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DOCTORAL THESIS

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Options under Stable Laws

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Dedicated to Nela and to the loving memory of my grandparents.

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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In .... date ....

Název práce: Možnosti se stabilními distribucemi.

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Abstrakt: Stabilní rozdělení jsou úzce spojena s problematikou konvergence součtu nekonečných řad nezávislých náhodných veličin. Hustoty těchto pravděpodobnostních rozdělení jsou dobře zkoumána za použití integralních transformací. Nejprve shrneme známé výsledky odvozené pomocí Fourierovi transformace, dále se zaměříme na méně častou Mellinovu transformaci. Pomocí této budeme vyšetřovat rozdělení součinu dvou nezávislých stabilních náhodných veličin. Ve čtvrté kapitole zobecníme model Louise Bacheliera za pomoci stabilních rozdělení a budeme diskutovat praktické aspekty spojené s finančními deriváty.

Klíčová slova: stabilní rozdělení, Mellinova transformace, součin nezávislých náhodných veličin, levy model, samoshodné plochy implikovaných volatilit

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Abstract: Stable laws play a central role in the convergence problems of sums of independent random variables. In general, densities of stable laws are represented by special functions, and expressions via elementary functions are known only for a very few special cases. The convenient tool for investigating the properties of stable laws is provided by integral transformations. In particular, the Fourier transform and Mellin transform are greatly useful methods. We first discuss the Fourier transform and we give overview on the known results. Next we consider the Mellin transform and its applicability on the problem of the product of two independent random variables. We establish the density of the product of two independent stable random variables, discuss the properties of this product density and give its representation in terms of power series and Fox's H-functions. The fourth chapter of this thesis is focused on the application of stable laws into option pricing. In particular, we generalize the model introduced by Louise Bachelier into stable laws. We establish the option pricing formulas under this model, which we refer to as the Lévy Flight Model, or more succinctly the levy model. We discuss the resulting self-similar volatility surfaces, give the approximative formulas of the volatility smiles and consider the hedging issues under levy model.

Keywords: one-dimensional stable laws, Mellin transform, product of independent random variables, levy model, self-similar volatility surfaces

Contents

1. Introduction

A random variable X with a probability distribution function $F(x)$ is characterized by the functionals describing, e.g., its shape, location, and scale. It is a very common problem to consider two random variables X_1, X_2, with probability distributions $F_1(x), F_2(x)$, respectively, each having different scale and location characteristics. If we are interested in whether the shape of their probability distributions is similar otherwise, we carry out a simple comparison called the normalization of a random variable. It is a transformation which shifts the expectations of the probability distributions to the same point of space, say, e.g., the origin of a Cartesian system, and it changes the scales to a specified unit. In applications it means that the transformed variable is in some sense "dimensionless", or expressed in dimensionless units. Dimensional reduction is widely used, e.g., in physics. See, e.g., [Bar96] for an introduction on the advantages of scaling and self-similarity properties.

The natural form of normalization used in Probability Theory and Mathematical Statistics, is given by $\tilde{X} = (X - m)/\sigma$, where m is the first moment and σ^2 is the variance of the probability distribution $F(x)$ of a random variable X. The assumption that a finite second moment exists for the probability distribution $F(x)$ is rather restricting in some applications. Depending on the nature of the problem, it can be more convenient to consider the transformation $(X - b)/a$, where $b \in \mathbb{R}$ is some change of location and $a > 0$ designates the units of measurement. The change of location can be performed by considering artificial centering, such as truncated moments or median. The new probability distribution of the random variable $\tilde{X}$ considered in dimensionless units is of the *same type* as the original probability distribution $F(x)$. By that we mean that for two probability distribution functions, say, e.g., $F_1(x), F_2(x)$, on the real line with densities $f_1(x), f_2(x)$, respectively, these two distributions $F_1(x), F_2(x)$ are of the same type if $F_2(x) = F_1(ax + b), a > 0, b \in \mathbb{R}$, where a is the scaling factor and b the centering constant. If for their densities we have $f_2(x) = af_1(ax + b)$, we understand that the two distributions $F_1(x), F_2(x)$ differ only by scale and location. See [Fel71].

Stable laws and infinitely divisible distributions were established by Paul Lévy [Lév54] and Alexander Y. Khintchine [Khi38]. They appeared in connection with problems on convergence of sums of independent random variables, and the related theory on central limit theorems. So, consider mutually independent identically distributed random variables $X, X_1, X_2, \ldots$, with a probability distribution function $F(x)$. We denote the sum of n elements as $S_n = X_1 + \ldots + X_n$. The distribution $F(x)$ is called *stable* (in the broad sense) if for each n there exists constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$S_n \stackrel{d}{=} a_n X + b_n. \tag{1.1}$$

By that we understand that the probability distribution function of the sum S_n differs only by the scale and/or by the location from the probability distribution $F(x)$. If b_n equals zero for all n, we call $F(x)$ the *strictly stable* distribution. The scaling constants for stable laws are given by $a_n = n^{1/\alpha}$, [Fel71]. The parameter $0 < \alpha \leq 2$ is called the stability parameter of stable law. If $F(x)$ is stable in the broad sense with stability parameter $\alpha \neq 1$, the centering constant b can be chosen such that $F(x+b)$ is strictly stable, see, e.g., [Fel71]. In this thesis we restrict our attention mainly to strictly stable distributions and often exclude case when $\alpha = 1$. As above, we consider the sum S_n of n independent random variables $X_1, \dots X_n$, all of them having the probability distribution $F(x)$. Let us assume that for the tails of the probability distribution function $F(x)$ and some positive real constants ν, C_1, C_2 holds:

$$1 - F(x) \sim C_1 x^{-\nu}, \quad x \to \infty,$$
$$F(x) \sim C_2 |x|^{-\nu}, \quad x \to -\infty.$$

Then the central limit theorem states, see, e.g., [Fel71, Zol86], that there exists sequences of positive real constants a_n and real numbers b_n, such that the probability distribution function of the centred and scaled sum $\tilde{S_n} = (S_n - b_n)/a_n$ converges weakly to the stable distribution with stability parameter α and some asymmetry parameter β. Here, $\alpha = \nu$ for $\nu < 2$, i.e. the case when the tails of probability distribution $F(x)$ are algebraically decaying. Otherwise, when $\nu \geq 2$, the limit distribution has the tails exponentially light and the stability parameter α equals 2. The latter case corresponds to Gaussian distribution. The asymmetry parameter is given by $\beta = (C_1 - C_2)/(C_1 + C_2)$. See, e.g., [Fel71, Zol86].

In general, the densities of stable laws do not have explicit expressions via elementary functions. Their properties are investigated indirectly, often by using the integral transformations. In the next two chapters we focus on the two well known integral transformations: the Fourier transform and Mellin transform. The Fourier transform is a basic tool in Probability Theory, so the following chapter gives overview on already established results. Among others, we discuss there parametrizations of the characteristic exponent of stable laws, originally introduced in [Zol86]. We combine two of these parametrizations in order to establish the parametrization which we use throughout the text. The choice of this particular parametrization is purely motivated by the need of explanatory parameters, that give clear practical information on the shape of the stable probability distribution function. Clearly, Fourier transformation is inevitably connected with the sums of independent random variables. Similarly, the Mellin transform is a useful tool for investigation of products of independent random variables. The objective of chapter three is to establish the density of the product of two independent stable random variables. The Mellin transform of stable laws was established in [Zol57], where multiplicative and divisibility properties of the stable laws are given. In [Zol86] examples are given where these multiplication and division theorems for stable laws are used to simplify the computation of probability distributions of statistics formed by random variables with stable laws. The explicit representation for distribution of the products of two or more independent stable random variables, however, has not been yet derived. We derive the density of the product of two independent strictly stable random variables, and represent the derived density in terms of its power series. This is done in section 3.4. The Mellin transform and the representation of stable densities in terms of Fox's H-function was considered in [Sch86]. In section 3.5 we discuss the connection between the derived product density and Fox's H-function. The third chapter is based on [Kared].

The Brownian motion was intuitively defined by Louis Bachelier in his pioneering work [Bac00] in order to model the fluctuations of the financial markets. Without doubt, Bachelier developed the first modern model for option pricing [Bac00], often referred to as the *normal model* by market practitioners. Based on physical arguments, Bachelier concluded that the price of a stock could be modelled as Brownian motion. As a natural outgrowth of Bachelier work, the European put and call options are usually priced using Martingale valuation theory, [HP81] *et seq.* Consider an asset whose forward price is modelled via the stochastic process $X = \{X(t), 0 \leq t \leq \tau\}$, and let $X(0) = x$ be the current forward price quoted in the market. Under Martingale valuation theory, the value of European put and call options on the asset are:

$$V_{put}(x,\tau) = \mathrm{E}\left\{ [K - X(\tau)]^+ \big| \, X(0) = x \right\} = \int_{-\infty}^{K} (K-y)\, g(y,\tau|\, x, 0)\mathrm{d}y,$$
$$V_{call}(x,\tau) = \mathrm{E}\left\{ [X(\tau) - K]^+ \big| \, X(0) = x \right\} = \int_{K}^{\infty} (y-K)\, g(y,\tau|\, x, 0)\mathrm{d}y,$$

where K is the option's *strike*, τ is the option's *exercise date*, and $g(y,\tau|x,0)$ is the *terminal risk-neutral density* for $X(\tau)$ at y, given that $X(0) = x$. Since we are taking the current date to be 0, τ is actually the time to expiry. Under this theory, X is a martingale in the risk-neutral measure, so:

$$\mathrm{E}\left\{X(\tau)|X(0) = x\right\} = \int_{-\infty}^{\infty} y g(y,\tau|\, x, 0)\mathrm{d}y = x.$$

Depending on the choice of the (terminal) risk-neutral density, different pricing models are obtained. For example, the risk-neutral density of the well known Black-Scholes Model has a log-normal distribution and when viewed at the current moment rather than at the terminal time of the option. The model is also referred to as the *log-normal* model by market practitioners. In Bachelier's model, at any future date τ the terminal risk-neutral density at $X(\tau) = y$ is Gaussian,

$$g(y-x,\tau) = \frac{1}{\sigma\sqrt{\tau}}\phi\left(\frac{y-x}{\sigma\sqrt{\tau}}\right).$$

Here, $\phi()$ is the density of the standard normal probability distribution. Measurements of asset prices show that they adhere much more closely to stable laws with stability parameters α between, say, 1.35 and 1.6 [MS95].

In the fourth chapter we extend the Bachelier's model to the stable laws with stability parameters $1 < \alpha \leq 2$. We refer to new model as the *Lévy Flight model*, or more succinctly the *levy model*. We establish the pricing formulas for option contracts, see section 4.2, and consider the related volatility surfaces, see section 4.3. Finally we discuss the practical aspects of mis-hedging of the financial instruments, see section 4.4, and concluding remarks for levy model are discussed in section 4.5. The self-similarity of the stable laws is an inherent property of these laws. This allows us to give the option pricing formulas and the resulting volatility surfaces in terms of a dimensionless variable, which we refer to as the *levy moneyness*. Therefore, our extension of Bachelier's model to stable laws, generalizes Bachelier's time scaling property "the square root of time", extensively used by the market practitioners, to the Lévy Flight rule "the fractional root α of time". The chapter four is based on the work established in [KHed].

In order to avoid confusion, we pointed out that the Black-Scholes Model is a different type of option pricing model from the Bachelier model. Efforts to extend the Black-Scholes Model to stable laws have already been made by many authors, see, e.g., discussion in [Man63, Fam65, Sam67]. More recently, the volatility surfaces were considered, e.g., by [CW03], and the exponential Lévy processes driven by stable laws were investigated, e.g., in [AL13]. Over more than the half century of existence of the discipline called 'Mathematical Finance', many option pricing models have been established, and modified by many authors. The Black-Scholes theory became the standard theory used by researchers, and comparing the work of Black and Scholes to Bachelier's work remained for a long time without attracting significant interest.

The last chapter of this thesis provides a list of problems which, in our opinion, deserve more exploration by other researchers interested in the field of Probability Theory, Mathematical Statistics and Mathematical Finance.

2. Fourier Transform Approach to Stable Laws

The objective of this chapter is to give overview on the basic results from the theory of stable laws. To name two basic texts, see, e.g., [Fel71, Zol86]. The stable laws are inevitably connected with the sums of random variables and so in the first section we give quick introduction into probability distributions of sums of independent random variables and infinitely divisible distributions. In particular we mention the representation of characteristic exponent given by Lévy [Lév54] and Khintchine [Khi38]. In the second section we discuss different parametrizations of the characteristic exponent of strictly stable laws considered by [Zol86]. We combine two of these parametrizations and use this modification throughout the text. In the third and fourth section we give overview on the representations and some basic properties of the densities and probability distribution functions of strictly stable laws. The last section gives quick summary of the most important parts of this chapter.

2.1 Sums of Independent Random Variables

The integral transforms are useful tool for studying the distributions obtained by applying the binary operations on the set of independent random variables. In particular, Fourier transform maps the convolution operation onto multiplication, and thus is greatly used for studying distributions of sums of independent random variables.

Consider two independent random variables X_1, X_2 with common distribution $F_1(x), F_2(x)$. The distribution of their sum $X_1 + X_2$ is given by:

$$F_1 * F_2(x) = \int_{-\infty}^{\infty} F_1(x-y)F_2\{\mathrm{d}y\}, \tag{2.1a}$$

where symbol $*$ designates the convolution. If distributions $F_1(x), F_2(x)$ have densities $f_1(x), f_2(x)$, respectively, then the density of the sum X_1+X_2 is obtained by:

$$f_1 * f_2(x) = \int_{-\infty}^{\infty} f_1(x-y)f_2(y)\mathrm{d}y. \tag{2.1b}$$

The convolution operation is commutative and associative among distributions and the continuity property and differentiability is preserved by convolution, see e.g. [Fel71].

Consider mutually independent random variables $X, X_1, X_2, \ldots$ with a common distribution $F(x)$ and denote the sum of n elements as $S_n = X_1 + \ldots + X_n$. The distribution of the sum S_n of these n mutually independent random variables with common distribution $F(x)$ is then $F^{n*}(x)$, where $F^{n*}(x)$ designates the n-th fold convolution of $F(x)$ with itself.

For stable distributions from (1.1) follows that $F^{n*}(x) = F(a_n x + b_n)$, i.e. the n-th fold convolution of $F(x)$ with itself differs from $F(x)$ only by the scale and the location. The condition on the stability of probability distribution implies that the distribution of sum of independent identically distributed random variables is of the same type as the distribution of each random variable in the sum. Stable laws belong to larger group of probability distributions called infinitely divisible. We say that the distribution $F(x)$ is infinitely divisible if for each n there exists a distribution $F_n(x)$ such that $F(x) = F_n^{n*}(x)$, see e.g. [Fel71].

The Fourier transform of a probability distribution $F(x)$ with a density $f(x)$ is defined for a real k as:

$$\varphi(k) = \int_{-\infty}^{\infty} e^{ikx} F\{\mathrm{d}x\} = u(k) + iv(k), \tag{2.2a}$$

and

$$u(k) = \int_{-\infty}^{\infty} \cos(kx) F\{\mathrm{d}x\}, \qquad v(k) = \int_{-\infty}^{\infty} \sin(kx) F\{\mathrm{d}x\}. \tag{2.2b}$$

The Fourier transform of a probability measure is commonly refered to as the characteristic function of a probability measure. For a random variable X with a density $f(x)$ we define:

$$\varphi(k) \equiv \mathrm{E}\left[e^{ikX}\right] = \int_{-\infty}^{\infty} e^{ikx} f(x) \mathrm{d}x. \tag{2.2c}$$

Whenever $\varphi(k)$ is integrable over $\mathbb{R}$, the inverse Fourier integral of density $f(x)$ exists and the density is expressible as:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \varphi(k) \mathrm{d}k, \tag{2.3}$$

see [Fel71]. If the probability distribution $F(x)$ has the finite second moment m_2, and lets denote its first moment as m_1, then

$$\varphi'(0) = im_1, \qquad \varphi''(0) = -m_2. \tag{2.4}$$

If $\varphi''(0)$ exists, then the second moment m_2 of a probability distribution $F(x)$ is finite, see [Fel71]. If $\varphi(k)$ is a real valued function of all k, then the related probability distribution is symmetric.

The characteristic function of probability distribution of the sum of two independent random variables X_1, X_2 with common distributions $F_1(x), F_2(x)$ is given by the product of their characteristic functions $\varphi_1(k)\varphi_2(k)$. So, for two random variables X_1, X_2 with densities $f_1(x), f_2(x)$, the density of the probability distribution of their sum is given by:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \varphi_1(k)\varphi_2(k) \mathrm{d}k, \tag{2.5}$$

where $\varphi_1(k), \varphi_2(k)$ are Fourier transforms of the densities $f_1(x), f_2(x)$, respectively.

The probability distribution of the sum S_n has the characteristic function $[\varphi(k)]^n$ and $F(ax+b)$ is transformed into $e^{ib}\varphi(ak)$, see e.g. [Fel71].

For strictly stable distributions we have relation:

$$[\varphi(k)]^n = \varphi(a_n k) = \varphi(n^{1/\alpha} k), \tag{2.6}$$

see, e.g., [Zol86]. For infinitely divisible distributions we define the infinitely divisible characteristic function, which is sometimes used as an alternative definition of the infinitely divisible distributions. The characteristic function $\varphi(k)$ is called infinitely divisible if for each n there exists a characteristic function $\varphi_n(k)$ such that

$$[\varphi_n(k)]^n = \varphi(k), \tag{2.7}$$

see, e.g., [Fel71].

Infinitely divisible distributions, and stable laws in particular, play crucial role in the theory of convergence of the sums of independent random variables. For an infinite sequence of probability distributions $\{F_n(x)\}$, there is a correspondence between a proper convergence of the probability distributions and the convergence of their characteristic functions. In order for a sequence $\{F_n(x)\}$ of probability distributions to converge properly to a probability distribution $F(x)$, it is necessary and sufficient that the sequence of their characteristic functions $\{\varphi_n(k)\}$ converges point-wise to a limit $\varphi(k)$ and $\varphi(k)$ needs to be continuous at the origin, see Continuity theorem in Feller [Fel71] XV.3.2, p.508. When the limit $\varphi(k) = \lim[\varphi_n(k)]^n$ of the sequence of characteristic functions is continuous everywhere on the real line, the limit distribution is infinitely divisible. The necessary and sufficient condition on the continuity of the limit is given by:

$$n[\varphi_n(k) - 1] \longrightarrow \psi(k) \tag{2.8}$$

where $\psi(k)$ is continuous function, and n approaches infinity. As a consequence, the characteristic function of the limit is given by:

$$\varphi(k) = e^{\psi(k)} \tag{2.9}$$

and $\psi(k)$ is called the *characteristic exponent* of infinitely divisible distribution, see [Fel71] XVII.1.1, p.555.

Lévy [Lév54] and Khintchine [Khi38] gave the representation of characteristic exponent $\psi(k)$, which is one of the fundamental results in the theory of infinitely divisible distributions. According to that the characteristic exponent $\psi(k)$ of infinitely divisible distribution has the representation:

$$\psi(k) = ibk - ck^2 + \int_{-\infty}^{\infty} \frac{e^{ikx} - 1 - ik\sin x}{x^2} \Omega\{\mathrm{d}x\}, \tag{2.10}$$

where constant b is real and $c > 0$. The measure $\Omega\{\mathrm{d}x\}$ is called the canonical measure and is defined on the whole real line such that it assigns every finite interval I a finite value and the integrals

$$\phi^+(x) \equiv \int_x^{\infty} y^{-2}\Omega\{\mathrm{d}y\}, \qquad \phi^-(-x) \equiv \int_{-\infty}^{-x} y^{-2}\Omega\{\mathrm{d}y\}, \tag{2.11}$$

are requested to converge for each $x > 0$, see [Fel71].

For $x > 0$ the canonical measure of strictly stable laws is given by: $\Omega\{(0,x)\} = Cq_1x^{2-\alpha}$ and $\Omega\{(-x,0)\} = Cq_2x^{2-\alpha}$, where $q_1, q_2 \geq 0$ and $q_1+q_2 = 1$, and $C > 0$, see [Fel71] on more rigorous introduction. As $\Omega\{\mathrm{d}x\} = C(2-\alpha)q_1x^{1-\alpha}\mathrm{d}x$ for $x > 0$, then

$$\psi(k) = C(2-\alpha)q_1 \int_0^\infty \frac{e^{ikx} - 1 - ikx}{x^2} x^{1-\alpha}\mathrm{d}x \tag{2.12a}$$

for $x > 0$ and $1 < \alpha \leq 2$. When $x < 0$ we have $\Omega\{\mathrm{d}x\} = C(2-\alpha)q_2|x|^{1-\alpha}\mathrm{d}x$ and so we rewrite the characteristic exponent as:

$$\psi(k) = C(2-\alpha)q_2 \int_{-\infty}^0 \frac{e^{ikx} - 1 - ikx}{x^2} |x|^{1-\alpha}\mathrm{d}x. \tag{2.12b}$$

We choose centering ikx rather then $ik\sin x$. This differs from (2.10). It is, however, more convinient for this case. In particular, it centers the stable law to 0. For $0 < \alpha < 1$, we have the characterictic exponent given by the integrals:

$$\psi(k) = C(2-\alpha)q_1 \int_0^\infty \frac{e^{ikx} - 1}{x^2} x^{1-\alpha}\mathrm{d}x \tag{2.12c}$$

for $x > 0$, and by

$$\psi(k) = C(2-\alpha)q_2 \int_{-\infty}^0 \frac{e^{ikx} - 1}{x^2} |x|^{1-\alpha}\mathrm{d}x \tag{2.12d}$$

for $x < 0$. The latter integrals are convergent. By working out these integrals, see, e.g. [Fel71], we obtain the characteristic exponent for any real k in the following form:

$$\psi(k) = C\frac{\Gamma(3-\alpha)\cos(\pi\alpha/2)}{\alpha(\alpha-1)}|k|^\alpha \Big[1 - i\,\mathrm{sgn}(k)(q_1-q_2)\tan(\pi\alpha/2)\Big], \tag{2.13}$$

see [Fel71]. Clearly, the weights q_1, q_2 influence the asymmetry of the probability distribution. For $q_1 = q_2$ the characteristic exponent takes only real values and so the probability distribution is symmetric.

2.2 Parametrizations of Characteristic Exponent of Stable Law

The form of characteristic exponent in (2.13) suggests that the characteristic function $\varphi(k)$ of a strictly stable law is represented by:

$$\log\varphi(k) = \begin{cases} zk^\alpha, & \text{for } k > 0, \\ \bar{z}|k|^\alpha, & \text{for } k < 0, \end{cases} \tag{2.14}$$

where z is a complex number, and where $\bar{z}$ designates the complex conjugate of complex number z. We require Re $z < 0$ in order to keep $\varphi(k)$ bounded. Clearly the characteristic exponent in (2.13) satisfies this condition. Depending on the form of complex number z in (2.14), different parametrizations have been proposed and studied in [Zol86]. In this section we discuss the parametrizations considered in [Zol86]. We modify one of these parametrizations in order to establish the parametrization which suits best to our purpose.

In the previous section we discussed the derivation of the characteristic exponent of strictly stable law using Lévy-Khintchine formula. There is an alternative and a rather natural way to derive the characteristic exponent of strictly stable law, see e.g. [Lév25]. The idea is based on taking the logarithm of (2.6). For positive k this yields the relation:

$$n\psi(k) = \psi(n^{1/\alpha}k), \tag{2.15}$$

where $\psi(k)$ is the characteristic exponent in (2.9) of strictly stable laws. The latter equation is a special case of Cauchy functional equation, see, e.g., [Acz66], and so the good candidate for the solution is:

$$\psi(k) = zk^{\alpha},$$

where z is a complex valued constant. In order for $\varphi(k)$ to be a characteristic function, we require Re $z < 0$. Similarly, we proceed for $k < 0$ and we obtain (2.14).

First we represent the complex number z in (2.14) in its algebraic form. We denote the real part of z by c, assume that $c > 0$ and rewrite the complex number z as $z = -c(1 + id)$. For characteristic exponent $\psi(k)$ we have:

$$\psi(k) = -c|k|^{\alpha} - icd|k|^{\alpha} \operatorname{sgn}(k). \tag{2.16a}$$

If $d = 0$, $\psi(k)$ is a real-valued function and the stable distribution is symmetric. Clearly, the choice of imaginary part of z influences the asymmetry of stable law. Next, we consider the constant z from (2.14) in its polar form, i.e. $z = -re^{i\theta}$ for $r > 0$. In order for $\varphi(k)$ to be bounded, we require $-\pi/2 < \theta < \pi/2$. We rewrite the characteristic exponent $\psi(k)$ as:

$$\psi(k) = -r\cos\theta|k|^{\alpha} - ir\sin\theta|k|^{\alpha} \operatorname{sgn}(k). \tag{2.16b}$$

The transformation relations between the algebraic and polar parametrization from (2.16a) and (2.16b) are given by:

$$r = c\sqrt{1 + d^2}, \tag{2.16c}$$

$$\theta = \arctan d. \tag{2.16d}$$

The representation of the characteristic exponent in (2.13) is obviously given in the algebraic form (2.16a) where for the parameters c, d we have:

$$c = -C\frac{\Gamma(3-\alpha)\cos(\alpha\pi/2)}{\alpha(\alpha-1)}, \tag{2.17a}$$

$$d = (q_2 - q_1)\tan(\alpha\pi/2), \tag{2.17b}$$

where constants C, q_1, q_2 were introduced in the previous section. In order to transform (2.13) into the polar coordinates (2.16c),(2.16d), we have the relations:

$$r = -C\frac{\Gamma(3-\alpha)\cos(\alpha\pi/2)}{\alpha(\alpha-1)}\sqrt{1 + (q_1 - q_2)^2\tan^2(\alpha\pi/2)}, \tag{2.17c}$$

$$\theta = \arctan[(q_2 - q_1)\tan(\alpha\pi/2)]. \tag{2.17d}$$

For θ we require: $-\pi/2 < \theta < \pi/2$, and so in order for relation (2.17d) to make sense we consider the principal branch of the tangent function. From the condition $0 \leq q_1, q_1 \leq 1$, we have: $-1 \leq q_1 - q_2 \leq 1$. Whenever $0 < \alpha < 1$, we have: $-\alpha\frac{\pi}{2} \leq \theta \leq \alpha\frac{\pi}{2}$. For $1 < \alpha < 2$, we use periodicity of tangent function and obtain $\tan(\alpha\pi/2) = \tan(\alpha\pi/2 - \pi) = -\tan[(2-\alpha)\pi/2]$ and so $-(2-\alpha)\frac{\pi}{2} \leq \theta \leq (2-\alpha)\frac{\pi}{2}$. Clearly the condition $r > 0$ is satisfied. The relations (2.17a)-(2.17d) gives the transformation equations containing the parameters of the canonical measure $\Omega\{\mathrm{d}x\}$ which was introduced in the previous section.

The discussion regarding the boundaries of parameter θ in (2.17d) motivates us to consider different expressions for the parameter θ. We start by taking the constant $-1 < \gamma_1 < 1$ and we define:

$$\theta_1 \equiv \begin{cases} \alpha\gamma_1\frac{\pi}{2} & \text{for } 0 < \alpha < 1, \\ (2-\alpha)\gamma_1\frac{\pi}{2} & \text{for } 1 < \alpha < 2. \end{cases} \tag{2.18a}$$

From (2.17d) we obtain the transformation equation between θ_1 and θ, and we express γ_1 as:

$$\gamma_1 = \begin{cases} \frac{2}{\alpha\pi}\arctan[(q_2-q_1)\tan(\alpha\pi/2)] & \text{for } 0 < \alpha < 1, \\ \frac{2}{(2-\alpha)\pi}\arctan[(q_2-q_1)\tan(\alpha\pi/2)] & \text{for } 1 < \alpha < 2. \end{cases}$$

Next, we rewrite the argument θ in (2.17d) as:

$$\theta_2 \equiv \begin{cases} \alpha\gamma_2\frac{\pi}{2}, & \text{where } -1 < \gamma_2 < 1 \text{ for } 0 < \alpha < 1, \\ \alpha\gamma_2\frac{\pi}{2}, & \text{where } -\frac{2-\alpha}{\alpha} < \gamma_2 < \frac{2-\alpha}{\alpha} \text{ for } 1 < \alpha < 2. \end{cases} \tag{2.18b}$$

Using (2.17d), we obtain relation for γ_2:

$$\gamma_2 = \frac{2}{\alpha\pi}\arctan[(q_2-q_1)\tan(\alpha\pi/2)],$$

where the domain of γ_2 varies depending on the value of α as discussed in (2.18b). With the closer examination of bounds for γ_2 we see that $|\gamma_2| < 1$. We take some $0 < \gamma_3 < 1$ such that, $\gamma_2 = \gamma_3 - (1-\gamma_3)$, i.e. $\gamma_3 = \frac{1+\gamma_2}{2}$. So we express γ_2 as the difference of two positive numbers γ_3 and $1-\gamma_3$ and the sum of these two positive numbers equals 1. From there we can formulate another parametrization for the angle of the complex number z in (2.14). It is defined as:

$$\theta_3 \equiv \begin{cases} \alpha\gamma_3\pi - \alpha\frac{\pi}{2}, & \text{where } 0 < \gamma_3 < 1 \text{ for } 0 < \alpha < 1, \\ \alpha\gamma_3\pi - \alpha\frac{\pi}{2}, & \text{where } \frac{\alpha-1}{\alpha} < \gamma_3 < \frac{1}{\alpha} \text{ for } 1 < \alpha < 2. \end{cases} \tag{2.18c}$$

To motivate the latter expression, recall that $e^{\alpha\pi/2} = i^\alpha$, and so:

$$\psi(k) = r(-i|k|)^\alpha e^{i\alpha\gamma_3\pi\,\mathrm{sgn}(k)}.$$

Again, using (2.17d), we express γ_3 as:

$$\gamma_3 = \frac{1}{2} + \frac{1}{\alpha\pi}\arctan[(q_2-q_1)\tan(\alpha\pi/2)],$$

where the domain of γ_3 depends on α and is given in (2.18c). Parameter θ_3 in (2.18c) also equals:

$$\theta_3 = \begin{cases} \alpha[\gamma_3 - (1-\gamma_3)]\frac{\pi}{2}, & \text{where } 0 < \gamma_3 < 1 \text{ for } 0 < \alpha < 1, \\ \alpha[\gamma_3 - (1-\gamma_3)]\frac{\pi}{2}, & \text{where } \frac{\alpha-1}{\alpha} < \gamma_3 < \frac{1}{\alpha} \text{ for } 1 < \alpha < 2. \end{cases} \tag{2.19}$$

Here we interpret γ_3 as the part of parameter γ_2, such that it is a weight influencing the asymmetry parameter of stable law. It rotates the modulus r only in the anti-clockwise direction. The remaining proportion $1 - \gamma_3$ is the second weight which rotates the modulus only in the clockwise direction.

The previous discussion provides us with a simple guide in parametrizations discussed by Zolotarev. So let us give overview on these parametrizations. Starting with the algebraic form with parameters given in (2.17a), (2.17b), Zolotarev [Zol86] refers to (2.16a) as the parametrization A and denotes it by:

$$\psi^A(k) = -\lambda|k|^\alpha + i\lambda\beta k|k|^{\alpha-1}\tan(\alpha\pi/2). \tag{2.20}$$

Here the positive real number λ corresponds to the scaling parameter of the stable law and $\beta \equiv q_1 - q_2$ designates the asymmetry parameter. Clearly for the asymmetry parameter β holds: $-1 \le \beta \le 1$. Next we consider the polar parametrizations. Zolotarev [Zol86] refers to the parametrization in (2.18a) as the form B and uses the notation:

$$\psi^B(k) = -\lambda|k|^\alpha e^{-i\beta K(\alpha)\frac{\pi}{2}\operatorname{sgn}(k)},$$

where $K(\alpha) = \alpha - 1 + \operatorname{sgn}(1-\alpha)$. Denote parameters α, β, λ under the parametrization B as $\alpha_B, \beta_B, \lambda_B$. Obviously, $\beta_B K(\alpha)\operatorname{sgn}(k) = \alpha\gamma_1$ and $\lambda_B = r$, where the modulus r is given by (2.17c) and the argument γ_1 is from (2.18a). Clearly $-1 < \beta_B K(\alpha) < 1$. Let us denote the parameters α, β, λ under the parametrization A as $\alpha_A, \beta_A, \lambda_A$. Then:

$$\alpha_A = \alpha_B, \quad \beta_A = \frac{\tan[\beta_B K(\alpha)\pi/2]}{\tan(\alpha\pi/2)}, \quad \lambda_A = \lambda_B \cos\beta_B.$$

In the virtue of formula (2.18b), Zolotarev [Zol86] introduced the parametrization C:

$$\psi^C(k) = -\lambda|k|^\alpha e^{-i\delta\alpha\frac{\pi}{2}\operatorname{sgn}(k)},$$

where $|\delta| \le \min\{1, 2/\alpha - 1\}$. We see, that $|\delta| < 1$ for $0 < \alpha < 1$ and $|\delta| < \frac{2}{\alpha} - 1$ for $1 < \alpha < 2$. Obviously, δ corresponds to γ_2 in (2.18b). The relation between the three parametrizations is:

$$\begin{aligned} \alpha_A &= \alpha_B = \alpha_C, \\ \delta &= \frac{\beta_B K(\alpha)}{\alpha} = \frac{2}{\alpha\pi}\arctan[\beta_A \tan(\alpha\pi/2)], \\ \lambda_C &= \lambda_B = \frac{\lambda_A}{\cos[\alpha\delta/K(\alpha)]}. \end{aligned}$$

Using the argument θ_3 in (2.18c), Zolotarev [Zol86] further modifies the parametrization C as:

$$\psi^C(k) = -\lambda(i|k|)^\alpha e^{-i\alpha\rho\pi\operatorname{sgn}(k)},$$

where $\rho = \frac{1+\delta}{2}$ and so:

$$\rho = \frac{1}{2} + \frac{\beta_B K(\alpha)}{2\alpha} = \frac{1}{2} + \frac{1}{\alpha\pi}\arctan[\beta_A \tan(\alpha\pi/2)].$$

We see, that $\rho = \gamma_3$ in (2.18c). The relation between the parameters α_C, λ_C and parameters α, λ under the parametrizations A and B, respectively, remains the same.

Based on the motivation from (2.19) we introduce the modification of Zolotarev's parametrizations B and C. We consider numbers p_1, p_2, such that $0 \leq p_1, p_2 \leq 1$ and $p_1 + p_2 = 1$ and we write z in (2.14) as: $z = -ce^{-i\alpha(p_1-p_2)\frac{\pi}{2}}$. Then we have:

$$\psi(k) = -c|k|^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}\operatorname{sgn} k}. \tag{2.21a}$$

Here $c > 0$ is some positive constant. In order to keep $\psi(k)$ bounded we have the following conditions on the asymmetry parameter p_1:

$$0 \leq p_1, p_2 \leq 1 \qquad \text{for } 0 < \alpha < 1, \tag{2.21b}$$

$$1 - \tfrac{1}{\alpha} \leq p_1, p_2 \leq \tfrac{1}{\alpha} \qquad \text{for } 1 < \alpha < 2. \tag{2.21c}$$

The relation between the asymmetry parameter p_1 and Zolotarev's parametrization C is simply: $p_1 = \rho$. In section 2.4 we will show that p_1 is the proportion of probability mass concentrated on the positive part of the real line and that $p_2 = 1 - p_1$ is the remaining proportion of probability mass concentrated on the negative part of the real line.

2.3 Stable Densities

Due to the absence of explicit expressions for stable densities and probability distribution functions in terms of elementary functions, determining the analytic properties of stable laws remains an active area of investigation. It has been shown in [Gaw84] that their densities are bell-shaped, i.e. that they are infinitely differentiable on the real line and its nth derivative possesses exactly n zeros on its support. The densities of stable laws belong to the class of special functions which can be expressed as Meijer's G-functions for α rational, [Zol94], and can be expressed for any α by Fox's H-functions, [Zol86, Sch86]. For definitions and properties of Meijer's G-functions and Fox's H-function, see, e.g., [BBEoNR53a, Fox61] or Appendix A.3.

Stable distributions are often investigated via their characteristic functions. The characteristic function of stable distribution is bounded, thus integrable over whole real line. Therefore, the inverse Fourier transform of stable characteristic function exists and the stable density is expressed by inverse Fourier integral as:

$$f(x; \alpha, p_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx - c|k|^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}\operatorname{sgn} k}}\mathrm{d}k. \tag{2.22}$$

Clearly,

$$\begin{aligned} f(x; \alpha, p_1, c) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx - c|k|^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}\operatorname{sgn} k}}\mathrm{d}k = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(kc^{1/\alpha})(xc^{-1/\alpha}) - |kc^{1/\alpha}|^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}\operatorname{sgn} k}}\mathrm{d}k = \\ &= c^{-1/\alpha} f(xc^{-1/\alpha}; \alpha, p_1, 1), \end{aligned}$$

and so we conclude that

$$f(x;\alpha,p_1,c) = c^{-1/\alpha} f(xc^{-1/\alpha};\alpha,p_1,1). \tag{2.23a}$$

We refer to this relation as the *scaling property* of stable law. For simplicity, we will assume that the scaling paramater $c = 1$, if not stated otherwise.

Further, for $x \geq 0$ we have:

$$\begin{aligned} f(x;\alpha,p_1) &= \frac{1}{2\pi}\int_0^\infty \left[e^{-ikx-k^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}} + e^{ikx-k^\alpha e^{i\alpha(p_1-p_2)\frac{\pi}{2}}}\right]\mathrm{d}k = \\ &= \frac{1}{2\pi}\int_0^\infty \left[e^{ik(-x)-k^\alpha e^{i\alpha(p_2-p_1)\frac{\pi}{2}}} + e^{-ik(-x)-k^\alpha e^{-i\alpha(p_2-p_1)\frac{\pi}{2}}}\right]\mathrm{d}k = \\ &= f(-x;\alpha,p_2), \end{aligned}$$

and so we get the *reflection property* of stable law:

$$f(x;\alpha,p_1) = f(-x;\alpha,p_2). \tag{2.23b}$$

We denote the integrand in (2.22) as $g(k,x;p_1)$, and $g(k;\alpha,p_1)$, respectively. We use the notation equivalently, depending on which parameter is of particular interest. Next we denote the complex conjugate of $g(k;\alpha,p_1)$ as $\overline{g(k;\alpha,p_1)}$. Obviously $g(k,x;p_1) + \overline{g(k,x;p_1)} = g(k,-x;p_2) + \overline{g(k,-x;p_2)}$. Therefore, without loss of generality, we assume $x \geq 0$. Clearly the following relation holds:

$$\begin{aligned} f(x;\alpha,p_1) &= \frac{1}{\pi}\mathrm{Re}\int_0^\infty e^{-ikx-k^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}}\mathrm{d}k = \\ &= \frac{1}{\pi}\mathrm{Re}\int_0^\infty e^{ikx-k^\alpha e^{i\alpha(p_1-p_2)\frac{\pi}{2}}}\mathrm{d}k. \end{aligned} \tag{2.23c}$$

The latter properties are the elementary properties of stable laws; see, e.g., [Zol86].

We split the real and imaginary part of the integrand $g(k,x;p_1)$ for $k > 0$:

$$\begin{aligned} g(k,x;p_1) &= e^{-ikx-k^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}} = \\ &= e^{-k^\alpha \cos[\alpha(p_1-p_2)\frac{\pi}{2}]}\cos\{kx - k^\alpha \sin[\alpha(p_1-p_2)\pi/2]\} - \\ &- ie^{-k^\alpha \cos[\alpha(p_1-p_2)\frac{\pi}{2}]}\sin\{kx - k^\alpha \sin[\alpha(p_1-p_2)\pi/2]\}, \end{aligned}$$

in order to represent the stable density by the integral

$$f(x;\alpha,p_1) = \frac{1}{\pi}\int_0^\infty e^{-k^\alpha \cos[\alpha(p_1-p_2)\frac{\pi}{2}]}\cos\{kx - k^\alpha \sin[\alpha(p_1-p_2)\pi/2]\}\,\mathrm{d}k. \tag{2.24}$$

From (2.24) is obvious that for $x > 0$ the integrand in (2.24) oscillates. The frequency of oscillation increases for larger values of x and so the representation (2.24) is not very convenient for a direct numerical implementation. See Figures 2.1-2.2. Using the Fast Fourier transform to evaluate the integral in (2.23c) is also problematic due to Gibbs phenomena and oscillation.

Power series of stable densities were firstly determined by [Ber52]. In our notation, they are given by the power series:

$$f(x;\alpha,p_1) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(k/\alpha+1)}{\Gamma(k+1)}\sin(p_2 k\pi)x^{k-1}. \tag{2.25a}$$

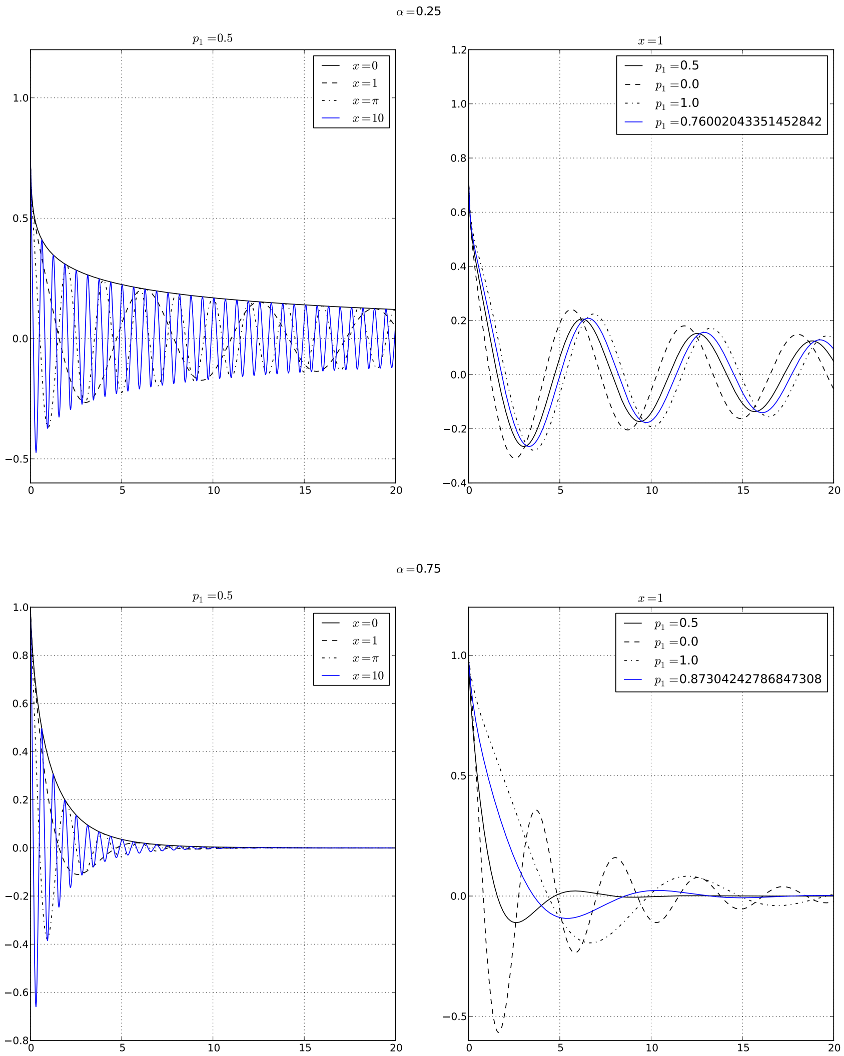


Figure 2.1: The integrand from (2.24) The pictures on the left depicts the dependence of the integrand on values of x for symmetric case. The pictures in the right column show the effect of the shift for different values of the asymmetry parameter p_1. The asymmetry parameter p_1 corresponds to values of the asymmetry parameter under Zolotarev's parametrization A: $\beta_A = 0, -1, 1, 0.5$.

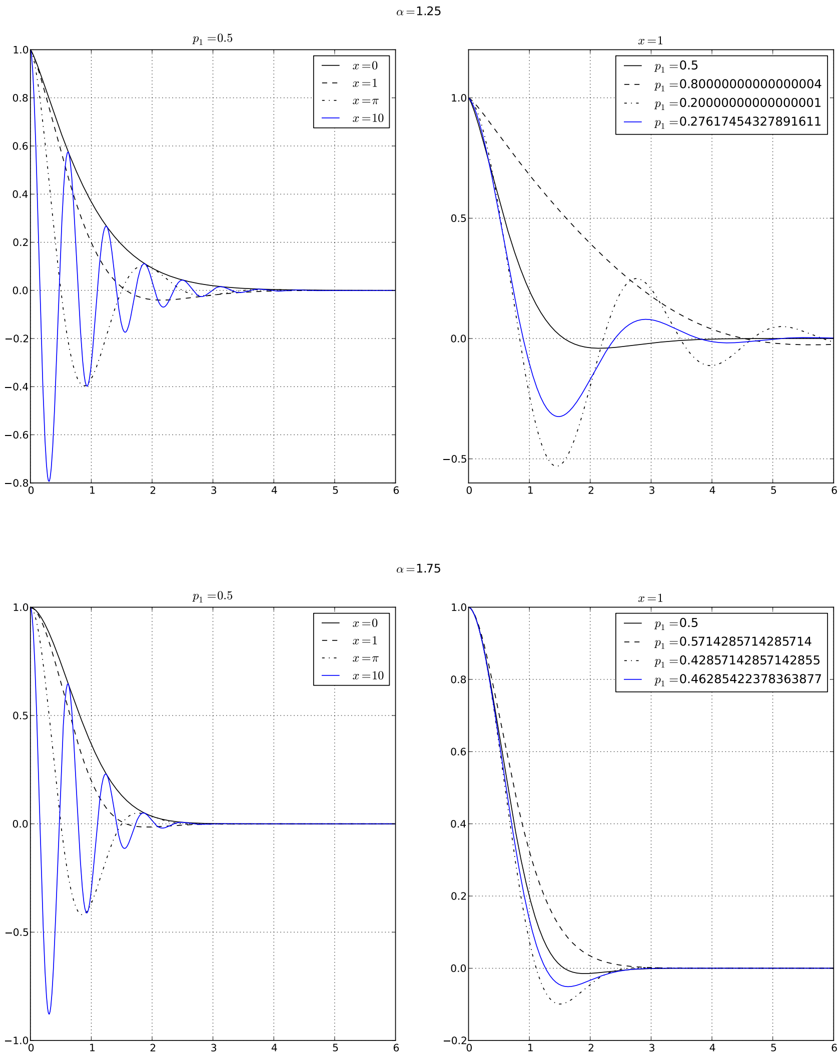


Figure 2.2: The integrand from (2.24). The pictures on the left depicts the dependence of the integrand on values of x for symmetric case. The pictures in the right column show the effect of the shift for different values of the asymmetry parameter p_1. The asymmetry parameter p_1 corresponds to the values of asymmetry parameter under Zolotarev's parametrization A: $\beta_A = 0, -1, 1, 0.5$.

It can be readily verified that the series is convergent everywhere on real line for $1 < \alpha \le 2$. For $\alpha = 2$, the sum of the power series equals the density of the normal distribution with variance 2. These series are absolutely and uniformly convergent in every finite domain. Although the series converges rapidly for any $|x| \le 1$, it converges rather slowly for larger values of x, as it takes many terms before the gamma functions can dominate the power x^k.

The asymptotic expansions for the stable densities and distribution functions are given by:

$$f(x;\alpha,p_1) \sim \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(\alpha k+1)}{\Gamma(k+1)}\sin(\alpha p_2 k\pi)x^{-\alpha k-1}, \tag{2.25b}$$

for $x \to \infty$. See [Ber52]. Although these series are everywhere divergent for $\alpha > 1$, the first few terms accurately approximate the density and probability distribution function for large enough x.

Further the series (2.25b) is convergent everywhere on the real line for $0 < \alpha < 1$, and the series (2.25a) is the asymptotic expansion around singularity at origin for $0 < \alpha < 1$. See [Ber52].

For moderate x, the slow convergence of the power series and the limited accuracy of the asymptotic expansion limit the utility of formulas (2.25a), (2.25b) for calculating stable densities and probability distribution functions. For these reasons, it is more convenient to use an integral representation of the stable law that was developed by [Zol66]. The integral representation of the density is given by:

$$f(x;\alpha,p_1) = \frac{\alpha x^{\alpha/(\alpha-1)}}{(\alpha-1)\pi}\int_0^{p_1\pi}\exp\{-x^{\alpha/(\alpha-1)}u(y,p_1)\}u(y,p_1)\mathrm{d}y = \tag{2.26a}$$

$$= \frac{1}{\pi}\int_0^{p_1\pi}\tfrac{\partial}{\partial x}\left[-\exp\{-x^{\alpha/(\alpha-1)}u(y,p_1)\}\right]\mathrm{d}y, \tag{2.26b}$$

where

$$u(y,p_1) = \frac{[\sin(p_1\pi - y)]^{1/(\alpha-1)}\sin[(\alpha-1)y + p_1\pi]}{\sin(\alpha y)^{\alpha/(\alpha-1)}}. \tag{2.26c}$$

From relation (2.24), the stable density can be estimated by:

$$f(x;\alpha,p_1) \le \frac{1}{\pi c^{1/\alpha}}\Gamma\left(1+\tfrac{1}{\alpha}\right)\{\cos[(p_1-p_2)\pi/2] + \sin[(p_1-p_2)\pi/2]\} =$$

$$= \frac{\sqrt{2}}{\pi c^{1/\alpha}}\Gamma\left(1+\tfrac{1}{\alpha}\right)\cos(\pi/4 - p_2\pi).$$

Consequently, the density of symmetric stable law, i.e. $p_1 = p_2$, has maximum at point $f(0;\alpha,1/2)$.

The n-th derivative of $f(x;\alpha,p_1)$ is given by its inverse Fourier transform as:

$$f^{(n)}(x;\alpha,p_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-ik)^n e^{-ikx-|k|^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}\operatorname{sgn} k}}\mathrm{d}k. \tag{2.27}$$

For $x \ge 0$ we have:

$$f^{(n)}(x;\alpha,p_1) = \frac{1}{2\pi}\int_0^{\infty}\left[e^{-i\frac{\pi}{2}n}k^n e^{-ikx-k^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}} + e^{i\frac{\pi}{2}n}k^n e^{ikx-k^\alpha e^{i\alpha(p_1-p_2)\frac{\pi}{2}}}\right]\mathrm{d}k.$$

Similarly to (2.23c) we get:

$$f^{(n)}(x;\alpha,p_1) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty e^{-i\frac{\pi}{2}n}k^n e^{-ikx-k^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}}\mathrm{d}k = \tag{2.28}$$
$$= \frac{1}{\pi}\mathrm{Re}\int_0^\infty e^{i\frac{\pi}{2}n}k^n e^{ikx-k^\alpha e^{i\alpha(p_1-p_2)\frac{\pi}{2}}}\mathrm{d}k,$$

and:

$$f^{(n)}(x;\alpha,p_1) = (-1)^n f^{(n)}(-x;\alpha,p_2). \tag{2.29}$$

The latter relation can be easily seen from:

$$f^{(n)}(x;\alpha,p_1) = \frac{1}{2\pi}\int_0^\infty \left[(-ik)^n e^{-ikx-k^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}} + (ik)^n e^{ikx-k^\alpha e^{i\alpha(p_1-p_2)\frac{\pi}{2}}}\right]\mathrm{d}k =$$
$$= \frac{(-1)^n}{2\pi}\int_0^\infty \left[(ik)^n e^{ik(-x)-k^\alpha e^{i\alpha(p_2-p_1)\frac{\pi}{2}}} + (-ik)^n e^{-ik(-x)-k^\alpha e^{-i\alpha(p_2-p_1)\frac{\pi}{2}}}\right]\mathrm{d}k.$$

The integral (2.28) gets substantially simplified for $x = 0$. For symmetric case holds:

$$f^{(n)}(0;\alpha,1/2) = \frac{\cos(n\pi/2)}{\pi}\int_0^\infty k^n e^{-k^\alpha}\mathrm{d}k = \frac{\cos(n\pi/2)}{\alpha\pi}\Gamma\left(\tfrac{n+1}{\alpha}\right). \tag{2.30}$$

For $p_1 \neq p_2$, we start with integral:

$$f^{(n)}(0;\alpha,p_1) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty e^{-i\frac{\pi}{2}n}k^n e^{-k^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}}\mathrm{d}k.$$

The exchange of the contour of integration is readily verified by Cauchy Theorem and Jordan's Lemma, and so we have:

$$\int_0^\infty e^{-i\frac{\pi}{2}n}t^n e^{-t^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}}\mathrm{d}t = e^{-i\frac{\pi}{2}n}e^{i(n+1)(p_1-p_2)\frac{\pi}{2}}\int_0^\infty t^n e^{-t^\alpha}\mathrm{d}t =$$
$$= ie^{-i(n+1)\pi p_2}\int_0^\infty t^n e^{-t^\alpha}\mathrm{d}t,$$

where we used $p_1+p_2 = 1$. By taking the real part of latter expression, we obtain:

$$f^{(n)}(0;\alpha,p_1) = \frac{1}{\alpha\pi}\Gamma\left(\tfrac{n+1}{\alpha}\right)\sin([n+1]p_2\pi) = \tag{2.31}$$
$$= \frac{(-1)^n}{\alpha\pi}\Gamma\left(\tfrac{n+1}{\alpha}\right)\sin([n+1]p_1\pi).$$

where we used (2.29) in the last equity. The computation of the integral (2.28) for $x > 0$ is more complicated. Nevertheless, we can easily compute estimates of the $n-th$ derivative. Using $|\exp\ z| = \exp(\mathrm{Re}\ z)$, we estimate:

$$f^{(n)}(x;\alpha,p_1) \le \frac{1}{2\pi}\int_{-\infty}^\infty |k|^n|e^{-ikx-|k|^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}\,\mathrm{sgn}\,k}}|\mathrm{d}k =$$
$$= \frac{1}{\pi}\int_0^\infty k^n e^{-k^\alpha\cos[\alpha(p_1-p_2)\frac{\pi}{2}]}\mathrm{d}k.$$

Substituting $t = k^\alpha\cos[\alpha(p_1-p_2)\frac{\pi}{2}]$, we obtain:

$$f^{(n)}(x;\alpha,p_1) \le \frac{1}{\alpha\pi}\Gamma\left(\tfrac{n+1}{\alpha}\right)\left(\cos[\alpha(p_1-p_2)\pi/2]\right)^{-\frac{n+1}{\alpha}}. \tag{2.32}$$

Comparing the terms in the series (2.25a) to the values of the $n-th$ derivatives at 0 given in (2.31), we conclude that the power series (2.25a) is the Maclaurin series of stable density for $1 < \alpha \le 2$.

2.4 Stable Distribution Function

In this section we give overview on the power series and the integral representation of the probability distribution function of stable strictly laws, see [Ber52, Zol66].

From (2.23b) we get:

$$
\begin{aligned}
F(x;\alpha,p_1) = \int_{-\infty}^{x} f(y;\alpha,p_1)\mathrm{d}y = \int_{-x}^{\infty} f(y;\alpha,p_2)\mathrm{d}y = \\
= 1 - F(-x;\alpha,p_2),
\end{aligned}
$$

therefore:

$$
F(x;\alpha,p_1) + F(-x;\alpha,p_2) = 1. \tag{2.33}
$$

From (2.26b) the integral representation of stable distribution function is easily derived by using Fubini theorem. In particular for $x = 0$ we have:

$$
\begin{aligned}
F(0;\alpha,p_1) &= 1 - \int_0^{\infty} f(x;\alpha,p_1)\mathrm{d}x = \\
&= 1 - \int_0^{\infty} \Big(\frac{1}{\pi}\int_0^{p_1\pi} \tfrac{\partial}{\partial x}\Big[-e^{-x^{\alpha/(\alpha-1)}u(y,p_1)}\Big]\mathrm{d}y\Big)\mathrm{d}x = \\
&= 1 - \frac{1}{\pi}\int_0^{p_1\pi}\mathrm{d}y \int_0^{\infty} \tfrac{\partial}{\partial x}\Big[-e^{-x^{\alpha/(\alpha-1)}u(y,p_1)}\Big]\mathrm{d}x = \\
&= 1 - \frac{1}{\pi}\int_0^{p_1\pi}\mathrm{d}y = 1 - p_1 = p_2.
\end{aligned}
$$

From (2.33), we have $F(0;\alpha,p_2) = p_1$. The parameter p_1 is equal to the integral of the density over the positive real line. As the value of the probability distribution function at point $x = 0$ equals $1-p_1$, the parameter p_1 represents the asymmetry of the stable law. When $p_1 = 0.5$, the stable law is symmetric. For any admissible $p_1 < 0.5$, the distribution is skewed to the left, whereas for any admissible $p_1 > 0.5$ the distribution is skewed to the right. The bounds on the parameters p_1, p_2 indicate the admissible proportion of the probability mass on the positive and negative parts of the real line, respectively. When $0 < \alpha < 1$ and $p_1 = 0$, the stable probability measure is concentrated only on the negative half-axis of the real line. Similarly for $p_2 = 0$ and $0 < \alpha < 1$ the support of the stable probability measure is located only on the positive half-axis of the real line. For $1 < \alpha \leq 2$, the boundaries of the asymmetry parameter p_1 are given by $1-1/\alpha \leq p_1, p_2 \leq 1/\alpha$ and so the support of stable measure is the entire real line. The shape of the boundaries of parameters p_1, p_2 implies that for $\alpha \to 2$, the stable distribution tends to a symmetric law. Figure 2.3 depicts the admissible values of parameters p_1, p_2. Obviously, for $\alpha = 2$, only $p_1 = 1/2$ is admissible, thus the Gaussian distribution is symmetric.

For $x > 0$ the stable distribution function is given by:

$$
F(x;\alpha,p_1) = 1 - \frac{1}{\pi}\int_0^{p_1\pi} e^{-x^{\alpha/(\alpha-1)}u(y,p_1)}\mathrm{d}y, \tag{2.34a}
$$

see [Zol66]. For $x < 0$ we use (2.33) and we get:

$$
F(x;\alpha,p_1) = 1 - F(|x|;\alpha,p_2) = \frac{1}{\pi}\int_0^{p_2\pi} e^{-|x|^{\alpha/(\alpha-1)}u(y,p_2)}\mathrm{d}y. \tag{2.34b}
$$

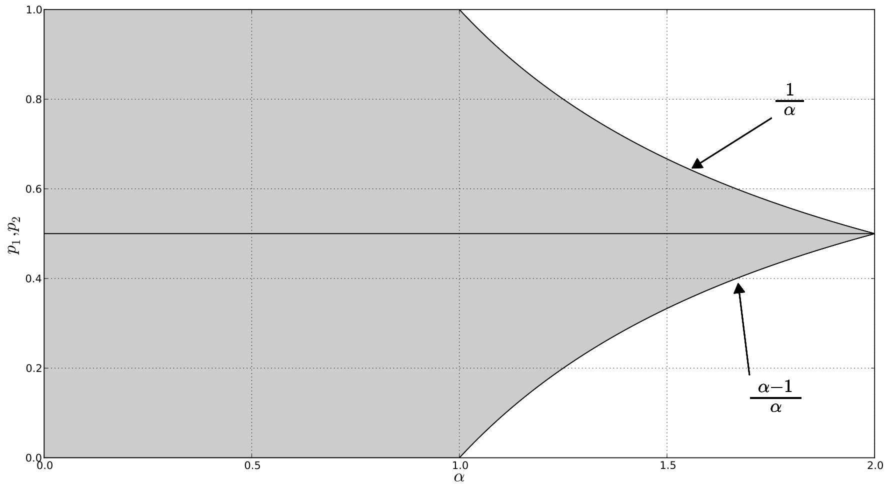


Figure 2.3: Set of admissible values of parameters p_1, p_2.

For $1 < \alpha < 2$ and $x > 0$, we integrate term-by-term the Maclaurin series (2.25a) of stable density,

$$F(x; \alpha, p_1) - F(0; \alpha, p_1) = \int_0^x f(y; \alpha, p_1) \mathrm{d}y,$$

and so we obtain:

$$F(x; \alpha, p_1) = p_2 + \frac{1}{\alpha\pi} \sum_{k=1}^{\infty} \frac{\Gamma(k/\alpha)}{\Gamma(k+1)} \sin(p_2 \pi k) x^k. \tag{2.35a}$$

Using reflection formula (A.3) we have:

$$F(x; \alpha, p_1) = p_2 + \frac{1}{\alpha} \sum_{k=1}^{\infty} \frac{\Gamma(k/\alpha)}{\Gamma(p_2 k)\Gamma(1 - p_2 k)} \frac{x^k}{k!}.$$

For $x < 0$ we use (2.33):

$$\begin{aligned} F(x; \alpha, p_1) &= p_1 + \frac{1}{\alpha\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(k/\alpha)}{\Gamma(k+1)} \sin(p_2 \pi k) x^k = \\ &= p_1 + \frac{1}{\alpha\pi} \sum_{k=1}^{\infty} \frac{\Gamma(k/\alpha)}{\Gamma(k+1)} \sin(p_1 \pi k) x^k. \end{aligned}$$

For $0 < \alpha < 1$ we integrate the power-series in (2.25b) term-by-term:

$$F(x; \alpha, p_1) = p_2 - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\Gamma(\alpha k)}{\Gamma(k+1)} \sin(\alpha p_2 \pi k) x^{-\alpha k}. \tag{2.35b}$$

The asymptotic expansion of stable distribution function $F(x; \alpha, p_1)$ is given by relation (2.35b) for $x \gg 1$ and $1 < \alpha < 2$. For $0 < \alpha < 1$, the series (2.35a) is the asymptotic expansion for $x \ll 1$. See [Ber52].

2.5 Summary

We discussed the different parametrizations of the characteristic exponent of strictly stable laws. We combined two of the parametrizations introduced by Zolotarev and we defined for the characteristic exponent of strictly stable law:

$$\log \varphi(k;\alpha,p_1,c) \equiv -c|k|^\alpha \exp\left\{-i\alpha(p_1-p_2)\tfrac{\pi}{2}\operatorname{sgn} k\right\},$$

where $c > 0$, $p_1+p_2 = 1$, $1-1/\alpha \le p_1, p_2 \le 1/\alpha$ for $1 < \alpha \le 2$ and $0 \le p_1, p_2 \le 1$ for $0 < \alpha < 1$. Here, α is the stability parameter, and p_1 is the asymmetry parameter. The asymmetry parameter equals $1 - F(0;\alpha,p_1)$, i.e. it exactly describes the proportion of the probability mass on the positive and negative part of the real line. The stability parameter α gives information on the power-law tail decay, i.e. we have information about the heaviness of the tails of the probability distribution. Parameter c is a scaling parameter.

Two important properties of stable laws are: reflection property and scaling property of their densities. These are given by:

$$\begin{aligned} f(x;\alpha,p_1,c) &= f(-x;\alpha,p_2,c),\\ f(x;\alpha,p_1,c) &= c^{-1/\alpha} f(xc^{-1/\alpha};\alpha,p_1,1). \end{aligned}$$

Stable densities are represented by their Maclaurin series:

$$f(x;\alpha,p_1) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(k/\alpha+1)}{\Gamma(k+1)}\sin(p_2k\pi)x^{k-1},$$

for $1 < \alpha \le 2$, and by power series:

$$f(x;\alpha,p_1) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(\alpha k+1)}{\Gamma(k+1)}\sin(\alpha p_2 k\pi)x^{-\alpha k-1},$$

for $0 < \alpha < 1$. The latter is also the asymptotic expansions for the stable densities when $x \gg 1$ and $1 < \alpha \le 2$. The distribution function of the stable law is obtained by integrating term-by-term the power series for the density. This yields for $1 < \alpha \le 2$ the Maclaurin series and the asymptotic expansion for $x \gg 1$, respectively:

$$\begin{aligned} F(x;\alpha,p_1) &= p_2 + \frac{1}{\alpha\pi}\sum_{k=1}^{\infty}\frac{\Gamma(k/\alpha)}{\Gamma(k+1)}\sin(p_2k\pi)x^{k},\\ 1-F(x;\alpha,p_1) &\sim \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(\alpha k)}{\Gamma(k+1)}\sin(\alpha p_2 k\pi)x^{-\alpha k}, \end{aligned}$$

and vice-versa for $0 < \alpha < 1$. The stable density is represented in terms of its integral representation as:

$$f(x;\alpha,p_1) = \frac{1}{\pi}\int_0^{p_1\pi}\tfrac{\partial}{\partial x}\left[-\exp\{-x^{\alpha/(\alpha-1)}u(y,p_1)\}\right]\mathrm{d}y,$$

where

$$u(y,p_1) = \frac{[\sin(p_1\pi-y)]^{1/(\alpha-1)}\sin[(\alpha-1)y+p_1\pi]}{\sin(\alpha y)^{\alpha/(\alpha-1)}}.$$

For a stable probability distribution function we have:

$$F(x;\alpha,p_1) = 1 - \frac{1}{\pi}\int_0^{p_1\pi} \exp\{-x^{\alpha/(\alpha-1)}u(y,p_1)\}\mathrm{d}y.$$

The numerical evaluation of this integral representation has been investigated by [Nol97], and is available in mathematical software packages, see e.g. [RN05]. The speed of convergence and quality of the numerical implementation of the power series representation have been investigated, e.g., in [Cro74]. Among others it makes it practical to use stable law solutions in financial modelling and we conclude that the above work provides a sound basis for formulating a practical theory of option pricing from the stable laws as we shall see in the fourth chapter.

3. Mellin Transform Approach to Stable Laws

The objective of this chapter is to establish the density of the product of two independent strictly stable random variables. This is done in Propositions 3.4.1-3.4.3. The theory connected with the sums of independent random variables is well established and has been explored by many researchers. Zolotarev [Zol86] discussed the generalization of the addition scheme connected with the infinitely divisible random variables to the scheme for multiplication of the independent random variables. Nevertheless, it seems that the theory of the products of independent random variables still provides an area for active investigation. The algebraic properties of Mellin transform makes this integral transformation suitable for studying this problem. In the first section we give introduction into the probability distributions of product of two independent random variables. The second section gives derivation of Mellin transform for strictly stable laws, already derived by [Zol57]. In the section three we discuss the representation of stable density by Mellin-Barnes integral and evaluate this integral by Residue Theorem, already done by [Sch86]. This is a simpler problem to solve, and so we use it as a motivating example for the section four. There we compute the density of the product of two independent strictly stable random variables. The derived density is represented in terms of its power series. In the section five we discuss the connection between the derived product density and Fox's H-functions. This chapter is based on the work presented in [Kared].

3.1 Product of Independent Random Variables

The Mellin transform maps the multiplicative convolution onto the multiplication operation, see e.g. [ML86], which provides a convenient tool for studying products of independent random variables. Of course, the product can be studied by taking the sum of logarithms of independent random variables. However, in general the computation of the probability distribution of logarithm of random variable can rise many technical difficulties. The Mellin transform provides a direct approach to study the product of independent random variables. The connection between Mellin transform and products of two independent random variables was discussed in [Eps48]. The method from [Eps48] was generalized in [ST66] to a computation of the product of n independent random variables and the tables for particular products of Cauchy and Gaussian random variables were provided. Consider two independent random variables X_1, X_2, with continuous probability density functions $f_1(x)$ and $f_2(x)$, respectively. The probability density function $g(x)$ of their product $X = X_1 \cdot X_2$ is given by:

$$g(x) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_1(x/y) f_2(y) \mathrm{d}y = \int_{-\infty}^{\infty} \frac{1}{|y|} f_2(x/y) f_1(y) \mathrm{d}y, \tag{3.1}$$

see e.g. [Eps48] for discussion. The Mellin transform is defined for a positive random variable X with continuous density $f(x)$ as:

$$M[f(x);s] \equiv F(s) = \mathrm{E}\left[X^{s-1}\right] = \int_0^\infty x^{s-1} f(x)\mathrm{d}x, \tag{3.2}$$

where s is a complex number, such that for some vertical strip $a_1 < \mathrm{Re}s < a_2$, the $F(s)$ is analytic in this strip. Therefore, the Mellin transform is defined by a pair: a function $F(s)$ and its strip of analyticity. The width of the strip is determined by the behaviour of $f(x)$ for x approaching the origin and for x going to infinity. A more detailed introduction to Mellin transform is given in Appendix A.2, or see, e.g., [ML86].

For the Mellin transform of a positive random variable Y, such that $Y = aX$, where $a > 0$ and X is a positive random variable with continuous density $f(x)$, we have:

$$G(s) = \mathrm{E}\left[Y^{s-1}\right] = \mathrm{E}\left[a^{s-1}X^{s-1}\right] = a^{s-1}F(s).$$

The Mellin transform of product $Y = X_1 \cdot X_2$ of two positive independent random variables X_1, X_2, with continuous densities $f_1(x), f_2(x)$, is given by:

$$\begin{aligned} G(s) &= \mathrm{E}\left[Y^{s-1}\right] = \mathrm{E}\left[X_1^{s-1}X_2^{s-1}\right] = \mathrm{E}\left[X_1^{s-1}\right]\mathrm{E}\left[X_2^{s-1}\right] = \\ &= F_1(s)F_2(s). \end{aligned} \tag{3.3}$$

The inverse Mellin transform is defined as:

$$f(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{-s}F(s)\mathrm{d}s, \tag{3.4}$$

for all positive x for which the function $f(x)$ is continuous, where the integration path lies within the strip of analyticity of $F(s)$, see e.g. [ML86] for more rigorous introduction.

Taking the inverse Mellin transform of (3.3) yields:

$$g(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{-s}F_1(s)F_2(s)\mathrm{d}s,$$

and it can be shown that probability density computed this way is exactly the density in (3.1), see [Eps48].

To extend this approach to random variable X which can be positive and negative, Epstein [Eps48] decomposed the random variable X with continuous density as $X = X^+ - X^-$, where X^+, X^-, designates the positive and negative part of random variable X, respectively. The density is then decomposed into two continuous functions $f_1(x), f_2(x)$, such that $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = f(x)$ for $x > 0$ and is 0 otherwise and $f_2(x) = f(x)$ for $x < 0$ and is 0 when x takes positive values. The Mellin transform is then:

$$F(s) = \mathrm{E}\left[X^{s-1}\right] = \mathrm{E}\left[(X^+)^{s-1}\right] + \mathrm{E}\left[(X^-)^{s-1}\right] \equiv F_1(s) + F_2(s),$$

where

$$F_1(s) = \mathrm{E}\left[(X^+)^{s-1}\right] = \int_0^\infty x^{s-1}f_1(x)\mathrm{d}x,$$
$$F_2(s) = \mathrm{E}\left[(X^-)^{s-1}\right] = \int_0^\infty x^{s-1}f_2(-x)\mathrm{d}x,$$

and $F_1(s)$ is the Mellin transform of X when $x > 0$ and $F_2(s)$ is the Mellin transform of X when $x < 0$. Applying the same procedure on the product of two real independent random variables X_1, X_2 with continuous densities $f_1(x), f_2(x)$, respectively, we decompose densities $f_1(x), f_2(x)$ into:

$$\begin{aligned} f_1(x) &= f_{11}(x) + f_{12}(x), \\ f_2(x) &= f_{21}(x) + f_{22}(x), \end{aligned}$$

where $f_{11}(x) = f_{21}(x) = 0$ for x negative and $f_{11}(x) = f_1(x)$, $f_{21}(x) = f_2(x)$ for $x > 0$, respectively. Similarly, for $x > 0$ we set $f_{12}(x) = f_{22}(x) = 0$ and $f_{12}(x) = f_1(x)$, $f_{22}(x) = f_2(x)$ for $x < 0$, respectively. Then substituting into (3.1), we have the probability density function $g(x)$ of the product $X = X_1 \cdot X_2$ given by:

$$\begin{aligned} g(x) = &\int_0^\infty \frac{1}{y} f_{11}(y/x) f_{21}(x) \mathrm{d}x + \int_0^\infty \frac{1}{y} f_{12}(-y/x) f_{22}(-x) \mathrm{d}x + \\ &+ \int_0^\infty \frac{1}{y} f_{11}(y/x) f_{22}(-x) \mathrm{d}x + \int_0^\infty \frac{1}{y} f_{12}(-y/x) f_{21}(x) \mathrm{d}x. \end{aligned}$$

Next, we decompose the density $g(x)$ of the product random variable X as $g(x) = g_1(x) + g_2(x)$, where

$$\begin{aligned} g_1(x) &= \int_0^\infty \frac{1}{y} f_{11}(y/x) f_{21}(x) \mathrm{d}x + \int_0^\infty \frac{1}{y} f_{12}(-y/x) f_{22}(-x) \mathrm{d}x, \\ g_2(x) &= \int_0^\infty \frac{1}{y} f_{11}(y/x) f_{22}(-x) \mathrm{d}x + \int_0^\infty \frac{1}{y} f_{12}(-y/x) f_{21}(x) \mathrm{d}x. \end{aligned}$$

Obviously, $g_1(x) > 0$ for $x > 0$ and is 0 otherwise, and $g_2(x) > 0$ for negative x and vanishes identically otherwise. The Mellin transform of the product density $g(x)$ yields:

$$G_1(s) = F_{11}(s) F_{21}(s) + F_{12}(s) F_{22}(s), \tag{3.5a}$$

$$G_2(s) = F_{11}(s) F_{22}(s) + F_{12}(s) F_{21}(s), \tag{3.5b}$$

where $F_{ij}(s),\ i, j = 1, 2$ denotes the Mellin transform of relevant $f_{ij}(x),\ i, j = 1, 2$. Taking the Mellin inverse of $G_1(s)$ yields $g_1(x)$, the density of the product of two independent real random variables X_1, X_2 for positive x, and taking the Mellin inverse of $G_2(s)$ yields $g_2(x)$, the density for negative x. See [Eps48, ST66].

3.2 Mellin Transform of Stable Laws

The Mellin transform of the density of stable law $f(x; \alpha, p_1)$ was given in [Zol57, Sch86]. The derivation is straightforward. So, we start with stable density $f(x; \alpha, p_1)$ given by its inverse Fourier integral (2.23c) and compute the Mellin transform directly from its definition. Assume $\alpha \neq 1$ and $0 < \mathrm{Re}\ s < 1$, then:

$$\begin{aligned} M[f(x; \alpha, p_1); s] &= \frac{1}{\pi} \int_0^\infty x^{s-1} \Big[\mathrm{Re} \int_0^\infty e^{-ikx - k^\alpha e^{-i\alpha(p_1 - p_2)\frac{\pi}{2}}} \mathrm{d}k \Big] \mathrm{d}x = \\ &= \frac{1}{\pi} \mathrm{Re} \int_0^\infty x^{s-1} e^{-ikx} \mathrm{d}x \int_0^\infty e^{-k^\alpha e^{-i\alpha(p_1 - p_2)\frac{\pi}{2}}} \mathrm{d}k. \end{aligned}$$

The change of order of integration is readily verified. Next, we rotate the contour of integration. So,

$$M[f(x;\alpha,p_1);s] = \frac{1}{\pi}\mathrm{Re}\Big[e^{-is\frac{\pi}{2}}\int_0^\infty x^{s-1}e^{-kx}\mathrm{d}x\int_0^\infty e^{-k^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}}\mathrm{d}k\Big] =$$
$$= \frac{1}{\pi}\mathrm{Re}\Big[e^{-is\frac{\pi}{2}}\Gamma(s)\int_0^\infty k^{-s}e^{-k^\alpha e^{-i\alpha(p_1-p_2)\frac{\pi}{2}}}\mathrm{d}k\Big],$$

and thus:

$$M[f(x;\alpha,p_1);s] = \frac{1}{\pi}\mathrm{Re}\Big[e^{-i\frac{\pi}{2}[s+(p_1-p_2)(s-1)]}\Big]\Gamma(s)\int_0^\infty k^{-s}e^{-k^\alpha}\mathrm{d}k =$$
$$= \frac{1}{\alpha\pi}\Gamma(s)\Gamma\left(\tfrac{1-s}{\alpha}\right)\mathrm{Re}\Big[-ie^{ip_1(1-s)\pi}\Big] =$$
$$= \frac{1}{\alpha\pi}\Gamma(s)\Gamma\left(\tfrac{1-s}{\alpha}\right)\sin(p_1[1-s]\pi).$$

The exchange of contour of integration is justified by Cauchy Theorem and Jordan's Lemma and by the condition $0 < \mathrm{Re}\ s < \alpha+1$, see [Zol86, Sch86]. This gives the strip of analyticity of the Mellin transform. Therefore, we conclude that the random variable X with stable density $f(x;\alpha,p_1)$ has the Mellin transform given by:

$$M[f(x;\alpha,p_1);s] = F(s;\alpha,p_1) = \frac{1}{\alpha\pi}\Gamma(s)\Gamma\left(\tfrac{1-s}{\alpha}\right)\sin(p_1[1-s]\pi) = \tag{3.6a}$$
$$= \frac{\Gamma(s)\Gamma\left(\frac{1-s}{\alpha}\right)}{\alpha\Gamma(p_1-p_1s)\Gamma(p_2+p_1s)},$$

for $x > 0$, and the strip of analyticity is: $0 < \mathrm{Re}\ s < \alpha+1$. The reflection property (2.23b) implies that for $x < 0$ the Mellin transform of $f(x;\alpha,p_1)$ equals $F(s;\alpha,p_2)$, i.e. for $x < 0$:

$$M[f(x;\alpha,p_1);s] = M[f(-x;\alpha,p_2);s] = F(s;\alpha,p_2). \tag{3.6b}$$

The scaling property of stable law (2.23b) implies that for the Mellin transform of density $f(x;\alpha,p_1,c)$ we have:

$$F(s;\alpha,p_1,c) = c^{(s-1)/\alpha}F(s;\alpha,p_1,1), \tag{3.6c}$$

see e.g. [Zol57, ML86].

3.3 Density of Stable Law

The probability density of stable law is represented by the Mellin inverse in the form of Mellin-Barnes integral, as:

$$f(x;\alpha,p_1) = \frac{1}{2\alpha\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Gamma(s)\Gamma\left(1-\frac{\alpha-1}{\alpha}-\frac{s}{\alpha}\right)}{\Gamma(1-p_2-p_1s)\Gamma(p_2+p_1s)}x^{-s}\mathrm{d}s, \tag{3.7}$$

for $x > 0$. For $x < 0$ the probability density of stable law is computed by using (3.6b) and (2.23b).

The density of stable law given by Mellin-Barnes integral in (3.7) can be evaluated using the Residue Theorem. We show this computation in more detail, because we use it as a motivation example. Consider the closed loop γ_l in complex domain given by:

$$\begin{aligned}&\gamma_l = \gamma_1 + \gamma_2, \text{ where:} &(3.8)\\ &\gamma_1(t) = a + t, t \in [-ir, ir], \text{ and } \gamma_2(t) = a + re^{it}, t \in [\pi/2, 3\pi/2],\end{aligned}$$

for $r > 0$ and $0 < a < 1$. Let n is the biggest integer smaller then r. Next we denote a complex valued function:

$$g(s) = \Gamma(s)\Gamma\left(\tfrac{1-s}{\alpha}\right)\sin(p_1[1-s]\pi)x^{-s} \tag{3.9}$$

and integrate $g(s)$ along the closed loop γ_l. Inside the loop γ_l the function $g(s)$ has only simple poles at all non-positive integers $-k$, where $0 \leq k < n$, and the value of residue at each $-k$ is:

$$\operatorname*{res}_{s=-k} g(s) = \frac{(-1)^k}{k!}\Gamma\left(\tfrac{1+k}{\alpha}\right)\sin(p_1[1+k]\pi)x^k.$$

The orientation of the closed loop γ_l is anticlockwise, and thus the index function has value 1. From Residue Theorem we have the equality:

$$\int_{\gamma_l} g(s)\mathrm{d}s = 2\pi i \sum_{k=0}^{n} \frac{(-1)^k}{k!}\Gamma\left(\tfrac{1+k}{\alpha}\right)\sin(p_1[1+k]\pi)x^k.$$

Using Jordan's Lemma and Stirling Formula to estimate Gamma function, we estimate the integral on the right-hand side of the equality. It is readily verified that when r approaches infinity, and $1 < \alpha < 2$, the integral over arc γ_2 goes to 0 and so:

$$\begin{aligned}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} &\Gamma(s)\Gamma\left(\tfrac{1-s}{\alpha}\right)\sin(p_1[1-s]\pi)x^{-s}\mathrm{d}s = \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\Gamma\left(\tfrac{1+k}{\alpha}\right)\sin(p_1[1+k]\pi)x^k.\end{aligned}$$

After rearranging the terms in the series and substituting back into (3.7) we obtain (2.25a). Thus we have obtained an alternative derivation of the power series (2.25a) and showed that the terms in the series (2.25a) are residues of the Mellin transform of the density of the stable law. These residues lie on the left-hand side of integration path in (3.7).

The complex valued function $g(s)$ in (3.9) also has simple poles on the right-hand side of the integration path in (3.7). These are located at all points $\alpha k + 1$, where k is a positive integer. The value of the residue of $g(s)$ at point $\alpha k + 1$ is:

$$\operatorname*{res}_{s=\alpha k+1} g(s) = \alpha\frac{(-1)^k}{k!}\Gamma(\alpha k + 1)\sin(\alpha p_1 k\pi)x^{-\alpha k-1}.$$

We consider the closed loop:

$$\begin{aligned}&\gamma_r = \gamma_1 - \gamma_2, \text{ where:} &(3.10)\\ &\gamma_1(t) = t, a + t \in [-ir, ir], \text{ and } \gamma_2(t) = a + re^{it}, t \in [-\pi/2, \pi/2],\end{aligned}$$

for $r > 0$ and $0 < a < 1$. Again let n is the biggest integer smaller then r. The orientation of the loop γ_r is clockwise, so the index function has value -1. The Residue Theorem gives:

$$\int_{\gamma_r} g(s)\mathrm{d}s = -2\alpha\pi i \sum_{k=1}^{n} \frac{(-1)^k}{k!}\Gamma(\alpha k+1)\sin(\alpha p_1 k\pi)x^{-\alpha k-1}.$$

From Jordan's Lemma it follows that when r approaches infinity and $0 < \alpha < 1$, the integral on the right hand-side of the previous equity is bounded. The integral over the arc γ_2 goes to 0, and so:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(s)\Gamma\left(\tfrac{1-s}{\alpha}\right)\sin(p_1[1-s]\pi)x^{-s}\mathrm{d}s =$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}\Gamma(\alpha k+1)\sin(\alpha p_1 k\pi)x^{-\alpha k-1}.$$

After re-arranging the terms in the series and dividing by α, we obtain (2.25b). We conclude that the terms in the series (2.25b) are residues of Mellin transform of density of stable law and these residues lie on the right-hand side of integration path in (3.7). At this point, we can represent densities of stable distributions via series expansions, where the coefficients can now be understood in terms of the Mellin-Barnes integral.

The Mellin transform of symmetric stable density simplifies to:

$$M[f(x,\alpha,1/2);s] = \frac{1}{\alpha\pi}\Gamma\left(\tfrac{1-s}{\alpha}\right)\Gamma(s)\cos(s\pi/2) = \tag{3.11}$$
$$= \frac{\Gamma\left(\frac{1-s}{\alpha}\right)\Gamma(s)}{\alpha\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}.$$

The above method can be used to investigate the densities of stable laws for a particular choice of parameters. The next example considers the simplest case.

Example 3.3.1. We start with investigation of simple case when $\alpha = 1/2$.

$$\Gamma(2z)\Gamma(1-z)\sin(z\pi/2)x^{z-1} = \Gamma(4s)\Gamma(1-2s)\sin(s\pi)x^{2s-1},$$

where we substituted $z = 2s$. We use Reflection and Duplicity formulas (A.3), (A.4), (A.5) to compute $\Gamma(1-2s)$:

$$\Gamma(1-2s) = \frac{\pi}{2\Gamma(2s)\sin(s\pi)\cos(s\pi)} = \frac{\pi\sqrt{\pi}}{2^{2s}\Gamma(s)\Gamma(s+1/2)\sin(s\pi)\cos(s\pi)} =$$
$$= \frac{\sqrt{\pi}\Gamma(1/2-s)}{2^{2s}\Gamma(s)\sin(s\pi)},$$

and Multiplication formula (A.6) for $\Gamma(4s)$:

$$\Gamma(4s) = \frac{4^{4s}}{4\pi\sqrt{2\pi}}\Gamma(s)\Gamma(s+1/4)\Gamma(s+1/2)\Gamma(s+3/4).$$

We have equity:

$$\Gamma(4s)\Gamma(1-2s)\sin(s\pi)x^{2s-1} = \frac{1}{4\sqrt{2}\pi x}\Gamma(s+1/4)\Gamma(s+1/2)\Gamma(1/2-s)(8x)^{2s}.$$

We need to evaluate the following integral:

$$I(s) := \frac{1}{4\sqrt{2}\pi^2 ix}\int_{-i\infty}^{i\infty}\Gamma(s+1/4)\Gamma(s+1/2)\Gamma(s+3/4)\Gamma(1/2-s)(8x)^{2s}\mathrm{d}s.$$

We make use of Residue Theorem and we evaluate residues on the left side from integration line. We obtain:

$$I(s) = \frac{1}{2\sqrt{2}\pi x}\Bigg\{\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma(1/4-k)\Gamma(1/2-k)\Gamma(k+3/4)(8x)^{-2k-1/2}+$$
$$+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma(-1/4-k)\Gamma(1/4-k)\Gamma(k+1)(8x)-2k-1+$$
$$+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma(-1/2-k)\Gamma(-1/4-k)\Gamma(k+5/4)(8x)^{-2k-3/2}\Bigg\}.$$

This simplifies to

$$I(s) = \frac{1}{2\sqrt{2}\pi x}\Bigg\{\frac{\Gamma(1/4)\Gamma(1/2)\Gamma(3/4)}{2\sqrt{2x}}\sum_{k=0}^{\infty}\frac{(3/4)_k}{(3/4)_k(1/2)_k}\frac{[-64x^2]^{-k}}{k!}-$$
$$-\frac{\Gamma(1/4)\Gamma(3/4)}{2x}\sum_{k=0}^{\infty}\frac{(1)_k}{(3/4)_k(5/4)_k}\frac{[-64x^2]^{-k}}{k!}+$$
$$+\frac{\Gamma(1/2)\Gamma(3/4)\Gamma(5/4)}{2x\sqrt{2x}}\sum_{k=0}^{\infty}\frac{(5/4)_k}{(5/4)_k(3/2)_k}\frac{[-64x^2]^{-k}}{k!}\Bigg\}=$$
$$=\frac{\Gamma(1/4)\Gamma(3/4)}{8x\sqrt{\pi x}}\,{}_0F_1[\frac{1}{2};-\frac{1}{64x^2}]-\frac{\Gamma(1/4)\Gamma(3/4)}{4\sqrt{2}x^2\pi}\,{}_1F_2[1;\frac{3}{4},\frac{5}{4};-\frac{1}{64x^2}]+$$
$$+\frac{\Gamma(1/4)\Gamma(3/4)}{32x^2\sqrt{\pi x}}\,{}_0F_1[\frac{3}{2};-\frac{1}{64x^2}].$$

Reflection formula gives $\Gamma(1/4)\Gamma(3/4) = \pi/(\cos[\pi/4]) = \pi\sqrt{2}$. Further using relations between Bessel functions of the First kind and generalized hypergeometric functions from [AS64] 9.1.69, p.362, and relation between Bessel functions and Spherical Bessel functions of the first kind [AS64] 10.1.1, 10.1.11, 10.1.12, p.437-8, we have:

$${}_0F_1[\frac{1}{2};-\frac{1}{64x^2}] = \sqrt{\frac{\pi}{8x}}J_{-1/2}(1/4x) = \cos[1/(4x)],$$
$${}_0F_1[\frac{3}{2};-\frac{1}{64x^2}] = \sqrt{2\pi x}J_{1/2}(1/4x) = 4x\sin[1/(4x)].$$

Next, from [BB08] 8.3.2.16, p.656, we have:

$${}_1F_2[1;\frac{3}{4},\frac{5}{4};-\frac{1}{64x^2}] = \sqrt{2\pi x}\Bigg[\sin[1/(4x)]S(1/\sqrt{2\pi x})+\cos[1/(4x)]C(1/\sqrt{2\pi x})\Bigg],$$

where $S(x), C(x)$ respectively, stands for Fresnel Integrals given by:

$$S(x) = \int_0^x \sin(t^2\pi/2)\mathrm{d}t, \quad C(x) = \int_0^x \cos(t^2\pi/2)\mathrm{d}t.$$

We obtain equity:

$$
\begin{aligned}
I(s) &= \frac{\pi}{4x\sqrt{2\pi x}}\cos[1/(4x)] - \frac{\pi}{2x\sqrt{2\pi x}}\sin[1/(4x)]S(1/\sqrt{2\pi x}) - \\
&- \frac{\pi}{2x\sqrt{\pi x}}\cos[1/(4x)]C(1/\sqrt{2\pi x}) + \frac{\pi}{4x\sqrt{2\pi x}}\sin[1/(4x)] = \\
&= \frac{\pi}{4x\sqrt{2\pi x}}\left\{\sin[1/(4x)][1-2S(1/\sqrt{2\pi x})] + \cos[1/(4x)][1-2C(1/\sqrt{2\pi x})]\right\}.
\end{aligned}
$$

We conclude, that:

$$
\begin{aligned}
f(x;1/2,1/2) = \frac{1}{2x\sqrt{2\pi x}}\Big\{ &\sin[1/(4x)][1-2S(1/\sqrt{2\pi x})] + \\
&+ \cos[1/(4x)][1-2C(1/\sqrt{2\pi x})]\Big\}.
\end{aligned}
$$

This particular expression has already been considered in [Zol86]. We provided the alternative brute force method how to derive the same result. A more discussion on representations of stable densities in terms of special functions in general is in, e.g. [Zol94, Sch86].

3.4 Product of Two Stable Random Variables

In this section we establish the density of product of two independent random variables X_1, X_2, with stable probability densities $f_1(x;\alpha_1,p_1)$ and $f_2(x;\alpha_2,q_1)$, respectively, in terms of their power series.

For $x > 0$ the Mellin transforms $F_1(s;\alpha_1,p_1)$, $F_2(s;\alpha_2,q_2)$ of the stable densities $f_1(x;\alpha_1,p_1)$ and $f_2(x;\alpha_2,q_1)$, respectively, are given in (3.6a). From Mellin transform of product density $g(x)$ in (3.5a), (3.5b) and the reflection property of stable density (2.23b) and (3.6b) we have:

$$
\begin{aligned}
G_1(s;\alpha_1,\alpha_2,p_1,q_1) = F_1(s;\alpha_1,p_1)F_2(s;\alpha_2,q_1) &+ \\
+ F_1(s;\alpha_1,p_2)F_2(s;\alpha_2,q_2)&
\end{aligned}
\tag{3.12a}
$$

for $x > 0$, and

$$
\begin{aligned}
G_2(s;\alpha_1,\alpha_2,p_1,q_1) = F_1(s;\alpha_1,p_1)F_2(s;\alpha_2,q_2) &+ \\
+ F_1(s;\alpha_1,p_2)F_2(s;\alpha_2,q_1)&
\end{aligned}
\tag{3.12b}
$$

for $x < 0$. The product density $g(x)$ is given by:

$$
g(x;\alpha_1,\alpha_2,p_1,q_1) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{-s}G_1(s)\mathrm{d}s, \tag{3.13a}
$$

$$
g(-x;\alpha_1,\alpha_2,p_1,q_1) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{-s}G_2(s)\mathrm{d}s, \tag{3.13b}
$$

where $x > 0$, and for simplicity $0 < a < 1$.

In the evaluation of Mellin-Barnes integral, clearly

$$\tilde{g}(x;\alpha_1,\alpha_2,p_1,q_1)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}F_1(s;\alpha_1,p_1)F_2(s;\alpha_2,q_1)x^{-s}\mathrm{d}s, \tag{3.13c}$$

is of particular interest. The power series of $\tilde{g}(x)$ is established in Lemmas 3.4.1, 3.4.2:

Lemma 3.4.1. *The function $\tilde{g}(x)$ defined by Mellin-Barnes integral as:*

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}[\Gamma(s)]^2\,\Gamma\left(\tfrac{1-s}{\alpha_1}\right)\Gamma\left(\tfrac{1-s}{\alpha_2}\right)\sin(p_1[1-s]\pi)\sin(q_1[1-s]\pi)x^{-s}\tfrac{\mathrm{d}s}{\alpha_1\alpha_2\pi^2}, \tag{3.14a}$$

where $0<a<1$, and $\alpha_1+\alpha_2<2\alpha_1\alpha_2$, has the power series representation as follows:

$$\begin{aligned}\tilde{g}(x)=&\tfrac{1}{\pi^2}\sum_{k=1}^{\infty}\Gamma\left(\tfrac{k}{\alpha_1}+1\right)\Gamma\left(\tfrac{k}{\alpha_2}+1\right)\sin(p_1k\pi)\sin(q_1k\pi)\xi(k)\frac{x^{k-1}}{(k!)^2} \\ &-\tfrac{p_1+q_1}{2\pi}\sum_{k=1}^{\infty}\Gamma\left(\tfrac{k}{\alpha_1}+1\right)\Gamma\left(\tfrac{k}{\alpha_2}+1\right)\sin([p_1+q_1]k\pi)\frac{x^{k-1}}{(k!)^2} \\ &+\tfrac{p_1-q_1}{2\pi}\sum_{k=1}^{\infty}\Gamma\left(\tfrac{k}{\alpha_1}+1\right)\Gamma\left(\tfrac{k}{\alpha_2}+1\right)\sin([p_1-q_1]k\pi)\frac{x^{k-1}}{(k!)^2},\end{aligned} \tag{3.14b}$$

where $\xi(k)$ designates the auxiliary function given by :

$$\xi(k)\equiv 2\psi(k)-\tfrac{1}{\alpha_1}\psi\left(\tfrac{k}{\alpha_1}\right)-\tfrac{1}{\alpha_2}\psi\left(\tfrac{k}{\alpha_2}\right)-\log x, \tag{3.14c}$$

and $\psi(k)$ is digamma function.

Proof of Lemma 3.4.1: We use the Residue Theorem for the evaluation of the integral (3.14a). Consider the closed loop γ defined as $\gamma=\gamma_1+\gamma_2$, where $\gamma_1(t)=a+t, t\in[-ir,ir]$ and $\gamma_2(t)=a+re^{it}, t\in[\pi/2,3\pi/2]$, $r>0$. The loop γ is formed by half-circle on the left-side of integration path of (3.14a) and matches the part of line of integration path. We denote the integrand in (3.14a) as

$$h(s)\equiv[\Gamma(s)]^2\,\Gamma\left(\tfrac{1-s}{\alpha_1}\right)\Gamma\left(\tfrac{1-s}{\alpha_2}\right)\sin(p_1[1-s]\pi)\sin(q_1[1-s]\pi)x^{-s}, \tag{3.15}$$

and integrate $h(s)$ along the loop γ. On the left-hand side of integration path in (3.14a) function $h(s)$ has poles of second order at all negative integer values. To evaluate the residue of $h(s)$ at $s=-k$, we rewrite $h(s)$ as:

$$h(s)=\frac{[\Gamma(s+k+1)]^2}{(s+k-1)^2\ldots s^2}\Gamma\left(\tfrac{1-s}{\alpha_1}\right)\Gamma\left(\tfrac{1-s}{\alpha_2}\right)\sin(p_1[1-s]\pi)\sin(q_1[1-s]\pi)x^{-s},$$

and we expand each term in $h(s)$ on the neighbourhood of point $s=-k$ as follows:

$$(s+n)^{-2}=\tfrac{1}{(k-n)^2}+\tfrac{2}{(k-n)^3}(s+k)+\ldots,\quad \text{for } 0\le n\le k-1,$$

and:

$$\begin{aligned}
&\left[\Gamma(s+k+1)\right]^2 = 1 + 2\Gamma'(1)(s+k) + \dots \\
&\Gamma\left(\tfrac{1-s}{\alpha_1}\right) = \Gamma\left(\tfrac{1+k}{\alpha_1}\right) - \tfrac{1}{\alpha_1}\Gamma'\left(\tfrac{1+k}{\alpha_1}\right)(s+k) + \dots, \\
&x^{-s} = x^k\left\{1 - (s+k)\log x + \dots\right\}, \\
&\sin(p_1[1-s]\pi) = \sin(p_1[1+k]\pi) - p_1\pi\cos(p_1[1+k]\pi)(s+k) + \dots.
\end{aligned}$$

Here we have used the Binomial Theorem, see e.g. 3.6.9 [AS64], and Taylor expansion, see e.g. 3.6.4 [AS64]. In order to express the expansion for $h(s)$, we multiply the above series together, and look for the coefficient associated with the first order term. It is given by:

$$\begin{aligned}
&\frac{x^k}{(k!)^2}\Gamma\left(\tfrac{k+1}{\alpha_1}\right)\Gamma\left(\tfrac{k+1}{\alpha_2}\right)\sin(p_1[1+k]\pi)\sin(q_1[1+k]\pi)\Big\{2\Gamma'(1)+ \\
&\quad + \sum_{n=1}^{k}\tfrac{2}{n} - \tfrac{1}{\alpha_1}\psi\left(\tfrac{k+1}{\alpha_1}\right) - \tfrac{1}{\alpha_2}\psi\left(\tfrac{k+1}{\alpha_2}\right) - \log x\Big\} \\
&\quad - \pi\frac{x^k}{(k!)^2}\Gamma\left(\tfrac{k+1}{\alpha_1}\right)\Gamma\left(\tfrac{k+1}{\alpha_2}\right)\Big\{p_1\cos(p_1[1+k]\pi)\sin(q_1[1+k]\pi) \\
&\quad + q_1\sin(p_1[1+k]\pi)\cos(q_1[1+k]\pi)\Big\},
\end{aligned} \tag{3.16}$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$ is digamma function; see e.g. [BBEoNR53a]. Using relation 6.3.2 in [AS64] for the digamma function $\psi(s)$, and 4.3.33 in [AS64] for circular functions, we obtain the value of the residue of $h(s)$ at $s = -k$. That is:

$$\begin{aligned}
\operatorname*{res}_{s=-k} h(s) = &\frac{x^k}{(k!)^2}\Gamma\left(\tfrac{k+1}{\alpha_1}\right)\Gamma\left(\tfrac{k+1}{\alpha_2}\right)\sin(p_1[1+k]\pi)\sin(q_1[1+k]\pi)\Big\{2\psi(k+1)- \\
&- \tfrac{1}{\alpha_1}\psi\left(\tfrac{k+1}{\alpha_1}\right) - \tfrac{1}{\alpha_2}\psi\left(\tfrac{k+1}{\alpha_2}\right) - \log x\Big\} \\
&- \tfrac{\pi}{2}\frac{x^k}{(k!)^2}\Gamma\left(\tfrac{k+1}{\alpha_1}\right)\Gamma\left(\tfrac{k+1}{\alpha_2}\right)\Big\{(p_1+q_1)\sin\left[(p_1+q_1)(1+k)\pi\right]+ \\
&- (p_1-q_1)\sin\left[(p_1-q_1)(1+k)\pi\right]\Big\}.
\end{aligned}$$

Let m denote the biggest integer smaller then r. The orientation of loop γ is anti-clockwise, so the index function has value 1. The residue Theorem gives:

$$\int_{a-ir}^{a+ir} h(t)\mathrm{d}t = 2\pi i\sum_{k=0}^{m}\operatorname*{res}_{s=-k} h(s) - \int_{\gamma_2} h(s)\mathrm{d}s.$$

Let us consider the convergence of integral:

$$\int_{\gamma_2} h(s)\mathrm{d}s = \int_{\pi/2}^{3\pi/2} h\left(re^{it}\right) ire^{it}\mathrm{d}t. \tag{3.17}$$

We use the asymptotic formula for the gamma function 6.1.39 in [AS64], and the Stirling formula for $s = u + iv = re^{it}$ to estimate the upper bound of factors of

$h(s)$. This yields $|\Gamma(s)| \leq \sqrt{2\pi} r^{u-1/2} e^{u-vt}$. Regarding the integral (3.17), the real part of s is negative, and so we estimate:

$$
\begin{aligned}
&|\Gamma(s)|^2 \leq 2\pi r^{2u-1} e^{-2u-2vt} \leq c_1 r^{-2r-1},\\
&\left|\Gamma\left(\tfrac{1-u}{\alpha_1} - i\tfrac{v}{\alpha_1}\right)\right| \leq \sqrt{2\pi} r^{-u/\alpha_1-(1/2-1/\alpha_1)} e^{u/\alpha_1+vt/\alpha_1} e^{-1/\alpha_1} \leq c_2 r^{r/\alpha_1},\\
&|\sin(p_1[1-s]\pi)| \leq c_3 \cosh(-p_1\pi vt),\\
&|x^{-s}| \leq c_4 x^r,
\end{aligned}
$$

where $c_i, i = 1,2,3,4$, are some real constants. So, we estimate:

$$
\left|\int_{\gamma_2} h(s)\mathrm{d}s\right| \leq \pi r \max_{s\in\gamma_2} |h(s)| \leq c r^{-r(2-1/\alpha_1-1/\alpha_2)},
$$

where again c is some real constant. In order for the integral (3.17) to vanish as r goes to infinity, we require: $\alpha_1 + \alpha_2 < 2\alpha_1\alpha_2$. Therefore, for any admissible α_1, α_2, we apply Jordan's Lemma on integral (3.17). Thus we have:

$$
\tfrac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} h(s)\mathrm{d}s = \sum_{k=0}^{\infty} \operatorname*{res}_{s=-k} h(s).
$$

After rearranging the terms in the series and substituting back into (3.14a) we obtain (3.14b). Q.E.D

Lemma 3.4.2. *The function $\tilde{g}(x)$ defined by Mellin-Barnes integral in (3.14a), with parameters satisfying $\alpha_1 + \alpha_2 > 2\alpha_1\alpha_2$ has the power series representation as follows:*

$$
\begin{aligned}
g(x) = &\tfrac{1}{\alpha_2\pi}\sum_{k=1}^{\infty} \tfrac{(-1)^{k+1}}{k!}\tfrac{[\Gamma(\alpha_1 k+1)]^2}{\Gamma(\alpha_1 k/\alpha_2+1)}\tfrac{\sin(\alpha_1 p_1 k\pi)\sin(\alpha_1 q_1 k\pi)}{\sin(\alpha_1/\alpha_2 k\pi)} x^{-\alpha_1 k-1} + \qquad (3.18a)\\
&+\tfrac{1}{\alpha_1\pi}\sum_{k=1}^{\infty} \tfrac{(-1)^{k+1}}{k!}\tfrac{[\Gamma(\alpha_2 k+1)]^2}{\Gamma(\alpha_2 k/\alpha_1+1)}\tfrac{\sin(\alpha_2 p_1 k\pi)\sin(\alpha_2 q_1 k\pi)}{\sin(\alpha_2/\alpha_1 k\pi)} x^{-\alpha_2 k-1},
\end{aligned}
$$

for $\alpha_1 \neq \alpha_2$. When $\alpha_1 = \alpha_2 = \alpha$ the power series is given by:

$$
\begin{aligned}
g(x) = &\tfrac{1}{\pi^2}\sum_{k=1}^{\infty} [\Gamma(\alpha k+1)]^2 \sin(\alpha p_1 k\pi)\sin(\alpha q_1 k\pi)\zeta(k)\frac{x^{-\alpha k-1}}{(k!)^2} + \qquad (3.18b)\\
&+\tfrac{p_1+q_1}{2\pi}\sum_{k=1}^{\infty} [\Gamma(\alpha k+1)]^2 \sin(\alpha[p_1+q_1]k\pi)\frac{x^{-\alpha k-1}}{(k!)^2} -\\
&-\tfrac{p_1-q_1}{2\pi}\sum_{k=1}^{\infty} [\Gamma(\alpha k+1)]^2 \sin(\alpha[p_1-q_1]k\pi)\frac{x^{-\alpha k-1}}{(k!)^2}.
\end{aligned}
$$

and we define:

$$
\zeta(k) \equiv \tfrac{2}{\alpha}\psi(k+1) - 2\psi(\alpha k+1) + \log x. \qquad (3.18c)
$$

Proof of Lemma 3.4.2: We proceed as in Proof of Lemma 3.4.1, only this time we consider the residues which lie on the right-side of integration path and use

Residue Theorem to evaluate the integral (3.14a). So, take the closed loop γ defined as $\gamma = \gamma_1 - \gamma_2$, where $\gamma_1(t) = a + t, t \in [-ir, ir]$ and $\gamma_2(t) = a + re^{it}, t \in [-\pi/2, \pi/2]$, $r > 0$. The loop γ is formed by the half-circle on the right-side of integration path of (3.14a) and coincides with the part of integration path. We denote the integrand in (3.14a) as $h(s)$ and integrate it along the loop γ. Whenever $\alpha_1 \neq \alpha_2$, function $h(s)$ has only simple poles on the right-side of integration path given in (3.14a). These are located at all points $\alpha_1 k + 1$ and $\alpha_2 k + 1$, where k is a positive integer. When $\alpha_1 = \alpha_2 = \alpha$, function $h(s)$ has second order poles at all points $\alpha k + 1$, and k being positive integer.

The evaluation of the simple poles is straightforward, and yields:

$$\operatorname*{res}_{s=\alpha_1 k+1} h(s) = \alpha_1 \tfrac{(-1)^{k+1}}{k!} \left[\Gamma(1+\alpha_1 k)\right]^2 \Gamma\left(-\tfrac{\alpha_1}{\alpha_2} k\right) \sin(\alpha_1 p_1 k\pi) \sin(\alpha_1 q_1 k\pi) x^{-\alpha_1 k - 1}.$$

Using the reflection formula for the Gamma function 6.1.17 [AS64], we have:

$$\operatorname*{res}_{s=\alpha_1 k+1} h(s) = \alpha_1 \pi \tfrac{(-1)^k}{k!} \tfrac{[\Gamma(\alpha_1 k+1)]^2}{\Gamma(\alpha_1 k/\alpha_2+1)} \tfrac{\sin(\alpha_1 p_1 k\pi)\sin(\alpha_1 q_1 k\pi)}{\sin(\alpha_1/\alpha_2 k\pi)} x^{-\alpha_1 k-1}.$$

The residue of the simple poles at $\alpha_2 k + 1$ with k being positive integer are computed similarly.

For $\alpha_1 = \alpha_2 = \alpha$, we rewrite $h(s)$ in (3.15) as:

$$h(s) = \frac{\alpha^2 \left[\Gamma\left(\frac{1-s}{\alpha} + k + 1\right)\right]^2}{\left(\frac{1-s}{\alpha} + k - 1\right)^2 \ldots \left(\frac{1-s}{\alpha}\right)^2} \left[\Gamma(s)\right]^2 \sin(p_1[1-s]\pi) \sin(q_1[1-s]\pi) x^{-s},$$

and expand the factors in $h(s)$ around $\alpha k + 1$:

$$\left(\tfrac{1-s}{\alpha} + n\right)^{-2} = \tfrac{1}{(k-n)^2} - \tfrac{2}{\alpha(k-n)^3}(s - \alpha k - 1) + \ldots, \qquad \text{for } 0 \le n \le k-1,$$

and:

$$\begin{aligned}
&\left[\Gamma\left(\tfrac{1-s}{\alpha} + k + 1\right)\right]^2 = 1 - \tfrac{2}{\alpha}\Gamma'(1)(s - \alpha k - 1) + \ldots \\
&\left[\Gamma(s)\right]^2 = \left[\Gamma(\alpha k + 1)\right]^2 \left\{1 + 2\psi(\alpha k + 1)(s - \alpha k - 1) + \ldots\right\}, \\
&x^{-s} = x^{-\alpha k - 1}\left\{1 - (s - \alpha k - 1)\log x + \ldots\right\}, \\
&\sin(p_1[1-s]\pi) = -\sin(\alpha p_1 k\pi) - p_1 \pi \cos(\alpha p_1 k\pi)(s - \alpha k - 1) + \ldots.
\end{aligned}$$

The residue of the second order pole of $h(s)$ is now given by the coefficient associated with the first order term of the expansion of $h(s)$, and so:

$$\begin{aligned}
\operatorname*{res}_{s=\alpha k+1} h(s) = &-\alpha \frac{x^{-\alpha k-1}}{(k!)^2} \left[\Gamma(\alpha k + 1)\right]^2 \sin(\alpha p_1 k\pi) \sin(\alpha q_1 k\pi) \Big\{ 2\psi(k+1) - \\
&- 2\alpha\psi(\alpha k + 1) + \alpha \log x \Big\} - \\
&- \tfrac{\alpha^2 \pi}{2} \frac{x^{-\alpha k-1}}{(k!)^2} \left[\Gamma(\alpha k + 1)\right]^2 \Big\{ (p_1 + q_1)\sin(\alpha[p_1 + q_1]k\pi) + \\
&- (p_1 - q_1)\sin(\alpha[p_1 - q_1]k\pi) \Big\}.
\end{aligned}$$

Let m denotes the biggest integer smaller then r. The orientation of loop γ is clockwise, so the index function has value -1. The Residue Theorem gives:

$$\int_{a-ir}^{a+ir} h(t)\mathrm{d}t = \int_{\gamma_2} h(s)\mathrm{d}s - 2\pi i \sum_{k=1}^{m} \operatorname*{res}_{s=\alpha_1 k+1} h(s) - 2\pi i \sum_{k=1}^{m} \operatorname*{res}_{s=\alpha_2 k+1} h(s),$$

for $\alpha_1 \neq \alpha_2$ and

$$\int_{a-ir}^{a+ir} h(t)\mathrm{d}t = \int_{\gamma_2} h(s)\mathrm{d}s - 2\pi i \sum_{k=1}^{m} \operatorname*{res}_{s=\alpha k+1} h(s),$$

when $\alpha_1 = \alpha_2 = \alpha$. To verify the Jordan's Lemma for integral along arc γ_2, we use the same estimates as in Proof of Lemma 3.4.1, only this time the real part of s is positive. So, denoting $s = u + iv = re^{it}$, we estimate $u \le r$ and we have:

$$\left| \int_{\gamma_2} h(s)\mathrm{d}s \right| \le \pi r \max_{s\in\gamma_2} |h(s)| \le c r^{r(2-1/\alpha_1-1/\alpha_2)},$$

where c is some real constant. The integral along arc γ_2 vanishes identically as the modulus r goes to infinity, whenever $\alpha_1 + \alpha_2 > 2\alpha_1\alpha_2$. This inequality is obviously satisfied for $0 < \alpha_1, \alpha_2 < 1$, or for pairs α_1, α_2 satisfying condition: $\alpha_1 > 2/3$ and $\frac{\alpha_1}{2\alpha_1 - 1} > \alpha_2$. In particular, when $\alpha_1 = \alpha_2 = \alpha$, we have $0 < \alpha < 1$. Therefore, for any admissible α_1, α_2, we apply Jordan's Lemma on the integral of $h(s)$ integrated along the arc γ_2. So, after substituting back into (3.14a), we obtain (3.18a), (3.18b), respectively. Q.E.D

The scaling property (2.23b) and (3.6c) implies the scaling property of product density:

$$g(x; \alpha_1, \alpha_2, p_1, q_1, c_1, c_2) = \frac{1}{c_1^{1/\alpha_1} c_2^{1/\alpha_2}} g\left(x c_1^{-1/\alpha_1} c_2^{-1/\alpha_2}; \alpha_1, \alpha_2, p_2, q_2 \right). \qquad (3.19a)$$

For the simplest case when random variables X_1, X_2 are identically distributed independent random variables, having symmetric stable density, i.e. $f_1(x) = f_2(x) \equiv f(x; \alpha, 1/2)$, we have: $G_1(s) = 2[F_1(s; \alpha, 1/2)]^2$, and $G_2(s) = G_1(s)$. Then the product density $g(x)$ simply equals $2\tilde{g}(x)$ for real x.

In the general case, close examination of (3.12a) and (3.12b) implies that:

$$g(x; \alpha_1, \alpha_2, p_1, q_1) = g(x; \alpha_1, \alpha_2, p_2, q_2), \qquad (3.19b)$$

$$g(x; \alpha_1, \alpha_2, p_1, q_2) = g(-x; \alpha_1, \alpha_2, p_1, q_1). \qquad (3.19c)$$

In the following Proposition we establish the power-series representation of $g(x)$.

Proposition 3.4.1. *Consider two independent random variables X_1, X_2, with strictly stable probability densities $f_1(x; \alpha_1, p_1)$ and $f_2(x; \alpha_2, q_1)$, respectively. Assume that the stability parameters α_1, α_2 are either $1 < \alpha_1, \alpha_2 \le 2$ or satisfy condition that $2/3 < \alpha_1 < 1$ and $\alpha_2 > \frac{\alpha_1}{2\alpha_1-1}$. The asymmetry parameters p_1, q_1 can have any admissible values.*

Then for $x \neq 0$, the density $g(x)$ of the random variable $X = X_1 \cdot X_2$ has the power series:

$$\begin{aligned} g(x; \alpha_1, &\alpha_2, p_1, q_1) = \qquad (3.20a) \\ &= \frac{2}{\pi^2} \sum_{k=1}^{\infty} \Gamma\left(\frac{k}{\alpha_1}+1\right) \Gamma\left(\frac{k}{\alpha_2}+1\right) \sin(p_1 k\pi) \sin(q_1 k\pi) \xi(k) \frac{x^{k-1}}{(k!)^2} + \\ &+ \frac{p_2 - q_1}{\pi} \sum_{k=1}^{\infty} \Gamma\left(\frac{k}{\alpha_1}+1\right) \Gamma\left(\frac{k}{\alpha_2}+1\right) \sin([p_1+q_1]k\pi) \frac{x^{k-1}}{(k!)^2} + \\ &+ \frac{p_1 - q_1}{\pi} \sum_{k=1}^{\infty} \Gamma\left(\frac{k}{\alpha_1}+1\right) \Gamma\left(\frac{k}{\alpha_2}+1\right) \sin([p_1-q_1]k\pi) \frac{x^{k-1}}{(k!)^2}, \end{aligned}$$

where $\xi(k)$ designates the auxiliary function given by :

$$\xi(k) \equiv 2\psi(k) - \tfrac{1}{\alpha_1}\psi\left(\tfrac{k}{\alpha_1}\right) - \tfrac{1}{\alpha_2}\psi\left(\tfrac{k}{\alpha_2}\right) - \log x, \tag{3.20b}$$

and $\psi(k)$ is a digamma function.

Proof of Proposition 3.4.1: We first discuss the condition on the stability parameters α_1, α_2. In order to apply results of Lemma 3.4.1, the parameters must satisfy $\alpha_1 + \alpha_2 < 2\alpha_1\alpha_2$. This inequality is satisfied for $1 < \alpha_1, \alpha_2 \le 2$ or for pairs α_1, α_2 satisfying condition: $\alpha_1 > 2/3$ and $\frac{\alpha_1}{2\alpha_1 - 1} < \alpha_2$.

From Lemma 3.4.1 and (3.12a), we compute the series:

$$g_1(x) = \tilde{g}(x; \alpha_1, \alpha_2, p_1, q_1) + \tilde{g}(x; \alpha_1, \alpha_2, p_2, q_2), \tag{3.21a}$$

where $\tilde{g}(x)$ is defined in (3.13c). From $p_1 + p_2 = 1$ and addition formulas for circular functions follows:

$$\begin{aligned} \sin(p_2 k\pi) &= (-1)^{k+1} \sin(p_1 k\pi), \\ \sin([p_2 + q_2]k\pi) &= -\sin([p_1 + q_1]k\pi), \\ \sin([p_2 - q_2]k\pi) &= -\sin([p_1 - q_1]k\pi). \end{aligned}$$

Therefore from (3.14b) we obtain:

$$\begin{aligned} \tilde{g}(x; \alpha_1, \alpha_2, p_2, q_2) = {}& \tilde{g}(x; \alpha_1, \alpha_2, p_1, q_1) + \\ & + \tfrac{1}{\pi} \sum_{k=1}^{\infty} \Gamma\left(\tfrac{k}{\alpha_1} + 1\right) \Gamma\left(\tfrac{k}{\alpha_2} + 1\right) \sin([p_1 + q_1]k\pi) \frac{x^{k-1}}{(k!)^2}, \end{aligned} \tag{3.21b}$$

and we proved the relation (3.20a).

Similarly, we have:

$$g_2(x) = \tilde{g}(x; \alpha_1, \alpha_2, p_1, q_2) + \tilde{g}(x; \alpha_1, \alpha_2, p_2, q_1),$$

and we have relations:

$$\begin{aligned} \sin([p_1 + q_2]k\pi) &= -\sin([p_2 + q_1]k\pi) = (-1)^k \sin([p_1 - q_1]k\pi), \\ \sin([p_1 - q_2]k\pi) &= -\sin([p_2 - q_1]k\pi) = (-1)^k \sin([p_1 + q_1]k\pi), \end{aligned}$$

and so:

$$\begin{aligned} \tilde{g}(x; \alpha_1, \alpha_2, p_1, q_2) = {}& \tilde{g}(-x; \alpha_1, \alpha_2, p_1, q_1) + \\ & + \tfrac{1}{2\pi} \sum_{k=1}^{\infty} \Gamma\left(\tfrac{k}{\alpha_1} + 1\right) \Gamma\left(\tfrac{k}{\alpha_2} + 1\right) \sin([p_1 + q_1]k\pi) \frac{(-x)^{k-1}}{(k!)^2} + \\ & + \tfrac{1}{2\pi} \sum_{k=1}^{\infty} \Gamma\left(\tfrac{k}{\alpha_1} + 1\right) \Gamma\left(\tfrac{k}{\alpha_2} + 1\right) \sin([p_1 - q_1]k\pi) \frac{(-x)^{k-1}}{(k!)^2}, \end{aligned}$$

and

$$\begin{aligned} \tilde{g}(x; \alpha_1, \alpha_2, p_2, q_1) = {}& \tilde{g}(-x; \alpha_1, \alpha_2, p_1, q_1) + \\ & + \tfrac{1}{2\pi} \sum_{k=1}^{\infty} \Gamma\left(\tfrac{k}{\alpha_1} + 1\right) \Gamma\left(\tfrac{k}{\alpha_2} + 1\right) \sin([p_1 + q_1]k\pi) \frac{(-x)^{k-1}}{(k!)^2} - \\ & - \tfrac{1}{2\pi} \sum_{k=1}^{\infty} \Gamma\left(\tfrac{k}{\alpha_1} + 1\right) \Gamma\left(\tfrac{k}{\alpha_2} + 1\right) \sin([p_1 - q_1]k\pi) \frac{(-x)^{k-1}}{(k!)^2}. \end{aligned}$$

Thus, the same power series gives us both $g_1(x)$ and $g_2(x)$, as $g_2(x) = g_1(-x)$. Q.E.D

This should have been expected since the power series for $g_1(x)$ converges uniformly in every bounded region of the complex plane and therefore densities are entire functions.

From (3.20a) we see that the commutative property of multiplication is satisfied.

For some specific choices of stability and asymmetry parameters of stable law, the sum given in (3.20a) simplifies. Consider the special case of $\alpha_1 = \alpha_2 = \alpha$, with $1 < \alpha \leq 2$. Let us consider the case when one stable law is symmetric, say, e.g. $q_1 = 1/2$, and the second stable density is asymmetric, i.e. $p_1 \neq 1/2$. Then the power series (3.20a) becomes:

$$g(x;\alpha,p_1,1/2) = \tfrac{2}{\alpha^2\pi^2}\sum_{k=0}^{\infty}(-1)^k\left[\Gamma\left(\tfrac{2k+1}{\alpha}\right)\right]^2\sin(p_1[2k+1]\pi)\xi(2k+1)\frac{x^{2k}}{[(2k)!]^2}-$$
$$-\tfrac{p_1-p_2}{\alpha^2\pi}\sum_{k=0}^{\infty}(-1)^k\left[\Gamma\left(\tfrac{2k+1}{\alpha}\right)\right]^2\cos(p_1[2k+1]\pi)\frac{x^{2k}}{[(2k)!]^2}.$$

The following Proposition is a consequence of Proposition 3.4.1 and summarizes the problem when X_1, X_2 are independent and identically distributed stable random variables with the density $f(x;\alpha,p_1)$ and $1 < \alpha \leq 2$.

Proposition 3.4.2. *Consider random variable $X = X_1 \cdot X_2$. The random variables X_1, X_2 are independent and identically distributed, with strictly stable density $f(x;\alpha,p_1)$. It is assumed that $1 < \alpha \leq 2$, and the asymmetry parameter p_1 has any admissible value.*

For $x \neq 0$ the density $g(x)$ of random variable X is represented by the power series:

$$g(x;\alpha,p_1) = \tfrac{2}{\pi^2}\sum_{k=1}^{\infty}\left[\Gamma\left(\tfrac{k}{\alpha}+1\right)\right]^2[\sin(p_1k\pi)]^2\,\tilde{\xi}(k)\frac{x^{k-1}}{(k!)^2} \qquad (3.23a)$$

for the asymmetric strictly stable law, i.e. $p_1 \neq 1/2$. For symmetric stable law, i.e. when $p_1 = 1/2$, we have:

$$g(x;\alpha,1/2) = \tfrac{2}{\alpha^2\pi^2}\sum_{k=0}^{\infty}\left[\Gamma\left(\tfrac{2k+1}{\alpha}\right)\right]^2\xi(2k+1)\frac{x^{2k}}{(2k!)^2}. \qquad (3.23b)$$

The auxiliary functions $\tilde{\xi}(k)$ and $\xi(k)$ are defined as follows:

$$\xi(k) = 2\psi(k) - \tfrac{2}{\alpha}\psi\left(\tfrac{k}{\alpha}\right) - \log x, \qquad (3.23c)$$
$$\tilde{\xi}(k) = \tfrac{1}{2}\xi(k) + (p_1-p_2)\psi(p_1k) - (p_1-p_2)\psi(1-p_1k). \qquad (3.23d)$$

Proof of Proposition 3.4.2: Let us assume that $x > 0$. First we consider $\tilde{g}(x;\alpha,p_1)$. When $p_1 \neq 1/2$, we rewrite (3.16) to obtain residue:

$$\operatorname*{res}_{s=-k} h(s) = \frac{x^k}{(k!)^2}\left[\Gamma\left(\tfrac{k+1}{\alpha}\right)\right]^2[\sin(p_1[1+k]\pi)]^2\Big\{2\psi(k+1)-$$
$$-\tfrac{2}{\alpha}\psi\left(\tfrac{k+1}{\alpha}\right) - 2p_1\pi\cot(p_1[1+k]\pi) - \log x\Big\},$$

For $\tilde{g}(x;\alpha,p_2)$ and $p_2 \neq 1/2$, we have:

$$\operatorname*{res}_{s=-k} h(s) = \frac{x^k}{(k!)^2}\left[\Gamma\left(\tfrac{k+1}{\alpha}\right)\right]^2[\sin(p_1[1+k]\pi)]^2\Big\{2\psi(k+1)- \\ -\tfrac{2}{\alpha}\psi\left(\tfrac{k+1}{\alpha}\right)+2(1-p_1)\pi\cot(p_1[1+k]\pi)-\log x\Big\},$$

and so for term in $g(x;\alpha,p_1)$ we get:

$$2\frac{x^k}{(k!)^2}\left[\Gamma\left(\tfrac{k+1}{\alpha}\right)\right]^2[\sin(p_1[1+k]\pi)]^2\Big\{\psi(k+1)- \\ -\tfrac{1}{\alpha}\psi\left(\tfrac{k+1}{\alpha}\right)-(p_1-p_2)\pi\cot(p_1[1+k]\pi)-\tfrac{1}{2}\log x\Big\}.$$

Using reflection formula for digamma function 6.3.7 [AS64], we obtain for term in $g(s;\alpha,p_1)$ the following expression:

$$2\frac{x^k}{(k!)^2}\left[\Gamma\left(\tfrac{k+1}{\alpha}\right)\right]^2[\sin(p_1[1+k]\pi)]^2\Big\{\psi(k+1)-\tfrac{1}{\alpha}\psi\left(\tfrac{k+1}{\alpha}\right)- \\ -(p_1-p_2)\psi(1-p_1[1+k])+(p_1-p_2)\psi(p_1[1+k])-\tfrac{1}{2}\log x\Big\}.$$

Thus, we have shown (3.23a). For $p_1 = 1/2$, the sum (3.23b) follows directly from (3.20a). Q.E.D

The series (3.23a) further simplifies for $\alpha = 2$. Using Duplication formulas for the gamma function 6.1.18 in [AS64], and the digamma function 6.3.8 in [AS64], respectively, we get:

$$g(x;2) = \tfrac{1}{\pi}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\Big\{\psi(k+1)-\log\tfrac{x}{4}\Big\}\left(\tfrac{x}{4}\right)^{2k}.$$

The power series representation for density of product of two stable random variables given by (3.20a)-(3.23b) assumed $1 < \alpha \leq 2$. The following two propositions cover the case for $0 < \alpha < 1$.

Proposition 3.4.3. *Consider two independent random variables X_1, X_2, with strictly stable probability densities $f_1(x;\alpha_1,p_1)$ and $f_2(x;\alpha_2,q_1)$, respectively. Assume that for the stability parameters α_1,α_2 holds that either $0 < \alpha_1,\alpha_2 < 1$ or stability parameters α_1,α_2 satisfy the condition: $\alpha_1 > 2/3$ and $\alpha_2 < \frac{\alpha_1}{2\alpha_1-1}$. The asymmetry parameters p_1, q_1 can have any admissible values.*

Then for $x > 0$, the density $g(x)$ of random variable $X = X_1 \cdot X_2$ is given by power series:

$$g(x;\alpha_1,\alpha_2,p_1,q_1) = \tfrac{1}{\alpha_2\pi}\sum_{k=1}^{\infty}\tfrac{(-1)^{k+1}}{k!}\tfrac{[\Gamma(\alpha_1k+1)]^2}{\Gamma(\alpha_1k/\alpha_2+1)}\phi(\alpha_1,\alpha_2,p_1,q_1)x^{-\alpha_1k-1}+ \\ +\tfrac{1}{\alpha_1\pi}\sum_{k=1}^{\infty}\tfrac{(-1)^{k+1}}{k!}\tfrac{[\Gamma(\alpha_2k+1)]^2}{\Gamma(\alpha_2k/\alpha_1+1)}\phi(\alpha_2,\alpha_1,p_1,q_1)x^{-\alpha_2k-1} \quad (3.24a)$$

for $\alpha_1 \neq \alpha_2$, where we have defined:

$$\phi(k;\alpha_1,\alpha_2,p_1,q_1) = \tfrac{\cos(\alpha[p_1+q_1]k\pi)-\cos(\alpha_1k\pi)\cos(\alpha_1[p_1-q_2]k\pi)}{\sin(\alpha_1/\alpha_2k\pi)}. \quad (3.24b)$$

For $\alpha_1 = \alpha_2 = \alpha$ we have the power series:

$$g(x;\alpha,p_1,q_1) = \tfrac{2}{\pi^2}\sum_{k=1}^{\infty}\left[\Gamma\left(\alpha k+1\right)\right]^2 \sin(\alpha p_1 k\pi)\sin(\alpha q_1 k\pi)\zeta(k)\frac{x^{-\alpha k-1}}{(k!)^2} + \tag{3.24c}$$
$$+ \tfrac{q_1-p_2}{\pi}\sum_{k=1}^{\infty}\left[\Gamma\left(\alpha k+1\right)\right]^2 \sin(\alpha[p_1+q_1]k\pi)\frac{x^{-\alpha k-1}}{(k!)^2} -$$
$$- \tfrac{p_1-q_1}{\pi}\sum_{k=1}^{\infty}\left[\Gamma\left(\alpha k+1\right)\right]^2 \sin(\alpha[p_1-q_1]k\pi)\frac{x^{-\alpha k-1}}{(k!)^2},$$

where:

$$\zeta(k) \equiv \tfrac{2}{\alpha}\psi(k+1) - 2\psi\left(\alpha k+1\right) + \log x. \tag{3.24d}$$

For computation of values for $x < 0$, we use reflection property (3.19c).

Proof of Proposition 3.4.3: Let us start with the series (3.24a), i.e. we assume that $\alpha_1 \neq \alpha_2$. We need to compute (3.21a). In order to prove the relation, we use formulas for circular functions 4.3.16, 4.3.17, 4.3.31, 4.3.33 in [AS64] and obtain:

$$\begin{aligned}\sin(\alpha_1 p_1 k\pi)\sin(\alpha_1 q_1 k\pi) + \sin(\alpha_1 p_2 k\pi)\sin(\alpha_1 q_2 k\pi) = \\ = \cos(\alpha_1[p_1-q_1]k\pi) - \tfrac{1}{2}\cos(\alpha_1[p_1+q_1]k\pi) - \tfrac{1}{2}\cos(\alpha_1[p_2+q_2]k\pi) = \\ = \cos(\alpha_1[p_1-q_1]k\pi) - \cos(\alpha_1 k\pi)\cos(\alpha_1[p_1-q_2]k\pi),\end{aligned}$$

and from Lemma 3.4.2 follows (3.24a).

Relation (3.24c) is established the same way as (3.20a) in Proposition 3.4.1.
Q.E.D

Remark 3.4.1. When X_1, X_2 are independent identically distributed stable random variables with density $f(x;\alpha,p_1)$ and $0 < \alpha < 1$, the density of the product $X_1 \cdot X_2$ reduces to:

$$g(x;\alpha,p_1) = \tfrac{2}{\pi^2}\sum_{k=1}^{\infty}\left[\Gamma\left(\alpha k+1\right)\right]^2\left[\sin(\alpha p_1 k\pi)\right]^2\tilde{\zeta}(k)\frac{x^{k-1}}{(k!)^2}, \tag{3.25a}$$

where

$$\tilde{\zeta}(k) = \tfrac{1}{2}\zeta(k) + (p_1-p_2)\psi(\alpha p_1 k) - (p_1-p_2)\psi(1-\alpha p_1 k), \tag{3.25b}$$

and $\zeta(k)$ is defined in (3.24d).

3.5 Product Density and Fox's H-function

Fox's H-function was introduced in [Fox61]. It is defined by the Mellin-Barnes integral as:

$$H_{pq}^{mn}\left[z\middle|\begin{array}{c}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{array}\right] = \frac{1}{2\pi i}\int_\gamma g(s)z^{-s}\mathrm{d}s,$$

where:

$$g(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)},$$

and where the integration path γ is a vertical line in complex plane indented to avoid the poles of the integrand $g(s)$.

Relation (3.7) indicates that the density of stable law can be written in terms of Fox's H-function as:

$$f(x;\alpha,p_1) = \tfrac{1}{\alpha}H_{22}^{11}\left[x\middle|\begin{array}{c}(\frac{\alpha-1}{\alpha},\frac{1}{\alpha}),(p_2,p_1)\\(0,1),(p_2,p_1)\end{array}\right], \tag{3.26}$$

for $x>0$, see [Sch86]. Further, the representation of stable densities by Fox's H-functions has been discussed in [Sch86] for some specific choices of the asymmetry parameter.

The Lemmas 3.4.1, 3.4.2, suggest that we can describe the power series in terms of Fox's H-function. Let us examine the particular choices of parameters for Fox's H-function in connection with the density of the product of two independent stable laws. Recalling (3.7), we represent function $g(x)$ in (3.13c) as:

$$\begin{aligned}&\tilde{g}(x;\alpha_1,\alpha_2,p_1,q_1) =\\ &= \frac{1}{2\alpha_1\alpha_2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{[\Gamma(s)]^2\Gamma\left(1-\frac{\alpha_1-1}{\alpha_1}-\frac{s}{\alpha_1}\right)\Gamma\left(1-\frac{\alpha_2-1}{\alpha_2}-\frac{s}{\alpha_2}\right)}{\Gamma(1-p_2-p_1s)\Gamma(p_2+p_1s)\Gamma(1-q_2-q_1s)\Gamma(q_2+q_1s)}x^{-s}\mathrm{d}s,\end{aligned}$$

for $x>0$. This can be written in terms of Fox's H-function as:

$$\tilde{g}(x;\alpha_1,\alpha_2,p_1,q_1) = \tfrac{1}{\alpha_1\alpha_2}H_{44}^{22}\left[x\middle|\begin{array}{c}(\frac{\alpha_1-1}{\alpha_1},\frac{1}{\alpha_1}),(\frac{\alpha_2-1}{\alpha_2},\frac{1}{\alpha_2}),(p_2,p_1),(q_2,q_1)\\(0,1),(0,1),(p_2,p_1),(q_2,q_1)\end{array}\right],$$

for $x>0$. From (3.21b) we have relations for sum:

$$\begin{aligned}&H_{44}^{22}\left[x\middle|\begin{array}{c}(\frac{\alpha_1-1}{\alpha_1},\frac{1}{\alpha_1}),(\frac{\alpha_2-1}{\alpha_2},\frac{1}{\alpha_2}),(p_2,p_1),(q_2,q_1)\\(0,1),(0,1),(p_2,p_1),(q_2,q_1)\end{array}\right]+\\ &+H_{44}^{22}\left[x\middle|\begin{array}{c}(\frac{\alpha_1-1}{\alpha_1},\frac{1}{\alpha_1}),(\frac{\alpha_2-1}{\alpha_2},\frac{1}{\alpha_2}),(p_1,p_2),(q_1,q_2)\\(0,1),(0,1),(p_1,p_2),(q_1,q_2)\end{array}\right]=\\ &=\alpha_1\alpha_2 g(x;\alpha_1,\alpha_2,p_1,q_1),\end{aligned}$$

where $g(x)$ is the density given in (3.20a)-(3.25a) and $x>0$. For difference of these two particular H-functions, we have:

$$\begin{aligned}&H_{44}^{22}\left[x\middle|\begin{array}{c}(\frac{\alpha_1-1}{\alpha_1},\frac{1}{\alpha_1}),(\frac{\alpha_2-1}{\alpha_2},\frac{1}{\alpha_2}),(p_2,p_1),(q_2,q_1)\\(0,1),(0,1),(p_2,p_1),(q_2,q_1)\end{array}\right]-\\ &-H_{44}^{22}\left[x\middle|\begin{array}{c}(\frac{\alpha_1-1}{\alpha_1},\frac{1}{\alpha_1}),(\frac{\alpha_2-1}{\alpha_2},\frac{1}{\alpha_2}),(p_1,p_2),(q_1,q_2)\\(0,1),(0,1),(p_1,p_2),(q_1,q_2)\end{array}\right]=\\ &=\tfrac{1}{\pi}\sum_{k=1}^{\infty}\Gamma\left(\tfrac{k}{\alpha_1}\right)\Gamma\left(\tfrac{k}{\alpha_2}\right)\sin([p_1+q_1]k\pi)\frac{x^{k-1}}{[(k-1)!]^2},\end{aligned}$$

and we assume $\alpha_1,\alpha_2>1$.

The other relations can be explored and established by combinations of results of previous section.

4. Lévy Flight Model

European put and call options are usually priced using Martingale valuation theory [HP81] *et seq.* Consider an asset whose forward price is modelled via the stochastic process $X = \{X(t), 0 \leq t \leq \tau\}$, and let $X(0) = x$ be the current forward price quoted in the market. Under Martingale valuation theory, the value of European put and call options on the asset is:

$$V_{put}(x,\tau) = \mathrm{E}\left\{ [K - X(\tau)]^{+} \,\middle|\, X(0) = x \right\} = \int_{-\infty}^{K} (K - y)\, g(y, \tau |\, x, 0)\mathrm{d}y, \quad (4.1a)$$

$$V_{call}(x,\tau) = \mathrm{E}\left\{ [X(\tau) - K]^{+} \,\middle|\, X(0) = x \right\} = \int_{K}^{\infty} (y - K)\, g(y, \tau |\, x, 0)\mathrm{d}y, \quad (4.1b)$$

where K is the option's *strike*, τ is the option's *exercise date*, and $g(y, \tau|x, 0)$ is the *terminal risk-neutral density* for $X(\tau)$ at y, given that $X(0) = x$. Since we are taking the current date to be 0, τ is actually the time to expiry. Under this theory, X is the martingale in the risk-neutral measure, so:

$$\mathrm{E}\left\{X(\tau)|X(0) = x\right\} = \int_{-\infty}^{\infty} yg(y, \tau |\, x, 0)\mathrm{d}y = x.$$

Consequently, European options satisfy put-call parity:

$$V_{put}(x,\tau) - V_{call}(x,\tau) = K - x. \quad (4.2)$$

This theory developed as a natural outgrowth of the pioneering work of Louis Bachelier, who developed the first modern model for option pricing [Bac00], often referred to as the *normal model* by market practitioners. We review the Bachelier model in the next section in more detail. Based on physical arguments, Bachelier concluded that the price of a stock X could be modelled as Brownian motion. As a consequence, at any future date τ the terminal risk-neutral density at $X(\tau) = y$ is Gaussian,

$$g(y - x, \tau) = \frac{1}{\sigma\sqrt{\tau}}\phi\left(\frac{y - x}{\sigma\sqrt{\tau}}\right).$$

Here, $\phi()$ is the density of the standard normal probability distribution. Measurements of asset prices shows that they adhere much more closely to stable laws with stability parameters α between, say, 1.35 and 1.6 [MS95].

In this chapter we extend the Bachelier's model to the stable laws with stability parameters $1 < \alpha \leq 2$. We refer to new model as the *Lévy Flight model*, or more succinctly the *levy model*. We give the option pricing formulas for the Lévy Flight model. Next we introduce concept of implied volatility and explore the implied volatility surfaces $\sigma(K, \tau)$ and the delta hedges generated by the levy model. This chapter is based on the work established in [KHed].

4.1 Bachelier Model

We start this section with a short historical excursion which provides an overview on the work of Louis Bachelier [Bac00]. In the part called "Probabilities In the

Operations of the Stock Exchange", Bachalier states the model assumptions in the very intuitive way: "It is obvious that the price considered by the market as the most likely is the current true price: if the market thought otherwise, it would quote not this price, but another one higher or lower.

In the remainder of this study, the true price corresponding to given epoch will be taken as the origin for the coordinates.

Prices can vary between $-x_0$ and ∞: x_0 being the current absolute price. It will be assumed that it can vary between $-\infty$ and ∞. The probability of a spread greater then x_0 being considered a priori entirely negligible.

Under these conditions, it may be admitted that the probability of a deviation from the true price is independent of the absolute level of this price, and that the probability curve is symmetrical with respect to the true price.

In what follows only relative prices will matter because the origin of the coordinates will always correspond to the current true price."

Therefore, the Bachelier models directly the stochastic process of stock price, rather then the process of yields over given time epoch, as became customary later, see e.g. [Man63]. The assumption on the spread greater then x_0 is a very reasonable assumption because in the stock exchange the price is not quoted as negative but capped by zero. The assumptions on the independence of the deviation, i.e. increments of the stochastic process, and the true price is a technical assumption. The symmetry of probability curve is a rather restricting and unnecessary assumption, see e.g. [HP81].

Bachelier established Brownian motion as the model of the stock price in three different ways. Firstly, he uses the law of joint probability and the assumption on the independence of the increments and he formulates what is now called Chapman-Kolmogorov equation: "Let $p_{x,t}\mathrm{d}x$ designates the probability that, at epoch t, the price is to be found in the elementary interval $x, x + \mathrm{d}x$.

We seek the probability that the price z be quoted at epoch $t_1 + t_2$, the price x having been quoted at epoch t_1.

By virtue of Principle of Compound Probabilities, the desired probability will be equal to the product of the probability that x be the quoted price at epoch t_1, that is to say, $p_{x,t_1}\mathrm{d}x$ multiplied by probability that x be the price quoted at epoch t_1, the current price z being quoted at epoch $t_1 + t_2$, that is to say, multiplied by $p_{z-x,t}\mathrm{d}z$.

The desired probability is therefore

$$p_{x,t_1} p_{z-x,t_2} \mathrm{d}x\mathrm{d}z.$$

At epoch t_1, the price could be located in any of the intervals $\mathrm{d}x$ between $-\infty$ and ∞, so the probability of the price z being quoted at epoch $t_1 + t_2$ will be

$$\int_{-\infty}^{\infty} p_{x,t_1} p_{z-x,t_2} \mathrm{d}x\mathrm{d}z.$$

The probability of this price z, at epoch $t_1 + t_2$, is also given by the expression p_{z,t_1+t_2}; we therefore have

$$p_{z,t_1+t_2}\mathrm{d}z = \int_{-\infty}^{\infty} p_{x,t_1} p_{z-x,t_2} \mathrm{d}x\mathrm{d}z$$

or

$$p_{z,t_1+t_2} = \int_{-\infty}^{\infty} p_{x,t_1} p_{z-x,t_2} \mathrm{d}x,$$

which is the equation for the condition which must be satisfied by the function p.

It can be seen that this equation is satisfied by the function

$$p = Ae^{-B^2x^2}."$$

With the basic probability text books in mind, it is obvious that Bachelier chose only one of the possible solutions. Any symmetrical Markov process would satisfy the equation as well.

The second way how Bachelier arrives to Gaussian distribution is by an intuitive sketch of central limit theorem. The third approach, considered by Bachelier, introduces the idea of Kolmogorov backward and forward equation, he formulates Kolmogorov backward equation for Brownian motion and solves it.

In the words of modern Mathematical Finance, the risk-neutral density considered by Bachelier has a normal distribution, i.e. stable distribution with $\alpha = 2$.

We conclude that for the forward price $X(t)$ of a stock at time t, with initial value $X(0) = x$ being the current market price, Bachelier considered the evolution of $X(t)$ over a finite time period $0 < t \leq \tau$, and assumed that during this time, it is overwhelmingly probable that the spread $|X(t) - x|$ is much smaller than x, with only a negligibly small probability that $|X(t) - x| = O(x)$. Under these conditions, it is reasonable to assume that for each $0 < t \leq \tau$, the spread $X(t) - x$ is independent of the initial condition x. Further, it is assumed that the increments of the process X are independent and stationary. From these assumptions Bachelier intuitively derived the Chapman-Kolmogorov equation and by taking only single set of solutions to the Chapman-Kolmogorov equations, he obtained the probability Gaussian densities:

$$f(X(\tau) - x, \tau) = \frac{1}{\sigma\sqrt{\tau}} \phi\left(\frac{X(\tau) - x}{\sigma\sqrt{\tau}}\right). \tag{4.3}$$

Bachelier concluded that the fair price for European call and put options with strike K and exercise date τ is:

$$V_{put}(x,\tau) = \mathrm{E}\left\{ [K - X(\tau)]^+ \middle| X(0) = x \right\} = \int_{-\infty}^{K} (K - y)\, f(y - x, \tau)\mathrm{d}y, \tag{4.4a}$$

$$V_{call}(x,\tau) = \mathrm{E}\left\{ [X(\tau) - K]^+ \middle| X(0) = x \right\} = \int_{K}^{\infty} (y - K)\, f(y - x, \tau)\mathrm{d}y. \tag{4.4b}$$

Working out the integrals, say, e.g., for put option:

$$V_{put}(x,\tau) = \int_{-\infty}^{K} \tfrac{K-y}{\sigma\sqrt{t}} \phi\left(\tfrac{y-x}{\sigma\sqrt{t}}\right) \mathrm{d}y = \sigma\sqrt{t} \int_{-\infty}^{(K-x)/\sigma\sqrt{t}} \left\{ \tfrac{K-x}{\sigma\sqrt{t}} - v \right\} \phi(v)\mathrm{d}v,$$

gives Bachelier's formulas:

$$V_{put}(x,\tau) = (K - x)\,\Phi\left(\frac{K - x}{\sigma\sqrt{\tau}}\right) + \sigma\sqrt{\tau}\phi\left(\frac{K - x}{\sigma\sqrt{\tau}}\right), \tag{4.5a}$$

$$V_{call}(x,\tau) = (x - K)\,\Phi\left(\frac{x - K}{\sigma\sqrt{\tau}}\right) + \sigma\sqrt{\tau}\phi\left(\frac{x - K}{\sigma\sqrt{\tau}}\right), \tag{4.5b}$$

where $\Phi()$ denotes the probability distribution function of the standard normal probability distribution.

Let us briefly examine Bachelier's formulas. For a normal model, the "normal moneyness" is a non-dimensional measure of the distance between the strike K and the current forward x, and is defined as:

$$z \equiv (K - x)/(\sigma\sqrt{\tau}). \tag{4.6a}$$

In terms of the moneyness, the value of a short position in the forward contract can be written as:

$$u \equiv (K - x) = z\sigma\sqrt{\tau}, \tag{4.6b}$$

and we can write the European option prices as:

$$V_{put}(u, \tau) = \sigma\sqrt{\tau} I_b(z), \tag{4.7a}$$

$$V_{call}(u, \tau) = \sigma\sqrt{\tau} I_b(-z), \tag{4.7b}$$

where we define:

$$I_b(z) = z\Phi(z) + \phi(z). \tag{4.7c}$$

Written this way, put-call parity is obvious, since $I_b(z) - I_b(-z) = z$. In terms of the moneyness, the put and call price can be expanded in a Taylor Series as:

$$V_{put}(u, \tau) = \frac{u}{2} - \sigma\sqrt{\frac{\tau}{2\pi}} \sum_{k=0}^{\infty} \frac{(-z^2/2)^k}{(2k-1)\Gamma(k+1)}, \tag{4.8a}$$

$$V_{call}(u, \tau) = -\frac{u}{2} - \sigma\sqrt{\frac{\tau}{2\pi}} \sum_{k=0}^{\infty} \frac{(-z^2/2)^k}{(2k-1)\Gamma(k+1)}, \tag{4.8b}$$

as shown by Bachelier. The above power-series are uniformly convergent over every bounded domain on the real line. From expansion (4.8a), we have approximative value of the put option given by:

$$\begin{aligned} V_{put}(u, t) &= \frac{u}{2} - \sigma\sqrt{\frac{t}{2\pi}} \sum_{k=0}^{\infty} \frac{(-u^2)^k}{2^k(2k-1)\Gamma(k+1)\sigma^{2k}t^k} = \\ &= \frac{u}{2} + \sigma\sqrt{\frac{t}{2\pi}} + \frac{u^2}{2\sqrt{2\pi t}}\sigma^{-1} - \frac{u^4}{24\sqrt{2\pi}t^{3/2}}\sigma^{-3} + \frac{u^6}{240\sqrt{2\pi}t^{5/2}}\sigma^{-5} - \\ &- \frac{u^8}{2688\sqrt{2\pi}t^{7/2}}\sigma^{-7} + \frac{u^{10}}{34560\sqrt{2\pi}t^{9/2}}\sigma^{-9} - \frac{u^{12}}{704 \cdot 6!\sqrt{2\pi}t^{11/2}}\sigma^{-11} + \dots \end{aligned}$$

Next, we sum the series (4.8a). We start by re-writing (4.8a) as:

$$V_{put}(u, \tau) = u/2 - \sigma\sqrt{\tau/(2\pi)} S,$$

where S designates the sum in (4.8a). The sum S has been worked out in [BBEoNR53b], p.135, 9.2(4):

$$S = \frac{|z|}{2\sqrt{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(z^2/2)^{k-1/2}}{k-1/2} = \frac{|z|}{2\sqrt{2}} \gamma(-1/2; z^2/2),$$

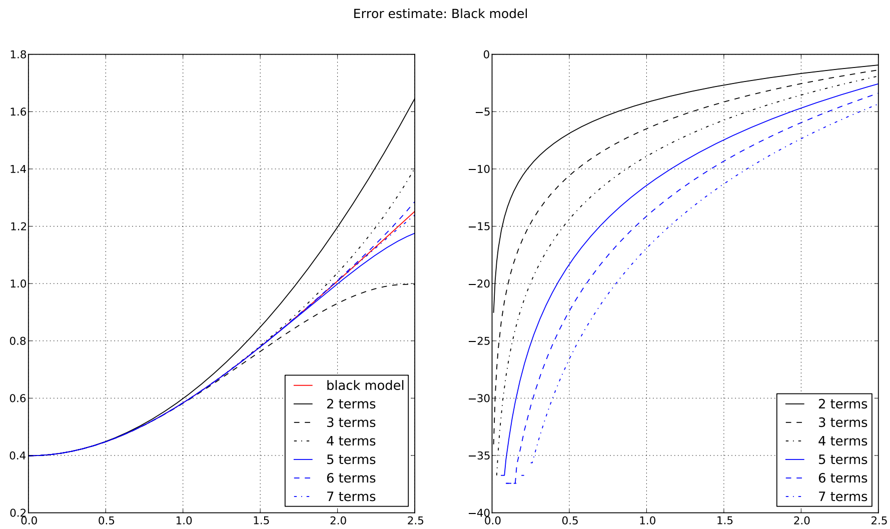


Figure 4.1: Exact Bachelier model in comparison with different number of terms in approximative formula (left). Logarithm of error between approximation and exact Bachelier model (right).

where $\gamma(a,z)$ denotes the incomplete gamma function, [BBEoNR53b]. Recalling the definition of normal moneyness in (4.6a), we conclude that the power series (4.8a) is equal to:

$$V_{put}(u,\tau) = \frac{u}{2} - \frac{|u|}{4\sqrt{\pi}}\gamma(-1/2, z^2/2).$$

But $\Gamma(-1/2) = -2\sqrt{\pi}$, so when $u > 0$ we have

$$V_{put}(u,\tau) = \frac{u}{4\sqrt{\pi}}\left[2\sqrt{\pi} - \gamma(-1/2, z^2/2)\right] = \frac{u}{4\sqrt{\pi}}\left[4\sqrt{\pi} + \Gamma(-1/2, z^2/2)\right].$$

Similarly, when $u < 0$ we have:

$$V_{put}(u,\tau) = \frac{|u|}{4\sqrt{\pi}}\left[-2\sqrt{\pi} - \gamma(-1/2, z^2/2)\right] = \frac{|u|}{4\sqrt{\pi}}\Gamma(-1/2, z^2/2).$$

To see that put price represented by incomplete gamma function is continuous at $u = 0$, we need to verify that $|z|\Gamma(-1/2, z^2/2)$ goes to $2\sqrt{2}$ as z approaches 0 from both, the left and the right. This follows from the expansion of S:

$$z\Gamma(-1/2, z^2/2) \sim |z|\Gamma(-1/2) + |z|\{2\sqrt{2}/z + 2z/\sqrt{2} + O(z^3)\}, \ |z| \to 0.$$

Therefore we conclude that the price of the put option equals $\sigma\sqrt{\tau/2\pi}$ at $u = 0$. The power series (4.8a), (4.8b) is written in terms of the incomplete Gamma function as follows:

$$V_{put}(u,\tau) = u^+ + |u|\Gamma(-1/2, z^2/2)/(4\sqrt{\pi}), \qquad (4.9a)$$

$$V_{call}(u,\tau) = u^- + |u|\Gamma(-1/2, z^2/2)/(4\sqrt{\pi}), \qquad (4.9b)$$

where u^+ and u^- designates the positive and negative parts, respectively, of u. The option prices given by the latter expressions are differentiable everywhere on the real line.

It can be easily verified by integrating (4.7c) by parts that:

$$I_b(z) = z^+ + |z|\Gamma(-1/2, z^2/2)/(4\sqrt{\pi}). \tag{4.10}$$

Clearly, the term $|u|\Gamma(-1/2, z^2/2)/(4\sqrt{\pi})$ corresponds to the time value of the European option in the normal model. This term can be re-written in terms of the moneyness as $\sigma\sqrt{\tau}\xi(z)$, where:

$$\xi(z) \equiv |z|\Gamma(-1/2, z^2/2)/(4\sqrt{\pi}). \tag{4.11}$$

We refer to the function $\xi()$ as the *reduced time value* of the European option.

The function $I_b(z)$ can be represented in terms of integral representation:

$$I_b(z) = -z + \frac{1}{\pi}\int_0^{\pi/2} \sin v\Gamma\left(1/2; [z^2/(2\sin v)^2]\right) \mathrm{d}v, \tag{4.12}$$

and z is normal moneyness.

To establish the formula (4.12), we start by proving that:

$$\tfrac{1}{\sqrt{2}}\phi(x/\sqrt{2}) = \frac{x}{2\pi}\int_0^{\pi/2} \exp\left\{-x^2/(4\sin^2 y)\right\}\frac{\mathrm{d}y}{\sin^2 y},$$

where $\phi()$ is the density of the standard normal distribution. We use the substitution $y = \arcsin\sqrt{x^2/(4z)}$ in the right-hand side of the latter relation, and using $\mathrm{d}y = -x/(4z\sqrt{z - x^2/4})\mathrm{d}z$ we get:

$$\frac{1}{2\pi}\int_{x^2/4}^{\infty} \frac{e^{-z}}{\sqrt{z - x^2/4}}\mathrm{d}z = \frac{1}{2\pi}e^{-x^2/4}\Gamma(1/2),$$

where we have substituted for $u = z - x^2/4$. Clearly, then,

$$\tfrac{1}{\sqrt{2}}\phi(x/\sqrt{2}) = \frac{1}{\pi}\int_0^{\pi/2} \tfrac{\partial}{\partial x}\left(-\exp\{-x^2/(4\sin^2 y)\}\right)\mathrm{d}y.$$

To show (4.12), we proceed in the same way as in proof of Proposition 4.2.1.

4.2 Lévy Flight Model

In this section we extend the Bachelier model to the stable laws, we establish the pricing formulas, and give the representation of option prices by its power series and its asymptotic expansions.

Let us outline the motivation for this work. We assume that the forward price of asset X follows a Markovian stochastic process $X = \{X(t), 0 \le t \le T\}$; see e.g. [Fel71]. For $0 < t_0 < t_1 \le \tau$ we define $p(y, t_1; x, t_0)$ to be the probability density at $X(t_1) = y$, given that $X(t_0) = x$ at the earlier time t_0. Following Bachelier, we assume that it is overwhelmingly probable that the spread $|X(t) - x|$ remains much smaller than the current value, x, over the time scales of interest. We also assume that these increments are independent of the absolute price level x. It then seems natural to model X as a stationary process with independent increments. These assumptions have been discussed in more detail by, e.g., [Man67, MS98]. Under these conditions, the probability density is a function only of the difference $y - x$ and the time lapse $\tau = t_1 - t_0$,

$$p(y, t_1; x, t_0) = g(y - x, \tau), \tag{4.13}$$

and does not depend on x, y, t_0, t_1 separately. In other words, the Markov process X is time-homogeneous and stationary. The Chapman-Kolmogorov equation now states that the probability density satisfies:

$$g(y, \tau) = \int_{-\infty}^{\infty} g(z, t) g(y - z, \tau - t) \mathrm{d}z, \tag{4.14}$$

for any $t \in (0, \tau)$, since the random process has to be at *some* state z at time t. After taking the Fourier transform:

$$\varphi(k, \tau) = E\left\{e^{ik[X(\tau) - x]}\right\} = \int_{-\infty}^{\infty} e^{iky} g(y, \tau) \mathrm{d}y, \tag{4.15}$$

the Chapman-Kolmogorov equation becomes:

$$\varphi(k, \tau) = \varphi(k, t)\varphi(k, \tau - t) \qquad \text{for each } 0 \leq t \leq \tau. \tag{4.16}$$

This is a functional equation; its only smooth solutions are:

$$\varphi(k, \tau) = e^{-\tau A(k)}, \tag{4.17}$$

see, e.g., [Acz66].

Following the results of the empirical studies carried out by [MS95], we now make one more assumption about the density $g(\tau, y)$. We assume that it has no intrinsic time scales or space scales; i.e., we assume that it is self-similar:

$$g(y, \tau) = \frac{1}{\tau^{1/\alpha}} f(y/\tau^{1/\alpha}), \tag{4.18}$$

for some parameter $\alpha > 0$. But now the Fourier transform of (4.15) yields:

$$\varphi(k, \tau) = \int_{-\infty}^{\infty} e^{izk\tau^{1/\alpha}} f(z) \mathrm{d}z, \tag{4.19}$$

showing that φ is a function only of the combination $k\tau^{1/\alpha}$. From (4.17), then, φ must be of the form

$$\varphi(k, \tau) = \begin{cases} e^{-b\tau k^\alpha} & \text{for } k > 0, \\ e^{-\bar{b}\tau |k|^\alpha} & \text{for } k < 0. \end{cases} \tag{4.20}$$

Here b and $\bar{b}$ must be complex conjugates in order for the probability density $g(y, \tau)$ to be real. Inverting the Fourier transform yields:

$$g(y, \tau) = \frac{1}{\pi} \mathrm{Re} \int_0^\infty e^{-iky} e^{-b\tau k^\alpha} \mathrm{d}k = \frac{1}{\pi\tau^{1/\alpha}} \mathrm{Re} \int_0^\infty e^{-iky/\tau^{1/\alpha}} e^{-bk^\alpha} \mathrm{d}k, \tag{4.21a}$$

and so

$$f(z) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty e^{-ikz}e^{-bk^\alpha}\mathrm{d}k. \qquad (4.21b)$$

We recognize that $f(z)$ is the probability density of a stable law with stability parameter α; this makes the process $X(t)$ a Lévy Flight.

Therefore, for τ being the expiry date of the option, and x being the current market price of the asset, we are assuming that the terminal risk-neutral probability density $g(y, \tau|x, 0)$, where $y = X(\tau)$, satisfies

$$g(y-x,\tau) = \frac{1}{\tau^{1/\alpha}}f([y-x]/\tau^{1/\alpha}) \equiv \frac{1}{\tau^{1/\alpha}}f([y-x]/\tau^{1/\alpha};\alpha,p_1,c), \qquad (4.22)$$

where $f(v;\alpha,p_1,c)$ is density of stable distribution. We refer to (4.22) as a *stable terminal risk-neutral density.* Recalling (4.1a), (4.1b), and using scaling property of stable law (2.23a), we compute the fair price of the European put:

$$V_{put}(x,\tau) = \int_{-\infty}^{K} \tfrac{K-y}{(c\tau)^{1/\alpha}} f\left(\tfrac{y-x}{(c\tau)^{1/\alpha}};\alpha,p_1\right)\mathrm{d}y =$$
$$= \int_{-\infty}^{(K-x)/(c\tau)^{1/\alpha}} \left\{\tfrac{K-x}{(c\tau)^{1/\alpha}} - v\right\} f(v;\alpha,p_1)\mathrm{d}v.$$

Analogously, for the fair price of European call option we have:

$$V_{call}(x,\tau) = \int_{K}^{\infty} \tfrac{y-K}{(c\tau)^{1/\alpha}} f\left(\tfrac{y-x}{(c\tau)^{1/\alpha}};\alpha,p_1\right)\mathrm{d}y =$$
$$= \int_{(K-x)/(c\tau)^{1/\alpha}}^{\infty} \left\{v - \tfrac{K-x}{(c\tau)^{1/\alpha}}\right\} f(v;\alpha,p_1)\mathrm{d}v.$$

Similarly to the normal model, we can re-write these prices more simply by introducing:

$$z \equiv (K-x)/(c\tau)^{1/\alpha}, \qquad (4.23a)$$

where we refer to z as the *levy moneyness*, the "moneyness" for stable laws. The value of the short position in the forward contract in terms of the levy moneyness is:

$$u = K - x = z\,(c\tau)^{1/\alpha}. \qquad (4.23b)$$

For $\alpha = 2$, the levy moneyness equals the normal moneyness.

In terms of the levy moneyness, the value of the European put and call options is:

$$V_{put}(u,\tau) = (c\tau)^{1/\alpha}\int_{-\infty}^{z}(z-v)\,f(v;\alpha,p_1)\mathrm{d}v, \qquad (4.24a)$$
$$V_{call}(u,\tau) = (c\tau)^{1/\alpha}\int_{z}^{\infty}(v-z)\,f(v;\alpha,p_1)\mathrm{d}v. \qquad (4.24b)$$

Note that the option prices depend only on the difference $u = K - x$. For brevity we will refer to option contracts with stable terminal risk-neutral densities as *levy options.*

As a consequence of the self-similarity of the stable terminal risk-neutral density, the European option prices in the Lévy Flight model can be expressed in forms similar to (4.7a), (4.7b) introduced for normal model.

Lemma 4.2.1. *For European put and call options with stable terminal risk-neutral density $g(v,\tau) = f(v;\alpha,p_1,c\tau)$, the option prices are given by:*

$$V_{put}(u,\tau) = c\tau^{1/\alpha} I_l(z,p_1), \tag{4.25a}$$
$$V_{call}(u,\tau) = c\tau^{1/\alpha} I_l(-z,p_2), \tag{4.25b}$$

where $I_l(z,p_1)$ is:

$$I_l(z,p_1) = \int_{-\infty}^{z} F(v;\alpha,p_1)\mathrm{d}v, \tag{4.25c}$$

and $F(v;\alpha,p_1)$ is the probability distribution function of the stable law with parameters α, p_1. Here z is the levy moneyness of (4.23a).

Proof. We start with establishing relations (4.25a) and (4.25c) from (4.24a). Integrating (4.24a) by parts yields:

$$\int_{-\infty}^{z} (z-v) f(v;\alpha,p_1)\mathrm{d}v = \lim_{z\to-\infty} zF(z;\alpha,p_1) + \int_{-\infty}^{z} F(v;\alpha,p_1)\mathrm{d}v,$$

The limit equals 0 because $F(z;\alpha,p_1) = O(|z|^{-\alpha})$ as $z \to \infty$, and since $1 < \alpha \leq 2$; see e.g. [Fel71].

Establishing (4.25b) from (4.24b) is similar. Q.E.D

As a consequence of reflection property of stable law, we have:

$$I_l(-z,p_2) = \int_{z}^{\infty} \left[1 - F(v;\alpha,p_1)\right] \mathrm{d}v. \tag{4.26}$$

Note that if $p_1 = p_2$, i.e., if the stable law is symmetric, the function $I_l(z,1/2)$ is a natural generalization of $I_b(z)$. In fact, due to the conditions on parameters p_1, p_2, for α approaching 2, $I_l(z,p_1)$ converges to $I_b(z)$.

By using the integral representation (2.26b) for the stable law density, one can derive an integral representation for option prices and their reduced time value. Since the integral representation is given only for $x > 0$, the reflection property (2.23b) is needed when the moneyness z takes negative values. The following proposition gives the resulting pricing formulas.

Proposition 4.2.1. *Consider a European option with a stable terminal risk-neutral density given by $g(v,\tau) = f(v;\alpha,p_1,c\tau)$. Then European put and call prices are given by the formulas*

$$V_{put}(u,\tau) = u^+ + (c\tau)^{1/\alpha}\xi(z;\alpha,p_1), \tag{4.27a}$$

$$V_{call}(u,\tau) = u^- + (c\tau)^{1/\alpha}\xi(z;\alpha,p_1), \tag{4.27b}$$

where z is the levy moneyness z and $\xi(z;\alpha,p_1)$ is the reduced time value.

For any real z the reflection property holds:

$$\xi(z;\alpha,p_1) = \xi(-z;\alpha,p_2). \tag{4.28a}$$

The reduced time value of the European option for $z = 0$ is given by

$$\xi(0; \alpha, p_1) = \frac{1}{\pi}\Gamma(1 - 1/\alpha) \sin p_1\pi, \tag{4.28b}$$

and for $z > 0$ it is given by the integral representation:

$$\xi(z; \alpha, p_1) = \frac{\alpha - 1}{\alpha\pi} \int_0^{p_1\pi} w(v, p_1)\Gamma(1 - 1/\alpha; [z/w(v, p_1)]^{\alpha/(\alpha-1)})\mathrm{d}v, \tag{4.28c}$$

where $w(v, p_1)$ is the auxilliary function:

$$w(v, p_1) \equiv \frac{\sin(\alpha v)}{[\sin(p_1\pi - v)]^{1/\alpha} \sin[(\alpha - 1)v + p_1\pi]^{(\alpha-1)/\alpha}}. \tag{4.28d}$$

Proof. The proof is by direct calculation. We first prove relation (4.28b), and then use Lemma 4.2.1 and (2.34a) to derive the remaining formulas.

We start with relation (4.24b), and recall that $z \equiv (K - x)/(c\tau)^{1/\alpha}$, (4.23a). For at-the-money options, $z = 0$ and (4.24b) becomes:

$$V_{call}(u, \tau) = (c\tau)^{1/\alpha} \int_0^{\infty} vf(v; \alpha, p_1)\mathrm{d}v \equiv (c\tau)^{1/\alpha} I_1.$$

To compute the integral I_1, recall from that:

$$I_1 = \int_0^{\infty} v \left[\frac{1}{\pi}\mathrm{Re} \int_0^{\infty} \exp\{-ikv - k^{\alpha}e^{-i\alpha(p_1-p_2)\pi/2}\}\mathrm{d}k\right] \mathrm{d}v.$$

Although the latter integral is obviously finite, we cannot change the order of integration. So let us consider a sequence of functions $g(v; \varepsilon) = ve^{-\varepsilon v}f(v; \alpha, p_1)$. It can be readily verified that the sequence converges uniformly to a function $vf(v; \alpha, p_1)$ as ε approaches 0. The functions $g(v; \varepsilon)$ are bounded, and integrals over the positive real half-axis exist. The exchange of limit and integral is, however, not straightforward as the integrals are considered over intervals of infinite length. Thus we take a sufficiently large but finite $b > 0$, and split the interval of integration into $[0, b]$ and $[b, \infty)$. The exchange of limit and integral sign is well justified on the first interval as a consequence of uniform convergence and integrability of series $g(v; \varepsilon)$, [Rud76]. We therefore need only to verify that the exchange is possible over $[b, \infty)$:

$$\left|\int_b^{\infty} g(v; \varepsilon)\mathrm{d}v - \int_b^{\infty} vf(v)\mathrm{d}v\right| \leq \int_b^{\infty} |vf(v)||e^{-\varepsilon v} - 1|\mathrm{d}v.$$

The tails of the stable density have the asymptotic behaviour $f(v; \alpha) \sim C|v|^{-1-\alpha}$ for some positive constant C. Integrating by parts, we estimate upper bound of the integral as:

$$C_1|b|^{1-\alpha}(e^{-\varepsilon b} - 1) + \varepsilon \int_b^{\infty} v^{1-\alpha}e^{-\varepsilon v}\mathrm{d}v \leq \varepsilon C_2 + \varepsilon^{\alpha-1}\Gamma(2 - \alpha).$$

So as ε approaches 0, we can make the difference between $g(v; \varepsilon)$ and $vf(v)$ be arbitrarily small.

Let us fix $\varepsilon > 0$ and substitute $\theta = (p_1 - p_2)\pi/2$. Then we have:

$$\begin{aligned}
&\int_0^\infty ve^{-\varepsilon v} v \int_0^\infty \exp\{-ikv - k^\alpha e^{-i\alpha\theta}\}\mathrm{d}k\mathrm{d}v = \\
&\quad = \int_0^\infty ve^{-\varepsilon v} e^{i\theta} \int_0^\infty \exp\{-ike^{i\theta}v - k^\alpha\}\mathrm{d}k\mathrm{d}v = \\
&\quad = e^{i\theta} \int_0^\infty \exp\{-k^\alpha\} \int_0^\infty v\exp\{-(\varepsilon + ie^{i\theta}k)v\}\mathrm{d}v\mathrm{d}k = \\
&\quad = e^{-i\theta} \int_0^\infty \frac{-\exp\{-k^\alpha\}}{(k - ie^{-i\theta}\varepsilon)^2}\mathrm{d}k,
\end{aligned}$$

where we used Fubini's Theorem in the second equality. Verification of the first and the third equality is based on Cauchy's Theorem and changing the contour of integration. Let us start with the first equality. Define the complex valued function $g(z) = \exp\{-izv - z^\alpha e^{-i\alpha\theta}\}$ and the closed loop γ, s.t. $\gamma = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$, where $\gamma_1(t) = t, t \in [r, R]$, $\gamma_2(t) = Re^{it}, t \in [0, \theta]$, $\gamma_3(t) = e^{i\theta}t, t \in [r, R]$ and $\gamma_4(t) = re^{it}, t \in [0, \theta]$. Then by Cauchy Integral Theorem, we have:

$$\int_r^R \exp\{-itv - t^\alpha e^{-i\alpha\theta}\}\mathrm{d}t + \int_{\gamma_2} g\mathrm{d}z = \int_r^R e^{i\theta}\exp\{-ite^{i\theta}v - t^\alpha\}\mathrm{d}t + \int_{\gamma_4} g\mathrm{d}z.$$

Taking r to zero and R to infinity, it is easily seen that the integrals over arcs γ_2 and γ_4 converge to 0 by Jordan's Lemma, and we have established the first equality.

To establish the third equality, we define the complex valued function $g(z) = ze^{-z}$, we denote $a = \mathrm{Re}(\varepsilon + ie^{i\theta}k)$ and $b = \mathrm{Im}(\varepsilon + ie^{i\theta}k)$, and consider the closed loop $\gamma = \gamma_1 + \gamma_2 - \gamma_3$, where $\gamma_1(t) = at, t \in [0, r]$, $\gamma_2(t) = ar + it, t \in [0, br]$ and $\gamma_3(t) = (a + ib)t, t \in [0, r]$. By using Cauchy's Theorem we have:

$$\int_0^r a^2te^{-at}\mathrm{d}t + \int_{\gamma_2} g\mathrm{d}z = \int_0^r (a + ib)^2te^{-(a+ib)t}\mathrm{d}t.$$

Taking r to infinity, we have:

$$(\varepsilon + ie^{i\theta}k)^2 \int_0^\infty t\exp\{-(\varepsilon + ie^{i\theta}k)t\}\mathrm{d}t = \int_0^\infty te^{-t}\mathrm{d}t = 1,$$

and so we established the third equity.

Next we need to verify that:

$$\lim_{\varepsilon\to 0^+} \int_0^\infty \frac{-\exp\{-k^\alpha\}}{(k - ie^{-i\theta}\varepsilon)^2}\mathrm{d}k = \Gamma(1 - 1/\alpha).$$

Under the assumption that we can exchange the limit and integral, we obtain:

$$\int_0^\infty -k^{-2}\exp\{-k^\alpha\}\mathrm{d}k = -\frac{1}{\alpha}\int_0^\infty t^{-1/\alpha-1}e^{-t}\mathrm{d}t.$$

Consider the extension of the Euler representation of the Gamma function to cases where the argument of Gamma function is negative; see e.g. [WW96]. Then the latter integral is finite and equals to $\Gamma(1-1/\alpha)$. It remains to verify that we can exchange limit and integral sign. But,

$$\left|\int_0^\infty \frac{\exp\{-k^\alpha\}}{(k-ie^{-i\theta}\varepsilon)^2}\mathrm{d}k - \int_0^\infty k^{-2}\exp\{-k^\alpha\}\mathrm{d}k \le \int_0^\infty \exp\{-k^\alpha\}\left|\frac{1}{(k-ie^{-i\theta}\varepsilon)^2}-\frac{1}{k^2}\right|\mathrm{d}k.\right.$$

Taking the absolute value of the complex number, using triangle inequality and Holder's inequality, we estimate the integral by majorant:

$$\int_0^\infty k^{-2}\exp\{-k^\alpha\}\frac{2\varepsilon k+\varepsilon^2}{k^2+\varepsilon^2}\mathrm{d}k \le \varepsilon C\int_0^\infty \frac{2k+\varepsilon}{k^2+\varepsilon^2}\mathrm{d}k.$$

Obviously the integral on the right-handside is finite and can be made arbitrarily small.

Let us return to $e^{-i\theta}$, and recall that $p_1+p_2=1$. Substituting this back into θ, we obtain $e^{-i\theta}=e^{i\pi/2-ip_1\pi}$. Taking the real part now yields $I_1=\frac{1}{\pi}\Gamma(1-1/\alpha)\sin p_1\pi$.

The intrinsic value of an at-the-money option is 0, so $I_1=\xi(0;\alpha,p_1)$ and we have established the relation (4.28b).

Because $p_1=1-p_2$, $\sin p_1\pi=\sin p_2\pi$, and thus:

$$\int_0^\infty vf(v;\alpha,p_1)\mathrm{d}v=\int_0^\infty vf(v;\alpha,p_2)\mathrm{d}v.$$

Using relation (2.23b), we can verify that the first moment of strictly stable law with $1<\alpha\le 2$ is 0, and so we conclude that (4.28a) also holds for $z=0$.

We now derive formulas (4.27a), (4.27b) and (4.28c), (4.28d). We assume that $z\neq 0$, where z is levy moneyness (4.23a). We start by deriving the formula for a European call option. Recall the integral representation of the probability distribution function of stable law given by (2.34a). The formula holds for $z>0$, and we can obtain the formula for $z<0$ by using the reflection property (2.23b). Let us assume that $z>0$. We substitute (2.34a) into (4.26) and get:

$$I_1=\frac{1}{\pi}\int_z^\infty\int_0^{p_1\pi}\exp\{-v^{\alpha/(\alpha-1)}u(t,p_1)\}\mathrm{d}t\mathrm{d}v,$$

where the function $u(t,p_1)$ is defined in (2.26c). Because the integral is convergent, we can use Fubini's Theorem to change the order of integration. Let us fix t in the interval $(0,p_1\pi)$ and substitute $k=v^{\alpha/(\alpha-1)}u(t,p_1)$. For clarity we denote $y=z^{\alpha/(\alpha-1)}u(t,p_1)$. Consider $\delta>0$ sufficiently small. The function $u(t,p_1)$ is continuous and bounded everywhere on $(\delta,p_1\pi]$, and $u(t,p_1)=O(t^{-\alpha/(\alpha-1)})$ as $t\to 0$. For t fixed in the interval $(\delta,p_1\pi-\delta)$, we have:

$$h(t):=\int_z^\infty\exp\{-v^{\alpha/(\alpha-1)}u(t,p_1)\}\mathrm{d}v=(1-1/\alpha)[u(t,p_1)]^{1/\alpha-1}\int_y^\infty k^{-1/\alpha}e^{-k}\mathrm{d}k= \\ =(1-1/\alpha)[u(t,p_1)]^{1/\alpha-1}\Gamma(1-1/\alpha,y).$$

Note that $h(t)\sim Ct\Gamma(1-1/\alpha,t^{-\alpha/(\alpha-1)})\sim Ct^{\alpha/(\alpha-1)}\exp\{-t^{-\alpha/(\alpha-1)}\}$ for $t\to 0$. In the neighbourhood of $p_1\pi$, the term $[u(t,p_1)]^{1/\alpha-1}=O([p_1\pi-t]^{-1/\alpha})$ and

$\Gamma(1-1/\alpha, y) \to \Gamma(1-1/\alpha)$. Thus the integrals of $h(t)$ over the intervals $(0, \delta)$, $(\delta, p_1\pi)$, for $1 < \alpha < 2$, are bounded and can be made arbitrarily small. So we have justified the following equality:

$$I_1 = \frac{\alpha-1}{\alpha\pi}\int_0^{p_1\pi}[u(t,p_1)]^{1/\alpha-1}\Gamma(1-1/\alpha, y)\mathrm{d}t,$$

where the integral I_1 is convergent. Let us introduce the auxiliary function $w(v, p_1)$:

$$w(v,p_1) = u(v,p_1)^{1/\alpha-1} = \frac{\sin(\alpha v)}{\sin(p_1\pi - v)]^{1/\alpha}(\sin[(\alpha-1)v + p_1\pi])^{(\alpha-1)/\alpha}}.$$

We re-write the relation for I_1 as:

$$I_1 = \frac{\alpha-1}{\alpha\pi}\int_0^{p_1\pi} w(v,p_1)\Gamma(1-1/\alpha, [z/w(v,p_1)]^{\alpha/(\alpha-1)})\mathrm{d}v, \tag{4.29}$$

where $z > 0$. For $z < 0$, we compute the call option by using put-call parity and (4.25c):

$$I_1 = -z + \frac{1}{\pi}\int_{-\infty}^{z}\int_0^{p_2\pi}\exp\{-|v|^{\alpha/(\alpha-1)}u(t,p_2)\}\mathrm{d}t\mathrm{d}v.$$

This equals

$$I_1 = -z + \frac{1}{\pi}\int_{|z|}^{\infty}\int_0^{p_2\pi}\exp\{-|v|^{\alpha/(\alpha-1)}u(t,p_2)\}\mathrm{d}t\mathrm{d}v,$$

and by proceeding the same way as we did for $z > 0$, we get

$$I_1 = -z + \frac{\alpha-1}{\alpha\pi}\int_0^{p_2\pi} w(v,p_2)\Gamma(1-1/\alpha, [|z|/w(v,p_2)]^{\alpha/(\alpha-1)})\mathrm{d}v. \tag{4.30}$$

Formula (4.29) for $z > 0$ is the price of an out-the-money European call option. Formula (4.30) for $z < 0$ gives the price for an in-the-money European call. This establishes (4.27b). The intrinsic value of an out-the-money call is 0 when the moneyness $z > 0$, and so $\xi(z; \alpha, p_1)$ equals (4.29).

To derive the pricing formula for a European put, we start with (4.25a) and (4.25c). For $z < 0$, we readily verify that

$$I_2 = \frac{\alpha-1}{\alpha\pi}\int_0^{p_2\pi} w(v,p_2)\Gamma(1-1/\alpha, [|z|/w(v,p_2)]^{\alpha/(\alpha-1)})\mathrm{d}v. \tag{4.31}$$

For $z > 0$ we re-write the price of the put using put-call parity and (4.26) to obtain:

$$I_2 = z + \int_{-\infty}^{-z} F(v;\alpha,p_2)\mathrm{d}v = z + \int_z^{\infty}[1 - F(v;\alpha,p_1)]\mathrm{d}v.$$

Proceeding as before we get

$$I_2 = z + \frac{\alpha-1}{\alpha\pi}\int_0^{p_1\pi} w(v,p_1)\Gamma(1-1/\alpha, [z/w(v,p_1)]^{\alpha/(\alpha-1)})\mathrm{d}v. \tag{4.32}$$

Formula (4.31) for $z < 0$ is the price of an out-the-money European put. The price of an in-the-money European put is given by formula (4.32). So we have established (4.27a). Q.E.D

For at-the-money levy European options,

$$V_{put}(0,\tau) = V_{call}(0,\tau) = \frac{1}{\pi}(c\tau)^{1/\alpha}\Gamma(1-1/\alpha)\sin p_1\pi, \tag{4.33}$$

as a consequence of (4.28a) and (4.28b).

Comparing the formulas (4.9a), (4.9b) and (4.27a), (4.27b) gives the generalization of I_b in (4.10):

$$I_l(z,p_1) = z^+ + \xi(z;\alpha,p_1). \tag{4.34a}$$

Written this way, put-call parity is obvious, since $I_l(z,p_1) - I_l(-z,p_2) = z$.

By combining expressions (4.25c) and (4.34a), we get:

$$I_l(z,p_1) = \xi(0;\alpha,p_1) + \int_0^z F(v;\alpha,p_1)\mathrm{d}v, \tag{4.34b}$$

which is the absolute difference between the option price and its at-the-money value in terms of levy moneyness.

By setting $\alpha = 2$ and choosing the only admissible parameter $p_1 = 1/2$, we have an integral representation for the reduced time value of the normal model:

$$\xi(z;2,1/2) = \frac{1}{\pi}\int_0^{\pi/2} \sin v\Gamma(1/2;[z/(2\sin v)]^2)\mathrm{d}v, \tag{4.35}$$

where z is clearly the normal moneyness. See formula (4.12).

Evaluating the integral in (4.28c) gives the most efficient numerical method for calculating the option values for general strikes. For strikes near the money, however, one can expand the option prices in a Taylor series in terms of the levy moneyness z. When $z > 0$, series (2.25a) yields the following formula for the price of a European in-the-money put option:

$$\begin{aligned} V_{put}(u,\tau) &= p_2 u + \frac{(c\tau)^{1/\alpha}}{\alpha\pi}\sum_{k=0,k\neq 1}^{\infty}\frac{\Gamma[(k-1)/\alpha]}{\Gamma(k+1)}\sin[p_2\pi(k-1)]z^k = \\ &= p_2(K-x) + (c\tau)^{1/\alpha}\Gamma(1-1/\alpha)\frac{\sin p_2\pi}{\pi} + \\ &+ \Gamma(1+1/\alpha)\frac{\sin p_2\pi}{2\pi}\frac{(K-x)^2}{(c\tau)^{1/\alpha}} + \cdots . \end{aligned} \tag{4.36a}$$

Since this expansion converges uniformly in every bounded region, it also gives the price of the put option when $z < 0$; i.e. when the put is out-of-the-money.

By Put-Call parity, one obtains the price for the European call option:

$$\begin{aligned} V_{call}(u,\tau) &= -p_1 u + \frac{(c\tau)^{1/\alpha}}{\alpha\pi}\sum_{k=0,k\neq 1}^{\infty}\frac{\Gamma[(k-1)/\alpha]}{\Gamma(k+1)}\sin[p_2\pi(k-1)]z^k = \\ &= p_1(x-K) + (c\tau)^{1/\alpha}\Gamma(1-1/\alpha)\frac{\sin p_2\pi}{\pi} + \\ &+ \Gamma(1+1/\alpha)\frac{\sin p_2\pi}{2\pi}\frac{(x-K)^2}{(c\tau)^{1/\alpha}} + \cdots . \end{aligned} \tag{4.36b}$$

This expansion is also valid for both $z < 0$ and $z > 0$.

Proof. Let us derive the power series (4.36a), (4.36b). Recall that (4.34b) gives the price of a European call option as an integral of the probability distribution of the stable law and (2.35a) gives the power series expansion of the probability distribution function of the stable law. As the series (2.35a) is absolutely and uniformly convergent, we can integrate it term-by-term over the finite integral. Let us assume that $z > 0$. Then by (4.27b) and (4.34b) we have

$$V_{call}(u,\tau) = (c\tau)^{1/\alpha} \int_0^{-z} F(v;\alpha,p_2)\mathrm{d}v + V_{atm}(0,\tau) =$$
$$= (c\tau)^{1/\alpha} \int_0^{z} [F(v;\alpha,p_1) - 1]\mathrm{d}v + V_{atm}(0,\tau) \equiv (c\tau)^{1/\alpha} I_1 + V_{atm}(0,\tau).$$

The computation of I_1 is straightforward:

$$I_1 = -z + p_2 z + \frac{1}{\alpha\pi} \int_0^z \sum_{k=1}^{\infty} \frac{\Gamma(k/\alpha)}{\Gamma(k+1)} \sin(p_2 k\pi) v^k \mathrm{d}v =$$
$$= -p_1 z + \frac{1}{\alpha\pi} \sum_{k=1}^{\infty} \frac{\Gamma(k/\alpha)}{\Gamma(k+2)} \sin(p_2 k\pi) z^{k+1}.$$

Adding the value of the at-the-money option we have:

$$V_{call}(u,\tau) = p_1(x-K) + \frac{(c\tau)^{1/\alpha}}{\alpha\pi} \sum_{k=0,k\neq 1}^{\infty} \frac{\Gamma[(k-1)/\alpha]}{\Gamma(k+1)} \sin[p_2\pi(k-1)] z^k,$$

where we omit the term, $k = 1$, as the function $\Gamma(v)$ has a singularity at 0.

For $z < 0$, we again use (4.27b) and (4.34b), we get:

$$V_{call}(u,\tau) = (c\tau)^{1/\alpha} \int_0^{|z|} F(v;\alpha,p_2)\mathrm{d}v + V_{atm}(0,\tau) \equiv (c\tau)^{1/\alpha} I_1 + V_{atm}(0,\tau).$$

By substituting (2.35a) for $F(v;\alpha,p_2)$ and integrating term-by-term, we obtain:

$$I_1 = p_1|z| + \frac{1}{\alpha\pi} \sum_{k=1}^{\infty} \frac{\Gamma(k/\alpha)}{\Gamma(k+2)} \sin(p_1 k\pi) |z|^{k+1}.$$

Adding the value of the at-the-money option, we have for $z < 0$,

$$V_{call}(u,\tau) = p_1(x-K) + \frac{(c\tau)^{1/\alpha}}{\alpha\pi} \sum_{k=0,k\neq 1}^{\infty} \frac{\Gamma[(k-1)/\alpha]}{\Gamma(k+1)} \sin[p_1\pi(k-1)] |z|^k.$$

Note that since $\sin[p_1\pi(k-1)]|z|^k = \sin[p_2\pi(k-1)]z^k$, this series is identical to the power series for $z > 0$. The power series for put options is derived similarly. For $z > 0$, we use (4.34b), obtaining

$$V_{put}(u,\tau) = p_2(K-x) + \frac{(c\tau)^{1/\alpha}}{\alpha\pi} \sum_{k=0}^{\infty} \frac{\Gamma[(k-1)/\alpha]}{\Gamma(k+1)} \sin[p_2\pi(k-1)] z^k.$$

For $z < 0$, we have:

$$V_{put}(u,\tau) = (c\tau)^{1/\alpha} \int_0^{|z|} [F(v;\alpha,p_2) - 1]\mathrm{d}v + V_{atm}(0,\tau),$$

and so the price of the put option for $z < 0$ is given by:

$$V_{put}(u,\tau) = p_2(K-x) + \frac{(c\tau)^{1/\alpha}}{\alpha\pi}\sum_{k=0}^{\infty}\frac{\Gamma[(k-1)/\alpha]}{\Gamma(k+1)}\sin[p_1\pi(k-1)]|z|^k.$$

As above we note that this expansion is identical to the expansion for $z > 0$. Q.E.D

For near-the-money, evaluating the option prices from the power series is more efficient, than using the general integral formulas. These expansions also provide extra intuition. In the first term, $u = K - x$ is the value of a short position in the forward contract, and p_1 and p_2 are the probabilities that $X(\tau) - x$ is positive or negative at the expiry date τ, i.e. $p_2 u = (K-x)\text{Prob}\{K \leq x\}$. The second term, $k = 0$, is the value of an at-the-money option, while the other terms of the power series represent increasingly subtle sensitivities of the at-the-money option values.

We can also use the asymptotic expansions (2.35b) to find the option prices for deep into the money and deep out of the money options.

For large positive values of the levy moneyness z, the call option is deep out of the money, and:

$$\begin{aligned} V_{call}(u,\tau) &\sim (c\tau)^{1/\alpha}\sum_{k=1}^{\infty}\frac{\Gamma(\alpha k-1)}{\Gamma(k+1)}\frac{\sin(\alpha p_2 k\pi)}{\pi z^{\alpha k-1}} = \\ &= (c\tau)\Gamma(\alpha-1)\frac{\sin(\alpha p_2\pi)}{\pi(K-x)^{\alpha-1}} + \tfrac{1}{2}(c\tau)^2\Gamma(2\alpha-1)\frac{\sin(2\alpha p_2\pi)}{\pi(K-x)^{2\alpha-1}} + \cdots . \end{aligned} \tag{4.37a}$$

The first term is proportional to $c\tau$. Similarly, when the levy moneyness $z \gg 1$, puts are deeply in-the-money, and their prices can be obtained by put-call parity:

$$\begin{aligned} V_{put}(u,\tau) &= K - x + V_{call}(u,\tau) \sim K - x + (c\tau)^{1/\alpha}\sum_{k=1}^{\infty}\frac{\Gamma(\alpha k-1)}{\Gamma(k+1)}\frac{\sin(\alpha p_2 k\pi)}{\pi z^{\alpha k-1}} = \\ &= K - x + c\tau\Gamma(\alpha-1)\frac{\sin(\alpha p_2\pi)}{\pi(K-x)^{\alpha-1}} + \cdots . \end{aligned} \tag{4.37b}$$

For the prices of deep out of the money puts we have large negative values of z, and so:

$$\begin{aligned} V_{put}(u,\tau) &\sim (c\tau)^{1/\alpha}\sum_{k=1}^{\infty}\frac{\Gamma(\alpha k-1)}{\Gamma(k+1)}\frac{\sin(\alpha p_1 k\pi)}{\pi|z|^{\alpha k-1}} = \\ &= c\tau\Gamma(\alpha-1)\frac{\sin(\alpha p_1\pi)}{\pi(x-K)^{\alpha-1}} + \tfrac{1}{2}(c\tau)^2\Gamma(2\alpha-1)\frac{\sin(2\alpha p_1\pi)}{\pi(x-K)^{2\alpha-1}} + \cdots . \end{aligned} \tag{4.37c}$$

As before, put-call parity then yields the call price:

$$\begin{aligned} V_{call}(u,\tau) &= x - K + (c\tau)^{1/\alpha}\sum_{k=1}^{\infty}\frac{\Gamma(\alpha k-1)}{\Gamma(k+1)}\frac{\sin(\alpha p_1 k\pi)}{\pi|z|^{\alpha k-1}} = \\ &= x - K + c\tau\Gamma(\alpha-1)\frac{\sin(\alpha p_1\pi)}{\pi(x-K)^{\alpha-1}} + \cdots . \end{aligned} \tag{4.37d}$$

Proof. Next, we derive the asymptotic expansions (4.37a)-(4.37d). Recall that equation (2.35a) gave the asymptotic expansion of the power law distribution function and recall that it is permissible to integrate asymptotic expansions term-by-term, [WW96]. We assume $z > 0$ and use (4.34b) to derive the asymptotic expansion for a European call:

$$V_{call}(u,\tau) \sim \frac{(c\tau)^{1/\alpha}}{\pi} \sum_{k=1}^{\infty} \frac{\Gamma(\alpha k - 1)}{\Gamma(k+1)} \sin(\alpha p_2 k\pi) z^{1-\alpha k},$$

for $z \gg 1$. When $z < 0$, we use relation (4.34b) and (2.33) to arrive at the formula:

$$V_{call}(u,\tau) \sim (x - K) + \frac{(c\tau)^{1/\alpha}}{\pi} \sum_{k=1}^{\infty} \frac{\Gamma(\alpha k - 1)}{\Gamma(k+1)} \sin(\alpha p_1 k\pi) z^{1-\alpha k},$$

which holds for $z \ll -1$.

To derive the price of the put option, we proceed similarly. For $z > 0$, we use relation (4.34b) and integrate series (2.35a), term-by-term, obtaining:

$$V_{put}(u,\tau) \sim (K - x) + \frac{(c\tau)^{1/\alpha}}{\pi} \sum_{k=1}^{\infty} \frac{\Gamma(\alpha k - 1)}{\Gamma(k+1)} \sin(\alpha p_2 k\pi) z^{1-\alpha k},$$

for $z \gg 1$. When $z < 0$, use relation (4.34b) to obtain:

$$V_{put}(u,\tau) \sim \frac{(c\tau)^{1/\alpha}}{\pi} \sum_{k=1}^{\infty} \frac{\Gamma(\alpha k - 1)}{\Gamma(k+1)} \sin(\alpha p_1 k\pi) z^{1-\alpha k},$$

for $z \ll -1$. Q.E.D

Examination of the previous formulas shows that the infinite sums corresponds to the asymptotic expansion of the reduced time value of the European options.

Proposition 4.2.2. *The asymptotic expansion of the reduced time value* $\xi(z;\alpha,p_1)$ *is given by:*

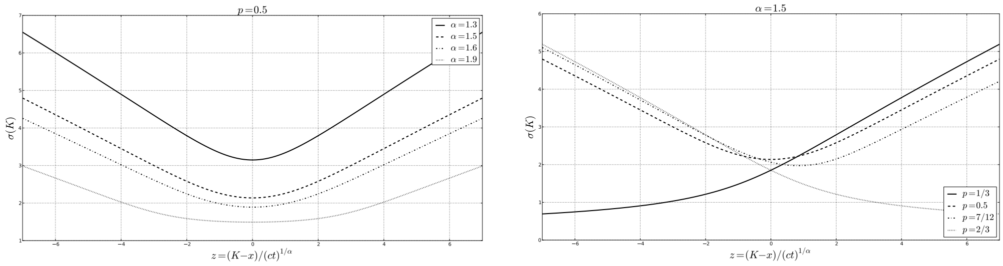
$$\xi(z;\alpha,p_1) \sim \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\Gamma(\alpha k - 1)}{\Gamma(k+1)} \sin(\alpha p_2 k\pi) z^{1-\alpha k}, \tag{4.38}$$

for $z \to \infty$.

Note that the asymptotic expansion for $z \to \infty$ can be obtained from (4.38) and the reflection property (2.23b).

4.3 Volatility Surfaces

On most trading desks, implied normal volatilities σ are used as a more intuitive surrogate for the option prices. So we investigate implied volatility surfaces generated by the option prices under the levy model. Next, we consider the delta hedges for both the normal model and the Lévy Flight model, and we discuss mis-hedging under the normal model.



(a) (b)

Figure 4.2: Implied normal volatilities for the Lévy Flight model. The parameters in z are $t = 1, c = 1$. Picture (a) depicts the relation for symmetric stable distributions varying in α. Picture (b) shows the influence of skewness parameter p_1 for the fixed value $\alpha = 1.5$.

Lets start with Bachelier model. From formulas (4.5a), (4.5b) is easily seen that the call and put prices are both increasing functions of the volatility σ. Because of this one-to-one relation, the pricing function can be inverted, and there is a unique value of σ for which the theoretical option prices in (4.5a), (4.5b) exactly match the market prices. This value of σ is often referred to as the implied normal (or absolute) volatility of the option. It has become customary in many markets to quote option prices in terms of the implied normal volatility σ. If Bachelier's theory was perfect, then the same volatility σ would work for all strikes K and expiries τ. To match actual market prices, however, each option generally requires it's own implied volatility, so σ is a function of both K and τ. One does not need implied volatilities for both puts and calls, because actual market prices and theoretical option prices both satisfy put-call parity given by equation (4.2).

Consider European option values under the levy model. For each strike K and exercise date τ, let $\sigma(K, \tau)$ be the implied normal volatility of this price. The relations for normal option prices (4.9a), (4.9b), and levy option prices (4.27a), (4.27b) imply that in order to compute implied normal volatility, we need to compare the time values of the options. Therefore, by equating (4.11) and (4.28c), we obtain:

$$\frac{|u|}{4\sqrt{\pi}}\Gamma\left(-\tfrac{1}{2}, \frac{u^2}{2\sigma^2\left(K,\tau\right)\tau}\right) = \left(c\tau\right)^{1/\alpha}\xi\left(\frac{u}{\left(c\tau\right)^{1/\alpha}}; \alpha, p_1\right), \tag{4.39}$$

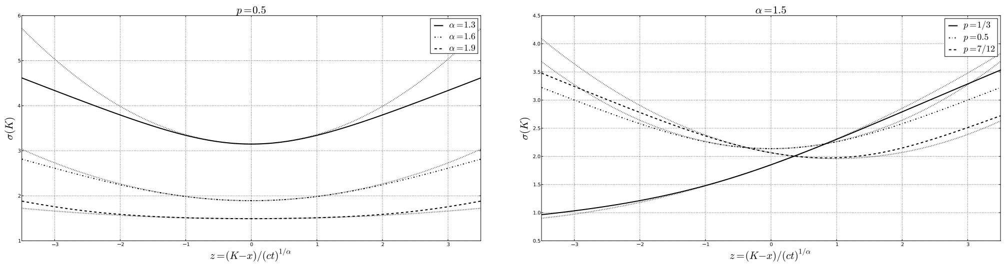
where $u = K - x$ is known difference between K and x, and $\sigma(K, \tau)$ is the unknown implied volatility.

For at-the-money case, i.e. $K = x$, the situation simplifies, as the left hand-side of the equation equals $\sigma(K, \tau)\sqrt{\tau/(2\pi)}$ and the right hand-side equals (4.33). The normal implied volatility is thus

$$\sigma_{atm}(K, \tau) = c^{1/\alpha}\tau^{1/\alpha - 1/2}\Gamma(1 - 1/\alpha)\sin p_1\pi\sqrt{2/\pi} \tag{4.40}$$

for $K = x$.

For $K \neq x$, we re-write equation (4.39) in terms of the levy moneyness z given



(a) (b)

Figure 4.3: Approximative formula for near-the-money options. We compare the approximative formula for S and the numerically computed exact S. The approximative formula is depicted by dotted lines. Picture (a) depicts the relation for symmetric stable distributions; picture (b) shows the influence of the skewness parameter p_1.

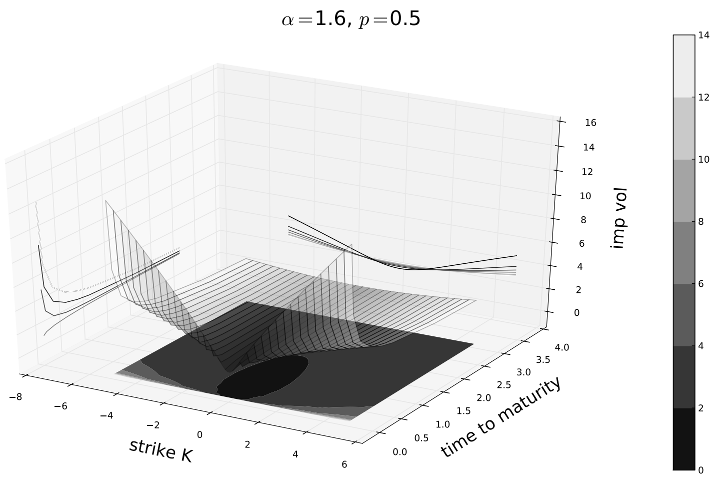


Figure 4.4: Normal volatility surfaces for the Lévy Flight model. Time to maturity ranges from one week to 3 years. This picture depicts the relation for symmetric stable distributions.

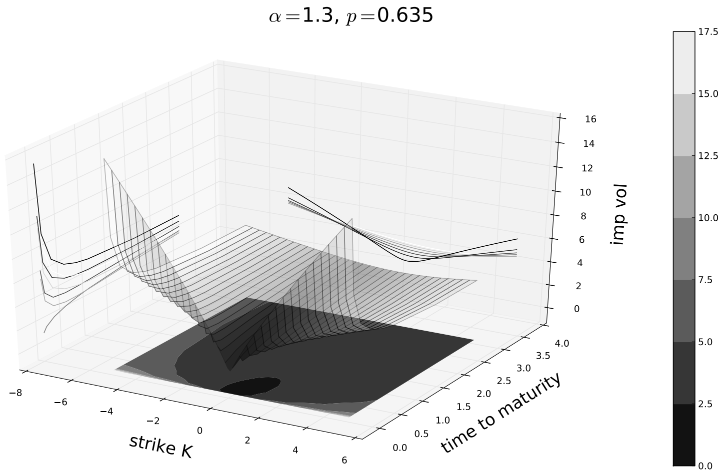


Figure 4.5: Normal volatility surfaces for the Lévy Flight model. Time to maturity ranges from one week to 3 years. Picture shows the influence of the skewness parameter p_1.

in (4.23a). This yields

$$\Gamma\left(-\tfrac{1}{2}, \frac{c^{2/\alpha}\tau^{2/\alpha-1}}{\sigma^2(K,\tau)}\frac{z^2}{2}\right) = \frac{4\sqrt{\pi}}{|z|}\xi\left(z;\alpha,p_1\right). \tag{4.41}$$

We see that the implied normal volatility surface has the functional form:

$$\sigma(K,\tau) = c^{1/\alpha}\tau^{1/\alpha-1/2}S(z;\alpha,p_1). \tag{4.42}$$

Thus the *shape* of the implied volatility surface is independent of the time to expiry, the width of the volatility surface spreads out like $(c\tau)^{1/\alpha}$, and its height increases like $\tau^{1/\alpha-1/2}$. If we denote $Q^{-1}(a,w)$ as the inverse of the incomplete Gamma function $\Gamma(a,z)=w$, we can symbolically express the functional relation $S(z;\alpha,p_1)$ as:

$$S(z;\alpha,p_1) = |z|\left|2Q^{-1}\left(-\tfrac{1}{2}, 4\sqrt{\pi}\xi(z;\alpha,p_1)/|z|\right)\right|^{-1/2}. \tag{4.43}$$

Figure 4.2 shows these universal implied volatility surfaces for different values of α and different values of p_1. The parameter α influences the power law decay of the tails of the terminal risk-neutral density given in (4.22), i.e. $g(y-x;\tau)\sim O(|x|^{-\alpha-1})$. So for heavier tails, the shape of the smile becomes more pronounced. The choice of the parameter p_1 determines the density's asymmetry; for $p_1=2/3$, say, the terminal density is skewed to the right with 2/3 of the probability mass on the positive real line. This causes the smile to be skewed to the right with its minimum located on the right-side of the at-the-money value. In all cases we used Brent's method to solve equation (4.39) for the implied volatility σ.

For near-the-money options, we can expand (4.41) to find:

$$\begin{aligned} S(z;\alpha,p_1) = \sqrt{2/\pi}\,\big\{&\Gamma(1-1/\alpha)\sin(p_2\pi) + \pi\left(p_2-\tfrac{1}{2}\right)z + \\ &+\tfrac{1}{2}\Gamma\left(1+1/\alpha\right)\left[\sin p_2\pi - \tfrac{1}{2}\frac{\alpha\sin\pi/\alpha}{\sin p_2\pi}\right]z^2 + \cdots. \end{aligned} \tag{4.44}$$

Proof. In (4.36a) we found that the value of the European put could be expanded as:

$$V_{put}(u,\tau) = \frac{(c\tau)^{1/\alpha}}{\pi}\left\{\Gamma\left(1-\tfrac{1}{\alpha}\right)\sin p_2\pi + \pi p_2 z + \sum_{k=1}^{\infty}\frac{\Gamma(1+k/\alpha)}{k\Gamma(k+2)}\sin(p_2\pi k)z^{k+1}\right\}$$

under the Levy flight model, where z is the levy moneyness given in (4.23a). In (4.8a) we found that the value of the put is:

$$V_{put}(u,\tau) = \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}}\left\{1+\sqrt{\frac{\pi}{2}}\frac{u}{\sigma\sqrt{\tau}} + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)\,\Gamma\,(k+1)}\left(\frac{u^2}{2\sigma^2\tau}\right)^k\right\}$$

under the Bachelier model. For each K, τ the implied normal volatility $\sigma(K,\tau)$ is the value of σ at which the Bachelier price in (4.8a) matches the levy price in (4.36a). Inspection of the two formulas shows that the implied normal volatility has the form:

$$\sigma(K,\tau) = \frac{(c\tau)^{1/\alpha}}{\sqrt{\tau}}S(z;\alpha,p_1),$$

as noted in (4.42). Substituting this into (4.8a) and equating (4.8a) and (4.36a) yields:

$$\begin{aligned} S + \sqrt{\frac{\pi}{2}}z + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)\,\Gamma\,(k+1)}\frac{z^{2k}}{2^k S^{2k-1}} \\ = \sqrt{\frac{2}{\pi}}\left\{\Gamma(1-1/\alpha)\sin p_2\pi + \pi p_2 z + \sum_{k=1}^{\infty}\frac{\Gamma(1+k/\alpha)}{k\Gamma(k+2)}\sin(p_2\pi k)z^{k+1}\right\}. \end{aligned}$$

Through $O(z^2)$, this yields:

$$\begin{aligned} S = \sqrt{2/\pi}\,\{\Gamma\,(1-1/\alpha)\sin p_2\pi + \pi\left(p_2-\tfrac{1}{2}\right)z \\ + \left[\tfrac{1}{2}\sin(2p_2\pi)\Gamma(1+1/\alpha) - \tfrac{1}{4}\sqrt{2\pi}/S\right]z^2 + \cdots . \end{aligned}$$

Substituting the at-the-money value (4.40) given by $\sqrt{2/\pi}\Gamma(1-1/\alpha)\sin p_2\pi$ for S in the last term, and using the reflection formula:

$$\Gamma(1+1/\alpha)\Gamma(1-1/\alpha) = \frac{\pi/\alpha}{\sin\pi/\alpha},$$

then yields the implied volatility surface:

$$\begin{aligned} S = \sqrt{2/\pi}\,\{\Gamma\,(1-1/\alpha)\sin p_2\pi + \pi\left(p_2-\tfrac{1}{2}\right)z \\ + \tfrac{1}{2}\Gamma(1+1/\alpha)\left[\sin(2p_2\pi) - \tfrac{1}{2}\frac{\alpha\sin\pi/\alpha}{\sin p_2\pi}\right]z^2 + \cdots \end{aligned}$$

through $O(\varepsilon^2)$. Q.E.D

The volatility smile can be also approximated by solving the quadratic and quartic equations, respectively. Denote:

$$s_n \equiv V_{put}(u,\tau) - \tfrac{1}{2}u. \tag{4.45}$$

Then, for the first two terms in power series of Bachelier's model, we have approximation of implied volatility given by:

$$\sigma^2 - s_n\sqrt{\frac{2\pi}{t}}\sigma + \frac{u^2}{2t} = 0. \tag{4.46}$$

The quadratic equation has two solutions:

$$\sigma_{1,2} = s_n\sqrt{\frac{\pi}{2t}} \pm \sqrt{\frac{\pi s_n^2}{2t} - \frac{u^2}{2t}},$$

the first of roots verifies the condition on At-The-Money volatility (4.33). Therefore, we have formula for implied volatility:

$$\sigma = s_n\sqrt{\frac{\pi}{2t}} + \sqrt{\frac{\pi s_n^2 - u^2}{2t}},$$

or the latter can be re-written as:

$$\sigma\sqrt{t} = s_n\sqrt{\pi/2} + \sqrt{\frac{\pi s_n^2 - u^2}{2}}. \tag{4.47}$$

Next, we derive approximation formula for implied volatility, where we consider first three terms of Black model and s_n defined in (4.45). We have relation:

$$\frac{\sigma\sqrt{t}}{\sqrt{2\pi}} + \frac{u^2}{2\sigma\sqrt{2\pi t}} - \frac{u^4}{24\sigma^3\sqrt{2\pi t^3}} = s_n,$$

which we rewrite as:

$$\sigma^4 - s_n\sqrt{\frac{2\pi}{t}}\sigma^3 + \frac{u^2}{2t}\sigma^2 - \frac{u^4}{24t^2} = 0,$$

For solving quartic equation, we follow Abramowitz [AS64], 3.8.3, p.17. We first find real root of cubic equation:

$$y^3 - \frac{u^2}{2t}y^2 + \frac{u^4}{6t^2}y + \frac{u^4(s_n^2\pi - u^2)}{12t^3} = 0.$$

Using notation in Abramowitz [AS64], 3.8.2, p.17, we have:

$$q = \frac{u^4}{36t^2},\ r = \frac{u^4}{6t^3}\left(\frac{5u^2}{36} - \frac{s_n^2\pi}{4}\right),\ q^3 + r^2 = \frac{u^8}{36t^6}\left\{\left[\frac{u}{6}\right]^4 + \left[\frac{5u^2}{36} - \frac{s_n^2\pi}{4}\right]^2\right\}.$$

Because $q^3 + r^2 > 0$, cubic equation has one real root and a pair of complex conjugate roots. The one real root for which we seek is:

$$y_1 = \frac{u^2}{6t} + \sqrt[3]{\frac{5u^6}{216t^3} - \frac{u^4 s_n^2\pi}{24t^3} + \frac{u^4}{6t^3}\sqrt{\left[\frac{u}{6}\right]^4 + \left[\frac{5u^2}{36} - \frac{s_n^2\pi}{4}\right]^2}} + \tag{4.48}$$

$$+ \sqrt[3]{\frac{5u^6}{216t^3} - \frac{u^4 s_n^2\pi}{24t^3} - \frac{u^4}{6t^3}\sqrt{\left[\frac{u}{6}\right]^4 + \left[\frac{5u^2}{36} - \frac{s_n^2\pi}{4}\right]^2}}.$$

We readily check that for $u = 0$, $y_1 = 0$. The four roots of quartic equation comes as a solution of two quadratic equations:

$$v^2 - \left[s\sqrt{\frac{\pi}{2t}} \pm \sqrt{\frac{s^2\pi}{2t} + y_1 - \frac{u^2}{2t}} \right] v + \frac{y_1}{2} \mp \sqrt{\frac{y_1^2}{4} + \frac{u^4}{24t^2}} = 0.$$

By setting $u = 0$ and having condition for At-The-Money volatility, we are left with formula of implied volatility:

$$\sigma = s\sqrt{\frac{\pi}{8t}} + \sqrt{\frac{s^2\pi}{8t} + \frac{y_1}{4} - \frac{u^2}{8t}} +$$
$$\sqrt{\frac{s^2\pi}{4t} + s\sqrt{\frac{s^2\pi^2}{16t^2} + \frac{\pi y_1}{8t} - \frac{\pi u^2}{16t^2}} - \frac{y_1}{4} - \frac{u^2}{8t} + \sqrt{\frac{y_1^2}{4} + \frac{u^4}{24t^2}}}$$

Recall relation (4.48) and denote:

$$\rho = u^2 + \sqrt[3]{5u^6 - 9u^4 s_n^2\pi + u^4\sqrt{u^4 + [5u^2 - 9s_n^2\pi]^2}} + \tag{4.49}$$
$$+ \sqrt[3]{5u^6 - 9u^4 s_n^2\pi - u^4\sqrt{u^4 + [5u^2 - 9s_n^2\pi]^2}},$$

then $y_1 = \rho/(6t)$. We rewrite aproximation formula for implied volatility and substitute for y_1 the later relation:

$$\sigma = s_n\sqrt{\frac{\pi}{8t}} + \sqrt{\frac{s_n^2\pi}{8t} + \frac{\rho}{24t} - \frac{u^2}{8t}} +$$
$$\sqrt{\frac{s_n^2\pi}{4t} + s_n\sqrt{\frac{s_n^2\pi^2}{16t^2} + \frac{\pi\rho}{48t^2} - \frac{\pi u^2}{16t^2}} - \frac{\rho}{24t} - \frac{u^2}{8t} + \sqrt{\frac{\rho^2}{144t^2} + \frac{u^4}{24t^2}}}.$$

We get relation:

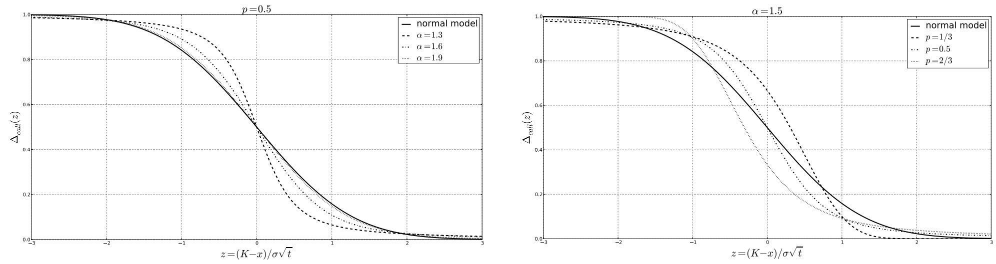
$$\sigma\sqrt{t} = s_n\sqrt{\frac{\pi}{8}} + \sqrt{\frac{s_n^2\pi}{8} + \frac{\rho}{24} - \frac{u^2}{8}} +$$
$$\sqrt{\frac{s_n^2\pi}{4} + s_n\sqrt{\frac{s_n^2\pi^2}{16} + \frac{\pi\rho}{48} - \frac{\pi u^2}{16}} - \frac{\rho}{24} - \frac{u^2}{8} + \sqrt{\frac{\rho^2}{144} + \frac{u^4}{24}}}.$$

4.4 Hedging

Risks are a key concern of trading desks. Indeed, often the main purpose of pricing deals is to understand the risks and hedges, since prices are usually readily obtainable from market quotes. The primary risk of an option is the delta risk, defined as $\mathrm{d}V/\mathrm{d}x$. Equations (4.7a)-(4.7c) show that for Bachelier's model, the delta risks of the put and call options are:

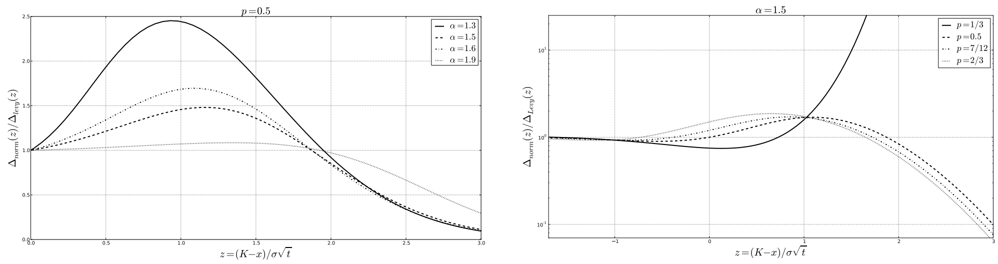
$$\Delta_{put} = \frac{\partial V_{put}}{\partial x} = -\Phi\left(\frac{u}{\sigma\sqrt{\tau}}\right), \tag{4.50a}$$

$$\Delta_{call} = \frac{\partial V_{call}}{\partial x} = \Phi\left(\frac{-u}{\sigma\sqrt{\tau}}\right), \tag{4.50b}$$



(a) (b)

Figure 4.6: Comparison of Δ_{call} for the normal model and the Lévy Flight model. The parameters in z are chosen as $t = 1, \sigma = 1$, where σ is the at-the-money volatility. Picture (a) compares the symmetric stable distributions with different α. Picture (b) shows the influence of the asymmetry parameter for the fixed value $\alpha = 1.5$.



(a) (b)

Figure 4.7: The ratio of Δ_{call}-hedges for the normal model and the Lévy Flight model. The parameters in z are chosen as $t = 1, \sigma = 1$. Picture (a) shows the comparison for symmetric stable distributions for different α. Picture (b) shows the influence of the asymmetry for the fixed value $\alpha = 1.5$. The ratios increase rather quickly due to the heavy tails of the asymmetric distribution. Note that the graphs are plotted with a logarithmic vertical axis.

where $\Phi()$ is the probability distribution function of the standard normal distribution and $u = K - x$. Similarly, formulas (4.25a)-(4.26) show that under the Lévy Flight model, the delta risk for the put option is

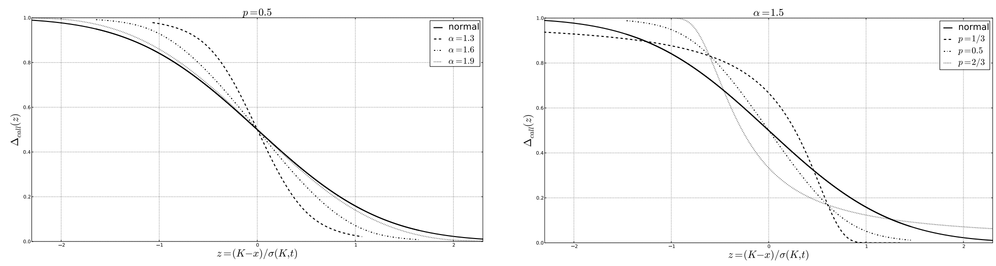
$$\Delta_{put}(z; \alpha, p_1) = \begin{cases} F(-z, \alpha, p_2) - 1 & \text{for } z < 0, \\ -F(z; \alpha, p_1) & \text{for } z > 0, \end{cases} \tag{4.51a}$$

and for the call option is:

$$\Delta_{call}(z; \alpha, p_1) = \begin{cases} F(-z, \alpha, p_2) & \text{for } z < 0, \\ 1 - F(z; \alpha, p_1) & \text{for } z > 0. \end{cases} \tag{4.51b}$$

Here z is levy moneyness and we have used (2.33) to obtain relations for negative moneyness. Relations (4.51a), (4.51b) clearly show the influence of the skewness of the distribution. For the symmetric case $p_1 = 1/2$, the formulas are natural generalisations of (4.50a), (4.50b), the deltas for the normal model.

Figure 4.6 shows the delta risks as a function of the strike K for both the normal and the levy models. We first fixed the time-to-expiry τ and the implied



(a) (b)

Figure 4.8: Comparison of Δ_{call} for the normal model and the Lévy Flight model. The parameters in z are chosen so that $\sigma(K)$ is the implied volatility for strike K. Picture (a) depicts symmetric stable distributions for different α. Picture (b) shows the influence of the asymmetry parameter for the fixed value $\alpha = 1.5$.

volatility σ, and graphed the normal model's delta as a function of the strike K. To compare this with the delta risks of the levy model for different values of α and p_1, for each α and p_1, we used the equation (4.40) to find the levy parameter c_{atm} that would match the levy price of the at-the-money option to the normal model's price of this option,

$$c_{atm} = \left[\sqrt{\pi/2}\frac{\sigma_{atm}(K,\tau)}{\Gamma(1-1/\alpha)\sin p_1\pi}\right]^{\alpha}\tau^{\alpha/2-1}. \tag{4.52}$$

We then plotted the Δ_{call} from equation (4.51b) as a function of the strike K. In figure 4a we graph the deltas for the symmetric distribution; i.e., $p_1 = 0.5$, at different values of α. In figure 4b we graph the deltas for $\alpha = 1.5$ for different values of the asymmetry parameter p_1. Note that near the money, the delta risks for the levy model are nearer to the $\Delta = 0$ and $\Delta = 1$ (all or nothing) delta risks from Bachelier's model. This reverses when we are far enough from the money.

Figure 4.7 shows the same delta, except here we have plotted the ratio of the delta hedges for the normal and levy models. Unless the call is well into the money (and thus Δ is 1), the delta hedges from the levy models are very disparate from the hedges from the normal model.

The actual mis-hedging is more subtle since trading desks use implied volatilities. Suppose the correct option model is the Lévy Flight model. Then for each strike K and expiry τ, the desk first calculates the implied normal volatility σ_N that matches the Bachelier price to the observed market price. Since we are assuming that the market price is given by the Lévy Flight model, this yields relation (4.42). The desk then calculates the delta risk from the Bachelier model using this implied volatility, which yields

$$\Delta_{put}(z;\alpha,p_1) = -\Phi\left(\frac{z}{S(z;\alpha,p_1)}\right), \tag{4.53a}$$

$$\Delta_{call}(z;\alpha,p_1) = \Phi\left(\frac{-z}{S(z;\alpha,p_1)}\right), \tag{4.53b}$$

as opposed to the true delta risks given by (4.51a), (4.51b). Figure 4.8 graphs (4.53b), showing the extent of the mis-hedging.

4.5 Discussion

The experimental data analysis in [MS95] shows that stock prices are modelled accurately by a stable law in the centre and near tails of the distribution; the observed probability density deviates from a stable law only in the extreme tails, where the observations suggest that the tails are exponentially truncated. Our model should accurately reflect the option prices and volatility smiles everywhere except these extreme tails. Consequently, the hedges determined from the stable law should be much defter than hedges determined from Bachelier's model, which should lower trading costs and reduce market friction.

In the extreme tail regions, the stable distribution has heavier tails than the actual distribution, so our formulas should provide upper bounds on the volatility surfaces and yield the conservative risks. In contrast, the Bachelier model has tails that are too thin, so using both models together should bound the risks for deep in the money and out of the money options.

The parameters α and p_1 give more precise information about the risk-neutral density than does the implied normal or log-normal volatilities. The scaling parameter α describes the decay of the tails of the risk-neutral density: for α closer to 1, the tails of the density decay slowly, and large jumps occur frequently; for α closer to 2, the price of the asset behaves more like Brownian motion. Parameter p_1 determines whether the density is skewed to the left or the right, and also gives the probability that each step is to the left or the right. It determines, for example, the probability that the option is profitable, but not the expected profitability.

5. Problems

In this last chapter I would like to formulate a list of some problems which I have not had time to solve. Some of the problems may be easy to answer, some of them require purely brute force techniques rather then elegant solutions. Some problems probably represent interesting research topics to explore in their own right.

There are two main significant results and contributions of this thesis. First is the power-series representation of density of the product of two independent strictly stable random variables, formulated in Propositions 3.4.1-3.4.3. The second is the generalization of Bachelier's model to stable laws presented in chapter four. Therefore, I formulate two different lists, each connected with relevant topic. In the first section are formulated problems connected with products of independent random variables. The section two considers some topics connected with Lévy Flight model.

5.1 Product of Independent Stable Random Variables

For convenience I will refer to the power-series derived in Propositions 3.4.1-3.4.3 as *power-series of stable product density.*

Problem 5.1.1. It is easy to verify that the integral in equation (3.1), where the considered densities are stable densities, is finite at the point 0. Expression for the value of the stable product density at the origin needs to be found.

Problem 5.1.2. Consider the multiplication and division theorems for stable random variables introduced in [Zol57] or discussed in [Zol86], 3.3, p.194. These theorems provide relations between stability and asymmetry parameters for products and ratios of strictly stable random variables. The theorems can be useful in exploring the properties of the power-series derived in Propositions 3.4.1-3.4.3, and hopefully can simplify some of these power-series. As a consequence, one may be able to find relations with the power-series (2.25a), (2.25b) and the power-series of stable product density (consider Proposition 3.4.2 and the scaling property for stable product density).

Problem 5.1.3. The methods in Proofs of Lemmas 3.4.1 and 3.4.2 can be generalized to the product of n independent strictly stable random variables. I suspect that the polygamma functions will occur in the expansions.

Problem 5.1.4. In section 2.3 we discussed the fact that the power-series (2.25a), (2.25b) for densities of stable laws derived by [Ber52], are Taylor Series which are expanded around origin. It is not clear whether the same holds for the power-series of stable product density.

Problem 5.1.5. The computational aspects for the power-series of stable product density has been left without any exploration. I suspect that the convergence of power-series of the product density is faster then for the power-series (2.25a), (2.25b). So, by combining this with the results of Problem 5.1.2, some computational improvement may be gained for the power-series of the density of stable random variables. Using the integral representations for stable densities (2.26b) in relation (3.1), and evaluating this integral numerically, would provide benchmark values.

Problem 5.1.6. Consider discrete schemes for the Mellin transform and the algorithm analogous to the Fast Fourier Transform. One naturally wonders whether this method can be applied effectively to numerical computation of the product of n independent stable random variables.

Problem 5.1.7. Consider Lagrange's Theorem for inverse function, see, e.g. [WW96], p.132. The results of Problem 5.1.3 used in the Lagrange's Theorem could be used to establish expansions for quantile functions of stable laws. The quantile functions are useful in, e.g., estimating the parameters of the stable laws.

Problem 5.1.8. Consider the Chi-squared distribution. Theorem 3.4.3 provides the possibility of generalizing the Chi-squared distribution to the sum of squared strictly stable random variables. I suspect it may prove useful for some problems in Mathematical Statistics.

Problem 5.1.9. Consider the stable Lévy motion $X = \{X(t), t \geq 0\}$ (a natural generalization of Brownian motion). Then the stochastic exponential Z is given by:

$$Z(t) = \prod_{0<s\leq t} [1 + \Delta X(s)] \exp\left\{-\Delta X(s) + \tfrac{1}{2}[X(s)]^2\right\},$$

for $0 < \alpha < 2$. Obviously, $\Delta X(s)$ is a random variable with stable distribution, and so the solution of Problem 5.1.3 may be found helpful in computing the probability distribution of the random variable $Z(t)$.

Problem 5.1.10. It would be interesting to examine the generator of the semigroup, in which the probability distribution is given by product of two or more independent random variables. In other words, what is the form of the Kolmogorov equations for the distribution of products of independent random variables.

5.2 Option Pricing under Lévy Flight Model

Problem 5.2.1. Clearly, the partial integro-differential equation and related boundary value problem for European Call option is as follows:

$$\begin{aligned} u_\tau(\tau, x) = c_1 \int_0^\infty \Big[u(\tau, x+y) - u(\tau, x) - y u_x(\tau, x)\Big] \frac{\mathrm{d}y}{y^{1+\alpha}} + \\ + c_2 \int_{-\infty}^0 \Big[u(\tau, x+y) - u(\tau, x) - y u_x(\tau, x)\Big] \frac{\mathrm{d}y}{|y|^{1+\alpha}} \end{aligned}$$

with $u(0, x) = (x - K)^+$ and $\tau = t_{ex} - t$ where $\tau \in [0, t_{ex}]$. The operator on the right-handside of the equation corresponds to operator which generates the stable semigroup, see, e.g. [Fel71]. Using the result of Kolmogorov Backward equation, the fundamental solution of the above equation is the density of stable distribution. The solutions to this particular problem are given in Proposition 4.2.1. The boundary condition $u(0, x)$ can be generalized and the problem can be investigated either by using analytically or by using numerical methods.

Problem 5.2.2. The computational aspects of option pricing formulas developed in Proposition 4.2.1 deserves a detailed numerical investigation.

Problem 5.2.3. The calibration procedure of the volatility surfaces to market data needs to be established.

A. Mellin Transform

Throughout the chapter three, we make use of the integral transform called Mellin transform and its inverse operation, which allows for representations of functions via Mellin-Barnes integrals. A rather formalized method for the evaluation of the integrals of Mellin-Barnes type is based on the Residue Theorem. The Mellin-Barnes integral involves integrand containing one or more gamma functions. In this Appendix we give overview on gamma function, digamma function, and incomplete gamma function. Then we recall basic properties of Mellin transform and introduce Mellin-Barnes integral and Fox's H-function.

A.1 Gamma Function and Related Functions

Gamma function appeared as an effort for natural extension of factorial. To recall its multiplicative origin we give following definition:

$$\Gamma(z) = 1/\Delta(z), \tag{A.1}$$

where $z \in$ and $\Delta(z)$ is Weierstrass function defined as:

$$\Delta(z) = ze^{\gamma z} \prod_{k=1}^{\infty} (1 + z/k) e^{-z/k},$$

where $\gamma = 0.5772156\ldots$ stands for Euler constant introduced such that $\Delta(1) = 1$. Gamma function satisfies the functional equation:

$$\Gamma(z+1) = z\Gamma(z), \tag{A.2}$$

with initial condition $\Gamma(1) = 1$. Other useful property is, that $\Gamma(z)$ is holomorphic and non-vanishing for all $z \in \setminus\{0, -1, -2, \ldots\}$ and has simple poles at every negative integer. Using functional equation, we can easily determine the residue:

$$\operatorname*{res}_{z=-n} \Gamma(z) = \lim_{z\to -n} (z+n)\Gamma(z) = \lim_{z\to -n} \frac{\Gamma(z+n+1)}{z(z+1)\ldots(z+n-1)} = \\ = \frac{\Gamma(1)}{-n(-n+1)\ldots(-1)} = \frac{(-1)^n}{n!}.$$

Gamma function is meromorphic on the entire complex plane. It is of particular interest to mention that $\frac{1}{\Gamma(z)}$ is holomorphic everywhere in complex domain, see [Rem98],2.2, p.39.

Let us recall two important formulae for gamma function: reflection and duplication. Reflection formula is given by:

$$\Gamma(z)\Gamma(1-z) = -z\Gamma(z)\Gamma(-z) = \frac{\pi}{\sin(z\pi)}, \tag{A.3}$$

see [Rem98], 2.2,p.40 for derivation. One can readily derive:

$$\Gamma(1/2+z)\Gamma(1/2-z) = \frac{\pi}{\cos(z\pi)}, \tag{A.4}$$

The other useful relation is Duplication formula:

$$\Gamma(2z) = \frac{1}{2\sqrt{\pi}} 2^{2z}\Gamma(z)\Gamma(z+1/2). \tag{A.5}$$

Duplication formula is a special case of Gauss' Multiplication formula which for k being integer gives relation:

$$\Gamma(kz) = (2\pi)^{(1-k)/2} k^{kz-1/2} \prod_{j=0}^{k-1} \Gamma(z + j/k). \tag{A.6}$$

For more details see [Rem98], 2.2, p.45 or [AS64], 6.1.17-6.1.20,p.256.

Gamma function is often defined by Euler's integral:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t, \tag{A.7}$$

where $z \in, \mathrm{Re} z > 0$. The path of integration is a positive real half-axis and t^{z-1} is considered to have its principal value. The integral on the right-hand side converges uniformly and absolutly to $\Gamma(z)$ in every strip $\{a_1 \leq \mathrm{Re} z \leq a_2\}$, where $0 < a_1 < a_2 < \infty$, see [Rem98] 2.3,p.49.

The Stirling's formula is rather helpful when investigating asymptotics and growth for large values of Gamma function:

$$\Gamma(z) = \sqrt{2\pi} z^{z-1/2} e^{-z} e^{\mu(z)}, \tag{A.8}$$

for $z \in^-$. We denote $z = x + iy = re^{i\theta}$ and compute the estimate:

$$|\Gamma(x+iy)| \leq \sqrt{2\pi} |z^{z-1/2} e^{-z}| = \sqrt{2\pi} r^{x-1/2} e^{-x-y\theta}. \tag{A.9}$$

For $|y| \to \infty$, the gamma function decays exponentially:

$$|\Gamma(x+iy)| \sim \sqrt{2\pi} |y|^{x-1/2} e^{-\frac{\pi}{2}|y|}. \tag{A.10}$$

For proof see e.g. [Rem98] 2.4 p.60.

Having Euler's integral for Gamma function, it is a very natural idea to investigate integrals:

$$\Gamma(z) = \int_0^x t^{z-1} e^{-t} \mathrm{d}t + \int_x^\infty t^{z-1} e^{-t} \mathrm{d}t,$$

where $x > 0$. We denote

$$\gamma(z,x) = \int_0^x t^{z-1} e^{-t} \mathrm{d}t, \qquad \Gamma(z,x) = \int_x^\infty t^{z-1} e^{-t} \mathrm{d}t. \tag{A.11}$$

Functions $\Gamma(z,x), \gamma(z,x)$ are called incomplete gamma functions and play significant role in the theory of special functions. Obviously:

$$\Gamma(z,x) = \Gamma(z) - \gamma(z,x). \tag{A.12}$$

In connection with Gamma function and hypergeometric functions is used notation of Pochhammer symbol. Let us write:

$$(a)_n = a(a+1)(a+2)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{A.13}$$

and in particular $(a)_0 = 1$. It can be readily verified, that:

$$(a-n)_n = (-1)^n(1-a)_n, \qquad \Gamma(a-n) = \Gamma(a)(-1)^n/(1-a)_n. \tag{A.14}$$

The generalized hypergeometric function is defined as:

$${}_pF_q[a_1, a_2, \ldots, a_p; b_1, \ldots, b_q; z] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \ldots (a_p)_k}{(b_1)_k(b_2)_k \ldots (b_q)_k} \frac{z^k}{k!}, \tag{A.15}$$

where any of parameters $a_1, \ldots, a_p, b_1, \ldots, b_q$ and variable z can be of real or complex value. We further require each parameter b_k in denominator to be positive integer, otherwise the hypergeometric function would not be well defined.

A.2 Mellin Transform

Next, we recall the definion of Mellin transform.

Definition A.2.1. Consider mapping f, which is absolutly integrable on all closed subintervals of a positive real axis, and for which exists strip $a_1 < \mathrm{Re}\, s < a_2$ in complex plane, such that $x^{s-1}f(x)$ is absolutly integrable with respect to x over $(0, \infty)$. The Mellin transform asigns a complex valued function $F(s)$ to each locally integrable f as follows:

$$M[f; s] := F(s) = \int_0^{\infty} f(x)x^{s-1}\mathrm{d}x. \tag{A.16}$$

The definition guarantees the existence of convergent integral $M[f; s]$ on the vertical strip of complex plane. Obviously for $\varepsilon > 0$ and f satisfying $f(x) = O(x^{-a_1-\varepsilon})$ for x approaching origin, $f(x) = O(x^{-a_2+\varepsilon})$ for x going to infinity, the function f is absolutly integrable and we call $a_1 < \mathrm{Re}\ s < a_2$ the strip of definition of Mellin transform. The Mellin transform is therefore defined by a pair: a function $F(s)$ and its strip of definition, where integral F converges.

Example A.2.1. Let us compute Mellin transform of exponential function:

$$f(x) = e^{-cx} \text{ for } c > 0.$$

By applying Definition A.2.1 we obtain:

$$M[e^{-cx}; s] = \int_0^{\infty} x^{s-1}e^{-cx}\mathrm{d}x = c^{-s}\Gamma(s).$$

Because Gamma function is analytic for $\mathrm{Re}\ s > 0$, we conclude that strip of definition is $(0, \infty)$.

Example A.2.2. Consider:

$$f(x) = e^{-x^a} \text{ for } a > 0.$$

Substituting $x = y^{1/a}$ in Definition A.2.1, we have:

$$M[e^{-x^a}; s] = \int_0^{\infty} x^{s-1}e^{-x^a}\mathrm{d}x = \frac{1}{a}\int_0^{\infty} y^{s/a-1}e^{-y}\mathrm{d}y = \frac{1}{a}\Gamma(s/a).$$

The strip of definition is obviously again $(0, \infty)$.

Example A.2.3. Next, let us compute Mellin transform of trigonometric functions $\cos(cx), \sin(cx)$. Using Euler's integral, we have:

$$M[e^{icx}; s] = \int_0^\infty x^{s-1} e^{icx} \mathrm{d}x = c^{-s}\Gamma(s) e^{i\frac{\pi}{2}s}.$$

To justify the later equity, we consider $g(z) = z^{s-1}e^{iz}$ for $z \in$, Re$z > 0$ and $s = a + ib \in$. We change the contour of integration as such: for $0 < \delta < r < \infty$ we take a loop $\gamma = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$, where $\gamma_1(t) = t, t \in [\delta, r]$, $\gamma_2(t) = re^{it}, t \in [0, \pi/2]$, $\gamma_3(t) = it, t \in [\delta, r]$ and $\gamma_4(t) = \delta e^{it}, t \in [0, \pi/2]$. Using Cauchy theorem on path integral $\int_\gamma g\mathrm{d}z$ and estimating $|\int_{\gamma_2} g\mathrm{d}z| \le \pi/2 r^a e^{-c_1 r}$, for r going to infinity we obtain:

$$\int_0^\infty t^{s-1} e^{ict} \mathrm{d}t = e^{i\frac{\pi}{2}s} \int_0^\infty t^{s-1} e^{-ct} \mathrm{d}t.$$

Taking real and imaginary part of $e^{i\frac{\pi}{2}s}$ we obtain:

$$M[\cos(cx); s] = c^{-s}\Gamma(s)\cos(s\pi/2) \text{ for } -1 < \text{Re } s < 1,$$
$$M[\sin(cx); s] = c^{-s}\Gamma(s)\sin(s\pi/2) \text{ for } 0 < \text{Re } s < 1.$$

The strip of definition is obtain by investigating $x^{s-1}\cos(cx)$, resp.$x^{s-1}\sin(cx)$ at 0 and infinity.

The results of previous motivation examples lead to formulating rules of operations with Mellin transform. We refer e.g to [ML86] for more rigorousness. We state the basic translation properties, which we use later in the computation:

$$M[f(cx); s] = c^{-s}F(s) \text{ for } c > 0, \tag{A.17}$$

$$M[f(x^a); s] = \frac{1}{a}F(s/a), \text{ where } a > 0, \tag{A.18}$$

$$M[x^a f(x); s] = F(s + a), \text{ for } a \in \mathbb{R}. \tag{A.19}$$

The Mellin transform of derivative and integral of function f can be found by using integration by parts:

$$M[f'; s] = \int_0^\infty f'(x) x^{s-1} \mathrm{d}x = \Big[x^{s-1} f(x)\Big]_0^\infty - (s-1)\int_0^\infty f(x) x^{s-2} \mathrm{d}x.$$

Adding condition that $x^{s-1}f(x)$ goes to 0 for x approaching both 0 and infinity, we have rule for derivative of Mellin transform:

$$M[f'; s] = -(s-1)F(s-1), \tag{A.20}$$

$$M[f^{(n)}; s] = (-1)^n (s-n)_n F(s-n). \tag{A.21}$$

For integral, it can be readily verified that:

$$M\Big[\int_x^\infty f(y)\mathrm{d}y; s\Big] = s^{-1}F(s+1). \tag{A.22}$$

The inversion formula for Mellin transform, denoted as $M^{-1}[F; s]$, is given by the following Theorem

Theorem A.2.1. *Assume that function $F(s)$ is analytic over strip $a_1 < Res < a_2$. Further assume that for some constant K $F(s)$ satisfies $|F(s)| \leq K|s|^{-2}$. Then*

$$M^{-1}[F;s] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[f;s]x^{-s}\mathrm{d}s, \tag{A.23}$$

for $a_1 < c < a_2$. $M^{-1}[F;s]$ converges to a continuous function $f(x)$ for $x > 0$, whose Mellin transform is $F(s) = M[f;s]$.

See [ML86], Theorem 11.1.1, p.228 for proof.

Parseval identity:

$$\begin{aligned}\int_0^\infty f(x)g(x)\mathrm{d}x &= \int_0^\infty \frac{1}{2\pi i}\left[\int_{c-i\infty}^{c+i\infty} M[f;s]x^{-s}\mathrm{d}s\right] g(x)\mathrm{d}x = \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[f;s]\left[\int_0^\infty x^{(1-s)-1}g(x)\mathrm{d}x\right]\mathrm{d}s = \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[f;s]M[g;1-s]\mathrm{d}s.\end{aligned} \tag{A.24}$$

As a consequence we obtain:

$$\int_0^\infty x^{a-1}f(x)g(x)\mathrm{d}x = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M[f;s]M[g;a-s]\mathrm{d}s. \tag{A.25}$$

A.3 Fox's H-function

In connection with inversion of Mellin transform, we introduce Mellin-Barnes Integral:

$$\frac{1}{2\pi i}\int_\gamma g(s)z^s\mathrm{d}s, \tag{A.26}$$

where integrand is composed of gamma functions:

$$g(s) = \frac{\prod_{j=1}^m \Gamma[b_j+\beta_j s]\prod_{j=1}^n \Gamma[a_j-\alpha_j s]}{\prod_{j=m+1}^q \Gamma[b_j-\beta_j s]\prod_{j=n+1}^p \Gamma[a_j+\alpha_j s]}, \tag{A.27}$$

$1 \leq m \leq p, 1 \leq n \leq q$ and coefficients α_j, β_j are assumed to be positive, whereas a_j, b_j are arbitrary. The integration path γ in complex plane is choosen, such that it is: either a closed path, or a vertical line indented to avoid a certian poles of the integrand, or a line between the two later paths avoiding certain poles of the integrand and aproaching infinity in certain fixed directions.

The convergence of integrals of Mellin-Barnes type is extensively discussed in [DF36], summary of this discussion is given in [BBEoNR53a], 1.19,p.49.

To emphasise the connection with inverse Mellin transform, in the following text we prefer to use Mellin-Barnes Integral in the form:

$$\frac{1}{2\pi i}\int_\gamma g(s)z^{-s}\mathrm{d}s, \tag{A.28}$$

with coefficient $g(s)$ of z^{-s} choosen as:

$$g(s) = \frac{\prod_{j=1}^{m} \Gamma[b_j + \beta_j s] \prod_{j=1}^{n} \Gamma[1 - a_j - \alpha_j s]}{\prod_{j=m+1}^{q} \Gamma[1 - b_j - \beta_j s] \prod_{j=n+1}^{p} \Gamma[a_j + \alpha_j s]}, \tag{A.29}$$

$1 \leq m \leq p, 1 \leq n \leq q$ and coefficients α_j, β_j are assumed to be positive, a_j, b_j are arbitrary. To see the advantage of this notation, recall reflexition formula (A.3).

The Mellin-Barnes integral with path $\gamma(t) = c + it, t \in (-\infty, \infty)$, is asociated with Fox's H-function, studied by [Fox61], p.408, denoted as:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s) z^{-s} \mathrm{d}s = H_{pq}^{mn} \left[z \left| \begin{array}{c} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{array} \right. \right], \tag{A.30}$$

with c choosen in a such a way that the integration line separates singularities in numerator of (A.29), such that poles of $\Gamma[b_j + \beta_j s], j = 1, \dots m$ lie on one side of integration path and poles of $\Gamma[1 - a_j - \alpha_j s], j = 1, \dots, n$ lie on the other side. Consider a special when all coefficients α_j and β_j equal 1, i.e.

$$g(s) = \frac{\prod_{j=1}^{m} \Gamma[b_j + s] \prod_{j=1}^{n} \Gamma[1 - a_j - s]}{\prod_{j=m+1}^{q} \Gamma[1 - b_j - s] \prod_{j=n+1}^{p} \Gamma[a_j - s]}, \tag{A.31}$$

$1 \leq m \leq p, 1 \leq n \leq q$ and a_j, b_j are arbitrary. Then Mellin-Barnes integral of (A.31) is called Meijer G-function, denoted as:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s) z^{-s} \mathrm{d}s = G_{pq}^{mn} \left[z \left| \begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_q \end{array} \right. \right], \tag{A.32}$$

with c choosen again such that the integration line separates singularities in numerator of (A.31), such that poles of $\Gamma[b_j + s], j = 1, \dots m$ lie on one side of integration path and poles of $\Gamma[1 - a_j - s], j = 1, \dots, n$ lie on the other side. The introduction to Meijer G-function can be found e.g. [BBEoNR53a] 5.2, p.206.

There is rather closed connection between Meijer G-function and generalized hypergeometric functions, namely:

$$G_{p,q+1}^{1,p} \left[-z \left| \begin{array}{c} 1 - a_1, \dots, 1 - a_p \\ 0, 1 - b_1, \dots, 1 - b_q \end{array} \right. \right] = \frac{\prod_{i=1}^{p} \Gamma(a_i)}{\prod_{i=1}^{q} \Gamma(b_i)} \, {}_pF_q[a_1, a_2, \dots, a_p; b_1, \dots, b_q; z]. \tag{A.33}$$

Example A.3.1. Consider function: $f(x) = e^{-x^a} \cos(bx),\ b > 0$. From Parseval identity we have:

$$\begin{aligned} \int_0^\infty e^{-x^a} \cos(bx) \mathrm{d}x &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M[e^{-x^a}; s] M[\cos(bx); 1 - s] \mathrm{d}s = \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{a} \Gamma[s/a] \Gamma[1 - s] \cos((1 - s)\pi/2) b^{-(1-s)} \mathrm{d}s = \\ &= \frac{1}{2\pi a i} \int_{c-i\infty}^{c+i\infty} \Gamma[s/a] \Gamma[1 - s] \sin(s\pi/2) b^{s-1} \mathrm{d}s = \\ &= \frac{1}{2abi} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma[s/a] \Gamma[1 - s]}{\Gamma[s/2] \Gamma[1 - s/2]} b^s \mathrm{d}s, \end{aligned}$$

where we used reflection formula (A.3) to obtain last equity.

It is now easy to express the integral in terms of H function as:

$$\int_0^\infty e^{-x^a}\cos(bx)\mathrm{d}x = \frac{\pi}{ab} H_{22}^{11}\left[b^{-1} \middle| \begin{array}{c} (0,1),(0,1/2) \\ (0,1/a),(0,1/2) \end{array} \right]. \tag{A.34}$$

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