Report on the memoir "Computational Homotopy Theory" by Marek Krčál

This report is devoted to the memoir *Computational Homotopy Theory* presented by Marek Krčál as PhD thesis.

First of all, I would like to tell some general words about the memoir. It is well structured and is written in a correct way. The style is very clear, the concepts and results are presented in an adequate way and the author's strategy of introducing the required preliminaries gradually makes the reading easy and pleasant. Moreover, the summaries included at the beginning of each chapter are also a good strategy to help the reader to understand the problems before going into the *hard* part of the text. Some details are sometimes skipped but the author transmits effectively his ideas, even in the most complicated aspects. The bibliography is thorough and current.

With respect to the theme of the memoir, the goal consists in studying the computational complexity of the problem of extending a map $f : A \to Y$ to a map $X \to Y$ and of the computation of $\pi_k(Y)$ and $[X, Y]$ for simplicial complexes $A \subseteq X$ and Y. The work is a very valuable contribution to the problem of computation in Algebraic Topology. To earlier researchers without a clear definition of computability, this problem was answered informally by algebraic constructions, principally spectral sequences, and this method allowed certain classical computations to be performed. With the intervening development of computability, more precise questions can be posed and machine computation brought to bear. Pioneering this approach we find Rolf Schön and Francis Sergeraert. Mr Krčál's thesis fits well into this approach and contributes deeply to its development.

The memoir is divided into four chapters. The first one contains some preliminaries and a brief and clear description of the main results of the thesis, which allow the reader to quickly understand the considered problems.

In Chapter 2 the author proves the undecidability of the extension problem outside the stable range (that is, when Y is d-connected and dim $X \geq 2d + 2$) even if we fix the dimension of either the target complex or the source complex. The proof requires an interesting use of different geometric and combinatorial techniques, and consists in defining a system of polynomial Diophantine equations (for which the existence of a non zero solution is known to be an undecidable problem) such that a solution of this system would correspond to the desired extension. With similar techniques, the author also proves hardness results for the computation of $[X, Y]$ and $\pi_n(Y)$.

Chapter 3 contains, in my opinion, the most striking contributions of the memoir. The author describes polynomial-time algorithms for the extension problem and the computation of $[X, Y]$ in the stable range. The algorithms are based on Postnikov systems, and a deep study is done showing that all ingredients involved in the construction can be computed in polynomial-time. Although some parts of the construction are not detailed (specially algorithms related with polynomial-time homology and polynomial time construction of a Postnikov system), the main ideas are well explained and the algorithms are sufficiently described.

As one of the ingredients for polynomial-time algorithms of Chapter 3, in Chapter 4 the author describes a polynomially bounded discrete vector field on the simplicial set $K(\mathbb{Z}, 1)$. The classical vector field defined by Eilenberg and MacLane for this space allowed them to construct a homotopy equivalence to the circle, but this vector field is exponentially (and not polynomially) bounded. Here a new discrete vector field is obtained with polynomial complexity, such that it produces a polynomial-time reduction from $K(\mathbb{Z}, 1)$ to S^1 which is then used by algorithms of Chapter 3.

Only as a remark, I would like to say that, being a thesis in Computational Topology, the implementation of (some parts of) the developed algorithms could have been a complement for the work. For example, it could be interesting to implement the polynomial-time discrete vector field for $K(\mathbb{Z}, 1)$ presented in Chapter 4 and compare some calculation times with the original (non polynomial) vector field. Moreover, in some parts of the memoir it could be useful to consider some particular examples (showing for instance the first steps of the Postnikov system for some particular simplicial set Y).

I find the complete work to be of high quality and scientifically sound. Along all the text, the author shows a very good knowledge of the subject. Moreover, Marek Krčál has showed his ability to work in a larger project together with other researchers.

As a conclusion, on the basis of the scientifically valuable work Marek Krčál has presented, **it deserves to be defended to obtain the PhD degree**.

Your sincerely,

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