Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

BAKALÁŘSKÁ PRÁCE

Dana Bartošová

Variace Brouwerovy věty o pevném bodě

Katedra matematické analýzy

Vedoucí bakalářské práce: prof. RNDr. Miroslav Hušek, DrSc.

Studijní program: Matematika, Obecná matematika

2006

Děkuji vedoucímu mé bakalářské práce, prof. RNDr. Miroslavu Huškovi, DrSc., za ochotnou spolupráci a cenné rady a připomínky.

Prohlašuji, že jsem svou bakalářskou práci napsala samostatně a výhradně s použitím citovaných pramen. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 31.5.2006

Dana Bartošová

Dana Bartošová

Contents

Název práce: Variace Brouwerovy věty o pevném bodě
Autor: Dana Bartošová
Katedra (ústav): Katedra matematické analýzy
Vedouc bakalářské práce: prof. RNDr. Miroslav Hušek, DrSc.
e-mail vedoucího: mhusek@karlin.mff.cuni.cz

Abstrakt: Tato práce se zabýva známou Brouwerovou větou o pevném bodě, která říká, že každé spojité zobrazení z uzavřené jednotkové koule v Euklidovském prostoru do sebe má pevný bod. Uvádíme několik alternativních formulací Brouwerovy věty o pevném bodě, které mají též bohaté využití. Zatímco důkaz jejich ekvivalence je poměrně jednoduchý, důkazy Brouwerovy věty bývají značně složité. Uvádíme čtyři různé důkazy jako ukázku rozmanitosti možných důkazových metod. Zabýváme se též základním rozšířením Brouwerovy věty do prostorů nekonečné dimenze. V poslední kapitole formulujeme Borsuk-Ulamovu větu, blízce příbuznou s Brouwerovou větou, a několik jejich důležitých důsledků. Nakonec uvádíma zajímavou aplikaci teorie pevných bodu v teorii grafů.

Klíčová slova: Brouwerova věta o pevném bodě, KKM princip, Borsuk-Ulamova věta

Title: Variations of Brouwer Fixed Point Theorem
Author: Dana Bartošová
Department: Katedra matematické analýzy
Supervisor: prof. RNDr. Miroslav Hušek, DrSc.
Supervisor's e-mail address: mhusek@karlin.mff.cuni.cz

Abstract: This thesis deals with the well-known Brouwer fixed point theorem, which states that every continuous mapping from the unit closed ball in n-dimensional Euclidean space to itself has at least one fixed point.

We present several alternative formulations to Brouwer fixed point theorem, that mathematicians have also found to be strong tools. While the proof of their equivalence is quite easy, proofs of Brouwer fixed point theorem are rather complicated. We give four different proofs as a sampling of the various proofs available. We also deal with the basic extensions of Brouwer fixed point theorem to infinite dimension. In the last chapter, we formulate Borsuk-Ulam theorem, closely related to Brouwer fixed point theorem, and a few of its important consequences. Finally, we show a surprising application of fixed point theory to graph theory.

Keywords: Brouwer fixed point theorem, KKM-principle, Borsuk-Ulam theorem

Chapter 1

Introduction

A fixed point of a mapping $f : Y \subset X \to X$ is a point $x \in X$ that satisfies $f(x) = x$.

Fixed point theorems provide us with answers to the following question: What assumptions must be passed on a topological space X, its subset Y and a mapping $f : Y \to X$ to guarantee that f has one or more fixed points?

Brouwer fixed point theorem is one of the oldest theorems in fixed point theory. In 1909, Brouwer [4] stated that every continuous mapping from a closed unit ball in n-dimensional normed linear space into itself has a fixed point. It immediately raised a great interest among mathematicians, who brought numerous quite different proofs and alternative formulations. Brouwer fixed point theorem and its reformulations were found to be a powerfull tool both in theoretical and in applied mathematics. They also turned out to be suitable for wide extensions and generalizations.

A comprehensive survey on fixed point theory is the matter of the book Fixed Point Theory by Andrzej Granas and James Dugundji [6] (1982) and its re-edition [7] (2003). These books served as a rich source of information for my thesis.

The thesis focuses on Brouwer fixed point theorem and closely related results.

In the first chapter, Examples From Real Life, we illustrate Brouwer fixed point theorem on examples in dimensions $1, 2, 3$.

The largest chapter, Brouwer Fixed Point Theorem, begins with five alternative formulations of Brouwer fixed point theorem, that are themselves widely used in mathematical sciences, and proofs of their equivalence. Further it contains four proofs (a combinatorial proof, a proof based on Sperner's lemma, an analytic proof, a proof applying topological degree), that present rather distinct approaches to the theorem.

The chapter Extensions deals with the following question: "Does Brouwer fixed point theorem hold in infinite dimension?" In 1935, Tychonoff [21] answered the question in a negative way, showing that the unit sphere in l_2 is a retract of the unit ball. Kakutani [10] gave another example: The continuous map $g : B^\infty \to B^\infty$ of the unit ball $B^\infty = \{x = \{x_i\} \in l_2, ||x|| = \sum_{i=1}^{\infty} {x_i}^2\}$ into itself, defined as $(x_1, x_2, \ldots) \mapsto (\sqrt{1 - (||x||^2)}, x_2, \ldots)$ is fixed point free.

Thus, to obtain any extension of Brouwer fixed point theorem to infinite dimension, we must either restrict the type of the self-map or the type of the normed space.

If we consider compact mappings in place of continuous mappings, we get a generalization of Brouwer fixed point theorem valid in any normed linear space. Obviously, this statement is precisely Brouwer fixed point theorem whenever a normed linear space is finite-dimensional, since any continuous self-map of such a space is compact. We present a slightly stronger result by Schauder.

We also introduce three well-known generalizations of Brouwer fixed point theorem preserving the assumption of continuous mapping (by Ky Fan, Tychonoff (1935), Markoff and Kakutani (1936)), applying Ky Fan's KKM principle [9] (1952). KKM principle generalizes Knaster-Kuratowski-Mazurkiewicz theorem [11] (1929), which is equivalent to Brouwer fixed point theorem (see section Proof Based on Sperner's Lemma).

This chapter is of illustrative character and the theorems are stated mostly without proofs.

The last chapter is devoted to applications of fixed point theory and closely related results.

In the first section, we formulate Borsuk-Ulam antipodal theorem [3] (1933), that easily implies Brouwer fixed point theorem (see section Combinatorial Proof). We apply the theorem to prove a few basic topological and algebraic results (e.g. Invariance of dimension number, Fundamental theorem of algebra).

In the second section, Fixed Point Spaces, we prove two basic observations on preserving fixed point property and we give an example of infinite-dimensional fixed point space, the Hilbert cube.

The third section, Evasiveness of Graph Properties, serves as an example of an application of fixed point theory to graph theory, that seems to have no link with topology at first sight.

Chapter 2

Examples from real life

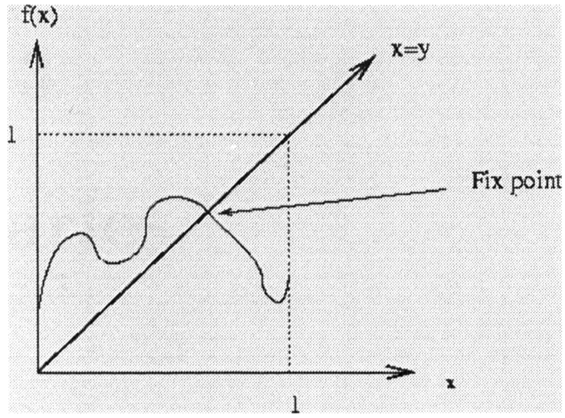
2.1 Monk Problem - dim. 1

One morning, exactly at sunrise, a Buddhist monk began to climb a tall mountain. The narrow path, no more than a foot or two wide, spiraled around the mountain to a glittering temple at the summit. The monk ascended the path at varying rates of speed, stopping many times along the way to rest and to eat the dried fruit he carried with him. He reached the temple shortly before sunset. After several days of fasting and meditation he began his journey back along the same path, starting at sunrise and again walking at variable speeds with many pauses along the way. His average speed descending was, of course, greater than his average climbing speed. Prove that there is a spot along the path that the monk will occupy on both trips at precisely the same time of day.

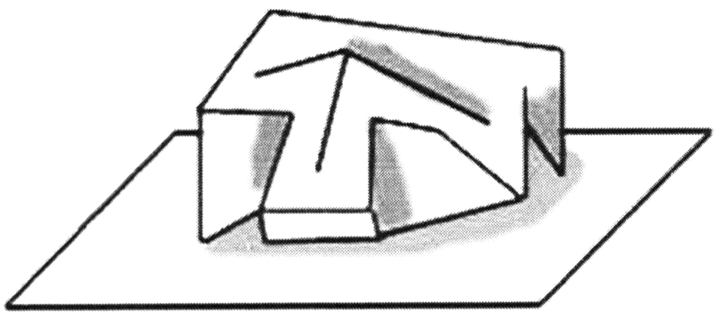
Here is an intuitive proof of the monk problem. Imagine that there are two monks, one going down and one going up, each beginning on the same day at sunrise. At some point in the day the hiker's must meet!

Geometrical Illustration in dim. 1

A continuous function $f : [0,1] \to [0,1]$ has a fixed point. It means that f must cross the diagonal.



2.2 Crumpled Sheet of Paper - dim. 2



Take two equal size sheets of graph paper with coordinate systems on them, lay one flat on the table and crumple up (but don't rip) the other one and place it any way you like on top of the first. Then there will be at least one point of the crumpled sheet that lies exactly on top of the corresponding point (i.e. the point with the same coordinates) of the flat sheet.

2.3 Cup of Coffee - dim. 3

Consider a cupful of coffee. Each point is somewhere in 3-dimensional space. Stir. At least one point ends up in the same place as it began.

Chapter 3

Preliminaries

3.1 Homotopy

Definition 3.1.1. Two continuous maps $f, g : X \to Y$ are called *homotopic* if there is a continuous map $H : X \times [0,1] \to Y$ such that for each $x \in X$ $H(x,0) = f(x)$ and $H(x,1) = g(x)$. The map H is called a *homotopy* or *continuous deformation* of f to g (written $H : f \simeq g$). A mapping $f : X \to Y$ homotopic to a constant map is called *nullhomotopic* (written $f \simeq 0$). A space X is called *contractible* if $\mathrm{id} : X \to X$ is nullhomotopic.

The relation of homotopy is an equivalence relation on the set of all continuous maps of X into Y. Classes of this equivalence relation are called *homotopy classes*.

3.2 C^1 Functions

Definition 3.2.1 (Derivation). Let G be an open subset of $\mathbb{R}^n$, and let $f = (f_1, f_2, \ldots, f_k) : G \subset \mathbb{R}^n \to \mathbb{R}^k$ be a mapping with components $f_i : G \to \mathbb{R}$, $i = 1, 2, \ldots, k$. We say that a linear mapping $L : G \to \mathbb{R}^k$ is a *derivation* of f in a point $a \in G$, if

$$lim_{h \to 0} \frac{f(a+h) - f(a) - L(h)}{||h||} = 0.$$

Derivation of the mapping f in the point a is denoted by $f'(a)$.

The matrix of the linear mapping $f'(a)$ is called *Jacobian matrix* and is of the form

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \dots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \dots & \frac{\partial f_2}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1}(a) & \dots & \frac{\partial f_k}{\partial x_n}(a) \end{pmatrix}.$$

If $k = n$, then the determinant of the Jacobian matrix in a is called *Jacobian determinant* and is denoted by $J_f(a)$.

Definition 3.2.2 (C^1 **functions**). Let G be an open subset of $\mathbb{R}^n$, and let $f : G \subset \mathbb{R}^n \to \mathbb{R}^k$ be a mapping. We say that f belongs to C^1 class of functions if all its partial derivatives $\frac{\partial f}{\partial x_i}$ $(i = 1, 2, \dots, n)$ are continuous.

Let $f : B^n \to \mathbb{R}^n$. We'll say that $f \in C^1(B^n)$ if there exists an open set B^{n^*} and $f^* \in C^1(B^{n^*})$ such that $f^*|_{B^n} = f$.

Definition 3.2.3 (**Regular Mapping**). Let G be an open subset of $\mathbb{R}^n$ and let $f : G \to \mathbb{R}^n$ be a mapping. We say that f is a *regular mapping* if $f \in C^1(G)$ and $J_f(x) \neq 0$ for all $x \in G$.

Theorem 3.2.4 (Special case of Stone-Weierstrass Theorem). *Let G be an open, bounded subset of $\mathbb{R}^n$, and let $C(G)$ be the space of continuous mappings from G to $\mathbb{R}^n$ with the sepreme norm. Then for every $\varepsilon > 0$ and every $f \in C$, there exists a $f^* \in C^1$ such that $||f - f^*|| < \varepsilon$.*

3.3 Simplicial Complex

Abstract Simplicial Complex

Definition 3.3.1. A *simplicial complex* is a finite collection K of sets such that

(1) $\forall X \in K, Y \subseteq X \Rightarrow Y \in K$ and

(2) $K \neq \emptyset$.

The sets in K are called *(abstract) simplices.* The elements of all sets in K are called *vertices of* K. The set of all vertices is denoted by $V(K)$.

Geometric Realization

Definition 3.3.2. A finite set of vectors in a linear normed space is called *affinely independent* if none of its elements lies in the affine hull of any subset of the others.

Definition 3.3.3 (Geometric Simplex, Face, Boundary). Let $V = \{p_0, \ldots, p_s\}$ be a finite set of $s+1$ affinely independent vectors in a linear normed space. The convex hull σ of V $\{\sum_{v \in V} \alpha_v v : \sum \alpha_v = 1, \alpha_v \geq 0\}$ is called the (closed) *s-simplex* and is denoted $[p_0, \ldots, p_s]$. Elements of V are called *vertices*. The dimension of the simplex σ is $|V| + 1$.

The convex hull of arbitrary subset of vertices of the simplex σ is called *face* of σ.

The *boundary* of the s-dimensional simplex σ is the union of all faces of dimensions $\leq s - 1$.

Let K be an abstract simplicial complex. We can construct a corresponding geometric realization $\widehat{K} \subseteq \mathbb{R}^{V(K)}$ in the following way: First we define the mapping $\hat{\ } : V(K) \to \mathbb{R}^{V(K)}$ such that no vertex of K is mapped to the affine hull of any subset of the others. Then we extend $\hat{\ }$ to the elements of K (sets of vertices): If $X \in K$ then $\widehat{X} = \{\sum_{v \in X} \alpha_v \widehat{v}, \sum \alpha_v = 1, \alpha_v \geq 0\}$. FinaSlly we extend $\hat{\ }$ to K: $\widehat{K} = \bigcup_{X \in K} \widehat{X}$.

Obviously for each $X \in K$, $\hat{X}$ is a simplex and each point $x \in \hat{K}$ has a unique representation $x = \sum_{v \in X} \alpha_v \hat{v}$ for some $X \in K$. Denote $\Delta x = \{v \in X : \alpha_v \neq 0\}$ the *support simplex of x*.

In correspondence with an abstract simplicial complex, we can define a geometric simplicial complex.

Definition 3.3.4 (Geometric simplicial complex). A collection $K = \{S_1, S_2, \ldots, S_n\}$ of simpleces is said to form a *geometric simplicial complex* if

(1) $\forall S_i, T$ a face of $S_i \Rightarrow T \in K$,

(2) $\forall S_i, S_j \in K, S_i \cap S_j \neq \emptyset \Rightarrow S_i \cap S_j$ is a face of both S_i and S_j

Definition 3.3.5 (Polyhedron). The union of all simplices in a geometric simplicial complex σ is the *polyhedron* of σ.

Definition 3.3.6 (Triangulation). Let X be a topological space. A simplicial complex σ such that X is homeomorphic $\hat{\sigma}$, if one exists, is called a *triangulation* of X.

Definition 3.3.7 (Barycentric subdivision). The *barycenter* of an n-simplex $\sigma = [p_0, \ldots, p_n]$ is the point $[\sigma] = \sum_{i=0}^{n} \frac{p_i}{n+1}$.

Let K be a geometric simplicial complex. The *first barycentric subdivision* of K is a simplicial complex $Sd^1(K)$ consisting of all simpleces $[[\sigma_0], \ldots, [\sigma_s]]$, where $\sigma_0 \subset \sigma_1 \subset \ldots \subset \sigma_s$ is a sequence of simplices of K.

The *m-th barycentric subdivision* is defined inductively by

$$Sd^0(K) = K, \quad Sd^m(K) = Sd^1(Sd^{m-1}(K))$$

Iterated barycentric subdivision can be used to construct arbitrarily fine triangulations of a given polyhedron.

Simplicial Mappings

Definition 3.3.8. Let K and L be two abstract simplicial complexes. A *simplicial mapping* is a mapping $f : V(K) \to V(L)$ that maps simplices to simplices, i.e. $f(X) \in L$ whenever $X \in K$.

Definition 3.3.9. Let K_1 and K_2 be two abstract simplicial complexes and σ_1 and σ_2 the corresponding geometric simplicial complexes. Let $f : V(K_1) \to V(K_2)$ be a simplicial mapping of K_1 into K_2. Then we define the mapping $\hat{f} : \hat{\sigma_1} \to \hat{\sigma_2}$ as follows:

$$\hat{f}(x) = \hat{f}(\sum_{v \in \Delta_x} \alpha_v \hat{v}) = \sum_{v \in \Delta x} \alpha_v \hat{f(v)}.$$

Chapter 4

Brouwer Fixed Point Theorem

Brouwer fixed point theorem is a very useful tool in many mathematical fields because it offers several different equivalent formulations, many different proofs, many extensions and generalizations and numeruous interesting applications.

4.1 Equivalent Formulations of Brouwer Fixed Point Theorem

Theorem 4.1.1 (Brouwer fixed point theorem). *Every continuous mapping $f : B^n \to B^n$ from the unit closed ball of n-dimensional Euclidean space to itself has at least one fixed point.*

Theorem 4.1.2. *The following statements are equivalent:*

(1) *Brouwer fixed point theorem.*

(2) *(**Non-retraction Theorem**). There is no retraction $r : B^n \to S^{n-1}$ from the unit closed n-dimensional ball to the unit $(n-1)$-dimensional sphere, i.e. there's no continuous mapping $r : B^n \to S^{n-1}$ which is identical on S^{n-1}.*

(3) *Every $f \in C^1(B^n)$ has at least one fixed point.*

(4) *There is no retraction $f \in C^1(B^n)$, $f : B^n \to S^{n-1}$.*

(5) *The unit (n-1)-dimensional sphere S^{n-1} is not contractible in itself.*

(6) *Poincaré-Miranda Theorem. (See Theorem 4.2.2)*

Proof. (1) $\Leftrightarrow$ (2). If there exists a retraction $r : B^n \longrightarrow S^{n-1}$, than the map $x \mapsto -r(x)$ is a continuous map of B^n into itself without a fixed point.

On the contrary, suppose that $f : B^n \longrightarrow B^n$ is continuous and has no fixed point. Since there is no $x \in B^n$, such that $f(x) = x$, for each x there is exactly one ray $(x, f(x))$ originating in $f(x)$ and going through x. So we can construct a continuous retraction $g : B^n \longrightarrow S^{n-1}$, where $g(x)$ as a point is an intersection of $(x, f(x))$ and S^{n-1}. (Detailed construction of g is the same as in the implication (4) $\Rightarrow$ (3) and is given in the section Analytic Proof.)

(1) $\Rightarrow$ (3). Each $f \in C^1(B^n)$ is continuous.

(3) $\Rightarrow$ (1). Let $f : B^n \longrightarrow B^n$ be a continuous mapping. Since B^n is a compact metric space, there exist $\alpha = \min_{x \in B^n} |f(x) - x|$ and $x_0 \in B^n$ such that $\alpha = |f(x_0) - x_0|$. We want to show that $\alpha = 0$.

Due to **??**, for each $\varepsilon > 0$ there exists $g \in C^1(B^n)$, $g : B^n \longrightarrow \mathbb{R}^n$ such that $||f - g|| \leq \varepsilon$. Thus $||g|| \leq ||f|| + ||f - g|| \leq 1 + \varepsilon$ and $(1 + \varepsilon)^{-1} g \in C^1(B^n)$ maps B^n into B^n. Due to (2) $(1 + \varepsilon)^{-1} g$ has a fixed point x_ε. We obtain following inequalities:

$$\begin{aligned} |f(x_\varepsilon) - x_\varepsilon| &= \left|f(x_\varepsilon) - (1+\varepsilon)^{-1} g(x_\varepsilon)\right| \\ &\leq |f(x_\varepsilon) + \varepsilon f(x_\varepsilon) - g(x_\varepsilon)| \, (1+\varepsilon)^{-1} \\ &\leq |f(x_\varepsilon) - g(x_\varepsilon)| + \varepsilon \, |f(x_\varepsilon)| \leq 2\varepsilon. \end{aligned}$$

For each $\varepsilon > 0$ we proved that $\alpha \leq 2\varepsilon$ and hence $\alpha = 0$.

(3) $\Leftrightarrow$ (4). Proof is given in the section Analytic Proof.

(2) $\Rightarrow$ (5). Suppose that S^{n-1} is contractible in itself, which means that there exists a homotopy $h : S^{n-1} \times [0, 1] \longrightarrow S^{n-1}$ such that $h(x, 0) = x$ and $h(x, 1) = x_0 \in S^{n-1}$. Then we can continuously extend $id_{S^{n-1}}$ to a retraction $c : B^n \longrightarrow S^{n-1}$:

$$c(x) = \begin{cases} x_0 & ||x|| \leq \frac{1}{2} \\ h\left(\frac{x}{||x||}, 2\,||x|| - 1\right) & ||x|| \geq \frac{1}{2} \end{cases}$$

(5) $\Rightarrow$ (1). Let $f : B^n \longrightarrow B^n$ be continuous without any fixed point, then we can define a continuous deformation $H : S^{n-1} \times [0, 1] \longrightarrow S^{n-1}$ of $\mathrm{id} : S^{n-1} \longrightarrow S^{n-1}$ to a constant.

$$H(x, t) = \begin{cases} \frac{x - 2tf(x)}{||x - 2tf(x)||} & 0 \leq t \leq \frac{1}{2} \\ \frac{(2-2t)x - f[(2-2t)x]}{||(2-2t)x - f[(2-2t)x]||} & \frac{1}{2} \leq t \leq 1 \end{cases}$$

(1) $\Leftrightarrow$ (6) The proof is given in the following section. □

4.2 Poincaré-Miranda Theorem

History of the statement in Brouwer fixed point theorem goes back to the 19th century when Bernard Bolzano (1781-1848), the outstanding Czech philosopher and mathematician, proved that if a function f, continuous in a closed interval $[a, b]$, changes signs at the endpoints, then f equals zero at some point of the interval. In 1883-1884, Henri Poincaré extended this result to finite families of continuous functions on n-dimensional cubes [15],[16].

"Let $f_1, \ldots, f_n$ be n continuous funcionts of n variables $x_1, \ldots, x_n$: the variable is subjected to vary between the limits $+a_i$ and $-a_i$. Let us suppose that for $x_i = a_i$, f_i is constantly positive, and that for $x_i = -a_i$, f_i is constantly negative; I say there will exist a system of values of x where all the $f's$ vanish."

In 1940, Miranda [14] proved that Poincaré theorem was equivalent to Brower fixed point theorem. That is the reason why Poincaré theorem is often called Poincaré-Miranda theorem.

Let us first show that Brouwer fixed point theorem implies Poincaré-Miranda theorem 4.2.2.

Proposition 4.2.1 (Theorem on partitions). *Let I^n denote the n-cube $\{(x_1, \ldots, x_n) |\, |x_i| \leq 1$ for $i = 1, 2, \ldots, n\}$. The i-th face $\{x \in I^n | x_i = 1\}$ of I^n will be denoted by I_i^+, and the opposite face $\{x \in I^n | x_i = -1\}$ by I_i^-. For each $i \in \{1, \ldots, n\}$ let A_i be a closed set separating I_i^+ and I_i^- (i.e. $I^n \setminus A_i = U_i^+ \cup U_i^-$, where the U_i^+, U_i^- are disjoint open sets and $I_i^+ \subset U_i^+$, $I_i^- \subset U_i^-$). Then $\bigcap_{i=1}^n A_i \neq \emptyset$.*

Proof. Since I^n is a connected set, $A_i \neq \emptyset$ for each i.

For each $i \in \{1, \ldots, n\}$ define

$$h_i(x) = \begin{cases} -d(x, A_i), & x \in U_i^+, \\ d(x, A_i), & x \in U_i^-, \\ 0 & x \in A_i. \end{cases}$$

We'll show that the mapping $h : x \mapsto x + (h_1(x), h_2(x), \ldots, h_n(x))$ maps I^n into itself, in other words, that $|x_i + h_i(x)| \leq 1$ for each $i \in \{1, \ldots, n\}$:

Let $a \in I^{n-1}$, each segment

$$X_i^a = \{x \in I^n | \, |x_i| \le 1, (\forall i < j)(x_j = a_j), (\forall j > i)(x_j = a_{j-1})\}$$

is a connected subset of I^n, so that either $X_i^a \subset U_i^+$, or $X_i^a \subset U_i^-$, or X_i^a meets A_i. Since $X_i^a \cap I_i^+ \neq \emptyset \neq X_i^a \cap I_i^-$, only $X_i^a \cap A_i \neq \emptyset$ is possible. For $x \in I^n$ and each $i = 1, 2, \ldots, n$, find a^i such that $x \in X_i^{a^i}$. If $x \in A_i$, then $h_i(x) = 0$ and $|x_i + h_i(x)| = |x_i| \le 1$. If $x \in U_i^+$, then $x_i - d(x, A_i) \le 1$, since $d(x, A_i) > 0$. As $x \in X_i^{a^i}$ for some $a^i \in I^{n-1}$, it follows that $d(x, A_i) \le d(x, I_i^-) \le d(x_i, -1)$ and consequently $x_i - d(x, A_i) \ge x_i - d(x_i, -1) \ge -1$. Similarly for $x \in U_i^-$.

We proved that the continuous mapping h maps I^n to itself. According to Brouwer fixed point theorem, h has a fixed point x_0, so that $h_i(x_0) = 0$ for all $x \in \{1, \ldots, n\}$ and consequently $x_0 \in \bigcap_{i=1}^n A_i$. □

Theorem 4.2.2 (Poincaré-Miranda). *Let $f_1, \ldots, f_n$ be continuous real-valued functions on I^n such that for each $i \in \{1, \ldots, n\}$*

$$f_i(x) \ge 0 \quad \text{for } x \in I_i^+,$$
$$f_i(x) \le 0 \quad \text{for } x \in I_i^-.$$

Then there exists $x' \in I^n$ such that $f_i(x') = 0$ for each $i \in \{1, \ldots, n\}$.

Proof. For $i = 1, 2, \ldots, n$ denote $A_i^+ = f_i^{-1}[0, \infty)$ and $A_i^- = f_i^{-1}(-\infty, 0]$. Obviously $A_i = A_i^+ \cap A_i^- \neq \emptyset$ is a closed set, $f_i(a) = 0$ for all $a \in A_i$ and $U_i^+ = A_i^+ \setminus A_i$, $U_i^- = A_i^- \setminus A_i$ open sets, such that $I_i^+ \subset U_i^+$ and $I_i^- \subset A_i^-$. Applying 4.2.1, there exists $x' \in A_1 \cap A_2 \cap \ldots \cap A_n$ which satisfies $f_i(x') = 0$ for each $i = 1, 2, \ldots, n$. □

Now we show how easily Poincaré-Miranda theorem implies Brouwer fixed point theorem.

Theorem 4.2.3 (Coincidence theorem). *Let $f, g : I^n \to I^n$ be continuous mappings. If $f(I_i^+) \subset I_i^+$ and $f(I_i^-) \subset I_i^-$ for each $i = 1, 2, \ldots, n$, then there exists a point $x \in I^n$ such that $g(x) = f(x)$.*

Proof. Define $h(x) = g(x) - f(x)$. The mapping h satisfies the assumptions of Miranda-Poincaré theorem and therefore there is a point x with $h(x) = 0$, which means that $g(x) = f(x)$. □

Theorem 4.2.4 (Brouwer fixed point theorem). *Every continuous mapping $g : I^n \to I^n$ has a fixed point.*

Proof. We just apply the Coincidence theorem to f, the identity map on I^n. □

4.3 Proofs of Brouwer Fixed Point Theorem

Brouwer [4] proved his theorem for $n = 3$ in 1909. A year later, Hadamar gave the first proof for arbitrary n. In 1912 Brouwer [5] presented a proof using the simplicial approximation technique. Simple and short proof was given by Knaster-Kuratowski-Mazurkiewicz in 1929 (based on Sperner's lemma [19] from 1928). First proof of analytic nature was presented by Milnor [13] in 1978. There have been given numerous various proofs using various techniques in language of various mathematical theories. We prove Brouwer fixed point theorem in four different ways to show that it cannot be considered as a part of a single mathematical theory.

Combinatorial Proof

We are going to prove the non-retraction theorem, which is equivalent with Brouwer fixed point theorem:

Theorem 4.3.1 (Non-retraction Theorem).
There is no retraction $r : B^n \to S^{n-1}$ from the unit closed n-dimensional ball to the unit $(n-1)$-dimensional sphere, i.e. there is no continuous mapping $r : B^n \to S^{n-1}$ which is identical on S^{n-1}.

The non-retraction Theorem is a direct consequence of one of the equivalent formulations of Borsuk-Ulam theorem:

Theorem 4.3.2. *There is no continuous antipodal mapping $f : S^n \to S^{n-1}$ (antipodal means $f(-x) = -f(x)$ for each $x \in S^n$).*

Proof of 4.3.2 ⇒ 4.3.1. Let $r : B^n \to S^{n-1}$ be a retraction of B^n onto S^{n-1}. Denote $S^n_+ = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1, x_{n+1} \geq 0\}$ the upper hemisphere of S^n and S^n_- the lower hemisphere of S^n. Clearly the projection $\pi : S^n_+ \to B^n$, $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$ is a homeomorphism and thus $h_+ = \pi \circ r$ is a retraction of S^n_+ onto S^{n-1}. Since $S^{n-1} = S^n_+ \cap S^n_-$, the extension $h : S^n \to S^{n-1}$ of h_+ to the whole sphere S^n

$$h(x) = \begin{cases} h_+(x) & x \in S^n_+ \\ -h_+(-x) & x \in S^n_-, \end{cases}$$

is a continuous and antipodal mapping from S^n into S^{n-1}. □

Existence of a mapping f from 4.3.2 contradicts Lusternik-Schnirelmann-Borsuk theorem:

Theorem 4.3.3 (Lusternik-Schnirelman-Borsuk). *Let $\{M_1, \ldots, M_{n+1}\}$ be a closed covering of S^n. Then there exists i such that $M_i \cap -M_i \neq \emptyset$.*

Proof of (4.3.3) $\Rightarrow$ (4.3.2). Let $f : S^n \to S^{n-1}$ be an antipodal mapping. $(n-1)$ simpleces in basic triangulation Σ^{n-1} of S^n provides decomposition of S^{n-1} into $n+1$ closed sets $A_1, \ldots, A_{n+1}$. Clearly, no A_i contains a pair of antipodal points.

Let $M_i = f^{-1}[A_i]$, $i = 1, \ldots, n+1$. $\{M_i\}_{i=1}^{n+1}$ is a collection of closed sets that covers S^n and $M_i \cap -M_i = \emptyset$. If for some i existed $x \in M_i \cap -M_i \neq 0$, then both $f(x) \in A_i$ and $f(-x) = -f(x) \in A_i$, which is a contradiction. □

Proof of 4.3.3. The proof is based on a simplicial approximation of S^n. Thus we consider the unit sphere in $\mathbb{R}^{n+1}$ with the norm $||x|| = \sum_{i=1}^{n+1} |x_i|$, which is homeomorphic with S^n in the Euclidean norm. In this norm the unit sphere can be regarded as a union of simpleces of a triangulation.

Definition 4.3.4 (Basic triangulation). The unit ball B^{n+1} is the convex hull of the set $\{e_1, -e_1, \ldots, e_{n+1}, -e_{n+1}\}$ and the set of all n-simpleces $[\pm e_1, \ldots, \pm e_{n+1}]$ and all their faces provides a triangulation of S^n. This triangulation is called *basic triangulation* and is denoted as Σ^n)

A special group of triangulations of S^n will be important for our purposes:

Definition 4.3.5 (Symmetric triangulation). A triangulation $\mathcal{S}^n$ of S^n is called *symmetric* if for each $k \leq n$ the complex of k-simpleces of $\mathcal{S}^n$ forms a triangulation of S^k and for each k-simplex $\sigma^k \in \mathcal{S}^n$ the set $-\sigma^k$ is a k-simplex of $\mathcal{S}^n$.

Definition 4.3.6. Let $f : \mathcal{S}^n \to \Sigma^n$ be a simplicial mapping, where $\mathcal{S}^n$ is a symmetric triangulation of S^n. An r-simplex $[p_0, \ldots, p_r]$ of $\mathcal{S}^n$ is called *positive* if $f([p_0, \ldots, p_r]) = [+e_{i_0}, -e_{i_1} \ldots, (-1)^r e_{i_r}]$ is an r-simplex of $\mathcal{S}^k$.

An r-simplex $[p_0, \ldots, p_r]$ of $\mathcal{S}^n$ is called *negative* if $f([p_0, \ldots, p_r]) = [+e_{i_0}, -e_{i_1} \ldots, (-1)^r e_{i_r}]$ is an r-simplex of $\mathcal{S}^k$.

An r-simplex $[p_0, \ldots, p_r]$ of $\mathcal{S}^n$ is called *neutral* if it's neither positive nor negative.

For any simplicial mapping $f : \mathcal{S}^k \to S^n$ and any subset $L \subseteq \mathcal{S}^n$, denote $p(f, L, r)$ the number of positive r-simpleces in L under f.

Theorem 4.3.7 (Combinatorial lemma). *Let $\mathcal{S}^n$ be a symmetric triangulation of S^n and let $f : \mathcal{S}^n \to \Sigma^n$ be an antipodal simplicial mapping. Then f maps an odd number of simpleces in $\mathcal{S}^n$ to $[e_1, -e_2, \ldots, (-1)^n e_{n+1}]$.*

Proof of combinatorial lemma.
It's enough to prove that $p(f, S^k, k) \equiv p(f, S^{k-1}, k-1) \mod 2$ for each $k \leq n$, because it directly implies that $p(f, S^n, n) \equiv p(f, S^0, 0) \mod 2$. Since S^0 consist of exactly two antipodal points, $f|_{S^0}$ maps them onto a pair of antipodal vertices and clearly $p(f, S^n, 0) = 1$.

Denote S^k_+ the upper hemisphere of S^k (recall that $S^{k-1} = S^k_+ \cap S^k_-$) and decompose the set of k-simpleces of $\mathcal{S}^n$ in S^k_+ into three disjoint classes:

$$\begin{aligned}
\mathcal{A}_+ &= \{s^k \subset S^k_+ | s^k \text{ is positive }\},\\
\mathcal{A}_- &= \{s^k \subset S^k_+ | s^k \text{ is negative }\},\\
\mathcal{A}_0 &= \{s^k \subset S^k_+ | s^k \text{ is neutral }\}.
\end{aligned}$$

Consider the sum

$$T = \sum_{s^k \in \mathcal{A}_+} p(f, s^k, k-1) + \sum_{s^k \in \mathcal{A}_-} p(f, s^k, k-1) + \sum_{s^k \in \mathcal{A}_0} p(f, s^k, k-1).$$

We'll count the parity of T in two ways:

Clearly the sum T contains all positive s^{k-1} in S^k_+. Because each $(k-1)$-simplex is a face of exactly two k-simpleces of $\mathcal{S}^n$, all positive s^{k-1} not in S^{k-1} occur in the sum T exactly twice and those which are in S^{k-1} exactly once and $T \equiv p(f, S^{k-1}, k-1) \mod 2$.

Let's count the parity of $N = \sum_{s_k \in \mathcal{A}_0} p(f, s^k, k-1)$. Since a neutral s^k can contribute to the sum N only if $\dim f(s^k) \geq k-1$, we can write $f(s^k) = [e_{i_0}, \ldots, e_{i_k}]$, where all the vertices are either distinct, or one occurs twice, and the signs do not alternate. As $(k-1)$-face of s^k is obtained by romoving one vertex from s^k, we get a positive $(k-1)$-face if and only if there is at most one pair of adjacent vertices with the same sign. If removing one of these vertices provides a positive face, so also will removal of the adjacent one. Thus for each $s^k \in \mathcal{A}_0$, $p(f, s^k, k-1)$ is even, so that

$$T \equiv \sum_{s^k \in \mathcal{A}_+} p(f, s^k, k-1) + \sum_{s^k \in \mathcal{A}_-} p(f, s^k, k-1) \mod 2.$$

As mentioned above, each positive $s^k \in \mathcal{A}_+$ has exactly one positive $(k-1)$-face and so does each negative $s^k \in \mathcal{A}_-$, so that

$$\sum_{s^k \in \mathcal{A}_+} p(f, s^k, k-1) = |\mathcal{A}_+|,$$

$$\sum_{s^k \in \mathcal{A}_+} p(f, s^k, k-1) = |\mathcal{A}_-|,$$

and therefore

$$T \equiv (|\mathcal{A}_+| + |\mathcal{A}_-|) \mod 2.$$

Because f is antipodal, $s^k \subseteq S^k_-$ is positive if and only if $-s^k \subseteq S^k_+$ is negative, which means that $|\mathcal{A}_-| = \left|\{s^k \subseteq S^k_- | s^k \in |\mathcal{A}_+|\}\right|$. Therefore $|\mathcal{A}_+| + |\mathcal{A}_-| = p(f, S^k, k)$ and the proof of combinatorial lemma is complete. □

The following result is well-known and we shall omit the proof; it can be found in numerous books, e.g. [6].

Theorem 4.3.8 (Lebesgue lemma). *Let (X, d) be a compact metric space and $\{G_1, \ldots, G_n\}$ its open covering. Then there exists a $\lambda > 0$ (a* Lebesgue number of covering*) with the property: if $A \subseteq X$, $diam X < \lambda$, then for some $i \in \{1, \ldots, n\} : A \subseteq G_i$.*

Combinatorial lemma and Lebesgue lemma helps us to prove the theorem which directly implies Lusternik-Schnirelman-Borsuk theorem.

Theorem 4.3.9. *Let $M_1, \ldots, M_{n+1}$ be closed subsets of S^n such that none of them contains a pair of antipodal points. If the collection of sets*

$$\{M_1, -M_1, \ldots, M_{n+1}, -M_{n+1}\}$$

is a covering of S^n, then $\bigcap_{i=1}^{n+1} M_i \neq \emptyset$.

Proof. 4.3.9. For contradiction assume that $\bigcap_{i=1}^{n+1} M_i = \emptyset$.

Define open sets $G_i = S^n - M_i$ for $i \in \{1, \ldots, n+1\}$. Let λ be a Lebesgue number of the open covering $\{G_1, \ldots, G_{n+1}\}$ (so that $(\forall A : \text{diam}(A) < \lambda)(\exists i \in \{1, \ldots, n+1\})(M_i \cap A = \emptyset))$.

Denote $-M_i := M_{-i}$. Because $M_i \cap M_{-i} = \emptyset$, $d(M_i, M_{-i}) = \varepsilon_i > 0$.

Let K_r be a symmetric triangulation of S^n such that the diameter of each simplex in K_r is $< r = \min\{\lambda, \varepsilon_1, \ldots, \varepsilon_{n+1}\}$. We first construct an abstract simplicial mapping $f : V(K_r) \to V(\Sigma^n)$ as follows:

For each vertex $p \in V(K_r)$, $f(p) = \text{sgn}(\text{j})(-1)^{\text{j}+1}\text{e}_{|\text{j}|}$, where j is the first index such that $p \in M_j$. Obviously f is antipodal. If f does not map two vertices p_k, p_l of a simplex in K_r to a pair of antipodal vertices, then f is also a vertex map: Assume that $f(p_k) = -f(p_l)$, it means that $p_k \in M_i$ and $p_l \in M_{-i}$ for some i or conversely. But $d(p_k, p_l) < r$ and $d(M_i, M_{-i}) \geq r$.

4.3.7 gives existence of a simplex $s^n = [p_1, \ldots, p_{n+1}]$ such that $f(s^n) = [e_1, -e_2, \ldots, (-1)^n e_{n+1}]$. Due to the definition of f, $p_i \in M_i$ $(i > 0)$, which means that $s^n \cap M_i \neq 0$ for $i = 1, \ldots, n+1$. But $\text{diam}(s^n) < \lambda$, so that for some $i_0 \in \{1, \ldots, n+1\}$, $M_{i_0} \cap s^n = \emptyset$. □

To end the proof of Lusternik-Schnirelman-Borsuk theorem 4.3.3, we simply apply 4.3.9:

Let $\mathcal{M} = \{M_1, \ldots, M_{n+1}\}$ be a closed covering of S^n, where none of the sets M_i contains a pair of antipodal points. Because the collection of sets $\{M_1, -M_1, \ldots, M_{n+1}, -M_{n+1}\}$ covers S^n, there exists $x_0 \in M_1 \cap M_2 \cap \ldots \cap M_{n+1}$. Because $\mathcal{M}$ covers S^n, there exists $j \in \{1, \ldots, n+1\}$ such that $-x_0 \in M_j$, which means that M_j contains both x_0 and $-x_0$.

□

Proof Based on Sperner's Lemma and KKM Theorem

In 1929, Knaster, Kuratowski and Mazurkiewicz applied Sperner's lemma to prove so called KKM-theorem, from which they deduced Brouwer fixed point theorem. The ease at which they obtained Brouwer's theorem brought a question if these three theorems were equivalent. This question remained open until 1974, when Yoseloff [22] showed that Brouwer fixed point theorem implies Sperner's lemma.

Equivalence of these three theorems is one of most important relationships Brouwer's theorem has with any results, since this trinity has a great number of applications.

The KKM theorem was extended to topological vector spaces by Ky Fan [9] in 1952.

Theorem 4.3.10 (Sperner's Lemma (1928)). *Let P be a subdivision of an n-dimensinal simplex $[p_0, \ldots, p_n]$, and let $V(P)$ be the set of all vertices of simplexes in P. If a labeling $h : V(P) \to \{0, 1, \ldots, n\}$ satisfies the condition*

$$h(v) \in \{i_0, i_1, \ldots, i_k\} \text{ whenever } [a_{i_0}, a_{i_1}, \ldots, a_{i_k}] \text{ is the support of } v\ ,$$

then the number of simpleces in P label by $\{0, 1, \ldots, n\}$ is odd.

Theorem 4.3.11 (KKM). *Let $X = \operatorname{conv}\{x_0, x_1, \ldots, x_m\}$ be an m-simplex. Suppose that $A_0, A_1, \ldots, A_m$ are closed subsets of X such that*

$$\operatorname{conv}\{x_{i_0}, x_{i_1}, \ldots, x_{i_k}\} \subseteq \bigcup_{j=0}^{k} A_{i_j}$$

holds for any subset $\{x_{i_j}\}$ of $\{x_i\}_{i=0}^{m}$. Then $\bigcap_{i=0}^{m} A_i \neq \emptyset$.

Proof of 4.3.10 $\Rightarrow$ 4.3.11. Let $X = \text{conv}\{x_0, x_1, \ldots, x_m\}$ be an m-simplex. For each $n \in \mathbb{N}$, we find a simplicial subdivision X^n of X such that the diameter of each m-subsimplex of X^n is less than $1/n$. We find a labeling of the vertices of simplices in X^n, that satisfies the assumptions in Sperner's lemma: each vertex $q \in X^n$ has its support simplex $[x_{i_0}, x_{i_1}, \ldots, x_{i_k}]$. By our assumption, $q \in A_{i_j}$ for some $0 \le j \le k$. By Sperner's lemma, there is an m-simplex in X^n labelled by $\{0, 1, \ldots, n\}$: $\text{conv}\{p_0^n, p_1^n, \ldots, p_m^n\}$. Upon relabeling if necessary, we can assume that $p_i^n \in A_i$. Compactness of X guarantees that for each $i \in \{0, 1, \ldots, m\}$ there is a convergent subsequence of $\{q_i^n\}_{n=1}^{\infty}$. Since the diameter of the simplexes in X^n goes to zero, all the subsequences must converge to a common point. Each A_i is closed in X, and thus this limit must be in $\bigcap_{i=0}^{m} A_i$. □

KKM-theorem easily implies Brouwer fixed point theorem:

Let $S = [s_0, s_1, \ldots, s_m]$ be an m-simplex and $f = (f_1, \ldots, f_m) : S \to S$ be a continuous mapping. Each $x \in S$ is represented as $x = \sum_{i=0}^{m} \alpha_i s_i$, where $\sum_{i=0}^{m} \alpha_i = 1$ and $x_i \ge 0$. Define closed sets

$$A_i = \{x \in S : f_i(x) \le x_i\}.$$

Applying KKM-theorem we get a point $y \in S$ such that $f_i(y) \le y_i$ for each i. Since $\sum y_i = 1 = \sum (f(y))_i$ we obtain $y_i = (f(y))_i$ for each i and consequently $f(y) = y$.

Analytic Proof

Until the 1970's, no proof based on analysis was known. In 1978, John Milnor published the first analytic proof of Brouwer fixed point theorem. Variations on Milnor's proof followed (e.g. Rogers [17]). In past decades, many other proofs analytic in nature appeared (e.g. Samelson [18], Su [20]).

We are going to prove the non-retraction theorem 4.3.1 for C^1 functions which is equivalent to Brouwer fixed point theorem.

Theorem 4.3.12. *The following statements are equivalent:*

(1) Each $f \in C^1(B^n)$, $f : B^n \to B^n$ has at least one fixed point.

(2) There is no retraction $f \in C^1(B^n)$, $f : B^n \to S^{n-1}$.

Proof. (1) $\Rightarrow$ (2). Let f be a mapping described in (2). Denote $\phi = -f$. Then $\phi \in C^1(B^n)$ and $\phi : B^n \to B^n$. Due to (1) there exists $x_0 \in B^n$ such that $\phi(x_0) = x_0$. Because $|\phi(x_0)| = |-f(x_0)| = 1$, $|x_0| = 1$ and $x_0 = f(x_0) = -f(x_0) = -x_0$. In other words $x_0 = 0$, which is a contradiction.

non(1) $\Rightarrow$ non(2). Let $\phi : B^n \to B^n$, $\phi \in C^1(B^n)$ without a fixed point. We're going to construct a retraction $f : B^n \to S^n$, $f \in C^1(B^n)$.

Let $B^{n^*} \supset B^n$ be an open ball, $\phi^* : B^{n^*} \to \mathbb{R}^n$, $\phi^* \in C^1(B^{n^*})$, $\phi^*|_{B^n} = \phi$.

We'll show that $\forall x \in B^n : \phi^*(x)x \neq 1$: $\phi^*(x)x \leq |\phi^*(x)|\,|x| \leq 1$. If there was $x \in B^n$, such that $\phi^*(x)x = 1$ then $|x| = |\phi^* x| = 1$ and there exists $c \in \mathbb{R}$ such that $\phi^*(x) = cx$ and thus $c = cx \cdot x = \phi^*(x)x = 1$ and $\phi^*(x) = x$.

So we can suppose that $\phi^*(x)x \neq 1$ on B^{n^*} and define

$$f_1^*(x) = x - \frac{1 - xx}{1 - \phi^*(x)x}\phi^*(x), x \in B^{n^*}.$$

Obviously $f_1^* \in C^1(B^{n^*})$.

f_1^* is nonzero on B^n: If there was $x \in B^n$ then there would exist $c \in \mathbb{R}$ such that $x = c\phi^*(x)$ and thus

$$0 = f_1^*(x) = \frac{x - (\phi^*(x)x)x - \phi^*(x) + (x \cdot x)\phi^*(x)}{1 - \phi^*(x)x} = \frac{x - \phi^*(x)}{1 - \phi^*(x)x},$$

so x would be the fixed point of ϕ. We showed that $f_1^*(x) \neq 0$ on B^n and we can suppose that $f_1^* \neq 0$ on B^{n^*}. $f^* = f_1^* / |f_1^*|$ is the required retraction: $f^* \in C^1(B^{n^*})$, $|f^*| = 1$ on B^{n^*}, $f_1^*(x) = x$ for each $x \in S^{(n-1)}$. □

Now if we prove non-retraction theorem for C^1 functions, Brouwer fixed point theorem will be proven.

Theorem 4.3.13. *There is no retraction $f \in C^1(B^n)$, $f : B^n \to S^{n-1}$.*

Proof. For contradiction we assume that there exists $f \in C^1(B^n)$, $f : B^n \to S^{n-1}$ such that $f(x) = x$ for all $x \in S^{n-1}$.

Let $B^{n^*} \supset B^n$ and $f^* : B^{n^*} \to \mathbb{R}^n$ such that $f^* \in C^1(B^{n^*})$ and $f^*|_{B^n} = f$. Define $g^* \in C^1(B^{n^*})$ as $g(x) = f^*(x) - x$, $x \in B^{n^*}$. We can suppose that there exists $k \in \mathbb{R}$, for which $||(g^*)'(x)|| \leq k$ for every $x \in B^{n^*}$. Mean value theorem gives us an inequality $|g^*(x) - g^*(y)| \leq k\,|x - y|$, $\forall x, y \in B^{n^*}$.

Define deformations $f_t^*(s) = x + tg^*(x)$, for $t \in [0, 1]$, $x \in B^{n^*}$. Clearly $f_t^*(x) = x$ for $x \in S^{n-1}$, $t \in [0, 1]$. We'll prove that for $t \in [0, 1/k)$, f_t^* is injective and regular on B^{n^*}:

Let $x, y \in B^{n^*}$ and suppose that $f_t^*(x) = f_t^*(y)$. Then $|x-y| = |t|\,|g^*(x) - g^*(y)| \le tk\,|x-y|$. From $tk < 1$ we see that $x = y$.

$(f_t^*)'(x) = I + t(g^*)'(x)$. Since $||t(g^*)'(x)|| \le tk < k(1/k) = 1$ for all $x \in B^{n^*}$, $(f_t^*)(x)$ is an injective linear mapping and f_t^* is regular.

Due to the inverse mapping theorem, f_t^* is an open mapping and thus $f_t^*(B_0^n)$, B_0^n is the interior of B^n, is an open set. For $x \in B^n$, $|f_t^*(x)| \le 1$, so that $f_t^*(B^n) \subseteq B^n$. Since f_t^* is a compact mapping, $f_t^*(B^n)$ is a closed subset of B^n. $f_t^*(B^n) = f_t^*(B_0^n) \cup f_t^*(S^{n-1}) = f_t^*(B_0^n) \cup S^{n-1}$. $f_t^*(B_0^n) \cap S^{n-1} = \emptyset$, because f_t^* is injective. It yields $f_t^*(B_0^n) = f_t^*(B^n) \cap B_0^n$ and so $f_t^*(B_0^n)$ is both open and closed subset of B_0^n. However B_0^n is a connected space, so that $f_t^*(B_0^n) = B_0^n$ and $f_t^*(B^n) = B^n$.

All the ssumptions of the substitution theorem are satisfied (λ^n is the Lebesgue measure in $\mathbb{R}^n$):

$$\lambda^n(B_0^n) = \lambda^n(f_t^*(B_0^n)) = \int_{B_0^n} |\det(f_t^*)'(x)|\,dx = \int_{B_0^n} \det(f_t^*)'(x)dx. \qquad (4.1)$$

The function $t \mapsto \det(f_t^*)'(x)$, $x \in B_0^n$ is a polynomial, which is 1 for $t = 0$ and non-zero for $t \in [0, 1/k)$ (because $(f_t^*)'$ is injective for $t \in [0, 1/k)$ and each $x \in B^{n^*}$). This verifies the last equality in (4.1).

From (4.1) we can see that the polynomial $t \mapsto \int_{B_0^n} \det(f_t^*)'(x)dx$ is constant on the interval $[0, 1/k)$ and thus for all $t \in [0, 1)$.

$$\lambda^n(B_0^n) = \int_{B_0^n} \det(f_1^*)'(x)dx = \int_{B_0^n} \det(f^*)'(x)dx.$$

If for some $x_0 \in B_0^n$ holds $\det(f^*)'(x_0) \ne 0$, then the image of some neighbourhood of x_0 is an open set. This cannot happen, beacuse $f^*(B^n) = S^{n-1}$. It means that $\lambda^n(B^n) = 0$, which is a contradiction. □

Proof Based on Topological Degree in $\mathbb{R}^n$

Definition 4.3.14. Topological Degree in $\mathbb{R}^n$. Denote by $\mathcal{D}$ the set of all triples (f, G, p), where G is an open set in $\mathbb{R}^n$, $f : \overline{G} \to \mathbb{R}^n$ is a continuous mapping and $p \in \mathbb{R}^n \setminus f(\partial G)$. A mapping $\deg : \mathcal{D} \to \mathbb{R}$ is a *Topological degree in* $\mathbb{R}^n$ if the following conditions are satisfied:

(1) If f is the identity on G and $p \in G$ then $\deg(f, G, p) = 1$.

(2) If $G_1, G_2 \subseteq G$ are two disjoint open sets and $p \notin f(\overline{G} \setminus G_1 \cup G_2)$ then $\deg(f, G, p) = \deg(f, G_1, p) + \deg(f, G_2, p)$.

(3) If $H : [0,1] \times \overline{G} \to R^n$ is a continuous mapping, $f_0(x) = H(0,x)$, $f_1(x) = H(1,x)$ and $H(t,x) \neq p$ for $t \in [0,1]$ and $x \in \partial G$, then $\deg(f_0, G, p) = \deg(f_1, G, p)$.

(4) If $\deg(f, G, p) \neq 0$, then there exists $x \in G$ such that $f(x) = p$.

Theorem 4.3.15 (Leray-Schauder [8]). *Topological degree in $\mathbb{R}^n$ exists and is unique.*

Proof of Brouwer's theorem. Let $f : B^n \to B^n$ be a continuous mapping without a fixed point (i.e. $\forall x \in B^n, f(x) \neq x$). Define the homotopy H as $H(t,x) = x - tf(x)$, for $x \in \mathbb{R}^n$ and $t \in [0,1]$. For $x \in S^{n-1}$ and $t \in [0,1]$, $H(t,x) \neq 0$: for $t = 1$ it follows from the assumption, for $t \in [0,1)$, $||tf(x)|| \leq t < 1$ and thus $x \neq tf(x)$. Denote B_o^n the unit open ball in $\mathbb{R}^n$, and $g_0(x) := H(0,x) = x$ and $g_1(x) := H(1,x) = x - f(x)$. Due to 4.3.14(3): $\deg(g_0, G, 0) = \deg(g_1, G, 0)$. Because g_0 is identical on G, due to 4.3.14(1) $\deg(g_0, G, 0) = 1$ and consequently $\deg(g_1, G, 1) = 1$. 4.3.14(4) gives us existence of $x \in G$ with $g_1(x) = 0$, which contradicts $x \neq f(x)$ for all $x \in B^n$. □

Remark. Some of the proofs of Brouwer fixed point theorem are elementary and thus long (such as our Combinatorial and Analytic proof). Some other proofs are very short but they require deeper knowledge of some mathematical theories (such as our proof based on topological degree). For example a proof using algebraic topology based on the theory of homology groups is very short. Having counted homology groups of the unit closed n-ball and its sphere, it is easy to prove that there is no retraction of the ball onto the sphere.

Chapter 5

Extensions

Brouwer fixed point theorem follows three assumptions - n-dimensional Eucleidean space, compact convex subset of this space and a continuous mapping from the subset to itself. The goal of the extensions is to relax some assumptions, while usually losing some generality of the others.

We show two extensions of Brouwer fixed point theorem to infinite dimensions. First valid for all normed spaced, but only for a subspace of continuous maps - compact maps. The main idea of this extension is approximation of compact maps between normed linear spaces by finite-dimensional maps.

Second part presents an extension of KKM theorem (being equivalent to Brouwer fixed point theorem) to infinite dimension, to so called KKM-principle. Three simple, but important applications of KKM-principle to fixed point theory follow without proofs.

5.1 Compact maps

Definition 5.1.1. Let X and Y be topological spaces. A continuous map $f : X \to Y$ is called *compact* if $f(X)$ is contained in a compact subset of Y. Let A be a subset of a metric space (X, d) and $F : A \to X$. Given $\varepsilon > 0$, any point $a \in A$ with $d(a, F(a)) < \varepsilon$ is called an *ε-fixed point* for F.

Proposition 5.1.2. *Let A be a closed subset of a metric space (X, d) and $F : A \to X$ a compact mapping. Then F has a fixed point if and only if it has ε-fixed point for each $\varepsilon > 0$.*

Proof. If F has a fixed point a_0, then a_0 is an ε-fixed point for each $\varepsilon > 0$.

For each $n \in \omega$, let a_n be a $1/n$-fixed point of F, so that $d(a_n, F(a_n)) < 1/n$. Since F is compact, there exist a subsequence $\{a_{n_k}\}_{k=1}^{\infty} \subseteq \{a_n\}_{n=1}^{\infty}$ such that $F(a_{n_k}) \to x \in \overline{F(A)}$. It follows that $a_n \to x \in A$. Since F is continuous, $F(a_n) \to F(x)$; consequently $x = F(x)$ and x is a fixed point of F. □

Theorem 5.1.3 (Schauder approximation theorem). *Let X be a topological space, let C be a convex subset of a normed linear space V and let $F : X \to C$ be a compact mapping. Then for each $\varepsilon > 0$, there exists a finite set $N = \{c_1, \ldots, c_n\} \subset F(X) \subset C$ and a finite-dimensional mapping $F_\varepsilon : X \to C$ such that:*

(i) $||F_\varepsilon - F(x)|| < \varepsilon$ *for all* $x \in X$,

(ii) $F_\varepsilon(X) \subset \mathrm{conv} N \subset C$.

Theorem 5.1.4 (Schauder fixed point theorem). *Let C be a convex (not necessarily closed) subset of a normed linear space V. Then each compact map $F : C \to C$ has at least one fixed point.*

Proof. According to 5.1.2, it is enough to prove that F has an ε-fixed point for each $\varepsilon > 0$.

Fix $\varepsilon > 0$. By 5.1.3, there is a $F_\varepsilon : C \to C$ with properties:

$$||F_\varepsilon(x) - F(x)|| < \varepsilon, \qquad x \in C$$
$$F_\varepsilon(C) \subset \mathrm{conv}(N) \subset C \quad \text{for some finite set} N \subset C.$$

Since $F_\varepsilon(\mathrm{conv}(N)) \subset \mathrm{conv}(N)$ and $\mathrm{conv}(N)$ is homeomorphic to a finite-dimensional closed ball, it follows from Brouwer fixed point theorem that F_ε has a fixed point x_ε. Because $||x_\varepsilon - F(x_\varepsilon)|| = ||F_\varepsilon(x_\varepsilon) - F(x_\varepsilon)|| \leq \varepsilon$, x_ε is the required ε-fixed point for F. □

5.2 KKM-Principle

KKM-principle extends KKM theorem that is equivalent to Brouwer fixed point theorem (see section Proof Based On Sperner's Lemma and KKM Theorem) to infinite-dimensional topological vector spaces (Ky Fan [9] (1952)). Further extensions and their applications have given rise to a branch of research known as KKM theory. Its results are widely used as a tool for fixed point theory, minimax problems, dimension theory, and mathematical economics.

Definition 5.2.1. Let V be a vector space and $X \subset V$ an arbitraty subset. A mapping $G : X \to 2^E$ is called a *Knaster-Kuratowski-Mazurkiewicz map* (or simply a KKM-map) if for each finite subset $\{x_1, \ldots, x_s\} \subset X$,

$$\text{conv}\{x_1, \ldots, x_s\} \subset \bigcup_{i=1}^{s} G(x_i)$$

We present Ky Fan's extension of KKM-theorem without proofs:

Theorem 5.2.2 (KKM-Map Principle). *Let X be an arbitraty subset of a vector space V, and let $G : X \to V$ be a KKM-map such that each $G(x)$ is finitely closed (i.e. for each finite-dimensional flat $L \subset V$, $G(x) \cap L$ is closed in the Euclidean topology of L). Then the family $\{G(x)|x \in X\}$ of sets has the finite intersection property.*

Theorem 5.2.3 (Ky Fan). *(Topological KKM-principle). Let X be an arbitraty subset of a topological vector space V and let $G : X \to 2^V$ be a KKM-map. If all the sets $G(x)$ are closed in V and if one of them is compact, then $\bigcap\{G(x)|x \in X\} \neq \emptyset$.*

The following lemma says that KKM-principle can be used to show the existence of the best approximations.

Lemma 5.2.4 (Ky Fan). *Let C be a compact convex subset of a normed space V and let $F : C \to V$ be continuous. Then there exists at least one point $y_0 \in C$ such that*

$$||y_0 - F(y_0)|| = \inf_{x \in C} ||x - F(y_0)|| .$$

Proof. Define a KMM-mapping $G : C \to 2^V$ by

$$G(x) = \{y \in C| \, ||y - F(y)|| \leq ||x - F(y)||\}$$

. ☐

The following results from fixed point theory generalizing Brouwer fixed point theorem can be obtained applying KKM-principle.

Theorem 5.2.5 (Ky Fan). *Let C be a compact convex subset of a normed space V. Let $F : C \to V$ be a continuous mapping such that for each $c \in C$ with $c \neq F(c)$ the line segment $[c, F(c)]$ contains at least two points of C. Then F has a fixed point.*

Theorem 5.2.6 (Schauder-Tychonoff). *Let C be a compact convex set in a locally convex topological space V. Then every continuous mapping $F : C \to C$ has a fixed point.*

Theorem 5.2.7 (Markoff-Kakutani). *Let C be a compact convex set in a locally convex linear space V and let $\mathcal{F}$ be a commuting family of continuous affine mappings of C into itself. Then $\mathcal{F}$ has a common fixed point.*

Markoff-Kakutani theorem has numeruous applications. Kakutani also proved that it implies Hahn-Banach theorem.

Chapter 6

Applications and Related Results

The application of fixed point theory is vast. It is used in all fields of both theoretical and applied mathematics, where equations in topological spaces appear. However it can also be applied to problems, that seem to be far away from topology, such as problems in combinatorics.

Brouwer fixed point theorem and its extensions are existential tools. They can be used in game theory to show the existence of a winner in some two player mathematical games, in theory of differential equations, in economics, in physics (any time there is a spot on the earth's surface where the winds are not blowing in any vertical direction - the flow lines of the winds on the earth's surface constitute a continuous mapping of that surface to another point thereon), to simplify existing proofs of many results, etc.

Among the most important applications of Brouwer fixed point theorem, KKM-principle is usually stated. Its extensions and their applications gave birth to a mathematical branch, KKM-theory, which has a wide range of applications in fixed point theory, minimax problems, dimension theory, and mathematical economics.

We prove several well-known fundamental results applying Borsuk-Ulam theorem (it implies Brouwer fixed point theorem), and one surprising application of fixed point theory in graph theory. Part Fixed Point Spaces covers two basic observations on preserving fixed point property and an example of an infinite-dimensional fixed point space.

6.1 Borsuk-Ulam Theorem

Borsuk-Ulam theorem, which easily implies Brouwer fixed point theorem (see 4.3.2), and its several equivalent formulations has a wide range of important and surprising applications in different parts of mathematics (see [12]).

Theorem 6.1.1 (Borsuk-Ulam). *Every continuous mapping $f : S^n \to \mathbb{R}^n$ sends at least one pair of antipodal points to the same point.*

Proof. Let $f : S^n \to \mathbb{R}^n$ be continuous and $f(x) \neq f(-x)$ for all $x \in S^n$. Define a continuous mapping $g : S^n \to S^{n-1}$ as follows

$$g(x) = \frac{f(x) - f(-x)}{||f(x) - f(-x)||}.$$

As $g(-x) = \frac{f(-x)-f(x)}{||f(-x)-f(x)||} = g(x)$, g is antipodal. It however contradicts 4.3.2 □

Theorem 6.1.2 (Invariance of dimension number). $\mathbb{R}^n$ *is not homeomorphic to* $\mathbb{R}^m$*, whenever* $n \neq m$.

Proof. Let $n > m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ continuous. $f|_{S^m} : S^m \to \mathbb{R}^n$ is continuous and 6.1.1 gives us existence of a point $x \in S^m$ with $f(x) = f(-x)$. It means that f is not injective. □

Theorem 6.1.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous involution (i.e. $f \circ f =$ id), then f has a fixed point.*

Sketch of proof. Suppose that $x \neq f(x)$ for every $x \in \mathbb{R}^n$. We will construct a continuous mapping $g : S^n \to \mathbb{R}^n$ that will contradict this assumption. The construction goes by induction on the dimension of the sphere. In dimension 1, $S^0 = \{-1, 1\}$ and we set $g_0(1) = x_0$ for any $x_0 \in \mathbb{R}^n$, and $g_0(-1) = f(x_0)$. We extend g_0 to a mapping g_1 continuous on S_1^+ and set $g_1(x) = f(g(-x))$ for each $x \in S_1^-$. In the next step, we extend g_1 to a g_2 continuous on S_2^+ and set $g_2(x) = f(g(-x))$ for $x \in S_2^-$. Finally we get $g_n = g : S^n \to \mathbb{R}^n$ such that for every $x \in S_n^-$ $g(x) = f(g(-x))$. Borsuk-Ulam theorem guarantees existence of a point $x \in S_+^n$ with $g(x) = g(-x) = f(g(x))$ and we get a fixed point $g(x)$ of the mapping f. □

Theorem 6.1.4 (Fundamental Theorem of Algebra). *Every polynomial has a complex root.*

Sketch of proof. Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial, $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$. We want to prove that there exists $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.

Set $\rho = |a_0| + |a_1| + \ldots + |a_{n-1}| + 1 + 1$ and define a continuous $f : \mathbb{C} \to \mathbb{C}$ as follows

$$f(z) = \begin{cases} z - \frac{P(z)}{\rho z^{n-1}} & |z| \geq 1 \\ z - \frac{P(z)}{\rho e^{i(n-1)\varphi r}} & |z| \leq 1, z = re^{i\varphi}. \end{cases}$$

We will show that f maps $B = \{z : |z| \leq \rho\}$ to itself. For $z \in B$, $|z| \geq 1$ we get the following inequalities

$$\begin{aligned} |f(z)| &= \left| z - \frac{z}{\rho} - \frac{a_{n-1}z^{n-1} + \ldots + a_0}{\rho z^{n-1}} \right| \\ &\leq |z| \left| 1 - \frac{1}{\rho} \right| + \left| \frac{1}{\rho}(a_{n-1} + \ldots + \frac{a_0}{z^{n-1}}) \right| \\ &\leq \rho \frac{\rho - 1}{\rho} + \frac{1}{\rho}(|a_{n-1}| + |a_{n-2}| \ldots + |a_0|) \leq \rho - 1. \end{aligned}$$

Similarly for $|z| \leq 1$

$$\begin{aligned} |f(z)| &\leq |z| + \frac{P(z)}{\rho} \leq 1 + \frac{1}{\rho} |z_n + \ldots + a_0| \\ &\leq 1 + \frac{1}{\rho}(1 + |a_{n-1}| + \ldots + |a_0|) \leq 1 + \frac{\rho - 1}{\rho} \leq \rho. \end{aligned}$$

According to Bouwer fixed point theorem, f has a fixed point, say z_0, and from the definition of f, $P(z_0) = 0$. ☐

6.2 Fixed Point Spaces

Definition 6.2.1. A topological space X is called a *fixed point space* if every continuous mapping $f : X \to X$ has a fixed point.

Observation 6.2.2. *Let X be a fixed point space, so also is every space homeomorphic to X.*

Proof. Suppose that X is a fixed point space and $h : X \to Y$ is a homeomorphism. Let $f : Y \to Y$ be continuous. Then $h_{-1} \circ f \circ h : X \to X$ is continuous and thus has a fixed point, say x. Then $f(h(x)) = h(x)$ and Y is a fixed point space. ☐

Observation 6.2.3 (Borsuk(1931) [1]). *If X is a fixed point space, so also is every retract of X.*

Proof. Let X be a fixed point space, and let $r : X \to Y \subseteq X$ a retraction. Suppose $f : Y \to Y$ is continuous. Then $f \circ r : X \to X$ is continuous and hence has a fixed point, i.e. for some $x \in X$, $f(r(x)) = x$. Since $x \in Y$, $r(x) = x$, so that x is a fixed point for f and consequently Y is a fixed point space. □

Proposition 6.2.4 (Borsuk(1932) [2]). *Let (X, d) be a compact metric space and assume that for each $\varepsilon > 0$, there is a continuous mapping f_ε such that*

(i) $d(x, f_\varepsilon(x)) < \varepsilon$ *for each* $x \in X$,

(ii) $f_\varepsilon(X)$ *is a fixed point space.*

Then X is a fixed point space.

Proof. It easily follows from 5.1.2. □

Example 6.2.5. *The Hilbert cube I^∞ is a fixed point space.*

Proof. The Hilbert cube can be considered as a metric space, indeed as a specific subset of a Hilbert space with countably infinite dimension. For our purposes, it is the best to think of it as a product

$$[0, 1] \times [0, 1/2] \times [0, 1/3] \times .....,$$

so that an element of the Hilbert cube is an infinite sequence (x_n), that satisfies $0 \leq x_n \leq 1/n$. Any such sequence belongs to the Hilbert space l_2, so the Hilbert cube inherits the metric from here.

We will define ε-mappings as in 6.2.4. For every $\varepsilon > 0$, we find an $n \in \omega$ satisfying $\sum_{k>n} 1/k^2 < \varepsilon$, and define $p_\varepsilon : I^\infty \to I^\infty$ as a projection to first n coordinates (i.e. $p_\varepsilon((x_n)_{n=1}^\infty) = (x_1, x_2, \ldots, x_n, 0 \ldots))$. $p_\varepsilon(I^\infty) = [0, 1] \times [0, 1/2] \times \ldots \times [0, 1/n]$ is an n-dimensional compact, convex subset of a normed space, and thus due to Brouwer fixed point theorem it is a fixed point space. As all the assumptions of 6.2.4 are satisified, I^∞ is a fixed point space. □

6.3 Evasiveness of Graph Properties

A number of important result in combinatorics, discrete geomtery and the theoretical computer science have been proved by surprising applications of topology. We show the link between monotone graph functions and simplicial complexes, that can be regarded both as a combinatorial and topological subject, which is the base of Kahn-Saks-Sturtevant technique. They proved that the simplicial complex associated with a non-evasive monotone function is contractible. They applied a fixed point theorem which states, that every continuous mapping from a finite contractible polyhedron to itself has a fixed point. For f a monotone graph property on graphs with a prime power number of nodes, they characterized all the possible fixed points sets of simplicial mappings to show that if f is non-trivial, no fixed point can exist, and thus f is evasive.

Aanderaa-Karp-Rosenberg Conjecture

Definition 6.3.1. A *simple decision three* for a function $f : \{0,1\}^n \to \{0,1\}$ is a binary tree with nodes labelled with elements of $\{0, 1, \ldots, n\}$, and leaves labelled by either 0, or 1. If a node has a label i, then the test performed at that node is to examine the i-th bit of the input. If the result is 0, one descends into the left subtree, whereas if the result is 1, one descends into the right subtree. The label of the leaf reached by this procedure is the value of the function on the input.

Definition 6.3.2. A function $f : \{0,1\}^n \to \{0,1\}$ is called *evasive*, if the depth of the simple decision tree of f is n.

We can represent arbitrary functions on graphs by encoding the adjecency matrix in the input to the function. For an undirected graph on n nodes, we let $x_{ij}^G : 1 \leq i < j \leq n$ represent the presence or absence of the edge (i, j) by taking value 1 or 0 respectively.

We restrict out attention to graph properties - boolean functions whose values are independent of the labeling of the nodes of the graph:

Definition 6.3.3. A boolean function $f : \{x_{ij} : 1 \leq i < j \leq n\} \to \{0,1\}$ is a *graph property* if for any permutation $\pi \in S_n$ and for any x

$$f(\ldots, x_{ij}, \ldots) = f(\ldots, x_{\pi(i)\pi(j)}, \ldots).$$

A graph property is called *monotone* if adding edges preserves the property.

Conjecture 6.3.4 (Aanderaa-Karp-Rosenberg Conjecture). *Any monotone, non-trivial graph property is evasive.*

Simplicial Complexes Associated with Monotone Boolean Functions

Lemma 6.3.5. *Let K be a simplicial complex. If for some $v \in K$, the polyhedrons given by subcomplexes $K \setminus v = \{X \in K : v \notin X\}$ and $K/v = \{X \in K : v \notin X, X \cup \{v\} \in K\}$ are contractible, then $\widehat{K}$ is contractible.*

A monotone boolean function $f \neq 1$ gives a simplicial complex

$$K_f = \{S \subseteq \{1, 2, \ldots, n\} : f(x^S) = 0\}.$$

Naturally

$$K_{f|_{x_i=0}} = \{S \subseteq \{1, \ldots, i-1, i+1, \ldots, n\} : S \in K_f\} = K_f \setminus i,$$
$$K_{f|_{x_i=1}} = \{S \subseteq \{1, \ldots, i-1, i+1, \ldots, n\} : S \cup \{i\} \in K_f\} = K_f / i.$$

Lemma 6.3.6 (Kahn-Saks-Sturtevant). *If $f \neq 1$ is monotone and non-evasive, then $\widehat{K}_f$ is contractible.*

Proof. Proof goes by induction on the number of nodes. □

Fixed Points of Simplicial Mappings

Let $\varphi : V(K) \to V(K)$ be a one-to-one simplicial mapping. We want to characterize all the fixed points for $\widehat{\varphi}$. Suppose that x is a fixed point with the support simplex Δ. Since $\widehat{\varphi(\Delta)}$ contains x, it contains $\widehat{\Delta}$. However, Δ and $\varphi(\Delta)$ are of the same size, which implies that $\varphi(\Delta) = \Delta$ and consequently φ premutes the vertices of the simplex Δ. Clearly the barycenter of $\widehat{\Delta}$ must also be a fixed point. If the cycles of the permutation induced by φ on Δ are $H_1, \ldots, H_k$, then $\widehat{H_i}$ is a face of $\widehat{\Delta}$ with the barycenter a fixed point. Also, any convex combination of these barycenters is a fixed point.

Let us show that the convex combinations of the barycenters of the faces corresponding to the cycles of φ are all the fixed points of $\widehat{\varphi}$. We showed that if x with the support simplex Δ is a fixed point, then $\varphi|_\Delta$ is a permutation of the vertices of Δ. Since

$$\sum_{v \in \Delta} \alpha_v \widehat{v} = x = \widehat{\varphi}(x) = \sum_{v \in \Delta} \alpha_v \widehat{\varphi(v)},$$

it follows that $\alpha_v = \alpha_{\varphi(v)}$ (x is a unique convex combination of vertices in Δ). So that for each cycle H_i of φ on Δ, we find β_i such that $u \in H_i \Rightarrow \alpha_u = \beta_i$ and we get

$$x = \sum_{v \in \Delta} \alpha_v \widehat{\Delta} = \sum_i \sum_{v \in H_i} \beta_i \widehat{v} = \sum_i \beta_i |H_i| w_i,$$

where w_i is the barycenter of H_i.

Application of Fixed Point Theory

We have everything prepared to a apply a consequence of Lefschetz-Hopf fixed point theorem, a generalization of Brouwer fixed point theorem, to show evasiveness of a non-trivial monotone boolean function invariant under a cyclic permutation of its imputs.

Theorem 6.3.7 (Lefschetz-Hopf). *A nullhomotopic map of any connected polyhedron always has a fixed point.*

We restrict out attention to contractible polyhedra, so for our purposes we refurmulate the theorem for this special case:

Theorem 6.3.8. *Every continuous map $f : \widehat{K} \to \widehat{K}$ from a contractible polyhedron $\widehat{K}$ into itself has a fixed point.*

Lemma 6.3.9. *Let $f : \{0,1\}^n \to \{0,1\}$ be a monotone, non-trivial function invariant under a cyclic permutation of its imputs. Then f is evasive.*

Proof. Let us assume without loss of generality that f is invariant under the permutation $\varphi(i) = i + 1 \mod n$ and assume that f is non-evasive, so that the simplicial complex $\widehat{K_f}$ is contractible. Since f is invariant under φ, φ is a simplicial map of K_f, so that φ has a fixed point. As discussed in the previous part, fixed points correspond to the cycles of φ contained in K_f. Since the only cycle of φ is $\{1, 2, \ldots, n\}$, this set must be a simplex of K_f. Thus $f(1, 1, \ldots, 1) = 0$, which contradicts that f is non-trivial. □

Applying previous lemma and a theorem on fixed points of groups of simplicial mappings of contractible polyhedrons, we can prove the main evasive result of this section:

Theorem 6.3.10. *Let Γ be a group of simplicial mappings of a contractible polyhedron $\widehat{K}$ onto itself. Let Γ_1 be a normal subgroup of Γ with $|\Gamma_1| = p^k$ for a prime p, and with Γ/Γ_1 cyclic. Then there exists an $x_0 \in \widehat{K}$ such that $\widehat{\varphi}(x_0) = x_0$ for each $\widehat{\varphi} \in \Gamma$.*

Theorem 6.3.11. *Let f be a non-trivial monotone graph property on graphs with a prime power p^k number of nodes. Then f is evasive.*

Proof. Let G be a graph of a prime power p^k number of nodes. Identify the nodes of G with $GF(p^k)$ ($GF(p^k)$ is the Galois field of order p^k). Consider the linear mapps $x \mapsto ax + b : GF(p^k) \to GF(p^k)$ as a group Γ, and the mappings $x \mapsto x + b$ as Γ_1. Clearly $|\Gamma_1| = p^k$, and its normality follows from $(((ax + b) + b') - b)/a = x + b/a$. The factor group Γ/Γ_1 is isomorphic to the group of mappings $x \mapsto ax : a \neq 0$, i.e. the multiplicative group of $GF(p^k) - \{0\}$, which is known to be cyclic. If f is non-evasive, then $\widehat{K_f}$ is contractible, and due to the preceding lemma, all the actions of Γ have a common fixed point on $\widehat{K_f}$. Since Γ is transitive on the edges, the only cycle of Γ consists of all of the edges. But that means that K_f must have as an element the set of all edges, and $f = 0$. $\square$

Bibliography

[1] K. Borsuk. Sur les retractes. *Fund. Math.*, 17:152–170, 1931.

[2] K. Borsuk. Einige Sätze über stetige Streckenbilder. *Fund. Math.*, 18:198–213, 1932.

[3] K. Borsuk. Drei Sätze über die n-dimensionale euklidische Sphäre. *Fund. Math.*, 20:177–190, 1933.

[4] L. E. J. Brouwer. On continuous one-to-one transformations of surfaces into themselves. *Proc. Kon. Ned. Ak. V. Wet. Ser. A*, 11:788–798, 1909.

[5] L. E. J. Brouwer. Invarianz des n-dimensionalen Gebiets. *Math. Ann.*, 71:305–313, 1912.

[6] J. Dugundji and A. Granas. *Fixed Point Theory, Vol. 1.* PWN - Polish Scientific Publishers, Warszawa, 1982.

[7] J. Dugundji and A. Granas. *Fixed Point Theory.* Springer-Verlag New York, 2003.

[8] J. Leray et J. Schauder. Topologie et équations fonctionnelles. *Ann. Sci. École Norm. Sup.*, 51:44–78, 1934.

[9] K. Fan. Fixed–point and minimax theorems in locally convex topological linear spaces. *Proc. Nat. Acad. Sci. USA*, 38, 1952.

[10] S. Kakutani. Topological properties of the unit sphere of a Hilbert space. *Proc. Imp. Acad. Tokyo*, 19:269–271, 1943.

[11] B. Knaster, K. Kuratowski, and S. Mazurkiewicz. Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe. *Fund. Math.*, 14:132–137, 1920.

[12] J. Matoušek. *Using the Borsuk-Ulam Theorem*. Springer-Verlag, Berlin Heidelberg, 2003.

[13] J. Milnor. Analytic proofs of the "hairy ball theorem" and the Brouwer fixed point theorem. *Amer. Math. Monthly*, 85:521–524, 1978.

[14] C. Miranda. Un'osservazione su un teorema di Brouwer. *Boll. Un. Math. Ital.*, 3:5–7, 1940.

[15] H. Poincaré. Sur certaines solutions particulierès du problème des trois corps. *C. R. Acad. Sci. Paris*, 97:251–252, 1883.

[16] H. Poincaré. Sur certaines solutions particulierès du problème des trois corps. *Bull Astronomique*, 1:63–74, 1884.

[17] C. A. Rogers. A less strange version of Milnor's proof of Brouwer's fixed point theorem. *Amer. Math. Monthly*, 87:525–527, 1980.

[18] H. Samelson. On the Brouwer fixed point theorem. *Portugal. Math.*, 22:189–191, 1963.

[19] E. Sperner. Neuer beweis für die Invarianz der Dimensionszahl und des Gebietes. *Abh. Math. Sem. Univ. Hamburg*, 6:265–272, 1928.

[20] F. E. Su. Borsuk-Ulam implies Brouwer: A direct construction. *Amer. Math. Monthly*, 104:855–859, 1997.

[21] A. Tychonoff. Ein Fixpunktsatz. *Math. Ann.*, 111:767–776, 1935.

[22] M. Yoseloff. Topological proofs of some combinatorial methods. *Math. Anal. Appl.*, 46:95–111, 1974.

PŘIJATO K OBHAJOBĚ

31-05-2006

PŘEDSEDA KOMISE PRO BSZZ
STUDIJNÍ PROGRAM MATEMATIKA