Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE

Stanislava Kucková

Anti-gaussovské kvadraturní formule

Katedra numerické matematiky
Vedoucí diplomové práce: Doc. RNDr. Josef Kofroň, CSc.
Studijní program: Matematika, Výpočtová matematika, Algoritmy

Děkuji Doc. RNDr. Josefu Kofroňovi, CSc. za cenné rady, připomínky, čas a trpělivost, které mi věnoval při psaní této práce. Děkuji také svým rodičům, sestře a přátelům za podporu ve studiu i v životě, kterou mi poskytli.

Prohlašuji, že jsem svou diplomovou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne 7.4.2004 Stanislava Kucková

Contents

Abstracts

Název práce: Anti-gaussovské kvadraturní formule
Autor: Stanislava Kucková
Katedra (ústav): Katedra numerické matematiky
Vedoucí diplomové práce: Doc. RNDr. Josef Kofroň, CSc.
E-mail vedoucího: Kofron@karlin.mff.cuni.cz
Abstrakt: Anti-gaussovská kvadraturní formule je $(n+1)$-bodová formule, jejiž algebraický stupeň přesnosti je $2n-1$ a pro polynomy až do stupně $2n+1$ je její zbytek v absolutní hodnotě roven zbytku n-bodové Gaussovy kvadraturní formule, ale má opačné znaménko. V práci podáme zevrubné důkazy významných vlastností anti-gaussovské kvadratury uvedených v [7] (zejména kladnost vah a postačujíci podmínky uzavřenosti pro speciální typy váhových funkcí), popíšeme algoritmus pro konstrukci anti-gaussovské formule, který využívá znalost ortogonalních polynomů potřebných pro konstrukci Gaussovy formule a předvedeme jeho implementaci v Maple. Dále dokážeme konvergenci anti-gaussovské kvadratury pro spojité funkce a odvodíme odhady chyb nejprve klasickou metodou využívající derivace vyšších řádů integrované funkce a následně pro analytické funkce spočteme odhad bez užítí derivací. Pro případ uzavřených anti-gaussovských formulí uvedeme tento odhad v zakončeném tvaru. V závěru srovnáme přesnost jednotlivých odhadů pro různé váhy a integrované funkce a porovnáme praktickou přesnost $(n+1)$-bodové Gaussovy formule a formule vzniké jako průměr anti-gaussovske formule a n-bodové Gaussovy formule.
Klíčová slova: anti-gaussovská kvadraturní formule, numerická integrace, odhady chyb

Title: Anti-Gaussian quadrature formulas
Author: Stanislava Kucková
Department: Department of Numerical Mathematics
Supervisor: Doc. RNDr. Josef Kofroň, CSc.
Supervisor's e-mail address: Kofron@karlin.mff.cuni.cz
Abstract: Anti-Gaussian quadrature formula is a $(n+1)$-point formula of degree $2n-1$ which integrates the polynomials of degree up to $2n+1$ with an error of the same magnitude as the the one of the n-point Gaussian formula but of the opposite sign. In this thesis we present detailed proofs of significant properties of the anti-Gaussian quadrature listed in [7] (in particular the positiveness of the weights and for certain weight functions sufficient conditions for the formula to be internal), we describe the algorithm for the construction of the anti-Gaussian formula using the knowledge of the orthogonal polynomials needed to construct Gaussian formula and we demonstrate its implementation in Maple. Next we prove the convergence of the the formula for continuous functions and derive the error estimates at first by the classical method involving the higher order derivatives of the integrated function and secondly an error estimate for analytic functions without use of the derivatives. For the case of internal anti-Gaussian formulas we present a finite form of this estimate. Finally we compare the accuracy of the estimates for different weight and integrated functions and confront the practical accuracy of the $(n+1)$-point Gaussian formula and a formula obtained as an average of the anti-Gaussian formula and the n-point Gaussian formula.
Keywords: anti-Gaussian quadrature formula, numerical integration, error estimates

1. Introduction

In this section we shall recall some basic notions in the theory of quadrature formulas and introduce the anti-Gaussian quadrature formula.

Definition 1.1. Let us consider a quadrature formula K for a functional F. If

$$Kf = Ff \ \forall f \in \mathbb{P}^m \tag{1.1}$$

where $\mathbb{P}^m$ denotes the space of all polynomials of degree less or equal m, we shall say that the algebraic degree of precision of the formula K is m.

Definition 1.2. Let w be a given function defined and nonnegative almost everywhere in the interval $\langle a, b\rangle$ for which the integral

$$\int_a^b w(x)dx$$

is finite. Let $G_w^{(n)}$ be a quadrature formula

$$G_w^{(n)} f := \sum_{i=1}^{n} w_i^{(n)} f(x_i^{(n)}) \tag{1.2}$$

then $G_w^{(n)}$ is n-point Gaussian quadrature formula for the integral

$$If := \int_a^b w(x)f(x)dx \tag{1.3}$$

if and only if

$$G_w^{(n)} p = Ip \ \forall p \in \mathbb{P}^{2n-1}. \tag{1.4}$$

In other words the algebraic degree of precision of the n-point Gaussian quadrature formula has to be the maximal possible which is $2n-1$.

Remark 1.3. In the definition 1.2 we shall say w is a *weight function* and $w_i^{(n)}$ and $x_i^{(n)}$ are *weights* and *nodes* of the quadrature formula $G_w^{(n)}$.

There are various questions of interest regarding the existence and other properties of quadrature formulas defined by a set of equations (e.g. (1.3)). To make the terminology precise we introduce the following definition.

Definition 1.4. We shall say

- the formula *exists*, if the set of the defining equations has a (possibly complex) solution.
- the formula is *real* if the points and weights are all real.
- a real formula is *internal* if all the nodes belong to the (closed) interval of integration. A node not belonging to the interval of integration is called an *exterior node*.
- the formula is *positive* if all the weights are positive.

It is known [10] that Gaussian formulas are internal and positive.

Often we need to estimate the error $If - G_w^{(n)}f$ where f is a function that has not been subjected to much analysis. The usual method is to use another quadrature formula A of degree greater than $2n-1$ and to estimate the error as $Af - G_w^{(n)}f$. Any such quadrature rule requires at least $n+1$ additional points. Imagine we want to keep the original Gaussian nodes $x_i,\ i = 1, \dots, n$ and add n-arbitrary nodes $y_i,\ i = 1, \dots, n$ so that it will be possible to find such weights A_i and B_i that the obtained formula will have the degree at least $2n-1$.

In other words we are looking for A_i, B_i and $y_i,\ i = 1, \dots, n$ for which

$$\sum_{i=1}^{n} A_i x_i^k + \sum_{i=1}^{n} B_i y_i^k = \int_a^b w(x)x^k dx =: I_k \ \forall k \in [0, 2n-1].$$

Or equivalently in the matrix form

$$\begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ x_1 & \dots & x_n & y_1 & \dots & y_n \\ x_1^2 & \dots & x_n^2 & y_1^2 & \dots & x_n^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ x_1^{2n-2} & \dots & x_n^{2n-2} & y_1^{2n-2} & \dots & y_n^{2n-2} \\ x_1^{2n-1} & \dots & x_n^{2n-1} & y_1^{2n-1} & \dots & y_n^{2n-1} \end{pmatrix} \begin{pmatrix} A_1 \\ \vdots \\ A_n \\ B_1 \\ \vdots \\ B_n \end{pmatrix} = \begin{pmatrix} I_0 \\ I_1 \\ \vdots \\ I_{2n-1} \end{pmatrix}.$$

One solution of this problem is to choose the weights A_i to be equal to the original weights of the Gaussian formula and $B_i = 0\ \forall\ i = 1, \dots, n$. But for any choice of y_i different to x_i the matrix above is regular, which implies that the solution is unique. Therefore we need to add at least $n+1$ additional points to the ones of the Gaussian formula to achieve a better degree of precision than $2n-1$.

In fact any additional $n+1$ points can be used to construct a formula of the degree at least $2n$. Determining the weights would lead to a similar matrix as the one above. And for any choice of y_i there will be a unique solution of the problem

$$\begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ x_1 & \dots & x_n & y_1 & \dots & y_n \\ x_1^2 & \dots & x_n^2 & y_1^2 & \dots & x_n^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ x_1^{2n-1} & \dots & x_n^{2n-1} & y_1^{2n-1} & \dots & y_n^{2n-1} \\ x_1^{2n} & \dots & x_n^{2n} & y_1^{2n} & \dots & y_n^{2n} \end{pmatrix} \begin{pmatrix} A_1 \\ \vdots \\ A_n \\ B_1 \\ \vdots \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} I_0 \\ I_1 \\ \vdots \\ I_{2n} \end{pmatrix}.$$

However the degree $2n$ is not the best we can get. For example if we choose y_i to be equal to the nodes of the $(n+1)$-point Gaussian formula we get the degree $2n+1$. Of course in this case the weights of the nodes of the n-point Gaussian formula will turn to be zeros. It has been shown by Kronrod [5] that for certain weight functions (including $w(x) = 1$) it is possible to find a $(2n+1)$-point quadrature formula containing the original n-points, with the degree at least $3n+1$. Unfortunately there are weight functions for which the Kronrod extension is not real.

Let us return back to the choices of y_i providing that there will exist a $(2n+1)$-point formula with the nodes $x_i,\ \imath = 1,\dots,n$ and $y_i,\ i = 1,\dots,n+1$ of degree at least $2n+1$. The concept of choosing the nodes of the $(n+1)$-point Gaussian formula is not the only option. In this work we are going to deal with a different possibility. Again we are looking for a $(2n+1)$-point formula with the degree at least $2n+1$ containing the n original nodes, but we shall add the condition that the weights of these "old" nodes are precisely the halves of the original ones. Equivalently we are looking for a $(n+1)$-point quadrature formula which integrates polynomials up to degree $2n+1$ with an error equal in magnitude to the one of the Gaussian formula, but with the opposite sign.

Let us denote such formula by $H_w^{(n+1)}$. As mentioned above the formula $H_w^{(n+1)}$ is precise for polynomials up to the degree $2n-1$ and the averaged formula

$$L_w^{(2n+1)} := \frac{1}{2}(H_w^{(n+1)} + G_w^{(n)}) \tag{1.5}$$

has degree at least $2n+1$ since the errors of $G_w^{(n)}$ and $H_w^{(n+1)}$ cancel each other.

The existence of $H_w^{(n+1)}$ is guaranteed by the fact that

$$(H_w^{(n+1)} - I)f = -(G_w^{(n)} - I)f \ \forall f \in \mathbb{P}^{2n+1}$$

or

$$H_w^{(n+1)} f = (2I - G_w^{(n)})f \ \forall f \in \mathbb{P}^{2n+1}.$$

Hence $H_w^{(n+1)}$ is actually a Gaussian quadrature rule for the linear functional $2I - G_w^{(n)}$. The existence of $H_w^{(n+1)}$ can now be easily deduced from the theorem for existence of Gaussian quadrature formulas.

The above mentioned rule $H_w^{(n+1)}$ is called the *anti-Gaussian* quadrature formula. After we have explained the existence we can step up to the definition.

Definition 1.5. Let $G_w^{(n)}$ be the Gaussian quadrature formula 1.2 for the integral 1.3. Let $H_w^{(n+1)}$ be $(n+1)$-point quadrature formula

$$H_w^{(n+1)} := \sum_{i=1}^{n+1} \lambda_i^{(n+1)} f(\xi_i^{(n+1)}) \tag{1.6}$$

such that

$$(H_w^{(n+1)} - I)f = -(G_w^{(n)} - I)f \ \forall f \in \mathbb{P}^{2n+1}. \tag{1.7}$$

Then $H_w^{(n+1)}$ is the *anti-Gaussian* quadrature formula for integral I and $\lambda_i^{(n+1)}$ and $\xi_i^{(n+1)}$ are its nodes and weights.

As we will see later anti-Gaussian formula has many advantageous practical and theoretical properties. In particular it can be constructed with only minimal additional costs from the data needed for the construction of the corresponding Gaussian formula, its weights are positive, nodes are always real and at most two of them may lie outside the interval of integration.

2. Construction of the Anti-Gaussian Formulas

An effective algorithm for generating anti-Gaussian formulas will be demonstrated in this chapter. We present a common procedure of constructing Gaussian quadrature rules described in [3] and show how this process has to be modified if we wish to get the anti-Gaussian rule. The reader is expected to be familiarized with the basic results of the theory of the orthogonal polynomials, particularly the definition, existence of 3-term recurrence relationship, Stieltjes formulas for the recurrence coefficients (e.g. [10]), the interlacing property and Christoffel-Darboux identity [1, p. 785].

2.1. **Algorithm.**

As we have already mention in section 1 from 1.7 we see that

$$H_w^{(n+1)}f = (2I - G_w^{(n)})f \ \forall f \in \mathbb{P}^{2n+1}. \tag{2.1}$$

By comparing 2.1 with the definition 1.2 we see that $H_w^{(n+1)}$ is in fact an $(n+1)$-point Gaussian formula for the linear functional $2I - G_w^{(n)}$. Therefore the weights and nodes of $H_w^{(n+1)}$ can be enumerated with use of the following theorem for the Gaussian formulas.

Theorem 2.1. *Denote $F := 2I - G_w^{(n)}$ and $\{p_i\}_{i=1}^{\infty}$ be a sequence of polynomials orthogonal with respect to F, let k_i be the leading coefficient of p_i $\forall i$ and $\{t_k\}_{k=1}^{n+1} \subset \langle a, b\rangle$ be the zeros of p_{n+1}, then*

$$Ff := \sum_{i=1}^{n+1} w_i^{(n+1)} f(t_i) \ \forall f \in \mathbb{P}^{2n+1} \tag{2.2}$$

where

$$w_i^{(n+1)} = -\frac{k_{n+2}}{k_{n+1}} \frac{1}{p_{n+2}(t_i)p'_{n+1}(t_i)}, \quad i = 1, 2 \ldots, n+1. \tag{2.3}$$

Unfortunately to apply the above theorem we would need to know the orthogonal polynomials of high degree and even their roots. One way of course is to determine the whole sequence of polynomials and then to find the desired zeros, but we shall present here a simpler way how to get the weights and nodes. To be precise we show that the nodes are the eigenvalues and weights are proportional to the squares of the first components of the orthonormal eigenvectors of a certain 3-diagonal matrix.

We know that every set $\{p_i\}_{i=1}^{n+1}$ of the polynomials orthogonal with respect to the linear functional F satisfies the 3-term recurrence relationship:

$$p_j(x) = (a_j x + b_j)p_{j-1}(x) - c_j p_{j-2}(x) \ j = 1, 2 \ldots, n+1 \tag{2.4}$$

where

$$a_j, c_j > 0, \quad p_{-1}(x) \equiv 0, \quad p_0(x) \equiv 1. \tag{2.5}$$

Later we will see how the recurrence coefficients can be obtained but now let us believe we know them.

Writing the formula 2.4 in the matrix form we get

$$
x\begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_n(x) \end{pmatrix} = \begin{pmatrix} \frac{-b_1}{a_1} & \frac{1}{a_1} & & \cdots & \mathbf{0} \\ \frac{c_2}{a_2} & \frac{-b_2}{a_2} & \frac{1}{a_2} & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & \frac{c_n}{a_n} & \frac{-b_n}{a_n} & \frac{1}{a_n} \\ \mathbf{0} & \cdots & & \frac{c_{n+1}}{a_{n+1}} & \frac{-b_{n+1}}{a_{n+1}} \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_n(x) \end{pmatrix} + \\
+ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{p_{n+1}(x)}{a_{n+1}} \end{pmatrix}. \tag{2.6}
$$

Let us denote the matrix above by $\mathbf{T}$, the vector $(p_0(x), p_1(x), \ldots, p_n(x))^T$ by $\mathbf{p}(x)$ and $\mathbf{e}_{n+1} = (0, \ldots, 0, 1)^T$. Now we can rewrite 2.6 in the form

$$
x\mathbf{p}(x) = \mathbf{T}\mathbf{p}(x) + \frac{1}{a_{n+1}} p_{n+1}(x)\mathbf{e}_{n+1} \tag{2.7}
$$

Thus $p_{n+1}(t_j) = 0$ if and only if

$$
t_j\mathbf{p}(t_j) = \mathbf{T}\mathbf{p}(t_j), \tag{2.8}
$$

so the eigenvalues of $\mathbf{T}$ correspond to the zeros of p_{n+1}.

If we multiply the formula 2.4 by $p_{j-2}(x)$ and apply the functional F on the both sides we get

$$
F(p_{j-2}(x)p_j(x)) = a_j F(xp_{j-1}(x)p_{j-2}(x)) + b_j F(p_{j-1}(x)p_{j-2}(x)) - c_j F(p_{j-2}^2(x)). \tag{2.9}
$$

As the polynomials p_i are orthogonal, the terms $F(p_{j-2}(x)p_j(x))$ and $F(p_{j-1}(x)p_{j-2}(x))$ are equal to zero. And hence

$$
c_j = \frac{a_j F(xp_{j-1}(x)p_{j-2}(x))}{F(p_{j-2}^2(x))}.
$$

Shifting the index j to $j-1$ in the formula 2.4 gives

$$
p_{j-1}(x) = (a_{j-1}x + b_{j-})p_{j-2}(x) - c_{j-1}p_{j-3}(x), \tag{2.10}
$$

and if we multiply it with p_{j-1}, apply the functional F and consider the orthogonality of the polynomials we arrive to

$$
F(p_{j-1}^2(x)) = a_{j-1}F(xp_{j-1}(x)p_{j-2}(x))
$$

which implies

$$
F(xp_{j-1}(x)p_{j-2}(x)) = \frac{F(p_{j-1}^2(x))}{a_{j-1}}
$$

and hence

$$c_j = \frac{a_j}{a_{j-1}} \frac{F(p_{j-1}^2)}{F(p_{j-2}^2)} \ . \tag{2.11}$$

Thus $\frac{c_j}{a_j} = \frac{1}{a_{j-1}}$ $\forall j$ (which means $\mathbf{T}$ is symmetric) if and only if there exists a positive constant K such that for every $i = 0, 1, \ldots, n+1$ $p_i = Kp_i^*$ where the polynomials $\{p_i^*\}_{i=0}^{n+1}$ are orhonormal, in other words if and only if the sequence $\{p_i\}_{i=0}^{n+1}$ is an uniform multiple of the set $\{p_i^*\}_{i=0}^{n+1}$. Soon we will see that this is precisely the property we need to get the weights. Obviously there exist such constants d_i that if we use the orthogonal system $\{d_i p_i\}_{i=0}^{n+1}$ instead of $\{p_i\}_{i=0}^{n+1}$ in 2.4 we would get a symmetric matrix at the place of $\mathbf{T}$ in the formula 2.8. It is easy to show that the (symmetric) matrix on the place of $\mathbf{T}$ will be $\mathbf{J} = D\mathbf{T}D^{-1}$ where

$$D = \begin{pmatrix} d_0 & & \cdots & \mathbf{0} \\ & d_1 & & \vdots \\ \vdots & & \ddots & \\ \mathbf{0} & \cdots & & d_n \end{pmatrix} \tag{2.12}$$

and hence

$$\mathbf{J} = \begin{pmatrix} \frac{-b_1}{a_1} & \frac{d_0}{d_1}\frac{1}{a_1} & & \cdots & \mathbf{0} \\ \frac{d_1}{d_0}\frac{c_2}{a_2} & \frac{-b_2}{a_2} & \frac{d_1}{d_2}\frac{1}{a_2} & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & \frac{d_{n-1}}{d_{n-2}}\frac{c_n}{a_n} & \frac{-b_n}{a_n} & \frac{d_{n-1}}{d_n}\frac{1}{a_n} \\ \mathbf{0} & \cdots & & \frac{d_n}{d_{n-1}}\frac{c_{n+1}}{a_{n+1}} & \frac{-b_{n+1}}{a_{n+1}} \end{pmatrix}. \tag{2.13}$$

Now we shall show how to find the constants d_i. The requirement of the symmetry leads to the following conditions for d_i:

$$\frac{d_{i-1}}{d_i}\frac{1}{a_i} = \frac{d_i}{d_{i-1}}\frac{c_{i+1}}{a_{i+1}}, \ \forall i = 1, \ldots, n. \tag{2.14}$$

Having these n conditions for $n+1$ constants d_i we can choose the first constant d_0 arbitrarily and compute the rest recurrently from the following relationship obtained by simple formatting of the equation 2.14:

$$d_i = \frac{d_{i-1}}{\left(\frac{c_{i+1}a_i}{a_{i+1}}\right)^{\frac{1}{2}}}. \tag{2.15}$$

As multiplying by a constant does not change the zeros of the $(n+1)$th polynomial, we can without the loss of generality consider the symmetric matrix $\mathbf{J}$ instead of $\mathbf{T}$.

To proceed further we will need the following lemma:

Lemma 2.2. *Let $\boldsymbol{p}(t_j)$ be the eigenvector of the matrix $\boldsymbol{J}$ corresponding with the eigenvalue t_j. Let w_j be of the same form as in the theorem 2.1. Then*

$$w_j[\boldsymbol{p}(t_j)]^T[\boldsymbol{p}(t_j)] = 1 \ \forall j = 1, 2, \ldots, n+1. \tag{2.16}$$

Proof. From the Christoffel-Darboux identity we have

$$\sum_{m=0}^{n+1} \frac{p_m(x)p_m(x_0)}{F(p_m^2(x))} = \frac{k_{n+1}}{k_{n+2}} \frac{p_{n+2}(x)p_{n+1}(x_0) - p_{n+2}(x_0)p_{n+1}(x)}{F(p_{n+1}^2(x))(x-x_0)}.$$

We make a limit passage for x tending to x_0 on the both sides and add and subtract $p_{n+2}(x_0)p_{n+1}(x_0)$ in the numerator on the right hand side. We get

$$\sum_{m=0}^{n+1} \frac{p_m^2(x_0)}{F(p_m^2(x_0))} = \frac{k_{n+1}}{k_{n+2}F(p_{n+1}^2(x_0))} \lim_{x\to x_0} \left[\frac{p_{n+2}(x)p_{n+1}(x_0) - p_{n+2}(x_0)p_{n+1}(x_0)}{(x-x_0)} - \right.$$
$$\left. - \frac{p_{n+2}(x_0)p_{n+1}(x) - p_{n+2}(x_0)p_{n+1}(x_0)}{(x-x_0)}\right].$$

We notice that the terms in the square brackets correspond to $p'_{n+2}(x_0)p_{n+1}(x_0)$ and $p_{n+2}(x_0)p'_{n+1}(x_0)$ respectively. This observation allows us to continue this way:

$$\sum_{m=0}^{n+1} \frac{p_m^2(x_0)}{F(p_m^2(x_0))} = \frac{k_{n+1}}{k_{n+2}F(p_{n+1}^2(x_0))}[p'_{n+2}(x_0)p_{n+1}(x_0) - p_{n+2}(x_0)p'_{n+1}(x_0)]$$

We remind the fact that the polynomials p_i are the uniform multiples of the orthonormal polynomials and so

$$F(p_m^2) = F(p_n^2)\ \forall m, n \in \mathbb{N},$$

which lets us to reduce these terms on the both sides.

Since x_0 was chosen arbitrarily, we can put $x_0 := t_j$ (note that $p_{n+1}(t_j) = 0$), which implies

$$\sum_{m=0}^{n+1} p_m^2(t_j) = [\mathbf{p}(t_j)]^T[\mathbf{p}(t_j)] = -\frac{k_{n+1}}{k_{n+2}} p_{n+2}(t_j)p'_{n+1}(t_j) = \frac{1}{w_j} \tag{2.17}$$

□

From the formula 2.8 we can see that if q_j is an orthonormal (with respect to the standard scalar product on $\mathbb{R}^{n+1}$) eigenvector of $\mathbf{J}$ corresponding to the eigenvalue t_j i.e.

$$Jq_j = t_j q_j$$

and

$$g_j^T g_j = 1$$

then the eigenvectors q_j are necessarily multiples of $\mathbf{p}(t_j)$ and hence

$$\forall j\ \exists b \in \mathbb{R} \text{ that } \ g_j = (q_j^0, \ldots, q_j^n) = b(p_0(t_j), \ldots, p_n(t_j)) \tag{2.18}$$

which together with 2.17 implies

$$1 = g_j^T g_j = b^2[\mathbf{p}(t_j)]^T[\mathbf{p}(t_j)] = \frac{b^2}{w_j} \tag{2.19}$$

especially $b = w_j$.

This yields

$$q_j^2 = w_j p_0^2(t_j) \tag{2.20}$$

and as $p_0 \equiv k_0$ and $\frac{1}{k_0^2} = \int_a^b w(x)dx$, we get

$$w_j = (q_j^0)^2 \frac{1}{k_0^2} = (q_j^0)^2 \int_a^b w(x)dx. \tag{2.21}$$

So the weights are proportional to the squares of the first components of the orthonormal eigenvectors of the symmetric matrix $\mathbf{J}$. Moreover the weights are positive, which will plays an important role in the convergence of these quadrature formulas.

2.2. **Development of the Coefficients of the Recurrence Relationship.**

Obviously the weak point of the above construction is in the proposed knowledge of the recurrence coefficients in the 3-term recurrence for the polynomials orthogonal with respect to the functional $2I - G_w^{(n)}$. Without the loss of generality we can assume that the coefficients $a_j \equiv 1\ \forall$j. The coefficients b_j and c_j are given by the Stieltjes formulas:

$$b_j = \frac{(2I - G_w^{(n)})(xp_{j-1}^2)}{(2I - G_w^{(n)})(p_{j-1}^2)},\ j = 1, 2, \ldots, n+1 \tag{2.22}$$

$$c_j = \frac{(2I - G_w^{(n)})(p_{j-1}^2)}{(2I - G_w^{(n)})(p_{j-2}^2)},\ j = 2, \ldots, n+1. \tag{2.23}$$

The coefficient c_1 can be any finite number, soon we will see that a convenient choice is $c_1 = (2I - G_w^{(n)})(p_0)$. Now we shall show how the required coefficients can be obtained from the corresponding coefficients for the original linear functional I. In many classical cases the coefficients for the functional I are known explicitly and tabulated (e.g. in [1]); in others software packages used to compute the Gaussian formulas compute them as a preliminary step.

Let $\{\varphi_j\}$ be the sequence of polynomials orthogonal with respect to the integral I, which satisfies the recurrence relation:

$$\begin{gathered}\varphi_j(x) = (x - \alpha_j)\varphi_{j-1}(x) - \beta_j\varphi_{j-2}(x)\ j = 1, 2, \ldots \\ \varphi_{-1}(x) \equiv 0, \varphi_0(x) \equiv 1.\end{gathered} \tag{2.24}$$

Same as before we choose $\beta_1 = I(p_0)$ and for the other coefficients holds:

$$\alpha_j = \tfrac{I(x\varphi_{j-1}^2)}{I(\varphi_{j-1}^2)} \quad j = 1, 2, \ldots \tag{2.25}$$

$$\beta_j = \tfrac{I(\varphi_{j-1}^2)}{I(\varphi_{j-2}^2)} \quad j = 2, 3, \ldots \tag{2.26}$$

For $p \in \mathbb{P}^{2n-1}$ is $(2I - G_w^{(n)})p = Ip$, therefore

$$b_j = \alpha_j\ j = 1, 2, \ldots, n \tag{2.27}$$

$$c_j = \beta_j\ j = 1, 2, \ldots, n \tag{2.28}$$

$$p_j = \varphi_j\ \ j = 1, 2, \ldots, n. \tag{2.29}$$

We only need to compute b_{n+1} and c_{n+1}. As the nodes of $G_w^{(n)}$ are the zeros of p_n, the result of applying $G_w^{(n)}$ to any product containing p_n is 0. So

$$b_{n+1} = \frac{(2I - G_w^{(n)})(xp_n^2)}{(2I - G_w^{(n)})(p_n^2)} = \frac{2I(xp_n^2)}{2I(p_n^2)} = \alpha_{n+1}. \tag{2.30}$$

Using the above argument as well as the fact that the degree of p_{n-1}^2 is less then $2n-1$ we find that

$$c_{n+1} = \frac{(2I - G_w^{(n)})(p_n^2)}{(2I - G_w^{(n)})(p_{n-1}^2)} = \frac{2I(p_n^2)}{I(p_n^2)} = 2\beta_{n+1}. \tag{2.31}$$

In other words we take precisely the same set of recurrence coefficients as when computing the Gaussian formula, except that the last coefficient c_{n+1} is doubled.

3. Implementation of the Algorithm for the Construction

In section 2 we have seen the algorithm for generating the nodes and weights for the anti-Gaussian formulas of the prescribed degree. In this section we show an example of the implementation of the algorithm in Maple programming language and give explicit values of the nodes and weights generated by the program for common weight functions. The source code of the programme is attached in section 9.

At the first line of the programm we initialize the linear algebra package by typing `with(linalg)` and set the number of digits Maple will use when performing the floating point operations by `Digits:=20:`. The program itself is divided into 7 sections. As I have mentioned above it is able to determine the anti-Gaussian rule of a given degree for a given weight function. The aim of the first part (Definition of the weight function, recurrence relations and degree) is therefore to define the weight function, the degree and the recurrence relationship.

Taking Legendre weight function as an example we would first describe the weight function and the interval of integration by

```
> w:= x->1: #Legendre weight function is w(x)=1
> a:=-1:b:=1: #interval of integration is (a,b)=(-1,1)
```

then we enter the desired number of nodes (The program computes the $(N+1)$-point anti-Gaussian formula and the degree of precision is therefore $2N-1$.)

```
> N:=2: #number of nodes for the Gaussian formula
```

and finally we put the formula for the leading coefficient $k(n)$ of the n-th orthogonal polynomial

```
> k := n-> 1./2^n*binomial(2*n,n):
```

and the recurrence coefficients $a1(n), \ldots, a4(n)$

```
> a1:= n-> (n+1);
> a2:= n-> 0;
> a3:= n-> (2*n+1);
> a4:= n-> n;
```

obtained from the recurrence relationship for the orthogonal polynomials $p_i(x)$ of the form

$$a1(n)p_{n+1}(x) = (a2(n) + a3(n)x)p_n(x) - a4(n)p_{n-1}(x). \tag{3.1}$$

Obviously this recurrence relationship is not compatible with the notation in the section 2. However in the classical cases the leading coefficients are tabulated (see section 9 or [1]) precisely in the form given above.

Deriving the recurrence relationships in the form used in the section 2 is the point of the second part of the program (Derivation of the recurrence relationships for "Gaussian" and "anti-Gaussian" orthogonal polynomials).

We need the recurrence of the form

$$P_n(x) = (x + b_n)P_{n-1}(x) - c_n P_{n-2}(x)$$

where P_i are polynomials orthogonal with respect to the weight function $w(x)$ with the leading coefficient equal to 1. Assuming $a1(n) \neq 0$ and shifting the index n to $n-1$ we can write the formula 3.1 as

$$p_n(x) = \left(\frac{a2(n-1)}{a1(n-1)} + \frac{a3(n-1)}{a1(n-1)}x\right) p_{n-1}(x) - \frac{a4(n-1)}{a1(n-1)}p_{n-2}(x).$$

So for the polynomials

$$P_n(x) = \frac{p_n(x)}{k(n)}$$

with the leading coefficients equal to 1 holds

$$P_n(x) = \left(\frac{a2(n-1)}{a1(n-1)} + \frac{a3(n-1)}{a1(n-1)}x\right) P_{n-1}(x)\frac{k(n-1)}{k(n)} - \frac{a4(n-1)}{a1(n-1)}P_{n-2}(x)\frac{k(n-2)}{k(n)}.$$

To simplify this expression we introduce the following notations:

$$ga(n) = -\frac{k(n-1)a2(n-1)}{k(n)a1(n-1)} \quad \text{and} \quad gb(n) = \frac{k(n-2)a4(n-1)}{k(n)a1(n-1)}. \tag{3.2}$$

Which in Maple would be represented by:

```
> ga:= n-> -(k(n-1)/k(n))*(a2(n-1)/a1(n-1)):
> gb:= n->  (k(n-2)/k(n))*(a4(n-1)/a1(n-1)):
```

Now the recurrence can be written in the form

$$P_n(x) = (x - ga(n))P_{n-1}(x) - gb(n)P_{n-2}(x), \tag{3.3}$$

which finally corresponds to the section 2.

We know that if the recurrence for the anti-Gaussian formula should be

$$Q_n(x) = (x - aga(n))Q_{n-1}(x) - agb(n)Q_{n-2}(x), \tag{3.4}$$

then the corresponding coefficients are equal except for the last one where $agb(N+1) = 2gb(N+1)$.

Hence

```
> aga:= n-> ga(n):#aga(n)=ga(n) for all n
> agb:= proc(n)
> if N+1>n then return(gb(n))fi: #agb(n)=gb(n) for n < N+1
> if N+1=n then return(2*gb(n)) fi: #agb(N+1)=2gb(N+1)
> end proc:
```

In the third section of the program (Symmetric matrix **J** for the "anti-Gaussian" case) the factors d_i (formula 2.15) are computed by

```
> d[0]:= 1:
> for i from 1 to N do d[i]:=d[i-1]/sqrt(agb(i+1)) od:
```

and the symmetrized matrix **J** (see formula 2.13) is built up as follows:

```
>matrixJ:= proc(i,j)
>   if i=j then return aga(i)
>   elif i=j-1 then return (d[i-1]/d[i]+d[j-1]/d[j-2]*agb(j))/2
>   elif i=j+1 then return (d[j-1]/d[j]+d[i-1]/d[i-2]*agb(i))/2
>   else return(0)
>   fi
> end proc:
> J:=matrix(N+1,N+1,matrixJ):
```

Here I have replaced the explicit expressions for $J_{i,i-1}$ and $J_{i-1,i}$ by their average since otherwise the rounding error would destroy the symmetry of **J**.

The fourth and perhaps the most important part (Weights and nodes for the anti-Gaussian formula) finally determines the nodes and weights of the desired anti-Gaussian formula.

In section 2 I have shown that nodes are the eigenvalues and weights can be gained by multiplying the squares of the first components of the orthonormal eigenvectors of the matrix **J** by $\int_a^b w(x)dx$. To obtain the eigenvalues and eigenvector of **J** I use the Maple function "eigenvectors"

```
>eigen_all:=eigenvectors(J):
```

which returns a sequence of lists of the form $[\lambda_i, m_i, \{v[1,i], \ldots, v[n_i,i]\}]$, where the λ_i is the eigenvalue, m_i its algebraic multiplicity, and $\{v[1,i], \ldots, v[n_i,i]\}$ is a set of orthonormal eigenvectors of $\mathbf{J}$ corresponding to λ_i.

The purpose of the next few commands is now to extract the important values from this structure.

```
> nodes:=vector(N+1,i->eigen_all[i][1]);
#i-th node is the first component of i-th list
> orthovects:= vector(N+1,i->eigen_all[i][3][1]):
#i-th orthogonal eigenvector is the first component of the third item of
the i-th list.
```

The line

```
>intw:=int(w(x),x=a..b);
```

is responsible for computing the value of $\int_a^b w(x)dx$ which allows us to define the weights by

```
>weights:=vector(N+1,j->(orthovects[j][1])^2.*intw):,
```

hence j-th weight is the square if the first component of the j-th orthogonal eigenvector multiplied by $\int_a^b w(x)dx$.

In this point the anti-Gaussian formula is ready. Its nodes and weights are stored in the arrays of the corresponding names. The rest of the program computes the according Gaussian formula and evaluate the integral for a given function.

In sections 5 (Symmetric matrix **JG** for the "Gaussian" case) and 6 (Weights and nodes for the Gaussian formula) of the programme the Gaussian formula is derived using the same tools as in the third and fourth part for the anti-Gaussian formula. As the result of this we get nodes and weights for the N-point Gaussian formula stored in the arrays *nodesG* and *weightG*.

In the last part (Integration of given function) we first enter the function to be integrated and compute the "exact" value of the integral $\int_a^b w(x)f(x)dx$ using the implicit Maple facility for integration,

```
> f:=x->sin(x):
> exact:=int(w(x)*f(x),x=a..b);
```

and then we evaluate the integral, absolute error and relative error by using the anti-Gaussian formula:

```
> resultA:=sum(weights[s]*f(nodes[s]),s=1..N+1);
> abserrA:=simplify(abs(resultA-exact));
> relerrA:=abs(abserrA/resultA);
```

the Gaussian formula:

```
> resultG:=sum(weightsG[s]*f(nodesG[s]),s=1..N);
> abserrG:=simplify(abs(resultG-exact));
> relerrG:=abs(abserrG/resultG);
```

and the averaged formula $(H_w^{(N+1)} + G_w^{(N)})/2$.

```
> result_average:=(resultA+resultG)/2.;
> abserr_average:=abs(result_average-exact);
> relerr_average:=abs(abserr_average/result_average);
```

All together the program uses the weight function, the interval of integration, the recurrence relationship and the expected number of nodes as the inputs and gives the nodes and weights for both the Gaussian and anti-Gaussian quadratures. Finally it computes the integral of a given function with use of the formulas just derived and the averaged formula $(H_w^{(N+1)} + G_w^{(N)})/2$ each accompanied with the relative and absolute errors.

Some examples of the gained nodes and weights for various weight functions can be found in section 9.

4. Theoretical Properties

In this section we will formulate and prove some assertions concerning theoretical properties of the anti-Gaussian formulas. First we (among other facts) derive a necessary and sufficient condition for the anti-Gaussian for being internal and after that we show its applications in several classical cases.

Theorem 4.1. *The anti-Gaussian quadrature formula* $H_w^{(n+1)} = \sum_{i=1}^{n+1} \lambda_i f(\xi_i)$ *has the following properties:*

a) The weights $\lambda_i \geq 0\ \forall i = 1, 2, \ldots, n+1$.

b) The nodes $\xi_i,\ i = 1, 2, \ldots, n+1$ *are real and are interlaced by the nodes of the Gaussian formula* $G_w^{(n)}$, *i.e.*

$$\xi_1 < x_1 < \xi_2 < \cdots < x_n < \xi_{n+1}. \tag{4.1}$$

c) The nodes $\xi_2, \ldots, \xi_n$ *belong to the integration interval.*

d) $\xi_1 \in \langle a, b\rangle \Leftrightarrow \frac{\varphi_{n+1}(a)}{\varphi_{n-1}(a)} \geq \beta_{n+1}$ *and* $\xi_{n+1} \in \langle a, b\rangle \Leftrightarrow \frac{\varphi_{n+1}(b)}{\varphi_{n-1}(b)} \geq \beta_{n+1}$ *where* φ_j, $j = 0, 1, \ldots, n+1$ *and* β_{n+1} *are the same as in the formulas 2.24 and 2.26.*

Proof. The proposition a) follows immediately from the construction (see the formula 2.21).

In b) the nodes are the eigenvalues of the real symmetric matrix thus real. From the theory of orthogonal polynomials we know that the zeros of an i-th orthogonal polynomial are interlaced by the zeros of the $(i+1)$-th one. As the polynomials are orthogonal with respect to the original integral I and $2I - G_w^{(n)}$ are equal up to the degree n we see that the roots of $p_n (= \varphi_n)$ are interlaced by the zeros of p_{n+1}. Finally the zeros of φ_n are the nodes of $G_w^{(n)}$ and zeros of p_{n+1} are the nodes of $H_w^{(n+1)}$ which proves the statement.

c) follows trivially from b) and the fact that the Gaussian formula is internal.

d) We derive a condition for ξ_{n+1} to belong to $\langle a, b\rangle$. ξ_1 would be treated similarly. As all the zeros of φ_i $\forall i$ belong to (a, b) and the limit of a polynomial with a positive leading coefficient is ∞ when $n \to \infty$ holds that $\varphi_{n-1}(b) > 0, \varphi_{n+1}(b) > 0$ and $p_{n+1}(b) > 0 \Leftrightarrow p_{n+1}$ does not have any zeros greater or equal to b (which means $\xi_{n+1} \in (a, b)$). Using the relationships 2.27, 2.28 and 2.29 we get

$$p_{n+1}(x) = (x - \alpha_{n+1})\varphi_n(x) - 2\beta_{n+1}\varphi_{n-1}(x) \tag{4.2}$$

$$\varphi_{n+1}(x) = (x - \alpha_{n+1})\varphi_n(x) - \beta_{n+1}\varphi_{n-1}(x). \tag{4.3}$$

Subtracting 4.2 - 4.3 gives:

$$p_{n+1}(x) = \varphi_{n+1}(x) - \beta_{n+1}\varphi_{n-1}(x) \tag{4.4}$$

and if we substitute $x := b$ then

$$\xi_{n+1} \leq b \Leftrightarrow p_{n+1}(b) \geq 0 \Leftrightarrow \varphi_{n+1}(x) - \beta_{n+1}\varphi_{n-1}(x) \geq 0 \Leftrightarrow \frac{\varphi_{n+1}(b)}{\varphi_{n-1}(b)} \geq \beta_{n+1}.$$

□

Remark 4.2. An alternative proof of the proposition b) could be held with help of the Cauchy interlace theorem [9], which says that if we delete i-th row and i-th column from a real symmetric matrix, then the eigenvalues of the "new" matrix interlace those of the original one. The recurrence coefficients for $G_w^{(n)}$ and $H_w^{(n+1)}$ are equal up to the index n so the nodes of $G_w^{(n)}$ are the eigenvalues of a matrix which we get by deleting the last row and column from the matrix in $\mathbf{T}$ in the formula 2.7. Concerning the fact that the nodes of $H_w^{(n+1)}$ are the eigenvalues of $\mathbf{T}$ and that $\mathbf{T}$ can be without the loss of generality considered to be symmetric and we can apply the Cauchy interlace theorem to obtain the desired result.

Theorem 4.3. *The anti-Gaussian formulas corresponding to the following weight functions are internal*

1) $w(x) = x^\alpha e^{-x}$ *in* $\langle 0, \infty)$ *where* $\alpha > -1$ (Generalized Laguere)

2) $w(x) = |x|^\alpha e^{-x^2}$ *in* $(-\infty, \infty)$ *where* $\alpha > -1$ (Generalized Hermite)

3) $w(x) = (1 - x^2)^\alpha$ *in* $\langle -1, 1\rangle$ *where* $\alpha \geq -\frac{1}{2}$ (Gegenbauer),

including special cases $\alpha = 0$ (Legendre), $\alpha = -\frac{1}{2}$ (Chebyshev) *and* $\alpha = \frac{1}{2}$ (Chebyshev, second kind).

Proof. For this proof and also later in this work we will work with the data in the tables 22.2 (Orthogonality Relations), 22.3 (Explicit Expressions), 22.4 (Special Values) and 22.7 (Recurrence Relations) in [1]. For more convenience a copy of the data, which we will need more than once can be found in section 9.

1) We need to verify the assumption of the theorem 4.1 d) at the point 0. In line with [1] let us denote the corresponding set of the orthogonal polynomials by $L_i^{(\alpha)}(x)$. From the table 22.3 in [1] we can see that the polynomials $L_i^{(\alpha)}(x)$ are considered to have the leading coefficient $\frac{(-1)^i}{i!}$ so the values $L_i^{(\alpha)}(0) = \binom{i+\alpha}{i}$ read from the table 22.4 in [1] need to be divided by $\frac{(-1)^i}{i!}$ for our purpose. In the following we shall write

$$l_i^{(\alpha)}(x) = L_i^{(\alpha)}(x)\frac{i!}{(-1)^i}. \tag{4.5}$$

Now

$$\frac{l_{n+1}^{(\alpha)}(0)}{l_{n-1}^{(\alpha)}(0)} = \frac{\binom{n+1+\alpha}{n+1}\frac{(n+1)!}{(-1)^{n+1}}}{\binom{n-1+\alpha}{n-1}\frac{(n-1)!}{(-1)^{n-1}}} = \frac{\frac{(n+1+\alpha)!}{(n+1)!\alpha!}(n+1)n}{\frac{(n-1+\alpha)!}{(n-1)!\alpha!}} = \frac{(n+1+\alpha)!}{(n-1+\alpha)!} = (n+1+\alpha)(n+\alpha). \tag{4.6}$$

From the table 22.7 in [1] we can deduce the recurrence relationship

$$(n+1)L_{n+1}^{(\alpha)}(x) = [(2n+\alpha+1)-x]L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x). \tag{4.7}$$

However we need the recurrence for $l_{n+1}^{(\alpha)}(x)$. Using 4.5 we can proceed as follows:

$$(n+1)\frac{(-1)^{n+1}}{(n+1)!}l_{n+1}^{(\alpha)}(x) = [(2n+\alpha+1)-x]\frac{(-1)^n}{n!}l_n^{(\alpha)}(x) - (n+\alpha)\frac{(-1)^{n-1}}{(n-1)!}l_{n-1}^{(\alpha)}(x)$$

dividing both sides by $\frac{(-1)^{n+1}}{n!}$ we get

$$l_{n+1}^{(\alpha)}(x) = [(2n+\alpha+1)-x]\frac{(-1)^n}{n!}\frac{n!}{(-1)^{n+1}}l_n^{(\alpha)}(x) - (n+\alpha)\frac{(-1)^{n-1}}{(n-1)!}\frac{n!}{(-1)^{n+1}}l_{n-1}^{(\alpha)}(x)$$

or

$$l_{n+1}^{(\alpha)}(x) = [x-(2n+\alpha+1)]l_n^{(\alpha)}(x) - n(n+\alpha)l_{n-1}^{(\alpha)}(x),$$

which is the recurrence we wanted and hence β_{n+1} from the theorem 4.1 b) is $n(n+\alpha)$. It remains to prove that

$$(n+l+\alpha)(n+\alpha) \geq n(n+\alpha), \tag{4.8}$$

but this is trivial since we assumed that $\alpha > -1$.

The case of the generalized Hermite weight is obvious since the interval of integration is whole real axis and the Gegenbauer's weight will be subsumed as a special case in the next theorem. ☐

Theorem 4.4. *The anti-Gaussian formula $H_w^{(n+1)}$ where $n \geq 1$ corresponding to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$ where $\alpha, \beta > -1$ in $\langle -1, 1\rangle$ is internal if and only if*

$$(2\alpha+1)n^2 + (2\alpha+1)(\alpha+\beta+1)n + \frac{1}{2}(\alpha+1)(\alpha+\beta)(\alpha+\beta+1) \geq 0 \tag{4.9}$$

$$(2\beta+1)n^2 + (2\beta+1)(\alpha+\beta+1)n + \frac{1}{2}(\beta+1)(\alpha+\beta)(\alpha+\beta+1) \geq 0 \tag{4.10}$$

Proof. Let us denote the appropriate set of the orthogonal polynomials according to the [1] by $P_i^{(\alpha,\beta)}$. Again from the table 22.3 in [1] we see that the leading coefficient is chosen as $\frac{1}{2^i}\binom{2i+\alpha+\beta}{i}$ therefore be

$$p_i^{(\alpha,\beta)} = \frac{2^i}{\binom{2i+\alpha+\beta}{i}}P_i^{(\alpha,\beta)} \tag{4.11}$$

set of the orthogonal polynomials with the leading coefficient 1. Let us for example show that there are no nodes of $H_w^{(n+1)}$ exceeding 1 (the proof for -1 would be analogous). The table 22.4 gives us that $P_i^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$ and hence

$$(4.12)\quad \frac{p_{n+1}^{(\alpha,\beta)}(1)}{p_{n-1}^{(\alpha,\beta)}(1)} = \frac{\binom{n+1+\alpha}{n+1}2^{n+1}\binom{2n-2+\alpha+\beta}{n-1}}{\binom{n-1+\alpha}{n-1}2^{n-1}\binom{2n+2+\alpha+\beta}{n+1}} = 4\frac{\frac{(n+1+\alpha)!}{(n+1)!\alpha!}\cdot\frac{(2n-2+\alpha+\beta)!}{(n-1)!(n-1+\alpha+\beta)}}{\frac{(n-1+\alpha)!}{(n-1)!\alpha!}\cdot\frac{(2n+2+\alpha+\beta)!}{(n+1)!(n+1+\alpha+\beta)}} =$$

$$= 4\frac{(n+1+\alpha)!(2n-2+\alpha+\beta)!(n+1+\alpha+\beta)!}{(n-1+\alpha)!(2n+2+\alpha+\beta)!(n-1+\alpha+\beta)} =$$

$$= 4\frac{(n+\alpha)(n+1+\alpha)(n+\alpha+\beta)(n+1+\alpha+\beta)}{(2n-1+\alpha+\beta)(2n+\alpha+\beta)(2n+1+\alpha+\beta)(2n+2+\alpha+\beta)}.$$

During our search for an appropriate recurrence relationship the following notation will be useful:

$$(4.13)\qquad a_b = \prod_{i=0}^{b-1}(a+i) \text{ where } a,b\in\mathbb{Z}.$$

Let us start with the recurrence from the table 22.7:

$$2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(x) =$$
$$= [(2n+\alpha+\beta)_3x+(2n+\alpha+\beta+1)]P_n^{(\alpha,\beta)}(x)-2(n+\alpha)(n+\beta)(2n+2+\alpha+\beta)P_{n-1}^{(\alpha,\beta)}(x).$$

If we substitute $P_i^{(\alpha,\beta)}(x)$ by $p_i^{(\alpha,\beta)}(x)\frac{1}{2^i}\binom{2i+2+\alpha+\beta}{n+1}$, divide the equality by the coefficient at the left hand side and reduce the factorials a bit, we arrive to

$$p_{n+1}^{(\alpha,\beta)}(x) = \frac{(2n+\alpha+\beta)!}{(2n+2+\alpha+\beta)!(2n+\alpha+\beta)}[(2n+\alpha+\beta)_3x+(2n+1+\alpha+\beta)]p_n^{(\alpha,\beta)}(x)-$$
$$-4.\frac{n(n+\alpha+\beta)(n+\alpha)(n+\beta)(2n+2+\alpha+\beta)(2n-2+\alpha+\beta)!}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)!}p_{n-1}^{(\alpha,\beta)}(x)$$

and after some more reducing we finally get

$$p_{n+1}^{(\alpha,\beta)}(x) = \left[\quad x+\frac{1}{(2n+2+\alpha+\beta)(2n+\alpha+\beta)}\right]p_n^{(\alpha,\beta)}(x) -$$
$$- \quad 4\frac{n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2(2n+1+\alpha+\beta)(2n-1+\alpha+\beta)}p_{n-1}^{(\alpha,\beta)}(x).$$

Hence the recurrence coefficient β_{n+1} from the theorem 4.1 b) is

$$4\frac{n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2(2n+1+\alpha+\beta)(2n-1+\alpha+\beta)}.$$

Now it remains to find when $\frac{p_{n+1}^{(\alpha,\beta)}(1)}{\beta_{n+1}p_{n-1}^{(\alpha,\beta)}(1)} \geq 1$. Reminding the formula 4.12 we see that

$$\begin{aligned}
\frac{p_{n+1}^{(\alpha,\beta)}(1)}{\beta_{n+1}p_{n-1}^{(\alpha,\beta)}(1)} &= \frac{\frac{(n+\alpha)(n+1+\alpha)(n+\alpha+\beta)(n+1+\alpha+\beta)}{(2n-1+\alpha+\beta)(2n+\alpha+\beta)(2n+1+\alpha+\beta)(2n+2+\alpha+\beta)}}{\frac{n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2(2n+1+\alpha+\beta)(2n-1+\alpha+\beta)}} = \\
&= \frac{\frac{(n+1+\alpha)(n+1+\alpha+\beta)}{(2n+2+\alpha+\beta)}}{\frac{n(n+\beta)}{(2n+\alpha+\beta)}} = \\
&= \frac{(n+1+\alpha)(n+1+\alpha+\beta)(2n+\alpha+\beta)}{(2n+2+\alpha+\beta)n(n+\beta)} = \\
&= \frac{(n+1+\alpha)(n+1+\alpha+\beta)2(n+\frac{\alpha+\beta}{2})}{2(n+1+\frac{\alpha+\beta}{2})n(n+\beta)} = \\
&= 1 + \frac{(n+1+\alpha)(n+1+\alpha+\beta)(n+\frac{\alpha+\beta}{2}) - (n+1+\frac{\alpha+\beta}{2})n(n+\beta)}{(n+1+\frac{\alpha+\beta}{2})n(n+\beta)}.
\end{aligned}$$

As the denominator of the last fraction is positive, the condition of the theorem 4.1 b) holds if and only if the numerator is positive.

This can be shown as follows

$$\begin{aligned}
& (n+\alpha+1)(n+\alpha+\beta+1)(n+\frac{\alpha+\beta}{2}) - n(n+\beta)(n+1+\frac{\alpha+\beta}{2}) = \\
= \; & (n+\alpha+1)(n+\alpha+\beta+1)n + (n+\alpha+1)(n+\alpha+\beta+1)\frac{\alpha+\beta}{2} - \\
& -n(n+\beta)(n+1) - n(n+\beta)\frac{\alpha+\beta}{2} = \\
= \; & \frac{\alpha+\beta}{2}[(n+\alpha+1)(n+\alpha+\beta+1) - n(n+\beta)] + \\
& +\alpha n(n+1+\alpha+\beta) + (n+1)n(n+1+\alpha+\beta) - n(n+\beta)(n+1) = \\
= \; & \frac{\alpha+\beta}{2}[(n+\alpha+1)(n+\alpha+\beta+1) - n(n+\beta)] + \\
& +\alpha n(n+1+\alpha+\beta) + \\
& +(n+1)n(n+\beta) + (n+1)n(\alpha+1) - (n+1)n(n+\beta) = \\
= \; & \frac{\alpha+\beta}{2}[n(n+\alpha+\beta+1) + (\alpha+1)(n+\alpha+\beta+1) - n(n+\beta)] + \\
& +\alpha n(n+1+\alpha+\beta) + (n+1)n(\alpha+1) = \\
= \; & \frac{\alpha+\beta}{2}[n(n+\alpha+\beta+1) + \\
& +(\alpha+1)n + (\alpha+1)(\alpha+\beta+1) - n(n+\beta)] + \\
& +\alpha n(n+1+\alpha+\beta) + (n+1)n(\alpha+1) =
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{\alpha+\beta}{2}[n(n+\alpha+\beta+1)+(\alpha+1)n-n(n+\beta)]+ \\
& +\frac{\alpha+\beta}{2}(\alpha+1)(\alpha+\beta+1)+ \\
& +\alpha n(n+1+\alpha+\beta)+(n+1)n(\alpha+1)= \\
= \quad & \frac{\alpha+\beta}{2}(2n+2n\alpha)+ \\
& +\frac{\alpha+\beta}{2}(\alpha+1)(\alpha+\beta+1)+ \\
& +\alpha n(n+1+\alpha+\beta)+(n+1)n(\alpha+1)= \\
= \quad & n[(\alpha+\beta)(\alpha+1)+\alpha(n+1+\alpha+\beta)+(n+1)(\alpha+1)]+ \\
& +\frac{\alpha+\beta}{2}(\alpha+1)(\alpha+\beta+1)= \\
= \quad & n[(n+1+\alpha+\beta)(\alpha+1)+\alpha(n+1+\alpha+\beta)]+ \\
& +\frac{\alpha+\beta}{2}(\alpha+1)(\alpha+\beta+1)= \\
= \quad & n(n+1+\alpha+\beta)(2\alpha+1)+\frac{\alpha+\beta}{2}(\alpha+1)(\alpha+\beta+1)= \\
= \quad & (2\alpha+1)n^2+(2\alpha+1)n(\alpha+\beta+1)+\frac{\alpha+\beta}{2}(\alpha+1)(\alpha+\beta+1).
\end{aligned}
$$

Hence all the nodes are smaller or equal 1 if and only if the last formula is positive, which is exactly what we wanted to prove. When proving the fact that there are no nodes smaller than -1 we get to the assumption 4.10.

□

Now some sufficient conditions for the anti-Gaussian formula for the Jacobi weight function to require an exterior node can be deduced. For example if $\alpha < -\frac{1}{2}$, then the formula needs an exterior node if n is large enough, because the coefficient in front of n^2 in 4.9 is negative. If $\alpha = -\frac{1}{2}$ and $\beta \in (-\frac{1}{2}, \frac{1}{2})$ then the formula requires an exterior node for every n since then the left hand side of 4.9 is independent of n and negative. If $\beta \in (-\frac{1}{2}, \frac{1}{2})$ then we can find α close enough to $-\frac{1}{2}$ such that the coefficients in front of n and n^2 will be small and the last term on left hand side of 4.9 will be negative and therefore for n sufficiently small an exterior node will be required.

Although the result stated in the previous theorem is precise we shall now show a (weaker) proposition for the anti-Gaussian formula for the Jacobi weight to be internal this time without the dependence on n.

Theorem 4.5. *The anti-Gaussian formula $H_w^{(n+1)}$ for the Jacobi weight function $w(x) = (1-x)^\alpha(1+x)^\beta$ in $\langle -1, 1\rangle$ is internal for every $n \geq 1$ if α and β satisfies the following conditions:*

$$\alpha \geq -\frac{1}{2}, \ \beta \geq -\frac{1}{2}$$

$$(2\alpha+1)(\alpha+\beta+2)+\frac{1}{2}(\alpha+1)(\alpha+\beta)(\alpha+\beta+1) \geq 0 \tag{4.14}$$

$$(2\beta+1)(\alpha+\beta+2)+\frac{1}{2}(\beta+1)(\alpha+\beta)(\alpha+\beta+1) \geq 0. \tag{4.15}$$

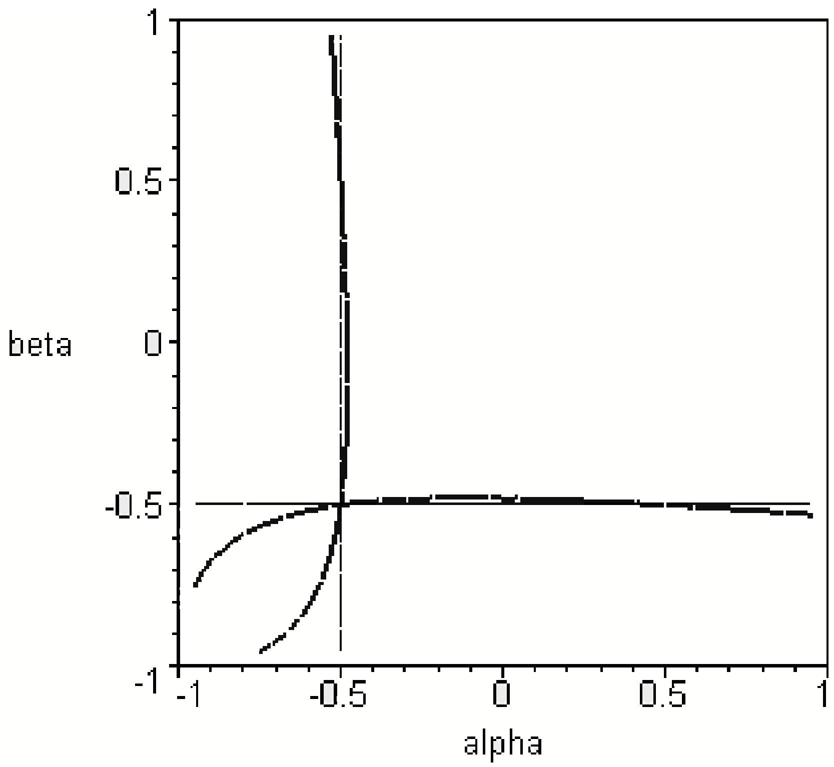

FIGURE 1. The anti-Gaussian formulas for the Jacobi weight function are internal for all n if α and β lie in the region to the north-east of the heavy lines.

Proof. If α and β are greater or equal to $-\frac{1}{2}$ then the coefficients in front of n and n^2 in the conditions 4.9 and 4.10 are nonnegative. Therefore the polynomials on their left hand sides are nondecreasing functions of n and it is enough to show that they are nonnegative for $n = 1$. As the conditions 4.14 and 4.15 were gained by substituting $n = 1$ in 4.9 and 4.10, the proof is finished. $\square$

Now let us remind the Gegenbauer's weight function from the theorem 4.3. When $\alpha = \beta \geq -\frac{1}{2}$ the conditions 4.14 and 4.15 reduce to $(2\alpha + 1)(\alpha + 1)(\alpha + 2) \geq 0$, which holds.

5. Convergence of the Anti-Gaussian Formulas

So far we have been considering the anti-Gaussian formulas only as a tool for integrating polynomials. In this section we shall show that with increasing n the formula $H_w^{(n+1)} f$ converges to $\int_a^b w(x)f(x)dx$ for any continuous function f.

First let us consider internal anti-Gaussian formulas. Probably the simplest way is to start with the Weierstrass approximation theorem (see e.g.[11]) claiming that every continuous function can be arbitrarily well approximated by a polynomial. To be precise that $\forall \varepsilon > 0$ and for any function f continuous in the interval of integration there exists a polynomial p such that

$$\|f - p\| < \varepsilon \tag{5.1}$$

where the norm is the maximum norm defined as follows:

$$\|f\| = \max_{x \in \langle a,b \rangle} |f(x)| \, \forall f \in C(\langle a, b \rangle)$$

Our task is now to show that $\forall \varepsilon$ positive and every continuous function f(x) there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$

$$\left| \int_a^b w(x)f(x)dx - H_w^{(n+1)} \right| < \varepsilon. \tag{5.2}$$

Let us have a given function f and a positive constant ε. By the Weierstrass theorem there exists a polynomial p such that

$$\|f - p\| < \frac{\varepsilon}{2 \int_a^b w(x)dx}.$$

The purpose of this complicated choice of the bound will be seen later.

From now on let $n > n_0$ where n_0 is so large that the formula $H_w^{(n_0+1)}$ integrates the polynomial p exactly. Now if we write $H_w^{(n+1)}$ in the form $\sum\limits_{i=1}^{n+1} \lambda_i^{(n)} f(\xi_i^n)$ and add and subtract $\int_a^b w(x)p(x)dx$ and $\sum_{i=1}^{n+1} \lambda_i^{(n)} p(\xi_i^n)$ inside the absolute value, we can write 5.2 as

$$\begin{aligned} \left| \int_a^b w(x)f(x)dx - H_w^{(n+1)} \right| &= \left| \int_a^b w(x)f(x)dx - \int_a^b w(x)p(x)dx + \int_a^b w(x)p(x)dx - \right. \\ &\left. - \sum_{i=1}^{n+1} \lambda_i^{(n)} p(\xi_i^n) + \sum_{i=1}^{n+1} \lambda_i^{(n)} p(\xi_i^n) - \sum_{i=1}^{n+1} \lambda_i^{(n)} f(\xi_i^n) \right| \end{aligned}$$

and using the triangle inequality

$$
(5.3) \quad \begin{aligned}
\left| \int_a^b w(x)f(x)dx - H_w^{(n+1)} \right| &\leq \left| \int_a^b w(x)f(x)dx - \int_a^b w(x)p(x)dx \right| + \\
&+ \left| \int_a^b w(x)p(x)dx - \sum_{i=1}^{n+1} \lambda_i^{(n)} p(\xi_i^n) \right| + \\
&+ \left| \sum_{i=1}^{n+1} \lambda_i^{(n)} p(\xi_i^n) - \sum_{i=1}^{n+1} \lambda_i^{(n)} f(\xi_i^n) \right|
\end{aligned}
$$

furthermore thanks to 5.1 the first and third absolute value on the right can be estimated by

$$\frac{\varepsilon}{2\int_a^b w(x)dx} \int_a^b w(x)dx = \frac{\varepsilon}{2}$$

and

$$\frac{\varepsilon}{2\int_a^b w(x)dx} \sum_{i=1}^{n+1} \lambda_i^{(n)} = \frac{\varepsilon}{2}$$

since

$$\sum_{i=1}^{n+1} \lambda_i^{(n)} = \int_a^b w(x)dx \text{ (The formula is exact for constants).}$$

The second absolute value on the left hand side of 5.3 is equal to zero as for n large enough the anti-Gaussian formula for p is exact. So finally

$$\left| \int_a^b w(x)f(x)dx - H_w^{(n+1)} \right| \leq \varepsilon.$$

and the convergence is proved.

An alternative way to achieve our goal is to use the well known Banach-Steinhaus theorem (sometimes also called Uniform boundedness theorem):

Theorem 5.1. *Let X be a Banach space, Y be a normed linear space. Let T and $\{T_n\}_{n=1}^{\infty}$ belong to $L(X,Y)$, which is the space of continuous linear operators from X to Y. Then $T_n x \to Tx \Leftrightarrow$ the sequence $\{\| T_n \|\}_{n=1}^{\infty}$ is bounded AND $T_n x \to Tx$ for all $x \in D \subset X$ where D is dense in X.*

For the proof of this theorem please refer to any textbook of the functional analysis e.g. [8].

Proof of the convergence of the anti-Gaussian formulas can be now performed by a direct application of the Banach-Steinhaus theorem. The space X is considered to be the space $C(\langle a,b\rangle)$ of the functions continuous in the (closed) interval of integration equipped with the maximum norm. It is known that the space $C(\langle a,b\rangle)$ is complete in the maximum norm and therefore it is a Banach space. Obviously the space Y will be the set of real numbers. If we put $T_n = H_w^{n+1}$ we have to prove that there exists a real constant M independent of n such that

$$\| H_w^{(n+1)} \| \leq M \ \forall n \in \mathbb{N}$$

.

Using the definition of the operator norm we can write

$$\| H_w^{(n+1)} \| = \sup_{\|f\|=1} \| H_w^{(n+1)} f \| = \sup_{\|f\|=1} \left| \sum_{i=1}^{n+1} \lambda_i^{(n+1)} f(\xi_i^{(n+1)}) \right| .$$

The sum (and therefore the whole expression) can be estimated by using the above facts as follows:

$$\sup_{\|f\|=1} \left| \sum_{i=1}^{n+1} \lambda_i^{(n+1)} f(\xi_i^{(n+1)}) \right| \leq \sup_{\|f\|=1} \sum_{i=1}^{n+1} | \lambda_i^{(n+1)} | . | f(\xi_i^{(n+1)}) | .$$

As the norm of f is considered to be 1 then $| f(\xi_i^{(n+1)}) | \leq 1 \ \forall \ i = 1, \ldots, n+1$. Therefore

$$\sup_{\|f\|=1} \sum_{i=1}^{n+1} | \lambda_i^{(n+1)} | . | f(\xi_i^{(n+1)}) | \leq \sum_{i=1}^{n+1} | \lambda_i^{(n+1)} |$$

and as the weights $\lambda_i^{(n+1)}$ are nonnegative

$$\sum_{i=1}^{n+1} | \lambda_i^{(n+1)} | = \sum_{i=1}^{n+1} \lambda_i^{(n+1)} = \int_a^b w(x)dx =: M < \infty.$$

Not to forget the second assumption of the theorem we remind the fact that the space of polynomials is a dense subset of X and can therefore serve as the set D. Obviously for any polynomial p there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ holds

$$H_w^{(n+1)} p = \int_a^b w(x)p(x)dx,$$

hence as all the assumptions of the Banach-Steinhaus theorem are fulfilled, we have proved the convergence of the anti-Gaussian formulas for any function continuous in the appropriate interval.

In case that the formula requires an exterior node, both proofs can be held in exactly the same way with the only difference that we will need to change the definition of norm and the interval where the function f is supposed to be continuous. In particular the continuousness will have to be assumed in a larger interval containing all the nodes and the maximum in the definition of the norm will have to be taken over the larger interval as well.

Finally let me mention one more interesting tool which can be used when examining the convergence of the linear functionals, even if it can not be use for our problem.

Theorem 5.2 (Korovkin). *Let $\{L_n\}$ be a sequence of positive linear functionals on $C[0,1]$ satisfying $L_n(1) \to 1$, $L_n(x) \to c$ and $L_n(x^2) \to c^2$ for some $c \in [0,1]$. Then $L_n(f(x)) \to f(c)$ for every $f \in C[0,1]$.*

Proof of this theorem can be found e.g. in [6].

Remark 5.3. Positiveness of the linear functional L defined on $C[a,b]$ means that $L(f) > 0$ for all $f \in C[a,b]$ where $f(x) > 0$ $\forall x \in [a,b]$.

Remark 5.4. In case we are working on an interval $[a,b]$ different to $[0,1]$ we can use following linear transform:

$$y = y(x) = \frac{x-a}{b-a} \; \forall x \in [a,b],$$

mapping the interval $[a,b]$ to $[0,1]$.

Conversely

$$x = y(b-a) + a,$$

which yields

$$f(x) = f(y(b-a)+a) = g(y),$$

where $g(y) \in C[0,1]$.

Finally if $L_n(1) \to k \neq 1$ we can use the sequence $\{\frac{L_n}{k}\}$ instead.

Unfortunately in our case there is no use for the theorem since the functionals $H_w^{(n+1)}$ do not fulfill the assumptions of the theorem at all (and even the claim of the theorem is not what we want to prove). This can be easily seen from the following considerations:

As for any $n \geq 2$ the anti-Gaussian formula is precise for x and x^2, it holds that

$$c := \lim_{n\to\infty} H_w^{(n+1)}(x) = H_w^{(3)}(x)$$

and

$$d := \lim_{n\to\infty} H_w^{(n+1)}(x^2) = H_w^{(3)}(x^2).$$

To satisfy the assumptions of the theorem it would have to be valid that $c^2 = d$ which means

$$(H_w^{(3)}(x))^2 = H_w^{(3)}(x^2)$$

hence

$$\left(\sum_{i=0}^{2} \lambda_i x_i\right)^2 = \sum_{i=0}^{2} \lambda_i x_i^2$$

which is obviously not true.

We shall end this section with the remark that the same methods can be used (and analogous results achieved) when examining the convergence of the sequence of averaged formulas $\frac{1}{2}(G_w^{(n)} + H_w^{(n+1)})$ since their weights are positive and they integrate constants exactly.

6. Error Estimates

In this section we derive error estimates for the anti-Gaussian and "averaged" formulas. First we present the classical estimate containing a higher order derivative $f^{(n)}(\xi)$ evaluated at some intermediate point of the interval of integration I. Since this point will usually be unknown, we shall employ the estimate $\max_{x\in I}\left|f^{(n)}(x)\right|$ instead. Such an estimate is valid for every function which is smooth enough which guarantees wide applicability of it but has several disadvantages as well. In the first place, the higher order derivatives might be difficult to obtain or not available at all (e.g. if the values of the integrated function are given by some measurements in a few points only) and estimating their maximum can be even more serious problem. Secondly, different rules can lead to estimates involving the derivatives of different order so it becomes impossible to compare the accuracy of the rules.

That is why we introduce an alternative error estimate based on the idea in [2]. Since in [2] the description of the process of deriving the estimate is not to detailed, we shall add all the missing details. The resulting estimate will consist of a multiplicative constant depending only on the used rule and a certain the norm of the integrated function. The clear advantages of this concept are that we do not need to know the derivatives of the function operated upon and in some cases even the norm can be estimated without the knowledge of the values in all points of the interval of integration. The price one has to pay for this is that the estimate only holds for analytic functions and we have to be able to estimate the values of the corresponding holomorphic function on the unit complex circumference. Finally we show, that for certain weight functions the infinite sum included in the estimate can be replaced by a "finite" expression, on the contrary to [4] we stress that this trick can not be used for any quadrature rule.

6.1. **Classical Approach.** Let Q_k be a quadrature rule with algebraic degree of precision equal to k and f be a function with $k+1$ continuous derivatives in the interval of integration $I = \langle a; b\rangle$. Then for any $x \in I$ we can write the function $f(x)$ in the terms of its Taylor polynomial in the point $c = \frac{(a+b)}{2}$ as:

$$f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2}(x-c)^2+\cdots+\frac{f^{(k)}(c)}{k!}(x-c)^k+\frac{f^{(k+1)}(\xi_x)}{(k+1)!}(\xi_x-c)^{k+1}, \tag{6.1}$$

where $\xi_x \in I$ depends on x.

To make everything more transparent we introduce the notations

$$f_1(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(k)}(c)}{k!}(x-c)^k$$

and

$$f_2(x) = \frac{f^{(k+1)}(\xi_x)}{(k+1)!}(\xi_x - c)^{k+1}. \tag{6.2}$$

Denoting the error by E_kf we can write:

$$E_kf = \left|\int_a^b w(x)f(x)dx - Q_kf\right| = \left|\int_a^b w(x)f_1(x)dx - Q_kf_1 + \int_a^b w(x)f_2(x)dx - Q_kf_2\right|.$$

The degree of f_1 is k so the quadrature Q_k is precise for f_1 and the above formula simplifies to

$$E_k f = \left| \int_a^b w(x) f_2(x) dx - Q_k f_2 \right| \leq \left| \int_a^b w(x) f_2(x) dx \right| + |Q_k f_2| \, .$$

If we substitute from 6.2 we get:

$$\left| \int_a^b w(x) f_2(x) dx \right| + |Q_k f_2| = \left| \int_a^b w(x) \frac{f^{(k+1)}(\xi_x)}{(k+1)!} (\xi_x - c)^{k+1} dx \right| + \left| Q_k \frac{f^{(k+1)}(\xi_x)}{(k+1)!} (\xi_x - c)^{k+1} \right| .$$

Since ξ_x belongs to I we can estimate $\left|(\xi_x - c)^{k+1}\right|$ from above by $\left(\frac{b-a}{2}\right)^{k+1}$ and $\left|f^{(k+1)}(\xi_x)\right|$ by $\max_{x \in I} \left|f^{(k+1)}(x)\right|$.

This gives

$$E_k f \leq 2 \, \frac{\max_{x \in I} \left|f^{(k+1)}(x)\right|}{(k+1)!} \left(\frac{b-a}{2}\right)^{k+1} \int_a^b w(x) dx \tag{6.3}$$

because $Q_k 1 = \int_a^b w(x) dx$.

This means, that the estimate only depends on the degree of the quadrature rule in question and not explicitly on its nodes and weights. Since the Gaussian formula $G_{(w)}^{(n+1)}$ and the averaged formula $(G_w^{(n)} + H_w^{(n+1)})/2$ are both of the degree $2n+1$ the theoretical estimate above is the same for both of them.

6.2. **Alternative approach.** In the alternative estimate we will assume, that the interval of integration is $[-1, 1]$. Otherwise we can use the linear transform

$$y = y(x) = \frac{2x - b - a}{b - a} \; \forall x \in [a, b],$$

which maps the interval [a,b] to [-1,1].

Then

$$x = \frac{y(b-a) + b + a}{2}$$

and hence

$$g(y) := f\left(\frac{y(b-a) + b + a}{2}\right) = f(x)$$

is the transform of the function f to the interval [-1,1]. The analogous transform needs to be performed with the weight function and the nodes of the quadrature formula and one has to keep in mind that the result will be proportional to the original one with the factor $\frac{b-a}{2}$. Formally we are just using the substitution

$$\int_a^b w(x) f(x) dx = \int_{-1}^1 w\left(\frac{y(b-a) + b + a}{2}\right) f\left(\frac{y(b-a) + b + a}{2}\right) \frac{b-a}{2} \, dy.$$

To avoid handling these complicated expressions, we will simply assume that the interval of integration $[a, b]$ is equal to $[-1, 1]$ in the following.

For any function f analytic in [-1,1] there exists the Taylor series

$$f(x) = \sum_{i=0}^{\infty} a_i x^i \ \forall x \in [-1, 1].$$

Any such function can be extended to a function holomorphic for all complex z in the unit circle $|z| \leq 1$ as

$$f(z) = \sum_{i=0}^{\infty} a_i z^i \ \forall z \in \mathbb{C} \text{ such that } |z| \leq 1.$$

Let us denote the boundary of the unit circle by C. By the Cauchy formula the function f possesses the representation

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} d\xi. \tag{6.4}$$

In the following we will already work with the anti Gaussian formula $H_w^{(n+1)} = \sum_{j=1}^{n+1} \lambda_j f(x_j)$. However the process of deriving the formula estimating the error does not anyhow depend on the used quadrature rule.

We shall try to estimate the error Ef defined as

$$Ef = \int_a^b w(x) f(x) dx - H_w^{(n+1)} f \tag{6.5}$$

where f is analytic in the interval $[-1, 1]$.
In the formula 6.5 we can write f in the form given by 6.4, which gives

$$Ef = \int_a^b w(x) \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - x} d\xi dx - \sum_{j=0}^{n+1} \lambda_j \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - x_j} d\xi. \tag{6.6}$$

The right hand side can be rewritten as

$$\frac{1}{2\pi i} \left(\int_a^b \int_C w(x) \frac{f(\xi)}{\xi - x} d\xi dx - \sum_{j=0}^{n+1} \int_C \lambda_j \frac{f(\xi)}{\xi - x_j} d\xi \right).$$

Using the Fubini theorem we can interchange the order of integration in the first term and the second term can be replaced by the integral of the sum. Hence

$$Ef = \frac{1}{2\pi i} \left(\int_C \int_a^b w(x) \frac{f(\xi)}{\xi - x} dx d\xi - \int_C \sum_{j=0}^{n+1} \lambda_j \frac{f(\xi)}{\xi - x_j} d\xi \right)$$

or

$$Ef = \frac{1}{2\pi i}\int_C \int_a^b w(x)\frac{f(\xi)}{\xi - x}dx - \sum_{j=0}^{n+1}\lambda_j \frac{f(\xi)}{\xi - x_j}\ d\xi.$$

$f(\xi)$ can be moved behind the first integration sign, which implies

$$Ef = \frac{1}{2\pi i}\int_C f(\xi)\left(\int_a^b w(x)\frac{1}{\xi - x}dx - \sum_{j=0}^{n+1}\lambda_j \frac{1}{\xi - x_j}\right) d\xi.$$

If we compare the term in the round brackets with the formula 6.5, we see that it is exactly $E(\frac{1}{\xi - x})$, where $\frac{1}{\xi - x}$ is considered as a function of x with the parameter ξ.

Therefore we can write

$$Ef = \frac{1}{2\pi i}\int_C f(\xi)E\left(\frac{1}{\xi - x}\right) d\xi. \tag{6.7}$$

The error can be now estimated as follows:

$$|Ef|^2 \leq \frac{1}{4\pi^2}\left|\int_C f(\xi)E\left(\frac{1}{\xi - x}\right) d\xi\right|^2 \tag{6.8}$$

If we use the parametrization $\varphi(t) = e^{it},\ t \in [0, 2\pi]$ of the circumference C we get

$$|Ef|^2 \ \leq \ \frac{1}{4\pi^2}\left|\int_0^{2\pi} f(\varphi(t))E\left(\frac{1}{\varphi(t) - x}\right)\varphi'(t)dt\right|^2.$$

The Hölder inequality allows us to estimate the right hand side from above by

$$|Ef|^2 \leq \frac{1}{4\pi^2}\left|\left(\int_0^{2\pi} |f(\varphi(t))|^2\, dt\right)^{\frac{1}{2}}\left(\int_0^{2\pi} \left|E\left(\frac{1}{\varphi(t) - x}\right)\varphi'(t)\right|^2 dt\right)^{\frac{1}{2}}\right|^2.$$

The linear (see 6.5) functional E "looks" at the function $\frac{1}{\varphi(t) - x}$ as at the function of x, where t is just a parameter. As the expression $\varphi'(t)$ does not depend on x we can use the linearity of E to get

$$|Ef|^2 \leq \frac{1}{2\pi}\|f(\varphi)\|^2\left|\left(\int_0^{2\pi}\left|E\left(\frac{\varphi'(t)}{\varphi(t) - x}\right)\right|^2 dt\right)^{\frac{1}{2}}\right|^2 \tag{6.9}$$

where the norm $\|.\|$ corresponds to the inner product

$$\langle u, v\rangle = \frac{1}{2\pi}\int_0^{2\pi} u(t)\overline{v(t)}dt, \tag{6.10}$$

where $\overline{z}$ denotes the complex conjugate of z. Hence

$$\|f(\varphi)\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\varphi(t))|^2\, dt\right)^{\frac{1}{2}}.$$

Since $\varphi(t) = e^{it}$ and $\varphi'(t) = ie^{it}$ it holds that

$$\begin{aligned}\int_0^{2\pi} \left| E\left(\frac{\varphi'(t)}{\varphi(t)-x}\right)\right|^2 dt &= \int_0^{2\pi} \left| E\left(\frac{ie^{it}}{e^{it}-x}\right)\right|^2 dt \\ &= \int_0^{2\pi} \left| E\left(\frac{1}{1-xe^{-it}}\right) i\right|^2 dt \\ &= \int_0^{2\pi} \left| E\left(\frac{1}{1-xe^{-it}}\right)\right|^2 dt\end{aligned}$$

hence

$$|Ef|^2 \le \frac{1}{2\pi} \|f(\varphi)\|^2 \int_0^{2\pi} \left| E\left(\frac{1}{1-xe^{-it}}\right)\right|^2 dt \tag{6.11}$$

The function $\frac{1}{1-xe^{-it}}$ (please note that the independent variable is x, not t) can be represented in the form of the sum of the following quadratic sequence:

$$\frac{1}{1-xe^{-it}} = \sum_{j=0}^{\infty} x^j e^{-itj}.$$

As the functional E is linear it holds that

$$E\left(\frac{1}{1-xe^{-it}}\right) = E\left(\sum_{j=0}^{\infty} x^j e^{-itj}\right) = \sum_{j=0}^{\infty} E(x^j)e^{-itj}.$$

The set $\{e^{-itj}\}$ is orthonormal with respect to the inner product 6.10, because

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-itm}\overline{e^{-itn}}dt = \frac{1}{2\pi}\int_0^{2\pi} e^{-it(m-n)}dt = \begin{cases} 0 \text{ when } m \neq n \\ 1 \text{ when } m = n \end{cases}, \tag{6.12}$$

which allows us to use the Parseval equality

$$\left\| E\left(\frac{1}{1-xe^{-it}}\right)\right\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \left| E\left(\frac{1}{1-xe^{-it}}\right)\right|^2 dt = \sum_{j=0}^{\infty} \left|E(x^j)\right|^2.$$

Hence

$$\int_0^{2\pi} \left| E\left(\frac{1}{\varphi(t)-x}\right)\right|^2 dt = 2\pi \sum_{j=0}^{\infty} \left|E(x^j)\right|^2.$$

Substituting this into the formula 6.11 yields

$$|Ef|^2 \le \|f(\varphi)\|^2 \sum_{j=0}^{\infty} \left|E(x^j)\right|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\varphi(t))|^2 \, dt \sum_{j=0}^{\infty} \left|E(x^j)\right|^2$$

hence

$$|Ef| \le \frac{1}{\sqrt{2\pi}} \left[\int_0^{2\pi} \left| f(e^{it}) \right|^2 dt \right]^{\frac{1}{2}} \left[\sum_{j=0}^{\infty} \left| E(x^j) \right|^2 \right]^{\frac{1}{2}}. \tag{6.13}$$

If we introduce the notation

$$\sigma := \frac{1}{\sqrt{2\pi}} \left[\sum_{j=0}^{\infty} \left| E(x^j) \right|^2 \right]^{\frac{1}{2}}, \tag{6.14}$$

we can rewrite the estimate 6.13 in the form

$$|Ef| \le \sigma \left[\int_0^{2\pi} \left| f(e^{it}) \right|^2 dt \right]^{\frac{1}{2}}. \tag{6.15}$$

Remark 6.1. In the beginning of this subsection we have assumed the function f to be analytic. In fact this requirement is not to restrictive since the quadrature formula only uses the values of the function in a finite number of points which can always be interpolated by an analytic function.

We will assume that we are able to estimate the expression $\left[\int_0^{2\pi} |f(e^{it})|^2 dt \right]^{\frac{1}{2}}$ by $2\pi \max_{t \in [0,2\pi]} |f(e^{it})|$ and we shall turn our attention to σ.

If all the nodes of the quadrature formula lie inside the interval (-1,1), which for the anti-Gaussian formula does not have to be true (see section 4), it is possible to write σ in the "finite" form.

Let us denote

$$\tau_j = \int_{-1}^{1} w(x) x^j dx.$$

Then

$$E(x^j) \quad = \quad \tau_j - \sum_{i=0}^{n} \lambda_i^{(n)} (x_i^{(n)})^j$$

and we can write

$$\begin{aligned}
2\pi\sigma^2 &= \sum_{j=0}^{\infty}\left[\tau_j - H_w^{(n+1)}(x^j)\right]^2 \\
&= \sum_{j=0}^{\infty}\left[\tau_j - \sum_{k=0}^{n}\lambda_k (x_k)^j\right]^2 \\
&= \sum_{j=0}^{\infty}\left[\tau_j^2 - 2\tau_j\sum_{k=0}^{n}\lambda_k (x_k)^j + \left(\sum_{k=0}^{n}\lambda_k (x_k)^j\right)^2\right] \\
&= \sum_{j=0}^{\infty}\tau_j^2 - 2\sum_{j=0}^{\infty}\tau_j\sum_{k=0}^{n}\lambda_k (x_k)^j + \sum_{j=0}^{\infty}\left(\sum_{k=0}^{n}\lambda_k (x_k)^j\right)^2 \\
&= \sum_{j=0}^{\infty}\tau_j^2 - 2\sum_{k=0}^{n}\lambda_k\sum_{j=0}^{\infty}\tau_j (x_k)^j + \sum_{j=0}^{\infty}\left(\sum_{k=0}^{n}\lambda_k\sum_{l=0}^{n}\lambda_l (x_k)^j (x_l)^j\right)
\end{aligned}$$

Now we use the assumption that all the nodes x_i belong to $(-1,1)$, which allows us to sum the quadratic sequence in the last term. After this we have

$$\begin{aligned}
2\pi\sigma^2 &= \sum_{j=0}^{\infty}\tau_j^2 - 2\sum_{k=0}^{n}\lambda_k\sum_{j=0}^{\infty}\tau_j (x_k)^j + \sum_{k=0}^{n}\lambda_k\sum_{l=0}^{n}\lambda_l\sum_{j=0}^{\infty}(x_k)^j (x_l)^j \\
&= \sum_{j=0}^{\infty}\tau_j^2 - 2\sum_{k=0}^{n}\lambda_k\sum_{j=0}^{\infty}\tau_j (x_k)^j + \sum_{k=0}^{n}\lambda_k\sum_{l=0}^{n}\lambda_l\frac{1}{1-x_k x_l}.
\end{aligned}$$

From the definition of τ_j we can easily sum the infinite sums in the first and second term. We get

$$\begin{aligned}
\sum_{j=0}^{\infty}\tau_j^2 &= \sum_{j=0}^{\infty}\int_{-1}^{1} w(y)y^j dy\int_{-1}^{1} w(x)x^j dx \\
&= \int_{-1}^{1}\int_{-1}^{1} w(y)w(x)\sum_{j=0}^{\infty} y^j x^j dx\ dy \\
&= \int_{-1}^{1}\int_{-1}^{1}\frac{w(y)w(x)}{1-xy}dx\ dy
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{\infty}\tau_j (x_k)^j &= \sum_{j=0}^{\infty}\int_{-1}^{1} w(x)x^j (x_k)^j dx \\
&= \int_{-1}^{1} w(x)\sum_{j=0}^{\infty} x^j (x_k)^j dx \\
&= \int_{-1}^{1}\frac{w(x)}{1-xx_k}dx.
\end{aligned}$$

This finally gives

$$2\pi\sigma^2 = \int_{-1}^{1}\int_{-1}^{1}\frac{w(y)w(x)}{1-xy}dx\,dy - 2\sum_{k=0}^{n}\lambda_k\int_{-1}^{1}\frac{w(x)}{1-xx_k}dx + \tag{6.16}$$
$$+\sum_{k=0}^{n}\lambda_k\sum_{l=0}^{n}\lambda_l\frac{1}{1-x_kx_l},$$

so the estimate 6.15 can be rewritten as

$$|Ef| \leq \frac{1}{\sqrt{2\pi}}\left(\int_{-1}^{1}\int_{-1}^{1}\frac{w(y)w(x)}{1-xy}dx\,dy - 2\sum_{k=0}^{n}\lambda_k\int_{-1}^{1}\frac{w(x)}{1-xx_k}dx + \right. \tag{6.17}$$
$$\left. +\sum_{k=0}^{n}\lambda_k\sum_{l=0}^{n}\lambda_l\frac{1}{1-x_kx_l}\right)^{\frac{1}{2}}\left[\int_0^{2\pi}\left|f(e^{it})\right|^2dt\right]^{\frac{1}{2}}.$$

If there exists one or more nodes not belonging to (-1,1) then the process we have just presented cannot be used. The logical idea is to perform another linear transform which would "compress" the nodes into (-1,1) by dividing them by an appropriate constant a. Unfortunately it crashes on the fact that the integrated function $f(x) = x^j$ will be transformed to $f(ax) = (ax)^j$ so finally in the quadrature formula we will have $(a\frac{x_i}{a})^j$ again.

7. Practical results

In this section we first give the particular forms of the classical error estimate 6.3 for several weight functions and demonstrate its useability for different integrated functions. Next we compare the values of σ in the estimate 6.15 for different quadrature rules. Finally suggest how the estimate 6.15 could be improved.

After substituting a, b and $\int_a^b w(x)dx$ into 6.3 we get these particular forms of the estimate for the chosen weight functions:

For Legendre and Jacobi (with $\alpha = 1,\ \beta = 0$) weight functions holds

$$E_{2n+1}f \leq \frac{4}{(2n+2)!}\max_{x\in[-1,1]}\left|f^{(2n+2)}(x)\right|,$$

(In both cases $\int_a^b w(x)dx = 2$, therefore the estimates are the same)

for Chebyshev weight function of the first kind (or Jacobi weight function with $\alpha = \beta = -\frac{1}{2}$)

$$E_{2n+1}f \leq \frac{\pi}{(2n+2)!}\max_{x\in[-1,1]}\left|f^{(2n+2)}(x)\right|$$

and for Chebyshev weight function of the second kind (or Jacobi weight function with $\alpha = \beta = \frac{1}{2}$)

$$E_{2n+1}f \leq \frac{2\pi}{(2n+2)!}\max_{x\in[-1,1]}\left|f^{(2n+2)}(x)\right|.$$

In the following tables one can see how (in)accurate the estimates are in practice. In each case we compare the error estimate with the actual errors of the $(n+1)$-point Gaussian formula and the averaged formula $L_w^{(2n+1)} = \frac{1}{2}(G_w^{(n)} + H_w^{(n+1)})$. The number of nodes n will be set to 5 and 6 each time. Presented results were obtained from the data generated by the programme discussed in section 3. To prevent confusing the decimal point and multiplication, we shall use the notation Ek instead of 10^k.

Legendre weight function ($w(x) = 1$),
n=5

f(x)	error estimate	error of $G_w^{(n+1)}$	error of $L_w^{(2n+1)}$
$sin(6x)$	18.17	$2.4E{-}19$	$4.8E{-}19$
$sin\left(\frac{x}{6}\right)$	$6E{-}19$	$8E{-}21$	$2E{-}19$
e^{6x}	7333.39	$6.2E{-}3$	$2.3E{-}5$
$e^{\frac{x}{6}}$	$1E{-}17$	$1E{-}19$	$1E{-}19$

n=6

f(x)	error estimate	error of $G_w^{(n+1)}$	error of $L_w^{(2n+1)}$
$sin(6x)$	3.6	$3.4E{-}19$	$1.5E - 19$
$sin\left(\frac{x}{6}\right)$	$9E{-}23$	$1.1E{-}20$	$1E{-}20$
e^{6x}	1450.56	$2.8E{-}4$	$6.2E{-}7$
$e^{\frac{x}{6}}$	$6.9E{-}22$	$1E{-}19$	$1E{-}19$

Jacobi weight function ($w(x) = (1-x)^{\alpha}(1+x)^{\beta}$) with $\alpha = 1,\ \beta = 0$,
n=5

f(x)	error estimate	error of $G_w^{(n+1)}$	error of $L_w^{(2n+1)}$
$sin(6x)$	18.17	$4E{-}4$	$3E{-}6$
$sin\left(\frac{x}{6}\right)$	$6E{-}19$	$5E{-}19$	$4.8E{-}19$
e^{6x}	7333.39	$2E{-}3$	$1E{-}5$
$e^{\frac{x}{6}}$	$1E{-}17$	$1.3E{-}18$	$1.2E{-}18$

n=6

f(x)	error estimate	error of $G_w^{(n+1)}$	error of $L_w^{(2n+1)}$
$sin(6x)$	3.6	$1.9E{-}5$	$3E - 8$
$sin\left(\frac{x}{6}\right)$	$9E{-}23$	$4.8E{-}19$	$4.8E{-}19$
e^{6x}	1450.56	$1E{-}4$	$3.2E{-}7$
$e^{\frac{x}{6}}$	$6.9E{-}22$	$1.3E{-}18$	$1.1E{-}18$

Chebyshev weight function of the first kind ($w(x) = (1-x^2)^{-\frac{1}{2}}$)
n=5

f(x)	error estimate	error of $G_w^{(n+1)}$	error of $L_w^{(2n+1)}$
$sin(6x)$	28.50	$5.1E{-}19$	$2.3E{-}19$
$sin\left(\frac{x}{6}\right)$	$9.9E{-}19$	$3.3E{-}20$	$2.8E{-}20$
e^{6x}	11519.26	$1.3E{-}2$	$1.3{-}8$
$e^{\frac{x}{6}}$	$7E{-}18$	$2E{-}19$	$2E{-}19$

n=6

f(x)	error estimate	error of $G_w^{(n+1)}$	error of $L_w^{(2n+1)}$
$sin(6x)$	5.6	$9.4E{-}19$	$1.6E - 19$
$sin\left(\frac{x}{6}\right)$	$1.5E{-}22$	$5.9E{-}20$	$1.5E{-}20$
e^{6x}	2278.53	$6.2E{-}4$	$4E{-}12$
$e^{\frac{x}{6}}$	$1E{-}21$	$6E{-}19$	$3E{-}19$

Chebyshev weight function of the second kind ($w(x) = (1 - x^2)^{\frac{1}{2}}$)
n=5

f(x)	error estimate	error of $G_w^{(n+1)}$	error of $L_w^{(2n+1)}$
$sin(6x)$	14.27	$3E{-}19$	$2.1E{-}19$
$sin\left(\frac{x}{6}\right)$	$4.9E{-}19$	$9.1E{-}21$	$9.4E{-}21$
e^{6x}	5759.63	$3E{-}3$	$6.2{-}11$
$e^{\frac{x}{6}}$	$3.56E{-}18$	0	$7E{-}18$

n=6

f(x)	error estimate	error of $G_w^{(n+1)}$	error of $L_w^{(2n+1)}$
$sin(6x)$	2.82	$9.8E{-}20$	$1.8E - 19$
$sin\left(\frac{x}{6}\right)$	$7.6E{-}23$	$1.7E{-}21$	$1.7E{-}20$
e^{6x}	1139.273	$1.4E{-}4$	$1.34E{-}14$
$e^{\frac{x}{6}}$	$5.4E{-}22$	$1E{-}19$	0

We can see that for the functions where the differentiation causes multiplying by a constant grater than 1, the estimate is very pessimistic. On the other hand if the differentiation generates a constant smaller than 1 the estimate is quite accurate. An explanation of this phenomenon is this: We are taking the maximum of the n-th derivative instead of the derivative in some point $\xi \in [a, b]$ which we do not know. If there are a big differences among the values of the appropriate order derivative is different points of (a,b), the error committed by estimating the value of the derivative in ξ by the maximum over the whole interval can be huge. On the other hand if the graph of the high order derivative is "flat" (i.e. the difference between its minimum and maximum over (a,b) is small), we would not be far from the truth by taking the value at any point (or the maximum).

The reason why in some of the tables above the actual errors are greater than their estimates is that in those cases the values are on the edge of the accuracy used by the software (which was set to 20 digits) and these anomalies are the consequence of the computational error.

Comparing the accuracy of the Gaussian and anti-Gaussian formulas of the corresponding degree we can observe that in some cases (mostly for the function e^{6x}) the results obtained by the anti-Gaussian formula are a bit better.

Let us investigate whether there is any theoretical reason for this.

As we have shown in section 4. The nodes of the anti-Gaussian formulas for Legendre, Jacobi (with $\alpha = 1,\ \beta = 0$) and Chebyshev (second kind) weight functions are internal. Moreover their nodes lie inside (-1,1) and hence in these cases we can apply the estimate 6.17.

The formula 6.16 lets us to compute following values of σ (see 6.14 for the definition):

Legendre weight function:

n	σ for $G_w^{(n+1)}$	σ for $L_w^{(2n+1)}$
4	0.6098	0.4721
5	0.0835	0.0375
6	0.0724	0.0317
7	0.0638	0.0275

Jacobi weight function ($\alpha = 1,\ \beta = 0$):

n	σ for $G_w^{(n+1)}$	σ for $L_w^{(2n+1)}$
4	0.1286	0.0590
5	0.1100	0.0490
6	0.0961	0.0419
7	0.0854	0.0366

We can see that even if the degree of precision if always the same for both formulas in each row of the above tables, the theoretical error estimate from the subsection 6.2 gives much better results for the anti-Gaussian formulas. Although there is no analytic reason why they should be more accurate, this can be a good reason to believe, that they actually are better.

Even if we have not included the term $\left[\int_0^{2\pi} |f(e^{it})|^2\, dt\right]^{\frac{1}{2}}$ in the tables above, we can see that for the functions where the classical error estimate were to pessimistic the alternative one can lead to satisfactory results.

Turning back to the estimate 6.15 we can see, that the estimate would not be zero if we apply it to a polynomial of a degree which the quadrature rule integrates accurately. An idea how to remove or at least reduce this imperfection can be replacing f by a function g such that the difference between f and g is a polynomial which is integrated exactly and the value of $\left[\int_0^{2\pi} |g(e^{it})|^2\, dt\right]^{\frac{1}{2}}$ is less than $\left[\int_0^{2\pi} |f(e^{it})|^2\, dt\right]^{\frac{1}{2}}$. The best result would be achieved if $f - g$ is in some sense the best approximation of f by a polynomial of a given degree. However we would not go to the details in thesis.

8. Conclusion

In this thesis we have proved that the weights of the anti-Gaussian formulas are positive and that at most two nodes can lie outside the interval of integration. In section 4 conditions for the formulas to be internal are stated. Concerning the construction of the $(n+1)$-point anti-Gaussian formulas we have shown that it is the question of determining the eigenvalues and eigenvectors of certain $(n+1) \times (n+1)$ matrix. This means that the costs of constructing the $(n+1)$-point anti-Gaussian formula $H_w^{(n+1)}$ is the same as the cost of constructing the $(n+1)$-point Gaussian formula $G_w^{(n+1)}$.

Practical testing has shown that the averaged formula $L_w^{(2n+1)} = (G_w^{(n)} + H_w^{(n+1)})/2$ gives in many cases better results than the (n+1)-point Gaussian formula $G_w^{(n+1)}$ even if the theoretical degree of precision is $2n+1$ in both cases. Moreover the error estimate derived in section 6.2 appears to be much better for the anti-Gaussian formula. In section 1 we have suggested the difference $G_w^{(n+1)} f - Af$ (where A is a quadrature formula of a higher degree) as a numerical approximation of the error $G_w^{(n+1)}$ applied to the function f.

The results presented in this thesis show that using the averaged formula $L_w^{(2n+1)}$ as the formula A would give more accurate estimate than using the formula $G_w^{(n+1)}$. Since deriving both formulas requires the same amount of operations (assuming we already have the n-point Gaussian formula) and the number of points where the integrated function f has to be evaluated are the same in both cases as well we can recommend using the averaged formula instead of the Gaussian formula in the above numerical estimate.

9. Appendices

Table 9.1 Orthogonal Polynomials

Name of the polynomial	$p_n(x)$	$w(x)$	a	b	k_n	Remarks
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$(1-x)^{\alpha}(1+x)^{\beta}$	-1	1	$\frac{1}{2^n}\binom{2n+\alpha+\beta}{n}$	$\alpha,\beta > -1$
Ultraspherical (Gegenbauer)	$C_n^{(\alpha)}(x)$	$(1-x^2)^{\alpha-\frac{1}{2}}$	-1	1	$\frac{2^n}{n!}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$	$\alpha > -\frac{1}{2}$
Chebyshev of the first kind	$T_n(x)$	$(1-x^2)^{-\frac{1}{2}}$	-1	1	2^{n-1}	
Chebyshev of the second kind	$U_n(x)$	$(1-x^2)^{\frac{1}{2}}$	-1	1	2^n	
Legendre (Spherical)	$P_n(x)$	1	-1	1	$\frac{(2n)!}{2^n(n!)^2}$	
Generalized Laguerre	$L_n^{(\alpha)}(x)$	$e^{-x}x^{\alpha}$	0	∞	$\frac{(-1)^n}{n!}$	$\alpha > -1$
Laguerre	$L_n(x)$	e^{-x}	0	∞	$\frac{(-1)^n}{n!}$	
Hermite	$H_n(x)$	e^{-x^2}	$-\infty$	∞	2^n	
Hermite	$He_n(x)$	$e^{-\frac{x^2}{2}}$	$-\infty$	∞	1	

Table 9.2 Recurrence Relations

$$a_1(n)p_{n+1}(x) = (a_2(n) + a_3(n)x)p_n(x) - a_4(n)f_{n-1}(x)$$

p_n	$a_1(n)$	$a_2(n)$	$a_3(n)$	$a_4(n)$
$P_n^{(\alpha,\beta)}(x)$	$2(n+1)(n+\alpha+\beta+1)$ $(2n+\alpha+\beta)$	$(2n+\alpha+\beta+1)(\alpha^2-\beta^2)$	$(2n+\alpha+\beta)_3$	$2(n+\alpha)(n+\beta)$ $(2n+\alpha+\beta+2)$
$C_n^{(\alpha)}(x)$	$n+1$	0	$2(n+\alpha)$	$n+2\alpha-1$
$T_n(x)$	1	0	2	1
$U_n(x)$	1	0	2	1
$P_n(x)$	$n+1$	0	$2n+1$	n
$L_n^{(\alpha)}(x)$	$n+1$	$2n+\alpha+1$	-1	$n+\alpha$
$L_n(x)$	$n+1$	$2n+1$	-1	n
$H_n(x)$	1	0	2	$2n$
$He_n(x)$	1	0	1	n

Table 9.3

Nodes and weights of $H_w^{(n+1)}$ where $w(x) \equiv 1$ (Legendre)

n	nodes	weights	n	nodes	weights
1	.816496580927726	1.000000000000000	8	.893581190017803	.164411441988856
	-.816496580927726	1.000000000000000		-.000000000000000	.368957072484166
				-.993888391435683	.034415599386995
2	-.930949336251263	.384615384615385		-.893581190017803	.164411441988856
	.930949336251263	.384615384615385		-.360613635820053	.344040703534754
	-.000000000000000	1.230769230769231		.993888391435683	.034415599386995
				-.672520467240063	.272653718847312
3	.964335275879562	.199826014447922		.360613635820053	.344040703534754
	.429352058315787	.800173985552078		.672520467240063	.272653718847312
	-.964335275879562	.199826014447922			
	-.429352058315787	.800173985552078	9	-.734696655470703	.223654050135981
				.995105205867138	.027569114782485
4	-.000000000000000	.693766937669377		-.995105205867138	.027569114782485
	.978315678013417	.121787277062268		.475285190219825	.290430742781218
	-.638731398345590	.531329254103044		-.475285190219825	.290430742781218
	.638731398345590	.531329254103044		.164365837601352	.325730290621930
	-.978315678013417	.121787277062268		-.914477642987090	.132615801678385
				.914477642987090	.132615801678385
5	-.280556681820821	.545769074217690		.734696655470703	.223654050135981
	.280556681820821	.545769074217690		-.164365837601352	.325730290621930
	-.752558388054789	.372395751222672			
	.752558388054789	.372395751222672	10	-.929795638911367	.109154362380246
	.985446820998315	.081835174559638		-.562678595062891	.246927255598589
	-.985446820998315	.081835174559638		-.780937965408210	.186329092356386
				-.294419959277147	.285581325610890
6	-.989564901331163	.058719277436163		.995991885381824	.022578391655128
	.820496210793208	.273663854856191		.780937965408210	.186329092356386
	-.820496210793208	.273663854856191		.000000000000000	.298859144797520
	.463335883847022	.426675691237058		-.995991885381824	.022578391655128
	.989564901331163	.058719277436163		.294419959277147	.285581325610890
	-.463335883847022	.426675691237058		.929795638911367	.109154362380246
	.000000000000000	.481882352941176		.562678595062891	.246927255598589
7	-.586467949432683	.338113373846498			
	-.864059339845500	.208912408709088			
	.992154829409481	.044164752444346			
	.586467949432683	.338113373846498			
	-.992154829409481	.044164752444346			
	-.207447135295099	.408809465000068			
	.207447135295099	.408809465000068			
	.864059339845500	.208912408709088			

Table 9.4

Nodes and weights of $H_w^{(n+1)}$ where $w(x) = (1-x)^{-\frac{1}{2}}$ (Chebyshev, first kind)

n	nodes	weights	n	nodes	weights
1	-1.000000000000000	1.570796326794897	8	-.707106781186548	.392699081698724
	1.000000000000000	1.570796326794897		-.000000000000000	.392699081698724
				.923879532511287	.392699081698724
2	-.000000000000000	1.570796326794897		1.000000000000000	.196349540849362
	-1.000000000000000	.785398163397448		-.382683432365090	.392699081698724
	1.000000000000000	.785398163397448		-1.000000000000000	.196349540849362
				.707106781186548	.392699081698724
3	-.500000000000000	1.047197551196598		-.923879532511287	.392699081698724
	-1.000000000000000	.523598775598299		.382683432365090	.392699081698724
	.500000000000000	1.047197551196598			
	1.000000000000000	.523598775598299	9	-.500000000000000	.349065850398866
				1.000000000000000	.174532925199433
4	-1.000000000000000	.392699081698724		.939692620785908	.349065850398866
	.707106781186548	.785398163397448		.173648177666930	.349065850398866
	1.000000000000000	.392699081698724		.766044443118978	.349065850398866
	-.707106781186548	.785398163397448		-1.000000000000000	.174532925199433
	-.000000000000000	.785398163397448		-.173648177666930	.349065850398866
				.500000000000000	.349065850398866
5	-.809016994374947	.628318530717959		-.766044443118978	.349065850398866
	.809016994374947	.628318530717959		-.939692620785908	.349065850398866
	-1.000000000000000	.314159265358979			
	-.309016994374947	.628318530717959	10	.000000000000000	.314159265358979
	.309016994374947	.628318530717959		.809016994374947	.314159265358979
	1.000000000000000	.314159265358979		.951056516295154	.314159265358979
				1.000000000000000	.157079632679490
6	.866025403784439	.523598775598299		-.951056516295154	.314159265358979
	-.500000000000000	.523598775598299		-.309016994374947	.314159265358979
	-1.000000000000000	.261799387799149		-1.000000000000000	.157079632679490
	.000000000000000	.523598775598299		-.809016994374947	.314159265358979
	.500000000000000	.523598775598299		.309016994374947	.314159265358979
	-.866025403784439	.523598775598299		.587785252292473	.314159265358979
	1.000000000000000	.261799387799149		-.587785252292473	.314159265358979
7	-.900968867902419	.448798950512828			
	-.623489801858734	.448798950512828			
	1.000000000000000	.224399475256414			
	.900968867902419	.448798950512828			
	.222520933956314	.448798950512828			
	.623489801858734	.448798950512828			
	-.222520933956314	.448798950512828			
	-1.000000000000000	.224399475256414			

Table 9.5
Nodes and weights of $H_w^{(n+1)}$ where $w(x) = (1 - x)^{\frac{1}{2}}$ (Chebyshev, second kind)

n	nodes	weights
1	.707106781186548	.785398163397448
	-.707106781186548	.785398163397448
2	.000000000000000	1.047197551196598
	.866025403784439	.261799387799149
	-.866025403784439	.261799387799149
3	-.382683432365090	.670379265333622
	.923879532511287	.115018898063826
	.382683432365090	.670379265333622
	-.923879532511287	.115018898063826
4	.587785252292473	.411239817295253
	-.951056516295154	.059999080743216
	.951056516295154	.059999080743216
	.000000000000000	.628318530717959
	-.587785252292473	.411239817295253
5	.707106781186548	.261799387799149
	-.707106781186548	.261799387799149
	-.258819045102521	.488524308328427
	.965925826289068	.035074467269872
	-.965925826289068	.035074467269872
	.258819045102521	.488524308328427
6	.433883739117558	.364310259621239
	-.433883739117558	.364310259621239
	.000000000000000	.448798950512828
	.974927912181824	.022222534076746
	.781831482468030	.174465894443050
	-.974927912181824	.022222534076746
	-.781831482468030	.174465894443050
7	-.555570233019602	.271489257084905
	.831469612302545	.121209824613819
	-.831469612302545	.121209824613819
	-.980785280403230	.014946218840648
	.555570233019602	.271489257084905
	-.195090322016128	.377752862858077
	.980785280403230	.014946218840648
	.195090322016128	.377752862858077
8	-.642787609686539	.204840249603193
	.642787609686539	.204840249603193
	.342020143325669	.308232902689759
	-.866025403784439	.087266462599716
	.866025403784439	.087266462599716
	-.984807753012208	.010525623305347
	-.342020143325669	.308232902689759
	.984807753012208	.010525623305347
	-.000000000000000	.349065850398866
9	.987688340595138	.007688024442412
	-.156434465040231	.306471240916567
	-.987688340595138	.007688024442412
	.891006524188368	.064750541154967
	.707106781186548	.157079632679490
	-.891006524188368	.064750541154967
	-.707106781186548	.157079632679490
	.156434465040231	.306471240916567
	.453990499739547	.249408724204012
	-.453990499739547	.249408724204012
10	.281732556841430	.262930389642668
	-.281732556841430	.262930389642668
	-.000000000000000	.285599332144527
	.909631995354518	.049285771941039
	.540640817455598	.202120791210338
	-.540640817455598	.202120791210338
	.989821441880933	.005784389841430
	-.909631995354518	.049285771941039
	.755749574354258	.122477154689710
	-.755749574354258	.122477154689710
	-.989821441880933	.005784389841430

Table 9.6
Nodes and weights of $H_w^{(n+1)}$ where $w(x) = e^{-x}$ (Laguere)

n	nodes	weights
1	5.236067977499790	.276393202250021
	.763932022500210	.723606797749979
2	.493358053613672	.386975717692728
	8.716021885347972	.016600486503632
	2.790620061038356	.596423795803641
3	12.309046017390413	.000718795204851
	2.000000000000000	.642857142857143
	5.324782088068325	.115944898786902
	.366171894541262	.240479163151104
4	4.035899241536990	.225707562416030
	.291555083447512	.163789573089150
	1.570446368284971	.597954531467579
	8.132847536700815	.012522283363778
	15.969251770029712	.000026049663464
5	19.674659672659734	.000000843349820
	11.116040824928635	.000972233216916
	.242345652236039	.118690314809966
	3.279645965112819	.308670750659650
	6.391127476449262	.040647934398461
	1.296180408613512	.531017923565187
6	.207408536677957	.089943926027562
	2.771738036768087	.360932400305636
	5.317741941298702	.079398971149898
	23.412876446490799	.000000025246690
	14.222348454865667	.000060902740029
	1.104762626924511	.464703732319837
	8.963123956974277	.004960042210348
7	.963192868381727	.405616564458865
	.181303717030789	.070503768824035
	2.404059915514005	.388658800369186
	4.571665521898387	.121290949042948
	7.589603084182166	.013460003015817
	27.176159939067557	.000000000713378
	17.420666094980109	.000003276693310
	11.693348858945259	.000466636882461
8	20.690763117090409	.000000157277689
	.161050277991333	.056746980623896
	2.124444015263906	.399056077613122
	30.959318414600767	.000000000019281
	6.610606785813053	.026527254145573
	14.544920817084766	.000036372342705
	10.037434401792262	.001716820826204
	.854083129890533	.354904800060654
	4.017379040472970	.161011537090875
9	8.834079630283873	.004307619099761
	.144875507219649	.046656228710126
	5.868902519022967	.043218857278315
	12.623231607779934	.000177643855176
	1.904135355837561	.397956067866459
	17.492955936005562	.000002456317432
	3.587124014598416	.195733688390548
	24.018668013446837	.000000006905056
	.767343192609217	.311947431576624
	34.758684223195985	.000000000000503
10	20.519795103187287	.000000148084787
	38.571560706765761	.000000000000013
	7.908255249718729	.008554779048588
	5.283983470112122	.062209886834183
	11.204969516479362	.000560285535228
	.131658172571087	.039035841257194
	1.725827648907708	.389566770803746
	.696695160544573	.275658401559781
	27.394286810766428	.000000000282141
	3.242430062765930	.224398248149325
	15.320538098181012	.000015638445015

Table 9.7

Nodes and weights of $H_w^{(n+1)}$ where $w(x) = e^{-x}x$ (Generalized Laguere with $\alpha = 1$)

n	nodes	weights
1	5.236067977499790	.276393202250021
	.763932022500210	.723606797749979
2	.493358053613672	.386975717692728
	8.716021885347972	.016600486503632
	2.790620061038356	.596423795803641
3	12.309046017390413	.000718795204851
	2.000000000000000	.642857142857143
	5.324782088068325	.115944898786902
	.366171894541262	.240479163151104
4	4.035899241536990	.225707562416030
	.291555083447512	.163789573089150
	1.570446368284971	.597954531467579
	8.132847536700815	.012522283363778
	15.969251770029712	.000026049663464
5	19.674659672659734	.000000843349820
	11.116040824928635	.000972233216916
	.242345652236039	.118690314809966
	3.279645965112819	.308670750659650
	6.391127476449262	.040647934398461
	1.296180408613512	.531017923565187
6	.207408536677957	.089943926027562
	2.771738036768087	.360932400305636
	5.317741941298702	.079398971149898
	23.412876446490799	.000000025246690
	14.222348454865667	.000060902740029
	1.104762626924511	.464703732319837
	8.963123956974277	.004960042210348
7	.963192868381727	.405616564458865
	.181303717030789	.070503768824035
	2.404059915514005	.388658800369186
	4.571665521898387	.121290949042948
	7.589603084182166	.013460003015817
	27.176159939067557	.000000000713378
	17.420666094980109	.000003276693310
	11.693348858945259	.000466636882461
8	20.690763117090409	.000000157277689
	.161050277991333	.056746980623896
	2.124444015263906	.399056077613122
	30.959318414600767	.000000000019281
	6.610606785813053	.026527254145573
	14.544920817084766	.000036372342705
	10.037434401792262	.001716820826204
	.854083129890533	.354904800060654
	4.017379040472970	.161011537090875
9	8.834079630283873	.004307619099761
	.144875507219649	.046656228710126
	5.868902519022967	.043218857278315
	12.623231607779934	.000177643855176
	1.904135355837561	.397956067866459
	17.492955936005562	.000002456317432
	3.587124014598416	.195733688390548
	24.018668013446837	.000000006905056
	.767343192609217	.311947431576624
	34.758684223195985	.000000000000503
10	20.519795103187287	.000000148084787
	38.571560706765761	.000000000000013
	7.908255249718729	.008554779048588
	5.283983470112122	.062209886834183
	11.204969516479362	.000560285535228
	.131658172571087	.039035841257194
	1.725827648907708	.389566770803746
	.696695160544573	.275658401559781
	27.394286810766428	.000000000282141
	3.242430062765930	.224398248149325
	15.320538098181012	.000015638445015

Table 9.8

Nodes and weights of $H_w^{(n+1)}$ where $w(x) = e^{-x^2}$ (Hermite)

n	nodes	weights	n	nodes	weights
1	1.000000000000000	.886226925452758	8	-.767093261812311	.430993129420261
	-1.000000000000000	.886226925452758		-2.441238396489535	.002453565951945
				-3.606368819529907	.000003860692187
2	-.000000000000000	1.417963080724413		1.563977927181141	.071475372891044
	1.581138830084190	.177245385090552		.767093261812311	.430993129420261
	-1.581138830084190	.177245385090552		-1.563977927181141	.071475372891044
				3.606368819529907	.000003860692187
3	-.602114101464426	.853956146188936		.000000000000000	.762601992994642
	-2.034074386254762	.032270779263822		2.441238396489535	.002453565951945
	.602114101464426	.853956146188936			
	2.034074386254762	.032270779263822	9	-3.852560035693396	.000000603397903
				.361029660612894	.635288343749770
4	.000000000000000	1.050343022758824		-1.861875888092628	.024933296378120
	1.074612544170356	.355476054592375		-.361029660612894	.635288343749770
	-2.417686472624545	.005579359480971		-2.713869218294079	.000586187968298
	-1.074612544170356	.355476054592375		3.852560035693396	.000000603397903
	2.417686472624545	.005579359480971		1.093513053818293	.225418493958667
				2.713869218294079	.000586187968298
5	1.475240917716105	.120659834409163		-1.093513053818293	.225418493958667
	-1.475240917716105	.120659834409163		1.861875888092628	.024933296378120
	-2.756238231186230	.000932833322973			
	2.756238231186230	.000932833322973	10	-.000000000000000	.685938300577191
	.476251034270315	.764634257720621		2.138862006542903	.008005478131748
	-.476251034270315	.764634257720621		-2.969558895028934	.000133954051144
				-2.138862006542903	.008005478131748
6	-1.828611210013403	.035886351703077		2.969558895028934	.000133954051144
	-.881604323971558	.413891949277428		.688554304791181	.431890198057097
	1.828611210013403	.035886351703077		-1.393823156221992	.103228051512604
	3.062507936082447	.000152291941665		4.085356687543293	.000000093411569
	-3.062507936082447	.000152291941665		-4.085356687543293	.000000093411569
	.881604323971558	.413891949277428		-.688554304791181	.431890198057097
	-.000000000000000	.872592665061177		1.393823156221992	.103228051512604
7	1.239870481811766	.184403570904643			
	-3.344197200038493	.000024419812028			
	.406782010086496	.692079585307343			
	-.406782010086496	.692079585307343			
	2.147927995035343	.009719349428744			
	-1.239870481811766	.184403570904643			
	-2.147927995035343	.009719349428744			
	3.344197200038493	.000024419812028			

Table 9.9

Nodes and weights of $H_w^{(n+1)}$ where $w(x) = e^{-\frac{x^2}{2}}$ (Hermite)

n	nodes	weights	n	nodes	weights
1	1.414213562373095	1.253314137315500	8	-3.452432449301447	.003469866245417
	-1.414213562373095	1.253314137315500		5.100175695498644	.000005459843251
				1.084833694459986	.609516328915756
2	2.236067977499790	.250662827463100		-.000000000000000	1.078482081185775
	-2.236067977499790	.250662827463100		2.211798795871730	.101081441718188
	.000000000000000	2.005302619704800		-2.211798795871730	.101081441718188
				3.452432449301447	.003469866245417
3	.851517928387080	1.207676363612255		-1.084833694459986	.609516328915756
	2.876615583917214	.045637773703246		-5.100175695498644	.000005459843251
	-.851517928387080	1.207676363612255			
	-2.876615583917214	.045637773703246	9	3.837990655018357	.000828994974867
				1.546460991341850	.318789891366064
4	.000000000000000	1.485409347929482		-1.546460991341850	.318789891366064
	-3.419124999151600	.007890405847344		-2.633090132396046	.035261005892605
	1.519731634261974	.502719057503415		.510573042457711	.898433391748466
	3.419124999151600	.007890405847344		-3.837990655018357	.000828994974867
	-1.519731634261974	.502719057503415		-5.448342652334175	.000000853333498
				2.633090132396046	.035261005892605
5	3.897909487674797	.001319225536783		5.448342652334175	.000000853333498
	-3.897909487674797	.001319225536783		-.510573042457711	.898433391748466
	-.673520671759293	1.081356137523587			
	.673520671759293	1.081356137523587	10	-4.199590463615580	.000189439635863
	2.086305713601847	.170638774255131		-.000000000000000	.970063247627417
	-2.086305713601847	.170638774255131		4.199590463615580	.000189439635863
				5.777566834655347	.000000132103908
6	4.331040258083033	.000215373329343		3.024807657697504	.011321455747199
	1.246776791607341	.585331608025175		-3.024807657697504	.011321455747199
	.000000000000000	1.234032381356800		-1.971163611078814	.145986510466473
	-1.246776791607341	.585331608025175		-.973762836266066	.610784975548349
	-4.331040258083033	.000215373329343		-5.777566834655347	.000000132103908
	2.586046773508430	.050750965282582		1.971163611078814	.145986510466473
	-2.586046773508430	.050750965282582		.973762836266066	.610784975548349
7	-3.037628901579833	.013745235779573			
	-.575276635593711	.978748335783191			
	.575276635593711	.978748335783191			
	4.729409035544568	.000034534829361			
	-4.729409035544568	.000034534829361			
	1.753441650964263	.260786030923375			
	-1.753441650964263	.260786030923375			
	3.037628901579833	.013745235779573			

Appendix 9.10

Programme for deriving the anti-Gaussian formulas

```
> reset:with(linalg):Digits:=20:
```

1. Definition of the weight function, recurrence relations and degree

```
> w:= x->1:  #Legendre
a:=-1:b:=1:  #interval of integration
N:=5:  #number of nodes for the Gaussian formula
> k := n-> 1./2^n*binomial(2*n,n):  #leading coefficient of the n-th
orthogonal polynomial
a1:= n-> (n+1):#coefficients of the recurrence relationship
a2:= n-> 0:
a3:= n-> (2*n+1):
a4:= n-> n:
```

2. Derivation of the recurrence relationships for "Gaussian" and "anti-Gaussian" orthogonal polynomials

```
> ga:= n-> -(k(n-1)/k(n))*(a2(n-1)/a1(n-1)):
> gb:= n->  (k(n-2)/k(n))*(a4(n-1)/a1(n-1)):
> aga:= n-> ga(n):
> agb:= proc(n)
> if N+1>n then return(gb(n))fi:
> if N+1=n then return(2*gb(n)) fi:
> end proc:
```

3. Symmetric matrix J for the "anti-Gaussian" case

```
> d[0]:= 1:
> for i from 1 to N do d[i]:=d[i-1]/sqrt(agb(i+1)) od:
> matrixJ:= proc(i,j)
> if i=j then return aga(i)
> elif i=j-1 then return (d[i-1]/d[i]+d[j-1]/d[j-2]*agb(j))/2
> elif i=j+1 then return (d[j-1]/d[j]+d[i-1]/d[i-2]*agb(i))/2
> else return(0)
> fi
> end proc:
> J:=matrix(N+1,N+1,matrixJ):
```

4. Weights and nodes for the anti-Gaussian formula

```
> eigen_all:=eigenvectors(J):
> nodes:=vector(N+1,i->eigen_all[i][1]):
> orthovects:= vector(N+1,i->eigen_all[i][3][1]):
> intw:=int(w(x),x=a..b):
> weights:=vector(N+1,j->(orthovects[j][1])^2.*intw):
> weights=weights():
```

5. Symmetric matrix JG for the "Gaussian" case

```
> e[0]:= 1:
> for i from 1 to N-1 do e[i]:=e[i-1]/sqrt(gb(i+1)) od:
> matrixJG:= proc(i,j)
> if i=j then return ga(i)
> elif i=j-1 then return (e[i-1]/e[i]+e[j-1]/e[j-2]*gb(j))/2
> elif i=j+1 then return (e[j-1]/e[j]+e[i-1]/e[i-2]*gb(i))/2
> else return(0)
> fi
> end proc:
> JG:=matrix(N,N,matrixJG):
```

6. Weights and nodes for the Gaussian formula

```
> vlastniG:=eigenvectors(JG):
> nodesG:=vector(N,i->vlastniG[i][1]):
> VG:=vector(N,i->vector(N,j->vlastniG[i][3][1][j])):
> orthovectsG:=[seq(VG[i],i=1..N)]:
> intwG:=int(w(x),x=a..b):
> weightsG:=vector(N,j->(orthovectsG[j][1])^2*intwG):
> weightsG=weightsG():
```

7. Integration of given function

```
> f:=x->sin(6*x):
> exact:=evalf(int(w(x)*f(x),x=a..b)):
> resultA:=sum(weights[s]*f(nodes[s]),s=1..N+1):
> abserrA:=simplify((resultA-exact)):
> relerrA:=abs(abserrA/resultA):
> resultG:=sum(weightsG[s]*f(nodesG[s]),s=1..N):
> abserrG:=simplify((resultG-exact)):
> relerrG:=abs(abserrG/resultG):
> result_average:=(resultA+resultG)/2.:
> abserr_average:=abs(result_average-exact):
> relerr_average:=abs(abserr_average/result_average):
```

References

[1] Abramowitz M., Stegun I.E.(1970): Handbook of mathematical functions with formulas, graphs and mathematical tables, National Bureau of standarts, Washington, D.C.

[2] Davis P.(1953): Errors of Numerical Approximation for Analytic Functions, J. rat. mech. anal. 2, 303-313

[3] Golub G.E., Welsch J.H.(1969): Calculation of Gauss Quadrature Rules, Math. Comput. 23, 221-230

[4] Kofroň J.(1972): Die Ableitungsfreien Fehlerabschtzungen Von Quadraturformeln I, Aplikace matematiky 17, 39-52

[5] Kronrod A.S.(1965): Nodes and Weights of Quadrature Formulas, Consultants Bureau, New York

[6] Kurtz L. C. (1975): Unique Hahn-Banach Extension and Korovkin's theorem, Proceedings of the American Mathematical Society, Vol.4, No.2, 413-416

[7] Laurie D.P.(1996): Anti-Gaussian quadrature formulas, Mathematics of computation 65, 739-747

[8] Lukeš J. (2002): Zápisky z funkcionální analýzy, Univerzita Karlova v Praze, Nakladatelství Karolinum.

[9] Mercer A. MCD., Mercer P.R.(2000): Cauchy's interlace theorem and lower bounds for the spectral radius, Internat. J. Math. & Math. Sci. Vol.23, No.8, 563-566 (http://www.emis.de/journals/IJMMS/volume-23/S016117120000257X.pdf)

[10] Segethová J.(1998): Základy numerické matematiky, Karolinum, Praha

[11] Weisstein Eric W.: Weierstrass Approximation Theorem, MathWorld–A Wolfram Web Resource
http://mathworld.wolfram.com/WeierstrassApproximationTheorem.html

ERRATA

In the line numbering only the text lines count.

(1) Page 5, formula (1.2) - add: *where f is a function uniquely defined in* $x_i,\ i = 1, \dots, n$

(2) Page 5, Remark 1.3 - add:
Note that there is no direct relationship between the weights $w_i^{(n)}$ *and the values of* $w(x_i)$.

(3) Page 5, line 15 - the formula *(1.4)* instead of (1.3) should be referenced.

(4) Page 6, line 6 - note that the nodes y_i are *distinct*.

(5) Page 7, line 14 and after - the Gaussian quadrature rule for the linear functional $2I - G_w^{(n)}$ is meant in the sense of the following definition:
We shall say that a linear functional G of the form

$$Gf = \sum_{i=1}^{n} \lambda_i(n) f(\xi_i(n))$$

is the n-point Gaussian formula for the linear functional F if

$$Gf = Ff \ \forall f \in \mathbb{P}^{2n-1}.$$

(6) Page 8, Theorem 2.1 and after - the polynomials orthogonal with respect to the linear functional $2I - G_w^{(n)}$ are defined as follows:
We shall say that the polynomials $\{p_i\}_{i=0}^{\infty}$ *are orthogonal with with respect to the linear functional* $2I - G_w^{(n)}$ *if*

$$2\int_a^b w(x)p_k(x)p_l(x)dx - \sum_{i=1}^{n} w_i^{(n)} p_k(x_i^{(n)}) p_l(x_i^{(n)}) = \begin{cases} 0 \text{ if } k \neq l \\ C_k \neq 0 \text{ if } k = l \end{cases}.$$

If $C_k = 1\ \forall k = 0, 1, \dots$ *we shall say the polynomials are orthonormal with respect to* $2I - G_w^{(n)}$.

(7) Page 8, line 7 - the citation (e.g. [10]) is related to all the preceding notions.

(8) Page 8, formula (2.2) - the sign := should be replaced by =.

(9) Page 9, formula (2.10) - the lower index at b should be $j - 1$

(10) Pages 9-10 formulas from (2.10) to (2.11) and page 10, line 2 - the range for j is $2, \dots, n + 2$.

(11) Page 11, line 8 to formula (2.21) - the range for j is $1, \dots, n + 1$.

(12) Page 11, last line - $b = \sqrt{w_j}$.

(13) Page 12, formula (2.20) - $(q_j^0)^2 = w_j p_0^2(t_j)$

(14) Page 12-13, formulas (2.22), (2.27) and (2.30) - missing minus on the right hand side.

(15) Page 18, line 5 - the references should be (2.30),(2.31) and (2.4)

(16) Page 19, line 6, page 20, line 7 and page 21, line 2 - should be 4.1 *d)* instead od 4.1 b).

(17) Pages 18-20, proofs of the theorems 4.3 and 4.4 - the factorials of positive non-integer values are meant to be defined by the Gamma function $\Gamma(p+1) = \int_0^\infty x^p e^{-x} dx = p!$.

(18) Page 19, formula (4.8) - $(n + 1 + \alpha)(n + \alpha) \geq n(n + \alpha)$

(19) Pages 19-20 - Note: The relevant data from the tables 22.2 in [1] and 22.3 in [1] are collected in the table 9.1 in section 9 and the data from table 22.7 in [1] in the table 9.2.

(20) Page 20, formula (4.12) -

$$\begin{aligned}
\frac{p_{n+1}^{(\alpha,\beta)}(1)}{p_{n-1}^{(\alpha,\beta)}(1)} &= \frac{\binom{n+1+\alpha}{n+1}2^{n+1}\binom{2n-2+\alpha+\beta}{n-1}}{\binom{n-1+\alpha}{n-1}2^{n-1}\binom{2n+2+\alpha+\beta}{n+1}} = 4\frac{\frac{(n+1+\alpha)!}{(n+1)!\alpha!}\cdot\frac{(2n-2+\alpha+\beta)!}{(n-1)!(n-1+\alpha+\beta)!}}{\frac{(n-1+\alpha)!}{(n-1)!\alpha!}\cdot\frac{(2n+2+\alpha+\beta)!}{(n+1)!(n+1+\alpha+\beta)!}} = \\
&= 4\frac{(n+1+\alpha)!(2n-2+\alpha+\beta)!(n+1+\alpha+\beta)!}{(n-1+\alpha)!(2n+2+\alpha+\beta)!(n-1+\alpha+\beta)!} = \\
&= 4\frac{(n+\alpha)(n+1+\alpha)(n+\alpha+\beta)(n+1+\alpha+\beta)}{(2n-1+\alpha+\beta)(2n+\alpha+\beta)(2n+1+\alpha+\beta)(2n+2+\alpha+\beta)}.
\end{aligned}$$

(21) Page 20, formula (4.13) - for $a \in \mathbb{R},\ b \in \mathbb{Z}$

(22) Page 20 the third, fourth and fifth formula - the second term in the square brackets should be multiplied by $(\alpha^2 - \beta^2)$

(23) Page 20, line 4 - the binomial should be $\binom{2i+\alpha+\beta}{i}$.

(24) Page 21, line 1 - add the reference:
Now it remains to find (Thm. 4.1 d)) when...

(25) Page 22, second paragraph - add:
Finally for $\alpha, \beta < -\frac{1}{2}$ the formula requires an exterior node for any n.

(26) Page 25, line 5 - should be *right* instead of left.

(27) Page 30, line 15 - Note: The interchanging of the order of integration can be legalized as follows: We replace the integral from -1 to 1 by the integral from $-\alpha$ to α where $0 < \alpha < 1$, we interchange the order of integration (since the the integrated function is continuous for x in $[-\alpha, \alpha]$) and we perform the limit passage for $\alpha \to 1$.

(28) Page 31, line 3 - $E\left(\frac{1}{\xi - x}\right)$ is meant in the sense of the formula (6.5) even if $\frac{1}{\xi-x}$ is not analytic. In case $x_i = \pm 1$ for some $i = 1, \ldots, n+1$ the analogous construction as in the previous item has to be used.

(29) Page 32, line 4 - should be *geometric* instead of quadratic

(30) Page 32, the third and fourth formula - holds $\forall x \in (-1, 1)$. $E(\pm 1) = 0$, since the quadrature formula integrates constants exactly. Therefore
$E\left(\frac{1}{1-xe^{-it}}\right) = \sum_{j=0}^{\infty} E(x^j)e^{-itj},\ x \in [-1, 1]$.

(31) Page 33, line 8 - should end by $2\pi \max\limits_{t\in[0,2\pi]} \left|f(e^{it})\right|$

(32) Page 34, the third formula - Note: The interchanging of the order of integration and summation can be legalized as follows: We replace the integrals from -1 to 1 by integrals from $-\alpha$ to α where $0 < \alpha < 1$, we interchange the order of integration and summation (since the geometric sequence converges for any x and y in $[-\alpha, \alpha]$) and we perform the limit passage for $\alpha \to 1$.

(33) Page 41, caption - $a_1(n)p_{n+1}(x) = (a_2(n) + a_3(n)x)p_n(x) - a_4(n)p_{n-1}(x)$

(34) Page 41 - for the definition of $(2n + \alpha + \beta)_3$ see the formula (4.13).