Stochastic Integrals Driven by Isonormal Gaussian Processes and Applications

Department of Probability and Mathematical Statistics

Supervisor of the master thesis: prof. RNDr. Bohdan Maslowski, DrSc.
Study programme: Mathematics
Specialization: Probability, Mathematical Statistics, and Econometrics

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Stochastické integrály řízené izonormálními gaussovskými procesy a aplikace

Autor: Bc. Petr Čoupek

Katedra: Katedra pravděpodobnosti a matematické statistiky

Vedoucí práce: prof. RNDr. Bohdan Maslowski, DrSc., Katedra pravděpodobnosti a matematické statistiky

Abstrakt: V diplomové práci je podrobně studován stochastický integrál deterministických funkcí s hodnotami v Hilbertově prostoru v případě, kdy řídící proces je tvaru \( \beta_t = \int_0^t K(t, s) dW_s \), kde \( W \) je Brownův pohyb a \( K \) je kvadraticky integrovatelné jádro. Takovéto procesy zobecnějí případ frakcionálního Brownova pohybu \( B^H \), definovaného pomocí Hurstova parametru \( H \in (0, 1) \). Na jádro \( K \) jsou uvažovány dvě sady podmínek, odpovídající regulárnímu a singulárnímu případu a studován byl konkrétně případ regulární. Hlavním výsledkem je, že prostor \( \beta \)-integrabilních funkcí lze vnořit do prostoru \( L^{\frac{2}{1+2\alpha}}([0, T]; V) \), což v případě frakcionálního Brownova pohybu odpovídá prostoru \( L^\frac{1}{H}([0, T]) \). Dále byl zaveden cylindrický gaussovský volterrovní proces a vůči němu stochastický integrál deterministických funkcí s hodnotami v prostoru lineárních operátorů. Výsledky byly dále aplikovány v teorii stochastických diferenciálních rovnic (SDR), konkrétně byla dokázána měřitelnost řešení dané SDR ve tvaru „mild“.

Klíčová slova: Isonormální gaussovský proces, stochastický integrál, stochastická diferenciální rovnice, frakcionální Brownův pohyb, volterrovní proces

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Author: Bc. Petr Čoupek

Department: Department of Probability and Mathematical Statistics

Supervisor: prof. RNDr. Bohdan Maslowski, DrSc., Department of Probability and Mathematical Statistics

Abstract: In this thesis, we introduce a stochastic integral of deterministic Hilbert space valued functions driven by a Gaussian process of the Volterra form \( \beta_t = \int_0^t K(t, s) dW_s \), where \( W \) is a Brownian motion and \( K \) is a square integrable kernel. Such processes generalize the fractional Brownian motion \( B^H \) of Hurst parameter \( H \in (0, 1) \). Two sets of conditions on the kernel \( K \) are introduced, the singular case and the regular case, and, in particular, the regular case is studied. The main result is that the space \( \mathcal{H} \) of \( \beta \)-integrable functions can be, in the strictly regular case, embedded in \( L^{\frac{2}{1+2\alpha}}([0, T]; V) \) which corresponds to the space \( L^\frac{1}{H}([0, T]) \) for the fractional Brownian motion. Further, the cylindrical Gaussian Volterra process is introduced and a stochastic integral of deterministic operator-valued functions, driven by this process, is defined. These results are used in the theory of stochastic differential equations (SDE), in particular, measurability of a mild solution of a given SDE is proven.

Keywords: Isonormal Gaussian Process, Stochastic Integral, Stochastic Differential Equation, Fractional Brownian Motion, Volterra Process
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# List of Symbols

## Sets and spaces

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<th>Description</th>
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<tr>
<td>$\mathbb{R}$</td>
<td>set of real numbers</td>
</tr>
<tr>
<td>$\mathbb{R}_+$</td>
<td>set of non-negative real numbers</td>
</tr>
<tr>
<td>$\mathbb{N}$</td>
<td>set of natural numbers $1, 2, 3, \ldots$</td>
</tr>
<tr>
<td>$L^p(D)$</td>
<td>$L^p$ space of functions on $D$ with real values</td>
</tr>
<tr>
<td>$L^p(D; V)$</td>
<td>$L^p$ space of $V$-valued functions on $D$</td>
</tr>
<tr>
<td>$H^1_0(D), H^1(D), H^2(D)$</td>
<td>Sobolev spaces over $D$</td>
</tr>
<tr>
<td>$\mathcal{L}(V)$</td>
<td>bounded linear operators from $V$ to $V$</td>
</tr>
<tr>
<td>$\mathcal{L}(U, V)$</td>
<td>bounded linear operators from $U$ to $V$</td>
</tr>
<tr>
<td>$\mathcal{L}_2(V)$</td>
<td>Hilbert-Schmidt operators from $V$ to $V$</td>
</tr>
<tr>
<td>$\mathcal{L}_2(U, V)$</td>
<td>Hilbert-Schmidt operators from $U$ to $V$</td>
</tr>
<tr>
<td>$(a, b)$</td>
<td>open interval from $a$ to $b$</td>
</tr>
<tr>
<td>$[a, b]$</td>
<td>closed interval from $a$ to $b$</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>set of pairs $(t, s) \in [0, T]^2$ such that $s &lt; t$</td>
</tr>
<tr>
<td>$\mathcal{B}(V)$</td>
<td>Borel $\sigma$-algebra on $V$</td>
</tr>
<tr>
<td>$\partial D$</td>
<td>border of $D$</td>
</tr>
<tr>
<td>$\text{Dom}(A)$</td>
<td>domain of operator $A$</td>
</tr>
<tr>
<td>${c_n}_{n=0}^\infty$</td>
<td>sequence ${c_0, c_1, c_2, \ldots} \subset \mathbb{R}$</td>
</tr>
</tbody>
</table>

## Functions and operators

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$\Gamma$</td>
<td>gamma function</td>
</tr>
<tr>
<td>Beta</td>
<td>beta function</td>
</tr>
<tr>
<td>$2F_1$</td>
<td>Gauss hypergeometric function</td>
</tr>
<tr>
<td>arg</td>
<td>argument function</td>
</tr>
<tr>
<td>log</td>
<td>principal value of natural logarithm</td>
</tr>
<tr>
<td>$\mathbf{1}_A$</td>
<td>indicator function of $A$</td>
</tr>
<tr>
<td>$\Re(s)$</td>
<td>real part of $s$</td>
</tr>
<tr>
<td>$\langle \cdot, \cdot \rangle$</td>
<td>inner (scalar) product</td>
</tr>
<tr>
<td>$| \cdot |$</td>
<td>norm on a functional space</td>
</tr>
<tr>
<td>$</td>
<td>\cdot</td>
</tr>
<tr>
<td>$\mathbb{E}$</td>
<td>expected value</td>
</tr>
<tr>
<td>$\text{Tr}(A)$</td>
<td>trace of $A$</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>Laplace operator</td>
</tr>
</tbody>
</table>

## Miscellaneous

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \mapsto b$</td>
<td>transformation of $a$ into $b$</td>
</tr>
<tr>
<td>$a := b$</td>
<td>$a$ is defined as $b$</td>
</tr>
<tr>
<td>$a \to b_{t \to s^+}$</td>
<td>$a$ converges to $b$ as $t$ goes to $s$ from the right</td>
</tr>
<tr>
<td>$\mathbb{P}$</td>
<td>probability measure</td>
</tr>
<tr>
<td>$</td>
<td>\mathcal{K}</td>
</tr>
<tr>
<td>$\subset$</td>
<td>subset or subspace, where appropriate</td>
</tr>
<tr>
<td>$a \wedge b$</td>
<td>minimum of $a$ and $b$</td>
</tr>
<tr>
<td>$a \vee b$</td>
<td>maximum of $a$ and $b$</td>
</tr>
<tr>
<td>$N(\mu, \sigma^2)$</td>
<td>Gaussian distribution; mean $\mu$, variance $\sigma^2$</td>
</tr>
<tr>
<td>$\mathbb{P} - \text{a.s.}$</td>
<td>$\mathbb{P}$-almost surely</td>
</tr>
<tr>
<td>$\otimes$</td>
<td>tensor product of Hilbert spaces</td>
</tr>
</tbody>
</table>
Introduction

Since the pioneering work of Itô, the theory of stochastic integration has been widely investigated. The principal example in the theory is the standard Brownian motion formally described by Wiener, which can be viewed as a limit of random walks and has significant use in modelling of a wide range of natural phenomena. As such, it plays a vital role in both pure and applied mathematics, namely the Black-Scholes option pricing model in mathematical finance (see [6]), models including random noise in filtering theory and by at last, but not least, the Brownian motion provided the basis for a rigorous path integral formulation of quantum field theory (see e.g. [9]).

One of the approaches to modelling natural phenomena is to describe a dynamical system by a set of differential equations which provide, in theory, a complete description of the time evolution of this system, given the initial conditions. Beginning with Newton, a deterministic infinitesimal calculus was developed by many others in order to formulate and solve such problems. Two main problems occur with this approach. Firstly, some systems cannot be solved due to their size, e.g. in thermodynamics, a mole of a gas consists of approximately $6.022 \times 10^{23}$ molecules or atoms, each one described by a differential equation. Therefore, a stochastic equation is used to describe the macroscopic behaviour of the system instead, with the small deviations from the ideal state modelled by a random noise process. Secondly, some systems are too imprecise, e.g. considering the standard heat equation the matter is assumed to be homogeneous. Similarly, as for the previous system of molecules, a stochastic approach is chosen with small inhomogeneities accounted for by a random noise process.

However, due to a unique nature of the Brownian motion, in particular the non-differentiability of its sample paths and their infinite total variation, the meaning to a stochastic differential cannot be given in the same way as in the Newtonian, or Lebesgue, calculus where firstly, the differential is given and then the integral is defined. Instead, another approach is chosen. Stochastic differential equations are given meaning in terms of stochastic integrals. The Itô stochastic calculus is defined with semimartingales as driving processes. Such processes are represented by the Brownian motion.

The Brownian motion is not the only stochastic process successfully used within various fields. Its generalisation, the standard fractional Brownian motion, has been widely investigated since the work of Mandelbrot and Van Ness [20]. It represents a family of Gaussian processes indexed by a one-dimensional parameter $H$, so-called the Hurst parameter. The fractional Brownian motion is currently used as a model in population dynamics (23, 24), mathematical finance (5, 13) and as well, used to model traffic of telecommunication (e.g. 26). Recently, stochastic calculus with respect to the fractional Brownian motion has been investigated (e.g. 1, 5, 8, 10, 12). This is indeed necessary, since the fractional Brownian motion is not a semimartingale for $H \neq \frac{1}{2}$. Stochastic differential equations with the fractional Brownian motion as a driving process are investigated in 13, 14 and 17. Alòs, Nualart and Mazet extended the definition of the standard fractional Brownian motion to a general class of processes, the Gaussian Volterra processes, defined by a kernel $K$ in 2.
In the thesis, the results of [2] are partly extended to Hilbert spaces and linear stochastic equations driven by Gaussian Volterra processes are investigated. In particular, two different conditions on the kernel \( K \) are given. These conditions correspond to the singular and regular case of the standard fractional Brownian motion. Then a stochastic integral with respect to a regular Gaussian Volterra process and its analogue, the stochastic integral with respect to a regular cylindrical Gaussian process, are defined. To apply the theoretical results, a stochastic linear differential equation in a Hilbert space is finally considered.

The thesis is organized as follows. In the first chapter, Preliminaries, some basic results on the fields of stochastic and functional analysis are presented, without proofs, and examples of Isornormal Gaussian processes are given. In particular, the fractional Brownian motion is presented together with its properties.

The second chapter is devoted to Gaussian Volterra processes. Firstly a general framework is introduced, the processes are defined and a reproducing kernel Hilbert space is presented. Then two cases of the condition on the kernel \( K \) are given - the singular and the regular case. Further, only the regular case is investigated. The main result shows that the space of \( V \)-valued deterministic functions integrable with respect to a regular Gaussian Volterra process can be continuously embedded into the space \( L^{\frac{2}{1+2\alpha}}([0,T]; V) \) for \( \alpha \in \left(0, \frac{1}{2}\right) \). Further, the cylindrical Gaussian Volterra process is defined and a corresponding stochastic integral is developed. The sufficient conditions for the existence of this stochastic integral are further given.

In the last chapter, a linear stochastic differential equation is approached. Firstly, as a motivation, the results for an analogous deterministic problem are presented. Then the stochastic Cauchy problem is formulated and its solution in the mild form is considered. It is shown that such a solution defines a measurable process which admits a version with sample paths almost surely in the space \( L^p([0,T]; V) \) for all \( 1 \leq p < \infty \) and for all \( T > 0 \). Finally, the results are demonstrated on examples of important stochastic differential equations, in particular the stochastic heat and wave equation.

It should be noted that these results correspond to the previous results for a standard Brownian and fractional Brownian motion. In particular, the stochastic parabolic equation on bounded domains in \( \mathbb{R}^d \), driven by a cylindrical Gaussian Volterra process, has a function space valued solution, if the dimension \( d \) of the underlying space equals 1 and 2. If, however, \( \alpha > \frac{1}{4} \), which in the case of the fractional Brownian motion means \( H > \frac{3}{4} \), then there is such a solution even in dimension \( d = 3 \). This corresponds to the previous results derived in [13] for the regular fractional Brownian motion, that is \( H > \frac{1}{2} \).
1. Preliminaries

1.1 Stochastic Preliminaries

In this section, some basic results in the field of stochastic analysis are collected. These statements are presented without proofs; however, with references to literature. We only present those results which are used in the following chapters. In particular, we present basic existence theorems such as Daniell-Kolmogorov Theorem and Kolmogorov-Chentsov Theorem and, from theory of stochastic integration, the Itô isometry.

1.1.1 Existence and Continuity of Stochastic Processes

Definition 1.1.1 ([25], p.49). Let \( T \) be the set of finite sequences \( \tilde{t} = (t_1, \cdots, t_n) \) of distinct, nonnegative numbers, where the length \( n \) of these sequences ranges over the set of positive integers. Suppose that for each \( \tilde{t} \) of length \( n \), we have a probability measure \( Q_{\tilde{t}} \) on \( (\mathbb{R}^n, B(\mathbb{R}^n)) \). Then the collection \( \{Q_{\tilde{t}}, \tilde{t} \in T\} \) is called a family of finite-dimensional distributions. This family is said to be consistent provided that the following two conditions are satisfied:

1. if \( \tilde{s} = (t_{i_1}, t_{i_2}, \cdots, t_{i_n}) \) is a permutation of \( \tilde{t} = (t_1, t_2, \cdots, t_n) \), then for any \( A_i \in B(\mathbb{R}) \), \( i = 1, \cdots, n \), we have that
   \[
   Q_{\tilde{t}} (A_1 \times A_2 \times \cdots \times A_n) = Q_{\tilde{s}} (A_{i_1} \times A_{i_2} \times \cdots \times A_{i_n});
   \]

2. if \( \tilde{t} = (t_1, t_2, \cdots, t_n) \), with \( n \geq 1 \), \( \tilde{s} = (t_1, t_2, \cdots, t_{n-1}) \), and \( A \in B(\mathbb{R}^{n-1}) \), then
   \[
   Q_{\tilde{t}} (A \times \mathbb{R}) = Q_{\tilde{s}} (A).
   \]

Denote by \( \mathbb{R}^{[0, \infty)} \) the set of real-valued functions on \( \mathbb{R}_+ \). Given a probability measure \( \mathbb{P} \) on \( (\mathbb{R}^{[0, \infty)}, B(\mathbb{R}^{[0, \infty)})) \), it is possible to define a consistent family of finite-dimensional distributions by

\[
Q_{\tilde{t}} (A) = \mathbb{P} \{ \omega \in \mathbb{R}^{[0, \infty)} : (\omega(t_1), \cdots, \omega(t_n)) \in A \},
\]

where \( A \in B(\mathbb{R}^n) \) and \( \tilde{t} = (t_1, t_2, \cdots, t_n) \in T \). The converse is provided by the famous Daniell-Kolmogorov consistency theorem.

Theorem 1.1.2 (Daniell (1918), Kolmogorov (1933), [25], p.50). Suppose that \( \{Q_{\tilde{t}}, \tilde{t} \in T\} \) is a consistent family of finite-dimensional distributions. Then there is a probability measure \( \mathbb{P} \) on \( (\mathbb{R}^{[0, \infty)}, B(\mathbb{R}^{[0, \infty)})) \), such that

\[
Q_{\tilde{t}} (A) = \mathbb{P} \{ \omega \in \mathbb{R}^{[0, \infty)} : (\omega(t_1), \cdots, \omega(t_n)) \in A \}, \quad A \in B(\mathbb{R}^n)
\]

holds for every \( \tilde{t} \in T \).

Daniell-Kolmogorov theorem can be used for the construction of important examples in the next section among whose the fractional Brownian motion is presented. Since we are mainly interested in Gaussian processes, their definition is further given.
Definition 1.1.3 (Gaussian process). A stochastic process \( X = (X_t, t \geq 0) \) is called Gaussian with a covariance function \( \varphi : \mathbb{R}_+^2 \to \mathbb{R} \) if all finite-dimensional distributions of \( X \) are Gaussian and \( \text{cov}(X_s, X_t) = \varphi(s, t) \) for \( s, t \in \mathbb{R}_+ \). The process \( X \) is called centered if \( \mathbb{E}X_t = 0 \) for all \( t \geq 0 \).

Definition 1.1.4. A function \( \varphi : \mathbb{R}_+^2 \to \mathbb{R} \) is called non-negative definite if for all \( n \in \mathbb{N} \), for all \( \tilde{t} = (t_1, \cdots, t_n) \) such that \( 0 \leq t_1 < t_2 < \cdots < t_n \) and for all \( \lambda_1, \cdots, \lambda_n \geq 0 \) the following holds:

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \varphi(t_i, t_j) \geq 0.
\]

As a corollary of Daniell-Kolmogorov theorem, Theorem 1.1.2, for every non-negative definite function \( \varphi \) there exists a Gaussian process \( X = (X_t, t \geq 0) \) with \( \varphi \) as a covariance function. Doob gives a general form of the following existence theorem ([11], p. 72).

Theorem 1.1.5 (Existence of Gaussian processes). For every real, symmetric, and non-negative definite function \( \varphi : \mathbb{R}_+^2 \to \mathbb{R} \) there is a real, centred Gaussian process \( X = (X_t, t \geq 0) \) such that

\[
\mathbb{E}(X_tX_s) = \varphi(t, s), \quad s, t \in \mathbb{R}_+.
\]

The following theorem, Kolmogorov-Chentsov Theorem, provides an excellent instrument in the theory of Gaussian processes since it assures the existence of a continuous version of a stochastic process. There are a few statements of this type, however, the condition in Kolmogorov-Chentsov Theorem, under which such a version exists, is easily verified for Gaussian processes.

Theorem 1.1.6 (Kolmogorov, Chentsov (1956), [25], p. 53). Let \( T > 0 \). Suppose that a process \( X = (X_t, t \in [0, T]) \) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfies the condition

\[
\mathbb{E}|X_t - X_s|^{\alpha} \leq C|t - s|^{1+\beta}, \quad 0 \leq s, t \leq T,
\]

for some positive constants \( \alpha, \beta \) and \( C \). Then there exists a continuous version \( \hat{X} = (\hat{X}_t, t \in [0, T]) \) of \( X \), which is locally Hölder-continuous with exponent \( \gamma \) for every \( \gamma \in \left(0, \frac{\beta}{\alpha}\right) \), i.e.,

\[
\mathbb{P} \left\{ \omega \in \Omega : \sup_{0 < t - s < h(\omega); s, t \in [0, T]} \frac{\left|\hat{X}_t(\omega) - \hat{X}_s(\omega)\right|}{|t - s|^{\gamma}} \leq \delta \right\} = 1,
\]

where \( h(\omega) \) is a \( \mathbb{P} \)-almost surely positive random variable and \( \delta > 0 \) is an appropriate constant.

1.1.2 Itô Calculus

Even though the Itô stochastic calculus has proven itself extremely useful in the theory of stochastic processes, for our purposes we only need some basic results. We start with the standard Brownian motion.
Brownian motion

**Definition 1.1.7** (Stochastic basis). Let \((\Omega, \mathcal{F})\) be a measurable space. A filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) of \(\mathcal{F}\) is, by definition, a non-decreasing sequence of \(\sigma\)-algebras such that \(\mathcal{F}_t \subset \mathcal{F}\) for each \(t \geq 0\). Given a probability measure \(P\) on \(\mathcal{F}\), the stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) is a probability space \((\Omega, \mathcal{F}, P)\) endowed with the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\).

**Definition 1.1.8** (Brownian motion, [25], p.47). A (standard, one-dimensional) Brownian motion is a continuous adapted process \(B = (B_t, \mathcal{F}_t, t \geq 0)\), defined on some stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\), with the following properties:

(i) \(B_0 = 0, \quad P\) - a.s.,

(ii) for \(0 \leq s < t\), the increment \(B_t - B_s\) is independent of \(\mathcal{F}_s\) and is normally distributed with mean zero and variance \(t - s\).

The condition stating the independence of increments of \(B\) means that the whole system of increments is independent, i.e., for all \(n \in \mathbb{N}\) and for all times \(0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < \infty\) the random variables \((B_{t_{i+1}} - B_{t_i}, i = 1, \cdots, n - 1)\) are independent. The explicit formula for the covariance function of \(B\) follows from Definition 1.1.8. The joint distribution of \((B_{t_1}, \cdots, B_{t_n})\) is normal, \(\text{var} B_{t_i} = t_i\) and \(\text{cov} (B_{t_i}, B_{t_j}) = t_i \wedge t_j\). Since this defines a non-negative definite function, by Theorem 1.1.5, we have that such a process exists. Further we use Kolmogorov-Chentsov Theorem, Theorem 1.1.6, to show that there exists a continuous version of \(B\) on \([0, T]\) for all \(T > 0\). Indeed, the condition (1.1) is satisfied with \(\alpha = 4\), \(\beta = 1\) and \(C = 3\). This follows from the properties of the normal distribution.

There are various ways of proving the existence of the process \(B\). One approach constructs its finite-distributions and then establishes a probability measure on an appropriate measurable space accordingly ([25]). This way it is verified that the process \(B\) is a random variable taking values on a measurable space \((\mathcal{C}[0, \infty), \mathcal{B}(\mathcal{C}[0, \infty]))\). Another proof exploits the idea of the \(*\)-weak limit of a sequence of random walks, Donsker’s invariance principle. For further reading and a thorough discussion on the topic it is referred to [25], Chapter 2 or [39], Chapter 1.

Further, some basic properties of the standard Brownian motion are presented. For consistency with the rest of the thesis, we provide, in particular, sample paths properties, i.e., those properties which hold with probability one.

**Theorem 1.1.9** (Strong law of large numbers, [25], p.104). Let \(B = (B_t, t \geq 0)\) be the standard Brownian motion. Then

\[
\lim_{t \to \infty} \frac{B_t}{t} = 0, \quad P\) - a.s.
\]

The following result is useful for defining the stochastic integral in the sense of Itô. It basically says that we can define an integral with respect to a quadratic variation of the standard Brownian motion because it is simply an integral with respect to the Lebesgue measure.

**Theorem 1.1.10** (Quadratic variation, [25], p. 105). Let \(\{\Delta_n\}_{n=1}^\infty\) be a sequence of partitions of the interval \([0, t]\) for some \(t > 0\) such that the norm of these
partitions goes to zero as \( n \to \infty \). Then

\[
V_t^{(2)}(\Delta_n) = \sum_{k=1}^{m_n} \left| B_{t_k}^{(n)} - B_{t_{k-1}}^{(n)} \right|^2 \xrightarrow{n \to \infty} L^2(\mathbb{P}) t,
\]

where \( m_n \) denotes the number of points of the partition \( \Delta_n \).

For almost all trajectories of the standard Brownian motion it holds that their total variation over interval \([0, t]\) is infinity for all \( t \geq 0 \). This property makes it impossible to define the integral with respect to the standard Brownian motion pathwise, i.e. for every \( \omega \in \Omega \) separately. In fact, if the quadratic variation of the path of a continuous process is finite it follows that the total variation is infinite.

Brownian motion presents an example of a mapping that produces continuous yet nowhere differentiable functions. The proof of the following theorem is due to Paley, Wiener and Zygmund (1933).

**Theorem 1.1.11** (Non-differentiability; Paley, Wiener and Zygmund (1933)). For almost every \( \omega \in \Omega \), the Brownian sample path \( t \mapsto B_t(\omega) \) is nowhere differentiable.

Theorem 1.1.11 is proved by showing that the set of trajectories having finite, agreeing right and left, lower and upper Dini derivatives at each point \( t > 0 \) is contained in the set of probability zero. For this particular proof see [25], p. 110. Theorem 1.1.11 can also be proved by an approach based on local time.

By Kolmogorov-Chentsov Theorem, Theorem 1.1.6 the trajectories of the standard Brownian motion are locally \((\frac{1}{2} - \varepsilon)\)-Hölder continuous for all \( \varepsilon \in (0, \frac{1}{2}) \). The following theorem provides a stronger result.

**Theorem 1.1.12** (Lévy modulus of continuity, P. Lévy (1937)). Let \( B = (B_t, t \geq 0) \) be the standard Brownian motion. Then, for almost all trajectories of \( B \) the following estimate holds:

\[
\lim_{\varepsilon \to 0^+} \sup_{|s-t| \leq \varepsilon} \frac{|B_t - B_s|}{\sqrt{2\varepsilon \log \frac{1}{\varepsilon}}} = 1.
\]

**Itô Integral**

In general, the Itô stochastic integral can be constructed for semimartingales as integrators; however, we do not attempt such a construction here, nor do we aim at a complete overview of this theory which can be found in various monographs such as [25], [33], [38] and [41]. An informal approach to the Itô integral with respect to continuous semimartingales can also be found in Bain’s lecture notes [4].

In this section, only the construction of an Itô stochastic integral with respect to the standard Brownian motion is given, following [25]. As integrands, we only consider deterministic \( V \)-valued functions, where \( V \) is a real separable Hilbert space. So let \( V \) be such a space. Denote by \( \mathcal{E} \) the class of \( V \)-valued step functions on \([0, T]\), that is,

\[
\mathcal{E} := \{ \varphi : [0, T] \to V, \varphi(s) = \sum_{i=1}^{n} \phi_i \mathbf{1}_{(t_i, t_{i+1}]}(s), \phi_i \in V, i \in \{1, \cdots, n\}, 0 = t_1 \leq t_2 \leq \cdots \leq t_{n+1} = T, n \in \mathbb{N} \}.
\]
It is well-known that $\mathcal{E}$ is a dense subset of $L^2([0,T];V)$, see e.g. [42]. Let therefore $f \in \mathcal{E}$ be a $V$-valued step function on $[0,T]$ for some $T > 0$ and define

$$\int_0^T f(s)dB_s := \sum_{i=1}^n f_i \left(B_{t_{i+1}} - B_{t_i}\right).$$

(1.2)

By Definition 1.1.8, the increment $B_{t_{i+1}} - B_{t_i}$ has the law of $N(0,t_{i+1} - t_i)$ and therefore the stochastic integral (1.2) is a $V$-linear combination of independent Gaussian random variables. As such, it is also Gaussian with the probability law $N(0,\sigma^2 t)$, where

$$\sigma^2_t = \sum_{i=1}^n |f_i|^2_v (t_{i+1} - t_i) = \int_0^T |f(s)|^2_v \, ds = \int_0^T |f(s)|^2_v \, dB_s.$$

Hence, we arrive at a simple version of the famous Itô isometry which says

$$\mathbb{E} \left| \int_0^T f(s)dB_s \right|^2_V = \int_0^T |f(s)|^2_v \, ds, \quad f \in \mathcal{E}. \quad (1.3)$$

Using the formula (1.3) and the fact that $\mathcal{E}$ is dense in $L^2([0,T];V)$, the integral (1.2) can be extended onto the whole space $L^2([0,T];V)$. Generally, we have that

$$\mathbb{E} \left< \int_t^s u(r)dB_r, \int_t^s v(r)dB_r \right>_V = \int_t^s \langle u(r), v(r) \rangle_v \, dr$$

for $u, v \in L^2([0,T];V)$ such that $\langle u, v \rangle_v \in L^1([0,T])$ and for $s, t \in [0,T]$. The Itô stochastic integral, defined as above, is a $V$-valued Gaussian random variable with mean zero and it can be proven that $\left(\int_0^T f(s)dB_s, T \geq 0\right)$ is a $V$-valued centred Gaussian process. Moreover, clearly it is a process that starts $\mathbb{P}$-almost surely at zero and the integral is a linear operator from $L^2([0,T];V)$. An important result is that it is also a $V$-valued martingale.

### 1.2 Functional Analysis Preliminaries

In this section some results in the field of functional analysis are collected. We start by a brief introduction into the fractional calculus for which the motivation lies in the theory surrounding the fractional Brownian motion and, in particular, we present Hardy-Littlewood inequality applied to a fractional integral which is a crucial instrument in proving embeddings of functional spaces in the second chapter. Furthermore, Hölder’s and Cauchy-Schwarz inequalities are also mentioned. The second part of this section is devoted to semigroups and their properties that provide us with a significant tool in the theory of linear differential equations in infinite dimensions. These results are widely used in the last chapter while investigating the stochastic evolution equations driven by a cylindrical Gaussian Volterra process.

Firstly we present Hölder inequality and Cauchy-Schwarz inequality in the form which will be used in the following chapters, that is, for a Lebesgue integral which is equivalent to a formulation in terms of $L^p$-norms. These inequalities were coherently presented by Hardy, Littlewood and Pólya in [21], p.16-30 and are due to Hölder [22].
Theorem 1.2.1 (Hölder inequality). Let \((E, \mathcal{E}, \mu)\) be a measurable space and let \(1 \leq p, q \leq \infty\) such that
\[
\frac{1}{p} + \frac{1}{q} = 1.
\]
Then for all \(f \in L^p(\mu)\) and \(g \in L^q(\mu)\) we have that
\[
\|fg\|_{L^1(\mu)} \leq \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)}
\]
The special case when \(p = q = 2\) is called the Cauchy-Schwarz inequality. In the sequel, the Hölder inequality is mainly used in the case when the measurable space \((E, \mathcal{E}, \mu)\) is \([0, T], \mathcal{B}([0, T]), \lambda\), \(\lambda\) being the Lebesque measure and \(T > 0\). In such a case we have that
\[
\int_0^T |f(x)g(x)| \, dx \leq \left( \int_0^T |f(x)|^p \, dx \right)^\frac{1}{p} \left( \int_0^T |g(x)|^q \, dx \right)^\frac{1}{q},
\]
p, q being the appropriate Hölder conjugates.

### 1.2.1 Fractional Calculus

Systematic and exhaustive survey of classical and modern results in the theory of fractional integrals can be found in [43]. Classical results can also be found in Hardy’s and Littlewood’s original work [19] and [20]. In this section only basic definitions and results are given. The original definition of fractional integrals and derivatives is, however, due to Riemann [40] and Liouville [27].

**Definition 1.2.2** (Left-sided fractional Riemann-Liouville integral). Suppose that \(f \in L^1([a, b])\), where \(a, b \in \mathbb{R}\) and let \(\alpha > 0\). The left-sided fractional Riemann-Liouville integral of \(f\) of order \(\alpha\) on \((a, b)\) is given for almost all \(x \in (a, b)\) by
\[
I^\alpha_a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha - 1} f(y) \, dy. \tag{1.4}
\]
Clearly, if \(\alpha = n \in \mathbb{N}\) then the left-sided Riemann-Liouville integral coincides with the usual \(n\)-th order integral. It can be easily verified that the following composition formula holds for any \(\alpha, \beta > 0\):
\[
I^\alpha_a \left( I^\beta_a f \right) = I^{\alpha + \beta}_a f.
\]
Having defined the fractional integral it is only natural to seek an analogue of the classical differential operator which is an inverse of the left-sided fractional Riemann-Liouville integral. Denote by \(I^\alpha_a(L^p([a, b]))\) the image of \(L^p([a, b])\) under the operator \(I^\alpha_a\).

**Definition 1.2.3** (Left-sided fractional Riemann-Liouville derivative). Suppose that \(f \in I^\alpha_a(L^p([a, b]))\), where \(a, b \in \mathbb{R}\), \(p > 1\) and \(\alpha \in (0, 1)\). The left-sided fractional Riemann-Liouville derivative of \(f\) of order \(\alpha\) is given for almost all \(x \in (a, b)\) by
\[
D^\alpha_a f(x) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x - y)^\alpha} \, dy. \tag{1.5}
\]
The right-sided analogues $I^\alpha_{a+}$ and $D^\alpha_{a+}$ can be defined similarly. The integrals in (1.4) and (1.5) converge point-wise for almost all $x \in (a,b)$ if $p = 1$ and in $L^p$-sense if $p > 1$, see [33], p. 355. By construction, it is clear that for a function $f \in I^\alpha_{a+}(L^p([a,b]))$ we have that

$$I^\alpha_{a+}(D^\alpha_{a+}f) = f,$$

and, for a general function $f \in L^1([a,b])$, we have that

$$D^\alpha_{a+}(I^\alpha_{a+}f) = f.$$

Similarly, there exist inversion formulas for the operators $I^\alpha_{b-}$ and $D^\alpha_{b-}$. The following theorem is due to Hardy-Littlewood’s inequality applied on fractional Riemann-Liouville integral and is crucial for the functional space embeddings while investigating the properties of Gaussian Volterra processes with regular kernels in the second chapter.

**Theorem 1.2.4** (Hardy-Littlewood inequality for $I^\alpha_{0+}$, [45], p.119). For a function $f \in L^p(0,\infty)$ we have that there is a finite constant $c_{\alpha,p} > 0$ depending only on $\alpha \in (0,1)$ and $p > 1$ such that

$$\|I^\alpha_{0+}f\|_{L^q(0,\infty)} \leq c_{\alpha,p}\|f\|_{L^p(0,\infty)}$$

where $1 < p < q < \infty$ and $p, q$ and $\alpha$ satisfy

$$\frac{1}{q} = \frac{1}{p} - \alpha.$$

### 1.2.2 Semigroups of Bounded Linear Operators

The following introduction is based on a monograph of Pazy [35], which provides a self-contained presentation of the theory of semigroups of bounded linear operators and its application to deterministic partial differential equations. Only the very basics are given here.

**Definition 1.2.5** (Strongly and uniformly continuous semigroup). Suppose that $X = (X, \| \cdot \|_X)$ is a Banach space. A one-parameter family $(S(t), t \geq 0)$ of bounded linear operators from $X$ to $X$ is called a semigroup of bounded linear operators on $X$ if

(S1) $S(0) = I$,

(S2) $S(t + s) = S(t)S(s)$ for every $t, s \geq 0$,

where $I$ denotes the identity operator on $X$. We further say that a semigroup $(S(t), t \geq 0)$ of bounded linear operators on $X$ is a strongly continuous semigroup of bounded linear operators if

(S3a) for every $x \in X$, $\lim_{t \to 0^+} \|S(t)x - x\|_X = 0$.

A strongly continuous semigroup of bounded linear operators is sometimes referred to as a $C_0$-semigroup. If, however, the semigroup $(S(t), t \geq 0)$ of bounded linear operators on $X$ satisfies the condition
\( \lim_{t \to 0^+} \|S(t) - I\|_{\mathcal{L}(X)} = 0, \)

it is called \textit{uniformly continuous}.

The conditions \([S1]\) and \([S2]\) are purely algebraical and justify the name semigroup whereas the condition \([S3a]\) provides the semigroup \((S(t), t \geq 0)\) with a topological structure in the sense that \(S(t)\) is continuous in the strong operator norm. Every semigroup corresponds to its infinitesimal generator \(A\) defined as follows:

\textbf{Definition 1.2.6} (Infinitesimal generator of semigroups, \[35\], p.1). The linear operator \(\mathcal{L}(X) \ni A : \text{Dom}(A) \to X\) defined by

\[ Ax = \lim_{t \to 0^+} \frac{S(t)x - x}{t} = \frac{d^+ S(t)x}{dt} \bigg|_{t=0} \]

for \(x \in \text{Dom}(A)\), where

\[ \text{Dom}(A) = \left\{ x \in X : \lim_{t \to 0^+} \frac{S(t)x - x}{t} \text{ exists} \right\}, \]

is called the \textit{infinitesimal generator} of the semigroup \((S(t), t \geq 0)\) and \(\text{Dom}(A)\) is the \textit{domain} of \(A\).

\textbf{Theorem 1.2.7} (Characterisation of infinitesimal generators, \[35\], p.2). A linear operator \(A\) is the infinitesimal generator of a uniformly continuous semigroup if and only if \(A\) is a bounded linear operator.

From Definition 1.2.6 it follows that every semigroup \((S(t), t \geq 0)\) has a unique infinitesimal generator. If, however, the semigroup \(S\) is uniformly continuous, the correspondence between its infinitesimal generator and the semigroup \(S\) is one-to-one, that is for every bounded linear operator \(A\) there is a unique uniformly continuous semigroup and vice versa. Two important results on \(C_0\)-semigroups follow.

\textbf{Theorem 1.2.8} (\[35\], p.4). Let \((S(t), t \geq 0)\) be a \(C_0\)-semigroup. Then there exist finite constants \(\omega \geq 0\) and \(M \geq 1\) such that for all \(t \geq 0\) we have that

\[ \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}. \]

As an immediate corollary of Theorem 1.2.8 we have that if \((S(t), t \geq 0)\) is a \(C_0\)-semigroup then for every \(x \in X\) the mapping \(t \mapsto S(t)x\) is continuous on \(\mathbb{R}_+\).

The assertions of the following theorem can be found in \[35\].

\textbf{Theorem 1.2.9} (Infinitesimal generators and differential equations). If \(A\) is the infinitesimal generator of a \(C_0\)-semigroup \((S(t), t \geq 0)\), then \(\text{Dom}(A)\), the domain of \(A\), is dense in \(X\) and \(A\) is a closed linear operator. Moreover,

\[ \frac{d}{dt} S(t)x = AS(t)x, \quad t > 0, \quad x \in \text{Dom}(A), \]

and hence, \(y(t) = S(t)x\) is a solution to a linear differential equation \(\dot{y}(t) = Ay(t)\) for \(t \geq 0\), with the initial condition \(y(0) = x\) for each \(x \in \text{Dom}(A)\).
1.3 Isonormal Gaussian Processes

This section introduces the general framework of this thesis. Firstly, an isonormal Gaussian process is introduced. This process, which presents a generalisation of Gaussian processes, can be viewed as a collection of random variables indexed by elements of a real separable Hilbert space. For this part, Nualart’s monograph \[33\] is closely followed. After the introduction, some examples of isonormal Gaussian processes are presented, namely the standard fractional Brownian motion. This process serves throughout the whole thesis as an important example on which the main ideas are demonstrated. For the properties of the standard fractional Brownian motion we refer to \[5\].

Suppose, in this section, that \((\Omega, \mathcal{F}, \mathbb{P})\) is a complete probability space and let \(H\) be a real separable Hilbert space with scalar product denoted by \(\langle \cdot, \cdot \rangle_H\).

**Definition 1.3.1 (H-isonormal Gaussian process).** We say that a stochastic process \(W = (W(h), h \in H)\) defined in \((\Omega, \mathcal{F}, \mathbb{P})\) is an \(H\)-isonormal Gaussian process (or a Gaussian process on \(H\)) if \(W\) is a centered Gaussian family of random variables such that

\[
\mathbb{E}(W(h)W(g)) = \langle h, g \rangle_H, \quad g, h \in H.
\]  

**Remark 1.3.2.** Isonormal Gaussian process is a mapping \(W : H \to L^2(\Omega)\). Equivalently, the process \(W = \{W(h), h \in H\}\) can be defined as a collection of zero-mean, Gaussian \(L^2(\Omega)\)-valued random variables such that the condition (1.6) holds. The mapping \(h \mapsto W(h)\) is a linear isometry from \(H\) onto a closed subspace of \(L^2(\Omega, \mathcal{F}, \mathbb{P})\) denoted by \(H_1\). Indeed, for any \(\alpha, \beta \in \mathbb{R}\), and \(h, g \in H\), it follows that

\[
\mathbb{E}\left((W(\alpha h + \beta g) - \alpha W(h) - \beta W(g))^2\right) =
\]

\[
= \|\alpha h + \beta g\|_H^2 + \alpha^2\|h\|_H^2 + \beta^2\|g\|_H^2
- 2\alpha(\alpha h + \beta g, h)_{H_1} - 2\beta(\alpha h + \beta g, g)_{H_1} + 2\alpha\beta(h, g)_{H_1} = 0.
\]

**Remark 1.3.3.** Given a real separable Hilbert space \(H\), a Gaussian process \(W\) and a probability space satisfying the condition (1.6) can always be constructed by Theorem 1.1.5. It would be more precise to call an isonormal Gaussian process in Definition 1.3.1 the pair \((W, H)\); however, only the first component is generally used for simplicity.

There is a clear correspondence between the standard Brownian motion and isonormal Gaussian processes. To see this, let \(B = (B_t, t \in [0, T])\) be, for some \(T > 0\), the standard Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\). Having set \(H = L^2([0, T])\) we denote

\[W(h) = \int_0^T h(s)dB_s, \quad h \in H,\]

where the stochastic integral is defined in the sense of Itô. Then \(W\) is a zero-mean Gaussian random variable with variance \(\int_0^T h^2(s)ds, h \in H\) and, in particular, from the Itô lemma we obtain

\[
\mathbb{E}\left(\int_0^T h_1(s)dB_s \cdot \int_0^T h_2(s)dB_s\right) = \int_0^T h_1(s)h_2(s)ds = \langle h_1, h_2 \rangle_H.
\]
for all $h_1, h_2 \in \mathbb{H}$. Since $1_{[0,t]} \in L^2([0,T])$ for all $t \in [0,T]$, we clearly have that

$$W(1_{[0,t]}) = \int_0^T 1_{[0,1]}(s)dB_s = B_t - B_0 = B_t.$$  

Thus, the standard Brownian motion indeed corresponds to an $\mathbb{H}$-isonormal, that is an $L^2([0,T])$-isonormal, Gaussian process. Note that for all $s, t \in [0,T]$ we have that

$$\mathbb{E}(B_s B_t) = \mathbb{E}(W(1_{[0,s]})W(1_{[0,t]})) = \langle 1_{[0,s]}, 1_{[0,t]} \rangle_H = s \wedge t.$$  

### 1.3.1 Fractional Brownian Motion

This section contains the definition and basic properties of the standard fractional Brownian motion which represents a generalisation of the standard Brownian motion. For a thorough survey and a comprehensive overview of the standard fractional Brownian motion and the stochastic calculus defined with respect to it, we refer to [5]. Other references include the original paper of Mandelbrot and Van Ness [29], where a stochastic integral representation for the fractional Brownian motion was given. For further reading see also [10] and [32].

**Definition 1.3.4 (Fractional Brownian Motion).** Let $T > 0$. We say that a real-valued stochastic process $B^H = (B^H_t, t \in [0,T])$ is a *standard fractional Brownian motion with Hurst parameter* $H \in (0,1)$ if it is a centered Gaussian process, $B^H_0 = 0, \ P - a.s.$ and whose covariance function is given by

$$\text{cov}(B^H_s, B^H_t) = \frac{1}{2} \left( s^{2H} + t^{2H} - |t-s|^{2H} \right), \ s, t \in [0,T]. \tag{1.7}$$

By Definition 1.3.4, the standard fractional Brownian motion is a process with stationary increments, i.e. the probability law of $B^H_{t+s} - B^H_t$ is the same as the law of $B^H_s$ for $s, t \in [0,T]$. The existence of such a process follows, by similar arguments as for the standard Brownian motion, from Theorem 1.1.5 and Kolmogorov-Chentsov theorem, Theorem 1.1.6. The process $B^H$ has a version with continuous sample paths. This version shall be hereafter considered.

Based on the value of the parameter $H$, so-called the Hurst parameter or Hurst index ([29]), the standard fractional Brownian motion branches out into three families, that is, for $H = \frac{1}{2}$ which represents the standard Brownian motion, for the singular case when $H < \frac{1}{2}$ and for the regular case when $H > \frac{1}{2}$. Only in the case of $H = \frac{1}{2}$ the increments of the process $B^H$ are independent. If $H \neq \frac{1}{2}$, it follows by (1.7) that the increments $B^H_{t+h} - B^H_t$ and $B^H_{t+2h} - B^H_{t+h}$ are correlated negatively in the singular case and positively in the regular case.

The standard fractional Brownian process represents a *self-similar* process (see e.g. [14]), that is, for every $a > 0$ it holds that

$$\text{Law} \left( B^H_{at}, t \geq 0 \right) = \text{Law} \left( a^H B^H_t, t \geq 0 \right).$$

The regularity of the sample path of the fractional Brownian motion also depends on the Hurst index $H$. This property follows from Kolmogorov-Chentsov Theorem, Theorem 1.1.6.
Theorem 1.3.5 (Regularity of the paths of fBm, [5], p. 11). Let $H \in (0, 1)$. The standard fractional Brownian motion $B^H$ admits a version whose sample paths are $\mathbb{P}$-almost surely Hölder continuous of order strictly less than $H$.

Since almost all sample paths of the standard Brownian motion are nowhere differentiable nor can be such a property expected in case of the fractional Brownian motion which is a straightforward generalisation.

Theorem 1.3.6 (Differentiability of sample paths of fBm, [29], p. 427). The sample path $B^H_t(\omega)$ is $\mathbb{P}$-almost surely not differentiable; in fact,

$$\limsup_{t \to t_0} \frac{|B^H_t(\omega) - B^H_{t_0}(\omega)|}{t - t_0} = \infty$$

with probability one.

In order to give the reader the general idea of the connection between the standard fractional Brownian motion and isonormal Gaussian processes defined at the beginning of this section, a brief informal outline is given. The precise construction is given in the following chapter for Gaussian Volterra processes which also includes the fractional Brownian motion. We start by taking the class $\mathcal{E}$ of real step functions on $[0, T]$ for some $T > 0$. Then for $H \in (0, 1)$ we define a space $\mathcal{H}^H$ as a completion of $\mathcal{E}$ with respect to the scalar product

$$\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}^H} := \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}), \quad s, t \in [0, T].$$

Further, define $B^H_t(1_{[0,t]}) = B^H_t$ for $t \in [0, T]$. Then, the family $\{B^H(h), h \in \mathcal{H}^H\}$ represents a $\mathcal{H}^H$-isonormal Gaussian process similarly as in the case of standard Brownian motion. Indeed, notice that

$$\mathbb{E} \left( B^H(1_{[0,t]}) B^H(1_{[0,s]}) \right) = \mathbb{E} \left( B^H_t B^H_s \right) = \langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}^H}$$

for $s, t \in [0, T]$ which is extended onto the whole space $\mathcal{H}^H$.

1.3.2 Liouville fractional Brownian motion

The Liouville fractional Brownian motion is a process whose properties reflect the properties of the standard fractional Brownian motion, however, they do not pose such a technical challenge since the formulation of the Liouville process is simpler and more symmetric around the critical value $H = \frac{1}{2}$. These properties thus allow a unified treatment of both the regular and the singular case. Only the definition is presented here. For a stochastic integration with respect to the Liouville fractional Brownian motion see [7] or [16].

Definition 1.3.7 (Liouville fractional Brownian motion, [7], p. 7). Let

$$R^H_L(s, t) = \frac{1}{\Gamma \left( H + \frac{1}{2} \right)^2} \int_0^{s \wedge t} (s - u)^{H - \frac{1}{2}} (t - u)^{H - \frac{1}{2}} du, \quad s, t \in [0, T].$$

A Liouville fractional Brownian motion of order $0 < H < 1$, indexed by $[0, T]$, is a Gaussian process $L^H = (L^H_t, t \in [0, T])$ such that

$$\mathbb{E} \left( L^H_s L^H_t \right) = R^H_L(s, t), \quad s, t \in [0, T].$$
2. Volterra Processes

In this chapter, a general type of a Gaussian process, which is further used as a driving process for stochastic integration, is introduced. This process is determined by its covariance function $R$ defined in terms of a general kernel $K$. Further, two sets of conditions on the kernel $K$ are considered and two corresponding types of the kernel $K$ are investigated - the singular case and the regular case.

2.1 Definition and Basic Properties

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $\beta = (\beta_t, t \in [0, T])$ be a real, zero-mean and continuous Gaussian process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance function

$$ R(t, s) := \mathbb{E}(\beta_t \beta_s) $$

for $t, s \in [0, T]$, such that $\beta_0 = 0$, $\mathbb{P}$ - a.s.. Suppose that $\mathcal{F}$ is generated by $\beta$. Define $\Delta = \{(t, s) \in [0, T]^2 : 0 \leq s < t \leq T\}$ and let us further assume that the function $R(t, s)$ can be expressed as

$$ R(t, s) = \int_0^{t \land s} K(t, r)K(s, r)dr, \quad (2.1) $$

where $K : \Delta \to \mathbb{R}$, is a kernel satisfying

$$ \sup_{t \in [0, T]} \int_0^t K(t, s)^2 ds < \infty. \quad (2.2) $$

We further suppose that the process $\beta$ can be expressed as

$$ \beta_t = \int_0^t K(t, s)dW_s, \quad \mathbb{P} - \text{a.s.,} \quad (2.3) $$

where $W = (W_t, t \in [0, T])$ is the standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Thus, we also assume that the probability space is rich enough to assure the existence of the standard Brownian motion. Processes that satisfy the form (2.3) are called Gaussian Volterra processes. The integral in (2.3) is defined in the sense of Itô.

Remark 2.1.1. Alòs, Mazet and Nualart claim ([2], p.770) that under the assumptions (2.1) and (2.2), the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ can be enlarged in such a way that we can find a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ such that the process $\beta$ can be expressed in the form (2.3) making this extra assumption unnecessary.

Let $V$ be a real separable Hilbert space. In order to investigate the behaviour of the process $\beta$ further, denote by $\mathcal{E}$ the set of $V$-valued step functions on $[0, T]$, that is

$$ \mathcal{E} := \{ \varphi : [0, T] \to V, \varphi(s) = \sum_{i=1}^{n} \varphi_i 1_{(t_i, t_{i+1})}(s), \varphi_i \in V, i \in \{1, \ldots, n\}, 0 = t_1 \leq t_2 \leq \cdots \leq t_{n+1} = T, n \in \mathbb{N}\}. $$

$\mathcal{E}$ is clearly a dense subset of $L^2([0, T]; V)$, (see [12]). Suppose further that the kernel $K$ satisfies the following condition:
(K) The function \( K(\cdot, s) : (s, T] \to \mathbb{R} \) has bounded variation on any interval \((u, T], \) for each \( u > s , \) where \( 0 < s < T . \)

By the condition \([K]\), the function \( K(\cdot, s) \) can be written as a difference of two non-decreasing functions \( K^+ \) and \( K^- \). Thus, for each \( s \in (0, T) \), one can define a Lebesgue-Stieltjes measure \( \mathcal{K}(\cdot, s) \) on \( B((s, T]) \) using the decomposition of \( K \) into these non-decreasing components. For a precise construction and further discussion we refer to [28], p. 186. We have that

\[
\mathcal{K}((s, t], s) = K(t+, s) - K(s+, s),
\]

where the + denotes a limit from the right-hand side. Consider the operator \( K^* : \mathcal{E} \to L^2([0, T]; V) \) defined by

\[
(K^* \varphi)(s) = \varphi(s)K(T, s) + \int_s^T (\varphi(t) - \varphi(s)) \mathcal{K}(dt, s).
\]

Further, the relationship between the operator \( K^* \) and the kernel \( K \) is discussed.

Notice that the kernel \( K \) defines an operator from \( L^2([0, T]; V) \) given by

\[
(Kh)(t) = \int_0^t K(t, s)h(s)ds,
\]

and that for \( \varphi \in \mathcal{E} \), the function \( (K^* \varphi)(\cdot) \) takes the form

\[
(K^* \varphi)(s) = \sum_{i=1}^n \phi_i 1_{(s_i, s_{i+1}]}(s)K(T, s) + \sum_{i=1}^{n-1} 1_{(s_i, s_{i+1}]}(s) \sum_{j=i+1}^n (\phi_j - \phi_i) (K(s_{j+1}, s) - K(s_j, s))
\]

where \( s \in [0, T] \), for \( T > 0 \). The expression \([2.6]\) together with the following lemma can be found in [2], p.771, where it is proved for a real step function \( \varphi \) and \( h \in L^2([0, T]). \) The statement, however, can be extended to a \( V \)-valued step function and \( h \in L^2([0, T]; V). \) Lemma \( 2.1.2 \) provides the relationship between \( K^* \) and \( K. \)

**Lemma 2.1.2** (Pairing of \( K \) and \( K^* \)). Let \( K^* \) be the operator defined by \([2.4]\) and \( K \) be a kernel satisfying \([2.1]\) and \([2.2]\). Then for a function \( \varphi \in \mathcal{E} \) and \( h \in L^2([0, T]; V) \), we have that

\[
\int_0^T (K^* \varphi)(t)h(t)dt = \int_0^T \varphi(t)(Kh)(dt).
\]

**Example 2.1.3.** Let \( T > 0 \) and \( B^H = (B^H_t, t \in [0, T]) \) be the standard fractional Brownian motion with Hurst parameter \( H \in (0, 1). \) The covariance function \( R_H \) of \( B^H \) is given by \([1.7]\). It has been shown (see [10] or [34]) that

\[
R(t, s) = \int_0^{\wedge} K_H(t, r)K_H(s, r)dr, \quad t, s \in [0, T],
\]

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for a square integrable kernel $K_H$, given by

$$K_H(t,s) = \Gamma \left( H + \frac{1}{2} \right)^{-1} (t-s)^{H-\frac{1}{2}} \, _2F_1 \left( H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s} \right),$$

where $0 \leq s < t \leq T$ and $F = _2F_1(a, b, c, z)$ is the Gauss hypergeometric function which can be defined for any $z$ such that $|\arg(1-z)| < \pi$ as

$$_2F_1(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{a} dt, \quad \Re(c) > \Re(b) > 0.$$  

Alòs, Mazet and Nualart ([2], p.798) give a more convenient expression. For $s < t$ the kernel $K_H$ can be expressed as

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H \left( \frac{1}{2} - H \right) \int_s^t (u-s)^{H-\frac{1}{2}} \left( 1 - \left( \frac{s}{u} \right)^{\frac{1}{2}-H} \right) du \quad (2.7)$$

c_H being a normalizing constant

$$c_H = \left( \frac{2H\Gamma \left( \frac{3}{2} - H \right)}{\Gamma \left( H + \frac{1}{2} \right) \Gamma(2H-2)} \right)^{\frac{1}{2}}. \quad (2.8)$$

Differentiating $K_H(t,s)$ in the first variable gives

$$\frac{\partial K}{\partial t}(t,s) = c_H \left( H - \frac{1}{2} \right) \left( \frac{s}{t} \right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}, \quad (2.9)$$

assuming that $H \neq \frac{1}{2}$ which corresponds to the case of the standard Brownian motion.

**Example 2.1.4.** Let $T > 0$ and $L^H = (L^H_t, t \in [0, T])$ be the Liouville fractional Brownian motion. By its very definition, the covariance function $R_L^H$ can be expressed in the form (2.1) with the kernel

$$K_L^H(t,s) = \frac{1}{\Gamma \left( H + \frac{1}{2} \right)} (t-s)^{H-\frac{1}{2}} \quad (2.10)$$

which, after differentiation of $K_L^H(\cdot, s)$, gives

$$\frac{\partial K_L^H}{\partial t}(t,s) = \frac{H - \frac{1}{2}}{\Gamma \left( H + \frac{1}{2} \right)} (t-s)^{H-\frac{3}{2}} = \frac{1}{\Gamma \left( H - \frac{1}{2} \right)} (t-s)^{H-\frac{3}{2}},$$

assuming again that $H \neq \frac{1}{2}$.

### 2.2 Reproducing Kernel Hilbert Space

Let $T > 0$ and let $\beta = (\beta_t, t \in [0, T])$ be a Gaussian process whose covariance function $R$ is of the form (2.1) which can be expressed as (2.3). Suppose further, that its kernel $K$ satisfies the conditions (2.2) and (K). In order to develop stochastic calculus with respect to the process $\beta$, the space $H$ of $V$-valued functions on $[0, T]$ is defined in this section. The elements of the space $H$ form
a set of deterministic functions for which the stochastic integral is defined. For \( x, y \in V \) define the function \( \langle \cdot, \cdot \rangle_H \) by

\[
\langle x_{[0,t]}, y_{[0,s]} \rangle_H := \langle x, y \rangle_V R(t,s), \quad (t, s) \in [0, T]^2.
\]

The inner product \( \langle \cdot, \cdot \rangle_H \) can be extended (by linearity) to \( E \) and hence \( (E, \langle \cdot, \cdot \rangle_H) \) forms a pre-Hilbert space. The completion of \( E \) with respect to the scalar product (2.11) is denoted by \( H \) and it forms a Hilbert space since \( E \) is a dense subset of \( L^2([0, T], V) \). Similar construction can be found in [30].

**Definition 2.2.1 (RKHS).** The Hilbert space \((H, \| \cdot \|_H, \langle \cdot, \cdot \rangle_H)\) defined as a completion of \( E \) under the scalar product (2.11) is called the reproducing kernel Hilbert space.

**Remark 2.2.2.** Notice that there is a clear relationship between the construction given in Definition 2.2.1 and previous works (namely [2], [13] and [33]) which can be explained in terms of a tensor product of Hilbert spaces (for discussion on tensor products of Hilbert spaces see [47], p. 47). Denote \( H_\mathbb{R} \) the real analogue to the Hilbert space \( H \) constructed as above, instead of \( V \), however, the space \((\mathbb{R}, | \cdot |)\) is considered. Then the reproducing kernel Hilbert space \( H \) can be expressed as

\[
H = V \otimes H_\mathbb{R},
\]

where \( \otimes \) denotes the tensor product of Hilbert spaces. The natural inner product is defined for \( \varphi_1, \varphi_2 \in V \) and \( \psi_1, \psi_2 \in H_\mathbb{R} \) as

\[
\langle \varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2 \rangle_H = \langle \varphi_1, \varphi_2 \rangle_V \langle \psi_1, \psi_2 \rangle_{H_\mathbb{R}}.
\]

Thus, it can be easily seen that

\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{H_\mathbb{R}} = R(t,s).
\]

In order to prove a crucial proposition for the existence of the stochastic integral with respect to Gaussian Volterra processes, we present the following lemma.

**Lemma 2.2.3.** For \( \varphi \in E \) and the operator \( K^* \) defined by (2.4) we have that

\[
\| \varphi \|^2_H = \| K^* \varphi \|^2_{L^2([0,T],V)}.
\]

**Proof.** Let \( \varphi \in E \). Then from the linearity of the scalar product we can write

\[
\| \varphi \|^2_H = \left\langle \sum_{i=1}^n \varphi_i 1_{(s_i, s_{i+1})}, \sum_{j=1}^n \varphi_j 1_{(s_j, s_{j+1})} \right\rangle_H = \sum_{i=1}^n \sum_{j=1}^n \langle \varphi_i 1_{(s_i, s_{i+1})}, \varphi_j 1_{(s_j, s_{j+1})} \rangle_H.
\]

On the other hand, using the linearity of the operator \( K^* \), we arrive at

\[
\| K^* \varphi \|^2_{L^2([0,T],V)} = \left\langle \sum_{i=1}^n K^* \varphi_i 1_{(s_i, s_{i+1})}, \sum_{j=1}^n K^* \varphi_j 1_{(s_j, s_{j+1})} \right\rangle_{L^2([0,T],V)} = \sum_{i=1}^n \sum_{j=1}^n \langle K^* \varphi_i 1_{(s_i, s_{i+1})}, K^* \varphi_j 1_{(s_j, s_{j+1})} \rangle_{L^2([0,T],V)}.
\]
Finally, the linearity of $K^*$ and the expansion $1_{[a,b]} = 1_{[0,b]} - 1_{[0,a]}$ for any $a, b \in [0,T]$ such that $a < b$ imply that
\[
\langle \varphi_i 1_{(s_i,s_{i+1})}, 1_{(s_j,s_{j+1})} \rangle_{\mathcal{H}} = \langle K^* \varphi_i 1_{(s_i,s_{i+1})}, 1_{(s_j,s_{j+1})} \rangle_{L^2([0,T],V)}
\]
for any $i, j \in \{1, \cdots, n\}$ which completes the proof. □

**Proposition 2.2.4** (Stochastic integral). Denote $H(\beta)$ the Gaussian space associated with the process $\beta$, i.e. the closed linear span of $\{\beta_t, t \in [0,T]\}$ in $L^2(\Omega, \mathcal{F}, P)$.

(i) The linear mapping $I : (\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{H}}) \to L^2((\Omega, \mathcal{F}, P), V)$ defined by
\[
x1_{[0,t]} \mapsto x\beta_t, \quad x \in V, \ t \in [0,T],
\]
can be extended to a bounded linear operator from $\mathcal{H}$ to $H(\beta)$ denoted again by $I : \varphi \mapsto \beta(\varphi)$.

(ii) For $\varphi \in \mathcal{H}$, the operator $I$ satisfies
\[
I(\varphi) = \int_0^T (K^* \varphi)(t) \, dW_t = \beta(\varphi), \quad P \text{- a.s.,} \tag{2.12}
\]
where $W = (W_t, t \in [0,T])$ is the standard Brownian motion.

**Proof.** (i) The statement follows directly from the linearity of the mapping $I$ and the fact that $\mathcal{E}$ is a dense subset of $L^2([0,T], V)$ since $\mathcal{H}$ is defined as a completion of $\mathcal{E}$ with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ defined by (2.11).

(ii) Let $\varphi \in \mathcal{E}$. Applying the linear mapping $I$ on $\varphi$ yields that
\[
\varphi = \sum_{i=1}^n \varphi_i \left(1_{(0,s_{i+1})} - 1_{(0,s_i)}\right) \quad \Rightarrow \quad \sum_{i=1}^n \varphi_i (\beta_{s_{i+1}} - \beta_{s_i}) = \beta(\varphi).
\]
Hence, by the formula (2.3), we have that
\[
\begin{align*}
\beta(\varphi) &= \varphi_0 \beta_T - \sum_{i=1}^{n-1} (\varphi_{i+1} - \varphi_i) \beta_{s_{i+1}} \\
&= \int_0^T \left( \varphi_0 K(T,s) - (\varphi_{i+1} - \varphi_i) 1_{(0,s_{i+1})}(s)K(s_{i+1},s) \right) \, dW_s \\
&= \int_0^T \left( \sum_{i=1}^n \varphi_i 1_{(s_i,s_{i+1})}(s)K(T,s) \\
&\quad + \sum_{i=1}^{n-1} (\varphi_{i+1} - \varphi_i) 1_{(0,s_{i+1})}(s) \left( K(T,s) - K(s_{i+1},s) \right) \right) \, dW_s \\
&= \int_0^T \left( \varphi(s) K(T,s) \\
&\quad + \sum_{i=1}^{n-1} 1_{(s_i,s_{i+1})}(s) \sum_{j=i+1}^n (\varphi_{i+1} - \varphi_i) \left( K(s_{j+1},s) - K(s_j,s) \right) \right) \, dW_s \\
&= \int_0^T (K^* \varphi)(s) \, dW_s
\end{align*}
\]
From Lemma 2.2.3 it follows that it is possible to represent $H$ as the closure of $E$ with respect to the norm

$$\|\varphi\|_H = \|K^*\varphi\|_{L^2([0,T],V)}.$$  

and by (i), the operator $I$ can be extended to the whole space $H$. Hence, the equality (2.12) can also be extended to the whole space $H$. 

\[\square\]

**Remark 2.2.5.** The process $\beta = (\beta(\varphi), \varphi \in H)$ is an isonormal Gaussian process in the sense of Definition 1.3.1. Indeed, for arbitrary $\varphi \in H$, the $V$-valued random variable $\beta(\varphi)$ is of the form (2.12) and the well-known properties of Itô integral imply that it has mean zero. Clearly, it is Gaussian and

$$\mathbb{E} \langle \beta(\varphi), \beta(\psi) \rangle_V = \mathbb{E} \left( \int_0^T (K^*\varphi)(s)dW_s, \int_0^T (K^*\psi)(s)dW_s \right)_V = \int_0^T \langle (K^*\varphi)(s), (K^*\psi)(s) \rangle_V ds = \langle K^*\varphi, K^*\psi \rangle_{L^2([0,T],V)} = \langle \varphi, \psi \rangle_H,$$

by the Itô isometry.

**Example 2.2.6.** In this example it is shown that the Liouville fractional Brownian motion can be expressed as (2.3). Consider the real analogues to the space $H$ and the operator $K^*$. Fix $t \in [0, T]$ and let $s \in [0, T]$. By the expression (2.6), we clearly have that

$$\left(K^*1_{[0,t]}\right)(s) = K(t, s)1_{[0,t]}(s)$$

and thus,

$$\langle K^*1_{[0,t]}, K^*1_{[0,s]} \rangle_{L^2([0,T])} = \langle K(t, \cdot)1_{[0,t]}, K(s, \cdot)1_{[0,s]} \rangle_{L^2([0,T])} = \int_0^T K(t, r)K(s, r)1_{[0,t]}(r)1_{[0,s]}(r)dr = \int_0^{\wedge s} K(t, r)K(s, r)dr.$$

Hence,

$$\langle K^*1_{[0,t]}, K^*1_{[0,s]} \rangle_{L^2([0,T])} = R(t, s) = \langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}_s}. \quad (2.14)$$

In the particular case of the Liouville fractional Brownian motion, the kernel $K_H^L$ is of the form (2.10) and the operator $\left(K_H^L\right)^*$, by its definition (2.4), can be expressed as

$$\left((K_H^L)^*\varphi\right)(s) = \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \left( \frac{\varphi(s)}{(T-s)^{\frac{1}{2}-H}} - \left(\frac{1}{2} - H\right) \int_s^T \frac{\varphi(s) - \varphi(t)}{(t-s)^{\frac{1}{2}-H}} dt \right),$$

which, however, coincides with the definition of a right-sided Riemann-Liouville fractional derivative which means that

$$\left((K_H^L)^*\varphi\right)(s) = \left(D_{T-}^{\frac{1}{2}-H}\varphi\right)(s).$$
The inverse operator defined in terms of a left-sided Riemann-Liouville fractional integral exists for all functions \( \varphi \in I_{\frac{1}{2}}^{1-H} (L^p) \). Therefore we have that

\[
((K_H^L)^*)^{-1} = I_{\frac{1}{2}}^{1-H}
\]

with a domain

\[
\text{Dom} \left( (K_H^L)^{-1} \right) = I_{\frac{1}{2}}^{1-H} (L^p).
\]

In particular, the inverse operator exists for a function \( \varphi = \mathbf{1}_{[0,t]} \) where \( t \in [0,T] \). Then, the inverse operator evaluated in \( \mathbf{1}_{[0,t]} \) takes the form

\[
((K_H^L)^*)^{-1} (\mathbf{1}_{[0,t]}) (s) = \frac{1}{\Gamma \left( \frac{1}{2} - H \right)} \int_s^T (r - s)^{-H - \frac{1}{2}} \mathbf{1}_{[0,t]}(r) dr.
\]

Therefore, define the stochastic process \( W = (W_t, t \in [0,T]) \) by

\[
W_t = \beta \left( (K_H^L)^{-1} (\mathbf{1}_{[0,t]}) \right), \quad t \in [0,T]. \tag{2.15}
\]

Then \( W \) is the standard Brownian motion. Indeed, \( W \) is a centred, Gaussian stochastic process with covariance function

\[
\mathbb{E}(W_s W_t) = \mathbb{E} \left( \beta \left( (K_H^L)^{-1} (\mathbf{1}_{[0,t]}) \right) \beta \left( (K_H^L)^{-1} (\mathbf{1}_{[0,s]}) \right) \right)
\]

\[
= \langle (K^*)^{-1} (\mathbf{1}_{[0,s]}), (K^*)^{-1} (\mathbf{1}_{[0,s]}) \rangle_{\mathcal{H}_R}
\]

\[
= \langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{L^2([0,T])}
\]

\[
= s \wedge t.
\]

By Kolmogorov-Chentsov Theorem, Theorem 1.1.6 we take its continuous version. Thus, from (2.15) it follows that \( L^H \) can be expressed as

\[
L^H_t = \int_0^t K^H(t, s) dW_s, \quad \mathbb{P} - \text{a.s.} \tag{2.16}
\]

It also follows from (2.2.6) that the process \( W \) is adapted to the filtration generated by the process \( L^H \). From (2.15) and (2.16) it follows that both processes generate the same filtration. This construction of \( W \) follows the construction for the standard fractional Brownian motion in [32], where the representation of \( B^H \) is proved using the same technique. This approach, however, gives the answer to the question when a general continuous Gaussian process whose covariance function satisfies the factorization (2.1) with some kernel \( K \) enjoying the condition (2.2). In the same way, we arrive at the expression (2.14), then show that the inverse of \( K^* \) is defined at least for the indicator function and define the process \( W \) as

\[
W_t = \beta \left( (K^*)^{-1} (\mathbf{1}_{[0,t]}) \right), \quad t \in [0,T].
\]

The crucial point of this construction is the inverse of the operator \( K^* \). This has been investigated by Erraoui and Essaky in [15], to whom we refer for further reading.
2.3 Processes with Singular Kernel

Having investigated the behaviour of the standard fractional Brownian motion \( B^H = (B^H_t, t \in [0, T]) \), where \( H \in (0, 1) \) is the Hurst parameter, it is natural to consider two types of a kernel \( K \) which correspond to the singular case \( (H < \frac{1}{2}) \) and to the regular case \( (H > \frac{1}{2}) \) of the standard fractional Brownian motion \( B^H \). In this section we introduce the conditions on kernels that define singular processes.

Let \( T > 0 \) and suppose that \( \beta = (\beta_t, t \in [0, T]) \) is a zero-mean Gaussian Volterra process with kernel \( K \), satisfying the condition (2.2), which corresponds to the covariance function \( R \) of the form (2.1). We say that the process \( \beta \) is singular or with a singular kernel on \( [0, T] \), if it satisfies the condition (K). We shall now give a more restrictive set of sufficient conditions for a process to be singular. Suppose that the kernel \( K \) satisfies the following:

\[ (C) \text{ For all } s \in (0, T) \text{ the function } K(\cdot, s): (s, T] \rightarrow \mathbb{R} \text{ is differentiable in the interval } (s, T) \text{ and both } K(t, s) \text{ and its derivative } \frac{\partial K}{\partial t}(t, s) \text{ are continuous at every } t \in (s, T). \]

\[ (S) \text{ Let } 0 < \alpha < \frac{1}{2}. \text{ Then there exists a finite constant } c > 0 \text{ such that} \]

\[ \left| \frac{\partial K}{\partial t}(t, s) \right| \leq c(t-s)^{-\alpha-1}, \quad (2.17) \]

\[ \int_s^t K(t, r)^2 dr \leq c(t-s)^{1-2\alpha} \quad (2.18) \]

for \( 0 < s < t < T. \)

**Proposition 2.3.1** (Sufficient conditions for singular kernels). Let \( K \) be a kernel satisfying one of the conditions \([C]\) or \([S]\). Then the condition \([K]\) holds.

**Proof.** For a fixed \( s \in [0, T] \), both the conditions \([C]\) and \([S]\) imply that the function \( K(\cdot, s): (s, T] \rightarrow \mathbb{R} \) is differentiable in the interval \([u, T] \subset \mathbb{R}, u > s\). Thus, the total variation \( V^T_u(K(\cdot, s)) \) can be expressed as

\[ V^T_u(K(\cdot, s)) = \int_u^T \left| \frac{\partial K}{\partial t}(r, s) \right| dr \]

which holds since the function \( \frac{\partial K}{\partial t}(\cdot, s) \) is Lebesgue integrable in \([u, T]\). Indeed, by condition \([C]\) it is continuous on any compact interval \([u, T]\), \( u > s \) and thus it is bounded there. By condition \([S]\) there is a dominating integrable function for the term \( \frac{\partial K}{\partial t}(\cdot, s) \). Hence, we can write

\[ V^T_u(K(\cdot, s)) = \int_u^T \left| \frac{\partial K}{\partial t}(r, s) \right| dr \leq (T - u) \sup_{t \in [u, T]} \left| \frac{\partial K}{\partial t}(r, s) \right| < \infty \]

in the case that the condition \([C]\) holds and

\[ V^T_u(K(\cdot, s)) = \int_u^T \left| \frac{\partial K}{\partial t}(r, s) \right| dr \leq c \int_u^T (r-s)^{-\alpha-1} dr < \infty \]

in the case when we only have the condition \([S]\).
Example 2.3.2. Let $T > 0$ and let $B^H = (B_t^H, t \in [0, T])$ be the standard fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$. The kernel $K_H$ is given by (2.7), i.e.

$$K_H(t, s) = c_H(t-s)^{H-\frac{1}{2}} + c_H \left(\frac{1}{2} - H\right) \int_s^t (u-s)^{H-\frac{1}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right) du$$

for $s < t$, where $c_H$ is given by (2.8). Its derivative, $\frac{\partial K}{\partial t}$, is thus of the form (2.9), i.e.

$$\frac{\partial K}{\partial t}(t, s) = c_H \left(H - \frac{1}{2}\right) \left(\frac{s}{t}\right)^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}}.$$ 

For the proof we refer to [33], p. 278-284, and for further reading to [2], p. 798. The conditions (C) and (S) are satisfied with $K = \frac{c}{H}$ where $c = c_H \left(\frac{1}{2} - H\right)$. To show (2.18) it is enough to notice that

$$\int_s^t K_H(t, r)^2 dr \leq \mathbb{E} \left| B_t^H - B_s^H \right|^2 = \mathbb{E} \left| B_t^H \right|^2 + \mathbb{E} \left| B_s^H \right|^2 - 2\mathbb{E} \left| B_t^H B_s^H \right|^2 = |t-s|^{2H}$$

for $s < t$, since $R(t, s)$ is of the form (1.7). As we shall see, by Proposition 2.3.4, the process $B^H$ has Hölder continuous paths up to the order $H$. This corresponds to previous results stated in Theorem 1.3.5.

Remark 2.3.3. Note that for the process with a singular kernel it can happen that

$$\int_s^T |K| (dt, s) = \infty$$

for every $s \in [0, T]$. As an example consider again the standard fractional Brownian motion $B^H = (B_t^H, t \in [0, T])$ with $H < \frac{1}{2}$. Then, for $s \in [0, T]$ the function $K_H(\cdot, s)$ is of the form (2.9). Hence,

$$\int_s^T |K| (dt, s) = \int_s^T \left| \frac{\partial K}{\partial t}(t, s) \right| dr = c_H \left(\frac{1}{2} - H\right) s^{\frac{1}{2}-H} s^{\frac{1}{2}-H} \int_s^T t^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}} dt.$$ 

The integral on the right diverges in the right neighbourhood of $s$ since $H < \frac{1}{2}$.

The continuity of the paths of the process $\beta$ is investigated in the next proposition. From Proposition 2.3.4 it is clear that the closer the parameter $\alpha$ to $\frac{1}{2}$ is, the rougher the paths become.

Proposition 2.3.4 (Hölder continuity of sample paths of singular processes). Let $\beta = (\beta_t, t \in [0, T])$ be a Gaussian Volterra process satisfying the condition (S) for some $0 < \alpha < \frac{1}{2}$. Then there exists a version of $\beta$ with Hölder continuous paths up to the order $\frac{1}{2} - \alpha$.

Proof. For $0 < s < t < T$ we obtain

$$\mathbb{E} |\beta_t - \beta_s|^2 = \mathbb{E} \left| \int_0^t K(t, r) dW_r - \int_0^s K(s, r) dW_r \right|^2 = \mathbb{E} \left| \int_0^s (K(t, r) - K(s, r)) dW_r + \int_s^t K(t, r) dW_r \right|^2$$
using the expression \((2.3)\). The stochastic integrals in the last term are independent random variables and thus, we arrive at

\[
\mathbb{E} |\beta_t - \beta_s|^2 = \int_s^t K(t,r)^2 dr + \int_0^s |K(t,r) - K(s,r)|^2 dr, \tag{2.19}
\]

by the Itô isometry. Since the kernel \(K\) enjoys the condition \((S)\) the first integral of \((2.19)\) can be estimated directly by \((2.18)\). Integrating the condition \((2.17)\) yields

\[
|K(t,s)| \leq \frac{c}{\alpha} (t-s)^{-\alpha} + C, \tag{2.20}
\]

where \(C\) is the integration constant. The expression \((2.20)\) is used to show that

\[
\mathbb{E} |\beta_t - \beta_s|^2 \leq c (t-s)^{1-2\alpha} + \frac{c^2}{\alpha^2} \int_0^s |(s-r)^{-\alpha} - (t-r)^{-\alpha}|^2 dr. \tag{2.21}
\]

Consider the integral from \((2.21)\). Using first the substitution \(u = s - r\) and then the substitution \(v = \frac{t-s}{u}\) we arrive at

\[
\int_0^s |(s-r)^{-\alpha} - (t-r)^{-\alpha}|^2 dr = \int_0^s \left(1 - \left(\frac{t-s}{u} + 1\right)^{-\alpha}\right)^2 u^{-2\alpha} du
\]

\[
= (t-s)^{1-2\alpha} \int_{\frac{1}{2}}^\infty (1 - (v + 1)^{-\alpha})^2 v^{2\alpha-2} dv.
\]

The last integral can be taken from 0 to \(\infty\) and as such, it can be evaluated using the Beta function, i.e. having the value of Beta\((1 - \alpha, 2\alpha - 1)\), provided that \(0 < \alpha < \frac{1}{2}\). This coincides with the condition \((S)\). Regardless of the precise value of this integral, we have that

\[
\mathbb{E} |\beta_t - \beta_s|^2 \leq (t-s)^{1-2\alpha} \left(c + \frac{c^2}{\alpha^2} \text{Beta}(1 - \alpha, 2\alpha - 1)\right) = C(t-s)^{1-2\alpha}.
\]

Since the process \(\beta\) is Gaussian, it is uniquely determined by its moments and thus, by Kolmogorov-Chencov theorem, Theorem 1.1.6, there exists a version of \(\beta\) with Hölder continuous paths of the order \(\frac{1}{2} - \alpha - \varepsilon\) for \(\varepsilon > 0\).

\[\square\]

### 2.4 Processes with Regular Kernel

In this section the conditions for a regular kernel are introduced and a process with a covariance function admitting the decomposition \((2.1)\) where \(K\) is a regular kernel is investigated. Let \(T > 0\).

**Definition 2.4.1** (Regular process). Suppose that \(\beta = (\beta_t, t \in [0,T])\) is a zero-mean Gaussian Volterra process which can be expressed in terms of a kernel \(K\) satisfying the condition \((2.2)\) which corresponds to the covariance function \(R\) of the form \((2.1)\). We say that the process \(\beta\) is regular or with a regular kernel on \([0,T]\), if it satisfies the condition

\((K')\) For all \(s \in [0,T]\), \(K(\cdot, s)\) has bounded variation on the interval \((s,T]\), and

\[
\int_0^T |K|((s,T],s)^2 ds < \infty.
\]
In order to provide a link between the processes with regular kernel and the standard fractional Brownian motion the sufficient conditions for the kernel \( K \) to be regular are given now. It is assumed that the kernel \( K \) enjoys the same condition \([C]\) as in the singular case and the condition \([R1]\) instead of \([S]\). This leads us to a definition of a strongly regular process.

**Definition 2.4.2 (Strongly regular process).** Suppose that \( \beta = (\beta_t, t \in [0, T]) \) is a zero-mean Gaussian Volterra process which can be expressed in terms of a kernel \( K \) satisfying the condition \((2.2)\) which corresponds to the covariance function \( R \) of the form \((2.1)\). By definition, the process \( \beta \) is strongly regular on \([0, T]\), if, for some \( \alpha \in (0, \frac{1}{2}) \), the kernel \( K \) satisfies the following:

\[ \left( C \right) \text{ For all } s \in (0, T) \text{ the function } K(\cdot, s) : (s, T] \to \mathbb{R} \text{ is differentiable in the interval } (s, T) \text{ and both } K(t, s) \text{ and its derivative } \frac{\partial K}{\partial t}(t, s) \text{ are continuous at every } t \in (s, T). \]

\[ \left( R1 \right) \text{ There exists a finite constant } c > 0 \text{ such that } \]

\[
\left| \frac{\partial K}{\partial t}(t, s) \right| \leq c(t - s)^{\alpha - 1}s^{-\alpha}, \quad (2.22)
\]

\[
\int_s^t K(t, r)^2 dr \leq c(t - s)^{2\alpha + 1} \quad (2.23)
\]

for \( 0 \leq s < t \leq T \).

Strongly regular kernels fulfil the condition \([C]\) and thus, by Proposition 2.3.1, they have bounded variation in the first variable on any interval \([u, T]\), \( u > s \). Definition 2.4.2 is justified by the following proposition.

**Proposition 2.4.3 (Strongly regular is regular).** Let \( K \) be a kernel enjoying the conditions \([C]\) and \((2.22)\). Then, the condition \([K']\) holds.

**Proof.** By \((2.22)\) the measure \( K \) is absolutely continuous with respect to the Lebesgue measure and thus,

\[
\int_0^T |K|(\cdot, (s, T], s)^2 \, ds = \int_0^T \left( \int_s^T \left| \frac{\partial K}{\partial t}(t, s) \right| \, dt \right)^2 \, ds \\
\leq c^2 \int_0^T s^{-2\alpha} \left( \int_s^T (t - s)^{\alpha - 1} \, dt \right)^2 \, ds \\
= \frac{c^2}{\alpha^2} \int_0^T s^{-2\alpha} (T - s)^{2\alpha} \, ds < \infty
\]

since \( \alpha \in (0, \frac{1}{2}) \).

Thus, the conditions \([C]\) and \([R1]\) are indeed sufficient for the process \( \beta \) to be regular. For the next section, it is useful to introduce even more restrictive conditions on the kernel \( K \).

**Definition 2.4.4 (Strictly regular process).** Suppose that \( \beta = (\beta_t, t \in [0, T]) \) is a zero-mean Gaussian Volterra process which can be expressed in terms of a kernel \( K \) satisfying the condition \((2.2)\) which corresponds to the covariance function \( R \) of the form \((2.1)\). By definition, the process \( \beta \) is strictly regular on \([0, T]\), if, for some \( \alpha \in (0, \frac{1}{2}) \), the kernel \( K \) satisfies the following:
(C) For all $s \in (0, T)$ the function $K(\cdot, s) : (s, T] \to \mathbb{R}$ is differentiable in the interval $(s, T)$ and both $K(t, s)$ and its derivative $\frac{\partial K}{\partial t}(t, s)$ are continuous at every $t \in (s, T)$.

(R2) There exist a constant $c > 0$ such that
\[
\left| \frac{\partial K}{\partial t}(t, s) \right| \leq c(t - s)^{\alpha - 1} \left( \frac{s}{t} \right)^{-\alpha}, \quad (2.24)
\]
and its derivative
\[
\int_s^t K(t, r)^2 dr \leq c(t - s)^{2\alpha + 1}
\]
for $0 \leq s < t \leq T$.

Remark 2.4.5. The only difference between Definition 2.4.2 and Definition 2.4.4 is in the condition (2.24). It is clear, that this condition implies the condition (2.22) and thus, every strictly regular Gaussian Volterra process is strongly regular and, by Proposition 2.4.3, it is regular as well.

Remark 2.4.6. The purpose that lies behind the definition of strongly regular processes is that one may wish derive an analogue of the classical Itô formula also for the integrals driven by Gaussian Volterra processes. In the work of Alòs, Nualart and Mazet (24), there is such a formula proposed for a stochastic integral defined as a divergence operator using the Malliavin Calculus (we refer to [33] for a thorough introduction into the topic). This formula is valid for a strongly regular Gaussian Volterra process. Since the class of strongly regular kernels is wider than the class of strictly regular kernels, it is natural to ask why the strictly regular process is defined. It shall be clear from Proposition 2.4.9 that the reproducing kernel Hilbert space of a strictly regular process can be embedded in the space $L^{\alpha + 2\alpha}([0, T]; V)$ where $V$ is a separable Hilbert space. Generally, if only the strong regularity is assumed, this is not true. There is also an important example of a strictly regular Gaussian Volterra process which is the standard fractional Brownian motion.

Example 2.4.7. Let $T > 0$ and let $B^H = (B^H_t, t \in [0, T])$ be the standard fractional Brownian motion with the Hurst parameter $H > \frac{1}{2}$. Then the kernel $K_H$ has a simpler expression (see [33], p. 278)
\[
K_H(t, s) = c_H \left( H - \frac{1}{2} \right) s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - \frac{1}{2}} u^{-\frac{1}{2}} du \quad (2.25)
\]
and its derivative is still of the form (2.9), i.e.
\[
\frac{\partial K_H}{\partial t}(t, s) = c_H \left( H - \frac{1}{2} \right) \left( \frac{s}{t} \right)^{\frac{1}{2} - H} (t - s)^{H - \frac{3}{2}}.
\]
Thus, the conditions (C) and (R2) hold with $\alpha = H - \frac{1}{2}$ and $c = c_H \left( H - \frac{1}{2} \right)$. Therefore, the standard fractional Brownian motion is a strictly regular Gaussian Volterra process.

Similarly, for the regular case of the Liouville fractional Brownian motion we have that
\[
\frac{\partial K_H^L}{\partial t}(t, s) = \frac{H - \frac{1}{2}}{\Gamma \left( H + \frac{3}{2} \right)} (t - s)^{H - \frac{3}{2}} \leq \frac{H - \frac{1}{2}}{\Gamma \left( H + \frac{3}{2} \right)} \left( \frac{t}{s} \right)^{H - \frac{3}{2}} (t - s)^{H - \frac{3}{2}},
\]
since $t > s$ and $H > \frac{1}{2}$.
In the following proposition the fractional Brownian motion and its kernel $K_H$ is used to show a result for the continuity of the paths of strictly regular Gaussian processes which is analogous to the singular case.

**Proposition 2.4.8** (Hölder continuity of strictly regular processes). Let $T > 0$ and $\beta = (\beta_t, t \in [0, T])$ be a Gaussian Volterra process satisfying the condition [R1] for some $0 < \alpha < \frac{1}{2}$. Then there exists a version of $\beta$ with Hölder continuous paths up to the order $\frac{1}{2} + \alpha$.

**Proof.** It is obvious, from the Example 2.4.7 that the standard fractional Brownian motion is a limiting case of the strictly regular Gaussian processes, with the kernel $K$ satisfying the condition (2.24) with equality if we set $\alpha = H - \frac{1}{2}$. Therefore, the assertion of Theorem 1.3.5 holds for all strictly regular processes. Thus, we have the existence of a version with Hölder continuous sample paths up to the order $H$, that is up to the order $\alpha + \frac{1}{2}$.

In the regular case, the operator $K^*$ defined by (2.4) can be expressed for some $\varphi \in \mathcal{E}$ and $0 < s < T$ as

$$(K^*\varphi)(s) = \varphi(s)K(s^+, s) + \int_s^T \varphi(t)K(dt, s),$$

(2.26)

where

$$K(s^+, s) = K(T, s) - K((s, T], s),$$

which is by the condition [KΩ] square integrable in $[0, T]$. Before we prove the main result of this section the space $|\mathcal{H}|$ is introduced. Let $\varphi : [0, T] \to V$ be a Borel measurable function and define the functional $\|\cdot\|_{|\mathcal{H}|}$ as

$$\|\varphi\|_{|\mathcal{H}|}^2 = \int_0^T \int_0^T |\varphi(t)||\varphi(r)||r - t|^{2\alpha - 1} dr dt.$$ 

Further, let $|\mathcal{H}|$ be the linear space of Borel measurable functions $\varphi : [0, T] \to V$ such that $\|\varphi\|_{|\mathcal{H}|}^2 < \infty$. If we let $V = \mathbb{R}$, it has been shown (see [33]) that $|\mathcal{H}|$ is a Banach space with the norm $\|\cdot\|_{|\mathcal{H}|}$ and $\mathcal{E}$ is dense in $|\mathcal{H}|$. On the other hand, this space equipped with the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is not complete and it is isometric to a subspace of $\mathcal{H}_\mathbb{R}$ (see [35]). Further, in this case, we have that

$$\|\varphi\|_{|\mathcal{H}|}^2 \leq c_\alpha \|\varphi\|_{L^{\frac{2\alpha}{\alpha-1}}(0, T)}^2.$$

We refer to [33] for further reading. Led by this motivation we are ready to prove the following theorem.

**Theorem 2.4.9** (Integrable functions). Let $K$ be a regular kernel (so it only satisfies the condition [KΩ]). Consider the seminorm

$$\|\varphi\|_{\mathcal{H}_R}^2 := \int_0^T |\varphi(s)|_V^2 K(s^+, s)^2 ds + \int_0^T \left( \int_s^T |\varphi(t)|_V |K|(dt, s) \right)^2 ds,$$  

(2.27)

defined on $\mathcal{E}$. Denote by $\mathcal{H}_R$ the completion of $\mathcal{E}$ with respect to $\|\cdot\|_{\mathcal{H}_R}$. Then $\mathcal{H}_R$ is continuously embedded in $\mathcal{H}$. 

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1. There exists a finite constant $c_1 > 0$ such that 
\[ \|\varphi\|_{\mathcal{H}} \leq c_1 \|\varphi\|_{L^\infty([0,T];V)} \]
for all $\varphi \in L^\infty([0,T];V)$.

Suppose further that $K(s^+,s) = 0$ for all $0 < s < T$. In such a case, the kernel $K$ is called vanishing. Hence, the next two assertions can be formulated as follows:

2. If $K$ is a strongly regular vanishing kernel (so it enjoys the conditions (C) and (R1)) then there exists a finite constant $c_2(\alpha) > 0$ such that 
\[ \|\varphi\|_{\mathcal{H}} \leq c_2(\alpha) \|u_{-\alpha}(\varphi)\|_{L^{\frac{2}{1+2\alpha}}([0,T];V)} \]
for all $\varphi$ such that $u_{-\alpha}(\varphi) \in L^{\frac{2}{1+2\alpha}}([0,T];V)$ where 
\[ u_{-\alpha}(\varphi)(s) = \varphi(s)s^{-\alpha}, \quad 0 < s < T. \]

3. If $K$ is a strictly regular vanishing kernel (so it enjoys the conditions (C) and (R2)) then there exists a finite constant $c_3(\alpha) > 0$ such that 
\[ \|\varphi\|_{\mathcal{H}} \leq c_3(\alpha) \|\varphi\|_{L^{\frac{2}{1+2\alpha}}([0,T];V)} \]
for all $\varphi \in L^{\frac{2}{1+2\alpha}}([0,T];V)$.

**Proof.** From Proposition 2.4.3 it follows that the first integral in (2.27) is well defined. Let $\varphi \in \mathcal{E}$. Then
\[
\|\varphi\|_{\mathcal{H}}^2 = \|K^*\varphi\|^2_{L^2([0,T],V)}
= \int_0^T \left| K(s^+,s)\varphi(s) + \int_s^T \varphi(t)|K|(dt,s) \right|^2_V ds
\leq 2 \int_0^T (K(s^+,s)|\varphi(s)|_V)^2 ds + 2 \int_0^T \left( \int_s^T |\varphi(t)|_V|K|(dt,s) \right)^2 ds
\]
which is
\[ \|\varphi\|_{\mathcal{H}}^2 \leq 2\|\varphi\|_{\mathcal{H}^R}^2. \]

Thus $\mathcal{H}_R$ is continuously embedded in $\mathcal{H}$ (analogous results can be found in [2]).

The first part of the theorem follows directly from \([K^*]\) and the following:
\[ \|\varphi\|_{\mathcal{H}}^2 \leq 2\|\varphi\|_{\mathcal{H}_R}^2 \leq 2\|\varphi\|^2_{L^\infty([0,T];V)} \left( \int_0^T K(s^+,s)^2 ds + \int_0^T |K|((s,T],s)^2 ds \right). \]

Suppose now that $K(s^+,s) = 0$ for all $0 < s < T$. To show the second part of the theorem we use the Fubini theorem to obtain
\[
\|\varphi\|_{\mathcal{H}}^2 = \int_0^T \left( \int_s^T \varphi(t)|K|(dt,s) \right)^2_V ds
\leq \int_0^T \left( \int_s^T |\varphi(t)|_V|K|(dt,s) \right) \left( \int_s^T |\varphi(r)|_V|K|(dr,s) \right) ds
= \int_0^T \int_0^T |\varphi(t)|_V|\varphi(r)|_V \left( \int_0^{t\wedge r} \left| \frac{\partial K}{\partial t}(t,s) \cdot \frac{\partial K}{\partial r}(r,s) \right| ds \right) dr dt.
\]
Hence by the condition (2.22) it follows that
\[
\| \varphi \|_H^2 \leq c^2 \int_0^T \int_0^T |\varphi(t)||\varphi(r)||V\left(\int_0^{t\wedge r} s^{-2\alpha}(t-s)^{\alpha-1}(r-s)^{\alpha-1} ds\right) dr dt.
\]
The last integral can be evaluated using the substitution
\[
z = \frac{t - s}{r - s}, \quad x = \frac{t}{r z}
\]
for \( r < t \). Hence
\[
\int_0^r s^{-2\alpha}(t-s)^{\alpha-1}(r-s)^{\alpha-1} ds = -(r-t)^{2\alpha-1} \int_t^\infty z^{\alpha-1}(t-zr)^{-2\alpha} dz = (tr)^{-\alpha}(r-t)^{2\alpha-1} \int_0^1 x^{\alpha-1}(1-x)^{-2\alpha} dx
\]
and thus, by the H"older’s inequality,
\[
\| \varphi \|_H^2 \leq c_0(\alpha) \int_0^T \int_0^T |\varphi(t)||\varphi(r)||V|(tr)^{-\alpha}(r-t)^{2\alpha-1} dr dt \leq c_0(\alpha) \left( \int_0^T \left( |\varphi(t)||Vt^{-\alpha}| \right)^{\frac{2}{1+2\alpha}} dt \right) \frac{1}{1+\alpha} \times \\
\times \left( \int_0^T \left( \int_0^T |\varphi(r)||Vr^{-\alpha}|r-t|^{2\alpha-1} dr \right)^{\frac{2}{1+2\alpha}} dt \right) \frac{1}{1-\alpha}
\]
The first integral coincides with the \( L^{\frac{2}{1+2\alpha}} \)-norm of the function \( u_{-\alpha}(|\varphi|_V) \) and the second factor, up to a multiplicative constant, is equal to the \( L^{\frac{2}{1-2\alpha}} \)-norm of the left-sided fractional integral \( I_{0+}^{2\alpha}(u_{-\alpha}(|\varphi|_V)) \). Hence, by Theorem 1.2.4, it follows that
\[
\| \varphi \|_H^2 \leq c_2(\alpha)\|u_{-\alpha}(|\varphi|_V)\|^{\frac{2}{L^{\frac{2}{1+2\alpha}}([0,T];V)}}.
\]
As it has been already mentioned in Remark 2.4.6 any strictly regular kernel is also strongly regular and thus it is regular. To prove the last part of the statement it is enough to notice that, following the same procedure as in the strongly regular case, (2.28) becomes
\[
\| \varphi \|_H^2 \leq c_0(\alpha) \int_0^T \int_0^T |\varphi(t)||\varphi(r)||V|r-t|^{2\alpha-1} dr dt = c_0(\alpha) \| \varphi \|_H^2
\]
and, by Theorem 1.2.4 we get
\[
\| \varphi \|_H^2 \leq c_0(\alpha)\|\varphi\|_H^2 \leq c_2(\alpha)\|\varphi\|^{\frac{2}{L^{\frac{2}{1+2\alpha}}([0,T];V)}}
\]
which completes the proof.
Remark 2.4.10. The condition $K(s^+, s) = 0$ for all $0 < s < T$, i.e. the vanishing property, is justifiable since it coincides with the case of the standard fractional Brownian motion $B^H$ for $H > \frac{1}{2}$. Indeed,

$$\lim_{t \to s^+} K_H(t, s) = c_H \left( H - \frac{1}{2} \right) s^{\frac{1}{2} - H} \lim_{t \to s^+} \int_s^t (u - s)^{H - \frac{3}{2}} u^{H - \frac{3}{2}} du = 0$$

by the Dominated Convergence Theorem and the expression (2.23) for $K_H$. Similarly, for the Liouville fractional Brownian motion it follows that

$$\lim_{t \to s^+} K_L(t, s) = \frac{1}{\Gamma \left( H + \frac{1}{2} \right)} \lim_{t \to s^+} (t - s)^{H - \frac{1}{2}} = 0,$$

since, again, $H > \frac{1}{2}$. In fact, the vanishing property says that the process $\beta$ has no Brownian component and that it is smoother than the standard Brownian motion. This follows from (2.12).

Remark 2.4.11. The standard fractional Brownian motion and the Liouville fractional Brownian motion are examples of strictly regular vanishing processes on $[0, T]$. Since from Theorem 2.4.9 it follows that the space $L^2([0, T]; \mathcal{V})$ consists of integrable functions with respect to strictly regular processes we shall mainly focus on these processes in the sequel. The class of strongly regular processes does not give, in general, such a space. This is, however, to be expected since the condition on the kernel, which defines the class of strongly regular processes, is weaker.

Corollary 2.4.12 (Integrability with respect to a strictly continuous vanishing process). Let $K$ be a kernel satisfying the most restrictive set of conditions from Theorem 2.4.9 that is:

1. For all $s \in (0, T)$ the function $K(\cdot, s) : (s, T) \to \mathbb{R}$ is differentiable in the interval $(s, T)$ and both $K(t, s)$ and its derivative $\frac{\partial K}{\partial t}(t, s)$ are continuous at every $t \in (s, T)$.

2. There exists a finite constant $c > 0$ such that for some $\alpha \in (0, \frac{1}{2})$

   (i) $\left| \frac{\partial K}{\partial t}(t, s) \right| \leq c(t - s)^{\alpha - 1} \left( \frac{t}{s} \right)^{-\alpha},$

   (ii) $\int_s^t K(t, r)^2 dr \leq c(t - s)^{2\alpha + 1}$

   holds for all $0 < s < t < T$.

3. $K(s^+, s) = 0$ for all $0 < s < T$.

Then

$$L^2([0, T]; \mathcal{V}) \subset L^{\frac{2}{1 + 2\alpha}}([0, T]; \mathcal{H}) \subset \mathcal{H} \subset \mathcal{H}_R \subset \mathcal{H},$$

all the embeddings being continuous.

Proof. Under these assumptions the inequalities from the proof of Theorem 2.4.9 can be written as

$$\|\varphi\|_{\mathcal{H}}^2 \leq c_0 \|\varphi\|_{\mathcal{H}_R}^2 \leq c_1(\alpha)\|\varphi\|_{\mathcal{H}}^2 \leq c_3(\alpha)\|\varphi\|_{L^{\frac{2}{1 + 2\alpha}}([0, T]; \mathcal{V})}^2 \leq c_4(\alpha)\|\varphi\|_{L^2([0, T]; \mathcal{V})}.$$  

The last inequality follows from the Hölder’s inequality since for $\alpha \in (0, \frac{1}{2})$ it holds that $\frac{2}{1 + 2\alpha} > 1$. 

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Example 2.4.13. Let $T > 0$ and let $B^H = (B^H_t, t \in [0, T])$ be the standard fractional Brownian motion with the Hurst parameter $H > \frac{1}{2}$. We have seen in Example 2.4.7 and in Remark 2.4.10 that $B^H$ is a strictly regular vanishing Gaussian Volterra process. As such, the kernel $K_H$, when $H > \frac{1}{2}$, enjoys the conditions from Corollary 2.4.12 and therefore, we have that

$$L^2([0, T]; V) \subset L^\frac{1}{H}([0, T]; V) \subset |H| \subset H \subset \mathcal{H}.$$

2.5 One-dimensional Stochastic Integral

For the convenience of the reader, the results for Gaussian Volterra processes with regular kernels are firstly summarized and then some further properties of the one-dimensional stochastic integral $I(\varphi)$ with respect to a regular Gaussian Volterra process are discussed. The integral $I(\varphi)$ will be used in the following section as a foundation for an infinite-dimensional stochastic integral of an operator-valued function.

Let $T > 0$ and $\beta = (\beta_t, t \in [0, T])$ be a Gaussian Volterra process with a kernel $K$ defined in the previous section, that is, the covariance function $R$ of the process $\beta$ admits the factorization

$$R(t, s) = \int_0^{t \wedge s} K(t, r)K(s, r)dr,$$

where $K : \Delta \to \mathbb{R}$ is a kernel satisfying

$$\sup_{t \in [0, T]} \int_0^t K(t, s)^2 ds < \infty.$$

We further suppose that the process $\beta$ can be expressed as

$$\beta_t = \int_0^t K(t, s)dW_s, \quad \mathbb{P} \text{- a.s.},$$

where $W = (W_t, t \in [0, T])$ is the standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. If the kernel $K$ is assumed to satisfy the continuity condition

(C) For all $s \in (0, T)$ the function $K(\cdot, s) : (s, T] \to \mathbb{R}$ is differentiable in the interval $(s, T)$ and both $K(t, s)$ and its derivative $\frac{\partial K}{\partial t}(t, s)$ are continuous at every $t \in (s, T)$.

and the condition:

(R1) There exists a finite constant $c > 0$ such that

$$\left| \frac{\partial K}{\partial t}(t, s) \right| \leq c(t - s)^{\alpha - 1}s^{-\alpha},$$

$$\int_s^t K(t, r)^2 dr \leq c(t - s)^{2\alpha + 1}$$

for $0 \leq s < t \leq T$.  

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in which case the process $\beta$ is called strongly regular on $[0, T]$, or even a more restrictive condition:

\[(R2)\, \text{There exists a finite constant } c > 0 \text{ such that}\]
\[
\left| \frac{\partial K}{\partial t} (t, s) \right| \leq c(t - s)^{\alpha - 1} \left( \frac{s}{t} \right)^{-\alpha},
\]
\[
\int_s^t K(t, r)^2 dr \leq c(t - s)^{2\alpha + 1}
\]

for $0 \leq s < t \leq T,$

by which it is called strictly regular on $[0, T]$, then the process $\beta$ also, by Proposition 2.4.3, enjoys the condition

\[(K')\, \text{For all } s \in [0, T), K(\cdot, s) \text{ has bounded variation on the interval } (s, T], \text{ and}\]
\[
\int_0^T |K| ((s, T], s)^2 ds < \infty.
\]

Thus, the process $\beta$ is a Gaussian Volterra process with a regular kernel on $[0, T]$. Let $V$ be a real separable Hilbert space and denote $E$ the set of $V$-valued step functions, i.e.

\[E := \{ f : [0, T] \rightarrow V, f(s) = \sum_{i=1}^n f_i 1(t_i, t_{i+1})(s), \]

\[f_i \in V, i \in \{1, \cdots, n\}, 0 = t_1 \leq t_2 \leq \cdots \leq t_{n+1} = T, n \in \mathbb{N}\}.\]

Then the operator $K^* : E \rightarrow L^2 ([0, T]; V)$ can be defined for $\varphi \in E$ by

\[(K^* \varphi)(s) = K(s+, s)\varphi(s) + \int_s^T \varphi(t)K(dt, s), \quad s \in [0, T].\]

If $K(s+, s) = 0$ for all $s \in [0, T]$ the kernel $K$ is called vanishing. The operator $K^*$ is the adjoint of $K$ in the sense provided by Proposition 2.1.2 which can be extended to a larger class of functions $\mathcal{H}$. This denotes the completion of $E$ under the inner product

\[\langle \varphi, \psi \rangle_{\mathcal{H}} = \langle K^* \varphi, K^* \psi \rangle_{L^2([0, T], V)}, \quad \varphi, \psi \in \mathcal{H}.\]

It is very important that by Proposition 2.2.4 the process $\beta$ can be expressed for $\varphi \in \mathcal{H}$ as

\[\beta(\varphi) = \int_0^T (K^* \varphi)(t)dW_t, \quad \mathbb{P} - \text{a.s.},\]

and hence, it provides a straightforward link to the Gaussian isonormal processes since for a fixed $\varphi \in \mathcal{H}$, $\beta(\varphi)$ represents a $V$-valued Gaussian random variable with zero mean. Taken as a process indexed by elements of $\mathcal{H}$ the collection of $V$-valued random variables $(\beta(\varphi), \varphi \in \mathcal{H})$ is a $V$-valued centered Gaussian process and thus it represents a $\mathcal{H}$-isonormal Gaussian process exactly as stated in Remark 2.2.5.

Proposition 2.2.4 gives the existence of a one-dimensional stochastic integral which takes $\beta$ as a driving process. Such an integral $I(\varphi)$ is understood to be an
extension of the mapping \( I : (\mathcal{E}, \langle \cdot, \cdot \rangle_\mathcal{H}) \to L^2((\Omega, \mathcal{F}, \mathbb{P}); V) \) defined by \( x \mathbf{1}_{[0,t]} \to x \beta_t \) for \( x \in V \) and \( t \in [0,T] \) onto the whole space \( \mathcal{H} \). Hence, the space \( \mathcal{H} \) represents the space of all \( \beta \)-integrable functions. From the proof of Proposition 2.2.4, the second part in particular, we have that

\[
\varphi = \sum_{i=1}^{n} \varphi_i \left( \mathbf{1}_{[0,t_{i+1}]} - \mathbf{1}_{[0,t_i]} \right) \overset{I}{\to} \sum_{i=1}^{n} \varphi_i \left( \beta_{t_{i+1}} - \beta_{t_i} \right) = I(\varphi)
\]

for \( \varphi \in \mathcal{E} \) which means that the integral \( I(\varphi) =: \int_0^T \varphi \, d\beta \) can be alternatively defined as

\[
\int_0^T \varphi \, d\beta := \sum_{i=1}^{n} \varphi_i \left( \beta_{t_{i+1}} - \beta_{t_i} \right), \tag{2.29}
\]

for some partition \( \{0 = t_1 \leq t_2 \leq \cdots \leq t_{n+1} = T\} \). The extension is therefore an \( L_2 \)-limit of (2.29) over partitions of \( [0,T] \) such that their norm \( \text{sup}_{1 \leq i \leq n} |t_{i+1} - t_i| \) tends to 0 as \( n \to \infty \) since \( \mathcal{E} \) is a dense subset of \( (\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H}) \). We shall hereafter identify the notation \( \int_0^T \varphi \, d\beta \) with \( I(\varphi) \). The following properties of the one-dimensional integral are direct corollaries of Theorem 2.4.9.

**Theorem 2.5.1.** Suppose that the process \( \beta \) is regular, i.e. its kernel satisfies only the condition \([K']\). Then \( L^\infty([0,T]; V) \subset \mathcal{H}_R \subset \mathcal{H} \) and the following inequality holds:

\[
\mathbb{E} \left| \int_0^T \varphi \, d\beta \right|^2_V \leq 2\|\varphi\|_{\mathcal{H}_R}^2, \quad \varphi \in \mathcal{H}_R.
\]

Moreover, there exists a finite constant \( c_1 > 0 \) such that

\[
\mathbb{E} \left| \int_0^T \varphi \, d\beta \right|^2_V \leq c_1 \|\varphi\|^2_{L^\infty([0,T]; V)}, \quad \varphi \in L^\infty([0,T]; V).
\]

Suppose further that \( K \) is a strictly regular vanishing kernel, i.e. it satisfies the assumptions \([C]\) and \([R2]\) with \( 0 < \alpha < \frac{1}{2} \) and \( K(s^+, s) = 0 \) holds for all \( 0 < s < T \). Then, namely, \( L^\infty([0,T]; V) \subset \mathcal{H} \) and there exists a finite constant \( c_3 > 0 \) such that

\[
\mathbb{E} \left| \int_0^T \varphi \, d\beta \right|^2_V \leq c_3 \|\varphi\|^2_{L^\infty([0,T]; V)}.
\]

**Proof.** By Proposition 2.2.4 and the Itô isometry for the standard Brownian motion we arrive at

\[
\mathbb{E} \left| \int_0^T \varphi \, d\beta \right|^2_V = \mathbb{E} \left| \int_0^T (K^* \varphi) \, dW \right|^2_V = \int_0^T \|(K^* \varphi) \|_V^2 \, ds = \|K^* \varphi\|^2_{L^2([0,T]; V)}
\]

for \( \varphi \in \mathcal{H} \), from which it follows that

\[
\mathbb{E} \left| \int_0^T \varphi \, d\beta \right|^2_V = \|\varphi\|_{\mathcal{H}}^2,
\]

by Lemma 2.2.3. The rest of the statement follows directly as a corollary of Theorem 2.4.9. \( \square \)
Proposition 2.5.2 (I(ϕ) as a random variable). The stochastic integral \( I(ϕ) \) defined for \( ϕ \in \mathcal{H} \) is a zero-mean \( V \)-valued Gaussian random variable with covariance operator \( q_T : V \to V \). Moreover, the operator \( q_T \) is of the form

\[
q_T x = \int_0^T \langle (K^* ϕ)(s), x \rangle_V (K^* ϕ)(s) ds, \quad x \in V.
\]  

(2.30)

Proof. Denote \( \mathcal{B}(V) \) the Borel \( σ \)-algebra on \( V \) and then consider the measurable space \((V, \mathcal{B}(V))\). For a fixed \( ϕ \in \mathcal{H} \), the mapping \( I(ϕ) : (Ω, \mathcal{F}, \mathbb{P}) \to (V, \mathcal{B}(V)) \) induces a measure \( μ_{I(ϕ)} \) in the following way

\[
μ_{I(ϕ)}(A) = (I^{-1} \circ \mathbb{P})(A), \quad A \in \mathcal{B}(V),
\]

which is a probability measure on \( \mathcal{B}(V) \). By definition, the covariance operator \( q_T \) is an operator \( q_T : V \to V \) such that

\[
\langle q_T x, y \rangle_V = \int_V \langle x, z \rangle_V \langle y, z \rangle_V dμ_{I(ϕ)}(z)
\]

for all \( x, y \in V \). Since the mapping \( I(ϕ) \) is \( \mathcal{F} \)-measurable, by substitution theorem, we obtain

\[
\int_V \langle x, z \rangle_V \langle y, z \rangle_V dμ_{I(ϕ)}(z) = \int_Ω (\langle x, \cdot \rangle_V \langle y, \cdot \rangle_V \circ I(ϕ)) (ω) d\mathbb{P}(ω)
\]

and hence we arrive at

\[
\langle q_T x, y \rangle_V = \mathbb{E} \langle x, I(ϕ) \rangle_V \langle y, I(ϕ) \rangle_V
\]  

(2.31)

By Proposition 2.2.4 we can write

\[
\int_0^T ϕ(s) dβ_s = \int_0^T (K^* ϕ)(s) dW_s, \quad \mathbb{P} \text{ - a.s.}
\]  

(2.32)

for \( ϕ \in \mathcal{H} \), which is an equality of two elements of \( L^2((Ω, \mathcal{F}, \mathbb{P}); V) \). Hence, since the term on the right-hand side of (2.32) is an \( V \)-valued Itô integral with respect to a standard Brownian motion, it follows immediately that \( I(ϕ) \) is Gaussian and zero-mean \( V \)-valued random variable. Thus, the probability measure \( μ_{I(ϕ)} \) is centred and Gaussian. By (2.31) and (2.32) we have that

\[
\langle q_T x, y \rangle_V = \mathbb{E} \left( \int_0^T (K^* ϕ)(s) dW_s, x \right)_V \left( \int_0^T (K^* ϕ)(s) dW_s, y \right)_V.
\]

Passing to the limit from step functions and by continuity of the scalar product it can be verified that

\[
\left( \int_0^T (K^* ϕ)(s) dW_s, x \right)_V = \int_0^T \langle (K^* ϕ)(s), x \rangle_V dW_s, \quad \mathbb{P} \text{ - a.s.},
\]

which leads to

\[
\langle q_T x, y \rangle_V = \mathbb{E} \left( \int_0^T \langle (K^* ϕ)(s), x \rangle_V dW_s \right) \int_0^T \langle (K^* ϕ)(s), y \rangle_V dW_s
\]

\[
= \int_0^T \langle (K^* ϕ)(s), x \rangle_V \langle (K^* ϕ)(s), y \rangle_V ds
\]

\[
= \int_0^T \langle (K^* ϕ)(s), x \rangle_V (K^* ϕ)(s), y \rangle_V ds
\]

\[
= \left( \int_0^T \langle (K^* ϕ)(s), x \rangle_V (K^* ϕ)(s), ds, y \right)_V.
\]

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Remark 2.5.3. The trace of the covariance operator $q_T$ can be expressed in terms of the norm $\| \cdot \|_H$. Indeed,

$$\text{Tr } q_T = E[I(\varphi)|_V^2 = \|K^* \varphi\|^2_{L^2([0,T];V)} = \| \varphi \|^2_H.$$  

### 2.6 Cylindrical Volterra Processes

The aim of the following section is to define an analogue to an infinitely dimensional Brownian motion, so called the cylindrical Brownian motion. The motivation for such a definition is to investigate, in the third chapter, the linear stochastic differential equation, formally written as

$$dX_t = AX_t dt + \Phi dB_t.$$  

The source of random perturbation in such an equation is a stochastic process $B = (B_t, t \in [0,T])$, so called the cylindrical Volterra process, in a separable Hilbert space $U$. Precise formulation of this problem is the content of the following chapter. In this section we only define the driving process. Similar construction can be found in [14].

**Definition 2.6.1** (Cylindrical Gaussian Volterra process). Let $K$ be a regular kernel on $[0,T]$, i.e. only the condition $[K]$ is satisfied. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $U$ be a real separable Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle_U$ and $T > 0$. The mapping $B : \Omega \times [0,T] \times U \to \mathbb{R}$ is called the cylindrical Gaussian Volterra process with a regular kernel on $[0,T]$ if

1. For all $h \in U, h \neq 0$, the mapping

$$(t, \omega) \mapsto \frac{B_t(h, \omega)}{|h|^2_U}$$  

is a $(\mathbb{R}^d$-valued) Gaussian Volterra process associated with the kernel $K$.

2. For all $\alpha_1, \alpha_2 \in \mathbb{R}$ and $h_1, h_2 \in U$ we have that

$$B_t(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 B_t(h_1) + \alpha_2 B_t(h_2), \quad \mathbb{P} \text{ - a.s.}$$  

**Remark 2.6.2.** Consider the example of the standard fractional Brownian motion with Hurst parameter $H$ and its kernel $K_H$. If we set $H = \frac{1}{2}$ we get the standard Brownian motion since then the covariance function is of the form $R(s,t) = s \wedge t$. In such a case, Definition 2.6.1 coincides with the usual definition of a cylindrical Brownian motion (see [16]).

**Remark 2.6.3.** The cylindrical Gaussian Volterra process with a regular kernel on $[0,T]$ is defined as a family of random linear functionals on $U$. Thus, the notation $B_t(h)$ can be interpreted as the evaluation of the functional $B_t$ at $h$ even though the process $B$ is not $U$-valued as we shall see in the following example.
Example 2.6.4. Let $\beta$ be a real centred continuous Gaussian process with covariance function $R$ and suppose further that this function can be expressed as (2.1) with a kernel $K$ satisfying (2.2). We further suppose that the process is Volterra, i.e. having a representation in the form (2.3). Let $U$ be a real separable Hilbert space with a complete orthonormal basis $\{e_n, n \in \mathbb{N}\}$. Consider further the sequence $\{\beta^n_t, n \in \mathbb{N}, t \in [0,T]\}$ of independent copies of $\beta$. In particular, their kernels are represented by the same function. Consider now the series
\[
\tilde{B}_t = \sum_{n=1}^{\infty} \beta^n_t e_n. \tag{2.33}
\]
Even though the sum (2.33) does not converge in $L^2(\mathbb{P})$ and therefore it does not define a $U$-valued random variable, the series (2.33) defines a cylindrical Gaussian Volterra process. Indeed, for $h \in U$, $h \neq 0$ set
\[
\tilde{B}_t(h) = \sum_{n=1}^{\infty} \beta^n_t \langle e_n, h \rangle_U.
\]
Linearity of $\tilde{B}_t$ follows from linearity of the inner product $\langle \cdot, \cdot \rangle_U$. To show that the mapping $(t, \omega) \mapsto B_t(h, \omega)|_{h^2_U}$ is a Gaussian Volterra process with kernel $K$ we only need to consider the covariance of $B_t$ and $B_s$ for $s < t$:
\[
\mathbb{E} \tilde{B}_t(h) \tilde{B}_s(h) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathbb{E} \beta^n_t \beta^m_s \langle e_n, h \rangle_U \langle e_m, h \rangle_U = \sum_{n=1}^{\infty} \mathbb{E} \beta^n_t \beta^n_s \langle e_n, h \rangle_U^2
\]
since the elements of the sequence $\{\beta^n, n \in \mathbb{N}\}$ are mutually independent. The term $\mathbb{E} \beta^n_t \beta^m_s$ in (2.34) does not depend on $n$ since all the processes $\beta^n$ have the same covariance function and thus, we arrive at
\[
\mathbb{E} \tilde{B}_t(h) \tilde{B}_s(h) = R(t, s) |h|_U^2,
\]
which yields the result. We have shown that given a real centred continuous Gaussian Volterra process with covariance function $R$ and a corresponding kernel $K$ a cylindrical Gaussian Volterra process can always be constructed.

Remark 2.6.5. From the Example 2.6.4 it is clear that the regularity assumption in Definition 2.6.1 is unnecessary; however, we are mainly interested in the regular processes and thus, the general framework is restricted only to these.

Remark 2.6.6. As stated in the Example 2.6.4 the sum (2.33) does not converge in $L^2(\mathbb{P})$ and therefore it does not define a $\tilde{U}$-valued random variable; however, for any Hilbert space $U$ embedded into $\tilde{U}$ where the linear embedding is a Hilbert-Schmidt operator, the series (2.33) defines a $\tilde{U}$-random variable. We refer to [37] for the case of standard Brownian motion and to [13] for the case of regular standard fractional Brownian motion.

Example 2.6.4 provides insight into the construction of the stochastic integral with respect to cylindrical Gaussian Volterra processes which is given further. The usual approach is followed (see e.g. [37]). Let $\{e_n, n \in \mathbb{N}\}$ be a complete orthonormal basis of the real separable Hilbert space $U$ and let $B$ be a cylindrical Gaussian Volterra process with a regular kernel on $[0,T]$. For $n \in \mathbb{N}$ we define
\[
\beta^n_t := B_t(e_n).
\]
Lemma 2.6.7 (Independence of $\beta^n$). The elements of the sequence of scalar Gaussian Volterra processes $(\beta^n_t, t \in [0, T], n \in \mathbb{N})$ are mutually independent.

Proof. Let $m,n \in \mathbb{N}$ such that $m \neq n$ and consider two elements $e_n, e_m$ of the orthonormal basis corresponding to $m$ and $n$. By definition the processes $\beta^n_t$ and $\beta^m_t$ are both Gaussian and thus, to show the desired independence, we only need to show that they are not correlated. Consider therefore the expression

$$\mathbb{E}\beta^n_t \beta^m_t = \frac{1}{2} \mathbb{E}(B_t(e_n) + B_t(e_m))^2 - \mathbb{E}B_t(e_n)^2 - \mathbb{E}B_t(e_m)^2$$

The first term on the right equals by Definition 2.6.1 to $\mathbb{E}(B_t(e_n + e_m))^2$ and since it is a Gaussian Volterra process with covariance function $R$ we have that

$$\mathbb{E}\beta^n_t \beta^m_t = \frac{1}{2} R(t,t) (\|e_n + e_m\|^2_U - \|e_n\|^2_U - \|e_m\|^2_U) = 0,$$

which yields the result. \qed

Having proved the previous lemma we are ready to define the stochastic integral with respect to cylindrical Gaussian Volterra processes now. Let $(V, \langle \cdot, \cdot \rangle_V)$ be a second real separable Hilbert space and let $G : [0, T] \to \mathcal{L}(U, V)$ be a deterministic operator-valued function. We define

$$\int_0^T G dB := \sum_{n=1}^{\infty} \int_0^T G e_n \, dB^n, \quad (2.35)$$

provided that the sum converges in $L^2(\Omega, V)$ and that $G(\cdot) e_n \in \mathcal{H}$ for all $n \in \mathbb{N}$ since the integral on the right-hand side of (2.35) is defined by Proposition 2.2.4. It would be useful to give precise conditions under whose the sum (2.35) converges in $L^2(\Omega, V)$. For that we need a short introduction.

Definition 2.6.8 (Hilbert-Schmidt operator). Let $X,Y$ be two Hilbert spaces with norms $|\cdot|_X$ and $|\cdot|_Y$ respectively. By definition, a linear operator $T : X \to Y$ is a Hilbert-Schmidt operator if

$$\|T\|_{\mathcal{L}_2(X,Y)} := \sqrt{\sum_{n=1}^{\infty} |Te_n|^2_X} < \infty,$$

where $\{e_n, n \in \mathbb{N}\}$ is an arbitrary complete orthonormal basis of $X$.

For the purposes of the following proposition define an operator $P$ in such a way that

$$P : U \to L^2([0, T]; V) \quad u \mapsto K^* G(\cdot) u \quad (2.36)$$

Proposition 2.6.9 (Existence of $\int_0^T G dB$). Let $P$ be an operator of the form (2.36). If $P$ is a Hilbert-Schmidt operator, i.e. $P \in \mathcal{L}_2(U, L^2([0, T]; V))$, then the stochastic integral $\int_0^T G dB$ defined by (2.35) is a well-defined $V$-valued Gaussian random variable with zero mean and with a covariance operator $Q_T$ given by

$$\langle Q_T x, y \rangle_V = \int_0^T \sum_{n=1}^{\infty} \langle Pe_n(s), x \rangle_V \langle Pe_n(s), y \rangle_V ds. \quad (2.37)$$
Proof. Let \( m, n \in \mathbb{N} \) such that \( m < n \) and consider the remainder of the series (2.35) in the form

\[
\mathbb{E} \left| \sum_{k=m+1}^{n} \int_{0}^{T} G(s) e_k d\beta_s^k \right|^2_V = \mathbb{E} \sum_{k, l=m+1}^{n} \left\langle \int_{0}^{T} G(s) e_k d\beta_s^k, \int_{0}^{T} G(s) e_l d\beta_s^l \right\rangle_V = \sum_{k=m+1}^{n} \mathbb{E} \left| \int_{0}^{T} G(s) e_k d\beta_s^k \right|^2_V
\]

which clearly follows from Lemma 2.6.7. Hence by Lemma 2.2.3 and Proposition 2.5.1 we infer that

\[
\mathbb{E} \left| \sum_{k=m+1}^{n} \int_{0}^{T} G(s) e_k d\beta_s^k \right|^2_V = \sum_{k=m+1}^{n} \left\| K^* G(\cdot) e_k \right\|^2_{L^2([0,T];V)}
\]

which can be made arbitrary small since \( P \) is a Hilbert-Schmidt operator. Clearly, from Lemma 2.6.7, the elements of the sequence \( \{ \int_{0}^{T} G e_n d\beta^n, n \in \mathbb{N} \} \) are mutually independent and by Proposition 2.5.2 we have that each of these terms has a covariance operator

\[
Q^n_T x = \int_{0}^{T} \langle (K^* G e_n)(s), x \rangle_V (K^* G e_n)(s) ds
\]

where \( x \in V \). Therefore, by the expression (2.35), the integral \( \int_{0}^{T} G dB \) is a zero-mean \( V \)-valued random variable which is Gaussian. Hence, by (2.35), i.e. the definition of \( \int_{0}^{T} G dB \), we infer that

\[
Q^n_T x = \int_{0}^{T} \sum_{n=1}^{\infty} \langle (P e_n)(s), x \rangle_V (P e_n)(s) ds
\]

(2.38)

which obviously converges since \( P \) is Hilbert-Schmidt. It follows from (2.38) that

\[
\langle Q^n_T x, y \rangle = \int_{0}^{T} \sum_{n=1}^{\infty} \langle (P e_n)(s), x \rangle_V (P e_n)(s), y \rangle_V ds.
\]

Remark 2.6.10. The trace of the covariance operator \( Q_T \) can be expressed in terms of the Hilbert-Schmidt norm of the operator \( P \). Indeed, for a complete orthonormal basis \( \{ f_i, i \in \mathbb{N} \} \) of \( V \) and a complete orthonormal basis \( \{ e_j, j \in \mathbb{N} \} \) of \( U \) we have that

\[
\text{Tr} Q_T = \sum_{j=1}^{\infty} \int_{0}^{T} |(P e_j)(s)|^2_V ds = \sum_{j=1}^{\infty} \left\| P e_j \right\|^2_{L^2([0,T];V)} = \left\| P \right\|^2_{L^2(U;L^2([0,T];V))}
\]

using the definition of a trace

\[
\text{Tr} Q_T = \sum_{i=1}^{\infty} \langle Q_T f_i, f_i \rangle_V
\]

and the expression (2.37) with \( x = y = f_i \).
It would be useful to obtain sufficient conditions for the operator $P$ from Proposition \ref{2.6.9}. This is summarized in the last part of this section.

**Proposition 2.6.11.** Let $P$ be an operator of the form (2.36) and suppose that $G : [0, T] \to \mathcal{L}(U, V)$ is an operator-valued function. Then $P$ is Hilbert-Schmidt if one of the following conditions holds:

(i) $K$ is a regular kernel on $[0, T]$ and $G$ can be written as $G(\cdot) = \psi(\cdot)R$, where $R \in \mathcal{L}_2(U, V)$ and $\psi \in L^\infty([0, T]; \mathcal{L}(V))$.

(ii) $K$ is a strongly regular vanishing kernel on $[0, T]$ and for all $s \in [0, T]$, $(u_{-\alpha}G)(s)$ is a Hilbert-Schmidt operator such that

$$\|u_{-\alpha}G(\cdot)\|_{\mathcal{L}_2(U, V)} \in L^{2\frac{2}{1+2\alpha}}(0, T).$$ (2.39)

(iii) $K$ is a strictly regular vanishing kernel on $[0, T]$ and for all $s \in [0, T]$, $G(s)$ is a Hilbert-Schmidt operator such that

$$\|G(\cdot)\|_{\mathcal{L}_2(U, V)} \in L^{2\frac{2}{1+2\alpha}}(0, T).$$ (2.40)

**Proof.** We prove that all the conditions (i)-(iii) are indeed sufficient for $P$ to be a Hilbert-Schmidt operator. In all the cases it suffices to show that

$$\sum_{k=1}^{\infty} \|K^*G(\cdot)e_k\|^2_{L^2([0, T]; V)} < \infty.$$

We begin by (i). Using the factorization property $G(\cdot) = \psi(\cdot)R$ and the first statement of Theorem \ref{2.4.9} we infer that

$$\sum_{k=1}^{\infty} \|K^*G(\cdot)e_k\|^2_{L^2([0, T]; V)} \leq c_0 \sum_{k=1}^{\infty} \|G(\cdot)e_k\|^2_{L^\infty([0, T]; V)}$$

$$= c_0 \sum_{k=1}^{\infty} \left( \sup_{s \in [0, T]} |\psi(s)Re_k|^2_V \right)$$

$$= c_0 \sum_{k=1}^{\infty} \left( \sup_{s \in [0, T]} |\psi(s)|^2_{L^2(V)} \right) |Re_k|^2_V = c \|R\|^2_{L^2(U, V)},$$

where $c_0$ is some finite positive constant. Since $R$ is Hilbert-Schmidt so it is $P$. Hence, (i) is indeed sufficient for $P$ to be Hilbert-Schmidt. The same conclusion is now shown for (ii). It follows from Theorem \ref{2.4.9} that there exists a constant $c_0(\alpha) < \infty$ for $\alpha \in \left(0, \frac{1}{2}\right)$ such that

$$\sum_{k=1}^{\infty} \|K^*G(\cdot)e_k\|^2_{L^2([0, T]; V)} \leq$$

$$\leq c_0(\alpha) \sum_{k=1}^{\infty} \int_0^T \int_0^T |G(t)e_k|^2_V |G(r)e_k|^2_V (|t-r|^{-\alpha} |t-r|^{2\alpha-1} drdt$$

$$\leq c_0(\alpha) \int_0^T \int_0^T \sum_{k=1}^{\infty} |G(t)e_k|^2_V |G(r)e_k|^2_V (|t-r|^{-\alpha} |t-r|^{2\alpha-1} drdt$$

$$\leq c_0(\alpha) \int_0^T \int_0^T \|G(t)\|_{\mathcal{L}_2(U, V)}\|G(r)\|_{\mathcal{L}_2(U, V)} (|t-r|^{-\alpha} |t-r|^{2\alpha-1} drdt$$

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which is justified by the Dominated Convergence Theorem and the Cauchy-Schwarz inequality. From Theorem 1.2.4, Hardy-Littlewood inequality applied to a left-sided Riemann-Liouville integral, we further infer (in the same way as in the proof of Theorem 2.4.9) that
\[
\sum_{k=1}^{\infty} \|K^*G(\cdot)e_k\|_{L^2([0,T];V)}^2 \leq C \left( \int_0^T (u-\alpha \|G(\cdot)\|_{L^2(U;V)}) \frac{2}{1+2\alpha} (s)ds \right)^{\alpha+\frac{1}{2}},
\]
where \(C\) is some finite positive constant. This is a finite expression since
\[
u-\alpha \|G(\cdot)\|_{L^2(U;V)} \in L^{\frac{2}{1+2\alpha}}(0,T).
\]
Similary, it can be shown that (iii) is also a sufficient condition for \(P\) to be Hilbert-Schmidt. Let us therefore assume (iii) instead of (ii) now. Since \(K\) admits the most restrictive set of conditions then, from the proof of Theorem 2.4.9, it follows that there exists a constant \(c_0(\alpha) < \infty\) for \(\alpha \in (0,\frac{1}{2})\) such that
\[
\sum_{k=1}^{\infty} \|K^*G(\cdot)e_k\|_{L^2([0,T];V)}^2 \leq c_0(\alpha) \sum_{k=1}^{\infty} \|G(\cdot)e_k\|_{H_k}^2.
\]
Using again the Cauchy-Schwarz inequality we infer that
\[
c_0(\alpha) \sum_{k=1}^{\infty} \|G(\cdot)e_k\|_{H_k}^2 = c_0(\alpha) \sum_{k=1}^{\infty} \int_0^T \int_0^T |G(t)e_k|_V |G(r)e_k|_V |t-r|^{2\alpha-1}drdt
\leq c_0(\alpha) \int_0^T \int_0^T \sum_{k=1}^{\infty} |G(t)e_k|_V |G(r)e_k|_V |t-r|^{2\alpha-1}drdt
\leq c_0(\alpha) \int_0^T \int_0^T \|G(t)\|_{L^2(U;V)} \|G(r)\|_{L^2(U;V)} |t-r|^{2\alpha-1}drdt
\]
which is justified by the Monotone Convergence Theorem. From Theorem 1.2.4, Hardy-Littlewood inequality applied to a left-sided Riemann-Liouville integral, we further infer (in the same way as in the proof of Theorem 2.4.9) that
\[
\sum_{k=1}^{\infty} \|K^*G(\cdot)e_k\|_{L^2([0,T];V)}^2 \leq C \left( \int_0^T \|G(s)\|_{L^2(U;V)}^{\frac{2}{1+2\alpha}} ds \right)^{\alpha+\frac{1}{2}},
\]
where \(C\) is some finite positive constant. This is a finite expression since
\[
\|G(\cdot)\|_{L^2(U;V)} \in L^{\frac{2}{1+2\alpha}}(0,T).
\]
This finishes the proof and shows that any of the conditions (i)-(iii) implies that \(P\) is a Hilbert-Schmidt operator.

\[\square\]

**Remark 2.6.12.** In Proposition 2.6.11 it would suffice only to assume that
\[
\int_0^T \int_0^T \|G(t)\|_{L^2(U;V)} \|G(r)\|_{L^2(U;V)} |t-r|^{2\alpha-1}drdt < \infty
\]
instead of the condition (2.39) which is intuitive, even though more restrictive, since it provides a standard class of functions for which the operator \(P\) is Hilbert-Schmidt. Similarly, instead of the condition (2.40), we could assume
\[
\int_0^T \int_0^T \|G(t)\|_{L^2(U;V)} \|G(r)\|_{L^2(U;V)} |t-r|^{2\alpha-1}drdt < \infty.
\]
Remark 2.6.13. Note that the less we require from the kernel $K$ in the Proposition 2.6.11 the stronger the conditions on $G$ need to be in order to obtain the Hilbert-Schmidt property of $P$. This also shows the connection between the kernel $K$ and the stochastic integrability of $G$ with respect to the process $B$. In the next chapter the focus shall be primarily on the conditions (i) and (iii) since there are important examples on whose the theory can be demonstrated.

Altering the proof of (i) in Proposition 2.6.11 we can obtain the following useful lemma.

Lemma 2.6.14. If $K$ is a regular vanishing kernel on $[0,T]$ which induces a non-atomic measure $\mathcal{K}$ and $G$ can be written as $G(\cdot) = \psi(\cdot)R$, where $R \in L_2(U,V)$ and $\psi \in L^\infty([0,T];L(V))$ then, for all $s \in (0,T)$,

$$
E \left| \int_s^t G(r)dB_r \right|^2_{V^*} \longrightarrow 0.
$$

Proof. Note that by Remark 2.6.10 we have that

$$
E \left| \int_s^t G(r)dB_r \right|^2_{V^*} = E \left| \int_0^t 1_{(s,t]} G(r)dB_r \right|^2_{V^*} = \text{Tr} Q_t,
$$

where $Q_t$ is the covariance operator defined by (2.37) with $G$ replaced by $1_{(s,t]}G$.

Thus, we evaluate the expression

$$
\sum_{n=1}^\infty \left\| K^* 1_{(s,t]} G(\cdot) e_n \right\|^2_{L_2([0,t];V)}.
$$

By the expression for $K^*$ in the regular vanishing case, (2.26), we have that

$$
\text{Tr} Q_t = \sum_{n=1}^\infty \left\| K^* 1_{(s,t]} G(\cdot) e_n \right\|^2_{L_2([0,t];V)}
\leq \sum_{n=1}^\infty \int_0^T \int_u^t |K| |K^* 1_{(s,t]} G(\cdot) e_n|_{V^*} |\mathcal{K}|(dr,u)du
$$

Since $G$ can be factorized into two components, as assumed, we can write

$$
\text{Tr} Q_t = \sum_{n=1}^\infty \int_0^T \int_u^T |\psi(r)Re_n|_{V^*} |\mathcal{K}|(dr,u)du
\leq \sup_{r \in [0,T]} \|\psi(r)\|_{L(V)}^2 \|R\|_{L_2(U,V)}^2 \int_0^T |\mathcal{K}|((u \vee t),u)du,
$$

which follows by the Cauchy-Schwarz inequality. The last integral can be split into two dividing the integration interval at $u = s$ which gives

$$
\int_0^t |\mathcal{K}|((u \vee s,t],u))du = \int_0^s |\mathcal{K}|((s,t],u)du + \int_s^t |\mathcal{K}|((u,t],u)du. \tag{2.41}
$$

Since $\mathcal{K}$ is non-atomic, it is therefore continuous (see [28], p.186, the non-atomic property is equivalent to continuity) and thus, the first integral goes to zero as $t \to s+$ since its integrand goes to zero. From the regularity condition $((\mathcal{K}))$ we also have the condition $((\mathcal{K}))$ and therefore the integrand of the second integral in (2.41) is bounded. Hence, as $t \to s+$ the area of integration gets smaller making the second integral zero. \qed
3. Stochastic Evolution Equation

This chapter is devoted to the study of a solution of a linear stochastic differential equation in a Hilbert space where the source of random perturbation is a cylindrical Gaussian Volterra process defined in the previous section. As an introduction into the topic, a deterministic system is considered in the first part. We take the inhomogeneous abstract Cauchy problem as a motivation for a stochastic system which is the content of the main section of this chapter. We conclude the theoretical part by examples at the end of the chapter.

3.1 Deterministic Cauchy Problem

In this section the inhomogeneous abstract initial value problem (also called the Cauchy problem) is taken as a lead-in into the topic of linear stochastic equations in a Hilbert space. We collect basic results in the theory which are presented without proofs. Van Neerven’s lecture notes ([46]) are closely followed in this section where there are more results and examples presented. Let \((V, \langle \cdot, \cdot \rangle_V)\) be a real separable Hilbert space. We begin by formulating the inhomogeneous abstract initial value problem which is a deterministic linear differential equation of the form

\[
\begin{aligned}
X'(t) &= AX(t) + f(t), \quad t \geq 0 \\
X(0) &= x,
\end{aligned}
\]

(DCP)

Here \(x \in V\) and \(A : \text{Dom}(A) \to V\) is an infinitesimal generator of a strongly continuous semigroup \((S(t), t \geq 0)\) of bounded linear operators on \(V\). It is further assumed that \(f \in L^1([0, T]; V)\).

**Definition 3.1.1** (Strong solution of DCP, [46], p.97). A strong solution of \((\text{DCP})\) is a function \(X \in L^1([0, T]; V)\) such that for all \(t \in [0, T]\) we have that \(\int_0^t X(s)ds \in \text{Dom}(A)\) and

\[
X(t) = x + A \int_0^t X(s)ds + \int_0^t f(s)ds.
\]

**Definition 3.1.2** (Weak solution of DCP, [46], p.97). A weak solution of \((\text{DCP})\) is a function \(X \in L^1([0, T]; V)\) such that for all \(t \in [0, T]\) and \(x^* \in \text{Dom}(A^*)\) we have that

\[
\langle X(t), x^* \rangle_V = \langle x, x^* \rangle_V + \int_0^t \langle X(s), A^*x^* \rangle_V ds + \int_0^t \langle f(s), x^* \rangle_V ds.
\]

Every weak solution is a strong solution and conversely, every strong solution is a weak solution. For details see [46] or [37]. The following theorem, which ensures that the system \((\text{DCP})\) admits a unique strong solution, gives the general idea how to approach the corresponding stochastic system and thus, it represents a justification for the way the stochastic system is treated.
Theorem 3.1.3 (Mild solution of DCP\cite{46}, p. 97). For all $x \in V$ and integrable functions $f \in L^1([0,T];V)$ the problem (DCP) admits a unique strong solution $X$, which is given by the convolution formula

$$X(t) = S(t)x + \int_0^t S(t-s)f(s)ds. \quad (3.1)$$

If $f \in L^p([0,T];V)$ with $1 \leq p < \infty$, then $X \in L^p([0,T];V)$.

To prove Theorem 3.1.3 it suffices to show that (DCP) admits a weak solution. Detail can be found again in \cite{46}. The solution \eqref{3.1} is sometimes called the mild form. Once it is shown that a system admits a solution in the mild form it is usually followed by investigation of regularity of the solution based on the properties of the function $f$. For basic results and general treatment of this problem we refer to \cite{35}, Chapter 4. This is also the approach we shall follow in the next section for the stochastic analogue of the deterministic abstract initial value problem.

3.2 Stochastic Cauchy Problem

In this section, a stochastic analogue of the deterministic initial value problem (DCP) from the previous section is investigated. Firstly, a solution in the mild form is considered and then some regularity properties are presented. We demonstrate the theory on important examples, namely the wave equation and the heat equation. In our approach, we firstly follow \cite{46}, Chapter 8, and then we follow \cite{13} where the initial value problem is investigated in the case that the driving process is a cylindrical fractional Brownian motion. We extend some of these ideas for cylindrical Gaussian Volterra processes with a regular kernel.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. In order to formulate the problem precisely, consider the linear stochastic differential equation on $(\Omega, \mathcal{F}, \mathbb{P})$ in the form

$$\begin{cases}
dX_t &= AX_t dt + \Phi dB_t, \quad t \geq 0 \\
X_0 &= x,
\end{cases} \quad \mathbb{P} - \text{a.s.} \quad \text{(SCP)}$$

where we model the random input by the cylindrical Gaussian Volterra process $B = (B_t, t \geq 0)$ in a separable Hilbert space $(U, \langle \cdot, \cdot \rangle_U)$ corresponding to a regular kernel $K$. The linear drift in (SCP) is associated with $A : \text{Dom}(A) \to V$, $\text{Dom}(A) \subset V$, an infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$ of bounded linear operators acting on a real separable Hilbert space $(V, \langle \cdot, \cdot \rangle_V)$. Finally, the diffusion coefficient is considered to be a bounded linear operator $\Phi \in \mathcal{L}(U,V)$ and $x \in V$. According to the approach taken in the deterministic system (DCP), the system (SCP) should admit a solution, a $V$-valued process $X = (X_t, t \in [0,T])$, in the mild form which is defined by

$$X_t = S(t)x + \int_0^t S(t-s)\Phi dB_s, \quad \mathbb{P} - \text{a.s.} \quad (3.2)$$

for $t \geq 0$. Firstly, it is useful to notice that the process $X = (X_t, t \geq 0)$ is a mild solution of the problem (SCP) if and only if the process $\tilde{X} = (X_t - S(t)x, t \geq 0)$ is
a mild solution of the corresponding problem with the initial value 0. Hence, for simplicity it can be assumed that \( x = 0 \). Thus, we need to assure the existence of the convolution integral on the right-hand side of (3.2) which is hereafter denoted by \( Z \). If we fix \( T > 0 \), \( Z_T \) represents a \( V \)-valued random variable if at least one of the conditions of Proposition 2.6.11 is satisfied with \( G \) by the convolution integral on the right-hand side of (3.2) which is hereafter denoted \( Z \). Schmidt. Therefore, assuming one of the conditions above, the stochastic integral (A1) For all \( T > 0 \), \( K \) is a regular vanishing kernel on \([0, T]\) which induces a non-atomic measure \( \mathcal{K} \) and \( \Phi \in L_2(U, V) \).

(A2) For all \( T > 0 \), \( K \) is a strictly regular vanishing kernel on \([0, T]\) and for all \( s \in [0, T] \), \( S(s)\Phi \) is a Hilbert-Schmidt operator such that

\[
\|S(\cdot)\|_{L_2(U, V)} \in L^{\frac{2}{1+2\alpha}}(0, T). \tag{3.3}
\]

In both these cases the operator \( P = K^*S(\cdot)\Phi : U \to L^2([0, T]; V) \) is Hilbert-Schmidt. Therefore, assuming one of the conditions above, the stochastic integral \( Z_T \) exists for all \( T > 0 \) and hence, the collection of \( V \)-valued random variables \( Z = (Z_T, T > 0) \) defines a \( V \)-valued random process. Generally, however, this approach assures neither the measurability of \( Z \) nor the existence of a version with continuous sample paths. Hence, we need to use the semigroup properties.

Remark 3.2.1. The following inequality is useful throughout this whole section. It follows the proof of Proposition 2.6.11 and provide us with a crucial estimate on the trace of the covariance operator of the stochastic integral \( \int_0^T GdB \) since by Proposition 2.6.9 and Remark 2.6.10 we have that

\[
\mathbb{E} \left| \int_0^T G(r)dB_r \right|^2 \leq \text{Tr} Q_T = \|P\|_{L_2([0, T]; V)}^2 = \|K^*G\|^2_{L_2(U; L^2([0, T]; V))}.
\]

Assuming the condition (A2) the kernel \( K \) is strictly regular and vanishing on every interval \([0, T]\) and thus, from Theorem 2.4.9 we have the following estimate:

\[
\|P\|_{L_2(U; L^2([0, T]; V))}^2 \leq c_0 \int_0^T \int_0^T \|G(t)\|_{L_2(U, V)} \|G(r)\|_{L_2(U, V)} |t - r|^{2\alpha - 1} \, dr \, dt, \tag{3.4}
\]

where \( c_0 > 0 \) is a finite constant. The right-hand side of (3.4) is finite which follows from (3.3). This estimate is used in the main proposition of this chapter.

Remark 3.2.2. If there is one \( T > 0 \) such that the for all \( s \in [0, T] \) the operator \( S(s)\Phi \) is Hilbert-Schmidt and (3.3) holds, then both (3.3) and the fact that for all \( s \in [0, T] \) the operator \( S(s)\Phi \) is Hilbert-Schmidt hold for all \( T > 0 \). Indeed, it is clear that both the conditions hold for every \( t \leq T \). Suppose therefore, that
Proposition 2.6.9, the integral
\[ \int_0^t |X(s)| \, ds \]
the process
\[ X \]
Proof. To show that there exists a version of the process
\[ X \]
where
\[ P \]
and its sample paths are
\[ X \]
Proposition 3.2.3 (Measurability of
\[ T > t \]
0. However, for a general kernel, this does not have to be true.

Hence, by the semigroup property again, we have that
\[ \int_0^t \| S(s) \phi \|^2_{L^2(U,V)} \, ds = \int_0^T \| S(s) \phi \|^2_{L^2(U,V)} \, ds + \int_T^t \| S(s) \phi \|^2_{L^2(U,V)} \, ds \]
Hence, by the semigroup property again, we have that
\[ \int_0^t \| S(s) \phi \|^2_{L^2(U,V)} \, ds \leq C + \int_T^t M^2 e^{2\omega s} \, ds, \]
which is a finite expression. Therefore, it is crucial that we control the behaviour of \( K \) on all the intervals \([0, T]\). This is true, for example, for the standard fractional Brownian motion since the behaviour of its kernel \( K_H \) is similar for all \( T > 0 \). However, for a general kernel, this does not have to be true.

Proposition 3.2.3 (Measurability of \( X \)). If either one of the conditions \([A1] \) and \([A2] \) holds, then the process \( X = (X_t, t \geq 0) \), defined by \((3.2)\), is measurable and its sample paths are \( \mathbb{P} \)-almost surely in \( L^2([0, T]; V) \) for all \( T > 0 \).

Proof. To show that there exists a version of the process \( X \) with measurable sample paths it follows from Proposition 3.6, \([3.7] \), that it is enough to show that the process \( X \) is mean square continuous. Under the condition \([A1] \) (resp. \([A2]\) ) the integral \( Z_t \) is a well-defined \( V \)-valued random variable for each \( t \geq 0 \) and by Proposition 2.6.9 \( X_t \) has the Gaussian law
\[ \mu_t^x = N(S(t)x, Q_t), \]
where \( Q_t \) is given by \((2.38)\) with
\[ \text{Tr} Q_t = \| P \|^2_{L^2(U; L^2([0, T]; V))} = \| K^* S(t - \cdot) \phi \|^2_{L^2(U, L^2([0, t]; V))}, \]
by Remark 2.6.10. Suppose now, that \( t > s \geq 0 \). To show the mean square continuity of \( Z \) consider the expression
\[ \mathbb{E} \left\| Z_t - Z_s \right\|^2_V = \mathbb{E} \left\| \int_s^t S(t - r) \phi dB_r - \int_0^s S(s - r) \phi dB_r \right\|^2_V \]
\[ = \mathbb{E} \left\| \int_s^t S(t - r) \phi dB_r - \int_0^s (S(t - s) - I) S(s - r) \phi dB_r \right\|^2_V \]
\[ \leq 2 \mathbb{E} \left\| \int_s^t S(t - r) \phi dB_r \right\|^2_V + 2 \mathbb{E} \left\| \int_0^s (S(t - s) - I) S(s - r) \phi dB_r \right\|^2_V. \]
Denote the integrals on the right-hand side $I_1$ and $I_2$ respectively. We want to show that both terms $I_1$ and $I_2$ tend to zero as $t \to s_+$. Consider first the expression $I_1$. We can write that

$$I_1 = \mathbb{E} \left| \int_s^t S(t - r) \Phi dB_r \right|^2_V = \mathbb{E} \left| \int_0^t 1_{(s,t)}(r) S(t - r) \Phi dB_r \right|^2_V$$

since the mapping $r \mapsto 1_{(s,t)}(r) S(t - r) \Phi$ is stochastically integrable in $(0, t)$ for all $s \leq t$ with respect to $B$. Therefore, $I_1$ equals to the trace of $Q_t$, that is

$$I_1 = \|K^* 1_{(s,t)} S(t - \cdot) \Phi\|^2_{\mathcal{L}_2(U,L^2([0,t];V))}.$$  

Then, regardless of the choice between (A1) and (A2) we have that

$$I_1 \xrightarrow{t \to s^+} 0. \quad (3.5)$$

Indeed, if (A1) is assumed, the assertion (3.5) holds by Lemma 2.6.14. Assuming the condition (A2) from (3.4) we obtain the estimate

$$I_1 \leq c_\alpha \int_s^t \int_s^t \|S(t - r) \Phi\|_{\mathcal{L}_2(U,V)} \|S(t - u) \Phi\|_{\mathcal{L}_2(U,V)} |u - r|^{2\alpha - 1} dr du$$

from which (3.5) clearly follows by Domination Convergence Theorem by the same argument as in Remark 3.2.1. To establish $I_2$ we use the strong continuity property of the semigroup $(S(t), t \geq 0)$ from which it follows that

$$\left| (S(t - s) - I) \int_0^s S(s - t) \Phi dB_r \right|_V \xrightarrow{t \to s^+} 0, \quad \mathbb{P} - \text{a.s.},$$

and hence, we can use the Domination Convergence Theorem to see

$$\mathbb{E} \left| (S(t - s) - I) \int_0^s S(s - t) \Phi dB_r \right|^2_V \xrightarrow{t \to s^+} 0,$$

since

$$\mathbb{E} \left| (S(t - s) - I) \int_0^s S(s - t) \Phi dB_r \right|^2_V \leq c \mathbb{E} \left| \int_0^s S(s - r) \Phi dB_r \right|^2_V = c \text{Tr} Q_s < \infty,$$

where $c > 0$ is a finite constant. We have shown the continuity of $Z$ in $L^2(\Omega; V)$ from the right. This is sufficient for the mean square continuity of $Z$. From Lemma 3.2.2 it follows clearly that for all $T > 0$ we have that

$$\sup_{t \in [0,T]} \mathbb{E} |Z_t|^2_V = \sup_{t \in [0,T]} \text{Tr} Q_t = \sup_{t \in [0,T]} \|K^* S(t - \cdot) \Phi\|^2_{\mathcal{L}_2(U,L^2([0,t];V))} < \infty,$$

and therefore,

$$\mathbb{E} \int_0^T |Z_t|^2_V dt \leq \int_0^T \sup_{t \in [0,T]} \mathbb{E} |Z_t|^2_V dt < \infty.$$  

Hence, the sample paths of $X$ lie, $\mathbb{P}$-almost surely, in $L^2([0,T]; V)$ for all $T > 0$ which completes the proof.  

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Since the solution \( X = (X_t, t \geq 0) \) is a \( V \)-valued Gaussian process, the immediate consequence of Proposition 3.2.3 is that there exists a version of \( X \) whose sample paths lie \( \mathbb{P} \)-almost surely in \( L^p([0,T];V) \) for \( 1 \leq p < \infty \) and for all \( T > 0 \). For details we refer to [37].

**Corollary 3.2.4** (Sufficient condition for \([A2]\)). If for all \( T > 0 \) there exist finite constants \( c \geq 0 \) and \( 0 \leq \gamma < \frac{1}{2} + \alpha \) such that

\[
\|S(t)\Phi\|_{L^2(U,V)} \leq ct^{-\gamma}, \quad t \in (0,T]
\]

then there exists a version of the process \( X \) defined by (3.2), where the process \( B \) is a cylindrical Gaussian Volterra process with a strictly regular vanishing kernel on \([0,T]\), with sample paths \( \mathbb{P} \)-almost surely in \( L^p([0,T];V) \) for \( 1 \leq p < \infty \).

**Proof.** We show that such assumptions imply the condition \([A2]\). This is indeed so since having fixed \( T > 0 \) we clearly see that for all \( t \in (0,T] \) the operator \( S(t)\Phi \) is Hilbert-Schmidt. The function \( f(t) := t^{-\gamma} \) is \( L^2 \) integrable if and only if

\[
-\frac{2\gamma}{1+2\alpha} > -1
\]

which holds since \( \gamma \) satisfies \( 0 \leq \gamma < \frac{1}{2} + \alpha \) by the initial assumption. Therefore, the assertion holds by Proposition 3.2.3.

### 3.3 Examples

**Example 3.3.1.** For the first example, suppose that the real separable Hilbert spaces \( U \) and \( V \) are of finite dimension, that is \( \dim U = n \) and \( \dim V = m \). Therefore the operator \( A \) in (SCP) is represented by a \( n \times n \) matrix and the operator \( \Phi \in \mathcal{L}(U,V) \) is represented by a \( n \times m \) matrix. Therefore its Hilbert-Schmidt norm is finite. The infinitesimal operator \( A \) generates a strongly continuous semigroup \( (S(t), t \geq 0) \) which is \( S(t) = e^{At} \) for all \( t \geq 0 \), where \( e^{A} \) is the matrix exponential. The strong continuity follows from this immediately. Therefore, the condition \([A1]\) is always satisfied. Thus, by Proposition 3.2.3 there exists a well-defined mild solution to the problem (SCP) defined in terms of the stochastic convolution integral (3.2), which admits a version with sample paths \( \mathbb{P} \)-almost surely in \( L^p([0,T];V) \) for all \( 1 \leq p < \infty \).

**Example 3.3.2.** In the second example the stochastic heat equation is considered. It is a parabolic equation formally written as

\[
\begin{aligned}
\frac{\partial u}{\partial t}(t,\xi) &= \Delta u(t,\xi) + \eta(t,\xi), \quad (t,\xi) \in \mathbb{R}_+ \times D, \\
u(0,\xi) &= x(\xi), \quad \xi \in D, \\
u|_{\mathbb{R}_+ \times \partial D} &= 0,
\end{aligned}
\qquad \text{(SHE)}
\]

where \( D \subset \mathbb{R}^d \) denotes a bounded domain with \( C^2 \)-smooth boundary. The random perturbation represents \( \eta \), a noise process which is a formal derivative of a space dependent Gaussian Volterra process. The system (SHE) is modelled by an infinite dimensional stochastic differential equation

\[
\begin{aligned}
\text{d}X_t &= AX_t \text{d}t + \Phi \text{d}B_t, \quad t \geq 0, \\
X_0 &= x, \quad \mathbb{P} \text{-a.s.},
\end{aligned}
\]
of the form \([\text{SCP}]\) where we set \(V = U = L^2(D)\) and the operator \(A := \Delta|_{\text{Dom}(A)}\) generates an analytic semigroup \((S(t), t \geq 0)\) on \(L^2(D)\) with

\[
\text{Dom}(A) = H^2(D) \cap H^1_0(D)
\]

where \(H^2\) and \(H^1_0\) denote appropriate Sobolev spaces. If, for example, we set \(d = 1\), \(D = (a, b)\) an interval, then the domain \(\text{Dom}(A)\) coincides with the space of functions \(\varphi : (a, b) \to \mathbb{R}\) such that \(\varphi, \varphi'\) are both absolutely continuous, \(\varphi'' \in L^2(a, b)\) and the boundary conditions are satisfied. We refer to [31] for a thorough discussion on the topic. The semigroup \(S\) is described in terms of the so-called Green function. That is, there exists a function \(G : L^2(0, T) \times D \times D \to \mathbb{R}\) such that for all \(y \in L^2(D)\) we have that

\[
S(t)y(\xi) = \int_D G(t, \xi, \eta)y(\eta)d\eta.
\]

The noise \(\eta\) is formally described as

\[
\frac{\partial}{\partial t}\Phi B(t, \omega) = \Phi \left(\frac{dB}{dt}\right)(t),
\]

where \(B\) is a cylindrical Gaussian Volterra process in \(L^2(D)\) corresponding to a regular kernel \(K\). The operator \(\Phi\), vaguely said, is related to the space covariance of the solution \(X\). It is further assumed that \(\Phi \in L(L^2(D))\). Two cases can be considered.

If we suppose that the operator \(\Phi\) is Hilbert-Schmidt, that is, \(\Phi \in L_2(L^2(D))\), then we only need to assume that the kernel \(K\) is regular on all \([0, T]\), vanishing and inducing a non-atomic measure. Then, by Proposition [3.2.3], it follows that there exists a solution \(X\) to (3.6), a \(L^2(D)\)-valued process, in the mild form (3.2) with sample paths \(\mathbb{P}\)-almost surely in \(L^p([0, T]; V)\) for \(1 \leq p < \infty\). This case corresponds to the condition \([A1]\).

If, however, the operator \(\Phi\) is only bounded, that is \(\Phi \in L(L^2(D))\), a stronger condition on the kernel \(K\) needs to be assumed. We need the kernel \(K\) to be strictly regular on every \([0, T]\) and vanishing. This way, the condition \([A2]\) can be used. By estimates on the Green function \(G\) it can be verified that

\[
\|S(t)\|_{L_2(L^2(D))} \leq ct^{\frac{d}{4}}, \quad t \in (0, T],
\]

for all \(T > 0\), where \(c > 0\) is a finite constant. For further reading on the Green function we refer to [3]. Corollary [3.2.4] can be used to show the existence of a solution to the problem (3.6) with sample paths \(\mathbb{P}\)-almost surely in \(L^p([0, T]; V)\) for \(1 \leq p < \infty\), by taking \(\gamma = \frac{d}{4}\). Hence, the inequality can only be fully satisfied with \(d = 1\) and \(d = 2\). If \(d = 3\) we need \(\alpha > \frac{1}{4}\), so the trajectories of the underlying Gaussian Volterra process \(\beta\) have to be relatively smooth which follows from Proposition [2.4.8].

These results correspond to the previous results in case of the standard fractional Brownian motion \(B^H\) with Hurst parameter \(H > \frac{1}{2}\) which is thoroughly investigated in [13].

**Example 3.3.3.** In the third example, we consider a similar, yet different, system describing the stochastic wave equation. The problem can be formally formulated
as follows:

$$\begin{cases}
\frac{\partial^2 u}{\partial t^2}(t, \xi) + a \frac{\partial u}{\partial t}(t, \xi) &= \Delta u(t, \xi) + \eta(t, \xi), \quad (t, \xi) \in \mathbb{R}_+ \times D \\
\frac{\partial u}{\partial t}(0, \xi) &= x_1(\xi), \quad \xi \in D \\
u(0, \xi) &= x_2(\xi), \quad \xi \in D, \\
u|_{\mathbb{R}_+ \times \partial D} &= 0,
\end{cases}
$$

(SWE)

where $a \geq 0$, $D \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary and $\eta$ represents a noise process which is a formal derivative of the space dependent Gaussian Volterra process. Similarly as in Example 3.3.2, an infinite dimensional stochastic equation in the form (SCP) is associated with the parabolic system (SWE). We have that

$$\begin{cases}
dX_t = AX_t dt + \Phi dB_t, \quad t \geq 0, \\
X_0 = (x_1, x_2), \quad \mathbb{P} - \text{a.s.}
\end{cases}
$$

(3.7)

To define the operator $A$ properly, denote by $\Lambda = \Delta|_{\text{Dom}(\Lambda)}$ a positive, self-adjoint operator, where

$$\text{Dom}(\Lambda) = H^1_0(D) \cap H^1(D),$$

which is again defined in terms of Sobolev spaces $H^1_0(D)$ and $H^1(D)$. Then, the domain of the operator $A$ is defined by

$$\text{Dom}(A) = \text{Dom}(\Lambda) \times \text{Dom}(-\Lambda)^{\frac{1}{2}},$$

and the operator $A$ itself, is hence taken as

$$A = \begin{pmatrix} 0 & I \\ \Lambda & -aI \end{pmatrix}.$$

The operator $A$ therefore generates a contraction, strongly continuous semigroup of bounded linear operators on $\text{Dom}(-\Lambda)^{\frac{1}{2}} \times L^2(D)$. We have that

$$S(t) \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \cos \sqrt{\Lambda} t & \frac{1}{\sqrt{\Lambda}} \sin \sqrt{\Lambda} t \\ -\sqrt{\Lambda} \sin \sqrt{\Lambda} t & \cos \sqrt{\Lambda} t \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

Hence, let $V = U = \text{Dom}(-\Lambda)^{\frac{1}{2}} \times L^2(D)$ and the stochastic process $B = (B_t, t \geq 0)$ be a standard Gaussian Volterra process associated with a kernel $K$ regular on all $[0, T]$, vanishing and inducing a non-atomic measure. The covariance term here is

$$\Phi = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\Phi} \end{pmatrix},$$

where $\tilde{\Phi}$ is a Hilbert-Schmidt operator on $L^2(D)$. In the case of the stochastic wave equation (3.7) we only may consider the condition (A1). Then, by Proposition 3.2.3 there exists a solution to the system (3.7) which has sample paths $\mathbb{P}$-almost surely in $L^p([0, T]; V)$ for $1 \leq p < \infty$ and which takes the form of (3.2).
Conclusion

The stochastic Gaussian Volterra processes provide a generalization of such profound processes as the Brownian motion and the fractional Brownian motion. Being a very general class of random processes, it is of great significance to create a theory which properly defines and treat Gaussian processes which can be represented by the stochastic integral

$$\beta_t = \int_0^t K(t, r) dW_r, \quad \mathbb{P} - \text{a.s.}$$

Certain regularity conditions were given in the scalar case, following the approach of Alòs, Mazet and Nualart in [2]. Under these conditions, a stochastic integral taking the general Gaussian Volterra process as the integrator is defined for Hilbert space $V$ valued functions which presents the extension of the cited work to Hilbert spaces. It has been proven that in the regular case the space of $\beta$-integrable $V$-valued functions can be continuously embedded in the space $L^{2+2\alpha}([0, T]; V)$. Further, the cylindrical Gaussian Volterra process has been introduced and a stochastic integral with respect to this process has been further used as a model of random perturbation in a linear stochastic differential equation system. The system has been shown to admit a solution in the mild form. Such a solution has been proven to be a well-defined measurable process with a version whose sample paths lie almost surely in $L^p([0, T]; V)$ for all $1 \leq p < \infty$ and for all $T > 0$. Such approach had been chosen in the previous works of Maslowski, Duncan and Pasik-Duncan, [13], for the regular case of the standard fractional Brownian motion. Our results extend these results to a wider class of Gaussian Volterra processes.

To suggest further direction of research, a natural next step is to use the factorization method to show that such a mild solution admits a version with almost surely continuous sample paths. In order to do this, the stochastic version of the Fubini theorem for Gaussian Volterra processes has to be proven first. One may also wish to find some conditions on the behaviour of the kernel $K$ when $T \to \infty$ under whose, there exists a limiting measure. As previous works suggest (cf. [13]), a sufficient condition on the semigroup $S$ might be its exponential stability. However, such research has yet to be conducted. In this thesis, it is assumed that the covariance function of the process $\beta$ can be factorized in terms of the kernel $K$. The conditions under which such a factorization is indeed possible remain yet unknown. The problem may be approached by investigating the covariance operator of the process $\beta$. 

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Bibliography


