Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

BAKALÁŘSKÁ PRÁCE

Petr Pošta

Variations of Banach fix point theorem

Katedra matematické analýzy

Vedoucí bakalářské práce: Prof. RNDr. Miroslav Hušek, DrSc.
Studijní program: Matematika
Obecná matematika
(Matematická analýza)

2006

Poděkování: Chtěl bych především poděkovat vedoucímu své bakalářské práce, prof. RNDr. Miroslavu Huškovi, DrSc., za věnovaný čas, podnětné poznámky a rady a nikoliv naposled za upozornění na řadu nepřesností, které se v průběhu prací na tezi vyskytly. Dále bych chtěl touto formou poděkovat všem, kteří jakkoliv přispěli k tomuto textu, především Mgr. Miroslavu Poštovi a Ing. Severinu Poštovi, PhD. za poskytnutou morální podporu a různé inspirativní návrhy. Stejný dík patří také Pavlu Ludvíkovi, Peteru Bellovi a dalším.

Prohlašuji, že jsem svou bakalářskou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce a jejím zveřejňováním.

V Praze dne 30. května 2006 Petr Pošta

Název práce: Variace Banachovy věty o pevném bodě
Autor: Petr Pošta
Katedra (ústav): Katedra matematické analýzy
Vedoucí bakalářské práce: prof. RNDr. Miroslav Hušek, DrSc.
e-mail vedoucího: mhusek@karlin.mff.cuni.cz

Abstrakt: V předložené práci studujeme rozličné důsledky a zobecnění Banachovy věty o pevném bodě. V první části studujeme důsledky klasického Banachova principu kontrakce: posloupnosti kontraktivních zobrazení, různé variace podmínky kontraktivnosti zobrazení, příklady použití v Banachových prostorech, diskrétní princip kontrakce (Eilenbergova a Jachymského verze) a otázku ekvivalence diskrétních vět s Banachovou větou. V druhé části jsou nastíněny možné přístupy k zobecnění Banachovy věty: jako příklady jsou dokázány různé věty o pevném bodě (autory jsou Edelstein, Bailey, Ćirić, Kirk a další), které zobecňují Banachovu větu.

Klíčová slova: Banachova věta o kontrakci, kontrakce, pevný bod, zobecněné kontrakce

Title: Variations of Banach fix point theorem
Author: Petr Pošta
Department: Department of Mathematical Analysis
Supervisor: prof. RNDr. Miroslav Hušek, DrSc.
Supervisor's e-mail address: mhusek@karlin.mff.cuni.cz

Abstract: In the present work we study various consequences and generalizations of Banach fixed point theorem. In the first part, we study consequences of classic contraction principle: sequences of contractive mappigs, several different variations of contractive conditions, several applications in Banach spaces and discrete contraction principle (versions of Eilenberg and Jachymski) and the question of equivalence between discrete principles and Banach theorem. In the second part, there are presented several ways how to generalize Banach theorem: as examples, various fixed point theorems are proven (Edelstein, Bailey, Ćirić, Kirk and others).

Keywords: Banach Contraction Principle, contraction, fixed point, generalized contractions

Contents

Chapter 1

Introduction

In this thesis, we would like to introduce some fixed point theorems which are consequences or extensions of famous Banach Contraction Principle.

Banach Contraction Principle. *Let* (Y, d) *be a complete metric space and* $F : Y \to Y$ *be contractive. Then* F *has a unique fixed point* u *and* $F^n y \to u$ *for each* $y \in Y$.

In the next chapter we shall make some necessary definitions and give a proof of Banach Contraction Principle. In Chapter 3, we shall derive some (almost direct) consequences of Banach Contraction Principle. At first, we shall investigate sequences of contractive mappings and continuity of map which assigns to every map of some family of contractions its fixed point. Then we shall give several examples how various conditions on contractivity of the map could be relaxed, especially when the metric space is compact. We shall give a short note about expansive mappings. In the last section we shall consider Banach space instead of general metric space and give several examples how a richer structure of Banach space implies interesting results. At the end of Chapter, we shall prove discrete version of Banach Contraction Principle.

In last Chapter, several extensions of the classic Banach Contraction Principle is derived. Above all, we shall modify the condition on contractivity

$$d(Fx, Fy) \leq \alpha\, d(x, y) \qquad \alpha \in (0, 1)$$

in several different ways. More details will be given later in Chapter 4. It is convenient to note that a survey written by Rhoades (already in 1977!) compares about 250 different generalized definitions of contraction. Therefore, instead of making a long survey, we shall try to introduce some interesting and useful techniques which could be used to derive significant extensions of Banach Contraction Principle.

Most of results which are presented in this thesis are mentioned (without proof) in Granas (2003) in Chapter 1 in Section 1.6 Miscellaneous Results and Examples.

Chapter 2

Preliminaries

Several well-known terms at the beginning.

Definition 1. Let (X, d) be a metric space. We call a map F *Lipschitz* or *Lipschitzian* when there is such a constant L so that the map satisfies a condition

$$d(Fx, Fy) \leq L\, d(x, y), \qquad \forall x, y \in X.$$

Then we call L a Lipschitz (or Lipschitzian) constant and (often) denote it by a symbol $L(F)$.

If $L(F) < 1$ then we call such a mapping as *contraction* or *contractive mapping* and $L(F)$ is called *contraction constant.*

Definition 2. Let X be any space and f a map of X, or of a subset of X, into X. A point $x \in X$ is called a *fixed point* for f if $x = f(x)$.

Definition 3. Let (X, d) be a metric space and $A \subset X$. A *diameter* of the set A is defined as

$$\operatorname{diam} A = \sup\{d(x, y) \mid x, y \in A\}.$$

We shall begin with a classic proof of Banach Contraction Principle. The advantage of the proof is that a useful estimate of error of n-th iteration is given.

Theorem A (Banach)**.** *Let (Y, d) be a complete metric space and $F : Y \to Y$ be contractive. Then F has a unique fixed point u and $F^n y \to u$ for each $y \in Y$.*

Proof. We denote α contraction constant for F.

Uniqueness: Let us assume there are two fixed points $x_0 = F(x_0)$ and $y_0 = F(y_0)$ such that $x_0 \neq y_0$. Then we have a contradiction

$$d(x_0, y_0) = d(F(x_0), F(y_0)) \leq \alpha d(x_0, y_0) < d(x_0, y_0).$$

Existence: Observe that for any $y \in Y$

$$d(F^n y, F^{n+1} y) \leq \alpha d(F^{n-1} y, F^n y) \leq \ldots \leq \alpha^n d(y, Fy).$$

It implies

$$\begin{aligned} d(F^n y, F^{n+p} y) &\leq \sum_{i=n}^{n+p-1} d(F^i y, F^{i+1} y) \\ &\leq (\alpha^n + \ldots + \alpha^{n+p-1} d(y, Fy) \leq \frac{\alpha^n}{1-\alpha} d(y, Fy). \end{aligned}$$

Since $\alpha < 1$, so that $\alpha^n \to 0$, $\{F^n y\}$ is a Cauchy sequence and, because (X, d) is complete, $F^n y \to u$ for some $u \in Y$.

By continuity of F, we must have

$$F(F^n y) \to F(u).$$

Because $\{F(F^n y)\} = \{F^{n+1} y\}$ is a subsequence of $\{F^n y\}$, we must have

$$F(F^n y) \to u.$$

Then $F(u) = u$ and F has at least one fixed point. $\square$

The immediate consequence which results from the proof is a useful estimation of errors: from

$$d(F^n y, F^{n+p} y) \leq \frac{\alpha^n}{1-\alpha} d(y, Fy)$$

if we send $p \to \infty$ we shall have an estimation of the error of the n-th iteration:

$$d(F^n y, u) \leq \frac{\alpha^n}{1-\alpha} d(y, Fy).$$

Chapter 3

Fixed Point Theorems in Complete Metric Spaces

We begin with a simple consequence of Banach Contraction Principle. We consider a map F (not necessarily continuous!) which has a property that F^N is contraction. This proposition was mentioned as an exercise in several different lecture notes about fixed point theorems, however, with a superfluous presumption on continuity of a map F.

Proposition 1 (Bryant, 1968)**.** *Let (X, d) be complete and $F : X \to X$ a map such that $F^N : X \to X$ is contractive for some $N \in \mathbb{N}$. Then F has a unique fixed point u and $F^n x \to u$ for each $x \in X$.*

Proof. *Uniqueness:* Let us assume that there are two fixed points $x_0 \neq y_0$ for F. Then we have

$$F^N(x_0) = F^{N-1}(F(x_0)) = F^{N-1}(x_0) = \ldots = x_0$$

and similarly $F^N(y_0) = y_0$. But since F^N is contractive and (by Banach Contraction Principle) has exactly one fixed point, this is a contradiction.

Existence: By Banach Contraction Principle, we know that F^N has one fixed point u and $\{F^{kN}y \xrightarrow{k\to\infty} u\}$ for every $y \in X$. Let us consider an arbitrary point $x \in X$ and a sequence

$$\{x, Fx, F^2x, F^3x, \ldots\}.$$

Then we can divide this sequence to N-subsequences: we denote $y_k = F^k x$ and we have

$$\begin{aligned}
&\{y_0, F^N y_0, F^{2N} y_0, \ldots\}\\
&\{y_1, F^N y_1, F^{2N} y_1, \ldots\}\\
&\{y_2, F^N y_2, F^{2N} y_2, \ldots\}\\
&\vdots\\
&\{y_{N-1}, F^N y_{N-1}, F^{2N} y_{N-1}, \ldots\}
\end{aligned}$$

Each of these subsequences converges to u. Hence, the sequence $F^n x \to u$ for any

$x \in X$.[1] Finally, we have to show that $F(u) = u$.[2] But if $F(u) = u' \neq u$, then we have

$$F^N(u) = u, \qquad F^N(u') = F^N(F(u)) = F(F^N(u)) = F(u) = u'.$$

This is a contradiction because F^N has exactly one fixed point. $\square$

The existence could be proven much easier: Since F^N is a contraction, then there is $x_0 \in X$ such that $F^N(x_0) = x_0$ and then

$$F^N(F(x_0) = F(F^N(x_0)) = F(x_0),$$

it implies $F(x_0)$ is also a fixed point of F^N and then $F(x_0) = x_0$.

Remark 1. A simple example that F really doesn't need to be continuous. Let us define $F : [0, 2] \to [0, 2]$,

$$F = \begin{cases} 0 & x \in [0, 1] \\ \frac{1}{2}x & x \in (1, 2] \end{cases}$$

Then $F^2(x) = 0$. $\square$

3.1 Sequences of contractive mappings

Several next theorems are concerned with convergent sequences of contractive mappings. Given such a sequence, there is a natural question:

If a sequence of contractive mappings converges, is the sequence of their fixed points convergent or not? If the answer is positive, is the limit a fixed point of the limit mapping?

We shall not investigate these questions in details. However, some basic facts are presented in two following propositions.

Proposition 2. *Let (X, d) be a complete metric space and $F_n : X \to X$ a sequence of continuous maps. Assume that each F_n has a fixed point x_n.*

(a) Let $F_n \rightrightarrows F$ on X.

- *(i) If $x_n \to x_0$ or $F(x_n) \to x_0$ then x_0 is a fixed point for F.*
- *(ii) If F is contractive then x_n converges to the unique fixed point of F.*

(b) Let $F_n \to F$ pointwise, with each F_n Lipschitzian, $L(F_n) \leq M < \infty$ for all n. Then

- *(i) F is Lipschitzian with $L(F) \leq M$;*
- *(ii) if $x_n \to x_0$, then x_0 is a fixed point for F;*
- *(iii) if $M < 1$, then $\{x_n\}$ converges to the unique fixed point of F.*

---

[1] Choose arbitrarily $\varepsilon > 0$: for each subsequence there is $k_i \in \mathbb{N}$ such that $d(F^{k_i N} y_i, u) < \varepsilon$. Then for $k = \max_i k_i$ we have $d(F^{kN} y_i, u) < \varepsilon$.

[2] Because F is not necessarily continuous (!) we can't do it in the same way as in Banach's theorem.

Proof. (a1): Assume $x_n \to x_0$. Then by triangle inequality and the fact

$$\sup_{t\in X} d(F_n(t), F(t)) \xrightarrow{n\to\infty} 0$$

(which is an easy consequence of uniform convergence $F_n \rightrightarrows F$) we have

$$\begin{aligned} d(F(x_n), x_0) &\le d(F(x_n), F_n(x_n)) + d(F_n(x_n), x_0) \\ &\le \sup_{t\in X} d(F(t), F_n(t)) + d(x_n, x_0) \xrightarrow{n\to\infty} 0. \end{aligned}$$

Thus the fact $x_n \to x_0$ implies $F(x_n) \to x_0$. F is obviously continuous (uniform convergence preserves continuity and F_n are contractive so they are especially continuous). Thus, by Heine theorem, we have

$$x_n \to x_0 \implies F(x_n) \to F(x_0)$$

so it implies $F(x_0) = x_0$.

Assume $F(x_n) \to x_0$. Then

$$d(x_n, x_0) \le d(F_n(x_n), F(x_n)) + d(F(x_n), x_0) \xrightarrow{n\to\infty} 0$$

because $F_n \rightrightarrows F$ and $F(x_n) \to x_0$. It implies $x_n \to x_0$ and we could complete the proof as above.

(a2): If F is contractive then (by Banach Contraction Principle) it has one unique fixed point x_0. It is sufficient to show $x_n \to x_0$. Let $\alpha < 1$ be the contractive constant for F.

Because of (a1) it is sufficient to show that $\{x_n\}$ is a convergent sequence and (because X is complete) it is sufficient to show that $\{x_n\}$ is a Cauchy sequence. For any $k > 0$ and $n > k$

$$\begin{aligned} d(x_n, x_k) &= d(F_n(x_n), F_k(x_k)) \\ &\le d(F_n(x_n), F(x_n)) + d(F(x_n), F(x_k)) + d(F(x_k), F_k(x_k)) \\ &\le d(F(x_n), F(x_k)) + \sup_{t\in X} d(F_k(t), F(t)) + \sup_{t\in X} d(F_n(t), F(t)) \end{aligned}$$

F is contractive with the contractive constant α. Thus we have

$$d(F(x_n), F(x_k)) \le \alpha\, d(x_n, x_k)$$

and the previous inequality gives

$$d(x_n, x_k) \le \frac{1}{1-\alpha} \sup_{t\in X} d(F_n(t), F(t)) + \frac{1}{1-\alpha} \sup_{t\in X} d(F_k(t), F(t)) \xrightarrow{k\to\infty} 0$$

(because of $F_k \rightrightarrows F$). Thus $\{x_n\}$ is a Cauchy sequence and converges to a point x_0 which is (by (a1)) fixed point of F.

(b1): We would like to show that F is Lipschitzian with $L(F) \le M$. For any $x, y \in X$, $x \ne y$ we have by triangle inequality

$$d(F(x), F(y)) \le d(F(x), F_n(x)) + d(F_n(x), F_n(y)) + d(F_n(y), F(y)).$$

We assume that $F_n \to F$ pointwise. For any $\varepsilon > 0$, there is $n_1 \in \mathbb{N}$ such that $d(F(x), F_n(x)) \leq \varepsilon$ for $n \geq n_1$ and $n_2 \in \mathbb{N}$ such that $d(F(y), F_n(y)) \leq \varepsilon$ for $n \geq n_2$. Let us take $n = \max(n_1, n_2)$. Since F_n is Lipschitzian, we have

$$d(F_n(x), F_n(y)) \leq L(F_n)\, d(x, y) \leq M\, d(x, y).$$

These facts and the previous inequality give

$$d(F(x), F(y)) \leq 2\varepsilon + M\, d(x, y)$$

for any $\varepsilon > 0$. Hence

$$d(F(x), F(y)) \leq M\, d(x, y)$$

which is precisely the proposition (b1).

(b2): Assume that $x_n \to x_0$. For any $\varepsilon > 0$ there is $n \in \mathbb{N}$ that $d(x_n, x_0) < \varepsilon$ and $d(F(x_0), F_n(x_0)) < \varepsilon$. So we have

$$\begin{aligned} d(F(x_0), x_0) &\leq d(F(x_0), F_n(x_0)) + d(F_n(x_0), F_n(x_n)) + d(F_n(x_n), x_0) \\ &\leq d(F(x_0), F_n(x_0)) + L(F_n)\, d(x_0, x_n) + d(x_n, x_0) \\ &\leq \varepsilon + M\varepsilon + \varepsilon = (2 + M)\varepsilon. \end{aligned}$$

The immediate consequence is that $d(F(x_0), x_0) = 0$ and thus $F(x_0) = x_0$.

(b3): By Banach Contraction Principle, we know that F has a unique fixed point x_0. Hence, it is sufficient to show that the sequence $\{x_n\}$ converges because by (b2) the limit is a fixed point for F. We have

$$\begin{aligned} d(x_0, x_n) &\leq d(F(x_0), F_n(x_0)) + d(F_n(x_0), F_n(x_n)) \\ &\leq d(F(x_0), F_n(x_0)) + M\, d(x_0, x_n) \end{aligned}$$

and, since F_n is lipschitzian with the Lipschitz constant $M < 1$ and F_n converges pointwise to F,

$$d(x_0, x_n) \leq \frac{1}{1 - M} d(F(x_0), F_n(x_0)) \xrightarrow{n \to \infty} 0.$$

□

Remark 2. The condition $L(F_n) \leq M < 1$ in the last proposition (b)(iii) cannot be relaxed to $L(F_n) < 1$ even if $L(F) < 1$. Define $F_n : l^2 \to l^2$ by

$$F_n(x_1, \ldots, x_n, \ldots) = (0, \ldots, \underbrace{(1 - 1/n)x_n + 1/n}_{n\text{-th place}}, 0, \ldots).$$

Then $L(F_n) = 1 - 1/n$ for each n and $\|F_n(0, \ldots, 1, 0, \ldots)\| = \|(0, \ldots, 1, 0, \ldots)\| = 1$, but F_n converges pointwise to the function $F \equiv 0$.

However, if we assume more about the complete metric space (X, d), the condition can be relaxed. One example is the next proposition.

Proposition 3 (Nadler (1968)). *Let (X, d) be a locally compact complete metric space and $F : X \to X$ be contractive. Assume $F_n : X \to X$ is a sequence of contractive maps converging pointwise to F. Let x_n (respectively $\hat{x}$) be the fixed point of F_n (respectively of F). Then x_n converges to $\hat{x}$.*

Proof. Let $\varepsilon > 0$ be sufficiently small so that

$$K(\hat{x}, \varepsilon) = \{x \in X \mid d(\hat{x}, x) \leq \varepsilon\}$$

is a compact subset of X.[3] F_n is an equicontinuous sequence of functions converging pointwise to F. That's because of

$$d(F_n(x), F_n(y)) < d(x, y).$$

Since $K(\hat{x}, \varepsilon)$ is compact, the sequence $\{F_n\}_{n=1}^{\infty}$ converges uniformly on $K(\hat{x}, \varepsilon)$ to F.[4] Let us denote α_n and α contractive constants of F_n and F. There exists $N \in \mathbb{N}$ such that for $n \geq N$ and every $x \in K(\hat{x}, \varepsilon)$ is $d(F_n(x), F(x)) < (1 - \alpha)\varepsilon$. Thus it holds

$$\begin{aligned} d(F_n(x), \hat{x}) &\leq d(F_n(x), F(x)) + d(F(x), \hat{x}) \\ &= d(F_n(x), F(x)) + d(F(x), F(\hat{x}) \\ &\leq (1 - \alpha)d(x, \hat{x}) + \alpha d(x, \hat{x}) = d(x, \hat{x}). \end{aligned}$$

Hence, every F_n for $n \geq N$ maps $K(\hat{x}, \varepsilon)$ into itself. It's an obvious consequence that $F_n|_{K(\hat{x},\varepsilon)}$ and $F|_{K(\hat{x},\varepsilon)}$ are contractions on $K(\hat{x}, \varepsilon)$ and thus have there their fixed points x_n, resp. $\hat{x}$. Using the proposition 2a(ii) $x_n \to \hat{x}$. □

3.2 Modified contractions

In several following theorems, the condition on contractivity is either relaxed or reformulated. If necessary, additional presumptions are given.

The first example is a simple extension, when the contractive constant α is varying.

Proposition 4 (Weissinger, 1952)**.** *Let (X, d) be complete and $\{\alpha_n\}$ be a sequence of nonnegative numbers with $\sum_{n=1}^{\infty} \alpha_n < \infty$. Let $F : X \to X$ be such that $d(F^n x, F^n y) \leq \alpha_n d(x, y)$ for all $x, y \in X$. Then F has a unique fixed point u and $F^n x \to u$ for each $x \in X$.*

Proof. Since $\alpha_n > 0$ and $\sum \alpha_n < \infty$, the sequence α_n converges to zero. Hence, there is $n_0 \in \mathbb{N}$ and $\alpha_{n_0} < 1$.

Uniqueness: Let us assume that there are two fixed points x, y for F. Thus x, y are fixed points for F^{n_0} and we have

$$d(x, y) = d(Fx, Fy) = d(F^{n_0}x, F^{n_0}y) \leq \alpha_{n_0} d(x, y) < d(x, y)$$

and it's a contradiction.

[3]) In a locally compact metric space, we could take a ball $B(\hat{x}, \varepsilon')$ which is definitely a neighbourhood of $\hat{x}$. Then we know there exists a compact neighbourhood in this ball and $\varepsilon > 0$ so that $B(\hat{x}, \varepsilon)$ is closed subset of this compact neighbourhood.

[4]) From Arzelá-Ascoli theorem. Since $K(\hat{x}, \varepsilon)$ is compact, F is bounded on $K(\hat{x}, \varepsilon)$. It's obvious that F_n are equally bounded because $F_n \to F$.

Existence: We could proceed in the same way as in the proof of Banach Contraction Principle. At first, we show that the sequence $\{F^n x\}$ is a Cauchy sequence

$$\begin{aligned} d(F^n x, F^{n+p+1}x) &\leq \sum_{i=n}^{n+p} d(F^n x, F^{n+1}x) \\ &\leq \sum_{i=n}^{n+p} \alpha_i d(x, Fx) \leq d(x, Fx) \cdot \sum_{i=n}^{\infty} \alpha_i \xrightarrow{n\to\infty} 0. \end{aligned}$$

($\sum_{i=n}^{\infty} \alpha_i$ is a residuum of a convergent series.) Thus $F^n x \to u$. F is lipschitzian and thus, especially, continuous. By continuity $F^n(Fx) \to Fu$ and it is obvious that $F^n(Fx) = F^{n+1}x \to u$. The proof is complete. □

The following theorem looks interesting at first sight. However, it is only a simple reformulation of Banach Contraction Principle in terms of shrinking sets. In the following chapter, we will introduce a technique of shrinking orbits[5] which looks similar.

Proposition 5 (H. Amann, 1973)**.** *Let (X, d) be complete and $F : X \to X$ be such that for any closed $A \subset X$ with* $\operatorname{diam}(A) \neq 0$*, we have* $\operatorname{diam}(F(A)) \leq \alpha \operatorname{diam}(A)$*, kde $0 \leq \alpha < 1$. Then F has a fixed point.*

Proof. F is a contraction. Assume that $A = \{x, y\}$. Then

$$d(F(x), F(y)) = \operatorname{diam}(F(A)) \leq \alpha \operatorname{diam}(A) = \alpha\, d(x, y).$$

This completes the proof. □

The following proposition is the last one which is dedicated to show a different version of condition on contractivity and it is probably the most interesting one.

Proposition 6. *Let (X, d) be complete and $F : X \to X$ be a map satisfying $d(Fx, Fy) < d(x, y)$ for $x \neq y$.*

(a) If for some $x_0 \in X$, the sequence $\{F^n x_0\}$ has a convergent subsequence, then F has a unique fixed point.

(b) If $\overline{F(X)}$ is compact (i.e., F is a compact map), then F has a unique fixed point u and $F^n x \to u$ for each $x \in X$.

Proof. (a) and (b): *Uniqueness:* If x, y are fixed points for F then

$$d(x, y) = d(Fx, Fy) < d(x, y)$$

easily gives contradiction.

(a) *Existence:* The sequence $\{F^n x_0\}$ has a convergent subsequence $F^{n_k} x_0 \to u$, $\{n_k\} \subset \mathbb{N}$. For arbitrary $\varepsilon > 0$, there exists $N \in \mathbb{N}$ and $N + m \in \mathbb{N}$ $(m > 0)$ so that

$$d(F^{N+m} x_0, u) < \varepsilon, \qquad d(F^N x_0, u) < \varepsilon.$$

[5]) it will be defined later

Thus it holds

$$\begin{aligned} d(u,Fu) &\le d(u,F^{N+m}x_0)+d(F^{N+m}x_0,F^m u)+d(F^m u,F^{m+1}u) \\ &\quad + d(F^{m+1}u,F^{m+1+N}x_0)+d(F^{m+1+N}x_0,Fu) \\ &\le d(u,F^{N+m}x_0)+d(F^N x_0,u)+d(Fu,F^2u) \\ &\quad + d(u,F^N x_0)+d(F^{m+N}x_0,u) \\ &\le 4\varepsilon + d(Fu,F^2u). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary

$$d(Fu,F^2u) \ge d(u,Fu).$$

But

$$Fu \ne F^2u \implies d(Fu,F^2u) < d(u,Fu).$$

Thus $Fu = F^2u$ and Fu is a fixed point for F.

(b) Existence: Let us take an arbitrary point $x \in X$. Since $\{F^n x\}_{n=1}^{\infty} \subset \overline{F(X)}$ and $\overline{F(X)}$ is compact, the sequence $\{F_n x\}$ contains an convergent subsequence.[6] Then we use part (a). □

This theorem has a familiar consequence which states: *If (X,d) is compact complete metric space, then a map F which satisfies a condition $d(Fx,Fy) < d(x,y)$ has unique fixed point.* In the following remark, we shall show that the condition

$$d(Fx,Fy) < d(x,y) \qquad (*)$$

is not strong enough (even with completness) to implicate the existence of fixed point of the map F.

The mentioned theorem was proved by Edelstein (1962). The proof was later simplified by Bennett and Fisher (1974).

Remark 3. There exists complete metric space (X,d) and a map $F : X \to X$ satisfying the inequality

$$d(Fx,Fy) < d(x,y)$$

without fixed points.

Proof. Let us consider the map $F(x) = \ln(1+e^x) \ : \ \mathbb{R} \to \mathbb{R}$. Then it holds for $x > y$

$$\ln(1+e^x) - \ln(1+e^y) < x - y.$$

One could see this from

$$\begin{gathered} \frac{1+e^x}{1+e^y} < e^{x-y} \\ 1+e^x < e^{x-y}+e^x \\ 1 < e^{x-y}. \end{gathered}$$

It implies the following inequality holds for any $x, y \in \mathbb{R}$

$$|F(x) - F(y)| < |x-y|.$$

F does not have a fixed point. That's a simple observation implied by

$$\ln(1+e^x) - x = \ln(1+e^x) - \ln e^x > 0 \qquad \forall x \in \mathbb{R}.$$

□

[6]) Known characteristic of compact metric spaces: every sequence has a convergent subsequence.

3.3 A short note about expansive mappings

Let (Y, d) be a metric space. We call map $F : Y \to Y$ *expansive* iff there is $\beta > 1$ and the following condition is valid for every $x, y \in Y$

$$d(Fx, Fy) \geq \beta d(x, y).$$

The following theorem is the simplest one of those which use this natural consequence: if the inverse mapping F^{-1} exists then it is obviously a contraction. This fact leads to one type of fixed point theorems for expansive mappings. One example follows.

Proposition 7. *Let (Y, d) be a complete metric space. A map $F : Y \to Y$ is surjective and* expanding *(i.e., $d(Fx, Fy) \geq \beta d(x, y)$ for some $\beta > 1$ and all $x, y \in Y$). Then F is bijective, F has a unique fixed point u and $F^{-n}y \to u$ for each $y \in Y$.*

Proof. F is injective (thus bijective): if $x \neq y$, then

$$d(Fx, Fy) \geq \beta d(x, y) > d(x, y) > 0$$

and it is obvious that $Fx \neq Fy$.

$F^{-n}y \to u$ and u is a unique fixed point of F: Uniqueness is obvious (the proof is similar as the one in Banach Contraction Principle). Since F is bijective, we can take the inverse F^{-1}. It simply follows from the condition on expansivity

$$d(Fx, Fy) \geq \beta d(x, y) \implies d(x, y) \geq \beta d(F^{-1}x, F^{-1}y) \implies$$

$$\implies d(F^{-1}x, F^{-1}y) \leq \frac{1}{\beta} d(x, y),$$

thus F^{-1} is a contraction, has a unique fixed point u and $F^{-n}y \to u$ for every $y \in Y$ (by Banach Contraction Principle). But

$$F^{-1}u = u \implies F(F^{-1}u) = Fu \implies u = Fu.$$

$\square$

3.4 Several examples in Banach spaces

In last Section of Chapter 3, we consider Banach spaces instead of complete metric spaces. Theorems about existence (and uniqueness) of solutions of operator equations in Banach spaces are maybe the best-known sort of applications of fixed point theorems in general. Several theorems of that kind are presented here.

Proposition 8. *Let $E, \|\cdot\|$ be a Banach space and $F : E \to E$ a linear operator such that $(I - F)^{-1}$ exists.*

(a) Let $G : E \to E$ be Lipschitzian with $\|(I - F)^{-1}\| L(G) < 1$. Then the map $x \mapsto Fx + Gx$, $x \in E$, has a unique fixed point.

(b) *Let r, λ be positive numbers with $\lambda < 1$, and let $K = K(0, r)$. Assume $G : K(0, r) \to E$ is a Lipschitzian map satisfying*

$$\|G(0)\| \leq (1 - \lambda)r / \|(I - F)^{-1}\|.$$

Then if $\|(I - F)^{-1}\| L(G) < \lambda$, then the map $x \mapsto Fx + Gx$, $x \in K$, has a unique fixed point.

Proof. (a): We consider an equation

$$x = Fx + Gx$$

which is equivalent (under assumptions) to

$$(I - F)x = Gx$$
$$x = (I - F)^{-1} G(x).$$

We define a map

$$T = (I - F)^{-1} G.$$

It holds

$$\begin{aligned} \|T(x) - T(y)\| &= \|(I - F)^{-1} G(x) - (I - F)^{-1} G(y)\| \\ &\leq \|(I - F)^{-1}\| \cdot \|G(x) - G(y)\| \\ &\leq \|(I - F)^{-1}\| L(G) \cdot \|x - y\|. \end{aligned}$$

According to the assumption $\|(I - F)^{-1}\| L(G) < 1$, the map T is a contraction on Banach space E and has unique fixed point. Hence the equation

$$x = (I - F)^{-1} G(x)$$

has unique solution and the same is valid for the equation

$$x = Fx + Gx.$$

The map $Fx + Gx$ has then a unique fixed point.

(b): Likewise in the part (a), we define

$$T = (I - F)^{-1} G, \qquad T : K \to E$$

We show several estimates that implicate $T(K) = K$ and T is a contraction on K. For this purpose, we consider

$$\|T(0)\| \leq \|(I - F)^{-1}\| \cdot \|G(0)\| \leq (1 - \lambda)r.$$

Then for arbitrary $x \in K$

$$\begin{aligned} \|T(x)\| - \|T(0)\| &\leq \|T(x) - T(0)\| = \|(I - F)^{-1} G(x) - (I - F)^{-1} G(0)\| \\ &\leq \|(I - F)^{-1}\| L(G) \cdot \|x - 0\| \leq \lambda \|x\| \leq \lambda r. \end{aligned}$$

Now we have

$$\|T(x)\| \leq \|T(0)\| + \lambda r \leq (1 - \lambda)r + \lambda r = r.$$

It is now obvious that $T(K) = K$. We shall proceed in a similar way as before

$$\begin{aligned} \|T(x) - T(y)\| &= \|(I - F)^{-1} G(x) - (I - F)^{-1} G(y)\| \\ &\leq \|(I - F)^{-1}\| L(G) \cdot \|x - y\| \leq \lambda \|x - y\| \end{aligned}$$

thus $T : K \to K$ is a contraction. Since K is a closed subset of Banach space E, K is complete and T has a unique fixed point. The rest is clear. $\square$

However, operator equations are not the only part of theory in Banach spaces that finds contractive mappings interesting and useful. Many other specific results could be derived for contractive mappings in Banach spaces. One geometric example follows.

Proposition 9. *Let $E = A \bigoplus B$ be a Banach space represented as a direct sum of two closed linear subspaces A and B with linear projections $P_A : E \to A$ and $P_B : E \to B$. Let $F : A \to E$ and $G : B \to E$ be two Lipschitzian maps, and let $f : A \to E$ and $g : B \to E$ be given by $a \mapsto a - F(a)$ and $b \mapsto b - G(b)$ respectively. If*

$$\|P_A\|L(F) + \|P(B)\|L(G) < 1,$$

then the intersection $f(A) \cap g(B)$ consists of at most one point.

Proof. At first, we should prove a useful lemma: *If $H : E \to E$ is contractive, then map $G : x \mapsto x - H(x)$ is a homeomorphism of E onto itself.* G is obviously continuous. G is bijective: if $y \in E$ is an arbitrary point then the equation

$$G(x) = y \iff x = y + H(x) =: G'(x)$$

has exactly one solution because G' is contractive.

Now we define a map

$$T = f \circ P_A + g \circ P_B, \quad T : E \to E.$$

$$T(x) = f(x_A) + g(x_B)$$

where we denote $x_A = P_A(x)$ and $x_B = P_B(x)$. Observe that

$$T(x) = x_A + x_B - F(x_A) - F(x_B) = x - F(x_A) - G(x_B)$$

and one can show that a map $K : x \mapsto x - T(x)$ is contractive:

$$\begin{aligned} \|K(x) - K(y)\| &\leq \|F(x_A) - F(y_A)\| + \|G(x_B) - G(y_B)\| \\ &\leq \Big\{\|P_A\|L(F) + \|P_B\|L(G)\Big\}\|x - y\|. \end{aligned}$$

By lemma mentioned at the beginning of the proof, it is obvious that T is homeomorphism E onto E.

If $f(A) \cap g(B)$ contains at least two points u, v then there has to be $y_1, z_1 \in A$ and $y_2, z_2 \in B$ such that

$$f(y_1) = u, \quad g(y_2) = u,$$

$$f(z_1) = v, \quad g(z_2) = v.$$

Then

$$T(y_1 + z_2) = f(y_1) + g(z_2) = g(y_2) + f(z_1) = T(y_2 + z_1)$$

and that is contradiction because T is one to one. □

Remark 4. One can show that $f(A) \cap g(B)$ consists of exactly one point.

3.5 Discrete contraction principle

We shall start with a theorem of Samuel Eilenberg[7] which has some applications in automata theory. Instead of metric space, we shall consider an abstract set X with sequence of equivalence relations.

(In fact, it is only a different way how to describe "the same" structure of the space. Instead of direct use of a metric, the structure is described through given neighbourhoods of the diagonal.)

Theorem 10 (Discrete Banach theorem, S. Eilenberg, 1978)**.** *Let Y be a set, and $\{R_n \mid n = 0, 1, \dots\} \subset Y \times Y$ a sequence of equivalence relations such that*

(a) $Y \times Y = R_0 \supset R_1 \supset \dots,$

(b) $\bigcap_{n=0}^{\infty} R_n =$ *the diagonal in* $Y \times Y$*,*

(c) if $\{y_n\}$ is any sequence in Y such that $(y_n, y_{n+1}) \in R_n$ for each n, then there is a $y \in Y$ such that $(y_n, y) \in R_n$ for each n.

Let $F : Y \to Y$ be a map such that whenever $(x, y) \in R_n$, then $(Fx, Fy) \in R_{n+1}$. Then F has a unique fixed point u and $(F^n y, u) \in R_n$ for each n and each $y \in Y$.

Proof. Uniqueness: let us assume that $u, v \in Y$ are two fixed points under F. Then by assumptions on F

$$(u, v) \in R_0 = Y \times Y \implies (u, v) = (F^n u, F^n v) \in R_n \qquad \forall n$$

and

$$(u, v) \in \bigcap_{n=0}^{\infty} R_n = \text{diagonal in } Y \times Y \implies u = v.$$

Existence: let us take an arbitrary point $y \in Y$. Then

$$(y, Fy) \in R_0 \implies (F^n y, F^{n+1} y \in R_n).$$

The sequence $\{F^n y\}$ has the property (c). Then

$$\exists u \in Y \quad (F^n y, u) \in R_n \quad \forall n.$$

Thus we have (from symmetry of equivalency and assumptions on F)

$$(u, F^n y) \in R_n, \quad (F^n y, F^{n+1} y \in R_n, \quad (F^{n+1} y, Fu) \in R_{n+1} \subset R_n$$

and from transitivity

$$(u, Fu) \in R_n \qquad \forall n.$$

Hence,

$$(u, Fu) \in \bigcap_{n=0}^{\infty} R_n = \text{diagonal in } Y \times Y \implies u = Fu.$$

□

[7]) As Jacek Jachymski told in his article from 2004, the theorem was presented by S. Eilenberg on his lecture at the University of Southern Carolina, Los Angeles, 1978

Remark 5. One can show that Eilenberg theorem is equivalent to Banach contraction principle restricted to ultrametric bounded metric spaces. (We call a metric space (Y, d) *ultrametric* if $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in Y$.) The idea of the proof is quite simple. Let us take two different points $x, y \in Y$. We define

$$d(x, y) = \alpha^{p(x,y)}, \quad \text{where } p(x, y) = \max_{n \in \mathbb{N}}\{(x, y) \in R_n\}.$$

One can show that (Y, d) is a bounded ultrametric space and a map F (which satisfies conditions in Eilenberg theorem) is α-contraction in that space. For the converse implication: if we have (Y, d) a bounded ultrametric space, then we shall define

$$R_n = \{(x, y) \in X \times X \ : \ d(x, y) \leq \alpha^n \operatorname{diam}(X)\}.$$

and verify that R_n are equivalences which satisfies conditions (i)-(iii). Then it is clear that a map F (satysfing conditions in the theorem) is an α-contraction in (Y, d).

However, this formulation is not equivalent to (unrestricted) Banach Contraction Principle. Jachymski (2004) proved an extension which is equivalent to Banach Contraction Principle. We mention it here without proof.

Let X be an abstract set and $(R_n)_{n\in\mathbb{Z}}$ a sequence of reflexive and symmetric relations in X such that

- *(i) given $n \in \mathbb{Z}$, if $(x, y) \in R_n$ and $(y, z) \in R_n$, then $(x, z) \in R_{n-1}$,*
- *(ii) $\bigcup_{n\in\mathbb{Z}} R_n = X \times X$, and $\ldots \supseteq R_{-1} \subseteq R_0 \subseteq R_1 \subseteq \ldots$,*
- *(iii) $\bigcap_{n\in\mathbb{Z}} R_n =$ the diagonal in $X \times X$,*
- *(iv) given a sequence $(x_n)_{n=1}^{\infty}$ such that $(x_n, x_{n+1}) \in R_n$ for all $n \in \mathbb{N}$, there is an $x \in X$ such that $(x_n, x) \in R_{n-1}$ for all $n \in \mathbb{N}$.*

If F is a self-map of X such that given $n \in \mathbb{Z}$ and $x, y \in X$, condition

$$(x, y) \in R_n \implies (Fx, Fy) \in R_{n+1}$$

is satisfied, then F has a unique fixed point x_*, and given $x \in X$, there is a $k \in \mathbb{N}$ such that $(F^{k+n}x, x_*) \in R_n$ for all $n \in \mathbb{N}$.

Chapter 4

Extensions of the Banach theorem

We shall begin with theorem of Michael Edelstein. The main question is here: Is Banach contraction principle still valid if the contractive condition is hold for near points only?

Theorem 11 (M. Edelstein, 1961). *A metric space (X, d) is* ε-chainable *if for each pair $x, y \in X$ there are finitely many points (ε-chain) $x = x_0, \ldots, x_{n+1} = y$ such that $d(x_i, x_{i+1}) < \varepsilon$ for all $0 \leq i \leq n$.*

Let (X, d) be complete, and let $F : X \to X$ be a map. Assume that there is an $\varepsilon > 0$ and a $0 \leq k < 1$ such that $d(Fx, Fy) \leq kd(x, y)$ whenever $d(x, y) < \varepsilon$. If (X, d) is ε-chainable, then F has a unique fixed point.

Proof. The proof is quite straightforward. Let us take arbitrary point $x \in X$ and a point $F(x) \in X$. There is an ε-chain

$$x = x_0, x_1, x_2, \ldots, x_n = F(x)$$

and it follows

$$d(x, F(x)) \leq \sum_{i=1}^{n} d(x_{i-1}, x_i) = n\varepsilon.$$

(Let us note that for fixed $x \in X$ is $n \in \mathbb{N}$ fixed either.) It is obvious that a finite sequence

$$F(x) = F(x_0), F(x_1), F(x_2), \ldots, F(x_n) = F^2(x)$$

is also an ε-chain because of condition of uniform local contractivity on F. By induction, a finite sequence

$$F^m(x) = F^m(x_0), F^m(x_1), F^m(x_2), \ldots, F^m(x_n) = F^{m+1}(x)$$

is an ε-chain for any $m \in \mathbb{N}$ and it follows by local contractivness

$$d(F^m x, F^{m+1} x) \leq \sum_{i=1}^{n} d(F^m x_{i-1}, F^m x_i) \leq \sum_{i=1}^{n} k^m d(x_{i-1}, x_i) = k^m n\varepsilon.$$

Hence, the sequence $\{F^m x\}_{m=1}^{\infty}$ is a Cauchy sequence and since (X, d) is a complete metric space, $F^m x \to u$. The rest of the proof is similar to the proof of Banach Contraction Principle.

Maybe we should note a bit more about uniqueness. If $u, v \in X$ are two different fixed points of F then there is an ε-chain

$$u = \xi_0, \xi_1, \ldots, \xi_n = v.$$

It follows for an arbitrary $N \in \mathbb{N}$

$$d(u, v) = d(F^N u, F^N v) \le \sum_{i=1}^{n} d(F^N \xi_{i-1}, F^N \xi_i) = k^m n\varepsilon \xrightarrow{N\to\infty} 0.$$

And that's an obvious contradiction. □

4.1 Generalized contractions I

The condition on the map F (to be a contraction) could be weakened in several ways. At first, we shall change classic contractive condition

$$d(Fx, Fy) \le \alpha\, d(x, y) \qquad \alpha < 1$$

with a condition of this kind

$$d(Fx, Fy) \le \varphi[d(x, y)]$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is an appropriate function. Several different types of functions can be used for substituting classic contractive condition.

Now, we shall give a proof of quite general principle in which F images the ball into itself if its center is sufficiently near. It will be useful later.

Theorem 12. *Let (X, d) be a complete metric space and $F : X \to X$ a map, not necessarily continuous. Assume*

(∗) *for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $d(x, Fx) < \delta$, then $F[B(x, \varepsilon)] \subset B(x, \varepsilon)$.*

Then, if $d(F^n u, F^{n+1} u) \to 0$ for some $u \in X$, the sequence $\{F^n u\}$ converges to a fixed point for F.

Proof. At first, one has to show that $\{F^n u\}$ converges and it is sufficient to show that $\{F^n u\}$ is Cauchy sequence. So choose an arbitrary $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that

$$d(F^N u, F^{N+1} u) < \delta(\varepsilon) \implies F[B(F^N u, \varepsilon)] \subset B(F^N u, \varepsilon) \implies F^{N+1} u \in B(F^N u, \varepsilon)$$

and then for every $k \in \mathbb{N}$

$$F^{N+k} u \in B(F^N u, \varepsilon).$$

Hence, for every $m, n \ge N$

$$d(F^m u, F^n u) \le d(F^m u, F^N u) + d(F^N u, F^n u) \le 2\varepsilon.$$

Thus, $\{F^n u\}$ is a Cauchy sequence and converges to some $y \in X$. Now, let us assume that y is not a fixed point for F. Then

$$\varepsilon' := d(y, Fy) > 0.$$

Choose such $n \in \mathbb{N}$ that $d(F^n u, y) < \varepsilon'/3$ and $d(F^n u, F^{n+1}u) < \delta(\varepsilon'/3)$. Since

$$F[B(F^n u, \varepsilon'/3)] \subset B(F^n u, \varepsilon'/3) \implies Fy \in B(F^n u, \varepsilon'/3),$$

one can show a contradiction with

$$d(Fy, F^n u) \geq d(Fy, y) - d(y, F^n u) \geq \frac{2}{3}\varepsilon' \implies Fy \notin B(F^n u, \varepsilon'/3).$$

□

This theorem is quite useful for deriving fixed point theorems based on completeness and contractionlike conditions mentioned above. As an example, we shall derive theorems of Matkowski (1975) and Browder (1968).

Proposition 13 (Matkowski, 1975)**.** *Let (X, d) be complete, and let $F : X \to X$ be a map satisfying*

$$d(Fx, Fy) \leq \varphi[d(x, y)],$$

where $\varphi : R^+ \to \mathbb{R}^+$ is any nondecreasing (not necessarily continuous) function such that $\varphi^n(t) \to 0$ for each fixed $t > 0$. Then F has a unique fixed point u and $F^n x \to u$ for each $x \in X$.

Proof. We would like to use the previous theorem. At first, for each $x \in X$

$$d(F^n x, F^{n+1}x) \leq \varphi^n[d(x, Fx)] \implies d(F^n x, F^{n+1}x) \to 0.$$

Now choose $\varepsilon > 0$ and choose $\delta(\varepsilon) = \varepsilon - \varphi(\varepsilon)$[1] and if $d(x, Fx) < \delta(\varepsilon)$ then for any $y \in B(x, \varepsilon)$

$$d(Fz, x) \leq d(Fz, Fx) + d(Fx, x) < \varphi[d(z, x)] + \delta \leq \varepsilon(\varepsilon) + \varepsilon - \varphi(\varepsilon) = \varepsilon.$$

The rest is easy. □

Proposition 14 (Browder (1968))**.** *Let (X, d) be complete and $F : X \to X$ a map satisfying $d(Fx, Fy) \leq \varphi[d(x, y)]$ for all $x, y \in X$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is any function such that*

(i) φ is nondecreasing,

(ii) $\varphi(t) < t$ for each $t > 0$,

(iii) φ is right continuous.

Then F has a unique fixed point u and $F^n x \to u$ for each $x \in X$.

Proof. If we show that $\varphi^n(t) \to 0$ for any $t > 0$, we shall be able to use the previous proposition. So choose fixed $t \in \mathbb{R}^+$. It is clear that $\{\varphi^n(t)\}$ is monotonic sequence and thus has a limit $y \in \mathbb{R}^+$. We denote $t_n = \varphi^n(t)$. Then

$$\lim_{n \to \infty} t_n = y$$

[1]) $\delta(\varepsilon) > 0$ because $\varphi(t) < t$ since φ is nondecreasing and $\varphi^n(t) \to 0$. Obviously, if $t \leq \varphi(t)$ then $\varphi(t) \leq \varphi(\varphi(t))$ etc.

and

$$t_1 > t_2 > t_3 > \ldots > t_n > \ldots > y$$

Since φ is right continuous, it has to be

$$\lim_{t_n \to y+} \varphi(t_n) = \varphi(y).$$

However, $\{\varphi(t_n)\} \subset \{t_n\}$ so

$$\varphi(y) = y.$$

Hence, $y = 0$ and the proof is complete. $\square$

4.2 Generalized contractions II

From a wide variety of generalized contractive conditions analyzed by Rhoades (1977), we shall take another example. Classic Banach's condition is now replaced by condition

$$d(Fx, Fy) \leq \text{something which operates with some terms of the set } \{d(x,y),\ d(x,Fx),\ d(y,Fy),\ d(x,Fy),\ d(y,Fx)\}$$

"Something" can have multifarious forms: e.g.

$$d(Fx, Fy) \leq a[d(x,Fx) + d(y,Fy)], \quad a \in (0, \tfrac{1}{2}) \qquad \text{(Kannan)}$$

$$d(Fx,Fy) \leq a_1 d(x,y) + a_2 d(x,Fx) + a_3 d(y,Fy) + a_4 d(x,Fy) + a_5 d(y,Fx), \quad \sum a_i < 1 \qquad \text{(Hardy, Rogers)}$$

and many other ones. Our first example is based on Ćirić's version of generalized contraction.

Proposition 15 (L. B. Ćirić, 1974)**.** *Let (X,d) be complete and $F : X \to X$ continuous. Assume that*

$$d(Fx,Fy) \leq k \max\{d(x,y),\ d(x,Fx),\ d(y,Fy),\ d(x,Fy),\ d(y,Fx)\} \qquad (*)$$

for some $k \in [0,1)$ and all $x, y \in X$. Then F has a unique fixed point u and $F^n x \to u$ for each $x \in X$.

Proof. Uniqueness: Let us consider u, v two different fixed points under F. The condition $(*)$ and equalities $Fu = u$, $Fv = v$ simply leads to contradiction

$$\begin{aligned} d(u,v) = d(Fu,Fv) &\leq k\ \max\{d(u,v),\ d(u,Fu),\ d(v,Fv),\ d(u,Fv),\ d(v,Fu) \\ &= k \max\{d(u,v),\ d(u,u),\ d(v,v),\ d(u,v),\ d(v,u)\} = k \cdot d(u,v) < d(u,v). \end{aligned}$$

Existence: We shall divide the proof into three steps. We define sets

$$O(x,n) := \{x, Fx, F^2x, \ldots, F^n x\},$$

$$O(x,\infty) := \{x, Fx, F^2x, \ldots, F^n x, \ldots\}.$$

At first, let us show this: *If* $n \in N$ *then for each* $x \in X$ *and all* $i, j \in \{1, \dots, n\}$ *is*

$$d(F^i x, F^j x) \le k \cdot \delta[O(x,n)].$$

Since F has the property $(*)$ and $F^i x, F^{i-1}x, F^j x, F^{j-1}x \in O(x,n)$, it follows

$$\begin{aligned} d(F^i x, F^j x) &= d(FF^{i-1}, FF^{j-1}x) \\ &\le k \cdot \max\big\{d(F^{i-1}x, F^{j-1}x),\ d(F^{i-1}x, F^i x),\ d(F^{j-1}x, F^j x), \\ &\quad d(F^{i-1}x, F^j x),\ d(F^i x, F^{j-1}x)\big\} \\ &\le k\delta[O(x,n)]. \end{aligned}$$

Second step: we shall show

$$\forall x \in X \qquad \delta[O(x,\infty)] \le \frac{1}{1-k} d(x, Fx).$$

Since $\delta[O(x,\infty) = \sup\{\delta[O(x,n)],\ n \in \mathbb{N}\}$, it is sufficient to show for all $n \in \mathbb{N}$

$$\forall x \in X \qquad \delta[O(x,n)] \le \frac{1}{1-k} d(x, Fx).$$

Choose $n \in \mathbb{N}$ arbitrarily. There exists $k \le n,\ k \in \mathbb{N}$ such that[2]

$$d(x, F^k x) = \delta[O(x,n)].$$

By triangle inequality and the first step of proof

$$\begin{aligned} d(x, F^k x) &\le d(x, Fx) + d(Fx, F^k x) \le d(x, Fx) + k \cdot \delta[O(x,n)] \\ &= d(x, Fx) + k \cdot d(x, F^k x). \end{aligned}$$

Therefore

$$\delta[O(x,n)] = d(x, F^k x) \le \frac{1}{1-k} d(x, Fx).$$

Last step: We shall show that $\{F^n x\}$ is a Cauchy sequence. Let be $n, m \in \mathbb{N}$, $n < m$. From the first step of the proof we see

$$d(F^n x, F^m x) = d(FF^{n-1}x, F^{m-n+1}F^{n-1}x) \le k \cdot \delta[O(F^{n-1}x, m-n+1)].$$

There is $k_1 \in \mathbb{N}$, $k_1 \le m-n+1$ such that

$$\delta[O(F^{n-1}x, m-n+1)] = d(F^{n-1}x, F^{k_1}F^{n-1}x).$$

Let's do the same thing once more. It follows

$$\begin{aligned} \delta[O(F^{n-1}x, m-n+1)] &= d(F^{n-1}x, F^{k_1}F^{n-1}x) \\ &= d(FF^{n-2}x, F^{k_1+1}F^{n-2}x) \\ &\le k \cdot \delta[O(F^{n-2}x, k_1+1)] \\ &\le k \cdot \delta[O(F^{n-2}x, m-n+2)]. \end{aligned}$$

[2]) The diameter of $O[x,n]$ is the distance of point x and one other point $F^k x \in O[x,n]$. This is an easy consequence of $(*)$ if we put $k = 1$ and apply the inequality on arbitrary points $d(F^k x, F^l x)$ as many times as it is necessary to achieve $F^0 x$ in one or the other coordinate.

Hence

$$d(F^n x, F^m x) \leq k \cdot \delta[O(F^{n-1}x, m-n+1)] \leq k^2 \cdot \delta[O(F^{n-2}x, m-n+2)]$$

and one could show in a similar way

$$d(F^n x, F^m x) \leq k^n \cdot \delta[O(x, m)].$$

The immediate consequence of the second step is

$$d(F^n x, F^m x) \leq k^n \cdot \frac{1}{1-k} d(x, Fx) \xrightarrow{n\to\infty} 0,$$

thus the sequence $\{F^n x\}$ is a Cauchy sequence. Since (X, d) is complete, there is $u \in X$ such that $F^n x \to u$. It holds

$$\begin{aligned} d(u, Fu) &\leq d(u, F^{n+1}x) + d(FF^n x, Fu) \\ &\leq d(u, F^{n+1}x) + k \cdot \max\{d(F^n x, u),\ d(F^n x, F^{n+1}x), \\ &\quad\ d(u, Fu),\ d(F^n x, Fu),\ d(F^{n+1}x, u)\} \\ &\leq d(u, F^{n+1}u) + k \cdot [d(F^n x, u) + d(F^n x, F^{n+1}x) \\ &\quad +d(u, Fu) + d(F^{n+1}x, u)] \end{aligned}$$

and thus

$$d(u, Tu) \leq \frac{1}{1-k}\left[kd(F^n x, u) + kd(F^n x, F^{n+1}x) + (1+k)d(F^{n+1}x, u)\right] \to 0$$

because $F^n x \to u$ and $\{F^n x\}$ is a Cauchy sequence. Then u is a fixed point under F and the proof is complete. □

As an example, how careful one must be in such a kind of statement, we shall mention a proposition of Pittnauer. Initially, we shall give a rather complicated proof.

Proposition 16 (F. Pittnauer, 1975)**.** *Let (X, d) be complete and $F : X \to X$ continuous. Assume that there exists and integer n and $0 \leq k < 1$ such that*

$$d(Fx, Fy) \leq k[d(x, F^n z) + d(y, F^n z)] \quad \forall x, y, z \in X. \tag{$*$}$$

Then F has a unique fixed point.

Proof. Let us choose $x \in X$ arbitrarily. The condition $(*)$ gives immediately

$$d(F^{n+1}x, F^{n+2}x) \leq k[d(F^n x, F^n z) + d(F^{n+1}x, F^n z)]$$

and if we choose $z = Fx$ we'll get

$$d(F^{n+1}x, F^{n+2}x) \leq k \cdot d(F^n x, F^{n+1}x).$$

In the same way, we could consider inequality

$$d(F^{n+2}x, F^{n+3}x) \leq k[d(F^{n+1}x, F^n z) + d(F^{n+2}x, F^n z)]$$

and if we choose $z = F^2x$ we'll get

$$d(F^{n+2}x, F^{n+3}x) \le k \cdot d(F^{n+1}x, F^{n+2}x) \le k^2 \cdot d(F^n x, F^{n+1}x).$$

By induction, one may show

$$d(F^{n+m}x, F^{n+m+1}x) \le k^m \cdot d(F^n x, F^{n+1}x).$$

(We remind that $n \in \mathbb{N}$ is fixed.) It follows

$$\begin{aligned} d(F^{n+m}x, F^{n+m+p}x) &\le \sum_{i=0}^{p-1} d(F^{n+m+i}x, F^{n+m+i+1}x) \\ &\le \sum_{i=0}^{p-1} k^{m+i} \cdot d(F^n x, F^{n+1}x) \\ &\le \frac{k^m}{1-k} d(F^n x, F^{n+1}x) \xrightarrow{m\to\infty} 0. \end{aligned}$$

Hence, the sequence $\{F^m\}_{m=0}^{\infty}$ is a Cauchy sequence and since (X, d) is complete, there exists $u \in X$ such that $F^m x \to u$. Since F is continuous, it has to be

$$F(F^m x) \to Fu, \qquad F^{m+1}x \to u.$$

Then $F(u) = u$ and F has at least one fixed point.

Uniqueness: Let us consider two different fixed points u, v under F. Then it follows by $(*)$ with $z = u$

$$\begin{aligned} d(u, v) = d(F^{n+1}u, F^{n+1}v) \le k[d(F^n u, F^n u) + d(F^n v, F^n u)] = \\ kd(F^n v, F^n u) = kd(v, u) < d(u, v). \end{aligned}$$

That is a contradiction. □

However, the condition $(*)$ only looks more generally than usual contractivity. It is obvious, that $(*)$ implies

$$d(Fx, Fy) \le k\, d(x, y) \qquad \text{for all } x, y \in \overline{F^n(X)}$$

and thus this result is a consequence of Banach Contraction Principle.[3]

4.3 Dugundji's approach

In the last section, we will aim our attention to an alternative technique how to construct fixed point theorems. It comes from James Dugundji and was introduced in 1975. The main "tool" of this method are theorems (17) and (20), both of them has a slightly different use and together implies a wide range of fixed point theorems. Our examples are mainly propositions which were derived earlier than Dugundji's theorems by (sometimes much) more difficult and sophisticated methods.

The first theorem is concentrating on minimizing sequences for a suitable function φ: vast majority of applications then use $\varphi(x) = d(x, Fx)$ and try to find suitable conditions in order to $\inf_{x\in X} \varphi(x)$ would be zero.

[3]) A proposition of Bryant (1) could be used, for example

Theorem 17 (Dugundji, 1975)**.** *Let (X, d) be complete metric space and $\varphi : X \to \mathbb{R}^+$ be an arbitrary (not necessarily continuous) nonnegative function. Assume that*

$$\inf\{\varphi(x) + \varphi(y) \mid d(x, y) \geq a\} = \mu(a) > 0 \qquad \forall a > 0.$$

Then each sequence $\{x_n\} \subset X$ for which $\varphi(x_n) \to 0$ converges to one and the same point $u \in X$.

Proof. Let $A_n = \{x \mid \varphi(x) \leq \varphi(x_n)\}$. These sets are nonempty and any finite family has a nonempty intersection. We show $\mathrm{diam}(A_n) \to 0$: given any $\varepsilon > 0$, choose N so large that $\varphi(x_n) < \frac{1}{2}\mu(\varepsilon)$ for all $n \geq N$; then for any $n \geq N$ and $x, y \in A_n$ we have $\varphi(x) + \varphi(y) < \mu(\varepsilon)$; therefore, the condition on φ gives $d(x, y) < \varepsilon$, so $\mathrm{diam}(A_n) \leq \varepsilon$. Thus, $\mathrm{diam}(A_n) \to 0$; because $\mathrm{diam}(\overline{A_n}) = \mathrm{diam}(A_n) \to 0$, we conclude from Cantor theorem that there is a unique $u \in \bigcap_n \overline{A_n}$ and, since $x_n \in \overline{A_n}$ for each n, that $x_n \to u$. For any other sequence $\{y_n\}$ satisfying $\varphi(y_n) \to 0$ we get $\varphi(x_n) + \varphi(y_n) \to 0$, so, from the condition above as before, $d(x_n, y_n) \to 0$ and therefore $y_n \to u$ also. □

It follows almost immediately:

Corollary 18. *Let (X, d) be complete metric space and $F : X \to X$ continuous. Assume that the function $\varphi(x) = d(x, Fx)$ has the property*

$$\inf\{\varphi(x) + \varphi(y) \mid d(x, y) \geq a\} = \mu(a) > 0 \qquad \forall a > 0$$

and

$$\inf_{x \in X} d(x, Fx) = 0.$$

Then F has a unique fixed point.

Proof. Since $\inf_{x \in X} \varphi(x) = 0$, there is a sequence $\{x_n\}$ such that $\varphi(x_n) \to 0$. Each sequence with that property converges to the same point. By the previous theorem $x_n \to \hat{x} \in X$ and it is obvious that

$$\varphi(\hat{x}) = 0 \iff d(\hat{x}, F(\hat{x})) = 0.$$

Then $F(\hat{x}) = \hat{x}$ and F has unique fixed point. □

It is worthy to note that Banach Contraction Principle is an easy consequence of (18). And as the promised example, we shall derive a fixed point theorem of Bailey.

Proposition 19 (D. F. Bailey, 1966)**.** *Let (X, d) be complete and $F : X \to X$ continuous. Assume that for each $\varepsilon > 0$ and each pair $x, y \in X$, there is an $n = n(x, y, \varepsilon)$ such that $d(F^n x, F^n y) < \varepsilon$. If the function $\varphi(x) = d(x, Fx)$ has the property*

$$\inf\{\varphi(x) + \varphi(y) \mid d(x, y) \geq a\} = \mu(a) > 0 \qquad \forall a > 0,$$

then F has a fixed point.

Proof. Choose $\varepsilon > 0$ and $x \in X$. For $y = F(x)$, there exists n such that

$$d(F^n x, F^{n+1} x) < \varepsilon.$$

Hence, for $X \ni u = F^n x$ we have

$$\varphi(u, Fu) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it must be

$$\inf_{x \in X} \varphi(x) = \inf_{x \in X} d(x, Fx) = 0.$$

Therefore, the proposition follows from (18). □

The second Dugundji theorem is similar to the first one. However, the condition is replaced: we have a suitable compact A (in applications, it is often a single point) and nonnegative function φ which has positive infimum out of the compact A. Then every sequence which minimizes φ contains a subsequence which converges to some point of A (mostly to the fixed point).

Theorem 20 (Dugundji, 1975)**.** *Let (X, d) be an arbitrary metric space, and let $A \subset X$ be compact. Let $\varphi : X \to \mathbb{R}^+$ be an arbitrary (not necessarily continuous) nonnegative function such that*

$$\inf\{\varphi(x) \mid d(x, A) \geq a\} > 0$$

for each $a > 0$. Then each sequence $\{x_n\}$ in X for which $\varphi(x_n) \to 0$ contains a subsequence converging to some point of A.

Proof. At first, let us assume that the sequence $\{x_n\}$ contains an infinitely many points of A; this means there is a subsequence $\{x_{n_k}\} \subset A$. But a well-known statement says we can choose a convergent subsequence from an arbitrary sequence in a compact metric space (or in a compact subset of a metric space).

Now assume that the sequence $\{x_n\}$ has only finitely many points in A. Then we can choose a subsequence (which we will denote $\{x_n\}$ as well) without any point in A. It is easy to show, that $d(x_n, A) \to 0$. Indeed, if $d(x_n, A) \geq a > 0$, then it is obvious by condition on φ that $\varphi(x_n) \geq K > 0$ for each $n \in \mathbb{N}$. Hence, we can once more choose a subsequence (which we still denote $\{x_n\}$) such that $d(x_n, A) \searrow 0$.

Choose a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k > 0$ and $\varepsilon_k \searrow 0$. For every $\varepsilon_k > 0$ there is x_k in our chosen subsequence $\{x_n\}$ such that $\varepsilon_{k+1} < d(x_k, A) < \varepsilon_k$ and there has to be $y_k \in A$ such that $2\varepsilon_{k+1} < d(x_k, y_k) < 2\varepsilon_k$. We constructed a sequence $\{y_k\} \subset A$ and A is compact. Let us assume that the sequence converges[4] and we denote the limit y. Obviously, $y \in A$ because A is compact and thus closed. It is also obvious[5] that $y \in \partial A$ because $d(y_k, X/A) < 2\varepsilon_k \to 0$.

We want to choose (hopefully, for the last time) a subsequence of the sequence $\{x_k\}$ which would converge to y. For every ε_k there has to be index $m(k) \geq k$ and a point $y_{m(k)}$ such that

$$d(y, y_{m(k)}) < \varepsilon_k.$$

[4]) otherwise, we can choose a convergent subsequence again

[5]) we denote ∂A a boundary of the set A.

But then for $x_{m(k)} \in \{x_k\}$ it holds

$$d(x_{m(k)}, y) \leq d(x_{m(k)}, y_{m(k)}) + d(y_{m(k)}, y) \leq \varepsilon_{m(k)} + \varepsilon_k < 2\varepsilon_k.$$

Hence, we shall construct a sequence $\{x_{m(k)}\}$ in the way as it is described.

Therefore, for arbitrary $\varepsilon > 0$ there is $\varepsilon_k < \frac{1}{2}\varepsilon$ and for every $m \geq m(k) \geq k$ it follows

$$d(x_m, y) \leq d(x_m, y_m) + d(y_m, y) < \varepsilon_m + \varepsilon_k < 2\varepsilon_k < \varepsilon.$$

This completes the proof. □

Maybe it would be better to repeat the idea of the proof once again. If the sequence $\{x_n\}$ has infinitely many points in A then we can choose a convergent subsequence immediately. If not then we use this idea: at first we choose a subsequence $\{x_n\}$ which approaches the set A so the distance $d(x_n, A)$ is decreasing monotonically. Then we use the boundary of A as a mirror – we define a point y_n as the point, which is "very near" to x_n but already in the set A. Then in the set A, we can choose a convergent subsequence. The rest is only a technical problem; it is pretty clear that if $y_n \to y$ and x_n are "very near" to y_n then the sequence x_n should converge to y (or at last there should be a converging subsequence).

The rest of this section contains several fixed point theorems which are derived by using this theorem. Almost every one of them is derived by using a suitable auxiliary function φ. We remind the fact that these theorems were known earlier and were proven with variety of different and more sophisticated techniques. The fact, that all of them are consequences of theorem (20), is truly noteworthy.

Proposition 21. *Let (X, d) be an arbitrary metric space and $F : X \to X$ a map satisfying $d(Fx, Fy) < d(x, y)$ whenever $x \neq y$. Assume that for some $z \in X$, the sequence $\{F^l z\}_{l=0}^{\infty}$ has a subsequence converging to a point u. Then u is a fixed point for F.*

Proof. We would like to use the previous theorem (20). Define $A := \{u\}$. Then A is obviously a compact set. Now we define a function

$$\varphi(x) = d(x, Fx) - d(Fx, F^2x) + d(x, u).$$

Because $d(x, Fx) > d(Fx, F^2x)$, it is obvious that φ is a nonnegative function. If $x \neq u$ and $\operatorname{dist}(x, A) = d(x, u) \geq a > 0$ then

$$\varphi(x) \geq d(x, u) \geq a > 0.$$

Now we want to show that $\varphi(F^{n_k}z) \to 0$ where $\{F^{n_k}z\}$ is the convergent subsequence of $\{F^l z\}$. At first, we shall derive several auxiliary inequalities. For any $\varepsilon > 0$, there are $n, m \in \mathbb{N}$ such that

$$d(F^n z, u) < \varepsilon, \qquad d(F^{n+m}z, u) < \varepsilon.$$

It holds:

$$\begin{aligned}
d(F^n z, F^{n+1}z) &\leq d(F^n z, F^{n+m}z) + d(F^{n+m}z, F^{n+m+1}z) \\
&\quad + d(F^{n+m+1}z, F^{n+1}z) \\
&\leq d(F^n z, F^{n+m}z) + d(F^{n+m}z, F^{n+m+1}z) + d(F^{n+m}z, F^n z) \\
&\leq 2\varepsilon + d(F^{n+m}z, F^{n+m+1}z) + 2\varepsilon
\end{aligned}$$

because

$$d(F^n z, F^{n+m} z) \leq d(F^n z, u) + d(u, F^{n+m} z) < 2\varepsilon.$$

Thus we have

$$\begin{aligned}\varphi(F^n z) &= d(F^n z, F^{n+1} z) - d(F^{n+1} z, F^{n+2} z) + d(F^n z, u) \\ &\leq 4\varepsilon + d(F^{n+m} z, F^{n+m+1} z) - d(F^{n+1} z, F^{n+2} z) + \varepsilon \\ &\leq 5\varepsilon,\end{aligned}$$

because $n + m \geq n + 1$ and it follows

$$d(F^{n+1} z, F^{n+2} z) \geq d(F^{n+2} z, F^{n+3} z) \geq \ldots \geq d(F^{n+m} z, F^{n+m+1} z))$$

which means

$$d(F^{n+m} z, F^{n+m+1} z)) - d(F^{n+1} z, F^{n+2} z) \leq 0.$$

Hence

$$\varphi(F^{n_k} z) \to 0.$$

Since $F^{n_k} z \to u$, it has to be

$$0 = \varphi(u) = d(u, Fu) - d(Fu, F^2 u).$$

It implies for $u \neq Fu$

$$d(Fu, F^2 u) < d(u, Fu) = d(Fu, F^2 u)$$

and that is a contradiction. Thus $u = F(u)$ and the proof is complete. □

Proposition 22. *If (X, d) is a metric space and $F : X \to X$ is a map, we denote the diameter of the orbit $\{F^n x \mid n = 0, 1, \ldots\}$ of $x \in X$ by $\delta(x)$. The map F is said to have* shrinking orbits *if for each x with $\delta(x) > 0$, there is an n with $\delta(F^n x) < \delta(x)$.*

Let (X, d) be a bounded metric space, and $F : X \to X$ a map satisfying $d(Fx, Fy) \leq d(x, y)$ for all $x, y \in X$. Assume that for some $z \in X$, the sequence $\{F^n z\}$ has a subsequence converging to a point u. If F has a shrinking orbits, then u is a fixed point for F.

Proof. At first, we shall show that the function $\delta(\cdot)$ is continuous. For that purpose, let us take arbitrary point x_0 and choose $\varepsilon > 0$. We assume that $d(x, x_0) < \varepsilon$. Then for each $n, k \in \mathbb{N}$ we have

$$\begin{aligned}d(F^n x, F^k x) &\leq d(F^n x, F^n x_0) + d(F^n x_0, F^k x_0) + d(F^k x_0, F^k x) \\ &\leq d(x, x_0) + \sup_{n,k \in \mathbb{N}} d(F^n x_0, F^k x_0) + d(x, x_0) \\ &\leq 2\varepsilon + \delta(x_0).\end{aligned}$$

Now we can apply supremum to the left side of inequality and it gives

$$\delta(x) \leq 2\varepsilon + \delta(x_0) \implies \delta(x) - \delta(x_0) \leq 2\varepsilon.$$

On the other side, we could write

$$\begin{aligned}d(F^n x_0, F^k x_0) &\leq d(F^n x_0, F^n x) + d(F^n x, F^k x) + d(F^k x, F^k x_0) \\ &\leq d(x, x_0) + \sup_{n,k \in \mathbb{N}} d(F^n x, F^k x) + d(x, x_0) \\ &\leq 2\varepsilon + \delta(x)\end{aligned}$$

and by applying supremum to the left side we get

$$\delta(x_0) - \delta(x) \leq 2\varepsilon.$$

Hence, the function $\delta(\cdot)$ is continuous.

Now we define a collection of auxiliary functions

$$\varphi(x) = \delta(x) - \delta(F^s x) + d(x,u), \qquad s \in \mathbb{N}.$$

It is obvious that $\delta(x) \geq \delta(F^s x)$ so φ is nonnegative and if $d(x,u) \geq a > 0$ then

$$\varphi(x) \geq d(x,u) \geq a > 0.$$

We would like to show that $\varphi(F^{n_k} z) \to 0$ for each $s \geq 1$. (Here is $\{F^{n_k} z\}$ the convergent subsequence of sequence $\{F^n z\}$ and we remind that $F^{n_k} z \to u$.) Let us take $s \in \mathbb{N}$ fixed and choose $\varepsilon > 0$. Then there is $K \in \mathbb{N}$ such that

$$d(F^{n_K} z, u) < \varepsilon$$

and for every $m \geq 0$ is

$$d(F^{n_K} z, F^{n_{K+m}} z) < \varepsilon.$$

(Convergent sequences are Cauchy sequences.) This implies (using the previous part of the proof)

$$\delta(F^{n_K} z) - \delta(F^{n_{K+m}} z) < 2\varepsilon.$$

We choose $m \in \mathbb{N}$ such that $n_{K+m} > n_K + s$. Then

$$\begin{aligned} \varphi(F^{n_K} z) &= \delta(F^{n_K} z) - \delta(F^{n_K + s} z) + d(F^{n_K} z, u) \\ &= \delta(F^{n_K} z) - \delta(F^{n_{K+m}} z) + \delta(F^{n_{K+m}} z) - \delta(F^{n_K + s} z) + d(F^{n_K} z, u) \\ &\leq 2\varepsilon + 0 + \varepsilon = 3\varepsilon \end{aligned}$$

because

$$n_K + s < n_{K+m} \implies \delta(F^{n_{K+m}} z) - \delta(F^{n_K + s} z) \leq 0.$$

This implies (for each $s \in \mathbb{N}$)

$$\varphi(F^{n_K} z) \to 0 \implies \varphi(u) = 0.$$

Thus we have

$$0 = \varphi(u) = \delta(u) - \delta(F^s u)$$

$$\delta(u) = \delta(F^s u) \qquad \forall s \in \mathbb{N}.$$

Since F has shrinking orbits, $\delta(u) = 0$ and then $u = F(u)$. $\square$

In this and the previous proof we use the fact that the function φ is continuous.

Proposition 23. *Let (X,d) be a metric space and $F : X \to X$ continuous. Assume that for some $z \in X$, the orbit $\{F^n z\}$ contains a convergent subsequence $\{F^{n_i} z\}$. If $d(F^{n_i} z, F^{1+n_i} z) \to 0$, then F has a fixed point.*

Proof. We denote $u = \lim F^{n_i} z$ and

$$\varphi(x) = d(x, u) + d(Fx, u).$$

Let us choose $\varepsilon > 0$. Since there is $n_i \in \mathbb{N}$ such that $d(F^{n_i} z, u) < \varepsilon$ and $d(F^{n_i} z, F^{n_i+1} z) < \varepsilon$, it holds

$$\begin{aligned} \varphi(F^{n_i} z) &= d(F^{n_i} z, u) + d(F^{n_i+1} z, u) \\ &\leq d(F^{n_i} z, u) + d(F^{n_i+1} z, F^{n_i} z) + d(F^{n_i} z, u) \\ &\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

Thus $\varphi(F^{n_i} z) \to 0$ and it implies if φ is continuous

$$\varphi(u) = 0.$$

Hence

$$0 = \varphi(u) = d(Fu, u) \implies F(u) = u.$$

It remains to show continuity of φ. For arbitrary point $x_0 \in x$, it holds

$$\begin{aligned} |\varphi(x) - \varphi(x_0)| &= |d(x, u) + d(Fx, u) - d(x_0, u) - d(Fx_0, u)| \\ &\leq |d(x, u) - d(x_0, u)| + |d(Fx, u) - d(Fx_0, u)| \\ &\leq d(x, x_0) + d(Fx, Fx_0) \xrightarrow{x \to x_0} 0 \end{aligned}$$

because F is continuous. The proof is complete. $\square$

Proposition 24. *Let (X, d) be a metric space and $F : X \to X$ continuous. Assume that there is a continuous nonnegative function $V : X \times X \to \mathbb{R}^+$ with $V^{-1}(0)$ contained in the graph of F, and such that $\inf\{V(x, x) \mid x \in X\} = 0$. Now*

(a) If the function $\varphi(x) = V(x, x)$ has the property in (20) relative to some compact $A \subset X$, then F has a fixed point.

(b) If (X, d) is complete and $\varphi(x) = V(x, x)$ has the property

$$\inf\{\varphi(x) + \varphi(y) \mid d(x, y) \geq a\} = \mu(a) > 0 \qquad \forall a > 0,$$

then F has a fixed point.

Proof. (a): Since $\inf_{x \in X} \varphi(x) = 0$, there is at least one sequence $\{x_n\}$ such that $\varphi(x_n) \to 0$. Let us choose an arbitrary sequence $\{x_n\}$ with this property. Then theorem (20) implies $x_n \to u$ where $u \in A$. It holds

$$\varphi(x_n) = V(x_n, x_n) \to 0$$

and continuity of V implies

$$V(u, u) = 0.$$

But $V^{-1}(0)$ is contained in the graph of F. Hence

$$F(u) = u.$$

(b): Since $\inf_{x \in X} \varphi(x) = 0$, there is at least one sequence $\{x_n\}$ such that

$$\varphi(x_n) \to 0.$$

Let us choose an arbitrary sequence $\{x_n\}$ with this property. Then theorem (17) implies $x_n \to u$ where $u \in X$ (and the point u is the same for every sequence with that property). The rest is similar to the part (a). It holds

$$\varphi(x_n) = V(x_n, x_n) \to 0$$

and continuity of V implies

$$V(u, u) = 0.$$

But $V^{-1}(0)$ is contained in the graph of F. Hence

$$F(u) = u.$$

□

Bibliography

[1] Amann H.: *Lectures on Some Fixed Point Theorems*, Inst. Mat. Pura Apl., Rio de Janeiro, 1973.

[2] Bailey D. F.: *Some theorems on contractive mappings*, J. London Math. Soc. **41** (1966) 101–106.

[3] Bennett D. G., Fisher B.: *On a Fixed Point Theorem for Compact Metric Spaces*, Mathematics Magazine, Vol. **47** No. 1 (Jan., 1974) 40–41

[4] Browder F. E.: *On the convergence of successive approximations for nonlinear functional equations*, Nederl. Akad. Wetensch. Proc. Ser. A 71=Indag. Math. **30** (1968) 27–35.

[5] Bryant V. W.: *A remark on a fixed-point theorem for iterated mappings*, Amer. Math. Monthly **75** (1968) 399–400.

[6] Ćirić L. B.: *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 267–273.

[7] Dugundji J.: *Aspects of fixed-point theory*, Proceedings of the Conference on Problems in Nonlinear Functional Analysis (Univ. Bonn, Bonn, 1975), pp. 56–58. Gesellsch. Math. Datenverarbeitung, St. Augustin, 1975.

[8] Dugundji J., Granas A.: *Fixed Point Theory*, Springer-Verlag, New York, 2003, 9–24

[9] Edelstein M.: *An extension of Banach's Contraction Principle*, Proc. Am. Math. Soc., Vol. **12** No. 1 (1961) 7–10

[10] Edelstein M.: *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962) 74–79

[11] Jachymski J.: *A discrete fixed point theorem of Eilenberg as a particular case of the contraction principle*, Fixed Point Theory Appl. 2004, no. **1**, 31–36.

[12] Matkowski J.: *Integrable solutions of functional equations*, Dissertationes Math. (Rozprawy Mat.) **127** (1975), 68 pp.

[13] Nadler S. B., Jr.: *Sequences of contractions and fixed points*, Pacific J. Math. **27** (1968) 579–585.

[14] Pittnauer F.: *Ein Fixpunktsatz in metrischen Räumen* (German), Arch. Math. (Basel) **26** (1975), no. 4, 421–426.

[15] Rhoades B. E.: *A comparison of Various Definitions of Contractive Mappings*, Transactions of Am. Math. Soc., Vol. **226** (1977) 257–290.