RIGOROUS THESIS

Numerical Modelling of Two-Prize Asymmetric Contests

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Declaration of Authorship

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Abstract

This thesis presents an analysis on a class of asymmetric imperfectly discriminating multi-prize contests with the aim to investigate when more than one prize becomes optimal prize allocation if the average effort is to be maximized. We present \( n \)-person model with heterogeneous contestants who compete for two, possibly different, prizes. The contestants may differ in their relative abilities, i.e., parameters affecting their probabilities to win either of the prizes. Two different numerical methods for finding pure strategy Nash equilibria are employed. Depending on particular distributions of the abilities, we find two possible scenarios when the second prize becomes optimal. Furthermore, we address an issue of existence and uniqueness of a pure strategy Nash equilibrium with respect to the returns to scale in effort parameter.

**JEL Classification**  C63, D72

**Keywords**  Imperfectly discriminating contests; Heterogeneous abilities; Multiple prizes; Numerical methods

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Abstrakt

Tato rigorózní práce analyzuje asymetrické nedokonale diskriminativní soutěže o více cen za účelem zjistit, kdy se v soutěži stává nabízení více než jedné ceny optimální strategií pro dosažení největšího průměrného úsilí všech hráčů. Představujeme model, ve kterém \( n \) heterogenních hráčů soutěží o dvě, možno různé, ceny. Soutěžící se mohou lišit ve svých relativních schopnostech, neboli parametrech, které ovlivňují jejich pravděpodobnost, že vyhrají jednu z cen. Ke hledání Nashovy rovnováhy v ryzích strategiích jsou použity dvě rozdílné numerické metody. V závislosti na konkrétní distribuci schopností hráčů nacházíme dva možné scénaře, ve kterých se druhá cena stává optimální. Práce se také zabývá otázkou existence a jednoznačnosti Nashovy rovnováhy v ryzích strategiích v závislosti na parametru udávající výnosy z rozsahu vynaloženého úsilí.

Klasifikace JEL C63, D72
Klíčová slova Nedokonale diskriminativní soutěže; Heterogenní schopnosti; Více cen; Numerické metody

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Acronyms

**CSF**  Contest Success Function  
**FOC**  First-order Condition  
**NE**  Nash Equilibrium  
**SCO**  Second-Order Condition
Introduction

There are many real life situations when people expend irreversible effort in order to get ahead of their opponents and win a prize. A financial reward is an obvious example of such a prize, but even more general concepts can also be used—the form of a public policy for lobbying parties, winning a dispute in a court, winning in a sport competition, receiving a tenure position, promotion, winning patent rights, winning a war, and many others. These all belong to situations that are studied in the field of contest theory.

Some contests, such as arms race or fighting over property rights that cannot be traded for, are not designed by a third party, and they exist as a result to the mere existence of a possibility of war or of these property rights. Others, on the other hand, are created by a contest designer, whose objective is to elicit some extra effort by the contestants. Among these contest designers we can include firms trying to motivate their workers by a promotion/relative reward scheme, universities trying to motivate their students by providing performance stipends, or organizers of a sport competition who want athletes to perform their best in order for the competition to be attractive to watch. This thesis presents an abstract model of the latter mentioned contests.

There are two different approaches how to look at the optimal design problem. Assuming that a contest designer is able to determine the shape of the contest success function (a function that relates the efforts exerted by the players to the probabilities of winning the prizes), the design problem becomes searching for an optimal contest success function while taking the number of the prizes and their valuations as given. Most of the literature on the optimal design follows this direction, which is supported by the great volume of literature devoted to contests with a single prize (see chapter one). Nevertheless, the assumption that a contest designer is able to determine the contest success function (CSF) is very likely to be violated. The other approach thus takes the CSF as given, and searches for the optimal prize allocation. The optimality of the second prize in a two-prize contest is the issue addressed by this thesis.
Whenever any model is to be set up, it always comes down to what CSF to work with, what CSF operating in the reality is. A benchmark case is represented by a deterministic function—the highest effort wins with certainty—that leads to so called perfectly discriminating contests. In reality, however, such situation hardly emerges, because noise disturbs the deterministic link between unobservable efforts and the probabilities. When the link between the efforts and the probabilities is not deterministic, we refer to so called imperfectly discriminating contests. Tullock (1980) addressed this kind of contests and presented a stochastic ratio-form function for a single prize contest that has been extensively used in contest literature since then. In this thesis, we examine a class of imperfectly discriminating contests employing a CSF that is based on Tullock (1980) work. Since we focus on contests with more than one prize, the question is how to generalize the Tullock (1980) CSF to multiple prize context. Unfortunately, the literature on extension of the Tullock (1980) CSF has not been sufficiently addressed yet. Among the rare works, Clark and Riis (1996, 1998c) propose a methodology that we follow. First, the contestants simultaneously expand efforts. Second, the winners are determined sequentially in several rounds, each round eliminating the winners of preceding rounds from the pool of contestants eligible to win the remaining prizes. It might be that some of our results happen not to be robust if the determination of the winners is more simultaneous than what we present. The determination of the contest success function always remains a topic for a discussion. This is, however, beyond the scope of this thesis. Examining the robustness of our result is thus a possible direction for future research.

Works on symmetric multi-prize imperfectly discriminating contests with the objective of maximizing the average effort find single prize being the optimal allocation. However, when asymmetries are introduced into the contest, this result may not hold any more. The purpose of this thesis is to examine whether asymmetries lead to optimality of multiple prizes and what effects they have on equilibrium efforts of the contestants. We set up an $n$-person contest with two, possibly different, prizes, in which contestants differ in their relative abilities (a parameter that is a part of CSF). Our setting is inspired by Szymanski and Valletti (2005), who examine a three-person contest with two, also possibly different, prizes and contestants that differ in their cost of effort.

Furthermore, we investigate the existence of a pure strategy Nash equilibrium in these multiple prize contests in response to changes in the Tullock returns to scales in effort parameter, which is a part of the CSF. From a single-
prize literature it is known that as this parameter exceeds some threshold value, a pure strategy Nash equilibrium does not exist, and the only possible equilibrium is then in mixed strategies. We examine whether a similar proposition can be made for multiple prize setting and how it is affected by introducing asymmetries into the contest.

Already changing the setting from a single prize to multiple prizes has brought great complexity in analytical expressions, which is amplified even more by introducing asymmetries into the contest. One of the purposes of this thesis has thus become to present a possible alternative approach of how to solve for equilibria in the contests when an analytical solution is rather difficult to obtain. We present two numerical methods designed for solving a system of \( n \) nonlinear equations in \( n \) unknowns that we obtain while solving for the maximization problems of the contestants. Since this approach is not limited by the complexity in expressions, we are able to provide a very thorough analysis on asymmetric multi-prize imperfectly discriminating contests for arbitrary chosen values of the parameters, which has not been provided before.

The thesis is divided as follows. Chapter one provides a brief literature overview of the contest theory with a focus on research following the work of Tullock (1980). Chapter two presents some basic definitions from the framework of the contest theory. Chapter three introduces the issue of the multiple prize (asymmetric) contests by presenting some already known results on the perfectly discriminating contests as limiting cases for the Tullock framework. Chapter four deals with the extension of the Tullock (1980) to a multi-prize setting and with the way how to introduce asymmetries into the contest. Chapter five presents a three-person asymmetric model with two prizes analytically solved for limiting cases. Chapter six covers the main part of this thesis, which is numerical modelling of an \( n \)-person asymmetric two-prize imperfectly discriminating contest. We set up the model, we describe the numerical methods employed to solve the model and we follow by discussion on the obtained results. Chapter seven concludes.
Chapter 1

Literature review

Contest theory, tournaments, or all-pay auctions is a very broad field that has developed enormously over the last forty years. The origins of the contest theory are dated back to the seminal work of Tullock (1967), who studied one specific contest called rent-seeking. The paper was followed by a number of other seminal works, including Tullock (1980), or Krueger (1974). The rent-seeking literature was then surveyed by Tollison (1982) and Brooks and Heijdra (1989). Several other works has been published that has put together the most important conclusions that had been made in this area. Buchanan, Tollison and Tullock (1980), or Lockard and Tullock (2001) are one of them.

The framework of rent-seeking contests can be generalized to many other situations. Voting, litigation, or awarding a prize are a few examples of them. The survey of alternative ways how to model rent-seeking contests is offered by Nitzan (1994). Konrad (2006) writes another, more detailed, survey which focuses on the theoretical insights into contests that had been developed in the literature. More recently, Corchon (2007) provides an introduction to the contest theory by presenting a basic model with its main properties from which numerous applications are derived.

An inseparable element of the contest theory is a contest success function, a function that relates the efforts exerted by the players to the probabilities of them winning the prize. Therefore, a great volume of the contest theory oriented literature are works that are primarily focused on this function, and its possible form. According to the properties of the contest success function, we can divide contests in perfectly and imperfectly discriminating contests. In the perfectly discriminating contest, the contestant who exert the highest effort
1. Literature review

wins the prize with certainty\(^1\). In the imperfectly discriminating contest, the contestant exerting the highest effort has the highest probability of winning the contest, but this probability is not necessarily equal to 1. In this thesis, we focus primarily on imperfectly discriminating contests.

For this kind of contests, Tullock (1980) presents a function which is in ratio-form (that is, the probabilities are determined on a relative scale of efforts of all contestants). The other camp is based on Hirshleifer (1989) who presents a function which is in difference-form (that is, the probabilities are determined on an absolute scale).\(^2\) The Tullock (1980) contest success function becomes the base for our model that is presented in this thesis.

One of the issues that emerges in the frame of contest theory is the optimal design problem. In most of the cases, the objective of the designer of the contest is to elicit the maximum effort (aggregate effort, average effort, maximum highest effort etc.). Most of the works concentrate on a case in which a single prize is available in the contest and then determining an optimal contest success function\(^3\). A great part of this thesis is also devoted to the optimal design problem. However, we take the contest success function as given and we search for an optimal allocation of a prize fund between two prizes from the point of view of average effort maximizer.

A single-prize symmetric Tullock (1980) contest as well as its specific cases allowing asymmetries among contestants have already been investigated very extensively. Allard (1988) Hillman and Riley (1989), and Nti (1999) analyze effects of different valuations that contestants associate with winning the reward.

\(^1\)For example, all-pay auction belongs to perfectly discriminating contests.

\(^2\)The two functions are axiomatized for fair contest (i.e. a contest in which "the probability of success is dependent only upon the players’ outlays, and not other inherent characteristics" (Clark and Riis (1998b)), which is a core of anonymity axiom) in Skaperdas (1996). Clark and Riis (1998) then extend the axiomatization to the unfair contest (i.e. a contest in which the players are treated differently, that is, the anonymity axiom is not employed). The results show that "the success function is uniquely characterized by Luce’s choice Axiom (implying independence of irrelevant alternatives) and homogeneity of degree zero (Clark and Riis (1998b)). Axiomatization of the contest success function enables a better understanding of any advantages or limitations it might have, unlike the previous practice that used certain classes of functional forms of contest success functions without any particular reason other than analytical convenience. Recently, Peeters (2011) presents an empirical assessment of Tullock and Hirshleifer contest success functions, using real-life contests (sport leagues). He finds that Tullock contest success function fits the data better than the Hirshleifer one for all the models he was testing.

\(^3\)Dasgupta and Nti (1998) and Nti (2004) determine the aggregate effort maximizing contest technologies, for the players with symmetric or asymmetric valuations of the reward, respectively. Michaels (1988) and Wang (2010) takes the Tullock (1980) power contest success function and examine the optimal accuracy level, again for symmetric and asymmetric contests, respectively.
Leininger (1993) and Kohli and Singh (1999) investigate effects of different relative abilities among contestants. Baik (1994) puts it together by examining two-player contest with asymmetries in valuations as well as in relative abilities, which Stein (2002) further extends to a more general N player model. An important finding of Hillman and Riley (1989) and Stein (2002) is that there is just a fraction of contestants active in the contests (those expending positive amount of effort) and that the fraction is determined endogenously from the parameters of the game.

Among real situations, however, not only single-prize contests but also contests in which multiple prizes are available are ubiquitous. The question is why and when it is optimal to allocate a reward into one single prize and why and when it is optimal to allocate a reward into more than one prize. Sisak (2009) presents a survey on multiple-prize contests looking at optimal allocation of a given fixed amount between number of prizes of different size.

With multiple-prize contests a question how to extend the contest success function developed in a single-prize contest literature to a multi-prize setting. In the framework of Tullock (1980) contest, Berry (1993) offers a generalization of this contest success function to multiple-prizes. Nevertheless, Clark and Riis (1996) criticize the function of Berry (1993) by showing that it can be transformed to a function corresponding to a contest with a single prize so that just one of the prizes act as an incentive to invest effort while the remaining prizes are awarded independently of the efforts. They propose a different approach to contest success function in multi-prize contests using a nested probability function, in which they work with a Tullock (1980) contest success function with constant returns to scale in efforts. This approach is further extended to Tullock (1980) function with variable returns to scale in efforts by Clark and Riis (1998c).

Clark and Riis (1998c) also examine the optimal allocation properties of a benchmark case assuming a symmetry in contestants’ characteristics and they conclude that “an income maximizing contest administrator obtains the most rent-seeking contributions when he makes available a single, large prize” (p.623). However, this might not be the case if an asymmetry in contestants’ characteristics is allowed as shown in Szymanski and Valletti (2005) for the imperfectly discriminating contest or in Moldovanu and Sela (2001) for the perfectly discriminating contest. In both models, the contestant differ in their cost of effort. In their paper, Moldovanu and Sela (2001) conclude that several
prizes might indeed be optimal\textsuperscript{4}.

This thesis enriches the literature on multiple-prize imperfectly discriminating contests by providing a similar analysis as Szymanski and Valleti (2005) perform in their paper. Nevertheless, instead of introducing asymmetries by different cost function of the contestants, the heterogeneity is introduced by differences in relative effectiveness among the contestants. To determine the success probability of winning any of the prizes, we follow the method proposed by Clark and Riis (1996) for multi-prize contests, but we allow for asymmetries in a nested Tullock (1980) function using methodology of Stein (2002) for a single-prize contest.

\textsuperscript{4}This happens to occur when the cost function is strictly convex
Chapter 2

Basic definitions

2.1 Framework

Definition 2.1 (Contest). A contest is defined by the following components:

- (finite) set of contestants \( N = \{1, \ldots, n\} \).
- prize which is to be awarded in the contest, its quantity \( V \in \mathcal{R} \) may depend on the actions taken by the contestants.
- a vector \( x = \{x_1, x_2, \ldots, x_n\} \in \mathcal{R}_+^n \) of efforts (actions, investments) made by the contestants before the prize is allocated. These efforts determine the particular allocation of the prize among the contestants (the probability of obtaining the prize for each of the contestants).
- contest success function (CSF); a function relating the vector of efforts \( x \) taken by the contestants into the individual probabilities of winning the contest

\[
p_i = p_i(x_1, x_2, \ldots, x_n) \quad \forall i \in \{1, \ldots, n\}:
\]

\[
p_i(x) \in [0, 1] \quad \forall i, \forall x
\]

\[
\sum_{i=1}^n p_i(x) = 1 \quad \forall x
\]

- individual value of winning the contest \( v_i \in \mathcal{R} \).
- individual cost function \( c_i(x_i) \); a function that describes \( i \)'s cost of exerting an effort \( x_i \).
The definition of the contest that we provide here is a very basic definition showing what elements constitute a contest. Later in the thesis, the models introduced slightly differ from this definition by what represents the value of winning, since the prize fund will be allocated not only to one, but to several prizes. Hence, we will later work with value of winning a particular prize out of several prizes instead of value of winning the whole contest. If the value of winning any prize but the first one is set to zero for all the participants, we are in the case described above. We will also change accordingly the probability of winning the contest to a probability of winning the particular prize. However, it will become clear what we mean by the value of winning, since we always explicitly state it at the beginning of each of the models.

Assuming that the contestants are risk-neutral, then, depending on the effort choices, the expected payoff function $E\pi_i$ of a contestant $i$ is given by

$$E\pi_i(x_1, \ldots, x_n) = p_i(x_1, \ldots, x_n)v_i - c_i(x_i).$$

(2.1)

Given this definition, a contest belongs to normal-form games. The strategies are the efforts and the payoffs are the expected utilities. An equilibrium concept used for solving of these games is the concept of Nash equilibrium (which may or may not exist).

**Definition 2.2 (Nash Equilibrium).** We say that the effort vector $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ constitutes a **Nash Equilibrium** (NE) if

$$E\pi_i(x_1^*, \ldots, x_i^*, \ldots, x_n^*) \geq E\pi_i(x_1^*, \ldots, x_i, \ldots, x_n^*) \quad \forall x_i \in \mathbb{R}_+^+, \forall i.$$

Whenever we refer to an equilibrium in the contest in following sections, we have in mind a Nash equilibrium.

### 2.2 Contest Success Function

Contest success function (CSF), which links the effort of players to their probabilities with which they win the prize, is an essential part of contests. We first define a CSF for a benchmark, deterministic case, which is the first-price all-pay auction. In reality, however, such situation hardly emerges, because noise disturbs the deterministic link between the efforts and the probabilities. We thus continue by defining Tullock CSF, one of the CSFs that is prevailing in the literature on stochastic case.
2. Basic definitions

2.2.1 Perfectly discriminating CSF

The perfectly discriminating CSF is a function that assigns the probability of one of winning the contest to a contestant who exerts the highest amount of effort out of all the efforts of all the participants. Hence, this kind of CSF is not continuous. A contest with the perfectly discriminating CSF corresponds to a situation in which the contestants’ efforts translate deterministically into another observable variable (quality, quantity), so that the designer of the contest is able to precisely recognize who exerted the highest effort and award him the prize. The perfectly discriminating CSF is thus equivalent to the first-prize all-pay auction.

Definition 2.3 (Perfectly discriminating CSF). A function describing the probability of winning of a contestant $i$ as

$$p_i(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } x_i > \max_{j \neq i} x_j \\ \frac{1}{\sum_{k \neq i, \sum x_k = x_i}} & \text{if } x_i = \max_{j \neq i} x_j \\ 0 & \text{if } x_i < \max_{j \neq i} x_j \end{cases}$$

is called the perfectly discriminating CSF.

Definition 2.4 (The perfectly discriminating contest). A contest in which the CSF is defined by the equation (2.2) is called the perfectly discriminating single-prize contest.

2.2.2 Tullock CSF

For imperfectly discriminating contests, Tullock (1980) presents a CSF that dominates in the subsequent literature. The value of probability of winning of each contestant is based on what is the level of the contestant’s effort relatively to the others. The relativity is what captures the stochastic character of the contest.

Definition 2.5 (Tullock CSF). A function describing the probability of winning of a contestant $i$ as

$$p_i(x_1, \ldots, x_n) = \begin{cases} \frac{x_i^r}{\sum_{j=1}^n x_j^r} & \text{if } \max\{x_1, \ldots, x_n\} > 0 \\ 1/n & \text{otherwise} \end{cases}$$

is called the Tullock CSF.
The parameter \( r \) in the Tullock CSF indicates what the marginal effect of a change in effort is. Mathematically,

\[
  r = \frac{\partial p_i(x)}{\partial x_i} \frac{x_i}{1-p_i(x)},
\]

That is, the parameter represents the elasticity of the odds of winning. It shows whether there are increasing \((r > 1)\), decreasing \((r < 1)\), or constant \((r=1)\) returns to scale in efforts. The function (2.3) converges toward the perfectly discriminating contest (all-pay auction) as \( r \to \infty \).

The Tullock CSF (2.3) can be rewritten as

\[
  p_i(x_1, \ldots, x_n) = \begin{cases} 
    \frac{1}{\sum_{j=1}^n (\frac{x_j}{x_i})^{r}} & \text{if } \max\{x_1, \ldots, x_n\} > 0 \\
    1/n & \text{otherwise}
  \end{cases}.
\]

We can see that the function (2.4) depends on a ratio of contestants’ efforts, therefore, the Tullock CSF is sometimes called the ratio-form (or logit-form) CSF.

**Definition 2.6 (The Tullock contest).** A contest in which the CSF is defined by the equation (2.3) is called the **Tullock single-prize contest**.
Chapter 3

Multi-prize perfectly discriminating contest

The model of perfectly discriminating contest apply when the ranking of the efforts exerted translates one-to-one into the ranking of the prizes awarded. The literature on this kind of contests refer to scenarios such as R&D races, lobbying, committee bribing, or worker incentive scheme. The use of this model is appropriate so far as the effort is more or less perfectly observable at least in ranking or it translates more or less perfectly to an observable variable that preserves the ranking. By being the limiting case to the Tullock (1980) contest it serves as a valuable benchmark case, even though these situations are hardly to be found in the reality. In this section, we present some of the already known results on this kind of contests, in which contestants are assumed to be risk neutral. Since the question of how attitude to risk affects an equilibrium is not the focus of this thesis, we do not present the literature addressing this issue, even though such literature exists.

3.1 Symmetric case

In the symmetric case, a non-continuity of the perfectly discriminating CSF causes a non-existence of a pure strategy equilibrium. Therefore, the only equilibria for this kind of contest are in mixed strategies. A symmetric multi-prize perfectly discriminating contest with complete information has been examined by Barut and Kovenock (1998). They analyze the case with \( n \) contestants and \( n \) prizes of valuations \( v_k \geq 0, \ k \in \{1, \ldots, n\} \) that are potentially different, but symmetric across the contestants (i.e., there is no pair of contestants that would
put different value on winning the same prize). A contestant who exerts the highest effort wins the first prize $v_1$ with certainty, a contestant who exerts the second highest effort wins the second prize $v_2$ with certainty, etc. The authors examine risk-neutral contestants with linear cost of effort and characterize all equilibria of the contest. The results show that, in absence of opportunity costs, the aggregate effort is maximized when value of the last prize $v_n$ is set to zero. Any distribution of the prize fund among the remaining $n - 1$ prizes is then optimal.

3.2 General case allowing differences

Siegel (2009) presents a very general model that allows introducing various asymmetries between the contestants, such as different costs of effort, prior investments, attitudes toward risk, and others. Moreover, their model allows for non-ordered cost functions, which arise when different contestants are disadvantaged differently relative to the others depending on the interval from which the effort is taken. The framework of the model is one of complete information.

In the model, there is $n$ contestants that compete for $m$ homogeneous prizes, $0 < m < n$. Each contestant chooses irreversible effort $x_i \in [x_i, \infty)$, where $x_i$ is his initial position. The contestants with $m$ highest efforts win one of the prizes. Each player has preferences represented by a Bernoulli utility function over lotteries $(x_i, W_i)$ where $W_i$ captures whether the contestants belong to the group of winners ($W_i = 1$) or losers ($W_i = 0$). Each contestant maximizes his expected payoff function of a form

$$E\pi_i(x) = P_i(x)v_i(x_i) - (1 - P_i(x))c_i(x_i),$$

where $x = (x_1, \ldots, x_n)$ is the vector of efforts of all contestants, $v_i : [x_i, \infty) \to \mathcal{R}$ is contestant’s valuation of winning, $c_i : [x_i, \infty) \to \mathcal{R}$ is contestant’s cost of losing and $P_i(x)$ is a contestant’s probability of winning

$$P_i(x) = \begin{cases} 
0 & \text{if } x_i > x_j \text{ for } m \text{ or more players } j \neq i, \\
1 & \text{if } x_i < x_j \text{ for } N - m \text{ or more players } j \neq i, \\
[0, 1] & \text{otherwise},
\end{cases}$$

such that $\sum_{j=1}^nP_j(x) = m$. For a special case of so called separable contests, when the effect of winning and losing on a player’s Bernoulli utility is additively
separable from that of the effort, the expected payoff (3.1) can be rewritten as

\[ E\pi_i(x) = P_i(x)v_i - c_i(x_i), \]

(3.3)

where \( v_i \) represents contestant’s valuation for a prize that is here independent of the effort chosen. For this case of separable contest, the contestants are assumed to be risk-neutral.

Siegel (2009) gives several definitions that are key in characterizing the equilibrium (Siegel (2009), p.78):

- Player \( i \)'s reach \( r_i \) is the highest effort at which his valuation for winning is 0. That is, \( r_i = \max \{ x_i \in [x_i, \infty) | v_i(x_i) = 0 \} \). Re-index players in (any) decreasing order of their reach, so that \( r_1 \geq r_2 \geq \ldots \geq r_n \).

- Player \( m+1 \) is the marginal player.

- The threshold \( T \) of the contest is the reach of the marginal player, \( T = r_{m+1} \).

- Player \( i \)'s power \( w_i \) is his valuation for winning at the threshold. That is, \( w_i = v_i(\max \{ x_i, T \}) \). In particular, the marginal player’s power is 0.

The author restricts the attention to a type of contest called generic contest, which is a contest that meets certain conditions on the players’ powers and costs. The contests that do not meet these conditions can always be perturbed slightly to meet them.

The first result that is captured in Theorem 1 says that "in any equilibrium of the generic contest, the expected payoff of every player equals the maximum of his power and 0" (Siegel (2009), p. 79). That is, all the contestants with reach below or equal the marginal player have the expected payoff of zero, while all the contestants with reach greater than that of the marginal player have strictly positive expected payoff. It does not say, however, that the first mentioned contestants cannot win the prize, it only states that in expectation their payoff is zero.

The second result that is captured in Theorem 2 concerns the contestants’ participation in an equilibrium. This theorem shows that in a generic all-pay auction only players \( 1, \ldots, m+1 \) participate, while the remaining players will decide not to participate.

\(^{1}\)"A player participate in an equilibrium of a contest if with strictly positive probability he chooses scores associated with strictly positive costs of losing." (Siegel (2009), p. 86)
3. Multi-prize perfectly discriminating contest

3.3 Effect of information

Asymmetries may be introduced into the contests by different costs of effort, different prior investments, different valuations. The effects that they have on equilibrium and on optimal prize allocation is greatly dependent on information the contestants have about these characteristics. When the characteristics are common knowledge, as in Siegel (2009), we can see how particular asymmetries directly affect the behavior of the contestants. The other scenario is the characteristics being private information to each of the contestants, distributed according to the same cumulative distribution function, but this function being a common knowledge. What is captured in the change of the behavior then is rather the effect of iid noise than effect of asymmetries themselves. We present both situations to see how these two concepts may induce distinct responses in the behavior. Nevertheless, in subsequent parts of the thesis, we continue with a complete information scheme only.

3.3.1 Complete information

As with the symmetric contestants, when the contest is one of complete information, a pure strategy Nash equilibrium does not exist either when asymmetries are introduced. The only equilibrium that the contest has is in mixed strategies.

Clark and Riis (1998a) present a model with $k$ prizes that are of identical value, but each individual places a different valuation on winning any of these prizes. The valuations are common knowledge. The personal valuations are assumed to be strictly ordered, $v_1 > v_2 > \ldots > v_n$, which prevents an existence of multiple equilibria. This setting can be equivalently expressed if each contestant has different costs of effort. Furthermore, the contestants are assumed to be risk neutral with identical linear costs of effort, in which the marginal cost of effort is set to one. Contestants simultaneously choose their effort $x_i$. Those with the highest $k$ efforts win one of the identical prizes.

The authors solve for an unique mixed strategy Nash equilibrium\footnote{They provide a proof of the uniqueness and the nonexistence of pure strategy equilibrium.}. In accordance with Siegel (2009), they find that only $k + 1$ contestants with the highest valuations participate\footnote{We refer to the same definition of participation as in model of Siegel (2009), p. 86: “A player participate in an equilibrium of a contest if with strictly positive probability he chooses scores associated with strictly positive costs of losing.”} in the equilibrium as they invest effort $x_i$, $i = \ldots$
1, \ldots, k+1, according to a cumulative probability distribution functions $F_i(x)$, while the remaining players do not participate in an equilibrium\(^4\). While the upper support of $F_i(x)$'s is same to all contestants, the value of the lower support decreases with the rank of the player (i.e., as valuations decrease) reaching zero for contestant $k$ and $k+1$. Each actively participating contestant $i = 1, \ldots, k+1$ has the expected net surplus equal to $v_i - v_{k+1}$. Since costs are assumed to be linear, the equilibrium expected effort can be expressed as the difference between the gross and the net surplus. Hence, for $i = 1, \ldots, k+1$, we have

$$x_i(k) = p_i(k)v_i - (v_i - v_{k+1}),$$

(3.4)

where $p_i(k)$ is the player $i$’s probability of winning any of the $k$ prizes\(^5\). Clark and Riis (1998a) do not further examine the optimal allocation of prizes of this simultaneous contest. Nevertheless, in the survey by Sisak (2009), page 100, an example based on Clark and Riis (1998s) and Hillman and Riley (1989) is provided showing that multiple prizes may be optimal with complete information and linear costs.

### 3.3.2 Incomplete information

The models of incomplete information, in which asymmetries are employed by different costs of effort, work with cost function as a function of two variables—the effort $x_i$ and the ability parameter, which influences the marginal cost of effort. The ability parameter is distributed according to a continuous cumulative distribution function in the population, which is a common knowledge, but the particular contestant’s type is a private information. Therefore, there is a continuum of types and incomplete information, which causes that pure strategy equilibria exist in this kind of contests (as opposed to the symmetric or asymmetric complete information scenario).

Moldovanu and Sela (2001) analyze the case of risk-neutral contestants for which they examine the solution when the cost functions are strictly concave, strictly convex, and linear. In the model, there are $n$ risk-neutral contestants and $p \leq n$ prizes of valuations $v_1 \geq v_2 \geq \ldots \geq v_p \geq 0$ that are potentially different, but symmetric across the contestants. A cost function of a contestant $i$ takes form of $c_i \gamma(x_i)$. It is assumed to be separable in effort $x_i$ and ability $c_i$, and strictly increasing with $\gamma(0) = 0$. The ability parameter $c_i$ is drawn

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\(^4\)For exact characterization of $F_i(x)$’s, see Clark and Riis (1998a).

\(^5\)The exact formulation of this probability can be found in Clark and Riis (1998a).
Multi-prize perfectly discriminating contest

independently from the interval \([m, 1]\), \(m > 0\), which is distributed according to a cumulative distribution function \(F(c)\), which is a common knowledge. Moldovanu and Sela (2001) put the main focus on case with two prizes of different values \(v_1 \geq v_2 \geq 0\) with \(n \geq 3\) contestants, but the extension to more prizes is analogous. Moreover, they restrict their attention to symmetric equilibria only. The equilibrium effort \(x(c)\) is solved from a differential equation arising from the first-order conditions of the maximization of the expected payoff function.

The symmetric equilibrium effort function \(x(c)\) takes form of

\[
x(c) = \gamma^{-1} [A(c)v_1 + B(c)v_2],
\]

where the weights \(A(c)\) and \(B(c)\) are given by

\[
A(c) = (n - 1) \int_c^1 \frac{1}{a} [1 - F(a)]^{n-2} F'(a) \, da
\]

and

\[
B(c) = (n - 1) \int_c^1 \frac{1}{a} [1 - F(a)]^{n-3} [(n - 1)F(a) - 1] F'(a) \, da.
\]

We can see, that the effort in equilibrium is a weighted sum of the two prizes, where the respective probabilities that a particular type wins a given prize influence those weights. The contest designer then maximizes the expected aggregate effort \(X(\alpha)\), that for two prizes takes a form of

\[
\max_{\alpha \in [1/2, 1]} X(\alpha) = n \int_m^1 \gamma^{-1} [A(c) + (1 - \alpha)(B(c) - A(c))] F'(c) \, dc
\]

where weights are characterized as above and \(\alpha \in [1/2, 1]\) is the share of the prize fund \(V\) allocated to the first prize (the remaining share \((1 - \alpha)\) is then allocated to the second prize).

Examining the obtained expression (3.8) for the expected aggregate effort Moldovanu and Sela (2002) conclude that in a case of linear as well as strictly concave cost function a single prize is optimal. For a case of strictly convex costs, however, they find a necessary and sufficient condition that makes more than one prize optimal. The optimality of the second prize (or further prizes) is increasing in the convexity of the cost function. The authors also note that the higher the number of contestant, the easier the condition of optimality of
more prizes is to be satisfied.

\section*{3.4 Concluding remarks}

In this section, we briefly surveyed some of the works on perfectly discriminating contests with risk-neutral contestants. Within a complete information framework, neither in symmetric nor in asymmetric setting a pure strategy Nash equilibrium exist. Since perfectly discriminating contests represent a limiting case to the Tullock contest when the Tullock scale to effort parameter goes to infinity, this finding is in accordance with the literature on Tullock single-prize contests. It states that as the Tullock parameter $r$ exceeds a certain threshold (that is no greater than 2.0), a pure strategy equilibrium does not exist in the contest\textsuperscript{6}. As we will later see, it is also in accordance with our findings on existence of pure strategy equilibria in multi-prize (asymmetric) contests. However, the nonexistence of a pure strategy equilibrium is not preserved under incomplete information framework, as it is shown by Moldovanu and Sela (2002).

Another observation, especially addressed by Siegel (2009), concerns active participation of the asymmetric contestants in an equilibrium. It is shown that when there is $k$ prizes in the contests, only $(k + 1)$ contestants actively participate, meaning that they have a positive probability of exerting strictly positive amount of effort. As we will see, asymmetries introduced into the Tullock contest (single- as well as multi-prize) also causes a fraction of contestants to exert nothing in an equilibrium, where the size of the fraction depends on the parameters of the model.

As to the optimal allocation of the prizes, in a symmetric case with linear costs, the allocation of the prize fund between the prizes does not matter so far as there is at least one prize less than the number of the contestants. In asymmetric complete information case with linear costs, the allocation of the prize fund matters and depends on the distribution of types. Single as well as multiple prizes can be found optimal. In asymmetric incomplete information case, the optimality of multiple prizes is found for strictly convex costs only; linear costs lead to optimality of a single prize.

\textsuperscript{6}A references about this literature are to be found in section 6.7.1 at page 59
Chapter 4

Extending asymmetric Tullock contests from single to multiple prizes

Contests with multiple prizes can be divided into different kinds along several dimensions. One dimension is whether contestants invest only one or more efforts. In the first case, all the prizes are then awarded solely on the basis of this one effort. In the latter case, the contestants are presented with subgroups of prizes for each of them they can choose different amount of effort to be expanded. Another possibility of how to divide multiple-prize contests is according to whether a contestant can win at most one prize or whether he is eligible to win more of the prizes.

In this thesis, we focus on single-efforts contests in which each contestant may win at most one prize. When modelling contests with more than one prize, there is a need to find how a CSF used for single-prize contests extend to a multi-prize context. Unfortunately, the literature on extension of the Tullock single-prize contest has not been sufficiently addressed yet. In this section, we present two works addressing this issue, one of which we follow in our subsequent work.

Both of these works assume symmetric contestants and linear costs of effort. Both of these works also come to the same conclusion, that a single prize is optimal. The research question is whether such statement remains valid when asymmetries are introduced. We thus continue by presenting a work on asymmetries in single-prize contests, whose methodology we further follow when setting up the multi-prize asymmetric model.
4.1 Extension of Tullock CSF to multi-prize contest

Berry (1993) presents the first attempt of generalization of a Tullock CSF to multiple prize contests. In his model, he works with constant returns to scale in efforts (Tullock parameter $r = 1$) and linear cost function $c(x_i) = x_i$. He assumes that the total amount of the reward $V$ is divided equally among $k$ available prizes, so that the value of winning any of the prizes is $v_j = \frac{V}{k}$, $\forall j = 1, \ldots, k$.

His proposed shape of the total probability of winning a prize of value $\frac{V}{k}$ for a contestant $n$ is

$$P_n(x_n, x_{-n}) = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} \times \left( \sum_{i=1}^{k-1} x_i + x_n + \sum_{i=1, i\neq 2}^{k} x_i + x_n + \ldots + \sum_{i=n-k+1}^{n-1} x_i + x_n \right) \left( \sum_{i=1}^{k} x_i + \sum_{i=1, i\neq 2}^{k+1} x_i + \ldots + \sum_{i=n-k+1}^{n} x_i \right) \frac{n}{k}. \quad (4.1)$$

The numerator is the sum of all efforts exerted by the winners in all combinations that include the contestant $n$ as a winner. If the contestant $n$ is one of the winners, then there is $\binom{n-1}{k-1}$ possible combinations of how to fill in the remaining “$k - 1$” winning spots out of remaining “$n - 1$” contestants. The denominator is the sum of all efforts exerted by the winners in all possible combinations of the winning group. There is $\binom{n}{k}$ ways how to pick “$k$” winners out of “$n$” contestants. The probability of winning is then based on the classical definition of probability\(^1\). The expected profit of a contestant $i$ to be maximized is thus given by

$$E\pi_i(x_i, x_{-i}) = P_i(x_i, x_{-i}) \frac{V}{k} - x_i. \quad (4.2)$$

Taking first order conditions and assuming an interior symmetric equilibrium

\(^1\text{Classical Definition of Probability: Given } n \text{ equally likely outcomes out of which } s \text{ represents the number of successful outcomes, the probability of success is } s/n.\)
(x_i = x \ \forall i) \text{ results in the total level of effort in the contest }, \ X(n, k), \text{ given by}

\[ X(n, k) = nx = \frac{n - kV}{n - k}. \quad (4.3) \]

The author concludes that, for this kind of contest, a single prize is optimal when maximizing the aggregate effort.

However, Clark and Riis (1996) show that the function (4.1) can be rearranged so that it takes a form of

\[ P_i(x_i, x_{-i}) = \frac{x_i}{\sum_{j=1}^{n} x_j} + \left(1 - \frac{x_i}{\sum_{j=1}^{n} x_j}\right) \frac{k - 1}{n - 1}. \quad (4.4) \]

A contest with the function (4.4) is equivalent to a standard Tullock single-prize contest with the value of the prize equal to \( \left(\frac{V}{k} - \frac{(k-1)V}{(n-1)k}\right) \). To see that, let substitute (4.4) for the probability in the equation (4.2). We get

\[ E\pi_i(x_i, x_{-i}) = \left(\frac{x_i}{\sum_{j=1}^{n} x_j} + \left(1 - \frac{x_i}{\sum_{j=1}^{n} x_j}\right) \frac{k - 1}{n - 1}\right) \frac{V}{k} - x_i \\
= \left[\frac{x_i}{\sum_{j=1}^{n} x_j} \left(1 - \frac{k - 1}{n - 1}\right)\right] \frac{V}{k} - x_i + \left(\frac{k - 1}{n - 1}\right) \frac{V}{k} \quad \text{constant with respect to } x_i \quad (4.5) \]

In other words, only the first prize is awarded on the basis of invested efforts (according to the standard Tullock CSF for a single-prize contest) while the remaining \( k - 1 \) prizes are awarded to each contestant with an equal probability \( \frac{1}{n - 1} \), given that the contestant does not win the first prize. The Berry (1993)’s conclusion about the optimality of a single prize is thus given by the specifics of the proposed CSF where only one of the prizes is awarded based on the efforts while others are awarded independently of the efforts.

Clark and Riis (1996) further propose a different method how to approach a CSF for multiple prize contests. They present a process for choosing the winners, in which all the prizes are awarded on a basis of invested efforts. First, all the contestants simultaneously exert their irreversible efforts which are valid for all \( k \) rounds in which winners of \( k \) prizes are determined. The winners are then chosen sequentially, each time eliminating the winners of preceding rounds from participation in the following rounds. Hence, contestants choose their efforts based upon the total probability of winning one of the rounds. The authors call this mechanism a nested game, since “the probability of winning
one contest is nested in the probability of not having won the previous contest” (Clark and Riis (1996), p.178).

Let $x_i$ be an effort invested by a contestant $i$, $i = 1, \ldots, n$, and denote $x = (x_1, \ldots, x_n)$ the vector of effort invested by all the contestants. Furthermore, let $P_i(x; k)$ denote the probability that a contestant $i$ wins in a $k$-round nested contest, and let $p^s_i(x; k)$ denote the probability that a contestant $i$ wins the $s$’th prize, conditional on losing each preceding prize. The assumptions made about $p^s_i(x; k)$ are that the function is homogeneous of degree zero in efforts of the contestants that are still in a game at round $s$ (not having won any preceding prize), increasing and strictly concave in $x_i$, and symmetric in all $x_j, j \neq i$. The focus is on symmetric equilibrium. The probability $P_i(x; k)$ that contestant $i$ wins a prize (he becomes a winner of one of the $k$ rounds) is given by:

$$P_i(x; 1) = p^1_i \quad \text{for } k = 1$$

$$P_i(x; k) = p^1_i + (1 - p^1_i) p^2_i + (1 - p^1_i) (1 - p^2_i) p^3_i + \ldots + \prod_{s=1}^{k-1} (1 - p^s_i) p^k_i = p^1_i + \sum_{j=1}^{k-1} \left[ \prod_{s=1}^{j} (1 - p^s_i) p^{j+1}_i \right] \quad \text{for } k > 1. \quad (4.6)$$

Assuming linearity of a cost function, $c(x_i) = x_i$, the expected payoff of contestant $i$ to be maximized is then

$$E\pi_i(x_i, x_{-i}; k) = P_i(x_i, x_{-i}; k)V(k) - x_i, \quad (4.7)$$

where $V(k)$ is the value of winning in a $k$-round nested contest.

Taking the FOCs and assuming interior symmetric pure strategy equilibrium, the authors are able to express the equilibrium aggregate level of effort in the contest. Examining this expression for a case, in which the conditional probability function is given by the Tullock CSF with constant returns to scale in efforts, $p^1_i(x) = \frac{x_i}{\sum_{j=1}^{x_i} x_j}$, and using the valuation form of Berry (1993), $V(k) = \frac{V}{k}$, they conclude that the aggregate effort is weakly decreasing in the number of prizes (while keeping prize sum constant and fixed number of contestants).

They work is further generalized by Clark and Riis (1998c), who consider the case for Tullock CSF with varying $r$. They state the conditions for the

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2The symmetric equilibrium is in fact the unique equilibrium of this nested game.

3In Berry (1993) the value of winning $V(k)$ is set to $\frac{V}{k}$, where $V$ is the total amount of the reward to be awarded in the contest.
existence of the symmetric pure strategy equilibrium. When these conditions are fulfilled, one of their findings shows that a single-prize is optimal when maximizing the total effort in the contest.

### 4.2 Single-prize asymmetric Tullock contest

Asymmetry can be introduced into contests by allowing contestants to have different valuations, different cost functions, and/or different relative ability (different relative effectiveness of translating of their efforts into the probability of winning). The strength of a contestant is evaluated by his probability of winning at the equilibrium solution. The most common finding of the works on asymmetric single-prize contests is that a strong contestant is one with higher valuation and/or high relative effectiveness.

Quite extensive literature has already been developed on asymmetry in single-prize contest represented by either kind of different characteristics of the contestants. Stein (2002) puts the partial results of different papers into a coherent whole. He considers an \( n \)-player imperfectly discriminating contest, \( n \geq 2 \), in which he allows for different valuations of winning, different relative effectiveness, or both.

Let \( x_i \) be an effort invested by a contestant \( i, i = 1, \ldots, n \), and denote \( \mathbf{x} = (x_1, \ldots, x_n) \) the vector of effort invested by all the contestants. Furthermore, let \( v_i, i = 1, \ldots, n \), be an individual (possibly different) valuation of winning the contest and let \( \lambda_i, i = 1, \ldots, n \), capture the relative effectiveness of exerted effort. It can be interpreted as the rate at which the effort \( x_i \) is translated into effective competing effort. The parameters \( \{v_i\}_{i=1}^n, \{\lambda_i\}_{i=1}^n \) are assumed to be a common knowledge. Without loss of generality, the parameters are assumed to be decreasing in their product: \( \lambda_1 v_1 \geq \lambda_2 v_2 \geq \ldots \geq \lambda_n v_n > 0 \). Assuming constant returns to scale in effort (Tullock parameter \( r \) set to 1), the probability of winning, \( p_i(\mathbf{x}) \), for a contestant \( i \) is given by

\[
p_i(x_i, x_{-i}) = \frac{\lambda_i x_i}{\sum_{j=1}^{n} \lambda_j x_j}.
\]  

(4.8)

Expected payoff, \( E\pi_i(\mathbf{x}) \), that a contestant \( i \) maximizes by choosing his level of effort, \( x_i \), takes a form of

\[
\max_{x_i \geq 0} E\pi_i(x_i, x_{-i}) = p_i(x_i, x_{-i}) v_i - x_i \overset{\text{(4.8)}}{=} \frac{v_i \lambda_i x_i}{\sum_{j=1}^{n} \lambda_j x_j} - x_i,
\]  

(4.9)
where a linear cost function, \( c_i(x_i) = x_i \) \( \forall i \) is assumed. Since the expected payoff is homogeneous of degree zero in \( \{\lambda_i\}_{i=1}^{n} \), for convenience \( 0 < \lambda_i \leq 1 \) is assumed.

From the FOCs of the problem (4.9) and after some further calculations, Stein (2002) is able to characterize the equilibrium effort strategies. He finds that there is an index \( p \in \{1, \ldots, n\} \) such that contestants 1, 2, \ldots, \( p \) (those having the highest values of a product \( \lambda_i v_i \)) exert strictly positive amount of effort in the equilibrium whereas the remaining contestants \( p + 1, \ldots, n \) exert nothing. The equilibrium effort of a contestant \( i \) (\( \forall i \leq p \)) takes form of

\[
x_i = \frac{(p - 1) \Gamma_p}{p \lambda_i} \left[ 1 - \frac{(p - 1) \Gamma_p}{p \lambda_i v_i} \right],
\]

(4.10)
giving a probability of winning for a contestant \( i \) (\( \forall i \leq p \)) by

\[
P_i = 1 - \frac{(p - 1) \Gamma_p}{p \lambda_i v_i},
\]

(4.11)

where \( \Gamma_p = \left[ \frac{1}{p} \sum_{i \leq p} \frac{1}{\lambda_i v_i} \right]^{-1} \) is the harmonic mean of the first \( p \) values of the sequence \( \{\lambda_i v_i\} \).

Furthermore, Stein (2002) shows that from equation (4.10) the value of \( p \) (the number of contestants active in the contest by exerting a strictly positive amount of effort) can be determined. It is found as the largest index \( p, 2 \leq p \leq n \), such that \( x_p > 0 \) or, equivalently, \( P_p > 0 \).

Examining the condition for the index \( p \) identifies a difference between 2-player contests and \( n \)-player contests, \( n > 2 \). In the first case, both contestants actively participate in the contest regardless of their valuations and relative effectiveness parameters. In the latter case, these parameters determine who actively participate and who does not, allowing for the possibility that some of the \( n \) contestants choose to exert zero effort.

We thus see that introducing asymmetry between risk-neutral contestants generates a discouragement effect for the contestants that are relatively weaker. In single-prize contests with \( n > 2 \) contestants, this discouragement may result in inactivity of some of the weaker contestants. The number of active participants (those exerting strictly positive amount of effort in equilibrium) is thus endogenously determined from the parameters of the problem.
Chapter 5

Analytical partial solution to two-prize asymmetric Tullock contest

Szymanski and Valletti (2005) provide the first attempt to examine a multi-prize contests with heterogeneous agents by analyzing a three-person contest and looking at the optimality of second prizes. The heterogeneity is introduced by different abilities of the contestants which can be represented as them having different cost of effort. They consider a case when two of the contestants are of equal ability while the third one is of different ability.

In the model, they suppose that there is a prize fund $V$ to be competed for, which is divided into two prizes; the fraction $1/2 \leq \alpha \leq 1$ is allocated to the first prize while the fraction $1 - \alpha$ is allocated to the second prize. The authors follow Clark and Riis (1998c) in the methodology of how to determine the success probability. The (risk-neutral) contestants maximize the following expected payoff function

$$\max_{x_i \geq 0} E\pi_i(x_i, x_{-i}) = p_i(x_i, x_{-i})\alpha V + \sum_{j \neq i} p_j(x_i, x_{-i})p_{ij}(x_i, x_{-i})(1 - \alpha)V - c_ix_i,$$

(5.1)

where $p_{ij}(x)$ is the conditional probability that contestant $i$ wins the second round given that contestant $j$ has won the first round and $p_i(x)$ ($p_j(x)$) is unconditional probability that contestant $i$ ($j$, respectively) wins the first round.
These probabilities are given by a Tullock formulation, that is,

\[ p_i(x_i, x_{-i}) = \frac{x_i^r}{\sum_{j=1}^{n} x_j^r}, \quad (5.2) \]

\[ p_{ij}(x_i, x_{-i}) = \frac{x_i^r}{\sum_{h=1, h \neq j}^{n} x_h^r}. \quad (5.3) \]

An important difference with respect to a symmetric contest is that we must distinguish who is the winner of the first prize when calculating the probability of winning the second prize. The reason is that the equilibrium efforts are likely to be asymmetric (unlike in the symmetric equilibrium considered in the section (4.1) in which this does not matter). In this model, the cost function is assumed to be linear with marginal cost \( c_i \) that differ across contestants.

Furthermore, two specific cases are considered in the paper: either there are two equally ‘strong’ contestant and one ‘weak’ contestant or there are two equally ‘weak’ contestants and one ‘strong’ contestant. Marginal costs of effort of a ‘weak’ contestant are \( c_w = 1 \) while marginal costs of effort of a ‘strong’ contestant are \( 0 < c_s < 1 \).

Szymanski and Valletti (2005) do not provide an explicit solution for the equilibrium efforts, however, they calculate implicit conditions for the equilibrium with which they analyze the limit behavior. Let \( p_s \) be the probability of a strong contestant winning the first prize, \( p_w \) be the probability of a weak contestant winning the first prize and \( q_{ws} \) be the probability of a weak contestant winning the second prize conditional on not having won the first prize.

In the case with two strong and one weak contestants, the implicit conditions for equilibrium strategies \( x_w \) and \( x_s \) of weak and strong contestant, respectively, are

\[ x_w = 2rV \left[ \alpha p_s (1 - 2p_s) + (1 - \alpha) p_s q_{ws} (2p_s - q_{ws}) \right] \quad (5.4) \]

\[ x_s = \frac{rV}{c_s} \left\{ \alpha p_s (1 - p_s) + (1 - \alpha) \left[ (1 - q_{ws}) (p_s q_{ws} - p_s^2) + \frac{(1 - 2p_s)^2}{4} \right] \right\}. \]

When \( c_s \), the marginal costs of strong contestants, approaches unity, the contest becomes a three-person symmetric contest and the results of Clark and Riis (1998c) apply; single prize is optimal when maximizing the aggregate effort. On the other hand, when \( c_s \) approaches zero, the contest reduces to a two-person (symmetric) contest, in which only the two strong contestants actively participate. Since their cost of effort is negligible, they both invest a very large amount of effort \( (x_s^{lim} = \lim_{c_s \to 0} x_s = \infty, \text{given } 1/2 < \alpha < 1) \)
Table 5.1: Case “two strong and one weak contestants”

<table>
<thead>
<tr>
<th>$c_s \rightarrow 1$</th>
<th>$c_s \rightarrow 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The contest becomes symmetric; $p_s = p_w = 1/3$, $q_{ws} = 1/2$ $\implies$ single prize is optimal.</td>
<td>$x_s \rightarrow \infty$, $x_w \rightarrow 0$ $\implies$ $p_s = 1/2$, $q_{ws} = 0$; reduction to two-player (symmetric) contest $\implies$ single prize is optimal.</td>
</tr>
</tbody>
</table>

resulting in their probability of winning equal to one half for each. The weak contestant is discouraged from active participating (plugging $p_s = 1/2$ and $q_{ws} = 0$ into equation for $x_w$ in (5.4) leads to $x_w = 0$). Single prize is thus optimal in this case as well. The summarization of the limit behavior for the case of two strong and one weak contestant is presented in the Table 5.1.

The second case with two weak and one strong contestants is more interesting. The implicit conditions for equilibrium strategies are

\[
x_w = rV \left\{ \alpha p_w (1 - p_w) + (1 - \alpha) \left[ \frac{(1 - 2p_w)^2}{4} + p_w q_{ws} (1 - p_w - q_{ws}) \right] \right\}
\]

\[
x_s = \frac{2rV}{c_s} \left[ \alpha p_w (1 - 2p_w) - (1 - \alpha) p_w (1 - q_{ws}) (1 - 2p_w - q_{ws}) \right].
\]

When $c_s$ approaches zero, the strong contestant invests an infinite amount of effort while the weak contestants invest effort equal to

\[
x_w^{\text{lim}} = \lim_{c_s \rightarrow 0} x_w = \frac{rV (1 - \alpha)}{4},
\]

given $1/2 < \alpha < 1$. In fact, this effort is equivalent to the equilibrium effort in a single-prize contest with two symmetric contestants and a prize of value $V(1 - \alpha)$. Therefore, the weak contestants focus solely on competing between themselves for the second prize, leaving out the (seemingly unrealistic) option of them winning the first prize. When $\alpha$ is set to 1, i.e., transforming the contest to a single-prize contest, the weak players do not actively participate (their effort is zero) and the strong player exert as much effort as is needed to keep them out.

Szymanski and Valletti (2005) further examine the effect of varying $\alpha$ on effort invested by the strong contestant in the equilibrium. The total differential
of the first-order condition of the strong contestant is

$$\frac{\partial^2 E_{\pi_s}}{\partial x_s^2} \ d\alpha + \frac{\partial^2 E_{\pi_s}}{\partial x_s \partial w} \ d\alpha + \frac{\partial^2 E_{\pi_s}}{\partial x_w \partial w} \ d\alpha = 0. \quad (5.7)$$

The direct effect (the second term in the equation) captures the feature that the effort of the strong contestant decreases as the first prize diminishes while keeping the efforts of the weak contestants constant. The strategic effect (the third term) captures the strong contestant’s response to the induced change in effort of the weak contestants. The equation (5.7) can be rearranged to

$$\frac{dx_s}{d\alpha} = -\frac{\frac{\partial^2 E_{\pi_s}}{\partial x_s \partial \alpha} > 0}{\frac{\partial^2 E_{\pi_s}}{\partial x_s \partial w} \ d\alpha + \frac{\partial^2 E_{\pi_s}}{\partial x_w \partial w} \ d\alpha} \quad (5.8)$$

The authors show that, for a sufficiently high $\alpha > \alpha^*$, $\frac{dx_s}{d\alpha} < 0$ meaning that the strategic effect dominates. They calculate the critical value of $\alpha^*$, which is given by

$$\alpha^* = \frac{2 + r}{2(1 + r)} \quad (5.9)$$

and they argue that $\alpha^*$ also maximize aggregate effort in the limiting case of $c_s \rightarrow 0$, because “in this limiting case, total effort is basically coincident with the effort supplied by the strong player” (Szymanski and Valletti (2005), p. 477). As $1/2 < \alpha^* < 1$, it implies that the second prize should never exceed the first prize for this case. As the contest becomes more discriminating ($r$ increases), the critical value of $\alpha^*$ decreases implying that the share of $V$ allocated to the first prize should decrease as well. Furthermore, Szymanski and Valletti (2005) investigate the case of a single-prize contest, i.e., $\alpha = 1$, for which they perform a comparative statics for varying $c_s$. They show that, especially for a high degree of asymmetry, introducing a second prize in the contest has positive effect on individual and total effort.
Chapter 6

Numerical modelling of two-prize asymmetric Tullock contests

In this section, we perform a similar analysis as Szymanski and Valletti (2005) perform in their paper. Instead of contestants having different costs, however, we introduce asymmetry by contestants having different relative abilities. We also extend the analysis to an arbitrary number of contestants and we analyze the contests also for milder values of the parameters, not just the limiting cases as Szymanski and Valletti (2005) present.

6.1 The general problem

We suppose that there is a total prize fund $V \in \mathcal{R}$ that is to be awarded in the contest, from which a fraction $1/2 \leq \alpha \leq 1$ is allocated to the first prize and a fraction $(1 - \alpha)$ is allocated to the second prize. If there is $n \in \mathcal{N}$ contestants in the contest, we suppose that the unconditional probability of winning the first prize, $p_i(x)$, is given by a Tullock formulation. That is, $\forall i \in \{1, \ldots, n\}$,

$$p_i(x) = \begin{cases} \frac{\lambda_i x_i^r}{\sum_{j=1}^{n} \lambda_j x_j} & \text{if } \max\{x_1, \ldots, x_n\} > 0 \\ \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j} & \text{otherwise} \end{cases},$$  \hspace{1cm} (6.1)$$

where $x_i$ is the effort exerted by a player $i$, $x = (x_1, \ldots, x_n)$ is a vector of efforts of all the contestants, $\lambda_i \geq 0$ is a parameter capturing the relative ability of a contestant $i$, $i \in \{1, \ldots, n\}$ and $r > 0$ is the Tullock parameter. Following the methodology of Stein (2002), we introduce asymmetry by allowing for the possibility that $\exists i, j \in \{1, \ldots, n\}: \lambda_i \neq \lambda_j$. 
To determine the probability of winning the second prize, we follow the method proposed by Clark and Riis (1996, 1998c) called the nested game\(^1\) that says: “If [contestant] \(i\) does not win this first rent, his contribution remains in the pool giving a chance of winning the second prize. The unconditional probability that \(i\) wins the second prize is the product of the probability that \(i\) does not win the first and the probability that he does win the second.” (p. 609). In our case, the unconditional probability that a contestant \(i\) wins the second prize can be thus defined as

\[
\sum_{j \neq i} p_j(x)p_{ij}(x),
\]

where

\[
p_{ij}(x) = \frac{\lambda_i x_i^r}{\sum_{h=1}^n h \neq j \lambda_h x_h^r}
\]

is the probability that a contestant \(i\) wins the second prize given that contestant \(j\) has won the first prize\(^2\).

Each contestant then maximizes his expected payoff, which is given by a function

\[
\max_{x_i \geq 0} E\pi_i(x) = p_i(x)\alpha V + \sum_{j \neq i} p_j(x)p_{ij}(x)(1 - \alpha)V - cx_i,
\]

where we assume that contestants have identical linear cost function of effort with \(c > 0\) marginal cost of effort. The first-order condition for the contestant \(i\)'s maximization problem is then

\[
\frac{\partial E\pi_i}{\partial x_i} = \frac{\partial p_i}{\partial x_i}\alpha V + \sum_{j \neq i} \left(\frac{\partial p_j}{\partial x_i}p_{ij} + p_j \frac{\partial p_{ij}}{\partial x_i}\right) (1 - \alpha)V - c = 0.
\]

\(^1\)See section 4.1, page 21.

\(^2\)Hence, \(j\) has been taken away from the pool of those competing for the second prize which is captured by \(h \neq j\) in the sum in the expression (6.3). This can be thought of as another contest when only \(n - 1\) contestants are competing for the second prize and \(j\) has already been eliminated from further participation, even though this is not a sequential model.
Let define $X \equiv \sum_{i=1}^{n} \lambda_i x_i^r$ and $X_{-j} \equiv \sum_{i \neq j} \lambda_i x_i^r$. Substituting

$$\frac{\partial p_i}{\partial x_i} = \frac{r \lambda_i x_i^r X - \lambda_i^2 x_i^{2r}}{X^2} \quad (6.6)$$

$$\frac{\partial p_j}{\partial x_i} = -\frac{r \lambda_i \lambda_j x_j^r x_i^r}{x_i X^2} \quad (6.7)$$

$$\frac{\partial p_{ij}}{\partial x_i} = \frac{r \lambda_i x_i^r X_{-j} - \lambda_i^2 x_i^{2r}}{x_i X_{-j}^2} \quad (6.8)$$

for the partial derivatives in the equation (6.5), multiplying the equation (6.5) by $x_i$ and rearranging the terms, we get

$$r \left\{ \frac{\lambda_i x_i^r X - \lambda_i^2 x_i^{2r}}{X^2} \alpha V + \sum_{j \neq i} \left( \frac{\lambda_j x_j^r \lambda_i x_i^r (XX_{-j} - \lambda_i x_i^r X - \lambda_i x_i^r X_{-j})}{X^2 X_{-j}^2} \right) (1 - \alpha) V \right\} - cx_i = 0. \quad (6.9)$$

In logit formulation of the probabilities, this expression reduces to

$$r \left\{ p_i (1 - p_i) \alpha V + \sum_{j \neq i} p_j p_{ij} (1 - p_i - p_{ij})(1 - \alpha) V \right\} - cx_i = 0. \quad (6.10)$$

Furthermore, we assume that the second-order condition is satisfied at equilibrium. To get this condition, let derivate equation (6.10) with respect to $x_i$ and further multiply the result by $x_i$.\footnote{Since $x_i$ is positive, multiplying the result by $x_i$ does not change the sign of the equation. Furthermore, since $\frac{\partial E_{\pi_i}}{\partial x_i} = 0$ at the equilibrium, second-order condition, $\frac{\partial^2 E_{\pi_i}}{\partial x_i^2}$, is equivalent to $\frac{\partial}{\partial x_i} \left( \frac{\partial E_{\pi_i}}{\partial x_i} x_i \right) = \frac{\partial^2 E_{\pi_i}}{\partial x_i^2} + \frac{\partial E_{\pi_i}}{\partial x_i} \frac{\partial x_i}{\partial x_i} = 0$, so that second-order condition corresponds to derivation of the equation (6.10).}

We get

$$\left( \frac{\partial p_i}{\partial x_i} (1 - 2p_i) x_i \right) \alpha V r - cx_i + (1 - \alpha) V r \cdot \left( \sum_{j \neq i} x_i \left( \frac{\partial p_j}{\partial x_i} p_{ij} + \frac{\partial p_{ij}}{\partial x_i} \right) (1 - p_i - p_{ij}) + p_j p_{ij} x_i \left( -\frac{\partial p_i}{\partial x_i} - \frac{\partial p_{ij}}{\partial x_i} \right) \right) < 0. \quad (6.11)$$

Substituting expressions (6.6), (6.7) and (6.8) for the derivatives and using the

logit formulation of the probabilities, the expression (6.11) can be rewritten as

\[(1 - 2p_i)p_i(1 - p_i)\alpha Vr^2 + (1 - \alpha)Vr^2.\]

\[\cdot \sum_{j \neq i} p_j p_{ij} [(1 - p_i - p_{ij})^2 - p_i(1 - p_i) - p_{ij}(1 - p_{ij})] - cx_i < 0.\] (6.12)

Overall, the equations (6.10) and (6.12) characterize the equilibrium of the contest\(^4\).

6.2 Symmetric contest

In this section, we display the conclusion by Clark and Riis (1998c) captured in their Proposition 2(d), in which they state that in a symmetric contest, maximum average effort is achieved when a single prize is made available.

In a symmetric contest \((\lambda_i \equiv \lambda \text{ for all } i \in \{1, \ldots, n\})\) in a symmetric equilibrium, each contestant invests the same amount of effort \((x_i \equiv x)\). Hence, each of them has also the same probabilities of winning the first \((p_i = p_j = 1/n)\) or the second prize \((p_{ij} = 1/(n - 1))\), respectively. Substituting for the probabilities in the first-order condition represented by the equation (6.10), we obtain

\[x = \frac{rV}{c} \left( \frac{1}{n} \frac{n-1}{n} \alpha + (n-1) \frac{1}{n} \frac{1}{n-1} \frac{n(n-1)}{n(n-1)} \frac{1}{n(n-1)} (1 - \alpha) \right),\]

which, after some rearranging, reduces to

\[x = \frac{rV}{c} \left( \frac{n-1}{n^2} - \frac{1 - \alpha}{n(n-1)} \right).\] (6.13)

Substituting for the probabilities in the second-order condition represented by the equation (6.12), we obtain

\[\alpha Vr^2 \frac{(n-2)(n-1)}{n^3} + (1 - \alpha)Vr^2(n-1) \frac{1}{n} \frac{1}{n-1} \cdot \left[ \left( \frac{1}{n} - \frac{1}{n-1} \right)^2 - \frac{n-1}{n^2} - \frac{n-2}{(n-1)^2} \right] - cx < 0.\]

When we further substitute for \(x\) from the equation (6.13) and rearrange the

\(^4\)We have arrived to the same expressions characterizing the equilibrium effort (6.10) and (6.12) as Szymanski and Valletti (2005). The difference is in the definition of probabilities and conditional probabilities in these expressions.
terms, the second-order condition to be met in the equilibrium becomes

\[ V r \frac{n-2}{n^3} \left[ \alpha r (n-1) + (1-\alpha) \frac{n^3 - 6n^2 + 4n - 1}{(n-1)^2} - \frac{n^3 - 2n^2 + \alpha n}{(n-1)(n-2)} \right] < 0. \]

(6.14)

To see what effect on the contestants’ effort introducing the second prize has, let differentiate the equation (6.13) with respect to \( \alpha \). We get,

\[ \frac{\partial x}{\partial \alpha} = \frac{r V}{c} \frac{1}{n(n-1)} > 0, \]

showing that contestant’s effort is increasing in \( \alpha \) at the equilibrium when one exists. Maximum effort is thus achieved when a single prize is made available.

### 6.3 Two-person asymmetric contest

Let there be \( n = 2 \) contestants with \( \lambda_1 \neq \lambda_2 \) parameters of their relative effectiveness. Let \( p_1 = \frac{\lambda_1 x_1^r}{\lambda_1 x_1^r + \lambda_2 x_2^r} \) be the probability of player one winning the first prize and \( p_2 = \frac{\lambda_2 x_2^r}{\lambda_1 x_1^r + \lambda_2 x_2^r} = (1 - p_1) \) be the probability of player two winning the first prize. Since there are only two contestants, the probability that a contestant \( i \) wins the second prize given that he has not won the first prize is thus identical to both contestants and equal to one, i.e., \( p_{ij} = 1 \). With this setting, the first-order conditions (6.10) become

\[ x_1 = \frac{r V}{c} (2\alpha - 1) p_1(x) p_2(x) = \frac{r V}{c} (2\alpha - 1) \frac{\lambda_1 x_1^r \lambda_2 x_2^r}{(\lambda_1 x_1^r + \lambda_2 x_2^r)^2}, \]

(6.15)

\[ x_2 = \frac{r V}{c} (2\alpha - 1) p_1(x) p_2(x) = \frac{r V}{c} (2\alpha - 1) \frac{\lambda_1 x_1^r \lambda_2 x_2^r}{(\lambda_1 x_1^r + \lambda_2 x_2^r)^2}. \]

(6.16)

**Proposition 6.1 (Two-person asymmetric contest).** In a two-person asymmetric contest, in which the asymmetry is captured by different relative effectiveness of the contestants \( \lambda_1 \neq \lambda_2 \), in the equilibrium:

i) both contestants exert identical level of effort \( x_i \equiv x = \frac{r V}{c} (2\alpha - 1) \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2} \),

ii) a second prize reduces the effort of each contestant,

iii) (except the case when the two prizes are identical) the effort of each contestant increases as the contest becomes more symmetric, reaching the maximum \( x_{\text{max}} = \frac{r V}{c} \frac{2\alpha - 1}{4} \) when the contest actually becomes symmetric.
Proof. i) The first point of the proposition can be seen directly from the first-order conditions (6.15) and (6.16). Since the RHS in both equation is the same, it implies that \( x_1 = x_2 \). Let define \( x \equiv x_1 = x_2 \). Then, from equation (6.15), we receive

\[
x = \frac{rV}{c} (2\alpha - 1) \frac{\lambda_1 \lambda_2 x^{2r}}{(\lambda_1 + \lambda_2)^2 x^{2r}} = \frac{rV}{c} (2\alpha - 1) \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2}.
\] (6.17)

ii) To see what effect on the contestants’ effort introducing the second prize has, let differentiate the equation (6.17) with respect to \( \alpha \). We get,

\[
\frac{\partial x}{\partial \alpha} = 2 \frac{rV}{c} \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2} > 0,
\] (6.18)
given that both parameters \( \lambda_i \), \( i = 1, 2 \) are different from zero. Hence, the equilibrium effort is increasing in \( \alpha \) (decreasing in the second prize) and thus maximized for \( \alpha = 1 \) (when single prize is awarded).

iii) Let define a new variable \( \lambda \equiv \frac{\lambda_1}{\lambda_2} \). Without loss of generality, let assume that \( 0 \leq \lambda_1 \leq \lambda_2 \). Hence, \( \lambda \in [0, 1] \) capturing the degree of asymmetry in the contest (\( \lambda = 1 \) means that the contest is symmetric. As \( \lambda \) decreases, the contest becomes more asymmetric). Using the new variable, the equation (6.17) can be rewritten as

\[
x = \frac{rV}{c} (2\alpha - 1) \frac{\lambda}{(\lambda + 1)^2}.
\] (6.19)

Now, let us differentiate equation (6.19) with respect to \( \lambda \). We get

\[
\frac{\partial x}{\partial \lambda} = \frac{rV}{c} \frac{\lambda^{\geq 0}}{(2\alpha - 1) \frac{1 - \lambda^2}{(\lambda + 1)^4} \geq 0} \in [0, 1]
\] (6.20)

with strict inequality if \( \alpha > 1/2 \) and \( \lambda < 1 \). The maximum is thus reached when \( \lambda = 1 \), which implies the effort of

\[
x^{max} = \frac{rV}{c} \frac{2\alpha - 1}{4},
\] (6.21)

which is the same result as we get when we substitute \( n = 2 \) into general equation for symmetric contest (6.13).

6.4 Methodology

As the number of contestants in the contest increases, the FOCs characterizing the equilibrium strategies become much more complicated. Let us rewrite the FOCs (6.10) for a general case. We have,

\[ x_i = \frac{rV}{c} \left( p_i(1 - p_i) + \sum_{j \neq i} p_j p_{ij}(1 - p_i - p_{ij})(1 - \alpha) \right), \]  

(6.22)

where

\[ p_i(x) = \frac{\lambda_i x_i^r}{\sum_{j=1}^{n} \lambda_j x_j^r}, \quad p_{ij}(x) = \frac{\lambda_i x_i^r}{\sum_{h=1, h \neq j}^{n} \lambda_h x_h^r}. \]  

(6.23)

This is a system of \(n\) nonlinear equations in \(n\) unknowns. Since solving the problem analytically is not possible in most of the cases, numerical methods are employed to solve this kind of problems. The method we employ here is one based on a direct iteration method.

**Direct iteration method for a system of (nonlinear) equations**

Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a mapping defined on a set \( \Omega \subset \mathbb{R}^n \). We want to find a vector \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Omega \) of real numbers \( \alpha_i \) such that

\[ F(\alpha) = 0. \]  

(6.24)

We begin by rewriting the equation \( F(\alpha) = 0 \) into a form

\[ x = \Phi(x). \]  

(6.25)

We assume that there is a set \( \Omega_0 \) which belongs to a domain of the
mapping \( \mathbf{F} \) as well as to a domain of the mapping \( \Phi \) and we assume that both of the mappings are continuous on this set. Moreover, we assume that this set contains a solution \( \mathbf{x}^* \) common to both of the equations and that conditions

a) \( \forall \mathbf{x} \in \Omega_0 : \Phi(\mathbf{x}) \in \Omega_0, \)

b) \( \exists q \in (0, 1] : \| \Phi(\mathbf{x}) - \Phi(\mathbf{y}) \| \leq q \| \mathbf{x} - \mathbf{y} \| \ \forall \mathbf{x}, \mathbf{y} \in \Omega_0 \)

are satisfied. Then a sequence \( \{ \mathbf{x}^{(k)} \} \) given by the following iteration relation (6.26) converges to \( \mathbf{x}^* \in \Omega_0 \) for any initial approximation \( \mathbf{x}^{(0)} \in \Omega_0 \).

Using the equation (6.25) we are able to construct a sequence of successive approximations \( \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots \) of the vector \( \mathbf{x}^* \) according to the following algorithm:

1. Choose a number \( \delta > 0 \) and an initial approximation \( \mathbf{x}^{(0)} \in \Omega_0 \).

2. The following approximation is determined by the iteration formula

   \[ \mathbf{x}^{(k)} = \Phi(\mathbf{x}^{(k-1)}), \quad k = 1, 2, 3, \ldots \] (6.26)

3. If \( \| \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} \| > \delta \), go to the point 2; otherwise, stop the process. \( \mathbf{x}^{(k)} \) is then considered the approximation of the solution vector \( \mathbf{\alpha} \).

(taken from Míka (1982))

In our case, the system (6.24) of \( n \) equations are the \( n \) derivatives of the expected payoff function \( \frac{\partial E\pi_i(\mathbf{x})}{\partial x_i} x_i \) given by the equation (6.5). The first-order conditions for finding a maximum imply setting these derivatives equal to zero, giving us the system (6.24), indeed. The form \( \mathbf{x} = \Phi(\mathbf{x}) \) of the system is then given by equation (6.22).

*Programming the direct iteration process*

To calculate the approximations of the solution of the system, we wrote a code “programme multiprize for changing alpha” and “programme multiprize for changing alpha with random initial values and random lambdas” in Fortran, see Appendix. We write a programme designed to solve the system of equations

\[ x_i = \Phi_i(x_1, x_2, \ldots, x_n; \lambda_1, \lambda_2, \ldots, \lambda_n; \alpha, r) \quad i = 1, \ldots, n \] (6.27)
where \( n \) is the number of the contestants, \( x_i \) is the effort of the contestant \( i \), \( \lambda_i \) is the parameter capturing the relative effectiveness of the contestant \( i \), \( \alpha \in [1/2, 1] \) is the parameter determining the division of the total budget \( V \) between the two prizes (where the fraction \( \alpha V \) is allocated to the first prize, the rest is allocated to the second prize) and \( r \) is the Tullock parameter. The system (6.27) is simply a general expression of the equation (6.22). Since \( x_i \) in the equation (6.22) is proportional to the term \( \frac{V}{\varepsilon} \), without loss of generality we set it equal to one, i.e., \( \frac{V}{\varepsilon} \equiv 1 \).

The method employed to solve the problem is the direct iteration method.

1. We choose the values of the parameters: \( n \)—the number of players, \( w \in [0, 1] \)—the relaxation parameter, \( r > 0 \)—the Tullock parameter of the returns to scale in efforts, \( \varepsilon > 0 \)—the value representing effort of zero if hit, \( \text{error} > 0 \).

2.a We choose the values of parameters of each contestant: \( \{\lambda_i\}_{i=1}^n \) and the initial approximations of the solution \( \{x_i^{(0)}\}_{i=1}^n \), or

2.b we randomly generate the values of parameters of each of the contestants \( \{\lambda_i\}_{i=1}^n \) and/or the initial approximations of the solution \( \{x_i^{(0)}\}_{i=1}^n \), generating the values from a given interval.

3. We employ the iteration formula: \( \forall i = 1, \ldots, n, \)

\[
x_i^{(k)} = wx_i^{(k-1)} + (1 - w) \Phi_i(x_1^{(k-1)}, \ldots, x_i^{(k-1)}, \ldots, x_n^{(k-1)}; \lambda_1, \ldots, \lambda_n; \alpha, r)
\]

(6.28)

4. If

\[
\left( \sum_{i=1}^n \left[ x_i^{(k)} - \left( wx_i^{(k-1)} + (1 - w) \Phi_i(x_1^{(k-1)}, \ldots, x_i^{(k-1)}, \ldots, x_n^{(k-1)}; \lambda_i; \alpha, r) \right) \right]^2 \right)^{1/2} \geq \text{error}
\]

we go to the point 3; otherwise we stop the process and the vector \( x^{(k)} = (x_1^{(k)}, \ldots, x_n^{(k)}) \) is considered the approximation to the solution \( x^* \).

5. We repeat the process for the values of \( \alpha \in [1/2, 1] \) with the step of 0.01.
We generalize the iteration method by using a relaxation parameter $w$ in the equation (6.28) in order to prevent big jumps in the iteration process that could cause problems with the convergence. If we set $w = 0$, we get the direct iteration method. Furthermore, for each of the solution vector we also calculate the value of the first and the second derivative of the expected payoff of each of the contestants that is given by the equation (6.4) to ensure that we have found the maximum, indeed. For the contestants with interior solution $(x_i^{(k)} > 0)$, the first derivative should equal to zero and the second derivative should be negative, while for the contestants with corner solution $(x_i^{(k)} = 0)$, the first derivative should be negative and there are no restriction on the value of the second derivative. The value of the expected payoff itself is also computed. It should be nonnegative for all contestants, otherwise, the FOCs and SOCs do not characterize the equilibrium, since the obtained solution is dominated by zero effort exerted by the contestants whose expected payoff is negative.

The code of the programmes, sample input and sample output files are attached in the Appendix.

The direct iteration process is a method by which we are able to find the solutions to our nonlinear system. However, especially for higher degrees of the Tullock parameter $r$, one problem arises. By the iteration process we are able to find maxima, but these need not be the global maxima. When we find the solution vector $x^* = (x_1^*, \ldots, x_n^*)$, it might be such that for some of the contestant $i$, his effort $x_i^*$ is the effort that locally maximizes his expected payoff $E\pi_i(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*)$, but it need not necessarily maximizes it globally. In such a case, the obtained solution fails to meet the definition of the Nash equilibrium (see Def. 2.2, page 9).

In order to overcome this problem, at least partially, we further apply one modified approach. We do not work with the first-order conditions of the maximization of expected payoff function any more. Instead, we work with the expected payoff functions themselves. We consider the system of $n$ equations in $n$ unknowns given by the $n$ maximization problems (6.4) as a discrete nonlinear dynamical system and we study the evolution of the effort variables in it. The main difference is in the timing of moves by the players.
Programming the discrete nonlinear dynamical system

To programme the evolution of the system, we use Fortran, for which the code “programme multiprize for changing alpha with zigzag maximizations” can be found in Appendix. We begin by specifying the values of $\lambda_i$’s and the number of iterations $k > 0$. We continue by randomly drawing the initial values of the effort vector $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$, $x_i^{(0)} \in [0, 1]$. For each of the contestant $i = 1, \ldots, n$, we fix the values $\mathbf{\pi}^{(0)}_{-i}$ of all other contestants and we search for $x_i^{(1)} \in [0, 1]$ that maximizes his expected payoff function $E\pi_i(x_i, \mathbf{\pi}^{(0)}_{-i})$. For each of the contestant, we save the new values $x_i^{(1)}$ and we repeat the same procedure using the new values $\mathbf{x}^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)})$ instead of $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$. Hence, the iteration formula is, $\forall i = 1, \ldots, n$:

$$
\begin{align*}
    x_i^{(k+1)} &= \max_{x_i \in [0, 1]} E\pi_i(x_1^{(k)}, \ldots, x_{i-1}^{(k)}, x_i, x_{i+1}^{(k)}, \ldots, x_n^{(k)}) \\
    x_1^{(0)}, \ldots, x_n^{(0)} \text{ given} &= \text{randomly drawn from } [0, 1],
\end{align*}
$$

where $E\pi_i(\mathbf{x})$ is given by equation (6.4). We then stop the iterations at $k$ chosen at the very beginning and we repeat the whole process for the selected values of $\alpha \in [1/2, 1]$. We are thus able to see how $x_i$’s evolve for a particular value of $\alpha$ and how the players react to each other. The evolution from $x_i^{(1)}$ to $x_i^{(k)}$ shows the game as if the contestants were playing sequentially several rounds, each round simultaneously reacting to the preceding round strategies of other contestants.

In general, there are various options how the solution of the discrete dynamical nonlinear system may evolve. It may converge to a stable solution, it may periodically fluctuate between several states, or it may even engender chaotic behavior. In our setting, if we find that the solution converges to a certain stationary state, this state then represents the Nash equilibrium of a simultaneous contest. Such a solution should also be one of the solutions that

---

5Since we have normalized $V \equiv 1$, we do not search the solution through values of $x_i > 1$, because a contestant’s effort will not exceed $V \overline{x}$ in the equilibrium. If $x_i > V \overline{x}$, the contestant would have received negative expected payoff, and thus exerting zero effort would have always yielded a higher expected profit.
we find by the direct iteration method, which we check that it indeed is the case.

*Example of various evolution paths*

To illustrate the possible evolution paths that a dynamical nonlinear system may induce, we present a very famous example of a *logistic map*. It is a system used by biologists to model the evolution of certain single species and it has already been very thoroughly investigated. This system is simpler than the system that is of our interest and even though it already shows the various iteration paths. Consider the logistic map $f_\mu : [0, 1] \to [0, 1]$ given by

$$x_{n+1} = f_\mu(x_n) = \mu x_n (1 - x_n),$$

where the parameter $\mu$ lies in the interval $[0, 4]$. Then, depending on the value of $\mu$, there are several different iteration paths of how the variable $x$ evolves. Figure 6.1 shows the various iteration paths and time series data representing the variable $x$ at time $n$. The variable can be attracted to a stationary stable point, they can vary periodically or they can behave in an erratic, unpredictable manner.

(taken from Lynch (2004))

As the example shows, when using this method we are faced with another potential problem. If the system converges to a stationary point, it implies that this point is the Nash equilibrium. However, no convergence does not imply that there is no Nash equilibrium in the contest. It might be that the stationary point (i.e., the Nash equilibrium) is unstable, so that in case the iteration process is not being initiated with the exact values corresponding to that point, the process may diverge from that point or periodically fluctuate between several different values around that point.

Therefore, when the issue of existence arises (for higher degrees of $r$), we combine the two previously proposed methods together. First, we find a solution by the direct iteration method with randomly drawn initial values $x_i^{(0)}$'s and we save the values of the obtained solution. We repeat the direct iteration
The values of $\mu$: (i) $\mu = 2$, (ii) $\mu = 3.2$, (iii) $\mu = 3.5$, (iv) $\mu = 4$

process several times (10 times minimally) receiving potential Nash equilibrium point(s). Next, we use the saved values of each of the potential Nash equilibrium points as the initial inputs with which we begin the computation of the evolution process. If the solution found by the direct iteration method is the Nash equilibrium (and not solely a local maximum), the evolution process should be then stationary from the very beginning. In theory, since these are already the values of the equilibrium, we should be able to find even the unstable equilibria. If we do not find any potential Nash equilibrium point with which the system would be stationary, it is very likely that there is no Nash equilibrium in the contest\(^6\). For the code to the programme, see Appendix, “united programmes random”.

### 6.5 Introduction to results

We proceed to analyze the results that we have obtained from the numerical simulations. For low values of the Tullock parameter \( r \) and low number of the contestants, we are able to find a pure-strategy Nash equilibrium. However, as \( r \) increases, we have obtained multiple solutions by the direct iteration method and no convergence in the evolution of the efforts from the point of view of dynamical nonlinear system. This demonstrates that the solutions found by the direct iteration method are likely local maxima and that there might not be any pure strategy equilibrium in the contest. Therefore, we first summarize the results for a three-person and a four-person contest\(^7\) when \( r = 1 \) (since for these values we have found pure strategy Nash equilibria) and only after that we discuss the effect of the Tullock parameter \( r \) on the existence of an equilibrium and on the characteristics of a pure-strategy equilibrium if one exists.

\(^6\)Nevertheless, since the system is discrete, the values obtained by the direct iteration process might be the closest, but not exact values to the maximum point. In such case, no convergence may still appear even for a stationary point. Therefore, we still cannot be sure that there is no Nash equilibrium, even though it is very unlikely.

\(^7\)We are able to find results for any \( n \geq 2 \). However, these two cases are sufficient when we want to illustrate how asymmetries effect the equilibrium efforts and what happens as the number of contestants increases in the contest.
6.6 Results: Asymmetries in three- and four-person contests, \( r = 1.0 \)

In our setting, the question of optimal prize allocation is represented by searching for \( \alpha \) that maximizes the average effort in the contest. Our aim is to find how equilibrium efforts of the contestants respond to changes in \( \alpha \). In the case of symmetric contestants, the answer is straightforward—the equilibrium effort of all contestants is increasing in \( \alpha \) resulting in optimality of a single prize, as shown in section 6.2. However, when heterogeneity between the contestants is introduced, the equilibrium behavior of the contestants becomes more diverse. Depending on the relative gap of abilities among the players, increasing the fraction \( \alpha \) allocated to the first prize may induce increase as well as decrease in equilibrium efforts of some of the contestants, leading even to inactivity in the extreme case. These opposite forces then lead to an ambiguity when questioning the effects of \( \alpha \) on the average effort in the contest.

There are two parameters that determine how \( \alpha \) affects the equilibrium behavior of the contestants: range of the contest, and strengths of the contestants. While the range primarily concerns the behavior of the lowest ability contestant, the strengths determine which of the two prizes a particular contestant is competing for. The question we want to answer is how range and strengths of contestants influence the effect of the second prize it has on the equilibrium behavior of the contestants. Before answering this question, let us first define the terms range and strength of the contestant more rigorously.

**Definition 6.1 (Range of \( \lambda \)'s).** For a contest \( c \) we define the range of the contest as an interval

\[
R_c = \left[ \frac{\min_i \lambda_i}{\max_i \lambda_i}, 1 \right].
\]

**Definition 6.2 (Strength of the contestant).** We define the strength of the contestant \( j \), \( S^j \), as

\[
S^j = \frac{\lambda_j - \min_i \lambda_i}{\max_i \lambda_i - \min_i \lambda_i}.
\]

From the above definition it follows that the lowest ability contestant is of strength 0 while the highest ability contestant is of strength 1. The value of the strength of other contestants then captures their relative position with respect
to these limiting contestants\(^8\). Note that the feature that the probabilities and thus the maximization problems of the contestants are homogeneous of degree zero in the relative ability parameters, is also captured as a feature of the range and the strengths. If we multiply all \(\lambda_i\)'s by the same positive number, neither the range nor any of the strengths change.

6.6.1 Legend to figures

In the following pages, we present various figures capturing our results, that we have drawn using the programme Gnuplot. For three-person contests, we present several figures in which the average equilibrium effort is captured as a function of two variables, \(\alpha\) and \(\lambda_2\), all other parameters being fixed. Furthermore, we present figures capturing the equilibrium behavior of the contestants (equilibrium effort, equilibrium probability of winning the first prize, and equilibrium probability of winning the second prize) as a function of \(\alpha\), all other parameters being fixed, or as a function of \(\lambda_2\), all other parameters being fixed.

In these figures, a thick red line represents the average effort of all the contestants. The individual contestants equilibrium behavior is then represented by different symbols, such as pluses, crosses, squares, triangles, etc. Above each of the graph there is indicated what values of the parameters \(\lambda_i\) have been chosen (the first number corresponds to \(\lambda_1\), the second one to \(\lambda_2\),...), what value of \(\alpha\) has been chosen if fixed as well as what value of the Tullock parameter \(r\) has been chosen. Without loss of generality, the contestants are ordered such that \(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0\).

6.6.2 Three-person contest

Let us begin our analysis with the simplest case of three contestants. In the graphs, red pluses (+) represent the contestant 1, green crosses (×) represent the contestant 2, violet squares (□) represent the contestant 3 and a red thick line represents the average effort of the three contestants.

Figure 6.2 shows the case of symmetric contestants (\(\lambda_1 = \lambda_2 = \lambda_3 = 1\)) when there are constant returns to scale in effort (Tullock parameter \(r = 1\)). It serves as a benchmark case from which we will deviate in different ways and

\(^8\)When three-person contests are examined, \(S^1 = 1\) and \(S^3 = 0\), which leaves the space to investigate the effects of varying strength of the contestant 2. As the number of players in the contest increases, the number of contestants whose strengths can be varied increases as well leading to higher number of directions of comparative statics.

Figure 6.2: Three symmetric contestants, \( r = 1 \)

thus show how asymmetries affect the equilibrium efforts. As expected, we can see that when all the contestants are symmetric, the equilibrium efforts are also symmetric and increasing for all the values of \( \alpha \in [1/2, 1] \). Therefore, the average effort reaches its maximum at \( \alpha^* = 1 \) indicating that a single prize is optimal, as it is stated in section 6.2.

We have computed numerous simulations of which we present several representative cases to demonstrate how the mentioned parameters (range \( R \), strengths \( S \)) operate. Figures 6.3 and 6.4 capture the results for a contest of range \( R^1 = [\frac{1}{100}, 1] \), while Figures 6.5 and 6.6 capture the results for a contest of range \( R^2 = [\frac{18}{100}, 1] \).

The first finding that emerges when looking at the figures is that the slopes of equilibrium efforts are ordered in the same way as the relative abilities are, that is,

\[
\frac{\partial x_1^*}{\partial \alpha} \geq \frac{\partial x_2^*}{\partial \alpha} \geq \frac{\partial x_3^*}{\partial \alpha}.
\] (6.30)

Note that some (or all) of these partial derivatives may be negative indicating that the individual equilibrium effort becomes decreasing in \( \alpha \). This is especially likely to happen for the lowest ability contestant. Intuitively, allocating greater fraction to the first prize causes the highest ability contestant to response the most aggressively since it is the first prize he is concerned with while it need not be the case for weaker contestants. This is supported by looking at the equilibrium probabilities of winning the prizes captured in the middle and the last columns of Figures 6.3 and 6.5. The highest ability contestant is always the most likely to win the first prize while the same cannot be said about his probability of winning the second prize.

There are distributions of abilities for which the equilibrium efforts of all
Figure 6.3: Effect of reduction of the second prize when $R^1 = [\frac{1}{100}, 1]$

Average effort

$\lambda_1 = 100, \lambda_3 = 1; r = 1.0$

<table>
<thead>
<tr>
<th>efforts</th>
<th>1st prize probability</th>
<th>2nd prize probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Strength $S^2 = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b) Strength $S^2 = 1/2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c) Strength $S^2 = 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 6.4: Effect of strength when $R^1 = [\frac{1}{100}, 1]$

Average effort

$\lambda_1 = 100, \lambda_3 = 1; r = 1.0$

<table>
<thead>
<tr>
<th>efforts</th>
<th>1st prize probability</th>
<th>2nd prize probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $\alpha = 0.5$</td>
<td><img src="a.png" alt="effort" /></td>
<td><img src="a.png" alt="1st prize probability" /></td>
</tr>
<tr>
<td>(b) $\alpha = 0.75$</td>
<td><img src="b.png" alt="effort" /></td>
<td><img src="b.png" alt="1st prize probability" /></td>
</tr>
<tr>
<td>(c) $\alpha = 1.0$</td>
<td><img src="c.png" alt="effort" /></td>
<td><img src="c.png" alt="1st prize probability" /></td>
</tr>
</tbody>
</table>
Figure 6.5: Effect of reduction of the second prize when $R^2 = \left[ \frac{18}{100}, 1 \right]$

Average effort

$\lambda_1 = 100, \lambda_3 = 18; r = 1.0$

<table>
<thead>
<tr>
<th>efforts</th>
<th>1st prize probability</th>
<th>2nd prize probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Strength $S^2 = 0$</td>
<td><img src="image1.png" alt="Graph" /></td>
<td><img src="image2.png" alt="Graph" /></td>
</tr>
<tr>
<td>(b) Strength $S^2 = 1/2$</td>
<td><img src="image3.png" alt="Graph" /></td>
<td><img src="image4.png" alt="Graph" /></td>
</tr>
<tr>
<td>(c) Strength $S^2 = 1$</td>
<td><img src="image5.png" alt="Graph" /></td>
<td><img src="image6.png" alt="Graph" /></td>
</tr>
</tbody>
</table>
Figure 6.6: Effect of strength when $R^2 = \left[ \frac{18}{100}, 1 \right]$

Average effort

$\lambda_1 = 100, \lambda_3 = 18; r = 1.0$

<table>
<thead>
<tr>
<th>efforts</th>
<th>1st prize probability</th>
<th>2nd prize probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $\alpha = 0.5$</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
</tr>
<tr>
<td>(b) $\alpha = 0.75$</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
</tr>
<tr>
<td>(c) $\alpha = 1.0$</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
</tr>
</tbody>
</table>
of the contestants are increasing in $\alpha$ resulting in optimality of a single prize. However, if the equilibrium effort of the lowest ability contestant is decreasing in $\alpha$, the optimality of a single prize becomes uncertain as the remaining contestants react to the fact that the contestant 3 is less and less engaged in the contest. The behavior of the lowest ability contestant is the key to optimal allocation rule and this behavior is influenced by the range as well as by the strength of the contestant 2.

A common effect of the range and the strength $S^2$ on contestant 3 is that widening the range or increasing the strength of the contestant 2 leads to decrease in the equilibrium effort of the lowest ability contestant, keeping $\alpha$ fixed. This effect can be best noticed if we compare Figures 6.3 and 6.5. Formally,

$$\frac{\partial x^*_3}{\partial \left(\frac{\lambda_4}{\lambda_1}\right)} \geq 0, \quad \frac{\partial x^*_3}{\partial \left(\frac{\lambda_2}{\lambda_1-\lambda_3}\right)} \leq 0.$$  \hspace{1cm} (6.31)

Note that the first derivative is nonnegative since widening range actually means decreasing the parameter $\frac{\lambda_4}{\lambda_1}$. The intuition is that the range determines whether and how intensively the contestant 3 is competing for the first prize, while the strength of the contestant 2 determines whether and how intensively the contestant 3 is competing for the second prize. As he becomes more asymmetric with respect to the contestant 1, he becomes less interested in the first prize. Analogically, as he becomes more asymmetric with respect to the contestant 2, he becomes less interested in the second prize. Getting less interested in either of the prizes then leads to investing less effort, which, in extreme case, results in investing nothing. Widening the range or increasing $S^2$ thus causes decreasing the span of $\alpha$’s for which the contestant 3 is actively participating in the contest.

Let us continue by examining the effect of the range separately. Widen- ing the range causes an overall (nonlinear) decrease for all $\alpha$’s in the level of the average effort, compare Figures 6.3 and 6.5 or Figures 6.4 and 6.6. That is, the average effort is maximized when the contestants are identical. Introducing asymmetries then leads to decrease in the average effort for all values of $\alpha$. As already noted, widening the range results in smaller interest of the lowest ability contestant to engage in competing for the first prize and thus he invests less effort. Since the competition becomes less fierce, the other contestants response by lowering their efforts as well. Out of all the contestants, the lowest ability contestant lowers his effort the most, in absolute terms. How much the
other contestants lower their efforts in response to the behavior of the lowest ability contestant is then somehow dependent on the relative relationship of their abilities and on a particular value of $\alpha^9$. Since this relative relationship of abilities is different for contestant 1 and contestant 2 (except when $\lambda_1 = \lambda_2$), each responses differently for any value of $\alpha$, leading to the mentioned nonlinearity. Particular distributions of abilities thus may cause a nonlinearity that leads to optimality of the second prize with varying range.

The mere act of widening the range, however, does not necessarily leads to optimality of the second prize. It further depends on what particular distribution of abilities (strengths of the contestant) there is in the contest. Table 6.1 summarizes the optimal prize allocation with varying range for two different distributions of abilities: when $S^2 = 0$, i.e., there are two equally weak contestants, and when $S^2 = 1$, i.e., there are two equally strong contestants. As we can see, in the case of two equally strong contestants captured in panel ii), there is only a limited interval $I = \left[\frac{48}{100}, \frac{58}{100}\right]$ of the values of the lower support of the range, for which the second prize is found as being optimal. Decreasing the lower support of the range below the value $\frac{48}{100}$ restores the optimality of a single prize again.

Therefore, let us turn to examining the **effect of the strength** $S^2$ of the contestant 2 as a parameter that captures the relative relationship of abilities. As already noted, widening the range decreases the effort of each of the contestants with respect to the symmetric case. This enables the mere possibility of the existence of optimality of the second prize. Keeping $\alpha$ fixed, the amount by which a contestant lowers his effort may differ for different contestant, depending on which of the two prizes a particular contestant is competing for. This is the moment when the strength of the contestant 2 comes into play.

Intuitively, when the range is sufficiently small, the lowest ability contestant still competes for the first prize leading to optimality of a single prize. When the range is sufficiently large (so that the lowest ability contestant is concerned about the second prize only), whether the second prize becomes optimal or not further depends on the strength of the contestant 2 as it determines which of

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9For instance, note that a decrease in $x^*_3$ enters into the probabilities of others multiplied by $\lambda_3$, so that there is some effect determined by $\lambda_j$ vs $\lambda_3$ relationship. Furthermore, behavior of the contestant 3 affects the probabilities differently, depending on whether it is probability to win the first or the second prize. In maximization problem, these probabilities are further multiplied by $\alpha$ and $(1 - \alpha)$, respectively. Therefore, the response of other contestants also depends on a particular value of $\alpha$ as “the effect of $\Delta x^*_3$ on the first prize probability times the effect of $\alpha$ plus the effect of $\Delta x^*_3$ on the second prize probability times the effect of $\alpha$” may differ for different $\alpha$’s. Not to mention that the change $\Delta x^*_3$ may differ for different $\alpha$’s.

the two prizes he is primarily competing for. If he is strong enough, he competes with the contestant 1 for the first prize, and, implicitly, for the second prize as well. This discourages the contestant 3 from any serious engagement in the contest. Such a situation resembles a two-player contest resulting in optimality of a single prize despite the decreasing force of the contestant 3. If the contestant is weaker, however, he competes with the lowest ability contestant for the second prize while “leaving” the first prize for the strong contestant. Such a situation then may lead to optimality of the second prize as it is shown in panel (a) of Figures 6.3 and 6.5.

Looking more closely at panel i) in Table 6.1, however, reveals that there are ranges for which the second prize is optimal even for case of two strong contestants (both thus competing for the first prize). It might seem that this does not really support what we have just said. Nevertheless, it just shows that there is one more possible scenario (to be add to scenario “contestant 2 and 3 competing for the second prize”) when optimality of the second prize may emerge. The optimality of the second prize for strength $S^2 = 1$ emerges with ranges for which the threshold value of $\alpha$, a value of $\alpha$ at which the contestant 3 stops playing at all, is close to unity. Therefore, the second prize may also become optimal when preventing the contestant 3 from nonactivity or getting him being more seriously engaged in the contest conditional on that a very small fraction allocated to the second prize suffices to do so. Figure 6.7 captures such situation.

The effect of the strength of the contestant 2 can be best detected by looking at Figures 6.4 and 6.6, which show the equilibrium behavior of the contestants as a function of $\lambda_2$. To see which of the two prizes the contestant 2 is competing for we can examine the probabilities of winning the first and the second prize that are presented in the middle and the last columns. The probabilities to win either of the prize of the contestant 3 are always decreasing which supports the earlier mentioned statement that the equilibrium effort of the contestant 3 is decreasing in $S^2$ (i.e, as the contestant 2 becomes stronger). The first prize probability of the contestant 2 is always increasing suggesting that as he is becoming stronger, he is getting concerned more and more with the first prize. What is interesting is then examining his second prize probability since (except for case $\alpha = 1.0$ when there is actually no second prize) it is nonmonotone in $\lambda_2$—increasing when he is of low strength and decreasing as his strength gets higher, reaching its peak for the same value of $\lambda_2$ no matter the value of $\alpha$. It suggests that the peak corresponds to the threshold value $\bar{S}^2$, at which the

Table 6.1: Varying range when $S^2 = 0$ or $S^2 = 1$, i.e., structure “two contestants of equal ability”

i) Two weak, $S^2 = 0$: $\lambda_1 = 100$  
ii) Two strong, $S^2 = 1$: $\lambda_1 = \lambda_2 = 100$

<table>
<thead>
<tr>
<th>$\lambda_2 = \lambda_3$</th>
<th>$\alpha^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.50</td>
</tr>
<tr>
<td>5</td>
<td>.50, .51</td>
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<tr>
<td>10</td>
<td>.58</td>
</tr>
<tr>
<td>15</td>
<td>.65</td>
</tr>
<tr>
<td>20</td>
<td>.70 - .73</td>
</tr>
<tr>
<td>25</td>
<td>.77 - .79</td>
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<tr>
<td>30</td>
<td>.83 - .85</td>
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<tr>
<td>35</td>
<td>.89 - .92</td>
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<tr>
<td>40</td>
<td>.95 - .98</td>
</tr>
<tr>
<td>41</td>
<td>.97 - .99</td>
</tr>
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<td>.98 - 1.0</td>
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<td>43</td>
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<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>1.0</td>
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</table>

<table>
<thead>
<tr>
<th>$\lambda_3$</th>
<th>$\alpha^*$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1.0</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>1.0</td>
</tr>
</tbody>
</table>

contestant 2 begins competing primarily for the first prize, given the range.

We conclude this section by comparing our results with the results for three-person contest with two identical contestants obtained by Szymanski and Valletti (2005). They state that in the case of two strong and one weak contestants, the contest reduces to a two-person asymmetric contest, for which a single-prize is optimal. In the case of two weak and one strong contestants, the optimality of the second prize emerges. However, since they were not able to solve the problem analytically, these conclusions were stated for a limiting case only. This would correspond in our setting to a contest whose range approaches $R_{lim} = [0, 1]$. As we diverge from the extreme cases, we are able to find optimality of multiple prizes for two strong and one weak contestants as well as optimality of a single prize for two weak and one strong contestants, see

---

10 In Proposition 3 (Szymanski and Valletti (2005), p. 476) they conclude that the optimal rule to adopt for allocating the prize fund is a 3:1 rule when $r = 1$. 

Figure 6.7: Preventing contestant 3 from leaving / More seriously engaging contestant 3 into the contest when $S^2 = 1$: Equilibrium efforts
Table 6.1. The optimality of the second prize for the case two strong and one weak contestant appears as a result of preventing the contestant 3 from leaving the contest. On the other hand, optimality of a single prize for the case of the weak and one strong contestant emerges when the range becomes sufficiently small so that the lowest ability contestant is also competing for the first prize (even when the second prize is made available).

### 6.6.3 Four-person contest

Let us proceed to a four-person contest, in which we again set $r = 1$. In the graphs, the contestants are represented (from the contestant 1 to the contestant 4, respectively) by red pluses (+), green crosses ($\times$), violet empty squares (□), and light blue filled squares (■). A red thick line represents the average effort of the four contestants.

Figure 6.8 shows the case of symmetric contestants ($\lambda_i = 1$ for all $i$). As in the three-person contest, the equilibrium is symmetric and increasing for all the selected values of $\alpha \in [1/2, 1]$. Therefore, the average effort reaches its maximum at $\alpha^* = 1$ indicating that a single prize is optimal. It confirms the result for a symmetric contest that is analyzed in section 6.2 showing that the optimality of a single prize in a symmetric contest is independent of the number of contestants.

As in the three-person contest, there are distributions of abilities for which an equal division ($\alpha^* = 1/2$), an unequal division ($\alpha^* \in (1/2, 1)$), as well as a single prize ($\alpha^* = 1$) is the optimal allocation of the prize fund. Examples are provided in Figure 6.9. From the graphs it becomes clear that the ordering of the slopes of equilibrium efforts from expression (6.30) does not necessarily
Figure 6.9: Optimal prize allocations: Four asymmetric contestants, \( r = 1 \)

<table>
<thead>
<tr>
<th>Efforts</th>
<th>1st Prize Probability</th>
<th>2nd Prize Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>i) Equal division</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ii) Unequal division</td>
<td></td>
<td></td>
</tr>
<tr>
<td>iii) Single prize</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
hold any more (see panel ii) in Figure 6.9). The difference is driven by the behavior of the contestant with the second highest ability. In three-person contests, his behavior resembles either the behavior of the strongest contestant or the behavior of the weakest contestant for all values of \( \alpha \). This suggests that he takes the same “strong/weak player status” independently of whether the contest is effectively a three- or a two-person contest. In four-person contests, on the other hand, his behavior might first resemble the behavior of the contestants of lower abilities. As soon as the lowest ability contestant becomes inactive (and the contest thus becomes effectively a three-person contest), his behavior might change. Hence, this suggests that there is operating something what can be called an “effective strength of an active contestant”. We define it analogically as the strength of a contestant except for \( \min_i \lambda_i \) being replaced by a minimal value of \( \lambda_i \) out of all actively participating contestants. Therefore, a contestant’s effective strength might suddenly changes as some contestant of lower ability drops out of the contest. If his ability is second to last ability out of all active players, with the effectively weakest contestant dropping out of the contest, his effective strength suddenly drops to 0. Otherwise, his effective strength suddenly jumps up. This jump is what causes that the ordering of the slopes need not hold any more\(^\text{11}\).

The number of parameters, along which the problem can be analyzed, increases with the number of contestants. The range of the contest and the strength of the contestant 2 remains as in the three-person contest. However, the strength of the contestant 3 must be considered now as well, since he is not the limiting contestant anymore. These parameters interact together determining how equilibrium behavior responses to changes in \( \alpha \). One of the effects that they determine is when the contestants of lower abilities stop playing, which causes noncontinuous changes in effective strengths of remaining players signifying a structural break in their equilibrium behavior. Since investigating how these parameters interact together would lead to great complexity, we do not provide a similar analysis to the three-person contest here. Instead, we decide to focus on examining contests of a special limiting structure “two different groups”.

Table 6.2 captures our results for the special case of two homogenous groups (of possibly different size) with varying range of the contest. When the group

---

\(^{11}\)In three-person contest, we do not encounter such a jump. If the lowest ability contestant becomes inactive, the effective strength of contestant drops to 0 since his ability is second to last.

Table 6.2: Varying range for cases “two homogeneous groups” in four-person contest

i) Group size 1:3: $\lambda_1 = 100$

$\Rightarrow I_1 \equiv \left[ \frac{1}{100}, \frac{58}{100} \right]$

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\alpha^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.50</td>
</tr>
<tr>
<td>10</td>
<td>.50</td>
</tr>
<tr>
<td>15</td>
<td>.50</td>
</tr>
<tr>
<td>20</td>
<td>.54 - .58</td>
</tr>
<tr>
<td>25</td>
<td>.60 - .64</td>
</tr>
<tr>
<td>30</td>
<td>.66 - .70</td>
</tr>
<tr>
<td>35</td>
<td>.73 - .75</td>
</tr>
<tr>
<td>40</td>
<td>.78 - .81</td>
</tr>
<tr>
<td>45</td>
<td>.83 - .87</td>
</tr>
<tr>
<td>50</td>
<td>.90 - .92</td>
</tr>
<tr>
<td>55</td>
<td>.96 - .98</td>
</tr>
<tr>
<td>56</td>
<td>.97 - .99</td>
</tr>
<tr>
<td>57</td>
<td>.97 - 1.0</td>
</tr>
<tr>
<td>58</td>
<td>.99, 1.0</td>
</tr>
<tr>
<td>59</td>
<td>1.0</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>1.0</td>
</tr>
</tbody>
</table>

ii) Group size 2:2: $\lambda_1 = \lambda_2 = 100$

$\Rightarrow I_2 \equiv \left[ \frac{41}{100}, \frac{68}{100} \right]$

<table>
<thead>
<tr>
<th>$\lambda_3 = \lambda_4$</th>
<th>$\alpha^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>40</td>
<td>1.0</td>
</tr>
<tr>
<td>41</td>
<td>.66</td>
</tr>
<tr>
<td>50</td>
<td>.78, .79</td>
</tr>
<tr>
<td>55</td>
<td>.84, .85</td>
</tr>
<tr>
<td>60</td>
<td>.89 - .92</td>
</tr>
<tr>
<td>65</td>
<td>.96</td>
</tr>
<tr>
<td>66</td>
<td>.96 - .98</td>
</tr>
<tr>
<td>67</td>
<td>.98</td>
</tr>
<tr>
<td>68</td>
<td>.98 - 1.0</td>
</tr>
<tr>
<td>69</td>
<td>1.0</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>1.0</td>
</tr>
</tbody>
</table>

iii) Group size 3:1: $\lambda_1 = \lambda_2 = \lambda_3 = 100$

$\Rightarrow I_3 \equiv \varnothing$

<table>
<thead>
<tr>
<th>$\lambda_4$</th>
<th>$\alpha^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>100</td>
<td>1.0</td>
</tr>
</tbody>
</table>
size (strong : weak) is 1:3 or 2:2, there are ranges for which the second prize is optimal, while when the group size is 3:1, no such range exists. Intuitively, increasing the size of the group intensifies its force it has on the average effort. When a member is added to a strong group, it intensifies the increasing effect of $\alpha$ on the average effort resulting in higher number of ranges in which a single prize is optimal, while when a member is added to a weak group, it intensifies the decreasing effect of $\alpha$ on the average effort. Comparing the three-person contests with the four-person contests indicates that it indeed is the case. For a case of two weak and one strong contestants—group size 1:2—deviating to group size 1:3 leads to increasing the number of the ranges for which the second prize is optimal. If we deviate to group size 2:2, the number of such ranges diminishes. On the other hand, deviating from the case of two strong and one weak contestants—group size 2:1—to the group size 3:1 leads to nonoptimality of the second prize for either of the range while deviating to the group size 2:2 again leads to widening increasing the number of the ranges for which the second prize is optimal.

6.7 Results for effects of Tullock $r$

6.7.1 Tullock $r$ and existence of an equilibrium

It is a well-known result from a contest literature on a single-prize, that as the Tullock parameter $r$ increases, a pure strategy Nash equilibrium may not exist. The only equilibrium is then in mixed strategies. Symmetric contests with two risk-neutral players and linear cost function find $r = 2.0$ being the threshold value. That is, in a single-prize Tullock contest where $r > 2.0$ a pure strategy Nash equilibrium does not exist. Increasing the number of contestants or introducing asymmetries (in valuations of the prize or in costs) in the contest further decreases this threshold value below 2.0 (Alcalde and Dahm (2008), Baye et al. (1994), Nti (1999, 2004), Wang (2010)).

Multi-prize contests are no exception—with higher $r$, it is very unlikely that the contest have a pure strategy Nash equilibrium. Nevertheless, when two prizes are available (i.e., $\alpha \neq 1$), we find situations, in which a pure strategy equilibrium exists even when $r > 2.0$.

In three-person contests that we examine in this section, two types of equilibria emerge as $r$ increases—when all three contestants are actively participating in the contest, and when one of the contestants stops playing. Keeping $r$ at
a particular value, as $\alpha$ gradually changes from one half to an unity, the type of equilibrium changes from active participation of all contestants to active participation of two contestants, with a possible nonexistence of a pure strategy equilibria for some of the middle values of $\alpha$. When $r$ exceeds a certain value, the two active contestants equilibrium disappears and the only pure strategy equilibrium is for low $\alpha$ with all three contestants active.

Figure 6.10 captures pure strategy equilibria for a three-person symmetric contest for combinations of $r \in [1.45, 2.45]$ and $\alpha \in [0.5, 1.0]$. As we can see, for $r \geq 2.0$, we have not found any pure strategy equilibria for $\alpha = 1$, which is consistent with literature on single-prize contests. Nevertheless, when two prizes are made available in the contest, pure strategy equilibria for values of the Tullock parameter even up to $r = 2.40$ exist.

Looking more closely at what effects the range and the strength of the contestant 2 have, let us examine Figures 6.11 and 6.12. These figures capture pure strategy equilibria for three-person asymmetric contests that are of different ranges and of different strengths of the contestant 2. The contest presented in Figure 6.11 is of range $R = \left[\frac{3}{4}, 1\right]$ with contestant 2 of strength $S_2 = 0.50$, while the contest presented in Figure 6.12 is of range $R = \left[\frac{9}{10}, 1\right]$ with contestant 2 of strength $S_2 = 0.15$.

Comparing these figures with the symmetric case, we can see that asymmetries captured by range and strength lead to changes in the value of $r$ for which at least one pure strategy equilibrium exists. In particular, for contest in Figure 6.11, the highest value of $r$, for which we are able to find any pure strategy equilibrium, is when $r = 2.20$. On the other hand, for contest in Figure 6.12, such value of $r$ is 1.70. Therefore, we can see, that range and strength of contestants affect the existence of a pure-strategy equilibrium in the contest. We do not go into more details about investigating how the quartet “$\alpha$—$r$—range—strengths” is related to the existence of a pure strategy equilibrium in this thesis as it is a very broad topic which a whole separate study may be devoted to.

Another question that emerges is when a pure strategy equilibrium exists whether it is unique pure strategy equilibrium. As we show, unique as well as multiple pure strategy equilibria can be found in these contests\textsuperscript{12}. In Figure 6.11, for instance, we have found several combinations of $r$—$\alpha$ for which multiple pure strategy equilibria exist. First, when $r = 1.20$ and $\alpha = 1.0$ or

\textsuperscript{12}By the question of uniqueness, we do not consider a possible existence of mixed strategies. We only ask whether there is only one or multiple pure strategy equilibria in the contest.
Figure 6.10: Existence of an equilibrium: Three symmetric contestants

$r = 1.25$ and $\alpha = 0.98$, equilibrium with all three active participants as well as equilibrium with the lowest ability contestant being inactive exist. More surprisingly, however, we have also found combinations of $r-\alpha$, for which multiple pure strategy equilibria exist, but in which the contestant with either the lowest or with the highest ability stays inactive. This happens, for instance, when $r = 1.90$ and $\alpha \in [0.92, 1.0]$. As the contest becomes relatively more asymmetric (strength $S^2$ and range $R$ decrease), an equilibrium with the highest ability contestant investing zero effort does not exist, though (see Figure 6.12).

The issue of existence of a pure strategy equilibrium in symmetric imperfectly discriminating multi-prize contests is extensively addressed in Clark and
Figure 6.11: Existence of an equilibrium: Three asymmetric contestants $\lambda_i = 8, 7, 6$
Figure 6.12: Existence of an equilibrium: Three asymmetric contestants $\lambda_i = 100; 30; 18$
Riis (1998c). They say that for high values of $r$, they cannot ensure the existence of pure strategy equilibria (the possible equilibrium is then in mixed strategies). Therefore, if the Tullock parameter $r$ is too high, the contest actually might not have any pure strategy equilibrium. This is in accordance with our findings for asymmetric contests provided in this section.

6.7.2 Tullock $r$ and changes of equilibrium efforts

In this section, we briefly discuss the effects of the Tullock parameter $r$ on the contest equilibrium efforts. As already noted, increasing the Tullock parameter $r$ may lead to nonexistence of any pure strategy equilibrium. Therefore, we work with examples for which we were able to find the Nash equilibria for all values of $\alpha$ while varying $r$.

The equilibrium efforts for a three-person contest with $r = 0.5$, $r = 1$, and $r = 1.30$ are captured in Figure 6.13. The first three panels present a symmetric contest while the next three panels present an asymmetric contest. In all of the presented cases, no matter the value of the Tullock parameter, asymmetry leads to a lower average effort in comparison with the symmetric case. Furthermore, looking at the panels it follows that increasing the Tullock parameter causes an increase in the efforts of all contestants (so far as the contestant is active) leading to a higher average effort. However, in asymmetric contest, an increase in $r$ may also result in inactivity of one of the contestants when the fraction of the prize fund allocated to the first prize exceeds a certain point, as it happens in panel b), case iii). Comparing the cases ii) and iii) in panel b), an increase in $r$ (causing inactivity of one of the contestants) may lead to different optimal allocation of the prize fund; $\alpha^* = 1.0$ in the case ii) while $\alpha^* = 0.94$ in the case iii).
Figure 6.13: Effect of increasing $r$ on equilibrium efforts: Three-person contest

a) Symmetric contestants: $\lambda_i = 1 \ \forall i$

i) $r = 0.50$

ii) $r = 1.00$

iii) $r = 1.30$

b) Asymmetric contestants: $\lambda_1 = 8, \lambda_2 = 7, \lambda_3 = 6$

i) $r = 0.50$

ii) $r = 1.00$

iii) $r = 1.30$
Chapter 7

Conclusion

In this thesis, we offer an analysis on a class of asymmetric imperfectly discriminating multi-prize contests that are an extension of the Tullock (1980) single-prize contest. We look at the optimality of the second prize in a contest offering two prizes, and we examine the issue of existence of an equilibrium with varying Tullock returns to scale in effort parameter \( r \). Since introducing asymmetries as well as multiple prizes lead to great complexity in analytical expressions, we employ a different approach to solve the problem—numerical methods for solving a system of \( n \) nonlinear equations in \( n \) unknowns.

To our knowledge, the Tullock multi-prize contest with heterogeneous contestants has been examined by Szymanski and Valletti (2005) only. They present a three-person contest with two prizes, in which two contestants are homogeneous while one contestant is different—the difference is captured by different marginal costs of effort. The complexity of the calculations, however, prevent them from providing an analysis for arbitrary chosen values of the parameters, so that they examine limiting cases only.

In this thesis, we choose a similar setting as one of Szymanski and Valletti (2005) with a slight change of introducing asymmetries into the contest. Following the methodology of Stein (2002), the contestants have the same marginal cost of effort, but they differ in the relative ability parameters that directly enter into the probability of winning one of the prizes. This modified concept should not cause very significant differences in the findings, however.

What makes our work original is the method by which we solve the system of nonlinear equations that are given by the maximization problems of the contestants. The method does not suffer from the problems with the complexity of the analytical analysis. Hence, it allows us to examine the contests not only
for some limiting cases, but also for milder values of the parameters, and, in principle, for a greater number of contestants.

We use two numerical methods. Since each of them has its own limitations, we complement each other, which allows us to partially overcome some of them. The first numerical method designed to solve for the equilibrium efforts is based on the direct iteration method that is generalized with a relaxation parameter. This method is used to solve the nonlinear system represented by the first-order conditions received from the expected payoff maximization of every contestant. The second numerical method, on the other hand, works with the expected payoff functions as they are. We consider the system of $n$ equations in $n$ unknowns given by the $n$ maximization problems as a discrete nonlinear dynamical system and we study the evolution of the effort variables.

We provide detailed discussion on three- and four-person contests, in which contestants compete for a prize fund $V$ that is allocated between two prizes; a fraction $\alpha \in [1/2, 1]$ is allocated to the first prize and a fraction $(1 - \alpha)$ is allocated to the second prize. Our aim is to investigate how equilibrium efforts of the contestants respond to changing $\alpha$. In the case of symmetric contestants, the answer is straightforward—the equilibrium efforts of all contestants are increasing in $\alpha$ resulting in optimality of a single prize. With heterogeneity being introduced, the responses of equilibrium efforts to changes of $\alpha$ become more diverse leading to an ambiguity when questioning the effects of $\alpha$.

We find that the behavior of the lowest ability contestant is the key to optimal allocation rule. This behavior is influenced by parameters that we call the range of the contest and the strengths of nonlimiting contestants, i.e., the strength of contestant 2 in the three-person case. The range captures the relative difference in abilities between the lowest and the highest ability contestants, while the strength of a contestant captures a relative position of his ability to these limiting contestants.

In a three-person contest, the range determines whether, and how intensively, the weakest contestant (contestant 3) is competing for the first prize, while the strength of the second weakest contestant (contestant 2) determines whether, and how intensively, the weakest contestant is competing for the second prize. Two possible scenarios leading to optimality of the second prize were found. In both scenarios, the range of the contest must be sufficiently large so that the contestant 3 is interested in competing for the second prize only. When the strength of the contestant 2 is small, he becomes competing with the contestant 3 for the second prize, which may then result in optimality
of the second prize. When the strength of the contestant 2 is high (thus the contestant 2 is primarily competing for the first prize), the second prize may become optimal when preventing the contestant 3 from nonactivity or getting him being more seriously engaged in the contest conditional on that a very small fraction allocated to the second prize suffices to do so. Comparing our results those obtained by Szymanski and Valleti (2005), we have thus found another possible scenario when the second prize emerges as optimal, which is not provided in the work of Szymanski and Valletti (2005).

The second part of our research addresses the issue of existence of a pure strategy Nash equilibrium as the Tullock parameter $r$ increases. From the literature on single-prize Tullock contests it is known that as $r$ increases, only a mixed-strategy equilibrium exists in the contest. We confirm such findings for multi-prize contests as well. However, the interval of the values of $r$ for which at least one pure strategy Nash equilibrium exists in the contest (i.e., there is $\alpha \in [1/2, 1]$ for which a pure strategy equilibrium exists) becomes broader than for a single-prize contests. Introducing asymmetries into the multi-prize contests then influences how large this interval is.

In the context of the issue of existence of a pure strategy equilibrium, we should also consider the closely related question of uniqueness. In the three-person contests that we examine, we find combinations of $r—\alpha$, for which a unique pure strategy Nash equilibrium exists, as well as combinations of $r—\alpha$, for which multiple pure strategy Nash equilibria exist. Surprisingly, one type of a pure strategy equilibrium is such that the contestant with the highest ability stays inactive in the contest. Such pure strategy equilibrium were found in a contest in which the contestants are still fairly homogeneous; as the contest becomes more asymmetric, such type of a pure strategy equilibrium does not exist anymore. Since investigating how asymmetries are related to the existence and uniqueness of a pure strategy equilibrium is a very broad topic, more comprehensive studies are postponed for further research.

The approach employing numerical methods presented by this thesis offers possible future research on asymmetric multi-prize contests. We model the problem when there are two prizes in the contest. Increasing the number of the prizes with number of the contestants is one possible extension to our work.
**Bibliography**


Appendix A

Programme codes

A.1 Fortran

c***************************************************************
programme multiprize_for_changing_alpha
c***************************************************************
c
version 2.0: April 18, 2013; 15:30 (@:CM&LM)
c

parameter(maxn=1000)
implicit real*8 (a-h,o-z)
dimension xlamb(1:maxn),x(1:maxn),f(1:maxn),p(1:maxn),
der(1:maxn),derpom(1:maxn)
open(11,file='output.txt')
open(12,file='fig.txt')
open(13,file='parameters.txt')
open(14,file='inputs.txt')
read(13,*) n,alph,r,w,eps,error
close(13)
write(11,'(i4,3f6.3,2e9.3)') n,alph,r,w,eps,error
do i=1,n
   read(14,*) xlamb(i),x(i)
   write(11,'(2f8.4)') xlamb(i),x(i)
   enddo
close(14)
dalph=0.01
alph=0.5-dalph
11 alph=alph+dalph
it=0
1 it=it+1
xx=0.0
do j=1,n
   xx=xx+xlamb(j)*x(j)**r
enddo

do j=1,n
  p(j)=x*lambda(j) * x(j)**r / xx
enddo

do j=1,n
  f(j)=alpha * p(j) *(1.0-p(j))*r
  do ip=1,n
    if(ip.eq.j) goto 2
    pjip=x*lambda(j) * x(j)**r / (xx-x*lambda(ip)*x(ip)**r)
    f(j)=f(j) + p(ip) * pjip * (1.0-p(j)-pjip) *(1.0-alpha)*r
  continue
 2 enddo
enddo

test=0.0
sum=0.0

do i=1,n
  rhs=w * x(i) + (1.0-w) * f(i)
  if(rhs.lt.eps) rhs=eps
  test=test+(x(i)-rhs)*(x(i)-rhs)
  sum=sum+rhs
  x(i)=rhs
enddo

if(sqrt(test).gt.error) goto 1

First and second derivative calculations and output

dx=0.001

do i=1,n
  der(i)=f(i)/x(i)-1.0
enddo

xx=0.0

do j=1,n
  xx=xx+x*lambda(j) * (x(j)+dx)**r
enddo

do j=1,n
  p(j)=x*lambda(j) * (x(j)+dx)**r / xx
enddo

do j=1,n
  f(j)=alpha * p(j) *(1.0-p(j))*r
  do ip=1,n
    if(ip.eq.j) goto 22
    pjip=x*lambda(j) * (x(j)+dx)**r / (xx-x*lambda(ip)*(x(ip)+dx)**r)
    f(j)=f(j) + p(ip) * pjip * (1.0-p(j)-pjip) *(1.0-alpha)*r
  continue
 22 enddo
enddo
do i=1,n
  derpom(i)=f(i)/(x(i)+dx)-1.0
enddo

do i=1,n
  dder=(derpom(i)-der(i))/dx
  write(11,'(i7,i4,f8.4,2e14.4)') it,i,x(i),der(i),dder
enddo

write(11,'(3f8.4)') alph,sum,sum/n
write(12,'(13f7.4)') alph,sum,sum/n,(x(i),i=1,n)
if(alph.lt.(0.999)) goto 11
close(11)
close(12)
stop
end

***************************************************************************
c***************************************************************************
programme multiprize_for_changing_alpha_with_random_initial_values_and_random_lambdas

version 3.0: May 2, 2013; 9:00 (©CM&LM)

parameter(maxn=1000)
implicit real*8 (a-h,o-z)
dimension xlamb(1:maxn),x(1:maxn),f(1:maxn),p(1:maxn),
der(1:maxn),derpom(1:maxn),e(1:maxn)
open(11,file='output.txt')
open(12,file='fig.txt')
open(13,file='parameters.txt')
open(14,file='inputs.txt')
read(13,*) n,alph,r,w,eps,error,xlamb1,xlamb2,x1,x2
close(13)
write(11,'(i4,3f6.3,2e9.3)') n,alph,r,w,eps,error
write(11,'(4f8.3)') xlamb1,xlamb2,x1,x2

call random_seed
do i=1,n
call random_number(xlamb(i))
xlamb(i)=xlamb(i)*abs(xlamb2-xlamb1)+xlamb1

call random_number(x(i))
x(i)=x(i)*abs(x2-x1)+x1
read(14,*) xlamb(i)
if(x(i).lt.eps) x(i)=eps
write(11,'(2f8.4)') xlamb(i),x(i)
enddo

close(14)
dalph=0.01
alph=0.5-dalph
11 alph=alph+dalph
it=0
1 it=it+1
xx=0.0
do j=1,n
xx=xx+xlamb(j)*x(j)**r
endo
do j=1,n
p(j)=xlamb(j)*x(j)**r/xx
endo
do j=1,n
f(j)=alph*p(j)*(1.0-p(j))*r
ende

Appendix A

\begin{verbatim}
  do ip=1,n
    if(ip.eq.j) goto 2
    pjip=xlamb(j)*x(j)**r/(xx-xlamb(ip)*x(ip)**r)
    f(j)=f(j)+p(ip)*pjip*(1.0-p(j)-pjip)*(1.0-alph)*r
    e(j)=e(j)+p(ip)*pjip*(1.0-alph)
  2 continue
enddo
enddo

test=0.0
sum=0.0
  do i=1,n
    rhs=w*x(i)+(1.0-w)*f(i)
    if(rhs.lt.eps) rhs=eps
    test=test+(x(i)-rhs)*(x(i)-rhs)
    sum=sum+rhs
    x(i)=rhs
  enddo
if(sqrt(test).gt.error) goto 1

  c
  c first and second derivative calculations and output
  c
  dx=0.001
  do i=1,n
    der(i)=f(i)/x(i)-1.0
  enddo
  xx=0.0
  do j=1,n
    xx=xx+xlamb(j)*(x(j)+dx)**r
  enddo
  do j=1,n
    p(j)=xlamb(j)*(x(j)+dx)**r/xx
  enddo
  do j=1,n
    f(j)=alph*p(j)*(1.0-p(j))**r
    do ip=1,n
      if(ip.eq.j) goto 22
      pjip=xlamb(j)*x(j+dx)**r/(xx-xlamb(ip)*x(ip+dx)**r)
      f(j)=f(j)+p(ip)*pjip*(1.0-p(j)-pjip)**(1.0-alph)**r
    22 continue
  enddo
enddo
enddo
  do i=1,n
    derpom(i)=f(i)/(x(i)+dx)-1.0
  enddo
  do i=1,n
\end{verbatim}

dder=(derpom(i)-der(i))/dx
write(11,'(i7,i4,f8.4,3e14.4)') it,i,x(i),der(i),dder,e(i)
enddo
write(11,'(3f8.4)') alph,sum,sum/n
write(12,'(13f7.4)') alph,sum,sum/n,(x(i),i=1,n)
if(alph.lt.(0.999)) goto 11
close(11)
close(12)
stop
end
programme multiprise_for_changing_alpha_with_zigzag_maximizations

version 2.0: April 27, 2013; 11:00 (© CM&LM)

parameter(maxn=1000)
implicit real*8 (a-h,o-z)
dimension xlamb(1:maxn),x(1:maxn),e(1:maxn),p(1:maxn),xpom(1:maxn)

open(11,file='output.txt')
open(12,file='fig.txt')
open(13,file='parameters.txt')
open(14,file='inputs.txt')
read(13,*) n,alph,r,w,eps,error,xlamb1,xlamb2,x1,x2,nit
close(13)
write(11,'(i4,3f6.3,2e9.3,i5)') n,alph,r,w,eps,error,nit
write(11,'(4f8.3)') xlamb1,xlamb2,x1,x2

call random_seed

do i=1,n
   read(14,*) xlamb(i),xpom(i)
call random_number(xlamb(i))
xlamb(i)=xlamb(i)*abs(xlamb2-xlamb1)+xlamb1
call random_number(xpom(i))
xpom(i)=xpom(i)*abs(x2-x1)+x1
   if(xpom(i).lt.eps) xpom(i)=eps
   write(11,'(2f8.4)') xlamb(i),xpom(i)
enddo
close(14)
dalph=0.01
alph=0.5-dalph
1   alph=alph+dalph
it=0
11  it=it+1
   do i=1,n
      x(i)=xpom(i)
   enddo
   do i=1,n
      x(i)=0.0
      xmax=0.0
      emax=0.0
   111 x(i)=x(i)+error
      if(x(i).lt.eps) goto 111
      xx=0.0
      do j=1,n
         xx=xx+xlamb(j)*x(j)**r
      enddo
   enddo
   write(11,'(2f8.4)') x1,x2
enddo
enddo
do j=1,n
  p(j)=xlambs(j)*x(j)**r/xx
enddo
ei=alph*p(i)-x(i)
do j=1,n
  if(j.eq.i) goto 2
  pji=xlambs(i)*x(i)**r/(xx-xlambs(j)*x(j)**r)
ei=ei+p(j)*pji*(1.0-alph)
  continue
enddo
if(ei.gt.emax) xmax=x(i)
if(ei.gt.emax) emax=ei
if(x(i).lt.1.0) goto 111
x(i)=xpom(i)
xpom(i)=xmax
e(i)=emax
enddo
do i=1,n
  write(11,'(i7,i4,f8.4,2e14.4)') it,i,x(i),e(i)
enddo
sum=0.0
do i=1,n
  sum=sum+x(i)
enddo
write(11,'(i7,3f8.4)') it,alph,sum,sum/n
if(it.lt.nit) goto 1
write(12,'(13f7.4)') alph,sum,sum/n,(x(i),i=1,n)
if(alph.lt.(0.999)) goto 11
close(11)
close(12)
stp
c************************************************************************
c************************************************************************
Appendix A

************************************************************************
united_programmes_random
************************************************************************

version 2.0: May 4, 2013; 21:00 (© CM&LM)

parameter(maxn=1000)
implicit real*8 (a-h,o-z)
dimension xlamb(1:maxn),x(1:maxn),f(1:maxn),p(1:maxn),
der(1:maxn),derpom(1:maxn),e(1:maxn),xpom(1:maxn)
open(11,file='output.txt')
open(13,file='parameters.txt')
open(14,file='inputs.txt')
read(13,*) n,alph,r,w,eps,error,xlamb1,xlamb2,x1,x2,nit
close(13)
write(11,'(i4,3f6.3,2e9.3)') n,alph,r,w,eps,error,nit
write(11,'(4f8.3)') xlamb1,xlamb2,x1,x2
 call random_seed
 do i=1,n
  read(14,*) xlamb(i)
 enddo
 close(14)

 do iexper=1,10
  close(14,*) xlamb(i)
 enddo
 close(14)

cexper
 do iexper=1,10
 cexper
 do i=1,n
  call random_number(xlamb(i))
  xlamb(i)=xlamb(i)*abs(xlamb2-xlamb1)+xlamb1
  call random_number(x(i))
  x(i)=x(i)*abs(x2-x1)+x1
  if(x(i).lt.eps) x(i)=eps
  write(11,'(2f8.4)') xlamb(i),x(i)
 enddo

 it=0
 1 it=it+1
  xx=0.0
  do j=1,n
   xx=xx+xlamb(j)*x(j)**r
  enddo
  do j=1,n
   p(j)=xlamb(j)*x(j)**r/xx
  enddo
  do j=1,n


Appendix A

\[ f(j) = \alpha p(j) (1.0 - p(j)) r \]
\[ \text{do } ip = 1, n \]
\[ \quad \text{if}(ip \text{ eq } j) \text{ goto } 2 \]
\[ \quad pjip = \lambda(j) x(j)^r / (xx - \lambda(ip) x(ip)^r) \]
\[ \quad f(j) = f(j) + p(ip) \cdot pjip \cdot (1.0 - p(j) - pjip) \cdot (1.0 - \alpha) \cdot r \]
\[ \text{2 continue} \]
\[ \text{enddo} \]
\[ \text{enddo} \]
\[ \text{test} = 0.0 \]
\[ \text{sum} = 0.0 \]
\[ \text{do } i = 1, n \]
\[ \quad \text{rhs} = w \cdot x(i) + (1.0 - w) \cdot f(i) \]
\[ \quad \text{if}(\text{rhs} \lt \text{eps}) \text{ rhs} = \text{eps} \]
\[ \quad \text{test} = \text{test} + (x(i) - \text{rhs}) \cdot (x(i) - \text{rhs}) \]
\[ \quad \text{sum} = \text{sum} + \text{rhs} \]
\[ \quad x(i) = \text{rhs} \]
\[ \text{enddo} \]
\[ \text{if} (\sqrt{\text{test}} \lt \text{error}) \text{ goto } 1 \]

C
C first and second derivative calculations and output
C
C
\[ dx = 0.001 \]
\[ \text{do } i = 1, n \]
\[ \quad \text{der}(i) = f(i) / x(i) - 1.0 \]
\[ \text{enddo} \]
\[ xx = 0.0 \]
\[ \text{do } j = 1, n \]
\[ \quad xx = xx + \lambda(j) \cdot (x(j) + dx)^r \]
\[ \text{enddo} \]
\[ \text{do } j = 1, n \]
\[ \quad p(j) = \lambda(j) \cdot (x(j) + dx)^r / xx \]
\[ \text{enddo} \]
\[ \text{do } j = 1, n \]
\[ \quad f(j) = \alpha p(j) (1.0 - p(j))^r \]
\[ \text{do } ip = 1, n \]
\[ \quad \text{if}(ip \text{ eq } j) \text{ goto } 22 \]
\[ \quad pjip = \lambda(j) \cdot (x(j) + dx)^r / (xx - \lambda(ip) \cdot (x(ip) + dx)^r) \]
\[ \quad f(j) = f(j) + p(ip) \cdot pjip \cdot (1.0 - p(j) - pjip) \cdot (1.0 - \alpha) \cdot r \]
\[ \quad \text{continue} \]
\[ \text{enddo} \]
\[ \text{enddo} \]
\[ \text{enddo} \]
\[ \text{do } i = 1, n \]
\[ \quad \text{derpom}(i) = f(i) / (x(i) + dx) - 1.0 \]
enddo
do i=1,n
  dder=(derpom(i)-der(i))/dx
write(11,'(i7,i4,f8.4,2e14.4)') it,i,x(i),der(i),dder
enddo
write(11,'(2f8.4)') sum, sum/n

C
C
C checking convergence on the solution obtained by the
C direct iteration method
C

do i=1,n
  xpom(i)=x(i)
enddo
ite=0
10 ite=ite+1
do i=1,n
  x(i)=xpom(i)
enddo
do i=1,n
  x(i)=0.0
  xmax=0.0
  emax=0.0
111 x(i)=x(i)+error
  if(x(i).lt.eps) goto 111
  xx=0.0
  do j=1,n
    xx=xx+xlamb(j)*x(j)**r
  enddo
  do j=1,n
    p(j)=xlamb(j)*x(j)**r/xx
  enddo
  ei=alph*p(i)-x(i)
  do j=1,n
    if(j.eq.i) goto 20
    pji=xlamb(i)*x(i)**r/(xx-xlamb(j)*x(j)**r)
    ei=ei+p(j)*pji*(1.0-alph)
  20 continue
  enddo
  if(ei.gt.emax) xmax=x(i)
  if(ei.gt.emax) emax=ei
  if(x(i).lt.1.0) goto 111
  x(i)=xpom(i)
Appendix A

XII


xpom(i)=xmax
e(i)=emax
endo
do i=1,n
   write(11,'(i7,i4,f8.4,2e14.4)') ite,i,x(i),e(i)
endo
sum=0.0
do i=1,n
   sum=sum+x(i)
endo
write(11,'(i7,3f8.4)') ite,alph,sum,sum/n
if(ite.lt.nit) goto 10
cexper
endo
cexper
   close(11)
   stop
end

c************************************************************************
c************************************************************************

A.2  Gnuplot

#***********Graphs***********
set term post eps color "Times-Roman" 22
set title 'lambda_1 = 100, lambda_3 = 75; alpha = 0.75; r = 1.0'
set nokey
#set dgrid3d 50,100
#set style data lines
#set pm3d
#set output 'effort_3D.ps'
set output 'effort.ps'
#set xlabel "alpha"
#set ylabel "lambda_2"
#set zlabel "effort"
set xlabel "lambda_2"
set ylabel "effort"
#splot [0.5:1] [75:100] [0:0.30] 'fig_x.txt' u 1:2:3 pal
plot [75:100] [0:0.25] 'fig_x.txt' u 1:3 w 1 lw 8, 'fig_x.txt' u 1:4 w p 1, 'fig_x.txt' u 1:5 w p 2,'fig_x.txt' u 1:6 w p 4
set output 'p1.ps'
set xlabel "lambda_2"
set ylabel "first prize probability"
plot [75:100] [0:1.0] 'fig_p1.txt' u 1:4 w p 1,'fig_p1.txt' u 1:5 w p 2, 'fig_p1.txt' u 1:6 w p 4#,'fig_p1.txt' u 1:7 w p 5,'fig_p1.txt' u 1:8 w p 6,
'fig_p1.txt' u 1:9 w p 7,'fig_p1.txt' u 1:10 w p 8,'fig_p1.txt' u 1:11 w p 9,
'fig_p1.txt' u 1:12 w p 10,'fig_p1.txt' u 1:13 w p 11
set output 'p2.ps'
set xlabel "lambda_2"
set ylabel "second prize probability"
plot [75:100] [0:1.0] 'fig_p2.txt' u 1:4 w p 1,'fig_p2.txt' u 1:5 w p 2,
'fig_p2.txt' u 1:6 w p 4#,'fig_p2.txt' u 1:7 w p 5,'fig_p2.txt' u 1:8 w p 6,
'fig_p2.txt' u 1:9 w p 7,'fig_p2.txt' u 1:10 w p 8,'fig_p2.txt' u 1:11 w p 9,
'fig_p2.txt' u 1:12 w p 10,'fig_p2.txt' u 1:13 w p 11

*****************************************************************************
Appendix B

Input files—examples

*********parameters.txt************
3 0.5 1.00 0.9 0.0001 0.0001 0.0 1.0 0.0 1.0 30
#_players alpha r weight epsilon error xlamb1 xlamb2 x1 x2 #_iterations
*****************************************************************

*********inputs.txt**************************
1.0 0.1
0.9 0.1
0.08 0.1
0.07 0.1
0.06 0.1
0.05 0.1
0.04 0.1
0.03 0.1
0.02 0.1
0.01 0.1
lambda x^(0)
********************************************************************