2 such that there is no edge between I_1 and I_2. An ordered graph is *nonseparable* if it is not separable.

We find that $\overline{\mathrm{R}}(\mathcal{G}; c)$ is exponential in c if $\mathcal{G}$ contains a nonseparable ordered graph with interval chromatic number 3, and polynomial otherwise. Moreover, there is only a finite number of minimal nonseparable ordered graphs with interval chromatic number at least 3. Therefore, the class of ordered graphs with polynomial ordered Ramsey numbers can be characterized by a finite number of forbidden ordered subgraphs. We use $n \cdot \mathcal{K}_{n,n}$ to denote the ordered graph consisting of n disjoint consecutive copies of $\mathcal{K}_{n,n}$.

Theorem 2.17. *Every ordered graph $\mathcal{G}$ on n vertices satisfies one of the following conditions.*

(i) We have $\mathcal{G} \subseteq n \cdot \mathcal{K}_{n,n}$ and $\overline{\mathrm{R}}(\mathcal{G}; c) \leq (2cn)^{n+1}$.

(ii) One of the ordered graphs from Figure 2.3 is an ordered subgraph of $\mathcal{G}$ and $\overline{\mathrm{R}}(\mathcal{G}; c) > 2^c$.

Proof. For part (i), let $\mathcal{G}$ be a given n-vertex ordered graph contained in $n \cdot \mathcal{K}_{n,n}$. For $N := (2cn)^{n+1}$, let the edges of $\mathcal{K}_N$ be colored with c colors. We find a monochromatic copy of $\mathcal{G}$.

For $t := cn$, we partition the vertex set of $\mathcal{K}_N$ into $2t$ intervals $A_1, B_1, \ldots, A_t, B_t$ in this order, such that each interval has size $K := (2cn)^n$. For every $i = 1, \ldots, t$, it follows from the pigeonhole principle that there is a color c_i that colors at least K^2/c edges of $\mathcal{K}_N[A_i \cup B_i]$.

By the Kővári–Sós–Turán theorem (Theorem 2.15), we have $Z(K, K; n, n) < 2nK^{2-1/n} = K^2/c$. Consequently, for every $i = 1, \ldots, t$, there is a copy of $\mathcal{K}_{n,n}$ of color c_i in $\mathcal{K}_N[A_i \cup B_i]$. By the pigeonhole principle, we have a monochromatic copy of $n \cdot \mathcal{K}_{n,n}$. Since $\mathcal{G} \subseteq n \cdot \mathcal{K}_{n,n}$, we have a monochromatic copy of $\mathcal{G}$ as well.

To prove part (ii), we first show that if $\mathcal{G}$ is not contained in $n \cdot \mathcal{K}_{n,n}$, then $\mathcal{G}$ contains one of the ordered graphs from Figure 2.3.

The ordered graph $\mathcal{G}$ contains a nonseparable ordered graph $\mathcal{H}$ with interval chromatic number $t \geq 3$, since $\mathcal{G}$ is not an ordered subgraph of $n \cdot \mathcal{K}_{n,n}$. Let

$I_1, \dots, I_t$ be a partitioning of the vertex set of $\mathcal{H}$ into t consecutive intervals such that there is no edge of $\mathcal{H}$ with both vertices in the same interval. Then $\mathcal{H}$ has an edge e between intervals I_1 and I_2 and an edge f between intervals I_2 and I_3. If e and f share a vertex, they form a monotone path on three vertices, which is the first ordered graph in Figure 2.3.

Assume that no vertex of I_2 has a neighbor in both I_1 and $I_3 \cup \cdots \cup I_t$. Then we partition I_2 into sets A_1, A_2, and A_3 such that every vertex of A_1 has a neighbor in I_1, no vertex in A_2 has a neighbor in $I_1 \cup I_3$, and every vertex of A_3 has a neighbor in I_3. If A_3 is to the left of A_1, then we can move some vertices of I_2 into I_1 and some into I_3 to obtain a partitioning of the vertex set of $\mathcal{H}$ into $t-1$ intervals such that there is no edge with both vertices in the same interval. This is impossible, as the interval chromatic number of $\mathcal{H}$ is t. Thus we can assume that the vertex in $e \cap I_2$ is to the left of the vertex in $f \cap I_2$ and that every vertex between $e \cap I_2$ and $f \cap I_2$ lies in A_2.

Since $\mathcal{H}$ is nonseparable, there is an edge g of $\mathcal{H}$ with one vertex to the left of $e \cap I_2$ and the other one to the right of $f \cap I_2$. The left vertex of g either lies to the left of $e \cap I_1$, or is in $e \cap I_1$, or lies between $e \cap I_1$ and $e \cap I_2$. Similarly, the right vertex of g is either to the right of $f \cap I_3$, or is in $f \cap I_3$ or lies between $f \cap I_3$ and $f \cap I_2$. This gives us nine pairwise nonisomorphic ordered graphs formed by the edges g, e, and f. Each of these ordered graphs is in Figure 2.3.

To finish the proof, note that every color in the coloring of $\mathcal{K}_{2^c}$ from the proof of Proposition 1.6 with $r_1 = \cdots = r_c = 2 = s_1 = \cdots = s_c$ induces an ordered subgraph of $2^c \cdot \mathcal{K}_{2^c,2^c}$. In particular, there is no monochromatic copy of $\mathcal{G}$. Therefore we have $\overline{\mathrm{R}}(\mathcal{G}; c) > 2^c$. □

2.5 Minimum ordered Ramsey numbers

By Theorem 2.1, there are arbitrarily large ordered matchings $\mathcal{M}$ on n vertices such that $\overline{\mathrm{R}}(\mathcal{M}) \geq n^{\Omega(\frac{\log n}{\log \log n})}$. In fact, Conlon et al. [CFLS14] showed that ordered Ramsey numbers of almost every ordered matching on n vertices satisfy this lower bound. In contrast, it is not difficult to find an ordered matching $\mathcal{M}$ on $2n$ vertices with $\overline{\mathrm{R}}(\mathcal{M})$ only linear in n. One such example is provided by the ordered matching $\mathcal{M}_{2n}$ on $[2n]$ with edges $\{i, 2n+1-i\}$ for $i = 1, \dots, n$, which is sketched in part (a) of Figure 2.4. Another example of an ordered matching $\mathcal{M}'_{2n}$ with $\overline{\mathrm{R}}(\mathcal{M}'_{2n})$ linear in n is sketched in part (b) of Figure 2.4. It follows easily from the pigeonhole principle that $\overline{\mathrm{R}}(\mathcal{M}_{2n}), \overline{\mathrm{R}}(\mathcal{M}'_{2n}) \leq 4n-2$.

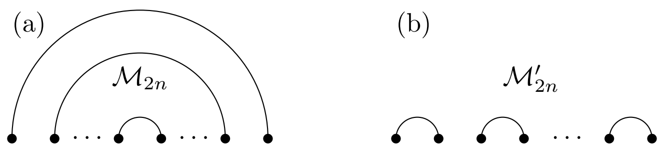


Figure 2.4: Two orderings of 1-regular graphs with the ordered Ramsey number linear in the number of vertices.

In particular, we see that every n-vertex graph with maximum degree 1 admits an ordering with the ordered Ramsey number linear in n. It is a natural question whether for every fixed positive integer Δ and every graph G on n vertices with maximum degree Δ there is an ordering $\mathcal{G}$ of G such that $\overline{\mathrm{R}}(\mathcal{G})$ is linear in n. Conlon et al. [CFLS14] consider this to be unlikely and pose the following problem.

Problem 2.18 ([CFLS14, Problem 6.7])**.** *Do random 3-regular graphs have superlinear ordered Ramsey numbers for all orderings?*

In this section, we study this problem from a more general perspective. For a graph G, we let min-$\overline{\mathrm{R}}(G)$ be the minimum of $\overline{\mathrm{R}}(\mathcal{G})$ over all orderings $\mathcal{G}$ of G. We call the parameter min-$\overline{\mathrm{R}}(G)$ the *minimum ordered Ramsey number of* G. Note that we have $\mathrm{R}(G) \leq$ min-$\overline{\mathrm{R}}(G) \leq \overline{\mathrm{R}}(\mathcal{G})$ for every graph G and every ordering $\mathcal{G}$ of G. Problem 2.18 then asks whether random 3-regular graphs have superlinear minimum ordered Ramsey numbers.

2.5.1 Lower bounds for bounded-degree graphs

Here, we give an affirmative answer to Problem 2.18. In fact, we solve the problem in a slightly more general setting, by extending the concept of d-regular graphs to non-integral values of d. For a real number $\rho > 0$ and a positive integer n with $\lceil \rho n \rceil$ even, we say that a graph G on n vertices is *ρ-regular*, if every vertex of G has degree $\lfloor \rho \rfloor$ or $\lceil \rho \rceil$ and the total number of edges of G is $\lceil \rho n \rceil/2$.

Note that when ρ is an integer, the above definition coincides exactly with the standard notion of regular graphs. We let $G(\rho, n)$ denote the random ρ-regular graph on n vertices drawn uniformly and independently from the set of all ρ-regular graphs on the vertex set $[n]$.

The graph $G(\rho, n)$ satisfies an event A *asymptotically almost surely* if the probability that A holds tends to 1 as n goes to infinity.

We may now state our main result in this subsection.

Theorem 2.19. *For every fixed real number $\rho > 2$, asymptotically almost surely*

$$\text{min-}\overline{\mathrm{R}}(G(\rho, n)) \geq \frac{n^{3/2-1/\rho}}{4 \log n \log \log n}.$$

In particular, almost every 3-regular graph G on n vertices satisfies min-$\overline{\mathrm{R}}(G) \geq \frac{n^{7/6}}{4 \log n \log \log n}$.

The main ingredient of the proof of Theorem 2.19 is the following technical result.

Theorem 2.20. *Let $\{\varepsilon_n\}_{n\geq 1}$, $\{\zeta_n\}_{n\geq 1}$, and $\{\rho_n\}_{n\geq 1}$ be sequences of real numbers satisfying these constraints:*

- $0 < \varepsilon_n$ *and* $0 < \zeta_n = o(1)$ *for every n large enough,*
- *there is a constant C such that $1 \leq \rho_n \leq C$ for all n,*
- $\lim_{n\to\infty} \left((\frac{1}{2} - \varepsilon_n)\rho_n - \zeta_n - 1 \right) n \log n = \infty$.

Then asymptotically almost surely the graphs $G(\rho_n, n)$ satisfy

$$\text{min-}\overline{\mathrm{R}}(G(\rho_n, n)) \geq \frac{\zeta_n n^{1+\varepsilon_n}}{2 \log (1/\zeta_n)}.$$

Proof of Theorem 2.19. For every $n \geq 2$, let $\rho_n := \rho$, $\varepsilon_n := 1/2 - 1/\rho - 1/\log n$, and $\zeta_n := 1/\log n$. Since $\rho > 2$, we have $0 < \varepsilon_n$, for n large enough, and the remaining assumptions of Theorem 2.20 are satisfied as well. Theorem 2.19 then follows directly from Theorem 2.20. $\square$

The next corollary of Theorem 2.20 shows that there are actually "almost 2-regular" graphs that have superlinear ordered Ramsey numbers for all orderings.

Corollary 2.21. *Asymptotically almost surely, graphs* $G_n := G(2 + \frac{9\log\log n}{\log n}, n)$ *satisfy*

$$\text{min-}\overline{\mathrm{R}}(G_n) \geq \frac{n \log n}{2 \log\log n}.$$

Proof. It suffices to set $\rho_n := 2 + \frac{9\log\log n}{\log n}$, $\varepsilon_n := \frac{2\log\log n}{\log n}$, and $\zeta_n := \frac{1}{\log n}$ and apply Theorem 2.20. The assumptions of Theorem 2.20 are satisfied, since

$$\begin{aligned}\left(\frac{1}{2} - \varepsilon_n\right)\rho_n - \zeta_n - 1 &= \frac{9\log\log n}{2\log n} - \frac{4\log\log n}{\log n} - 18\left(\frac{\log\log n}{\log n}\right)^2 - \frac{1}{\log n} \\ &= \frac{\log\log n}{2\log n} - O\left(\frac{1}{\log n}\right).\end{aligned}$$

The corollary follows. $\square$

We now prove Theorem 2.20.

Lemma 2.22. *Let M, n, s, t be positive integers with $M \leq \binom{t}{2} + t$ and let $\rho > 0$ and $\delta > 0$ be real numbers such that $\lceil \rho n \rceil$ is even and*

$$t^n \cdot \binom{t^2}{M} \cdot \binom{s^2 M}{\lceil \rho n \rceil/2} < \delta D \tag{2.1}$$

where D is the number of ρ-regular graphs on $[n]$. Then, with probability at least $1 - \delta$, the graph $G(\rho, n)$ satisfies the following statement: for every partition of $V(G(\rho, n))$ into sets $X_1, \ldots, X_t$, each of size at most s, there are more than M pairs (X_i, X_j) with $1 \leq i \leq j \leq t$ and $e_{G(\rho,n)}(X_i, X_j) > 0$.

Proof. Let $X_1, \ldots, X_t$ be a partition of the set $[n]$ such that X_i contains at most s elements for every $1 \leq i \leq t$. Let S be a set of M pairs (X_i, X_j) for some $1 \leq i \leq j \leq t$. We let h denote the number of graphs H on $[n]$ with $\lceil \rho n \rceil/2$ edges such that $e_H(X_i, X_j) = 0$ for every pair (X_i, X_j) with $1 \leq i \leq j \leq t$ that is not contained in S. Since the size of every X_i is at most s, we have

$$h \leq \binom{s^2 M}{\lceil \rho n \rceil/2}$$

for every set S. Since every ρ-regular graph on n vertices contains $\lceil \rho n \rceil/2$ edges, the probability that there are no edges of $G(\rho, n)$ between X_i and X_j for every $(X_i, X_j) \notin S$ is at most h/D.

The number of partitions $X_1, \ldots, X_t$ of $[n]$ is at most t^n. The number of choices for the set S is at most $\binom{t^2}{M}$. Altogether, the expected number of partitions of $[n]$

into sets $X_1, \dots, X_t$ of size at most s with at most M pairs (X_i, X_j), $1 \le i \le j \le t$, that satisfy $e_{G(\rho,n)}(X_i, X_j) > 0$ is at most

$$\frac{t^n}{D} \cdot \binom{t^2}{M} \cdot \binom{s^2 M}{\lceil \rho n \rceil / 2}.$$

By our assumption this term is bounded from above by δ. By Markov's inequality, the probability that $G(\rho, n)$ satisfies the statement from the lemma is at least $1 - \delta$. $\square$

Bender and Canfield [BC78] and independently Wormald [Wor78] showed that the number D of d-regular graphs on n vertices satisfies

$$D = (1 + o(1)) \frac{(dn)!}{2^{\frac{dn}{2}} \left(\frac{dn}{2}\right)! (d!)^n} \exp\left(\frac{1 - d^2}{4}\right)$$

for a fixed integer d and a sufficiently large n such that dn is even. Bender and Canfield [BC78] also proved an asymptotic formula for the number of labeled graphs with a given degree sequence. In the case of ρ-regular graphs, their result gives the following estimate.

Corollary 2.23 ([BC78, Theorems 1 and 2])**.** *For a real number $\rho \ge 2$ with $\lceil \rho n \rceil$ even, the number of ρ-regular graphs with the vertex set $[n]$, for n sufficiently large, is at least*

$$\frac{\lceil \rho n \rceil !}{2^{\lceil \rho n \rceil / 2} \left(\frac{\lceil \rho n \rceil}{2}\right)! (d!)^n (d+1)^{\lceil \gamma n \rceil} e^{d^2}},$$

where $d := \lfloor \rho \rfloor$ and $\gamma := \rho - d \in [0, 1)$.

Proof of Theorem 2.20. Let ε_n, ζ_n, and ρ_n be sequences satisfying the assumptions of the theorem. Note that these assumptions imply $\varepsilon_n < 1/2$ for a sufficiently large n. We may also assume $\zeta_n = \omega(1/n^{\varepsilon_n})$, as otherwise the statement is trivial. We set $m := \lceil \rho_n n \rceil$. That is, m is the sum of degrees of every ρ_n-regular graph on n vertices.

We set $M := \zeta_n n$, $s := n^{\varepsilon_n}$, $t := \frac{\zeta_n n}{2 \log(1/\zeta_n)}$, and $\delta := 2^{(\rho_n(\varepsilon_n - 1/2) + 1 + \zeta_n) n \log n}$. Note that by our assumptions on ε_n, ζ_n, and ρ_n, the value of δ tends to zero as n goes to infinity. We show that the parameters δ, M, n, s, t satisfy (2.1).

By Corollary 2.23, the number D of ρ_n-regular graphs on n vertices is asymptotically at least

$$\frac{m!}{2^{\frac{m}{2}} \left(\frac{m}{2}\right)! (d_n!)^n (d_n + 1)^{\lceil \gamma_n n \rceil} e^{d_n^2}},$$

where $d_n := \lfloor \rho_n \rfloor$ and $\gamma_n := \rho_n - d_n$.

Recalling that the sequence d_n is bounded, and using the estimate $(k/e)^k \le k! \le k^k$ for a positive integer k, we obtain that D is at least

$$\frac{\left(\frac{m}{e}\right)^m}{2^{m/2} \left(\frac{m}{2}\right)^{m/2} d_n^{d_n n} (d_n + 1)^{\lceil \gamma_n n \rceil} e^{d_n^2}} > \frac{m^{m/2}}{2^{O(n)}}.$$

By the choice of M, s, t and by $D > \frac{m^{m/2}}{2^{O(n)}}$, the left side of (2.1) divided by D is at most

$$\frac{2^{O(n)}}{m^{m/2}} \left(\frac{\zeta_n n}{2 \log(1/\zeta_n)}\right)^n \cdot \binom{\left(\frac{\zeta_n n}{2 \log(1/\zeta_n)}\right)^2}{\zeta_n n} \cdot \binom{\zeta_n n^{2\varepsilon_n + 1}}{m/2}.$$

Applying the estimates $\zeta_n = o(1)$ and $\binom{a}{b} \leq \left(\frac{ea}{b}\right)^b$, we bound this term from above by

$$\frac{1}{m^{m/2}} \cdot n^n \cdot \left(\frac{e\zeta_n n}{4\log^2(1/\zeta_n)}\right)^{\zeta_n n} \cdot \left(\frac{2e\zeta_n n^{2\varepsilon_n+1}}{m}\right)^{m/2}.$$

Using elementary calculations, we see that for n large enough, this is less than

$$\begin{aligned} n^{-m/2+n+\zeta_n n+(2\varepsilon_n+1)m/2-m/2} &\leq 2^{(-\rho_n+1+\zeta_n+(\varepsilon_n+1/2)\rho_n)n\log n} \\ &= 2^{(\rho_n(\varepsilon_n-1/2)+1+\zeta_n)n\log n} \\ &= \delta. \end{aligned}$$

The inequality follows from $m = \lceil \rho_n n \rceil$ and $\varepsilon_n < 1/2$. We also have $M \leq \binom{t}{2} + t$ due to $\zeta_n = \omega(1/n^{\epsilon_n})$ and $\epsilon_n < 1/2$. That is, the assumptions of Lemma 2.22 are satisfied. Since δ tends to zero as n goes to infinity, the statement of the lemma holds asymptotically almost surely for this choice of δ, M, s, t.

Let $\mathcal{R}$ be the ordered complete graph with loops and with the vertex set $[t]$. Let c be a coloring of $\mathcal{R}$ that assigns either a red or a blue color to every edge of $\mathcal{R}$ independently at random with probability $1/2$. Let $I_1, \dots, I_t$ be a partition of the vertex set of $\mathcal{K}_{st}$ into consecutive intervals, each of size s. We define the coloring c' of $\mathcal{K}_{st}$ such that the color $c'(e)$ of an edge e of $\mathcal{K}_{st}$ is $c(\{i,j\})$ if one vertex of e lies in I_i and the other one in I_j. Note that, since $\mathcal{R}$ contains loops, every edge of $\mathcal{K}_{st}$ receives some color via c'.

By Lemma 2.22, asymptotically almost surely, there are more than M pairs (X_i, X_j) with $1 \leq i \leq j \leq t$ and $e_{G(\rho_n,n)}(X_i, X_j) > 0$ in every partition of $V(G(\rho_n, n))$ into sets $X_1, \dots, X_t$ of size at most s. We show that if $G(\rho_n, n)$ satisfies this condition, then we have $\overline{\mathrm{R}}(\mathcal{G}) \geq st$ for every ordering $\mathcal{G}$ of $G(\rho_n, n)$.

Let $\mathcal{G}$ be an arbitrary ordering of $G(\rho_n, n)$. We show that the probability that there is a red copy of $\mathcal{G}$ in c' is less than $1/2$. Suppose there is a red copy $\mathcal{G}_0$ of $\mathcal{G}$ in c'. For $i = 1, \dots, t$, let $J_i := V(\mathcal{G}_0) \cap I_i$. Then $J_1, \dots, J_t$ induces a partition of the vertices of $\mathcal{G}$ into t (possibly empty) intervals of size at most s. The number of such partitions of $\mathcal{G}$ is at most

$$\binom{n+t-1}{t-1} \leq \left(\frac{e(n+t-1)}{t-1}\right)^{t-1} \leq \left(\frac{3n}{t}\right)^t = \left(\frac{6\log(1/\zeta_n)}{\zeta_n}\right)^t < 2^{2t\log(1/\zeta_n)},$$

where the last inequality follows from $\zeta_n = o(1)$, as then $6\log(1/\zeta_n) < 1/\zeta_n$.

By Lemma 2.22, there are more than M pairs (J_i, J_j) with $1 \leq i \leq j \leq t$ and $e_{\mathcal{G}_0}(J_i, J_j) > 0$. From the choice of c', the red copy $\mathcal{G}_0$ corresponds to an ordered subgraph $\mathcal{H}$ of $\mathcal{R}$ with more than M edges that are all red in c. Such ordered graph $\mathcal{H}$ appears in $\mathcal{R}$ with probability at most 2^{-M-1}.

The edges of $\mathcal{H}$ are determined by the partition $J_1, \dots, J_t$ of $\mathcal{G}$. Thus, by the union bound, the probability that there is a red copy of $\mathcal{G}$ in c' is less than

$$2^{2t\log(1/\zeta_n)} \cdot 2^{-M-1} = 2^{\zeta_n n - \zeta_n n - 1} = 1/2.$$

From symmetry, the probability that there is a blue copy of $\mathcal{G}$ in c' is less than $1/2$. Thus the probability that c' contains no monochromatic copy of $\mathcal{G}$ is positive. It follows that there is a coloring of $\mathcal{K}_{st}$ with no monochromatic copy of $\mathcal{G}$ and we have $\overline{\mathrm{R}}(\mathcal{G}) \geq st$.

Since $\mathcal{G}$ is an arbitrary ordering of $G(\rho_n, n)$, we obtain that asymptotically almost surely the graph $G(\rho_n, n)$ satisfies min-$\overline{\mathrm{R}}(G(\rho_n, n)) \geq st$. □

For the upper bounds in the case of larger maximum degree, a simple corollary of Theorem 2.9, states that every graph on n vertices with constant maximum degree admits an ordering $\mathcal{G}$ with $\overline{\mathrm{R}}(\mathcal{G})$ polynomial in n.

Corollary 2.24. *For a positive integer Δ, every graph G with n vertices and with maximum degree Δ satisfies* min-$\overline{\mathrm{R}}(G) \leq O(n^{(\Delta+1)\lceil\log(\Delta+1)\rceil+1})$.

Proof. Since the maximum degree of G equals Δ, there is a coloring of $V(G)$ by $\Delta + 1$ colors. Consider an ordering $\mathcal{G}$ of G that is obtained by placing the $\Delta + 1$ color classes as $\Delta + 1$ disjoint intervals. The interval chromatic number of $\mathcal{G}$ is then $\Delta + 1$ and the corollary follows from Theorem 2.9. □

Note that the gap between the upper bounds from Corollary 2.24 and the lower bounds from Theorem 2.19 is rather large. It would be interesting to close it at least for $\Delta = 3$.

2.5.2 Upper bounds for 2-regular graphs

In contrast to the lower bounds from Subsection 2.5.1, we show that the trivial linear lower bound is asymptotically the best possible for graphs of maximum degree two.

Theorem 2.25. *There is an absolute constant C such that every graph G on n vertices with maximum degree 2 satisfies* min-$\overline{\mathrm{R}}(G) \leq Cn$.

In fact, the following stronger Turán-type statement is true when G is bipartite.

Theorem 2.26. *For every real $\varepsilon > 0$, there is a constant $C(\varepsilon)$ such that, for every integer n, every bipartite graph G on n vertices with maximum degree 2 admits an ordering $\mathcal{G}$ of G that is contained in every ordered graph with $N := C(\varepsilon)n$ vertices and with at least εN^2 edges.*

The rest of the subsection is devoted to the proofs of Theorems 2.25 and 2.26. Since every graph of maximum degree two is a union of vertex disjoint paths and cycles, it suffices to prove these results for 2-regular graphs. We make no serious effort to optimize the constants.

Every 2-regular graph G is a union of pairwise vertex disjoint cycles. Thus a natural approach for proving Theorem 2.25 is to find an ordering of every cycle C_k with the ordered Ramsey number linear in k and then place these ordered cycles such that their vertex sets partition the vertex set of G into consecutive intervals in the ordering of G. However, this attempt fails in general, as shown by the following example.

Let $\mathcal{G}$ be such ordering of a 2-regular graph that consists of the cycle $C_{n/3}$ and $2n/9$ copies of C_3. Let $N := (n/3 - 1)4n/9$ and let c be the following coloring of $\mathcal{K}_N$. Partition the vertex set of $\mathcal{K}_N$ into consecutive intervals $I_1, \ldots, I_{4n/9}$ of size $n/3 - 1$ and color all edges between vertices from the same interval I_i blue. Then color all remaining edges red. The coloring c contains no blue copy of $\mathcal{G}$, as the longest cycle $C_{n/3}$ has more vertices than any interval I_i. There is also no red copy of $\mathcal{G}$ in c, as no two vertices of any red ordered 3-cycle are in the same interval I_i. Altogether, we have $\overline{\mathrm{R}}(\mathcal{G}) \geq \Omega(n^2)$.

Our approach in the proofs of Theorems 2.25 and 2.26 is based on the alternating paths $(P_n, \triangleleft_{alt})$ that are introduced in Section 1.3. See Figure 1.3 for an example of the alternating path on seven vertices. We recall that, by Proposition 1.9, for every real $\varepsilon > 0$ every ordered graph on $N \geq n/\varepsilon$ vertices with at least εN^2 edges contains $(P_n, \triangleleft_{alt})$ as an ordered subgraph.

Let $\mathcal{M}_{2n}$ be the ordered matching from part (a) of Figure 2.4.

Corollary 2.27. *Let $\varepsilon > 0$ be a real constant. Then, for every integer n, every ordered graph on $N \geq 2n/\varepsilon$ vertices with at least εN^2 edges contains $\mathcal{M}_{2n}$ as an ordered subgraph.*

Proof. This result follows easily from Proposition 1.9, as $\mathcal{M}_{2n}$ is an ordered subgraph of $(P_{2n}, \triangleleft_{alt})$. □

For positive integers k and n, we use $\mathcal{P}_n^k$ to denote the ordered graph that is obtained from the alternating path on n vertices by replacing every edge with a copy of $\mathcal{K}_{k,k}$. Formally, let $(P_n, \triangleleft_{alt})$ be the alternating path with the vertex set $[n]$ and let $B_1, \dots, B_n$ be a collection of consecutive intervals, each of size k. Then $\cup_{i=1}^k B_i$ is the vertex set of $\mathcal{P}_n^k$ and a pair $\{u, v\}$ with $u \in B_i$ and $v \in B_j$ is an edge of $\mathcal{P}_n^k$ if and only if $\{i, j\}$ is an edge of $(P_n, \triangleleft_{alt})$; see Figure 2.5. We call the ordered graph $\mathcal{P}_n^k$ the *k-blow-up of* $(P_n, \triangleleft_{alt})$ and we call the intervals $B_1, \dots, B_n$ the *blocks of* $\mathcal{P}_n^k$. Note that $\mathcal{P}_n^k$ has kn vertices and that $\mathcal{P}_n^1 = (P_n, \triangleleft_{alt})$.

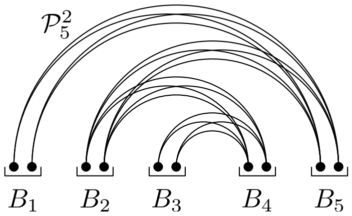


Figure 2.5: The 2-blow-up $\mathcal{P}_5^2$ of $(P_5, \triangleleft_{alt})$.

If $\mathcal{H}$ is an ordered graph with the vertex set partitioned into consecutive intervals $I_1, \dots, I_m$, then we say that $\mathcal{P}_n^k$ is an *ordered subgraph of $\mathcal{H}$ respecting the partitioning* $I_1, \dots, I_m$ if $\mathcal{P}_n^k \subseteq \mathcal{H}$, every block of $\mathcal{P}_n^k$ is contained in some interval I_i, and no two blocks of $\mathcal{P}_n^k$ are contained in the same interval I_i.

The following result is a variant of Proposition 1.9 for k-blow-ups of $(P_n, \triangleleft_{alt})$.

Lemma 2.28. *Let $\varepsilon > 0$ be a real constant and let k and $d \geq k(2/\varepsilon)^k$ be positive integers. Then, for every integer n, every ordered graph $\mathcal{H}$ with $N \geq 2d^{2k+1}\varepsilon^{-1}n$ vertices partitioned into consecutive intervals $I_1, \dots, I_{N/d}$, each of size d, and with at least εN^2 edges contains $\mathcal{P}_n^k$ as an ordered subgraph respecting the partitioning $I_1, \dots, I_{N/d}$.*

Proof. For integers k and $m \geq k$, the Kővári–Sós–Turán Theorem [KST54] (Theorem 2.15) says that every bipartite graph with color classes of size m, which contains no $K_{k,k}$ as a subgraph, has fewer than $k^{1/k}m^{2-1/k} + km \leq 2k^{1/k}m^{2-1/k}$ edges. Since the ordering $\mathcal{K}_{m,m}$ of $K_{m,m}$ is uniquely determined up to isomorphism, we see that the Kővári–Sós–Turán theorem is true in the ordered setting. That is, every ordered graph, which is contained in $\mathcal{K}_{m,m}$ and which contains no $\mathcal{K}_{k,k}$ as an ordered subgraph, has fewer than $2k^{1/k}m^{2-1/k}$ edges.

Let $\mathcal{H}$ be an ordered graph on $N \geq 2d^{2k+1}\varepsilon^{-1}n$ vertices partitioned into consecutive intervals $I_1, \ldots, I_{N/d}$, each of size d, and with at least εN^2 edges.

There are at least $\frac{\varepsilon N^2}{2d^2}$ pairs $\{i,j\} \in \binom{[N/d]}{2}$ with $e_{\mathcal{H}}(I_i, I_j) \geq 2k^{1/k}d^{2-1/k}$. Otherwise there are fewer than

$$\frac{N}{d}\binom{d}{2} + (1-\varepsilon)\binom{N/d}{2} \cdot 2k^{1/k}d^{2-1/k} + \frac{\varepsilon N^2}{2d^2} \cdot d^2$$
$$\leq \frac{N^2}{2d^2} \cdot 2k^{1/k}d^{2-1/k} + \frac{\varepsilon N^2}{2d^2} \cdot d^2 = \varepsilon N^2 \left(\frac{1}{\varepsilon}\left(\frac{k}{d}\right)^{1/k} + \frac{1}{2}\right) \leq \varepsilon N^2$$

edges in $\mathcal{H}$, which contradicts our assumptions. The last inequality follows from $d \geq k(2/\varepsilon)^k$. By the Kővári–Sós–Turán Theorem (Theorem 2.15), there are at least $\frac{\varepsilon N^2}{2d^2}$ pairs $\{I_i, I_j\}$ that induce a copy of $\mathcal{K}_{k,k}$ in $\mathcal{H}$.

Let $\mathcal{R}$ be an ordered graph with the vertex set $[N/d]$ such that $\{i,j\}$ is an edge of $\mathcal{R}$ if and only if there is a copy of $\mathcal{K}_{k,k}$ in $\mathcal{H}$ with one part in I_i and the other one in I_j. For every edge $\{i,j\}$ of $\mathcal{R}$, we fix one such copy $\mathcal{K}$ of $\mathcal{K}_{k,k}$ and say that $\mathcal{K}$ *represents* the edge $\{i,j\}$. The *type of the left color class* of $\mathcal{K}$ is the image of the left color class of $\mathcal{K}$ via the bijective mapping $I_i \to [d]$ that preserves the ordering of I_i. Similarly, the *type of the right color class* of $\mathcal{K}$ is the image of the right color class via the bijective mapping $I_j \to [d]$ that preserves the ordering of I_j.

We know that $\mathcal{R}$ contains at least $\frac{\varepsilon N^2}{2d^2}$ edges. Let T be a set of all pairs (A,B) of k-tuples $A, B \in \binom{[d]}{k}$. Note that the size of T is $\binom{d}{k}^2 \leq d^{2k}$. We assign a pair (A,B) to every edge $\{i,j\}$ of $\mathcal{R}$ if the type of the left and the right color class of the copy of $\mathcal{K}_{k,k}$ that represents $\{i,j\}$ is A and B, respectively. By the pigeonhole principle there are at least $\frac{\varepsilon N^2}{2d^{2k+2}}$ edges of $\mathcal{R}$ with the same pair (A_0, B_0).

Let $\mathcal{R}'$ be the ordered subgraph of $\mathcal{R}$ consisting of edges that were assigned the pair (A_0, B_0). By our observations, $\mathcal{R}'$ contains N/d vertices and at least $\frac{\varepsilon}{2d^{2k}}(\frac{N}{d})^2$ edges. Since $N/d \geq 2d^{2k}\varepsilon^{-1}n$, Proposition 1.9 implies that $(P_n, \lhd_{alt})$ is an ordered subgraph of $\mathcal{R}'$. This alternating path corresponds to a monochromatic copy of $(P_n, \lhd_{alt})$ in $\mathcal{R}$. It follows from the construction of $\mathcal{R}$ that $\mathcal{P}_n^k$ is an ordered subgraph of $\mathcal{H}$ respecting the partitioning $I_1, \ldots, I_{N/d}$. □

We now introduce orderings of cycles that we use in the proofs of Theorems 2.25 and 2.26. The final ordering of a given 2-regular graph will be obtained by constructing a union of these ordered cycles. For a positive integer n, let $(P_n, \lhd_{alt})$ be the alternating path on vertices $u_1 \lhd_{alt} \cdots \lhd_{alt} u_n$.

For $n \geq 3$, the even *alternating cycle* $\mathcal{C}_{2n-2} = (C_{2n-2}, \prec)$ is obtained from $(P_n, \lhd_{alt})$ as follows. First, for every $i \in [n] \setminus \{1, \lceil\frac{n+1}{2}\rceil\}$, we replace each vertex u_i with two consecutive vertices $v_i \prec w_i$. For $i \in \{1, \lceil\frac{n+1}{2}\rceil\}$, we set $v_i := u_i$ and $w_i := u_i$. Then, for every edge $\{u_i, u_j\}$ of $(P_n, \lhd_{alt})$, we place edges $\{v_i, w_j\}$ and $\{w_i, v_j\}$ into $\mathcal{C}_{2n-2}$. The *blocks of* $\mathcal{C}_{2n-2}$ are the sets $\{v_i, w_i\}$ for $i = 1, \ldots, n$. The edge that contains the $(n-1)$th and the nth vertex of $\mathcal{C}_{2n-2}$ is the *inner edge of* $\mathcal{C}_{2n-2}$; see part (a) of Figure 2.6.

For $n \geq 1$, the odd alternating cycle $\mathcal{C}_{2n+1} = (C_{2n+1}, \prec)$ is constructed similarly. For every $i \in [n] \setminus \{\lceil\frac{n+1}{2}\rceil\}$, we replace each vertex u_i with two consecutive vertices $v_i \prec w_i$. For $i = \lceil\frac{n+1}{2}\rceil$, we set $v_i := u_i$ and $w_i := u_i$. Then we place edges $\{v_i, w_j\}$ and $\{w_i, v_j\}$ for every edge $\{u_i, u_j\}$ of $(P_n, \lhd_{alt})$. Additionally, we insert two new vertices $v_0 \prec w_0$ to the left of v_1 and add edges $\{v_0, w_0\}$, $\{v_0, w_1\}$, and $\{w_0, v_1\}$ into $\mathcal{C}_{2n+1}$. The *blocks of* $\mathcal{C}_{2n+1}$ are the sets $\{v_i, w_i\}$ for $i = 1, \ldots, n$. The edge that contains the $(n+2)$th and the $(n+3)$th vertex of $\mathcal{C}_{2n+1}$ is the *inner edge of* $\mathcal{C}_{2n+1}$ and the edge $\{v_0, w_0\}$ is the *outer edge of* $\mathcal{C}_{2n+1}$; see part (b) of Figure 2.6.

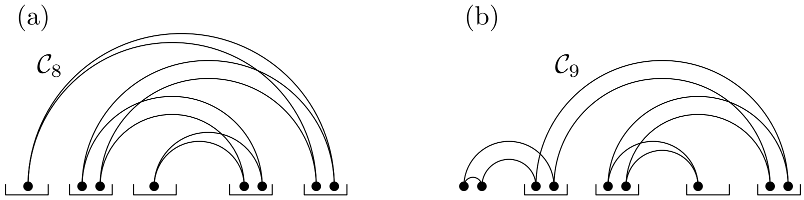


Figure 2.6: (a) The alternating cycle $\mathcal{C}_8$. (b) The alternating cycle C_9.

Using Lemma 2.28, we can now easily prove Theorem 2.26. We note that no such Turán-type result holds when the given graph is not bipartite. For example, the ordered graph $\mathcal{K}_{N/2,N/2}$ contains $N^2/4$ edges, while no ordered odd cycle is an ordered subgraph of $\mathcal{K}_{N/2,N/2}$.

Proof of Theorem 2.26. It is sufficient to prove the statement for 2-regular graphs, as every bipartite graph on n vertices with maximum degree 2 is a subgraph of a bipartite 2-regular graph on at most $4n$ vertices.

Let G be a given bipartite 2-regular graph partitioned into even cycles $C_{n_1}, \ldots, C_{n_m}$ with $n_1, \ldots, n_m \geq 4$ and $n_1 + \cdots + n_m = n$. First, we order each cycle C_{n_i} as the alternating cycle $\mathcal{C}_{n_i}$. The ordering $\mathcal{G}$ of G is then constructed by placing the vertex set of $\mathcal{C}_{n_i}$ between the vertices of the inner edge of $\mathcal{C}_{n_{i-1}}$ for every $i = 2, \ldots, m$; see Figure 2.7.

Let $\mathcal{H}$ be a given ordered graph with $N \geq 2^{16}\varepsilon^{-11}n$ vertices and with at least εN^2 edges. We show that $\mathcal{G}$ is an ordered subgraph of $\mathcal{H}$.

We apply Lemma 2.28 for $k := 2$, $d := 2(2/\varepsilon)^2$, and $\mathcal{H}$ that is partitioned into consecutive intervals $I_1, \ldots, I_{N/d}$, each of size d. Since $N \geq 2d^{2k+1}\varepsilon^{-1}n$, this gives us a copy of $\mathcal{P}_n^2$ in $\mathcal{H}$. To finish the proof, we observe that $\mathcal{G}$ is an ordered subgraph of $\mathcal{P}_n^2$. It suffices to greedily map the vertices of $\mathcal{G}$ into blocks of $\mathcal{P}_n^2$ such that every block of $\mathcal{C}_{n_i}$ is contained in a block of $\mathcal{P}_n^2$. Again, see Figure 2.7 for an illustration. Note that we might not use all blocks of $\mathcal{P}_n^2$ in the process. $\square$

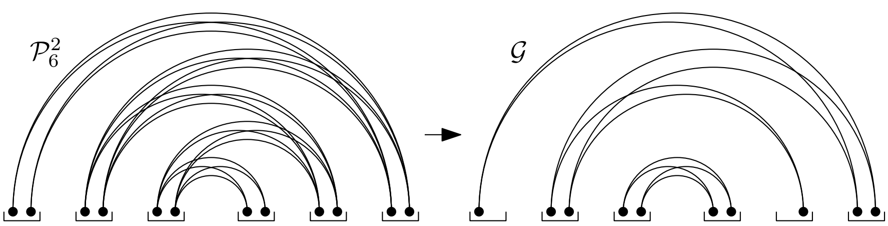


Figure 2.7: The ordering $\mathcal{G}$ of a 2-regular graph consisting of $\mathcal{C}_6$ and $\mathcal{C}_4$ is an ordered subgraph of $\mathcal{P}_6^2$.

For the rest of the subsection, we deal with the case when G is not bipartite. To do so, we first introduce an auxiliary ordered graph.

For integers k and n, let $B_1, \ldots, B_n$ be consecutive intervals, each of size k. Consider the ordered matching $\mathcal{M}_{2n} = (M_{2n}, \prec)$ from part (a) of Figure 2.4 and let $u_1 \prec \cdots \prec u_n$ and $v_1 \prec \cdots \prec v_n$ be the vertices of the left and the right color class of $\mathcal{M}_{2n}$, respectively. We place $\mathcal{M}_{2n}$ to the left of B_1. For every $i \in [n]$, we then add the edges $\{u_i, w\}$ and $\{v_{n+1-i}, w\}$ for every vertex $w \in B_i$ and use $\mathcal{T}_n^k$ to denote the resulting ordered graph on $(k+2)n$ vertices; see part (a) of Figure 2.8.

The intervals $B_1, \ldots, B_n$ are the *blocks of* $\mathcal{T}_n^k$. Let $\mathcal{H}$ be an ordered graph with the vertex set partitioned into consecutive intervals $I_1, \ldots, I_m$. We say that $\mathcal{T}_n^k$ is an *ordered subgraph of* $\mathcal{H}$ *respecting the partitioning* $I_1, \ldots, I_m$ if $\mathcal{T}_n^k \subseteq \mathcal{H}$, every block of $\mathcal{T}_n^k$ is contained in some interval I_i and no two blocks of $\mathcal{T}_n^k$ are contained in the same interval I_i.

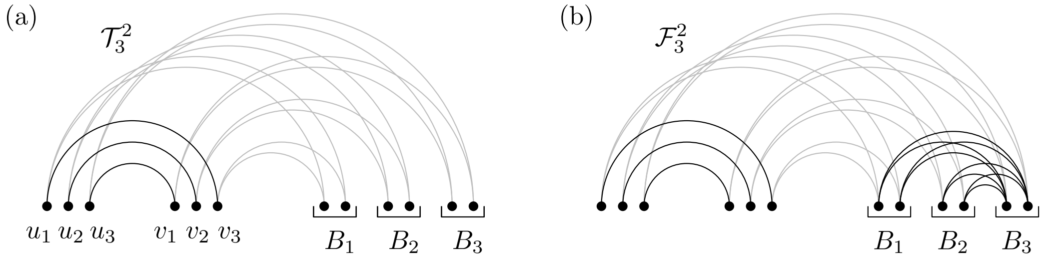


Figure 2.8: (a) The ordered graph $\mathcal{T}_3^2$. (b) The ordered graph $\mathcal{F}_3^2$.

Lemma 2.29. *Let $\varepsilon > 0$ be a real constant and let k and $d \geq 8\varepsilon^{-2}k$ be positive integers. Then, for every integer n, every ordered graph $\mathcal{H}$ with $N \geq 2^{14}\varepsilon^{-8}dn$ vertices partitioned into consecutive intervals $I_1, \ldots, I_{N/d}$, each of size d, and with at least εN^3 copies of $\mathcal{K}_3$ contains $\mathcal{T}_n^k$ as an ordered subgraph respecting the partitioning $I_1, \ldots, I_{N/d}$.*

Proof. We assume that the set $[N]$ is the vertex set of $\mathcal{H}$. A copy of $\mathcal{K}_3$ is called a *triangle*. Let $u < v < w \in [N]$ be the vertices of a triangle $\mathcal{K}$ in $\mathcal{H}$. The vertex u is the *leftmost* vertex of $\mathcal{K}$ and w is the *rightmost* vertex of $\mathcal{K}$. We call the edges $\{u, v\}$ and $\{v, w\}$ the *left leg* and the *right leg* of $\mathcal{K}$, respectively. We recall that the length of an edge $\{u, v\}$ of $\mathcal{H}$ equals $|u - v|$.

First, there is a set T of at least $\varepsilon N^3/2$ triangles from $\mathcal{H}$ with the right leg of length at least $\varepsilon N/2$. Otherwise the total number of copies of $\mathcal{K}_3$ in $\mathcal{H}$ is less than

$$\frac{\varepsilon N^3}{2} + \frac{\varepsilon N^2}{2} \cdot N = \varepsilon N^3,$$

as there are at most $\varepsilon N^2/2$ edges of length smaller than $\varepsilon N/2$ in $\mathcal{H}$ and for each such edge there are at most N choices for the leftmost vertex of a triangle in $\mathcal{H}$.

For $i = 2, \ldots, N-1$, let T_i be the set of triangles from T with the rightmost vertex in $\{i+1, \ldots, N\}$ and with the remaining two vertices in $\{1, \ldots, i\}$. Then we have

$$\frac{\varepsilon^2 N^4}{4} \leq \frac{\varepsilon N}{2}|T| \leq |\{(i, \mathcal{K})\colon \mathcal{K} \in T_i, i = 2, \ldots, N-1)\}| = \sum_{i=2}^{N-1} |T_i|,$$

where the second inequality follows from the fact that every $\mathcal{K} \in T$ is contained in at least $\varepsilon N/2$ sets T_i. It follows that there is a set T_j of size at least $\varepsilon^2 N^3/4$.

We let $A := \{1, \ldots, j\}$ and $B := \{j+1, \ldots, N\}$. Every left leg of a triangle from T_j is an edge of $\mathcal{H}[A]$. There is a set S of at least $\varepsilon^2 N^2/8$ edges of $\mathcal{H}[A]$ such each edge from S is the left leg of at least $\varepsilon^2 N/4$ triangles from T_j. Otherwise the total number of triangles in T_j is less than

$$\frac{\varepsilon^2 N^2}{8} \cdot N + \frac{N^2}{2} \cdot \frac{\varepsilon^2 N}{4} = \frac{\varepsilon^2 N^3}{4}.$$

Let $\mathcal{R}$ be the ordered graph with the vertex set A and with the edge set S. We know that $\mathcal{R}$ contains $|A|$ vertices and at least $\varepsilon^2 N^2/8 \geq \varepsilon^2 |A|^2/8$ edges. Corollary 2.27 implies that $\mathcal{R}$ contains a copy $\mathcal{M}$ of $\mathcal{M}_{2N'}$ for an integer $N' := \varepsilon^2|A|/16$. Since $\binom{|A|}{2} \geq \varepsilon^2 N^2/8$, we obtain $|A| \geq \varepsilon N/2$ and $N' \geq \varepsilon^3 N/2^5$.

We use $u_1 < \cdots < u_{N'}$ and $v_1 < \cdots < v_{N'}$ to denote the vertices of the left and the right color class of $\mathcal{M}$, respectively. Let B' be the set $\{d \cdot i \colon i \in [N/d], I_i \cap B \neq \emptyset\}$. The set B' contains at most $\lceil |B|/d \rceil \leq N/d$ elements and is disjoint with A.

Let $\mathcal{R}'$ be the ordered graph with the vertex set $\{v_1, \dots, v_{N'}\} \cup B'$ and place an edge $\{v_i, d \cdot l\}$ in $\mathcal{R}'$ if there are at least k triangles with the vertex set $\{u_i, v_i, w\}$ for some $w \in I_l \cap B$ in $\mathcal{H}$.

Since every edge $\{u_i, v_i\}$ of $\mathcal{M}$ is the left leg of at least $\varepsilon^2 N/4$ triangles from T_j, every vertex v_i of $\mathcal{R}'$ has degree at least

$$\left(\frac{\varepsilon^2 N}{4} - k|B'|\right)\frac{1}{d} \geq \left(\frac{\varepsilon^2 N}{4} - \frac{kN}{d}\right)\frac{1}{d} \geq \frac{\varepsilon^2 N}{8d}$$

in $\mathcal{R}'$. The last inequality follows from $d \geq 8\varepsilon^{-2}k$.

The number $|V(\mathcal{R}')|$ of vertices of $\mathcal{R}'$ trivially satisfies $\varepsilon^3 N/2^5 \leq N' \leq |V(\mathcal{R}')| \leq N$. Thus $\mathcal{R}'$ contains at least

$$\frac{\varepsilon^3 N}{2^5} \cdot \frac{\varepsilon^2 N}{8d} = \frac{\varepsilon^5 N^2}{2^8 d} \geq \frac{\varepsilon^5}{2^8 d}|V(\mathcal{R}')|^2$$

edges. By Corollary 2.27, there is a copy $\mathcal{M}'$ of $\mathcal{M}_{2n}$ in $\mathcal{R}'$, since $|V(\mathcal{R}')| \geq \varepsilon^3 N/2^5 \geq 2n/(\varepsilon^5/2^8 d)$. It follows from the constructions of $\mathcal{R}$ and $\mathcal{R}'$ that there is a copy of $\mathcal{T}_n^k$ in $\mathcal{H}$ as an ordered subgraph respecting the partitioning $I_1, \dots, I_{N/d}$. □

Let k and n be positive integers. By combining the ordered graphs $\mathcal{P}_n^k$ and $\mathcal{T}_n^k$, we obtain an ordered graph that is used later to embed orderings of 2-regular graphs.

Let $B_1, \dots, B_n$ be the blocks of $\mathcal{T}_n^k$ and let $\mathcal{P}_n^k$ be the k-blow-up of $(P_n, \lhd_{alt})$ with blocks $B_1, \dots, B_n$. The ordered graph $\mathcal{F}_n^k$ is the union of $\mathcal{T}_n^k$ and $\mathcal{P}_n^k$; see part (b) of Figure 2.8. The intervals $B_1, \dots, B_n$ are the *blocks of* $\mathcal{F}_n^k$. Observe that we have $\mathcal{F}_n^k \subseteq \mathcal{F}_{n'}^{k'}$ for every $k \leq k'$ and $n \leq n'$.

Lemma 2.30. *For every $\varepsilon > 0$, there is a constant $\delta = \delta(\varepsilon) > 0$ such that every ordered graph with n vertices and with at least $(1/4 + \varepsilon)n^2$ edges contains at least δn^3 copies of $\mathcal{K}_3$.*

Proof. The Triangle Removal Lemma [RS78] states that for every $\varepsilon' > 0$ there is a $\delta' = \delta'(\varepsilon') > 0$ such that every graph on n vertices with at most $\delta' n^3$ copies of K_3 can be made K_3-free by removing at most $\varepsilon' n^2$ edges.

For a given $\varepsilon > 0$, let $\delta > 0$ be the parameter $\delta'(\varepsilon)$. Let H be the underlying graph in the given ordered graph on n vertices with at least $(1/4 + \varepsilon)n^2$ edges.

Suppose for contradiction that H contains fewer than δn^3 copies of K_3. By the Triangle Removal Lemma, we delete fewer than εn^2 edges of H and obtain a K_3-free graph with more than $(1/4+\varepsilon)n^2 - \varepsilon n^2 = n^2/4$ edges. However, this contradicts Turán's theorem (Theorem 1.5).

Thus H contains at least δn^3 copies of K_3. Since all orderings of complete graphs are isomorphic, the statement follows. □

Having all auxiliary results, we can now prove Theorem 2.25.

Proof of Theorem 2.25. Again, it is sufficient to prove the statement for 2-regular graphs, as every graph on n vertices with maximum degree 2 is a subgraph of a 2-regular graph on at most $3n$ vertices.

Let G be a given 2-regular graph consisting of cycles $C_{n_1}, \dots, C_{n_m}$ with $n_1, \dots, n_m \geq 3$ and $n_1 + \cdots + n_m = n$. First, we order each cycle C_{n_i} as the alternating cycle $\mathcal{C}_{n_i}$. For every $i = 1, \dots, m$, let $v_1^i, \dots, v_{n_i}^i$ be the vertices of $\mathcal{C}_{n_i}$ in this order.

For $i = 2, \dots, m$, the ordering $\mathcal{G}$ of G is then constructed iteratively as follows. For n_i even, we place the vertex set of $\mathcal{C}_{n_i}$ between the vertices of the inner edge of $\mathcal{C}_{n_{i-1}}$ (if $n_{i-1} = 3$, then we place the vertex set of $\mathcal{C}_{n_i}$ to the right of $\mathcal{C}_{n_{i-1}}$). For n_i odd, we place the outer edge of $\mathcal{C}_{n_i}$ between the vertices of the outer edge of a previous odd cycle $\mathcal{C}_{n_j}$. If $\mathcal{C}_{n_i}$ is the first odd cycle in the process, then we let v_1^i and v_2^i be the first two vertices in the ordering. Then we place the vertices $v_3^i, \dots, v_{n_i}^i$ between the vertices of the inner edge of $\mathcal{C}_{n_{i-1}}$ (if $n_{i-1} = 3$, then we place $v_3^i, \dots, v_{n_i}^i$ to the right of $\mathcal{C}_{n_{i-1}}$); see Figure 2.9, for an illustration.

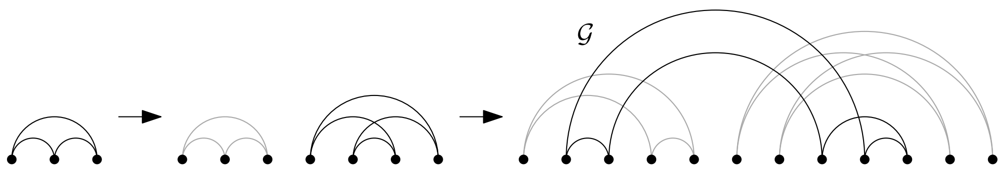


Figure 2.9: The iterative construction of the ordering $\mathcal{G}$ of a 2-regular graph G with parameters $m = 3$, $n_1 = 3$, $n_2 = 4$, and $n_3 = 5$.

The ordering $\mathcal{G}$ of G is an ordered subgraph of $\mathcal{F}_n^2$. For $i = 1, \dots, m$, it suffices to greedily map $v_1^i, \dots, v_{n_i}^i$ for n_i even and $v_3^i, \dots, v_{n_i}^i$ for n_i odd into blocks of $\mathcal{F}_n^2$ such that every block of $\mathcal{C}_{n_i}$ is contained in a block of $\mathcal{F}_n^2$. Then, for every odd n_i, we map v_1^i and v_2^i to an edge $\{j, n+1-j\}$ of $\mathcal{F}_n^2$ such that j and $n+1-j$ are adjacent to vertices of the block of $\mathcal{F}_n^2$ that contains the first block of $\mathcal{C}_{n_i}$. Note that we might not even use all the blocks of $\mathcal{F}_n^2$ in the process; see Figure 2.10.

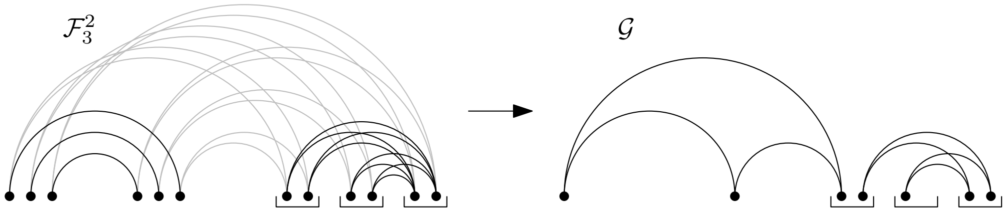


Figure 2.10: The ordered graph $\mathcal{F}_3^2$ contains the ordering $\mathcal{G}$ of a 2-regular graph consisting of $\mathcal{C}_3$ and $\mathcal{C}_4$.

Let $\delta \in (0,1)$ be the parameter $\delta(1/8)$ from Lemma 2.30. Let $N := 2^{163}\delta^{-10}n$ and let c be a red-blue coloring of $\mathcal{K}_N$. We show that c contains a monochromatic copy of $\mathcal{F}_n^2$. The rest then follows, as $\mathcal{G}$ is an ordered subgraph of $\mathcal{F}_n^2$.

Since $\mathrm{R}(K_3) = 6$, every six-tuple of vertices of $\mathcal{K}_N$ induces a coloring of $\mathcal{K}_6$ that contains a monochromatic copy of $\mathcal{K}_3$. Every such copy is contained in $\binom{N-3}{3}$ six-tuples of vertices of $\mathcal{K}_N$. This gives us at least $\binom{N}{6}/\binom{N-3}{3} = \frac{N(N-1)(N-2)}{120}$ monochromatic copies of $\mathcal{K}_3$ in c. Without loss of generality, we assume that at least half of them are red. Therefore we have at least $\frac{N(N-1)(N-2)}{240} > N^3/2^8$ red copies of $\mathcal{K}_3$ in c.

We let $N_1 := 2^{54}\delta^{-8}n$, $\varepsilon_1 := 1/2^8$, $k_1 := 2^{12}\delta^{-2}$, and $d_1 := 2^{31}\delta^{-2}$ and apply Lemma 2.29 with parameters ε_1, k_1, and d_1 for the ordered subgraph of $\mathcal{K}_N$ that is formed by red edges in c. Since there are at least $\varepsilon_1 N^3$ red copies of $\mathcal{K}_3$ in c, $d_1 \geq 8\varepsilon_1^{-2}k_1$, and $N \geq 2^{14}\varepsilon_1^{-8}d_1N_1$, this gives a red copy of $\mathcal{T}_{N_1}^{k_1}$ in c. Let $B_1, \dots, B_{N_1}$ be the blocks of this red copy of $\mathcal{T}_{N_1}^{k_1}$ and let $\mathcal{H}$ be the induced ordered subgraph $\mathcal{K}_N[B_1 \cup \cdots \cup B_{N_1}]$ of $\mathcal{K}_N$ with the red-blue coloring induced by c. Note that the sets $B_1, \dots, B_{N_1}$ are consecutive intervals of size k_1 that partition the vertex set of $\mathcal{H}$. In particular, $\mathcal{H}$ has k_1N_1 vertices.

Now we distinguish two cases. First, we assume that at least $\binom{k_1N_1}{2} - 3k_1^2N_1^2/8 > k_1^2N_1^2/2^4$ edges of $\mathcal{H}$ are red in c. Then we apply Lemma 2.28 with $\varepsilon_2 := 1/2^4$, $k_2 := 2$, and $d_2 := k_1$ for the red ordered subgraph of $\mathcal{H}$ that is partitioned into $B_1, \dots, B_{N_1}$. Since $d_2 = k_1 = 2^{12}\delta^{-2} > k_2(2/\varepsilon_2)^{k_2}$, $k_1N_1 = 2^{66}\delta^{-10}n > 2d_2^{2k_2+1}\varepsilon_2^{-1}n$, and $\mathcal{H}$ has at least $\varepsilon_2(k_1N_1)^2$ edges, we obtain a red copy of $\mathcal{P}_n^2$ as an ordered subgraph of $\mathcal{H}$ respecting the partitioning $B_1, \dots, B_{N_1}$. Together with the red copy of $\mathcal{T}_{N_1}^{k_1}$ in c, this gives a red copy of $\mathcal{F}_n^2$ in $\mathcal{H}$.

Otherwise there are more than $3k_1^2N_1^2/8$ blue edges of $\mathcal{H}$ in c. By Lemma 2.30, the ordered graph $\mathcal{H}$ contains at least $\delta k_1^3N_1^3$ blue copies of $\mathcal{K}_3$. We apply Lemma 2.29 with $\varepsilon_3 := \delta$, $k_3 := 2^9$, $d_3 := k_1$, and $N_3 := 2^{40}n$ for the blue ordered subgraph of $\mathcal{H}$ that is partitioned into $B_1, \dots, B_{N_1}$. Since $d_3 = k_1 \geq 8\varepsilon_3^{-2}k_3$ and $k_1N_1 = 2^{66}\delta^{-10} \geq 2^{14}\varepsilon_3^{-8}d_3N_3$, this gives us a blue copy of $\mathcal{T}_{N_3}^{k_3}$ as an ordered subgraph $\mathcal{H}$ respecting the partitioning $B_1, \dots, B_{N_1}$. Let $B'_1, \dots, B'_{N_3}$ be the blocks of this copy of $\mathcal{T}_{d_3}^{k_3}$ and let $\mathcal{H}'$ be the ordered graph with the vertex set $B'_1 \cup \cdots \cup B'_{N_3}$ formed by edges of the more frequent color in the coloring c of $\mathcal{H}[B'_1 \cup \cdots \cup B'_{N_3}]$. Again, observe that $B'_1, \dots, B'_{N_3}$ are consecutive intervals of size k_3 that partition the vertex set of $\mathcal{H}'$.

The ordered graph $\mathcal{H}'$ has k_3N_3 vertices and at least $\binom{k_3N_3}{2}/2 > k_3^2N_3^2/8$ edges. As the last step, we apply Lemma 2.28 with $\varepsilon_4 := 1/8$, $k_4 := 2$, $d_4 := k_3$ for $\mathcal{H}'$ partitioned into $B'_1, \dots, B'_{N_3}$. Since $d_4 = k_3 = 2^9 = k_4(2/\varepsilon_4)^{k_4}$ and $k_3N_3 = 2^{49}n \geq 2d_4^{2k_4+1}\varepsilon_4^{-1}n$, we obtain a monochromatic copy of $\mathcal{P}_n^2$ as an ordered subgraph of $\mathcal{H}'$ respecting the partitioning $B'_1, \dots, B'_{N_3}$. Together with either the red copy $\mathcal{T}_{N_1}^{k_1}$ of or with the blue copy of $\mathcal{T}_{N_3}^{k_3}$, we obtain a monochromatic copy of $\mathcal{F}_n^2$ in c. □

2.6 Open problems

There are many interesting open problems that arose in the study of ordered Ramsey numbers. Here, we would like to draw attention to some of them.

Corollary 2.8 and Theorem 2.9 imply that ordered Ramsey numbers of ordered graphs that have bounded degeneracy and bounded interval chromatic number are polynomial in the number of vertices. However, we have no nontrivial lower bounds.

Problem 2.31. *Is there an absolute constant $c > 0$ such that for every fixed Δ there is a sequence $\{\mathcal{G}_n\}_{n\geq 1}$ of ordered Δ-regular graphs $\mathcal{G}_n$ with n vertices and interval chromatic number 2 such that $\overline{\mathrm{R}}(\mathcal{G}_n) \geq n^{c\Delta}$?*

Similarly, it would be interesting to find some nontrivial lower bounds on ordered Ramsey numbers of ordered graphs of bounded bandwidth. For positive integers p and n, let $\mathcal{P}_n^{(p)}$ be the ordered graph on n vertices $v_1, \ldots, v_n$, in this order, such that $\{v_i, v_j\}$ is an edge if and only if $0 < |i - j| \leq p$. In particular, $\mathcal{P}_n^{(1)} = (P_n, \lhd_{mon})$. Note that every ordered graph with n vertices and with bandwidth at most p is an ordered subgraph of $\mathcal{P}_n^{(p)}$.

Problem 2.32. *For an integer $p \geq 2$, what is the growth rate of $\overline{\mathrm{R}}(\mathcal{P}_n^{(p)})$ with respect to n?*

By Proposition 1.9, the ordered Ramsey numbers of alternating paths are linear with respect to the number of vertices. Is it true that these orderings minimize ordered Ramsey numbers of ordered paths?

Problem 2.33. *For some positive integer n, is there an ordering $\mathcal{P}_n$ of the path P_n on n vertices such that $\overline{\mathrm{R}}(\mathcal{P}_n) < \overline{\mathrm{R}}((P_n, \lhd_{alt}))$?*

By Theorem 2.1, there are ordered matchings $\mathcal{M}$ on n vertices with $\overline{\mathrm{R}}(\mathcal{M}) \geq n^{\Omega(\frac{\log n}{\log \log n})}$. This bound is asymptotically almost tight, since, by Theorem 2.9, every ordered matching $\mathcal{M}$ on n vertices satisfies $\overline{\mathrm{R}}(\mathcal{M}) \leq n^{O(\log n)}$. It would be interesting to close the gap between these bounds.

Problem 2.34 ([CFLS14, Problem 6.2])**.** *Close the gap between the lower and upper bounds for ordered Ramsey numbers of ordered matchings.*

Similarly, if we restrict ourselves to ordered matchings with interval chromatic number 2, then there is still a gap between the upper bound from Observation 2.2 and the lower bound from Theorem 2.3.

Problem 2.35 ([CFLS14, Problem 6.3])**.** *Close the gap between the lower and upper bounds for ordered Ramsey numbers of ordered matchings with interval chromatic number 2.*

Concerning the non-diagonal case, Conlon et al. [CFLS14] showed that there are ordered matchings $\mathcal{M}$ on n vertices that satisfy $\overline{\mathrm{R}}(\mathcal{M}, \mathcal{K}_3) \geq \Omega((n/\log n)^{4/3})$. On the other hand, we have $\overline{\mathrm{R}}(\mathcal{M}, \mathcal{K}_3) \leq \mathrm{R}(K_n, K_3) \leq O(n^2/\log n)$ for every ordered n-vertex matching $\mathcal{M}$, where the last inequality was proved by Ajtai, Komlós, and Szemerédi [AKS80]. It is not clear which bound is closer to the truth.

Problem 2.36 ([CFLS14, Problem 6.1])**.** *Does there exist an $\varepsilon > 0$ such that every ordered matching $\mathcal{M}$ on n vertices satisfies $\overline{\mathrm{R}}(\mathcal{M}, \mathcal{K}_3) \leq O(n^{2-\varepsilon})$?*

We mostly considered the case of two colors and, in general, not much is known about ordered Ramsey numbers for more than two colors. Conlon et al. [CFLS14] proved that for $q \geq 2$ every ordered matching $\mathcal{M}$ on n vertices satisfies $\overline{\mathrm{R}}(\mathcal{M}; q) \leq n^{(2\log n)^{q-1}}$. They also believe that two following stronger upper bound holds.

Problem 2.37 ([CFLS14, Problem 6.8])**.** *Show that for every integer $q \geq 3$ there is a constant $c = c(q)$ such that every ordered matching $\mathcal{M}$ on n vertices satisfies $\overline{\mathrm{R}}(\mathcal{M}) \leq n^{c\log n}$.*

3. Applications

In this chapter we show some (and by no means all) examples, in which Ramsey-type problems on ordered hypergraphs appear. The examples consist of both classical and new results. We use some results from Chapters 1 and 2.

In Section 3.1, we show a connection between ordered Ramsey numbers of monotone paths and the well-known Erdős–Szekeres lemma on monotone subsequences. We then present a new proof of the famous Erdős–Szekeres theorem by Moshkovitz and Shapira [MS14]. This elegant proof is based on ordered Ramsey numbers of monotone hyperpaths. In Section 3.3 we refute a conjecture of Peters and Szekeres [PS06] about a possible strengthening of the Erdős–Szekeres conjecture to ordered hypergraphs. Finally, in Section 3.4 we show some applications of ordered Ramsey numbers in the theory of geometric Ramsey numbers that were introduced by Károlyi et al. [KPT97, KPTV98]. In particular, we provide the exact formula for geometric Ramsey numbers of cycles and show that convex geometric Ramsey numbers of outerplanar graphs are at most quasipolynomial in the number of vertices.

3.1 The Erdős–Szekeres Lemma

The *Erdős–Szekeres lemma* is a well-known fact proved by Erdős and Szekeres [ES35]. It states that, for every positive integer k, every sequence of at least $(k-1)^2+1$ distinct integers contains a decreasing or an increasing subsequence of length k. It is easy to see that the bound $(k-1)^2+1$ is sharp. The Erdős–Szekeres lemma has many proofs [Ste95] and we show that it is a special case of a Ramsey-type result for ordered graphs.

Given a sequence $S=(s_1,\dots,s_N)$ of distinct integers, we construct an ordered graph $\mathcal{K}_N=(K_N,\prec)$ with the vertex set S and the ordering of the vertices given by their positions in S. That is, for $s_i,s_j\in S$, we have $s_i\prec s_j$ if $i<j$. Then we color an edge $\{s_i,s_j\}$ with $i<j$ red if $s_i<s_j$ and blue otherwise. Afterwards, red monotone paths in $\mathcal{K}_N$ correspond to increasing subsequences of S and blue monotone paths in $\mathcal{K}_N$ to decreasing subsequences of S. Note that for every $N\geq 3$ there are red-blue colorings of $\mathcal{K}_N$ that cannot be obtained in this way. The Erdős–Szekeres lemma now follows from Proposition 1.8, which shows $\overline{\mathrm{R}}((P_k,\lhd_{mon}))=(k-1)^2+1$.

3.2 The Erdős–Szekeres Theorem

The *Erdős–Szekeres theorem* [ES35] is, without an exaggeration, one of the most important results in Ramsey theory. It is also one of the earliest results that contributed to the development of Ramsey theory [GN02]. It says that for every integer $k\geq 2$ there is a least number $\mathrm{ES}(k)$ such that every set of $\mathrm{ES}(k)+1$ points in the plane in *general position* (no three points lie on a common line and all x-coordinates are distinct) contains k points in convex position.

Points $p_1,\dots,p_k\in\mathbb{R}^2$ with increasing x-coordinates form a *k-cap* if the slopes of the lines $\overline{p_1p_2},\dots,\overline{p_{k-1}p_k}$ are decreasing. The points $p_1,\dots,p_k$ form a *k-cup* if the slopes are increasing. For integers $a, u \geq 2$, let $\mathrm{N}(a, u)$ be the maximum size of a set of points in the plane in general position with no a-cap and no u-cup. Note that vertices of every k-cap and k-cup are in convex position and thus we have $\mathrm{ES}(k) \leq \mathrm{N}(k, k)$.

Erdős and Szekeres [ES35] proved the bound $\mathrm{ES}(k) \leq \binom{2k-4}{k-2}$ by showing

$$\mathrm{N}(a, u) = \binom{a+u-4}{a-2} = \binom{a+u-4}{u-2} \tag{3.1}$$

for all integers $a, u \geq 2$.

Peters and Szekeres [PS06] and Fox et al. [FPSS12] considered a generalization of the Erdős–Szekeres theorem in terms of ordered hypergraphs. Fox et al. [FPSS12] suggested the following framework using monotone hyperpaths. We use the term k-*path* as an abbreviation for "3-uniform monotone hyperpath on k vertices" and, for integers $a, u \geq 2$, we use $\widehat{\mathrm{N}}(a, u)$ to denote the number $\overline{\mathrm{R}}((P^3_a, \triangleleft_{mon}), (P^3_u, \triangleleft_{mon})) - 1$. That is, $\widehat{\mathrm{N}}(a, u)$ is the maximum N such that there is a red-blue coloring of the ordered complete 3-uniform hypergraph $\mathcal{K}^3_N$ with no red a-path and no blue u-path. We assume that a 2-path consists of two isolated vertices.

We now observe that $\mathrm{N}(a, u) \leq \widehat{\mathrm{N}}(a, u)$ for all integers $a, u \geq 2$. Let P be a point set in the plane in general position. We color every triple T of points from P, ordered by x-coordinates, red if T is oriented clockwise and blue if T is oriented counter-clockwise. Every coloring of $\mathcal{K}^3_N$ obtained in this way from some point set of size N is called *realizable*. The inequality follows, as, for every $k \geq 3$, k-caps and k-cups in P are in one-to-one correspondence with red and blue, respectively, k-paths in the realizable coloring obtained from P.

Moshkovitz and Shapira [MS14] discovered a connection between ordered Ramsey numbers of monotone hyperpaths and high-dimensional integer partitions and introduced the following new proof of (3.1).

Theorem 3.1 ([FPSS12, MS14]). *For all integers $a, u \geq 2$, we have*

$$\widehat{\mathrm{N}}(a, u) = \binom{a+u-4}{a-2}.$$

Proof. We assume $a, u \geq 3$, as the cases with $a = 2$ or $u = 2$ are trivial. First, we prove the upper bound. For two elements $x = (x_1, x_2)$ and $y = (y_1, y_2)$ from $[a-2] \times [u-2]$, we let $x \preceq y$ if $x_1 \leq y_1$ and $x_2 \leq y_2$. We say that a subset D of $[a-2] \times [u-2]$ is a *down-set* if $y \in D$ implies $x \in D$ for every $x \in [a-2] \times [u-2]$ such that $x \preceq y$.

Claim 3.2. *The number of down-sets in $[a-2] \times [u-2]$ is $\binom{a+u-4}{a-2}$.*

To prove the claim, we bijectively map every down-set $D \subseteq [a-2] \times [u-2]$ to an integer sequence $(n_1, \dots, n_{a-2})$ that satisfies $u - 2 \geq n_1 \geq \cdots \geq n_{a-2} \geq 0$. For every i with $1 \leq i \leq a-2$, we let $n_i := \max\{y \colon (i, y) \in D\}$, where the maximum over an empty set is 0. Since D is a down-set in $[a-2] \times [u-2]$, we have $u - 2 \geq n_1 \geq \cdots \geq n_{a-2} \geq 0$. It is easy to verify that this mapping is a bijection. There is a one-to-one correspondence between the sequences $(n_1, \dots, n_{a-2})$ and lattice paths in $\mathbb{Z}^2$ that start in $(0, u-2)$, end in $(a-2, 0)$, and that go either down or to the right in each step. Since the steps in which the path moves down determine the path, the number of such lattice paths is exactly $\binom{a+u-4}{a-2}$. This finishes the proof of the claim.

Assume there is a red-blue coloring of $\mathcal{K}_N^3 = (K_N^3, <)$ with no red a-path and no blue u-path. We show $N \leq \binom{a+u-4}{a-2}$. For two vertices $u < v$ of $\mathcal{K}_N^3$, let $C(u,v)$ be a pair $(t_1 - 1, t_2 - 1)$ where t_1 is the number of vertices of the longest red monotone hyperpath that ends in u, v and t_2 is the number of vertices of the longest blue monotone hyperpath that ends in u, v. Note that $C(u,v) \in [a-2] \times [u-2]$. For a vertex v of $\mathcal{K}_N^3$, let $D(v) := \{x \in [a-2] \times [u-2] \colon x \preceq C(u,v) \text{ for some } u < v\}$. By definition, $D(v)$ is a down-set in $[a-2] \times [u-2]$.

Suppose for contradiction that there are vertices $u < v$ of $\mathcal{K}_N^3$ such that $D(u) = D(v)$. By definition, $C(u,v) \in D(v)$ and, since $D(u) = D(v)$, we have $C(u,v) \in D(u)$. From the definition of $D(u)$, there is a vertex $t < u$ of $\mathcal{K}_N^3$ such that $C(u,v) \preceq C(t,u)$. However, if $\{t,u,v\}$ is red, then we can extend the longest red monotone hyperpath that ends in t, u to a longer one that ends in u, v. Similarly if $\{t,u,v\}$ is blue. In both cases we have $C(u,v) \not\preceq C(t,u)$, a contradiction.

Since all the sets $D(v)$ are distinct, Claim 3.2 together with the pigeonhole principle imply $N \leq \binom{a+u-4}{a-2}$.

Now, we show the lower bound. Let $N := \binom{a+u-4}{a-2}$ and assume that the vertex set of $\mathcal{K}_N^3$ is the set of integer sequences $(n_1, \dots, n_{a-2})$ that satisfy $u - 2 \geq n_1 \geq \cdots \geq n_{a-2} \geq 0$ and that are ordered in the lexicographic ordering $\lessdot$. For two such sequences $A \lessdot B$, let $\delta(A,B)$ be the minimum position where they differ. For $i \in [a-2]$ and a vertex A of $\mathcal{K}_N^3$, we use A_i to denote the ith element of A. For vertices $A \lessdot B \lessdot C$, we color the edge $\{A,B,C\}$ of $\mathcal{K}_N^3$ red if $\delta(A,B) < \delta(B,C)$ and blue otherwise.

We now prove by induction on $k \geq 2$ that if $B \lessdot C$ are the last two vertices of a k-path $\mathcal{P}$ in $\mathcal{K}_N^3$, then $\delta(B,C) > k-2$ if $\mathcal{P}$ is red and $C_{\delta(B,C)} > k-2$ if $\mathcal{P}$ is blue. This will imply that there is no red a-path and no blue u-path in the coloring of $\mathcal{K}_N^3$.

If two vertices $A \lessdot B$ form a 2-path, then $\delta(A,B) > 0$ and $B_{\delta(A,B)} > 0$. For $k > 2$, let $A \lessdot B \lessdot C$ be the last three vertices of $\mathcal{P}$. From the inductive hypothesis, we have $\delta(A,B) \geq k-2$ if $\mathcal{P}$ is red and $B_{\delta(A,B)} \geq k-2$ if $\mathcal{P}$ is blue. If $\mathcal{P}$ is red, then $k-2 \leq \delta(A,B) < \delta(B,C)$. If $\mathcal{P}$ is blue, then we have $\delta(A,B) \geq \delta(B,C)$. Since the vertices are non-increasing sequences that are ordered lexicographically, we obtain $k - 2 \leq B_{\delta(A,B)} \leq B_{\delta(B,C)} < C_{\delta(B,C)}$. □

The proof of Theorem 3.1 can be strengthened for more than two colors using high-dimensional integer partitions [MS14]. This gives a generalization of Proposition 1.8 to 3-uniform monotone hyperpaths. Moshkovitz and Shapira [MS14] further generalized Theorem 3.1 to monotone hyperpaths of uniformity higher than three. In particular, they showed $\overline{\mathrm{R}}((P_n^k, \lhd_{mon})) = t_{k-1}((2-o(1))n)$ for every $k \geq 3$ and every sufficiently large n. We recall that $(P_n^k, \lhd_{mon})$ denotes the k-uniform monotone hyperpath on n vertices and t_h denotes the tower function of height h defined by $t_1(x) = x$ and $t_h(x) = 2^{t_{h-1}(x)}$ for $h \geq 2$.

We note that the coloring, which is constructed in the proof of the lower bound in Theorem 3.1, is realizable. In fact, it is the coloring obtained by coloring triples of points according their orientation in the point set constructed by Erdős and Szekeres [ES35] in their proof of the lower bound in (3.1).

3.3 A conjecture of Peters and Szekeres

In their seminal paper, Erdős and Szekeres proved the upper bound $\mathrm{ES}(k) \leq \binom{2k-4}{k-2}$ by showing (3.1). They also posed the *Erdős–Szekeres conjecture*, stating that $\mathrm{ES}(k) = 2^{k-2}$ for every $k \geq 2$. In spite of considerable efforts, this famous conjecture is still open.

In the 1960s, Erdős and Szekeres [ES61] supported their conjecture with the lower bound $\mathrm{ES}(k) \geq 2^{k-2}$. In 1998, Chung and Graham [CG98] improved the upper bound $\mathrm{ES}(k) \leq \binom{2k-4}{k-2}$ by 1 for $k \geq 4$, obtaining the first improvement after sixty years. Shortly after, Kleitman and Pachter [KP98] showed $\mathrm{ES}(k) \leq \binom{2k-4}{k-2} - 2k + 6$ for $k \geq 4$. A better estimate $\mathrm{ES}(k) \leq \binom{2k-5}{k-2} + 1$ for $k \geq 5$ was found by Tóth and Valtr [TV98] few months later. In 2005, Tóth and Valtr [TV05] further improved this bound by 1, obtaining $\mathrm{ES}(k) \leq \binom{2k-5}{k-2} = \frac{1}{2}\binom{2k-4}{k-2}$ for $k \geq 5$, which was the best known upper bound for ten years. Recently, Mojarrad and Vlachos [MV15] improved the upper bound to $\binom{2k-5}{k-2} - \binom{2k-8}{k-3} + 1$ for $k \geq 6$. This gives $\limsup_{k\to\infty} \mathrm{ES}(k)/\binom{2k-5}{k-2} \leq \frac{7}{8}$, which was also independently proved by Norin and Yuditsky [NY16]. All the upper bounds on $\mathrm{ES}(k)$ mentioned so far are of asymptotic order $\Theta(4^k/\sqrt{k})$. Very recently, Suk [Suk16] obtained a major asymptotic improvement by showing that $\mathrm{ES}(k) \leq 2^{k+o(k)}$.

The Erdős–Szekeres conjecture is known to hold for $k \leq 6$ and is open for $k > 6$. The case $k = 6$ was proved by Peters and Szekeres [PS06] who carried out a clever computer-aided proof.

By Theorem 3.1, we have $\widehat{\mathrm{N}}(a, u) = \mathrm{N}(a, u)$ for all $a, u \geq 2$. That is, the formula for the maximum size of a set with no a-cap and no u-cup remains the same even when we consider much more general setting for ordered hypergraphs. Peters and Szekeres [PS06] conjectured that a similar phenomenon occurs for the Erdős–Szekeres conjecture. We state this formally.

If P is a point set in the plane in general position, then every k-tuple of points from P in convex position is a union of an a-cap and a u-cup with common endpoints where a and u are some integers satisfying $a + u - 2 = k$. Using this fact, Peters and Szekeres [PS06] generalized the notion of convex position to the hypergraph setting as follows. For $k \geq 2$, an ordered 3-uniform hypergraph $\mathcal{H}$ on k vertices is called a *(convex) k-gon* or a *polygon* if $\mathcal{H}$ is a union of a red monotone hyperpath and a blue monotone hyperpath that are vertex disjoint except for the two common end-vertices. In this definition, we allow hyperpaths in $\mathcal{H}$ with two vertices and no edges.

Note that in a k-gon, the leftmost vertex and the rightmost vertex lie in both hyperpaths and each of the $k-2$ other vertices lies either in the red hyperpath or in the blue hyperpath. Therefore there are exactly 2^{k-2} pairwise non-isomorphic k-gons for every $k \geq 2$.

Let $\widehat{\mathrm{ES}}(k)$ be the maximum number N such that there is a red-blue coloring of $\mathcal{K}^3_N$ with no k-gon.

If P is a set of points in the plane in general position, then k-tuples of points from P in convex position are in one-to-one correspondence with k-gons in the realizable coloring of $\mathcal{K}^3_{|P|}$ obtained from P. Thus we have $2^{k-2} \leq \mathrm{ES}(k) \leq \widehat{\mathrm{ES}}(k)$ for every $k \geq 2$. On the other hand, every monochromatic k-path is a k-gon, thus $\widehat{\mathrm{ES}}(k) \leq \widehat{\mathrm{N}}(k, k)$ and, by Theorem 3.1, $\widehat{\mathrm{ES}}(k) \leq \binom{2k-4}{k-2}$.

For $2 \leq k \leq 5$, Peters and Szekeres [PS06] showed $\widehat{\mathrm{ES}}(k) = 2^{k-2}$. For $k = 5$

this was shown by their computer-aided proof. Peters and Szekeres conjectured that this equality is satisfied for every $k \geq 2$. We call this conjecture the *Peters–Szekeres conjecture* and, as a main result of this section, we refute it by showing the following result.

Theorem 3.3. *We have* $\widehat{\mathrm{ES}}(7) > 32$*,* $\widehat{\mathrm{ES}}(8) > 64$*, and* $\widehat{\mathrm{ES}}(9) > 128$*.*

Although we have found counterexamples to the Peters–Szekeres conjecture for values $k = 7, 8, 9$, we have no construction that would provide counterexamples to this conjecture for arbitrarily large values of k. For the remaining cases, our guess is that $\widehat{\mathrm{ES}}(6) = 16$ and $\widehat{\mathrm{ES}}(k) > 2^{k-2}$ for every $k \geq 10$.

To prove Theorem 3.3, we first discuss an equivalent version of the Peters–Szekeres conjecture that we use later in a search for a counterexample. Our approach is based on the following equivalent version of the Erdős–Szekeres conjecture introduced by Erdős, Tuza, and Valtr [ETV96].

For integers $a, u, k \geq 2$, let $\mathrm{N}(a, u; k)$ be the maximum number N such that there is a set of N points in the plane in general position with no a-cap, no u-cup, and no k points in convex position. We are mostly interested in the value of $\mathrm{N}(a, u; k)$ in cases when $\max\{a, u\} \leq k \leq a + u - 2$. Therefore it is useful to define a set of integer triples

$$D := \{(a, u, k) : a, u \geq 2 \text{ and } \max\{a, u\} \leq k \leq a + u - 2\}.$$

For three integers $(a, u, k) \in D$, we set

$$\mathrm{S}(a, u; k) := \sum_{i=k-a+2}^{u} \mathrm{N}(i, k + 2 - i) = \sum_{i=k-a+2}^{u} \binom{k-2}{i-2}.$$

Conjecture 3.4 ([ETV96])**.** *For all triples* $(a, u, k) \in D$*,*

$$\mathrm{N}(a, u; k) = \mathrm{S}(a, u; k).$$

Erdős, Tuza, and Valtr [ETV96] showed that Conjecture 3.4 is equivalent with the Erdős–Szekeres conjecture and obtained the following inequality.

Proposition 3.5 ([ETV96])**.** *For all triples* $(a, u, k) \in D$*,*

$$\mathrm{N}(a, u; k) \geq \mathrm{S}(a, u; k).$$

Depending on the values of a, u, k, the best known upper bound on $\mathrm{N}(a, u; k)$ is either the trivial estimate $\mathrm{N}(a, u; k) \leq N(a, u) = \binom{a+u-4}{a-2}$, or it is obtained by combining the trivial estimate $\mathrm{N}(a, u; k) \leq \mathrm{ES}(k)$ with the best known upper bound on $\mathrm{ES}(k)$ [MV15, Suk16].

For $k \geq a + u - 3$, every k-gon contains an a-cap or a u-cup which implies $\mathrm{N}(a, u; k) = \mathrm{N}(a, u)$ in this case. It follows that Conjecture 3.4 holds for $k \in \{a + u - 2, a + u - 3\}$, as for $k = a + u - 2$ we get

$$\mathrm{N}(a, u; k) = \mathrm{N}(a, u) = \binom{a+u-4}{u-2} = \mathrm{S}(a, u; k)$$

and for $k = a + u - 3$ we get

$$\mathrm{N}(a, u; k) = \mathrm{N}(a, u) = \binom{a+u-4}{u-2} = \binom{a+u-5}{u-3} + \binom{a+u-5}{u-2} = \mathrm{S}(a, u; k).$$

The gap between known bounds for $\mathrm{N}(a, u; k)$ appears first for $k = a + u - 4$. By a more careful analysis of this first nontrivial case, we improve the best known upper bound by 1 for the case $a = 4$.

Proposition 3.6. *For every integer $k \geq 3$, we have*

$$\binom{k}{2} - k + 2 \leq \mathrm{N}(4, k; k) \leq \binom{k}{2} - 1.$$

Proof. The lower bound follows from Proposition 3.5, as $\mathrm{S}(4, k; k) = \binom{k}{2} - k + 2$.

For the upper bound, let P be a set of points in general position with no 4-cap and no k points in convex position. Then, in particular, P does not contain a k-cup. Let U be a set of points of P that are the rightmost point of some $(k-1)$-cup in P. Then the set $P \setminus U$ contains no 4-cap, no $(k-1)$-cup, and no k points in convex position. Thus the size of $P \setminus U$ satisfies

$$|P \setminus U| \leq \mathrm{N}(4, k-1; k) = \binom{k-1}{2}.$$

We show that U contains no 3-cap and no $(k-1)$-cup. This will conclude the proof, as then we have $|U| \leq k - 2$ and

$$|P| = |U| + |P \setminus U| \leq k - 2 + \binom{k-1}{2} = \binom{k}{2} - 1.$$

If there is a 3-cap in U, then, by the choice of U, its leftmost point is also the rightmost point of a $(k-1)$-cup in P. Then either the 3-cap or the $(k-1)$-cup can be extended to a 4-cap or a k-cup, respectively. However, neither is possible from the choice of P.

Suppose for contradiction that there is a $(k-1)$-cup C in U and let p_1 be its leftmost point. From the choice of U, the point p_1 is also the rightmost point of some $(k-1)$-cup C' in P. Let p_2 be the second leftmost point of C and let q_{k-2} be the second rightmost point of C'.

Then either every point from $C \setminus \{p_1, p_2\}$ lies above the line $\overline{q_{k-2}p_2}$ or every point from $C' \setminus \{q_{k-2}, p_1\}$ lies above the line $\overline{q_{k-2}p_2}$. Otherwise there are points $p \in C \setminus \{p_1, p_2\}$ and $q \in C' \setminus \{q_{k-2}, p_1\}$ below $\overline{q_{k-2}p_2}$ and we have a 4-cap in P formed by q, q_{k-2}, p_2, p; see Figure 3.1.

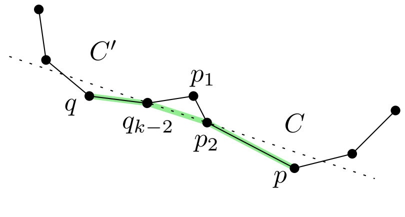


Figure 3.1: Points $p \in C \setminus \{p_1, p_2\}$ and $q \in C' \setminus \{q_{k-2}, p_1\}$ below $\overline{q_{k-2}p_2}$ force a 4-cap.

If the first case occurs, i. e., if every point from $C \setminus \{p_1, p_2\}$ lies above the line $\overline{q_{k-2}p_2}$, then we consider the triple of points q_{k-2}, p_1, p_{k-1}, where p_{k-1} is the rightmost point of C. If the triple is a 3-cup then $C' \cup \{p_{k-1}\}$ is a k-cup; see

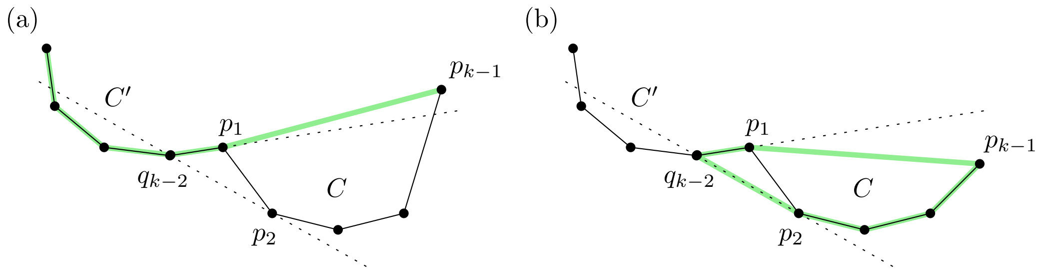


Figure 3.2: (a) The point p_{k-1} above $\overline{q_{k-2}p_1}$ forces a k-cup. (b) The point p_{k-1} below $\overline{q_{k-2}p_1}$ forces k points in convex position.

part (a) of Figure 3.2. Otherwise $C \cup \{q_{k-2}\}$ is a set of k points in convex position; see part (b) of Figure 3.2.

The other case, where every point from $C' \setminus \{q_{k-2}, p_1\}$ lies above the line $\overline{q_{k-2}p_2}$, is symmetric. Thus we obtain a contradiction and, consequently, we see that U contains no $(k-1)$-cup. □

Now, we introduce a version of Conjecture 3.4 for the hypergraph setting. For integers $a, u, k \geq 2$ and N, an $(a, u; k)$-*coloring* of $\mathcal{K}_N^3$ is a red-blue coloring of $\mathcal{K}_N^3$ with no red a-path, no blue u-path, and no k-gon. Let $\widehat{\mathrm{N}}(a, u; k)$ be the maximum number N such that there is an $(a, u; k)$-coloring of $\mathcal{K}_N^3$.

Conjecture 3.7. *For all triples $(a, u, k) \in D$,*

$$\widehat{\mathrm{N}}(a, u; k) = \mathrm{S}(a, u; k).$$

We note that the argument from Proposition 3.6 applies in the hypergraph setting and gives $\widehat{\mathrm{N}}(4, k; k) \leq \binom{k}{2} - 1$ for every $k \geq 3$.

A generalization of the approach of Erdős, Tuza, and Valtr [ETV96] gives the following statement.

Proposition 3.8. *Conjecture 3.7 is equivalent with the Peters–Szekeres conjecture.*

Before proving this statement, we first introduce the following auxiliary result.

Lemma 3.9. *Let $\alpha, \psi, \alpha', \psi', \kappa \geq 2$ be integers with $\alpha + \psi' \leq \kappa + 1$. If we set $\alpha'' := \max\{\alpha + 1, \alpha'\}$ and $\psi'' := \max\{\psi, \psi' + 1\}$, then*

$$\widehat{\mathrm{N}}(\alpha'', \psi''; \kappa) \geq \widehat{\mathrm{N}}(\alpha, \psi; \kappa) + \widehat{\mathrm{N}}(\alpha', \psi'; \kappa).$$

Proof. Set $N_1 := \widehat{\mathrm{N}}(\alpha, \psi; \kappa)$ and $N_2 := \widehat{\mathrm{N}}(\alpha', \psi'; \kappa)$. For $I_1 := \{1, \ldots, N_1\}$ and $I_2 := \{N_1 + 1, \ldots, N_1 + N_2\}$, we define a red-blue coloring c of $\mathcal{K}^3_{N_1+N_2}$ with the vertex set $[N_1 + N_2]$ as follows. We color the complete hypergraph induced by I_1 with an $(\alpha, \psi; \kappa)$-coloring. We color the complete hypergraph induced by I_2 with an $(\alpha', \psi'; \kappa)$-coloring. Finally, every edge $\{x, y, z\}$ of $\mathcal{K}^3_{N_1+N_2}$ with $x < y < z$, $x \in I_1$, and $z \in I_2$ is colored red if $y \in I_1$ and blue if $y \in I_2$.

We claim that c is an $(\alpha'', \psi''; \kappa)$-coloring. Let R be a red monotone hyperpath in c and let B be a blue monotone hyperpath in c. By the construction, if all vertices of R lie in I_2 then R has at most $\alpha' - 1$ vertices. Otherwise R has at most one vertex in I_2, since every edge with exactly two vertices in I_2 is blue. Then R

has at most $(\alpha - 1) + 1 = \alpha$ vertices. Similarly, B has at most $\psi - 1$ vertices if all vertices of B lie in I_1, and it has at most ψ' vertices otherwise.

It remains to prove that there is no κ-gon in c. Clearly, there is no κ-gon with all vertices in I_1 and no κ-gon with all vertices in I_2. We show that every polygon intersecting both I_1 and I_2 has fewer than κ vertices. Let Z be a polygon with the leftmost vertex u lying in I_1 and with the rightmost vertex v lying in I_2. The polygon Z is a union of a red monotone hyperpath R and a blue monotone hyperpath B, both connecting u with v. By the discussion above, R has at most α vertices and B has at most ψ' vertices. It follows that Z has at most $\alpha + \psi' - 2 < \kappa$ vertices. □

With Lemma 3.9 in hand, we can prove Proposition 3.8 quite easily.

Proof of Proposition 3.8. If Conjecture 3.7 is true, then, in particular, we have $\widehat{\mathrm{N}}(k, k; k) = \mathrm{S}(k, k; k)$ and thus

$$\widehat{\mathrm{ES}}(k) = \widehat{\mathrm{N}}(k, k; k) = \mathrm{S}(k, k; k) = \sum_{i=2}^{k} \binom{k-2}{i-2} = 2^{k-2},$$

and the Peters–Szekeres conjecture is also true.

On the other hand, we show that if Conjecture 3.7 is false, then the Peters–Szekeres conjecture is false as well. Suppose Conjecture 3.7 is false. That is,

$$\widehat{\mathrm{N}}(a, u; k) > \mathrm{S}(a, u; k)$$

for some integers $(a, u, k) \in D$.

For technical reasons, we set

$$\widehat{\mathrm{N}}(k, 1; k) = \mathrm{S}(k, 1; k) = \widehat{\mathrm{N}}(1, k; k) = \mathrm{S}(1, k; k) := 0,$$

for any integer $k \geq 2$.

We first show that $\widehat{\mathrm{N}}(a, k; k) \geq \widehat{\mathrm{N}}(k - u + 1, k; k) + \widehat{\mathrm{N}}(a, u; k)$. If $u = k$ then the inequality follows from $\widehat{\mathrm{N}}(1, k; k) = 0$. Otherwise an application of Lemma 3.9 with $(\alpha, \psi, \kappa) = (k - u + 1, k, k)$ and $(\alpha', \psi', \kappa) = (a, u, k)$ gives the inequality, since we have $a = \max\{k - u + 2, a\}$ and $k = \max\{k, u + 1\}$ due to $(a, u, k) \in D$ and $u \neq k$.

We further show that $\widehat{\mathrm{N}}(k, k; k) \geq \widehat{\mathrm{N}}(a, k; k) + \widehat{\mathrm{N}}(k, k - a + 1; k)$. If $a = k$ then the inequality follows from $\widehat{\mathrm{N}}(k, 1; k) = 0$. Otherwise an application of Lemma 3.9 with $(\alpha, \psi, \kappa) = (a, k, k)$ and $(\alpha', \psi', \kappa) = (k, k - a + 1, k)$ gives the inequality, since we have $k = \max\{a + 1, k\}$ and $k = \max\{k, k - a + 2\}$ due to $(a, u, k) \in D$ and $a \neq k$.

Since $\widehat{\mathrm{ES}}(k) = \widehat{\mathrm{N}}(k, k; k)$, the two inequalities proved above imply

$$\widehat{\mathrm{ES}}(k) \geq \widehat{\mathrm{N}}(k - u + 1, k; k) + \widehat{\mathrm{N}}(a, u; k) + \widehat{\mathrm{N}}(k, k - a + 1; k).$$

Since $(k - u + 1, k, k) \in D$ for $u \neq k$, we have $\widehat{\mathrm{N}}(k - u + 1, k; k) \geq \mathrm{S}(k - u + 1, k; k)$ by Proposition 3.5. Analogously, we obtain $\widehat{\mathrm{N}}(k, k - a + 1; k) \geq \mathrm{S}(k, k - a + 1; k)$. From $\widehat{\mathrm{N}}(a, u; k) > \mathrm{S}(a, u; k)$, we thus get a strict inequality

$$\widehat{\mathrm{ES}}(k) > \mathrm{S}(k - u + 1, k; k) + \mathrm{S}(a, u; k) + \mathrm{S}(k, k - a + 1; k)$$

$$= \sum_{i=u+1}^{k} \binom{k-2}{i-2} + \sum_{i=k-a+2}^{u} \binom{k-2}{i-2} + \sum_{i=2}^{k-a+1} \binom{k-2}{i-2} = \sum_{i=2}^{k} \binom{k-2}{i-2} = 2^{k-2},$$

which shows that the Peters–Szekeres conjecture is false. □

The main profit gained by considering $\widehat{\mathrm{N}}(a, u; k)$ is that Conjecture 3.7 is, in a certain sense, finer than the Peters–Szekeres conjecture. This allows us to employ an exhaustive computer search for larger values of k in order to find an $(a, u; k)$-coloring of $\mathcal{K}_N^3$ for some suitable integers a, u, k, such that $N > \mathrm{S}(a, u; k)$. This will disprove Conjecture 3.7 and, by Proposition 3.8, the Peters–Szekeres conjecture.

The exhaustive search for extremal colorings is performed by SAT solvers. We use a SAT encoding of the following problem: for given integers $a, u, k, N \geq 3$, is there an $(a, u; k)$-coloring of $\mathcal{K}_N^3$? The SAT encoding of this problem is described in Appendix A.

In our experiments we use the *Glucose* SAT solver [AS13], the winner of the *certified UNSAT* category of the SAT 2013 competition [BBHJ13]. All experiments were conducted on a computer equipped with Intel Xeon E5-1620 CPU running at 3.60GHz and 63GB of RAM.

Proof of Theorem 3.3. Using the SAT solver, we found a $(4, 7; 7)$-coloring c of $\mathcal{K}_{17}^3$ which is recorded on the webpage [BV]. This refutes Conjecture 3.7 for $a = 4$ and $u = k = 7$, as it shows that $\widehat{\mathrm{N}}(4, 7; 7) \geq \mathrm{S}(4, 7; 7) + 1 = 17$. It follows from the proof of Proposition 3.8 that the coloring c can be extended to a coloring of $\mathcal{K}_{33}^3$ with no 7-gon, therefore we refute the Peters–Szekeres conjecture as well. Our experiments showed that every coloring of $\mathcal{K}_{18}^3$ contains either a red 4-path or a 7-gon, i. e., $\widehat{\mathrm{N}}(4, 7; 7) = 17$.

By running additional tests, we obtained further counterexamples to Conjecture 3.7. We found colorings that give $\widehat{\mathrm{N}}(5, 6; 7) \geq 26$, $\widehat{\mathrm{N}}(5, 7; 7) \geq 27$, $\widehat{\mathrm{N}}(6, 6; 7) \geq 31$, $\widehat{\mathrm{N}}(6, 7; 7) \geq 32$, and even $\widehat{\mathrm{N}}(7, 7; 7) \geq 33$. We also obtained colorings that provide bounds $\widehat{\mathrm{N}}(4, 8; 8) \geq 23$ and $\widehat{\mathrm{N}}(4, 9; 9) \geq 30$. By the proof of Proposition 3.8, these colorings can be extended to counterexamples to the Peters–Szekeres conjecture for $k = 8$ and $k = 9$.

We remark that all the above seven inequalities are of the form $\widehat{\mathrm{N}}(a, u; k) \geq \mathrm{S}(a, u; k) + 1$ but we conjecture that the difference $\widehat{\mathrm{N}}(a, u; k) - \mathrm{S}(a, u; k)$ may be arbitrarily large. □

For larger values of a, u, and k than those mentioned in the above proof of Theorem 3.3, the input formulas become too large for the SAT solver.

Our experiments verify Conjecture 3.7 for $k = 6$ and for all possible values of a and u, except for the case $a = u = k$. In this case the solver did not terminate on 17 vertices even after 266 hours of computation.

We also ran tests to explore the validity of Conjecture 3.4. Our approach is based on a restriction of the setting of Conjecture 3.7 to *pseudolinear colorings of* $\mathcal{K}_N^3$. A red-blue coloring c' of $\mathcal{K}_N^3$ is pseudolinear if every 4-tuple of vertices of $\mathcal{K}_N^3$ induces a realizable coloring of $\mathcal{K}_4^3$ in c'. Clearly, every realizable coloring is pseudolinear.

Peters and Szekeres [PS06] call pseudolinear colorings 'signatures that satisfy geometric constraints'. However, we feel that the term 'pseudolinear coloring' is more fitting, as such colorings of $\mathcal{K}_N^3$ are in one-to-one correspondence with *pseudolinear x-monotone drawings* of K_N; see Theorem 4.9. In particular, there are pseudolinear colorings of $\mathcal{K}_N^3$ for $N \geq 9$ that are not realizable [Knu92, Figure 1, page 26].

Considering only pseudolinear colorings in our experiments, we verified Conjecture 3.4 in the cases $a = 4$, $u = k = 7$ and $a = 4$, $u = k = 8$. That is, we have $\mathrm{N}(4, 7; 7) = 16$ and $\mathrm{N}(4, 8; 8) = 22$. For pseudolinear colorings, all our results matched the values from Conjecture 3.4. All colorings obtained by our experiments can be found on a separate webpage [BV]. A complete list of known values of $\widehat{\mathrm{N}}(a, u; 6)$, $\widehat{\mathrm{N}}(a, u; 7)$, and $\mathrm{N}(a, u; 7)$ can be found in Tables 3.1, 3.2, and 3.3, respectively.

$\widehat{\mathrm{N}}(a, u; 6)$	2	3	4	5	6
2					1
3				4	5
4			6	10	11
5		4	10	14	15
6	1	5	11	15	[16,70]

Table 3.1: The known values and estimates of $\widehat{\mathrm{N}}(a, u; 6)$. The entry [16, 70] in the position of $\widehat{\mathrm{N}}(6, 6; 6)$ means that the best known lower and upper bounds on $\widehat{\mathrm{N}}(6, 6; 6)$ are 16 and 70, respectively.

$\widehat{\mathrm{N}}(a, u; 7)$	2	3	4	5	6	7
2						1
3					5	6
4				10	15	**17**
5			10	20	[**26**,35]	[**27**,56]
6		5	15	[**26**,35]	[**31**,70]	[**32**,126]
7	1	6	**17**	[**27**,56]	[**32**,126]	[**33**,210]

Table 3.2: The known values and estimates of $\widehat{\mathrm{N}}(a, u; 7)$. The bold values are larger (by 1) than the values from Conjecture 3.7.

$\mathrm{N}(a, u; 7)$	2	3	4	5	6	7
2						1
3					5	6
4				10	15	16
5			10	20	[25,35]	[26,56]
6		5	15	[25,35]	[30,70]	[31,112]
7	1	6	16	[26,56]	[31,112]	[32,112]

Table 3.3: The known values of $\mathrm{N}(a, u; 7)$.

We define a *$\geq k$-gon* as any l-gon with $l \geq k$. For every positive integer N, if c is a realizable (or more generally pseudolinear) coloring of $\mathcal{K}_N^3$ with no k-gon, then c contains no $\geq k$-gon. The following proposition shows that this is no longer true for general colorings of $\mathcal{K}^3_N$.

Proposition 3.10. *There is a coloring of $\mathcal{K}^3_6$ with a 6-gon and with no 5-gon.*

Proof. We denote the vertices of $\mathcal{K}^3_6$ by $1, 2, \dots, 6$, whereas $1 < 2 < \cdots < 6$. First, consider the coloring of $\mathcal{K}^3_6$ such that an edge is colored red if it contains an even number of vertices from $\{1, 2, 3\}$, and it is colored blue otherwise. Observe that there is no monochromatic 4-path. To obtain a desired coloring we switch the colors of the two edges 145 and 236. The obtained coloring contains a unique red 4-path 1456 and a unique blue 4-path 1236. Their union is a 6-gon.

It remains to verify that there is no 5-gon. Since there is no monochromatic 5-path, a 5-gon would be the union of a monochromatic 4-path in one color and a 3-path (edge) in the other color. However, none of the monochromatic 4-paths 1456 and 1236 can be completed to a 5-gon of this type, as the edges 126 and 136 are red and the edges 146 and 156 are blue. □

Somewhat abusing the notation, we use $\widehat{\mathrm{ES}}(\geq k)$ to denote the maximum N such that there is a red-blue coloring of $\mathcal{K}^3_N$ with no $\geq k$-gon. Similarly, $\widehat{\mathrm{N}}(a, u; \geq k)$ denotes the maximum N such that there is a red-blue coloring of $\mathcal{K}^3_N$ with no red a-path, no blue u-path, and no $\geq k$-gon. Obviously, $\mathrm{ES}(k) \leq \widehat{\mathrm{ES}}(\geq k) \leq \widehat{\mathrm{ES}}(k)$. Theorem 3.3 can be strengthened as follows.

Theorem 3.11. *We have $\widehat{\mathrm{ES}}(\geq 7) > 32$, $\widehat{\mathrm{ES}}(\geq 8) > 64$, and $\widehat{\mathrm{ES}}(\geq 9) > 128$.*

Proof. Let $k \in \{7, 8, 9\}$. Consider a $(4, k; k)$-coloring c of $\mathcal{K}^3_{\mathrm{S}(4,k;k)+1}$ obtained in the proof of Theorem 3.3. We first prove that c contains no $\geq k$-gon. By the choice of c, it contains no k-gon. Suppose there is an l-gon G with $l > k$ in c. Since there is no red 4-path, the red monotone hyperpath of G has at most 3 vertices and therefore the blue monotone hyperpath has at least k vertices. Its subpath on k vertices is a k-gon, which is impossible. Thus, c contains no $\geq k$-gon.

Now, observe that a slight modification of the proof of Lemma 3.9 gives the following modification of the lemma. Under the same assumptions as in Lemma 3.9, the following inequality holds:

$$\widehat{\mathrm{N}}(\alpha'', \psi''; \geq \kappa) \geq \widehat{\mathrm{N}}(\alpha, \psi; \geq \kappa) + \widehat{\mathrm{N}}(\alpha', \psi'; \geq \kappa).$$

Applying this modification of Lemma 3.9 with $(\alpha, \psi, \kappa) = (4, k, k)$ and $(\alpha', \psi', \kappa) = (k, k-3, k)$, we get

$$\widehat{\mathrm{N}}(k, k; \geq k) \geq \widehat{\mathrm{N}}(4, k; \geq k) + \widehat{\mathrm{N}}(k, k-3; \geq k).$$

Since

$$\widehat{\mathrm{N}}(4, k; \geq k) \geq 1 + \mathrm{S}(4, k; k) = 1 + \sum_{i=k-2}^{k} \binom{k-2}{i-2}$$

and

$$\widehat{\mathrm{N}}(k, k-3; \geq k) \geq \mathrm{S}(k, k-3; k) = \sum_{i=2}^{k-3} \binom{k-2}{i-2},$$

we get

$$\widehat{\mathrm{ES}}(\geq k) = \widehat{\mathrm{N}}(k, k; \geq k) \geq 1 + \sum_{i=k-2}^{k} \binom{k-2}{i-2} + \sum_{i=2}^{k-3} \binom{k-2}{i-2} = 1 + 2^{k-2}. \qquad \square$$

The following strengthening of the Erdős–Szekeres conjecture, introduced by Peters and Szekeres [PS06], remains open: for every $k \geq 2$, is it true that every pseudolinear coloring of $\mathcal{K}_N^3$ with $N = 2^{k-2} + 1$ contains a k-gon? Similarly, Goodman and Pollack [GP81] conjectured that for every $k \geq 2$ the number $\mathrm{ES}(k)$ equals the maximum N for which there is a pseudolinear coloring of $\mathcal{K}_N^3$ with no k-gon. Note that the conjecture of Goodman and Pollack might be true even if the previous strengthening is not.

None of the counterxamples for Conjecture 3.7 is pseudolinear. If there was a pseudolinear coloring c that refutes Conjecture 3.7, then we could use the proof of Proposition 3.8 and extend c to a counterexample to the strengthening above. If c was realizable, then it would give a counterexample even to the Erdős–Szekeres conjecture.

Another possible direction for further research is to improve the bounds for $\widehat{\mathrm{ES}}(k)$ and, possibly, to recognize some structure behind the colorings that we found. For a sufficiently large k, this could lead to a general construction of colorings of $\mathcal{K}_N^3$ with no k-gon for $N > 2^{k-2} + 1$.

3.4 Geometric Ramsey numbers

In discrete geometry, *geometric Ramsey numbers* [CGK$^+$15, KPT97, KPTV98] are natural analogues of ordered Ramsey numbers. For a finite set of points $P \subset \mathbb{R}^2$ in general position, we denote as K_P the *complete geometric graph on P*, which is a complete graph drawn in the plane so that its vertices are represented by the points in P and the edges are drawn as straight-line segments between the pairs of points in P. The graph K_P is *convex* if P is in convex position.

The *geometric Ramsey number* of a graph G, denoted by $\mathrm{Rg}(G)$, is the smallest N such that every complete geometric graph K_P on N vertices with edges colored by two colors contains a noncrossing monochromatic drawing of G. If we consider only convex complete geometric graphs K_P in the definition, then we get so-called *convex geometric Ramsey number* $\mathrm{Rc}(G)$. Note that these numbers are finite only if G is outerplanar and that $\mathrm{Rc}(G) \leq \mathrm{Rg}(G)$ for every outerplanar graph G. It follows from a result of Gritzmann et al. [GMPP91] (see also Lemma 2.1 in [CGK$^+$15]) that if G is an outerplanar graph on n vertices and P is a set of n points in the plane in general position, then K_P contains a noncrossing copy of G.

For the cycles C_n with $n \geq 3$, Károlyi et al. [KPTV98] showed the upper bound $\mathrm{Rg}(C_n) \leq 2n^2 - 6n + 6$ and also observed that $\mathrm{Rc}(C_n) \geq (n-1)^2 + 1$. We first show that convex geometric Ramsey numbers of cycles equal ordered Ramsey numbers of monotone cycles.

Observation 3.12. *For every $n \geq 3$, we have* $\mathrm{Rc}(C_n) = \overline{\mathrm{R}}((C_n, \lhd_{mon}))$.

Proof. Consider a set of n points in convex position. Order the points $v_1 \prec \cdots \prec v_n$ in the clockwise order starting at an arbitrary vertex. The observation follows from the fact that a cycle with the vertex set $\{v_1, \ldots, v_n\}$ is noncrossing if and only if it is the monotone cycle with respect to $\prec$. □

By combining this simple observation together with Theorem 1.11, we obtain the exact formula for geometric and convex geometric Ramsey numbers of cycles.

Theorem 3.13. *For every integer $n \geq 3$, we have* $\mathrm{Rc}(C_n) = \mathrm{Rg}(C_n) = 2n^2 - 6n + 6$.

Proof. We recall that $\mathrm{Rc}(G) \leq \mathrm{Rg}(G)$ for every outerplanar graph G. The upper bound $\mathrm{Rg}(C_n) \leq 2n^2 - 6n + 6$ was proved by Károlyi et al. [KPTV98]. The lower bound $2n^2 - 6n + 6 \leq \mathrm{Rc}(C_n)$ follows from Observation 3.12 and Theorem 1.11. □

We also mention a problem with an application in the theory of geometric Ramsey numbers.

A *crossing* in an ordered graph $(G, \prec)$ is a pair of edges $\{v_i, v_k\}$, $\{v_j, v_l\}$ such that $v_i \prec v_j \prec v_k \prec v_l$. An ordered graph is *noncrossing* if it contains no crossing.

Let $\overline{\mathrm{Rnc}}(n)$ be the largest ordered Ramsey number of a noncrossing ordered graph on n vertices. Since noncrossing ordered graphs are outerplanar, they are 2-degenerate, and thus, by Theorem 2.9, we have $\overline{\mathrm{Rnc}}(n) \leq n^{O(\log n)}$.

Problem 3.14. *What is the growth rate of* $\overline{\mathrm{Rnc}}(n)$*? In particular, is it polynomial in n?*

It is an open problem whether there is a general polynomial upper bound for geometric Ramsey numbers of outerplanar graphs [CGK+15]. The following theorem shows that Problem 3.14 is equivalent to the question of determining the asymptotics of the maximum convex geometric Ramsey number of an outerplanar graph on n vertices.

Theorem 3.15. *Let* $\mathrm{Rc}(n)$ *be the maximum convex geometric Ramsey number of an outerplanar graph on n vertices. For every $n \geq 2$, we have*

$$\mathrm{Rc}(n) \leq \overline{\mathrm{Rnc}}(n) \leq \mathrm{Rc}(4n - 4).$$

Proof. Let G be an outerplanar graph drawn in the plane so that its vertices are the vertices of a convex n-gon, and the edges are drawn as straight-line segments with no crossings. Let $v_1 \prec v_2 \prec \cdots \prec v_n$ be a clockwise ordering of the vertices of G along the n-gon with v_1 chosen arbitrarily. In this way, we obtain a noncrossing ordered graph $\mathcal{G}$. If we find a monochromatic copy of $\mathcal{G}$ in every 2-coloring of $\mathcal{K}_N$ for some N, we can find a monochromatic noncrossing copy of the graph G in every 2-coloring of the complete convex geometric graph on N vertices. This proves of the first inequality.

Now we prove the second inequality. The case $n = 2$ is trivial, so we assume that $n \geq 3$. Since adding edges to an ordered graph never decreases its ordered Ramsey number, we know that $\overline{\mathrm{Rnc}}(n)$ is attained by a noncrossing ordered graph $\mathcal{G}$ with vertices $v_1 \prec \cdots \prec v_n$ that contains the Hamiltonian cycle $v_1, v_2, \ldots, v_n, v_1$. We form an outerplanar graph H as follows. We take four unordered copies $G^{(1)}, \ldots, G^{(4)}$ of $\mathcal{G}$. For every $i \in [4]$, let $v_1^{(i)}, \ldots, v_n^{(i)}$ be the set of vertices of $G^{(i)}$. We identify $v_n^{(1)}$ with $v_n^{(2)}$, $v_1^{(2)}$ with $v_1^{(3)}$, $v_n^{(3)}$ with $v_n^{(4)}$, and $v_1^{(4)}$ with $v_1^{(1)}$; see Figure 3.3. The resulting graph H is Hamiltonian and thus there is only one planar straight-line drawing of H on a given set of $4n-4$ points in convex position, up to rotation and mirroring.

Let K be a complete geometric graph whose vertices $u_1, u_2, \ldots, u_N$ form, in this order, the vertices of a convex polygon. In every noncrossing copy of H in K, at least three of the graphs $G^{(i)}$, where $i \in [4]$, satisfy the property that the images of the vertices $v_1^{(i)}, \ldots, v_n^{(i)}$ form a monotone sequence in the ordering

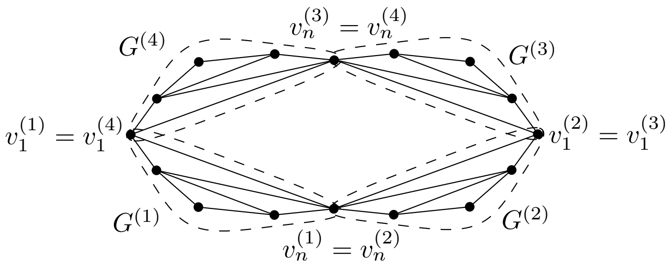


Figure 3.3: Construction of the graph H in the proof of Theorem 3.15.

$u_1 \prec u_2 \prec \cdots \prec u_N$. Consequently, in at least one $G^{(i)}$, the vertices form an increasing sequence. If $N \geq \mathrm{Rc}(4n-4)$, every 2-coloring of the complete convex geometric graph on N vertices contains a monochromatic noncrossing copy of H. Therefore, every 2-coloring of $\mathcal{K}_N$ contains a monochromatic copy of the ordered graph $\mathcal{G}$. □

Combining this theorem with results from Chapter 2 we can derive a quasipolynomial upper bound on $\mathrm{Rc}(n)$, improving the previous exponential bound (see, e.g., [CGK+15]).

Corollary 3.16. *We have* $\mathrm{Rc}(n) \leq n^{O(\log n)}$.

Proof. By the first inequality in Theorem 3.15 and Theorem 2.9 by Conlon et al. [CFLS14], the upper bound $\overline{\mathrm{Rnc}}(n) \leq n^{O(\log n)}$ gives $\mathrm{Rc}(n) \leq n^{O(\log n)}$. □

4. Crossing numbers of K_n

4.1 Graph drawings and crossing numbers

Let G be a graph with no loops or multiple edges. In a *drawing D of a graph G* in the plane, the vertices are represented by distinct points and each edge is represented by a simple continuous arc connecting the images of its endpoints. As usual, we identify the vertices and their images, as well as the edges and the arcs representing them. We require that the edges pass through no vertices other than their endpoints. We also assume for simplicity that any two edges have only finitely many points in common, no two edges *touch* at an interior point and no three edges meet at a common interior point.

A *crossing in D* is a common interior point of two edges where they properly cross. The *crossing number* $\mathrm{cr}(D)$ *of a drawing D* is the number of crossings in D. The *crossing number* $\mathrm{cr}(G)$ *of a graph G* is the minimum of $\mathrm{cr}(D)$, taken over all drawings D of G. A drawing D is called *simple* if no two adjacent edges cross and no two edges have more than one common crossing. It is well-known and easy to see that every drawing of G which minimizes the crossing number is simple.

According to the famous conjecture of Hill [Guy60, HH63] (also known as Guy's conjecture), the crossing number of the complete graph K_n on n vertices satisfies $\mathrm{cr}(K_n) = Z(n)$, where

$$Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

Despite several attempts over the years, Hill's conjecture remains open. It has been verified for $n \leq 10$ by Guy [Guy72] and recently for $n \leq 12$ by Pan and Richter [PR07]. Moreover for each n, there are drawings of K_n with exactly $Z(n)$ crossings [BK64, Guy60, HH63, Har02]. Current best asymptotic lower bound, $\mathrm{cr}(K_n) \geq 0.8594Z(n)$, follows from the lower bound on the crossing number of the complete bipartite graph [dKPS07] by an elementary double-counting argument [RT97].

Note that the given definition of a drawing of a graph is quite general. By considering more restricted classes of drawings, one may derive many variants of the crossing number. See a recent survey by Schaefer [Sch14] for an encyclopedic treatment of all known variants of the crossing numbers. We mainly consider three particular classes of drawings and the corresponding variants of the crossing number in this chapter.

A curve α in the plane is *x-monotone* if every vertical line intersects α in at most one point. A drawing of a graph G in which every edge is represented by an x-monotone curve and no two vertices share the same x-coordinate is called *x-monotone* (or *monotone*, for short). The *monotone crossing number* $\mathrm{mon\text{-}cr}(G)$ *of a graph G* is the minimum of $\mathrm{cr}(D)$, taken over all monotone drawings D of G.

A *rectilinear drawing* of G is a drawing of G in which the vertices are represented by points in *general position* (no three points lie on a common line and all x-coordinates of vertices are distinct) and each edge is represented by a straight-line segment. The *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ *of a graph G* is the smallest number of crossings in a rectilinear drawing of G.

A drawing D of a graph G is *pseudolinear* if the edges of D can be extended to unbounded simple curves that cross each other exactly once, thus forming an *arrangement of pseudolines*. The *pseudolinear crossing number* $\widetilde{\mathrm{cr}}(G)$ *of* G is the minimum number of crossings in a pseudolinear drawing of G.

Vertices of a pseudolinear drawing D of the complete graph K_n together with the $\binom{n}{2}$ pseudolines extending the edges are said to form a *pseudoarrangement of points*. Note that the pseudoarrangement of points extending D is usually not unique as there is a certain freedom in choosing where the pseudolines extending disjoint noncrossing edges of D cross.

It is well-known that every arrangement of pseudolines can be made x-monotone by a suitable isotopy of the plane (this follows, for example, by the duality transform established by Goodman [Goo80, GP84]). Therefore, every pseudolinear drawing of a graph G is isotopic to an x-monotone pseudolinear drawing of G. Every rectilinear drawing of G is x-monotone and pseudolinear, but there are pseudolinear drawings of K_n that cannot be "stretched" to rectilinear drawings. Altogether, we have $\mathrm{cr}(G) \leq \text{mon-cr}(G) \leq \widetilde{\mathrm{cr}}(G) \leq \overline{\mathrm{cr}}(G)$ for every graph G.

The monotone crossing number has been introduced by Valtr [Val05] and recently further studied by Pach and Tóth [PT12], who showed that $\text{mon-cr}(G) < 2\mathrm{cr}(G)^2$ holds for every graph G. On the other hand, they showed that the monotone crossing number and the crossing number are not always the same: there are graphs G with arbitrarily large crossing numbers such that $\text{mon-cr}(G) \geq \frac{7}{6}\mathrm{cr}(G) - 6$.

The pseudolinear crossing number of K_n is known to be asymptotically larger than $Z(n)$: this follows from the lower bound $\widetilde{\mathrm{cr}}(K_n) \geq (277/729)\binom{n}{4} - O(n^3)$ by Ábrego et al. [ACFM+12, AFMLS08] and from the simple upper bound $Z(n) \leq \frac{3}{8}\binom{n}{4} + O(n^3)$. It is not known whether $\overline{\mathrm{cr}}(K_n) = \widetilde{\mathrm{cr}}(K_n)$ for all values of n, but there are graphs for which the rectilinear crossing number is strictly larger than the pseudolinear one [HVLnS14]. More detailed discussion on the bounds for $\widetilde{\mathrm{cr}}(K_n)$ and $\overline{\mathrm{cr}}(K_n)$ can be found in Section 4.4.

Generalizing the notion of a simple drawing, we call a drawing of a graph G *semisimple* if adjacent edges do not cross but independent edges may cross more than once. The *monotone semisimple odd crossing number* of G (called *monotone odd +* by Schaefer [Sch14]), denoted by $\text{mon-ocr}_+(G)$, is the smallest number of pairs of edges that cross an odd number of times in a monotone semisimple drawing of G. Note that we have $\text{mon-ocr}_+(G) \leq \text{mon-cr}(G)$ for every graph G. We also remark that for this notion of the crossing number, optimal drawings do not have to be simple.

In Section 4.2, we prove Hill's conjecture for monotone crossing number and for monotone semisimple odd crossing number. We also very briefly survey further progress on this conjecture that has been made in the last three years. Then we show a combinatorial characterization of simple and semisimple monotone drawings of K_n in Section 4.3. In Section 4.4, a similar combinatorial characterization of pseudolinear drawings of K_n is used to obtain current best asymptotic lower bound on $\widetilde{\mathrm{cr}}(K_n)$. We finish this chapter by mentioning some open problems, one of which is a strengthening of Hill's conjecture; see Section 4.5.

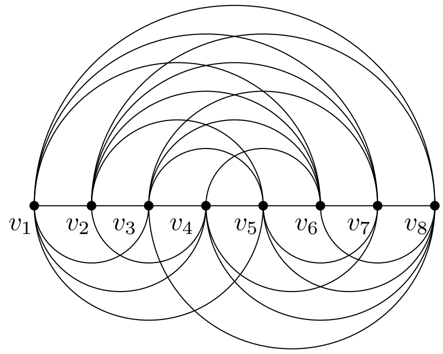


Figure 4.1: An example of a 2-page book drawing of K_8 with $Z(8) = 18$ crossings obtained by Blažek and Koman [BK64].

4.2 Hill's conjecture and monotone drawings

The drawings of complete graphs with $Z(n)$ crossings obtained by Blažek and Koman [BK64] (see also [Har02]) are 2-*page book* drawings. In such drawings the vertices are placed on a line l and each edge is fully contained in one of the half-planes determined by l. An example of such drawing can be found in Figure 4.1. Since 2-page book drawings may be considered as a strict subset of x-monotone drawings, we have $\text{mon-cr}(K_n) \leq Z(n)$ for every positive integer n.

Here, we prove Hill's conjecture for monotone semisimple drawings by providing a lower bound on $\text{mon-ocr}_+(K_n)$ that matches the upper bound obtained from the drawings of K_n by Blažek and Koman [BK64].

Theorem 4.1. *For every positive integer n, we have*

$$\text{mon-ocr}_+(K_n) = \text{mon-cr}(K_n) = Z(n).$$

4.2.1 Proof of Theorem 4.1

Let P denote a set of n points in the plane in general position and let k be an integer satisfying $0 \leq k \leq n-2$. The line segment joining a pair of points p and q in P is a k-*edge* ($\leq$$k$-*edge*) if there are exactly (at most, respectively) k points of P in one of the open half-planes defined by the line $\overline{pq}$.

Ábrego and Fernández-Merchant [AFM05] and Lovász et al. [LVWW04] discovered a relation between the numbers of k-edges (or $\leq$$k$-edges) in P and the number of convex 4-tuples of points in P, which is equal to the number of crossings of the complete geometric graph with vertex set P. This relation transforms every lower bound on the number of $\leq$$k$-edges to a lower bound on the number of crossings. Using this method, many incremental improvements on the rectilinear and pseudolinear crossing number of K_n have been achieved [ABFM+08, ACFM+12, AFM05, AGOR07, BS06, LVWW04].

Ábrego et al. [AAFM+13] showed that every 2-page book drawing D of K_n satisfies $\text{cr}(D) \geq Z(n)$. To prove this, Ábrego et al. [AAFM+13] generalized the notion of k-edges to arbitrary simple drawings of complete graphs. They also introduced the notion of $\leq\leq$$k$-edges, which capture the essential properties of 2-page book drawings better than $\leq$$k$-edges. We show that the approach using $\leq\leq$$k$-edges can be generalized to arbitrary semisimple x-monotone drawings.

For a semisimple drawing D of K_n and distinct vertices u and v of K_n, let γ be the oriented arc representing the edge $\{u, v\}$. If w is a vertex of K_n different from u and v, then we say that w is *on the left (right) side of* γ if the topological triangle uvw with vertices u, v, and w traced in this order is oriented counter-clockwise (clockwise, respectively). This generalizes the definition introduced by Ábrego et al. [AAFM+13] for simple drawings.

A k*-edge* in D is an edge uv of D that has exactly k vertices on the same side (left or right). Since every k-edge has $n-2-k$ vertices on the other side, every k-edge is also an $(n-2-k)$-edge and so every edge of D is a k-edge for some integer k where $0 \le k \le \lfloor n/2 \rfloor - 1$.

Analogously to the case of point sets, an i-edge in D with $i \le k$ is called a $\le$*k-edge*. Let $E_i(D)$ be the number of i-edges in D and $E_{\le k}(D)$ the number of $\le k$-edges in D. Clearly, $E_{\le k}(D) = \sum_{i=0}^{k} E_i(D)$. Similarly, the *number* $E_{\le\le k}(D)$ *of* $\le\le k$*-edges* of D is defined by the following identity for every $0 \le k \le \lfloor n/2 \rfloor - 1$.

$$E_{\le\le k}(D) := \sum_{j=0}^{k} E_{\le j}(D) = \sum_{i=0}^{k} (k+1-i) E_i(D). \tag{4.1}$$

For convenience, we define $E_{\le\le -1}(D) := 0$ and $E_{\le\le -2}(D) := 0$.

Considering the only three different simple drawings of K_4 up to a homeomorphism of the plane, Ábrego et al. [AAFM+13] showed that the number of crossings in a simple drawing D of K_n can be expressed in terms of the number of k-edges in the following way.

Lemma 4.2 ([AAFM+13])**.** *Every simple drawing D of K_n satisfies*

$$\operatorname{cr}(D) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k(n-2-k) E_k(D), \tag{4.2}$$

which can be equivalently rewritten as

$$\operatorname{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\le\le k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor \frac{n-2}{2} \right\rfloor - \frac{1}{2}\left(1 + (-1)^n\right) E_{\le\le \lfloor n/2 \rfloor - 2}(D).$$

Proof. We claim that every simple drawing of K_4 is plane-homeomorphic to one of the drawings in Figure 4.2. In the proof of this claim, we use A, B, and C to denote these three homeomorphism classes.

First, we show that there is at most one crossing in every simple drawing of K_4. Assume that two edges ac and bd of a simple drawing D' of K_4 cross at a point x. Since D' is simple, no edge crosses the closed curve abx. In particular, the edges ab and cd do not cross. Similarly, bc and ad share no crossing. Every other pair of edges consists of two adjacent edges that do not cross, as D' is simple.

Assuming that only edges ac and bd can cross in D', the edges ab, bc, cd, and ad form a hamiltonian cycle C of non-crossed edges. Once C is drawn, the edges ac and bd can be drawn either both in the interior of C, both in the exterior of C or one in the interior and the other in the exterior of C. These three possibilities correspond to drawings B, C, and A in Figure 4.2, respectively.

The drawing A is not plane-homeomorphic to B nor C, as it has no crossing. The drawings B and C are not plane-homeomorphic, since such homeomorphism of

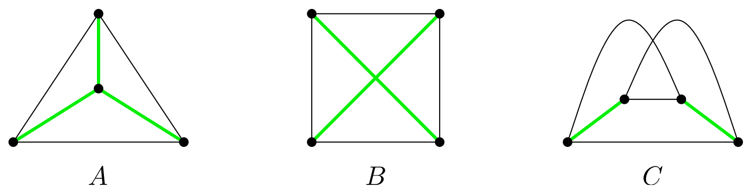


Figure 4.2: The three homeomorphism classes of simple drawings of K_4. The green edges are 1-edges.

the plane would map a compact set (closure of the interior of C) to a non-compact set (closure of the exterior of C), which is impossible.

Let D be a simple drawing of K_n. A *separation* in D is an unordered triple $\{ab, c, d\}$, where ab is an edge of D, c, d are vertices of D distinct from a, b, and the orientations of the two triangles abc and abd are opposite. Observe that $\{ab, c, d\}$ is a separation in D if and only if ab is a 1-edge in the complete subgraph of D induced by the vertices a, b, c, d. The total number of separations in D relates to both the crossing number and the numbers of k-edges in the following way.

(i) Every k-edge in D belongs to exactly $k(n-k-2)$ separations.

(ii) Every 4-tuple of vertices inducing a crossing contributes two separations, and every 4-tuple of vertices inducing a planar drawing of K_4 contributes three separations. In particular, for every drawing D' that is induced by 4 vertices of D we have the equality $\mathrm{cr}(D') + E_1(D') = 3$.

Fact (i) is a direct consequence of the definitions. Fact (ii) is easily seen by inspecting the three homeomorphism classes of simple drawings of K_4 in the plane.

Let n_A, n_B, and n_C denote the number of 4-tuples of vertices of D that induce a drawing of K_4 contained in the homeomorphism class A, B, and C, respectively. Clearly, $n_A + n_B + n_C = \binom{n}{4}$.

We use double-counting on the number s of separations in D. On one hand, we have $s = 3n_A + 2n_B + 2n_C$ by Fact (ii). On the other hand, Fact (i) implies $s = \sum_{k=0}^{\lfloor n/2 \rfloor -1} k(n-2-k)E_k(D)$. Putting these equations together and subtracting $n_A + n_B + n_C = \binom{n}{4}$ three times, we obtain

$$n_B + n_C = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor -1} k(n-2-k)E_k(D).$$

The first part of Lemma 4.2 now follows from the equality $\mathrm{cr}(D) = n_B + n_C$.

To finish the proof, it remains to rewrite the expression of $\mathrm{cr}(D)$ into the form that is in the second part of the lemma. To do so, we first observe that $E_k(D) = E_{\leq k}(D) - E_{\leq k-1}(D)$ and $E_{\leq k}(D) = E_{\leq\leq k}(D) - E_{\leq\leq k-1}(D)$ for every $1 \leq k \leq \lfloor n/2 \rfloor - 1$. Consequently, we have $E_k(D) = E_{\leq\leq k}(D) - 2E_{\leq\leq k-1}(D) + E_{\leq\leq k-2}(D)$ for $k \geq 2$. Note that the last equation is true even for $k \in \{0, 1\}$, since $E_{\leq\leq -1}(D) = 0$ and $E_{\leq\leq -2}(D) = 0$.

Using these equalities, the sum in the expression of $\mathrm{cr}(D)$ can be rewritten as follows.

$$
\begin{aligned}
&\sum_{k=0}^{\lfloor n/2\rfloor-1} k(n-2-k)E_k(D)\\
&= \sum_{k=0}^{\lfloor n/2\rfloor-1} k(n-2-k)\left(E_{\leq\leq k}(D)-2E_{\leq\leq k-1}(D)+E_{\leq\leq k-2}(D)\right)\\
&= \sum_{k=0}^{\lfloor n/2\rfloor-3} \left(k(n-2-k)-2(k+1)(n-3-k)+(k+2)(n-4-k)\right)E_{\leq\leq k}(D))\\
&+(\lfloor n/2\rfloor-1)(n-1-\lfloor n/2\rfloor)E_{\leq\leq\lfloor n/2\rfloor-1}(D)+(-2(\lfloor n/2\rfloor-1)(n-1-\lfloor n/2\rfloor)\\
&+(\lfloor n/2\rfloor-2)(n-\lfloor n/2\rfloor))E_{\leq\leq\lfloor n/2\rfloor-2}(D)\\
&= -2\sum_{k=0}^{\lfloor n/2\rfloor-3} E_{\leq\leq k}(D)+(\lfloor n/2\rfloor-1)(n-1-\lfloor n/2\rfloor)E_{\leq\leq\lfloor n/2\rfloor-1}(D)\\
&+(-2(\lfloor n/2\rfloor-1)(n-1-\lfloor n/2\rfloor)+(\lfloor n/2\rfloor-2)(n-\lfloor n/2\rfloor))E_{\leq\leq\lfloor n/2\rfloor-2}(D).
\end{aligned}
$$

Since $E_{\leq\leq\lfloor n/2\rfloor-1}(D)=E_{\leq\leq\lfloor n/2\rfloor-2}(D)+E_{\leq\lfloor n/2\rfloor-1}(D)=E_{\leq\leq\lfloor n/2\rfloor-2}(D)+\binom{n}{2}$, we have

$$
\begin{aligned}
\operatorname{cr}(D) = 3\binom{n}{4} &- \sum_{k=0}^{\lfloor n/2\rfloor-1} k(n-2-k)E_k(D) = 3\binom{n}{4}+2\sum_{k=0}^{\lfloor n/2\rfloor-3} E_{\leq\leq k}(D)\\
&+(n+1-2\lfloor n/2\rfloor)E_{\leq\leq\lfloor n/2\rfloor-2}(D)-(\lfloor n/2\rfloor-1)(n-1-\lfloor n/2\rfloor)\binom{n}{2}\\
&= 2\sum_{k=0}^{\lfloor n/2\rfloor-3} E_{\leq\leq k}(D)-\frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor+\begin{cases} E_{\leq\leq\lfloor n/2\rfloor-2}(D) & \text{if } n \text{ is even},\\ 2E_{\leq\leq\lfloor n/2\rfloor-2}(D) & \text{if } n \text{ is odd}.\end{cases}
\end{aligned}
$$

□

Lemma 4.2 generalizes the relation found by Ábrego and Fernández-Merchant [AFM05]. We further generalize it to semisimple drawings D of K_n where $\operatorname{cr}(D)$ is replaced by $\operatorname{ocr}(D)$, which counts the number of pairs of edges that cross an odd number of times in D.

Lemma 4.3. *Every semisimple drawing D of K_n satisfies*

$$
\operatorname{ocr}(D) = 2\sum_{k=0}^{\lfloor n/2\rfloor-2} E_{\leq\leq k}(D)-\frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor-\frac{1}{2}\left(1+(-1)^n\right)E_{\leq\leq\lfloor n/2\rfloor-2}(D).
$$

We recall that a *face* of a drawing D in the plane is a connected component of the complement of all the edges and vertices of D in $\mathbb{R}^2$. The *outer face* of D is the unbounded face of D.

Proof. To generalize Lemma 4.2 to semisimple drawings, we observe that semisimple drawings of K_4 can be classified analogously as the simple drawings of K_4. In particular, the following claim implies that the equality $\operatorname{ocr}(D)+E_1(D)=3$ is still satisfied for every semisimple drawing D of K_4.

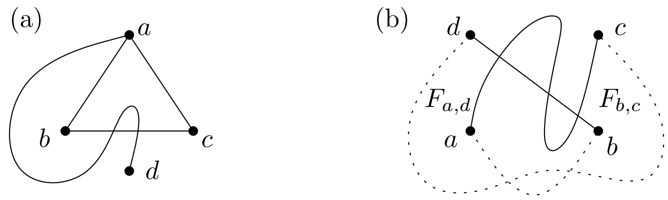


Figure 4.3: Illustrations to the proof of Lemma 4.3.

Claim. *A semisimple drawing D of K_4 has at most one pair of edges crossing an odd number of times. Moreover, D has three separations if* $\mathrm{ocr}(D) = 0$ *and two separations if* $\mathrm{ocr}(D) = 1$.

In the rest of the proof we prove the claim. Let D be a semisimple drawing of K_4. Suppose that $\mathrm{ocr}(D) = 0$. Let abc be a triangle in D and let d be the fourth vertex of D; see part (a) of Figure 4.3. If the edge ad crosses bc, then either d and b share no face in the drawing of the subgraph with edges ab, ac, ad, bc, or d and c share no face in the drawing of the subgraph with edges ab, ac, ad, bc. This means that one of the edges bd or cd either crosses an adjacent edge or crosses another edge an odd number of times. Therefore, the edge da has no crossing with the triangle abc. Analogous argument for the edges db and dc shows that D has no crossings at all. In particular, D has three separations; see Figure 4.2, left.

Now suppose that $\mathrm{ocr}(D) \geq 1$ and let ac and bd be two edges that cross an odd number of times. Since all the other edges are adjacent to both ac and bd, the vertices a, b, c, d share a common face F in the drawing of the subgraph with edges ac, bd. Moreover, the cyclic order of the vertices along the boundary of F is a, b, c, d, either clockwise or counter-clockwise; see part (b) of Figure 4.3.

We show that at most one more pair of edges can cross, either ab and cd, or ad and bc, but only an even number of times. For example, in the drawing of the subgraph with edges ac, bd, ab, the vertices c and d belong to the same face, and the edge cd is allowed to cross only the edge ab, each time switching faces. If ab and cd cross, then a and d share a unique face $F_{a,d}$ in the drawing of the graph K with edges ac, bd, ab, cd, and c and b share a unique face $F_{b,c}$ different from $F_{a,d}$. Since the edges ad and bc are adjacent to all edges of K, the edge ad lies completely in $F_{a,d}$, the edge bc lies completely in $F_{b,c}$ and thus ad and bc cannot cross. A symmetric argument shows that if ab and cd are disjoint, then ad and bc are either disjoint or cross an even number of times. In any case, we have $\mathrm{ocr}(D) \leq 1$.

It remains to show that every semisimple drawing D of K_4 with $\mathrm{ocr}(D) = 1$ has exactly two 1-edges. More precisely, we show that the two 1-edges always form a perfect matching.

Let e be an edge in D incident with the outer face. An *edge flip* is an operation where the portion of e incident with the outer face is redrawn along the other side of the drawing; see Figure 4.4. For drawings on the sphere, the edge flip is just a homeomorphism of the sphere. For every bounded face F of D, there is a sequence of edge flips that makes F the outer face.

If D is a semisimple drawing of K_4, then every edge flip of an edge e changes the orientation of the two triangles adjacent to e. Consequently, exactly the four edges adjacent to e, forming a 4-cycle, change from 1-edges to 0-edges or vice versa. Also observe that the edge flip of e can be performed only if e is a 0-edge.

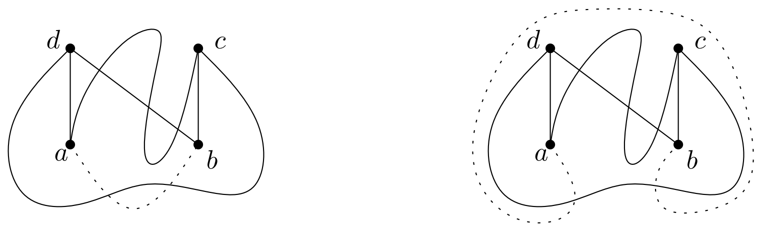


Figure 4.4: An edge flip of ab.

It follows that 1-edges form a perfect matching in D if and only if they form a perfect matching in the drawing obtained by the edge flip.

Let D be a semisimple drawing of K_4 with $\mathrm{ocr}(D) = 1$. Let ac and bd be the two edges that cross an odd number of times. By performing edge flips, we may assume that all the vertices are adjacent to the outer face of the drawing of the subgraph H with edges ac and bd. Each edge e of the remaining four edges can be drawn in two essentially different ways with respect to H, which differ just by an edge flip of e in $H + e$; see Figure 4.4. In total, there are 16 possible combinations. We cannot, however, assume any particular combination, since not all edge flips are always available. Observe that the orientations of all triangles are determined by the four binary choices for the edges ab, bc, cd, ad. Also, changing the choice for one edge e has the same effect on the orientations of the triangles as the edge flip of e. For one particular choice, for example the one yielding the middle drawing in Figure 4.2, the 1-edges form a perfect matching. Changing the choice for a subset of edges yields either a perfect matching of 1-edges or a complete graph of 1-edges. However, the latter option is excluded by the fact that in every semisimple drawing the edges incident with the outer face are 0-edges. This finishes the proof of the claim and the lemma. $\square$

Considering $\leq k$-edges, Ábrego and Fernández-Merchant [AFM05] and Lovász et al. [LVWW04] proved that for rectilinear drawings of K_n, the inequality $E_{\leq k} \geq 3\binom{k+2}{2}$ together with (4.2) gives $\overline{\mathrm{cr}}(G) \geq Z(n)$. However, there are simple x-monotone (even 2-page book) drawings of K_n where $E_{\leq k} < 3\binom{k+2}{2}$ for $k = 1$ [AAFM+13]. Ábrego et al. [AAFM+13] showed that the inequality $E_{\leq\leq k} \geq 3\binom{k+3}{3}$, which is implied by inequalities $E_{\leq j} \geq 3\binom{j+2}{2}$ for $j \leq k$, is satisfied by all 2-page book drawings. We show that the same inequality is satisfied by all x-monotone semisimple drawings of K_n.

Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_n. Note that we can assume that all vertices in an x-monotone drawing lie on the x-axis. We also assume that the x-coordinates of the vertices satisfy $x(v_1) < x(v_2) < \cdots < x(v_n)$.

The following observation describes the structure of k-edges incident to vertices on the outer face in semisimple drawings of K_n; see part (a) of Figure 4.5.

Observation 4.4. *Let D be a semisimple drawing of K_n, not necessarily x-monotone. Let v be a vertex incident to the outer face of D and let γ_i be the ith edge incident to v in the counter-clockwise order so that γ_1 and γ_{n-1} are incident to the outer face in a small neighborhood of v. Let v_{k_i} be the other endpoint of γ_i. Then for every i, j, $1 \leq i < j \leq n-1$, the triangle $v_{k_i} v v_{k_j}$ is oriented clockwise. Consequently, for every k with $1 \leq k \leq \lfloor n/2 \rfloor$, the edges γ_k and γ_{n-k} are $(k-1)$-edges.*

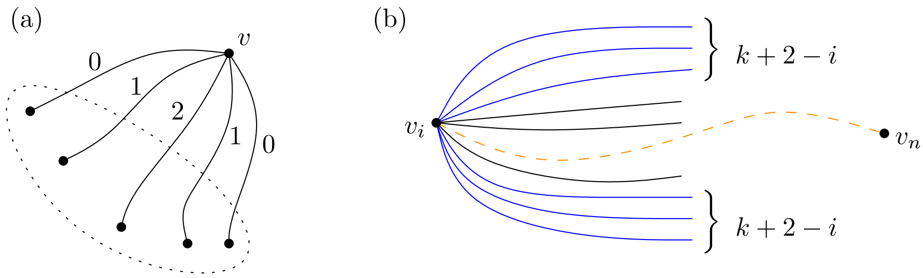


Figure 4.5: (a) k-edges incident with a vertex on the outer face. (b) After removing v_n, at least $k+2-i$ right edges at v_i are invariant $\leq k$-edges.

For a semisimple x-monotone drawing D of K_n, we use Observation 4.4 for the vertex v_n and the drawing D and then for each i, for the vertex v_i and the drawing of the subgraph induced by $v_i, v_{i+1}, \dots, v_n$.

The following definitions were introduced by Ábrego et al. [AAFM+13] for 2-page book drawings. Let D be a semisimple x-monotone drawing of K_n and let D' be the drawing obtained from D by deleting the vertex v_n together with its adjacent edges. A k-edge in D is a *(D, D')-invariant k-edge* if it is also a k-edge in D'. It is easy to see that every $\leq k$-edge in D' is also a $\leq(k+1)$-edge in D. If $0 \leq j \leq k \leq \lfloor n/2 \rfloor - 1$, then a (D, D')-invariant j-edge is called a *(D, D')-invariant $\leq k$-edge*. Let $E_{\leq k}(D, D')$ denote the number of (D, D')-invariant $\leq k$-edges.

For $i < j$, the edge $v_i v_j$ is called a *right edge at v_i*. The right edges at v_i have a natural vertical order, which coincides with the order of their crossings with an arbitrary vertical line separating v_i and v_{i+1}. The set of j *topmost* (*bottommost*) right edges at v_i is the set of j right edges at v_i that are above (below, respectively) all other right edges at v_i in their vertical order.

Lemma 4.5. *Let D be a semisimple x-monotone drawing of K_n and let k be a fixed integer such that $0 \leq k \leq (n-3)/2$. For every $i \in \{1, 2, \dots, k+1\}$, the $k+2-i$ bottommost and the $k+2-i$ topmost right edges at v_i are $\leq k$-edges in D. Moreover, at least $k+2-i$ of these $\leq k$-edges are (D, D')-invariant $\leq k$-edges.*

Proof. See part (b) of Figure 4.5. The first part of the lemma follows directly from Observation 4.4. If the edge $v_i v_n$ is one of the $k+2-i$ topmost right edges at v_i, then the $k+2-i$ bottommost right edges at v_i are (D, D')-invariant $\leq k$-edges. Otherwise the $k+2-i$ topmost right edges at v_i are (D, D')-invariant $\leq k$-edges. □

Corollary 4.6. *We have*

$$E_{\leq k}(D, D') \geq \sum_{i=1}^{k+1} (k+2-i) = \binom{k+2}{2}.$$

The following theorem gives a lower bound on the number of $\leq\leq k$-edges in a semisimple x-monotone drawing of K_n.

Theorem 4.7. *Let $n \geq 3$ and let D be a semisimple x-monotone drawing of K_n. Then for every k satisfying $0 \leq k < n/2 - 1$, we have*

$$E_{\leq\leq k}(D) \geq 3\binom{k+3}{3}.$$

Proof. The proof proceeds by induction on n and k starting at $n = 3$ and $k = -1$. The case $n = 3$ is trivially true, and the case $k = -1$ is taken care of by setting $E_{\leq\leq -1}(D) := 0$ for every drawing D. Let $n \geq 4$ and let D be a semisimple x-monotone drawing of K_n. For the induction step we remove the point v_n together with its adjacent edges to obtain a drawing D' of K_{n-1}, which is also semisimple and x-monotone.

Using Observation 4.4 we see that, for $0 \leq i \leq k < n/2-1$, there are two i-edges adjacent to v_n in D and together they contribute with $2\sum_{i=0}^{k}(k+1-i) = 2\binom{k+2}{2}$ to $E_{\leq\leq k}(D)$ by (4.1).

Let γ be an i-edge in D'. If $i \leq k$, then γ contributes with $(k - i)$ to the sum

$$E_{\leq\leq k-1}(D') = \sum_{i=0}^{k-1}(k-i)E_i(D').$$

We already observed that γ is either an i-edge or an $(i + 1)$-edge in D. If γ is also an i-edge in D (that is, γ is a (D, D')-invariant i-edge), then it contributes with $(k + 1 - i)$ to $E_{\leq\leq k}(D)$. This is a gain of $+1$ towards $E_{\leq\leq k-1}(D')$. If γ is an $(i + 1)$-edge in D, then it contributes only with $(k - i)$ to $E_{\leq\leq k}(D)$. Therefore we have

$$E_{\leq\leq k}(D) = 2\binom{k+2}{2} + E_{\leq\leq k-1}(D') + E_{\leq k}(D, D').$$

By the induction hypothesis we know that $E_{\leq\leq k-1}(D') \geq 3\binom{k+2}{3}$ and thus we obtain

$$E_{\leq\leq k}(D) \geq 3\binom{k+3}{3} - \binom{k+2}{2} + E_{\leq k}(D, D').$$

The theorem follows by plugging the lower bound from Corollary 4.6. □

We now show how to derive Theorem 4.1. Let us recall that it remains to show the lower bound $\text{mon-ocr}_+(K_n) \geq Z(n)$, as the upper bound $\text{mon-cr}(K_n) \leq Z(n)$ follows from drawings obtained by Blažek and Koman [BK64].

Let D be a semisimple x-monotone drawing of K_n for $n \geq 3$. By Lemma 4.3 and Theorem 4.7, we have

$$\begin{aligned}\text{ocr}(D) &\geq 2\sum_{k=0}^{\lfloor n/2\rfloor-2} 3\binom{k+3}{3} - \frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor - \frac{3}{2}\left(1+(-1)^n\right)\binom{\lfloor n/2\rfloor+1}{3}\\ &= 6\binom{\lfloor n/2\rfloor+2}{4} - \frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor - \frac{3}{2}\left(1+(-1)^n\right)\binom{\lfloor n/2\rfloor+1}{3}\end{aligned}$$

The last expression can be rewritten as $(n-1)^2(n-3)^2/64$ if n is odd and as $n(n-2)^2(n-4)/64$ if n is even. In other words, we obtain $\text{ocr}(D) \geq Z(n)$. This finishes the proof of Theorem 4.1.

4.2.2 Further progress on Hill's conjecture

Although Hill's conjecture remains open, some progress has been made in the last three years. Here, we very briefly survey this recent development and we introduce the current state of knowledge on this famous problem.

Pedro Ramos [Ram13] introduced the term *shellable drawings* of K_n for which the crossing number is always at least $Z(n)$. Ábrego et al. [AAFM+14a] later observed that a still more general condition, s-shellability for some $s \geq \lfloor n/2 \rfloor$, is sufficient, since the depth of the recursion in the proof of Theorem 4.1 is only $n/2$. For integers n and $s \geq \lfloor n/2 \rfloor$, a drawing of the complete graph with vertex set V of size n is called *s-shellable* if there is a subset $\{v_1, v_2, \dots, v_s\}$ of vertices from V such that for every pair i, j with $1 \leq i < j \leq s$, the vertices v_i and v_j are on the outer face of the drawing induced by $V \setminus \{v_1, v_2, \dots, v_{i-1}, v_{j+1}, v_{j+2}, \dots, v_s\}$. This is slightly more restrictive compared to the original definition [AAFM+14a], where v_1 and v_s are not required to be incident with the outer face. The notions of a shellable drawing of K_n and an n-shellable drawing of K_n coincide.

The class of s-shellable drawings includes, for example, all drawings with a crossing-free cycle of length s, with at least one edge of the cycle incident with the outer face [AAFM+14a]. In particular, the class of s-shellable drawings of K_n includes x-bounded drawings of K_n, which also form a subclass of shellable drawings and generalize x-monotone drawings. A drawing of a graph is *x-bounded* if no two vertices share the same x-coordinate and every interior point of every edge uv lies in the interior of the strip bounded by two vertical lines passing through the vertices u and v.

It is not a priori clear that shellable drawings are essentially different from monotone or x-bounded drawings, since the conditions for shellability and x-boundedness are very similar at first sight. Balko, Fulek, and Kynčl [BFK15] showed that simple shellable drawings are indeed more general than simple monotone drawings.

In 2014, essentially only two classes of drawings of K_n with $Z(n)$ crossings were known: Hill's so-called cylindrical drawings [HH63] and 2-page book drawings by Blažek and Koman [BK64]. Ábrego et al. [AAFM+14a] showed that Hill's drawings of K_n are $\lfloor n/2 \rfloor$-shellable and that 2-page book drawings of K_n are n-shellable. Thus it was natural to ask whether all drawings of sufficiently large cardinality n with $Z(n)$ crossings are s-shellable for some $s \geq \lfloor n/2 \rfloor$. Ábrego et al. [AAFM+14b] answered this question in the negative and constructed drawings $D_{m,m,1}$ of K_{2m+1} for every $m \geq 5$ such that $\mathrm{cr}(D_{m,m,1}) = Z(2m+1)$ and every edge is crossed by at least one other edge in $D_{m,m,1}$. In particular, the drawing $D_{m,m,1}$ is not s-shellable for any $s \geq \lfloor n/2 \rfloor$, as every face of $D_{m,m,1}$ contains at most one vertex on its boundary.

Very recently, Ábrego et al. [AAFM+15] introduced a more general variant of shellability. For $s \in \mathbb{N}_0$, a drawing D of K_n is *s-bishellable* if there is a face F of D and subsets $a_0, \dots, a_s$ and $b_s, \dots, b_0$ of vertices of D, each consisting of distinct vertices of K_n, such that the following conditions are satisfied:

(1) for every $i \in \{0, \dots, s\}$, the vertex a_i is incident with the face of $D - \{a_0, \dots, a_{i-1}\}$ that contains F,

(2) for every $i \in \{0, \dots, s\}$, the vertex b_i is incident with the face of $D - \{b_0, \dots, b_{i-1}\}$ that contains F,

(3) for every $i \in \{0, \dots, s\}$, the intersection $\{a_0, \dots, a_i\} \cap \{b_{s-i}, \dots, b_0\}$ is empty.

If a drawing D is s-shellable with a witnessing subset $v_1, \dots, v_s$, then D is $(s-2)$-bishellable with subsets $a_0, \dots, a_{s-2}$ and $b_{s-2}, \dots, b_0$, where $a_i := v_{i+1}$ and

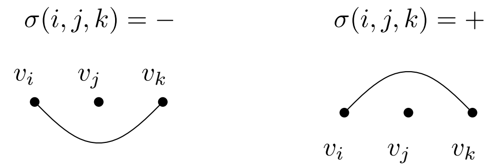


Figure 4.6: The negative and the positive signature $\sigma(i, j, k)$.

$b_i := v_{s-i}$. Also, if D is s-bishellable, then D is also $(s-1)$-bishellable. We say that a drawing is *bishellable* if it is $(\lfloor n/2 \rfloor - 2)$-bishellable.

Ábrego et al. [AAFM+15] proved a version of Theorem 4.1 for currently most general class of drawings by showing that if D is a simple bishellable drawing of K_n, then $\text{cr}(D) \geq Z(n)$. They also provided a drawing of K_{11} with $Z(11)$ crossings that is bishellable, but not s-shellable for any $s \geq 5$. However, the drawings $D_{m,m,1}$ are not bishellable. In fact, it seems likely that the portion of bishellable drawings of K_n with $Z(n)$ crossings vanishes as n grows [AAFM+15].

Theorem 4.1 can be strengthened also for other notions of the crossing number. We call a drawing of a graph *weakly semisimple* if every pair of adjacent edges cross an even number of times; independent edges may cross arbitrarily. The *monotone weakly semisimple odd crossing number* of a graph G, denoted by $\text{mon-ocr}_{\pm}(G)$, is the smallest number of pairs of edges that cross an odd number of times in a monotone weakly semisimple drawing of G. Note that we have $\text{mon-ocr}_{\pm}(G) \leq \text{mon-ocr}_{+}(G)$ for every graph G.

Balko, Fulek, and Kynčl [BFK15] proved a stronger version of Theorem 4.1 by showing $\text{mon-ocr}_{\pm}(K_n) \geq Z(n)$ for every positive integer n. They also extended Theorem 4.1 by showing that every semisimple shellable drawing D of K_n satisfies $\text{ocr}(D) \geq Z(n)$.

4.3 Monotone drawings of K_n

In this section we develop a combinatorial characterization of simple and semisimple x-monotone drawings based on the signature functions introduced by Peters and Szekeres [PS06] as generalizations of order types of planar point sets. Let T_n be the set of ordered triples (i, j, k) with $i < j < k$, from the set $[n]$ and let Σ_n be the set of *signature functions* $\sigma \colon T_n \to \{-, +\}$. The set T_n may be also regarded as the set $\binom{[n]}{3}$ of all unordered triples, since we write all the triples in the increasing order of their elements.

Let D be an x-monotone drawing of the complete graph $K_n = (V, E)$ with vertices $v_1, v_2, \ldots, v_n$ such that their x-coordinates satisfy $x(v_1) < \cdots < x(v_n)$. We assign a signature function $\sigma \in \Sigma_n$ to the drawing D according to the following rule. For every edge $e = \{v_i, v_k\} \in E$ and every integer $j \in (i, k)$, let $\sigma(i, j, k) = -$ if the point v_j lies above the arc representing the edge e and $\sigma(i, j, k) = +$ otherwise; see Figure 4.6. Note that if the drawing D is also semisimple, then a triangle $v_i v_j v_k$, with $j \in (i, k)$, is oriented clockwise (counter-clockwise) if and only if $\sigma(i, j, k) = -$ ($\sigma(i, j, k) = +$, respectively).

It is easy to see that, for every signature function $\sigma \in \Sigma_n$, there exists an x-monotone drawing that induces σ. However, some signature functions are induced only by drawings that are not semisimple. We show a characterization of

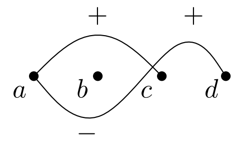


Figure 4.7: A 4-tuple (a, b, c, d) of the form $+-+\xi$ forces two adjacent edges to cross.

simple and semisimple x-monotone drawings by small forbidden configurations in the signature functions.

For integers $a, b, c, d \in [n]$ with $a < b < c < d$, signs $\xi_1, \xi_2, \xi_3, \xi_4 \in \{-, +\}$ and a signature function $\sigma \in \Sigma_n$, we say that the 4-tuple (a, b, c, d) *is of the form* $\xi_1\xi_2\xi_3\xi_4$ *in* σ if

$$\sigma(a, b, c) = \xi_1,\ \sigma(a, b, d) = \xi_2,\ \sigma(a, c, d) = \xi_3, \text{ and } \sigma(b, c, d) = \xi_4.$$

For a sign $\xi \in \{-, +\}$ we use $\overline{\xi}$ to denote the opposite sign, that is, if $\xi = +$ then $\overline{\xi} = -$ and conversely, if $\xi = -$ then $\overline{\xi} = +$.

Theorem 4.8. *A signature function $\sigma \in \Sigma_n$ can be realized by a semisimple x-monotone drawing if and only if every 4-tuple of indices from $[n]$ is of one of the forms*

$$++++, ----, ++--, --++, -++-, +--+,$$
$$---+, +++-, +---, -+++$$

in σ. The signature function σ can be realized by a simple x-monotone drawing if, in addition, there is no 5-tuple (a, b, c, d, e) with $1 \le a < b < c < d < e \le n$ such that

$$\sigma(a, b, e) = \sigma(a, d, e) = \sigma(b, c, d) = \overline{\sigma(a, c, e)}.$$

Proof. Let σ be a signature function with a *forbidden* 4*-tuple*, that is, an ordered 4-tuple (a, b, c, d) whose form in σ is not listed in the statement of the theorem. Such a 4-tuple (a, b, c, d) is one of the forms $\xi_1\overline{\xi_1}\xi_1\xi_2$ or $\xi_2\xi_1\overline{\xi_1}\xi_1$ where $\xi_1, \xi_2 \in \{-, +\}$. If (a, b, c, d) is of the form $+-+\xi$ where $\xi \in \{-, +\}$ is an arbitrary sign, then the edges v_av_c and v_av_d are forced to cross between the vertical lines going through v_b and v_c; see Figure 4.7. However this is not allowed in a semisimple drawing and we have a contradiction. The other cases are symmetric.

On the other hand, let σ be a signature function such that every 4-tuple is of one of the ten allowed forms in σ. We will construct a semisimple x-monotone drawing D of K_n that induces σ. We use the points $v_i := (i, 0)$, $i \in [n]$, as vertices and connect consecutive pairs of vertices by straight-line segments.

For $m \in [n]$, let L_m be the vertical line containing v_m. In every x-monotone drawing, the line L_m intersects every edge v_iv_j with $1 \le i < m \le j \le n$ exactly once. To draw the edges of K_n, it suffices to specify the positions of their intersections with the lines L_m and to draw the edges as polygonal lines with bends at these intersections. Instead of the absolute position of these intersections on L_m, we only need to determine their vertical total ordering, which we represent by a total ordering $\prec_m$ of the corresponding edges. The edges whose right endpoint is v_m will be ordered by $\prec_m$ according to their vertical order in the left neighborhood of v_m. The edges with left endpoint v_m are not considered in $\prec_m$.

The idea of the construction is to interpret the signature function as the set of above/below relations for vertices and edges and take a set of orderings $\prec_m$ that obey these relations and minimize the total number of crossings. In the rest of the proof we show a detailed, explicit construction of the orderings $\prec_m$ which induce an x-monotone semisimple drawing.

For $i \in [n]$, we define an ordering $\lessdot_i$ of the edges with a common left endpoint v_i (that is, the right edges at v_i) in the following way. If $e = v_iv_j$ and $f = v_iv_k$, $i < j, k$, are two such edges, then we set $e \lessdot_i f$ if either $j < k$ and $\sigma(i,j,k) = +$, or $k < j$ and $\sigma(i,k,j) = -$. Clearly, the relation $\lessdot_i$ is irreflexive, antisymmetric, and for every pair of right edges e, f at v_i either $e \lessdot_i f$ or $f \lessdot_i e$. To show that $\lessdot_i$ is a total ordering, it remains to prove that it is transitive. Suppose for contrary that there are three edges $e = v_iv_j$, $f = v_iv_k$, and $g = v_iv_l$ with $i < j < k < l$ such that $e \lessdot_i f$, $f \lessdot_i g$ and $g \lessdot_i e$. Then $\sigma(i,j,k) = +$, $\sigma(i,k,l) = +$ and $\sigma(i,j,l) = -$, so the 4-tuple i,j,k,l is of the form $+{-}{+}\xi$ in σ, which is forbidden. Similarly, if $f \lessdot_i e$, $e \lessdot_i g$ and $g \lessdot_i f$, then the 4-tuple i,j,k,l is of the form $-{+}{-}\xi$ in σ, which is forbidden as well. The remaining four cases can be treated similarly, we always obtain one of the previous two forbidden forms in σ.

We proceed by induction on m. In the case $m = 1$ the ordering $\prec_1$ is empty. For $m = 2$ the ordering $\prec_2$ compares only edges with the common endpoint v_1, so we can set $\prec_2 := \lessdot_1$. Since all the edges are drawn by line segments starting in a common endpoint, no crossings appear between L_1 and L_2.

Let $m > 2$. For the inductive step we consider the following sets $S_1, \ldots, S_6$ of edges which intersect L_{m-1} and L_m (see Figure 4.8):

$$
\begin{aligned}
S_1 &:= \{v_iv_j \mid \sigma(i, m-1, j) = -, \sigma(i, m, j) = -\},\\
S_2 &:= \{v_{m-1}v_j \mid \sigma(m-1, m, j) = -\},\\
S_3 &:= \{v_iv_j \mid \sigma(i, m-1, j) = +, \sigma(i, m, j) = - \text{ or } j = m\},\\
S_4 &:= \{v_iv_j \mid \sigma(i, m-1, j) = -, \sigma(i, m, j) = + \text{ or } j = m\},\\
S_5 &:= \{v_{m-1}v_j \mid \sigma(m-1, m, j) = +\},\\
S_6 &:= \{v_iv_j \mid \sigma(i, m-1, j) = +, \sigma(i, m, j) = +\}.
\end{aligned}
$$

The edges within sets S_2 and S_5 are ordered according to $\lessdot_{m-1}$ and the edges in each of the remaining sets S_k according to $\prec_{m-1}$. For $e \in S_k$ and $f \in S_l$ where $k < l$, we set $e \prec_m f$. Observe that $\prec_m$ is a total ordering.

We show that the drawing D determined by the orders $\prec_m$ is semisimple. Suppose for contradiction that two adjacent edges $e = v_iv_j$ and $f = v_iv_k$, with $i < j, k$ and $e \lessdot_i f$, cross. Their leftmost crossing occurs between lines L_{m-1} and L_m, where $i < m - 1$ and $m \le j, k$. There are three cases:

(i) $e \in S_6$ and $f \in S_3$,

(ii) $e \in S_4$ and $f \in S_1$, or

(iii) $e \in S_4$ and $f \in S_3$.

We analyze the cases (i) and (iii) together, case (i) and case (ii) are symmetric. If $j < k$ then $\sigma(i, m, k) = -$ and by the definition of the relation $\lessdot_i$, we have $\sigma(i, j, k) = +$. This further implies that $m < j$ and $\sigma(i, m, j) = +$. Thus (i, m, j, k)

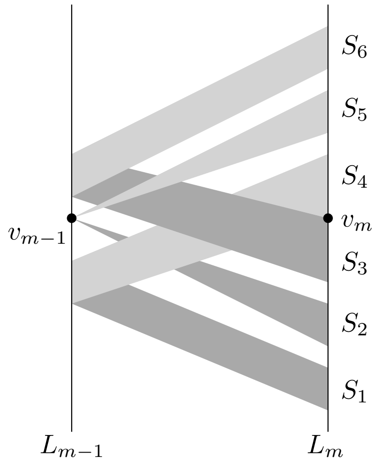


Figure 4.8: Placing edges and minimizing the number of crossings.

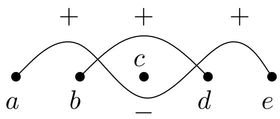


Figure 4.9: A forbidden 5-tuple (a, b, c, d, e) forces at least two crossings.

forms a forbidden 4-tuple. If $k < j$, then $\sigma(i, m, j) = +$, $\sigma(i, k, j) = -$, which implies that $m < k$ and $\sigma(i, m, k) = -$, and so we obtain a forbidden 4-tuple (i, m, k, j).

Now suppose that two adjacent edges $e = v_i v_k$ and $f = v_j v_k$, with $i, j < k$, cross. Their leftmost crossing occurs between lines L_{m-1} and L_m, where $i, j \leq m-1$ and $m < k$. We may assume that $f \prec_m e$ and $e \prec_{m-1} f$. There are five cases:

(i) $e \in S_6$ and $f \in S_3$,

(ii) $e \in S_4$ and $f \in S_1$,

(iii) $e \in S_4$ and $f \in S_3$,

(iv) $e \in S_4$ and $f \in S_2$, or

(v) $e \in S_5$ and $f \in S_3$.

Case (i) and case (ii) are symmetric, as well as case (iv) and case (v). Therefore it is sufficient to consider cases (i), (iii), and (v). In all these three cases $\sigma(j, m, k) = -$ and $\sigma(i, m, k) = +$. If $j < i$, then $\sigma(j, i, k) = +$ since $e \prec_{m-1} f$ and the edges e and f do not cross to the left of L_{m-1}. Hence (j, i, m, k) forms a forbidden 4-tuple in σ. If $i < j$, then analogously $\sigma(i, j, k) = -$ and (i, j, m, k) forms a forbidden 4-tuple in σ. This finishes the proof that D is semisimple.

It remains to show the second part of the theorem. If D is a drawing with a signature function σ with a *forbidden* 5-*tuple* (a, b, c, d, e), then D is not simple as the edges $v_a v_e$ and $v_b v_d$ are forced to cross at least twice; see Figure 4.9.

Given a signature function σ with no forbidden 4-tuples and 5-tuples we apply the same construction as before to obtain a semisimple x-monotone drawing D. We show that D is, in addition, simple. Since D is semisimple, no two crossing edges have an endpoint in common. By the construction of D, every crossing c of

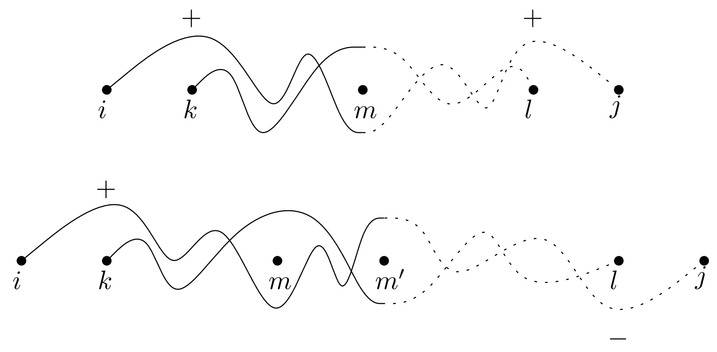


Figure 4.10: Edges v_iv_j and v_kv_l crossing twice imply a forbidden 5-tuple or 4-tuple; case $l < j$.

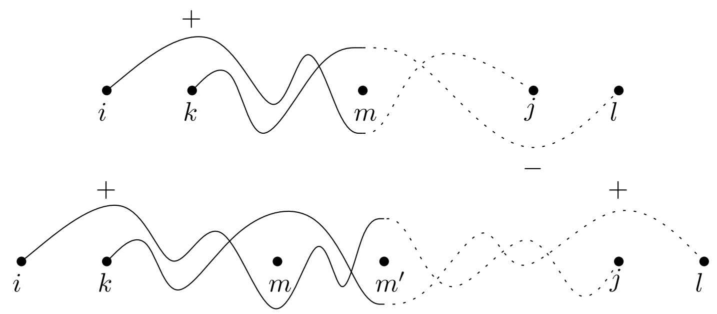


Figure 4.11: Edges v_iv_j and v_kv_l crossing twice imply a forbidden 5-tuple or 4-tuple; case $j < l$.

two edges e and f occurs between lines L_m and L_{m+1} for some $m \in [n-1]$ and we say that v_{m+1} is the *right neighbor* of c. The right neighbor is either an endpoint of e or f or it separates the crossings of L_{m+1} with e and f. Suppose that there are edges $e = v_iv_j$ and $f = v_kv_l$ with $i < k < j, l$ that cross at least twice. We show that then there is always a forbidden 4-tuple or a forbidden 5-tuple in σ.

Let v_m be the right neighbor of the leftmost crossing and $v_{m'}$ the right neighbor of the second leftmost crossing of e and f. Observe that $i, k < m < m' \leq j, l$.

First assume that $l < j$. Refer to Figure 4.10. If $\sigma(i,k,j) = \sigma(i,l,j) = \xi$ for some $\xi \in \{-,+\}$, then $\xi = \sigma(k,m,l) = \overline{\sigma(i,m,j)}$ and so (i,k,m,l,j) forms a forbidden 5-tuple. If $\sigma(i,k,j) = \overline{\sigma(i,l,j)} = \xi$ for some $\xi \in \{-,+\}$, then e and f cross at least three times and so $m' < l, j$. We have $\xi = \sigma(k,m,l) = \overline{\sigma(i,m,j)} = \overline{\sigma(k,m',l)} = \sigma(i,m',j)$. If $\sigma(k,m,m') = \overline{\xi}$, then (k,m,m',l) forms a forbidden 4-tuple. If $\sigma(k,m,m') = \xi$, then (i,k,m,m',j) forms a forbidden 5-tuple.

Conversely let $j < l$. Refer to Figure 4.11. Assume that $\sigma(i,k,j) = \overline{\sigma(k,j,l)} = \xi$ for some $\xi \in \{-,+\}$. Then $\xi = \sigma(k,m,l) = \overline{\sigma(i,m,j)}$. If $\sigma(k,m,j) = \xi$, we get a forbidden 4-tuple (i,k,m,j), otherwise $\sigma(k,m,j) = \overline{\xi}$ and we get a forbidden 4-tuple (k,m,j,l). Finally, assume that $\sigma(i,k,j) = \sigma(k,j,l) = \xi$ for some $\xi \in \{-,+\}$. The proof in this case is identical to the proof of the case $l < j$ and $\sigma(i,k,j) = \overline{\sigma(i,l,j)} = \xi$ in the previous paragraph. □

4.4 Pseudolinear crossing number of K_n

All x-monotone pseudolinear drawings of K_n can be characterized in a combinatorial way by forbidden 4-tuples in the corresponding signature function, by further restricting the conditions on the signatures in Theorem 4.8. In fact, the conditions in Theorem 4.8 are precisely the *geometric constraints* that Peters and Szekeres [PS06] used to restrict the set of signature functions in their investigation of the Erdős–Szekeres conjecture.

Theorem 4.9 ([FW01, BFK15]). *A signature function $\sigma \in \Sigma_n$ can be realized by a pseudolinear x-monotone drawing if and only if every ordered 4-tuple of indices from $[n]$ is of one of the forms*

$$++++,+++-,++--,+---,$$
$$----,---+,--++,-+++$$

in σ.

Theorem 4.9 is a dual analogue of a result by Felsner and Weil [FW01]. A direct, self-contained proof of Theorem 4.9 was later found by Balko, Fulek, and Kynčl [BFK15].

We recall that $\widetilde{\mathrm{cr}}(G)$ and $\overline{\mathrm{cr}}(G)$ denote the pseudolinear and the rectilinear crossing number of a graph G, respectively. We also recall that $\widetilde{\mathrm{cr}}(G) \leq \overline{\mathrm{cr}}(G)$ for every graph G, as every rectilinear drawing is pseudolinear.

Both the rectilinear and the pseudolinear crossing number of K_n have attracted a lot of attention; see the survey by Ábrego, Fernández-Merchant, and Salazar [AFMS13]. For the pseudolinear crossing number of K_n, the best known lower bound $\widetilde{\mathrm{cr}}(K_n) > 0.379972\binom{n}{4} - O(n^3)$ is due to Ábrego et al. [ACFM+12].

Ábrego and Fernández-Merchant [AFM07] proved $\overline{\mathrm{cr}}(K_n) < 0.380559\binom{n}{4} + O(n^3)$ using an iterative procedure that generates arbitrarily large rectilinear drawings with few crossings from a given base drawing. In his *Rectilinear Crossing Number Project*, Aichholzer [Aic] published a list of best known rectilinear drawings of K_n for $n \leq 100$ (this list has been updated recently). Ábrego et al. [ACFM+10] used one of these drawings to improve the upper bound to $\overline{\mathrm{cr}}(K_n) < 0.380544\binom{n}{4} + O(n^3)$ and also produced new drawings yielding the upper bound $\overline{\mathrm{cr}}(K_n) < 0.380488\binom{n}{4} + O(n^3)$. Fabila-Monroy and López [FML14] recently improved this bound to $\overline{\mathrm{cr}}(K_n) < 0.380473\binom{n}{4} + O(n^3)$ using a simple heuristic, moving the vertices of an initial rectilinear drawing one by one by a vector chosen randomly from a product of exponential distributions. The bound of Fabila-Monroy and López [FML14] has been the best known upper bound on $\widetilde{\mathrm{cr}}(K_n)$.

Here, we improve the upper bound on $\widetilde{\mathrm{cr}}(K_n)$. We note that this is the first time the leading constant in the best upper bound on $\widetilde{\mathrm{cr}}(K_n)$ is strictly smaller than in the best upper bound on $\overline{\mathrm{cr}}(K_n)$.

Theorem 4.10. *For every positive integer n, we have*

$$\widetilde{\mathrm{cr}}(K_n) < 0.380448\binom{n}{4} + O(n^3).$$

To prove Theorem 4.10, we follow Fabila-Monroy's and Lopez's [FML14] approach and adapt it to pseudolinear drawings. We perform small random changes in an initial pseudolinear drawing D of K_n for small n while keeping track of $\mathrm{cr}(D)$. To search the space of configurations efficiently, we use *simulated annealing* [Č85, KGJV83], a probabilistic metaheuristic motivated by the thermodynamics of annealing in metallurgy. In most of the cases (where $n \geq 42$), we obtain a pseudolinear drawing of K_n with fewer crossings than the current best known upper bounds. The improvement on the leading constant in $\widetilde{\mathrm{cr}}(K_n)$ is derived using a pseudolinear version of the construction by Ábrego and Fernández-Merchant [AFM07].

According to our knowledge, all previous upper bounds on $\widetilde{\mathrm{cr}}(K_n)$ follow from upper bounds on $\overline{\mathrm{cr}}(K_n)$. It is possible that some of our configurations are not stretchable; this would hint at the possibility that $\widetilde{\mathrm{cr}}(K_n) < \overline{\mathrm{cr}}(K_n)$ for large values of n. This it true for general graphs, see [HVLnS14] for a construction of graphs for which the rectilinear crossing number is strictly larger than the pseudolinear one.

Some improvements on the leading constant of $\widetilde{\mathrm{cr}}(K_n)$ are illustrated in Figure 4.12. The results of our experiments can be found on a separate webpage [BK].

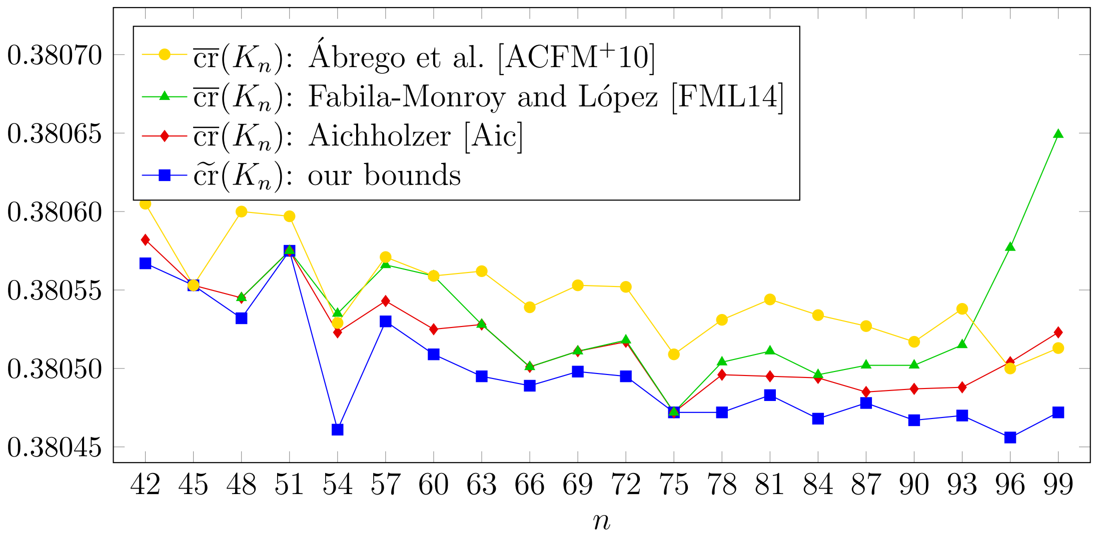


Figure 4.12: Improvements on the leading constant in $\overline{\mathrm{cr}}(K_n)$ and $\widetilde{\mathrm{cr}}(K_n)$.

We apply Theorem 4.9 and represent pseudolinear drawings of K_n by signatures, as they permit a straightforward implementation and allow fast prototyping. It is a simple observation that in every pseudolinear drawing D of K_n the number of crossings $\mathrm{cr}(D)$ equals the number of 4-tuples of one of the forms $++++$, $----$, $++--$, $--++$ in the n-signature realized by D.

4.4.1 Crossing minimization via simulated annealing

Simulated annealing is a probabilistic metaheuristic for minimizing a given objective function over vast configuration spaces, first described by Kirkpatrick, Gelatt, and Vecchi [KGJV83] and independently by Černý [Č85]. Here we describe our implementation of this method.

An n-signature σ is *realizable* if there is a pseudolinear drawing D of K_n that realizes σ in the sense described in Section 4.3. A *switch of a sign* $\sigma(i, j, k)$,

$(i,j,k) \in T_n$, is the change of $\sigma(i,j,k)$ to $\overline{\sigma(i,j,k)}$.

The input of the algorithm consists of a realizable n-signature σ_0, a parameter $T_0 \in \mathbb{R}$, called the *temperature*, and an upper bound $t_{\max} \in \mathbb{N}$ on the total number of steps.

In every step $t \in \{0, \dots, t_{\max}-1\}$, we find all triples from the set T_n of triples (i,j,k) with $1 \leq i < j < k \leq n$, whose signs can be switched in σ_t so that the resulting n-signature is still realizable. Such triples are called *switchable in* σ_t. Let (i,j,k) be a switchable triple in σ_t chosen uniformly and independently at random from the set of all switchable triples in σ_t. We use σ' to denote the realizable n-signature obtained from σ_t by the switch of $\sigma_t(i,j,k)$. Let D_t and D' be the pseudolinear drawings of K_n that realize σ_t and σ', respectively.

We set $\sigma_{t+1} := \sigma'$ if $\widetilde{\mathrm{cr}}(D') < \widetilde{\mathrm{cr}}(D_t)$. In the case $\widetilde{\mathrm{cr}}(D') \geq \widetilde{\mathrm{cr}}(D_t)$, we set $\sigma_{t+1} := \sigma'$ with the *acceptance probability* $p(\widetilde{\mathrm{cr}}(D_t), \widetilde{\mathrm{cr}}(D'), T_t)$ and set $\sigma_{t+1} := \sigma_t$ otherwise. At the end of step t, we update the temperature by setting $T_{t+1} := h(T_t, t)$ where h is so-called *cooling function*. In one of our implementations of the algorithm, we set $p(a,b,T) := e^{(a-b)/T}$ and $h(T,t) := T/\log(t+1000)$.

The algorithm terminates after $t_{\max}$ steps and outputs the realizable n-signature $\sigma_{t_{\max}}$.

Note that we allow switches that increase the number of crossings. By being able to move to a worse drawing, it is possible to jump out of local minima and, possibly, find a better drawing.

It follows from Theorem 4.9 that in every realizable n-signature σ, a triple $(i,j,k) \in T_n$ is switchable in σ if and only if σ contains no 4-tuple $\{a,i,j,k\} \in \binom{[n]}{4}$ of one of the following forms:

(i) $\bullet\circ\circ\circ$ or $\bullet\bullet\circ\circ$ for $a < i$,

(ii) $\circ\circ\circ\circ$ or $\bullet\circ\circ\circ$ for $i < a < j$,

(iii) $\circ\circ\circ\circ$ or $\circ\circ\circ\bullet$ for $j < a < k$,

(iv) $\circ\circ\circ\bullet$ or $\circ\circ\bullet\bullet$ for $k < a$.

Here $\circ \in \{-,+\}$ denotes the sign $\sigma(i,j,k)$ and $\bullet \in \{-,+\}$ is $\overline{\sigma(i,j,k)}$.

After the switch of $\sigma_t(i,j,k)$ in step t, only triples contained in a 4-tuple $F \in \binom{[n]}{4}$ with $i,j,k \in F$ can become or stop being switchable in σ_t. It can be shown that exactly one triple in F becomes and exactly one stops being switchable in σ_t. Therefore we can update the list of switchable triples in σ_t in time $O(n)$.

We can update the number of crossings in time $O(n)$, as it is sufficient to check only 4-tuples from $\binom{[n]}{4}$ that contain i, j, and k. Thus the time complexity of a single step of our algorithm is $O(n)$.

We obtained new pseudolinear drawings of K_n for small values of n, improving the previously best pseudolinear drawings in many instances; see Table 4.1. In particular, we found a pseudolinear drawing of K_{216} with 33 260 204 crossings. The proof of Theorem 4.10 is based on this drawing. Signatures of all the new drawings can be found in [BK].

Compared to the method of Fabila-Monroy and López [FML14], our approach is, in a certain sense, finer. Fabila-Monroy and López perturb a single vertex of a rectilinear drawing D of K_n in every step. However, this may lead to a rather large change in the number of crossings (quadratic in the worst case), as

n	$\overline{\text{cr}}(K_n)$ bounds	$\widetilde{\text{cr}}(K_n)$ bounds	n	$\overline{\text{cr}}(K_n)$ bounds	$\widetilde{\text{cr}}(K_n)$ bounds
42	40 590 [Aic]	40 588	73	403 180 [Aic]	403 166
44	49 370 [Aic]	49 366	74	426 398 [Aic]	426 391
46	59 463 [Aic]	59 459	76	475 773 [Aic]	475 758
48	71 010 [Aic]	71 007	77	502 011 [Aic]	501 997
50	84 223 [Aic]	84 219	78	529 278 [Aic]	529 242
52	99 161 [Aic]	99 158	79	557 741 [Aic]	557 723
54	115 975 [Aic]	115 953	80	587 280 [Aic]	587 251
56	134 917 [Aic]	134 901	81	617 930 [Aic]	617 908
57	145 164 [Aic]	145 158	83	682 976 [Aic]	682 958
58	156 042 [Aic]	156 040	84	717 276 [Aic]	717 222
59	167 506 [Aic]	167 490	85	752 971 [Aic]	752 963
60	179 523 [Aic]	179 514	86	789 911 [Aic]	789 892
63	219 659 [Aic]	219 637	87	828 125 [Aic]	828 107
64	234 447 [Aic]	234 441	88	867 887 [Aic]	867 862
65	249 962 [Aic]	249 938	89	908 940 [Aic]	908 914
66	266 151 [Aic]	266 142	90	951 379 [Aic]	951 323
67	283 238 [Aic]	283 230	91	995 478 [Aic]	995 430
68	301 057 [Aic]	301 043	92	1 040 946 [Aic]	1 040 897
69	319 691 [Aic]	319 679	93	1 087 899 [Aic]	1 087 843
70	339 252 [Aic]	339 241	94	1 136 586 [Aic]	1 136 565
71	359 645 [Aic]	359 635	96	1 238 646 [ACFM+10]	1 238 490
72	380 925 [Aic]	380 900	99	1 404 552 [ACFM+10]	1 404 386

Table 4.1: Best upper bounds on $\overline{\text{cr}}(K_n)$ and on $\widetilde{\text{cr}}(K_n)$ for $n < 100$. We list the values of n for which the bounds on $\widetilde{\text{cr}}(K_n)$ are better than the bounds on $\overline{\text{cr}}(K_n)$.

such perturbation can switch more signs in the n-signature realized by D. In our approach we perform only a single switch per step. Moreover, we use simulated annealing to avoid being trapped in local minima.

4.4.2 Blowing up pseudolinear drawings

The general approach for bounding $\overline{\text{cr}}(K_n)$ is to blow up a given base drawing of K_{n_0}, for some $n_0 \in \mathbb{N}$, with few crossings so that the blown-up drawing also contains few crossings. Currently, the best constructions of this type are due to Ábrego et al. [AFM07, ACFM+10] and give the following lower bound (there are two constructions, one for each parity of n_0).

Theorem 4.11 ([AFM07, ACFM+10])**.** *Let D be a rectilinear drawing of K_{n_0} that contains a halving matching. Then there is a rectilinear drawing of K_{2n_0} that contains a halving matching and has $16\,\text{cr}(D) + (n_0/2)(2n_0^2 - 7n_0 + 5)$ crossings.*

Here, a *halving line* in a rectilinear drawing D of K_n is a line that intersects two vertices u and v of D such that the edge uv is a $\lfloor (n-2)/2 \rfloor$-edge. A *halving matching* in D is an injection from the vertices of D to the set of halving lines of D such that every vertex v of D is mapped to a halving line incident with v.

Ábrego et al. [AFM07, ACFM+10] showed that an iterative application of Theorem 4.11 with the base drawing D of K_{n_0} implies that $\overline{\text{cr}}(K_n)$ is at most

$$\frac{24\,\text{cr}(D) + 3n_0^3 - 7n_0^2 + \frac{30}{7}n_0}{n_0^4}\binom{n}{4} + O(n^3) \tag{4.3}$$

for every sufficiently large n.

To prove Theorem 4.10, we now translate this method into the pseudolinear setting. First, we generalize the notion of a halving matching. Let i, j be elements of $[n]$ with $i < j$ and let σ be an n-signature realized by a pseudolinear drawing D of K_n. We say that an element a from $[n] \setminus \{i, j\}$ is *on the left side of the edge* ij *in* σ if one of the following cases occurs: $\sigma(a, i, j) = +$ for $a < i$, $\sigma(i, a, j) = -$ for $i < a < j$, or $\sigma(i, j, a) = +$ for $j < a$. The pair $\{i, j\}$ is a *halving pair in* σ if there are exactly $\lfloor (n-2)/2 \rfloor$ or $\lceil (n-2)/2 \rceil$ elements of $[n] \setminus \{i, j\}$ on the left side of ij. That is, if the edge ij is a $\lfloor (n-2)/2 \rfloor$-edge in D.

The drawing D contains a *halving matching* if there is a mapping $f\colon [n] \to [n]$ such that for every element i from $[n]$ the pair $\{i, f(i)\}$ is a halving pair in σ and there are no distinct $i, j \in [n]$ with $f(i) = j$ and $f(j) = i$.

Proposition 4.12. *Let D be a pseudolinear drawing of K_{n_0} that contains a halving matching. Then there is a pseudolinear drawing D' of K_{2n_0} that contains a halving matching and satisfies*

$$\operatorname{cr}(D') = 16 \operatorname{cr}(D) + 2n_0 \left(\left\lceil \frac{n_0}{2} \right\rceil^2 + \left\lfloor \frac{n_0}{2} \right\rfloor^2 \right) - \frac{7n_0^2}{2} + \frac{5n_0}{2}.$$

Proof. Let D be a pseudolinear drawing of K_{n_0} with vertices $v_1, \ldots, v_{n_0}$ ordered by their x-coordinates. We assume without loss of generality that D is x-monotone. Let $f\colon [n_0] \to [n_0]$ be the halving matching in the n_0-signature realized by D. To construct D', we first replace each vertex v_i of D with two vertices w_{2i-1} and w_{2i} that are placed on the pseudoline $v_i v_{f(i)}$ within a small neighborhood of v_i. Assuming that pseudolines of D are oriented from left to right, we place w_{2i} after w_{2i-1} on $v_i v_{f(i)}$.

Pseudolines in D are replaced with bundles of pseudolines. Each pseudoline $v_i v_{f(i)}$ is replaced with the bundle from part (a) of Figure 4.13. Every other pseudoline has the form $v_i v_j$ for $i \neq f(j)$ and $j \neq f(i)$ and is replaced with the bundle from part (b) of Figure 4.13. This method can be also described in the terms of $2n_0$-signatures, which we used in our implementation of the construction.

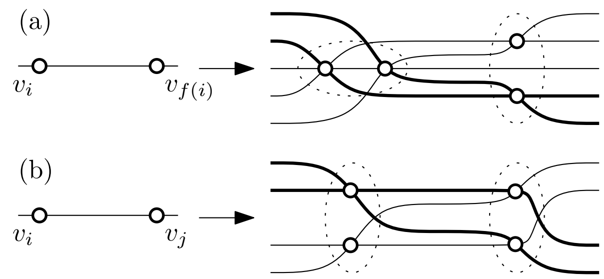


Figure 4.13: Replacing pseudolines of D with bundles.

If we draw the new pseudolines very close to the original pseudolines, then it can be shown that the drawing D' is indeed pseudolinear. Observe that D' contains a halving matching $f'\colon [2n_0] \to [2n_0]$ defined as $f'(2i-1) := 2i$ and $f'(2i) := 2f(i)$ for every $i \in [n_0]$. To finish the proof, it remains to count the number of crossings in D'. This is done similarly as in [AFM07].

We consider several types of 4-tuples of vertices of D'. Four-tuples $\{w_i, w_j, w_k, w_l\}$ with $1 \leq \lceil i/2 \rceil < \lceil j/2 \rceil < \lceil k/2 \rceil < \lceil l/2 \rceil \leq n_0$ add $16\operatorname{cr}(D)$ crossings to $\operatorname{cr}(D')$.

This is because no two points of such 4-tuple are in the same neighborhood and, by the construction, this 4-tuple forms a crossing if and only if the 4-tuple $\{v_{\lceil i/2\rceil}, v_{\lceil j/2\rceil}, v_{\lceil k/2\rceil}, v_{\lceil l/2\rceil}\}$ forms a crossing in D.

Four-tuples $\{w_i, w_j, w_k, w_l\}$ that satisfy $\lceil i/2\rceil = \lceil j/2\rceil$ and $\lceil k/2\rceil = \lceil l/2\rceil$ add $\binom{n_0}{2} - n_0$ crossings to $\mathrm{cr}(D')$, since every such 4-tuple with $\lceil k/2\rceil \neq f(\lceil i/2\rceil)$ adds exactly the crossing in the bundle for the pseudoline $v_{\lceil i/2\rceil}v_{\lceil k/2\rceil}$ and every such 4-tuple with $\lceil k/2\rceil = f(\lceil i/2\rceil)$ adds no crossing; see Figure 4.13.

In the remaining case w_i and w_j are in the same neighborhood while w_k and w_l are in distinct neighborhoods. That is, $\lceil i/2\rceil = \lceil j/2\rceil$ and $\lceil k/2\rceil \neq \lceil l/2\rceil$. The points w_k and w_l lie on the same side of the halving pseudoline w_iw_j, if they form a crossing. Otherwise the pseudoline w_kw_l would have to cross the neighborhood that contains w_i and w_j. We divide this case into two subcases.

First, assume that $\lceil k/2\rceil, \lceil l/2\rceil \neq f(\lceil i/2\rceil)$. To add a crossing to $\mathrm{cr}(D')$, we can choose the points w_k and w_l in $4\binom{\lceil n_0/2\rceil-1}{2} + 4\binom{\lfloor n_0/2\rfloor-1}{2}$ ways. Together with the choice of $\lceil i/2\rceil$, we see that this subcase adds $4n_0\left(\binom{\lceil n_0/2\rceil-1}{2} + \binom{\lfloor n_0/2\rfloor-1}{2}\right)$ crossings to $\mathrm{cr}(D')$.

In the second subcase we have $\lceil i/2\rceil = \lceil j/2\rceil$, $\lceil k/2\rceil = f(\lceil i/2\rceil)$, and $\lceil k/2\rceil \neq \lceil l/2\rceil$. We have $2(\lfloor n_0/2\rfloor - 1)$ choices for w_l for one choice of w_k from the neighborhood of v_k and $2(\lceil n_0/2\rceil - 1)$ choices for w_l for the other choice of w_k. Taking into account the choice of $\lceil i/2\rceil$, this adds $2n_0(\lceil n_0/2\rceil + \lfloor n_0/2\rfloor - 2) = 2n_0^2 - 4n_0$ crossings to $\mathrm{cr}(D')$.

In total, the number of crossings in D' is $16\mathrm{cr}(D) + \frac{n_0}{2}(2n_0^2 - 7n_0 + 5)$ for n_0 even and $16\mathrm{cr}(D) + \frac{n_0}{2}(2n_0^2 - 7n_0 + 7)$ for n_0 odd. □

For n_0 even, Proposition 4.12 produces drawings with the same number of crossings as we have in Theorem 4.11. In Figure 4.12 we illustrate the bound on the leading constant in $\widetilde{\mathrm{cr}}(K_n)$ obtained by iteratively applying Proposition 4.12 to the new drawings of K_n for $n < 100$.

Proof of Theorem 4.10. Let n_0 be a positive even integer and let D be a pseudolinear drawing of K_{n_0} with a halving matching. Iterative application of Proposition 4.12 with the base drawing D bounds $\widetilde{\mathrm{cr}}(K_n)$ from above by (4.3). Therefore it suffices to choose the base drawing D so that the leading constant in (4.3) is minimized.

We let D be the pseudolinear drawing of K_{216} with 33 260 204 crossings found by our experiments. Since D contains a halving matching [BK], we can set $n_0 := 216$ and $\mathrm{cr}(D) := 33\,260\,204$ in (4.3) and obtain $\widetilde{\mathrm{cr}}(K_n) \leq \frac{120\,772\,213}{317\,447\,424}\binom{n}{4} + O(n^3) < 0.380448\binom{n}{4} + O(n^3)$. □

4.5 Open problems

It would be interesting to see if techniques similar to those used in the proof of Theorem 4.1 can be used to prove Hill's conjecture for general drawings of complete graphs. We note that the same approach does not generalize to all drawings. For example, a particular planar realization of the so-called *cylindrical drawing* [Guy60, HH63] of K_{10}, with $Z(10)$ crossings, does not satisfy the lower bound on $\leq\leq 1$-edges from Theorem 4.7; see part (b) of Figure 4.14. Part (a) of Figure 4.14 shows an even smaller example, but this drawing of K_6 is not crossing

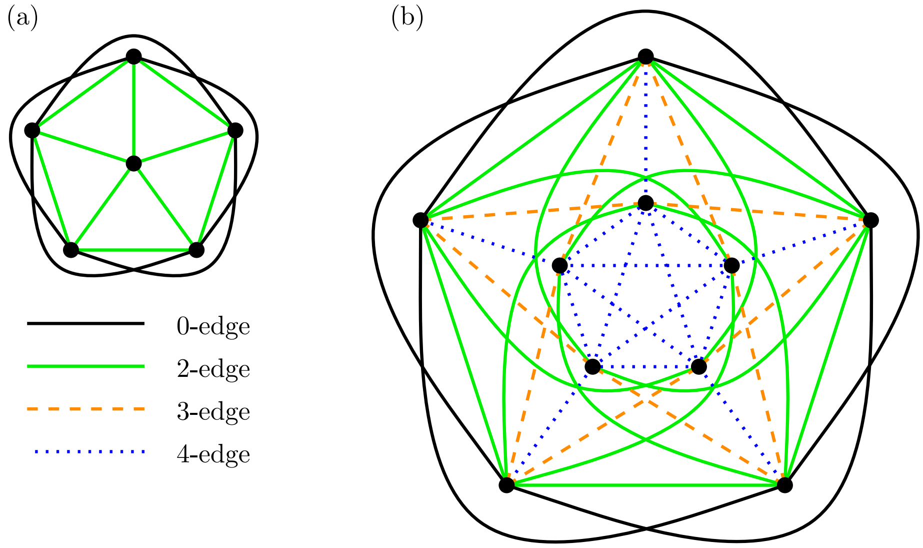


Figure 4.14: (a) A general simple drawing of K_6. (b) A cylindrical drawing of K_{10} (right) where $E_0 = 5$ and $E_1 = 0$, hence $E_{\leq\leq 1} = 10 < 12 = 3\binom{1+3}{3}$.

optimal. Analogous cylindrical drawings of K_{4k+6}, for $k \geq 2$, violate the lower bound on $\leq\leq k$-edges from Theorem 4.7.

Extrapolating the definitions of $\leq k$-edges and $\leq\leq k$-edges, we define the *number of* $\leq\leq\leq k$*-edges*, $E_{\leq\leq\leq k}(D)$, by the following identity.

$$E_{\leq\leq\leq k}(D) := \sum_{j=0}^{k} E_{\leq\leq j}(D) = \sum_{i=0}^{k} \binom{k+2-i}{2} E_i(D).$$

In our context, using $\leq\leq\leq k$-edges seems to be even more natural than using $\leq\leq k$-edges, since the formula from Lemma 4.2 can be rewritten in the following compact form:

$$\operatorname{cr}(D) = 2E_{\leq\leq\leq\lfloor n/2\rfloor-2}(D) - \frac{1}{8}n(n-1)(n-3) \quad \text{for } n \text{ odd, and}$$

$$\operatorname{cr}(D) = E_{\leq\leq\leq\lfloor n/2\rfloor-3}(D) + E_{\leq\leq\leq\lfloor n/2\rfloor-2}(D) - \frac{1}{8}n(n-1)(n-2) \quad \text{for } n \text{ even.}$$

We conjecture that the following lower bound on $\leq\leq\leq k$-edges is satisfied by all simple drawings of complete graphs.

Conjecture 4.13. *Let $n \geq 3$ and let D be a simple drawing of K_n. Then for every k satisfying $0 \leq k < n/2 - 1$, we have*

$$E_{\leq\leq\leq k}(D) \geq 3\binom{k+4}{4}.$$

Conjecture 4.13 is stronger than Hill's conjecture. Theorem 4.7 implies Conjecture 4.13 for all simple x-monotone drawings. All our examples of simple drawings of complete graphs, including the cylindrical drawings, also satisfy Conjecture 4.13. We note that Conjecture 4.13 is trivially satisfied for $k = 0$, since every simple drawing of a complete graph with at least three vertices has at least three 0-edges—those incident with the outer face.

We have no counterexample even to the following conjecture, which further generalizes Conjecture 4.13 to arbitrary graphs.

Conjecture 4.14. *Let $k \geq 0$ and let D be a simple drawing of a graph with at least $\binom{2k+3}{2}$ edges. Then*

$$E_{\leq\leq\leq k}(D) \geq 3\binom{k+4}{4}.$$

Note that in a drawing of a general graph with n vertices, a k-edge contained in t triangles is also a $(t-k)$-edge, but not necessarily an $(n-2-k)$-edge. Thus, for example, in every drawing of a triangle-free graph, every edge is a 0-edge. This suggests that it might be easier to prove Conjecture 4.14 for non-complete graphs. Also, Conjecture 4.14 or some still stronger variant might be susceptible to a proof by induction on the number of edges.

Further, it would be interesting to generalize Theorem 4.1 to arbitrary monotone drawings, where adjacent edges are also allowed to cross oddly. For such drawings, two notions of the crossing number are of interest. The *monotone odd crossing number*, mon-ocr(G), counting the minimum number of pairs of edges crossing an odd number of times, and the *monotone independent odd crossing number*, mon-ocr$_-(K_n)$, counting the number of pairs of nonadjacent edges crossing an odd number of times. By definition, for every graph G we have mon-ocr$_-(G) \leq$ mon-ocr$(G) \leq$ mon-ocr$_\pm(G)$.

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A. SAT encoding for Theorem 3.3

Here, we describe in detail our SAT encoding of the following decision problem: given integers $a, u, k, N \geq 3$, is there a red-blue coloring of $\mathcal{K}_N^3$ with no red a-path, no blue u-path, and no k-gon? See Sections 3.2 and 3.3 for definitions.

For every triple $\{i, j, k\}$ of integers with $1 \leq i < j < k \leq N$, we define a boolean variable $x_{i,j,k}$. The variable $x_{i,j,k}$ represents the edge $\{i, j, k\}$ of $\mathcal{K}_N^3$. The red color of $\{i, j, k\}$ is represented with the logical value TRUE of $x_{i,j,k}$ and the blue color with the value FALSE.

We define a formula $\Phi^-(a, N)$ that detects an occurrence of a red a-path in a coloring of $\mathcal{K}_N^3$. For every $A = \{v_1, \ldots, v_a\} \in \binom{[N]}{a}$ with $v_1 < \cdots < v_a$, let C_A^- be a clause defined as $C_A^- := \bigvee_{i=1}^{a-2} \neg x_{v_i,v_{i+1},v_{i+2}}$. The formula $\Phi^-(a, N)$ is defined as

$$\Phi^-(a, N) := \bigwedge_{A \in \binom{[N]}{a}} C_A^-.$$

To detect blue u-paths, we first define clauses $C_U^+ := \bigvee_{i=1}^{u-2} x_{w_i,w_{i+1},w_{i+2}}$ for every $U = \{w_1, \ldots, w_u\} \in \binom{[N]}{u}$ with $w_1 < \cdots < w_u$. Then we set

$$\Phi^+(u, N) := \bigwedge_{U \in \binom{[N]}{u}} C_U^+.$$

It remains to detect an occurrence of a k-gon in a coloring of $\mathcal{K}_N^3$. We define a collection of 2^{k-2} clauses for every k-tuple $K \in \binom{[N]}{k}$ such that each clause corresponds to a particular k-gon with the vertex set K. Let K^* be the $(k-2)$-tuple obtained from K by deleting the minimum element m and the maximum element M. Let $G = \{v_1, \ldots, v_{|G|}\}$ be a (possibly empty) subset of K^* with $v_1 < \cdots < v_{|G|}$ and let $w_1 < \cdots < w_{k-|G|-2}$ be the elements of $K^* \setminus G$. We define a clause C_G^K as

$$C_G^K := \left(\bigvee_{i=0}^{|G|-1} \neg x_{v_i,v_{i+1},v_{i+2}} \right) \vee \left(\bigvee_{j=0}^{k-|G|-3} x_{w_j,w_{j+1},w_{j+2}} \right)$$

where we set $v_0 := m$, $v_{|G|+1} := M$, $w_0 := m$, and $w_{k-|G|-1} := M$.

The final formula $\Phi(a, u, k, N)$ is then defined as

$$\Phi(a, u, k, N) := \left(\bigwedge_{K \in \binom{[N]}{k}} \bigwedge_{G \subseteq K^*} C_G^K \right) \wedge \Phi^-(a, N) \wedge \Phi^+(u, N).$$

Clearly, $\Phi(a, u, k, N)$ is satisfiable if and only if there exists a red-blue coloring of $\mathcal{K}_N^3$ with no red a-path, no blue u-path, and no k-gon.

A restriction to pseudolinear colorings of $\mathcal{K}_N^3$ is ensured as follows in our SAT encoding. For every 4-tuple $T = \{i, j, k, l\} \in \binom{[N]}{4}$ with $i < j < k < l$, let C_T be a clause defined as

$$\begin{aligned} C_T := &(x_{i,j,k} \vee \neg x_{i,j,l} \vee x_{i,k,l}) \wedge (\neg x_{i,j,k} \vee x_{i,j,l} \vee \neg x_{i,k,l}) \wedge \\ &(x_{i,j,l} \vee \neg x_{i,k,l} \vee x_{j,k,l}) \wedge (\neg x_{i,j,l} \vee x_{i,k,l} \vee \neg x_{j,k,l}) \wedge \\ &(x_{i,j,k} \vee \neg x_{i,j,l} \vee \neg x_{i,k,l} \vee x_{j,k,l}) \wedge (\neg x_{i,j,k} \vee x_{i,j,l} \vee x_{i,k,l} \vee \neg x_{j,k,l}). \end{aligned}$$

The formula C_T is satisfiable if and only if the copy of $\mathcal{K}_4^3$ that is induced by T is colored by one of the eight colorings of $\mathcal{K}_4^3$ from Theorem 4.9. By this theorem, a coloring of $\mathcal{K}_N^3$ is pseudolinear if and only if every clause C_T with $T \in \binom{[N]}{4}$ is satisfied.

Altogether, there is a pseudolinear coloring of $\mathcal{K}_N^3$ with no red a-path, no blue u-path, and with no k-gon if and only if the formula

$$\Phi(a, u, k, N) \wedge \left(\bigwedge_{T \in \binom{[N]}{4}} C_T \right)$$

is satisfiable.