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Technique of operator algebras in quantum structures

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To my family
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Chapter 1

Introduction

The theory of operator algebras was initiated in the paper [57] published in 1930 by von Neumann. He introduced, motivated by quantum physics, certain algebras of bounded operators acting on a Hilbert space that are closed in the weak operator topology (nowadays known as von Neumann algebras). In the same paper, von Neumann proved the important approximation theorem called double commutant theorem describing an interplay between an algebraic and a topological structure of such algebras. The considerable progress in the theory of von Neumann algebras was given by Murray and von Neumann in the famous series of papers [54, 55, 59, 60]. In particular, they classified factors into different types and also constructed examples of these types.

The investigation of $C^*$-algebras, under the name *-rings by Gelfand and Neumark [35] was the next milestone in the development of operator algebras. They uncovered the structure of unital commutative as well as noncommutative $C^*$-algebras. It was proved, among other things, that every unital $C^*$-algebra is isomorphic to a $C^*$-subalgebra of the set of all bounded operators on a Hilbert space. The proof of this theorem was based on ideas which were later used by Segal [70] to construct representations of (not necessarily unital) $C^*$-algebras via linear functionals called states. This important construction, known as GNS construction, has become a fundamental tool in the theory of operator algebras. Moreover, Gelfand and Neumark showed in [35] that every unital commutative $C^*$-algebra is isomorphic to a $C^*$-algebra of all continuous complex-valued functions on a compact Hausdorff topological space. Accordingly, the theory of $C^*$-algebras is sometimes called noncommutative topology.

Currently, the theory of operator algebras is a vast discipline involving

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1The term $C^*$-algebra was first used by Segal [70].
$C^*$-algebras, von Neumann algebras, Jordan algebras, etc. Moreover, it has a strong impact to many branches of modern mathematics. We have remarked that the theory of $C^*$-algebras can be viewed as a noncommutative analogue of topology. Similarly, a correspondence between commutative von Neumann algebras and spaces $L^\infty(\Gamma, \mu)$ leads naturally to the fact that the theory of von Neumann algebras is regarded as a noncommutative version of measure theory. The noncommutative geometry and the theory of operator spaces are other examples of areas of mathematics which are strongly influenced by the theory of operator algebras.

Besides the influence on modern mathematics, operator algebras have also found an important field of applications in foundations of physics. In particular, they have provided a deep insight into fundamental questions concerning the structure of quantum theory.

The first comprehensive formulation of principles of quantum theory was given by Dirac [27] in 1930. A self-consistent mathematical description of this framework was later formulated by von Neumann in his famous book [58]. The essential concept of this traditional approach is a Hilbert space, whose elements correspond to the states of a physical system. Observables and the time evolution of the system are then described by operators on the Hilbert space.

The language of operator algebras allows to formulate another approach which is based on axiomatization of algebraic properties of observables. Since a suitable mathematical structure for this description is a (unital) $C^*$-algebra, it is sometimes called $C^*$-algebraic approach. Its basic axioms can be formulated as follows [13, 73]. The set of observables is described by the set of self-adjoint elements of a unital $C^*$-algebra. Positive linear functionals of norm one on the algebra represent physical states of the system and the time evolution is realized by *-automorphisms of the algebra.

An advantage of $C^*$-algebraic approach is a possibility to incorporate both classical and quantum physics into a common mathematical framework. The difference between these physical theories is given by the structure of the underlying $C^*$-algebra. This algebra is commutative in the case of classical physics, whereas in the quantum case it is noncommutative. Furthermore, this approach has important applications to the description of physical systems with infinite number of degrees of freedom. It lies at the heart of Haag-Kastler axiomatic framework of quantum field theory [4, 33, 37] and it is also used in statistical mechanics [19, 33].

The aim of this work is the study of quantum structures by means of operator algebras. More specifically, in this vast field, our attention is focused

\footnote{This framework is also known under the name algebraic quantum field theory.}
on the well known Bell inequalities and a certain partial order on operator algebras called the star order.

Bell inequalities was first studied by Bell [10]. They provide an upper bound on the strength of correlations between measurements performed on physical systems. The constraints predicted by Bell inequalities can be experimentally tested [5, 6] and made it possible to put a discussion on the local hidden-variables theories on the experimental ground.

From the mathematical point of view, Bell inequalities describe correlations of noncommutative random variables which, in the $C^*$-algebraic approach, are modeled by self-adjoint elements in a $C^*$-algebra.

A beautiful analysis of Bell inequalities in the context of algebraic quantum field theory was done by Summers and Werner in the series of papers [74, 75, 76, 77, 78, 79]. They showed that easily formulated Bell inequalities are surprisingly connected with deep structural results and difficult concepts of the theory of von Neumann algebras. Bell-type correlation constraints have provided useful invariants for studying operator algebras.

Recently, Bell inequalities have been intensely studied especially in a context of quantum information theory [61]. The reason for this lies in a relation between a violation of Bell inequalities and entanglement [82]. The entanglement is one of the key ingredients of quantum information theory and so Bell inequalities have found significant applications in many areas of this theory such as quantum cryptography [1, 2], communication complexity [22], quantum game theory [48, 71], estimates of a bound for the dimension of the underlying Hilbert space [20, 21, 81], etc. All these aspects motivate effort to understand the structure of Bell inequalities and their violation. The deep results in this field have been recently obtained by applying the powerful techniques of operator space theory [43, 44, 66, 67].

The next goal of this thesis is the investigation of the star order introduced by Drazin [30]. This partial order has been studied mainly on matrix algebras where a number of interesting facts were obtained [8, 9, 41, 42, 51]. Recently, it has also been considered on the set of all bounded operators on a Hilbert space [3, 28]. This has not only brought new infinite dimensional results but it has also put older facts on the star order for matrices into a new perspective.

The star order is related to the well known Gudder order [36]. It turns out that the Gudder order, which can be interpreted as a logical order on bounded quantum observables, is in fact a restriction of the star order to the self-adjoint part of the set of all bounded operators on a Hilbert space. This observation gives, for example, the solution of preserver problem [29] and infimum and supremum problem [36, 68] for the star order on bounded self-adjoint operators.
CHAPTER 1. INTRODUCTION

This thesis is organized as follows. In the next chapter, we briefly summarize the basic notions and facts of the theory of operator algebras. The results are presented without proofs. For the proofs and a more detailed treatment of this subject, we refer the reader to standard monographs [12, 40, 46, 50, 53, 64, 69, 80].

Chapter 3 deals with Bell inequalities. More concretely, we investigate the CHSH version of Bell inequality and its quantum form called Cirel’son inequality. The Cirel’son inequality is generalized to (real and complex) linear spaces endowed with a pseudo inner product. It is also found a relationship between elements in which the bound in the inequality is attained. This result leads to some interesting statements concerning maximal violation of the Bell inequality in the framework of \(*\)-algebras and Jordan algebras.

In Chapter 4, we introduce the star order on appropriate \(*\)-algebras and investigate its properties. Special effort is devoted to the infimum and supremum problem for the star order on a \(*\)-algebra \(C(X)\) of all continuous complex-valued functions on a Hausdorff topological space \(X\). It turns out that every upper bounded subset of \(C(X)\) has the infimum and the supremum whenever \(X\) is locally connected or extremely disconnected.

In Chapter 5, we discuss star order isomorphisms between various subsets of \(C^*\)-algebras. The main goal of this chapter is the study of nonlinear continuous star order isomorphisms between normal parts of von Neumann algebras without Type I\(_2\) direct summand. If these isomorphisms satisfy a certain additional condition, then we completely describe their structure.

The last chapter summarizes the main results presented in this thesis and outlines possibilities of further research.
Chapter 2

Operator algebras

In this chapter, we recall the basic notions and results of the theory of operator algebras for the convenience of the reader. In particular, we discuss some concepts of the theory of $C^*$-algebras and their representations. Further, we present selected topics in the theory of von Neumann and Jordan algebras. A comprehensive treatment of the theory of operator algebras is given in the standard monographs \[12, 40, 46, 50, 53, 64, 69, 80\]. A brief summary and discussion of some special topics can also be found in \[38\].

Throughout this thesis, all Hilbert spaces are considered to be complex unless otherwise specified. Furthermore, we do not consider the trivial case in which the algebra contains only the zero element.

2.1 Basic definitions and facts

The notion of $C^*$-algebra is motivated by the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators acting on a Hilbert space $\mathcal{H}$. The following definitions naturally generalize the well known concepts from $\mathcal{B}(\mathcal{H})$ to the case of associative algebras.

**Definition 2.1.1.** Let $\mathcal{A}$ be an associative complex algebra. A map $a \mapsto a^*$ from $\mathcal{A}$ into itself is called an **involution** if the following properties are satisfied

(i) $(a + b)^* = a^* + b^*$,

(ii) $(\lambda a)^* = \overline{\lambda} a^*$,

(iii) $(ab)^* = b^*a^*$,

(iv) $(a^*)^* = a$
for all \(a, b \in \mathcal{A}\) and \(\lambda \in \mathbb{C}\). An associative complex algebra endowed with an involution is called a \(*\)-algebra.

**Definition 2.1.2.** A normed algebra is an associative complex algebra \(\mathcal{A}\) equipped with a norm \(\|\cdot\|\) such that

\[
\|ab\| \leq \|a\| \|b\|
\]

for any \(a, b \in \mathcal{A}\). If a normed algebra is complete with respect to the norm, then it is called a Banach algebra.

**Definition 2.1.3.** A \(C^*\)-algebra is a Banach algebra together with an involution such that

\[
\|a^*a\| = \|a\|^2
\]

for any \(a \in \mathcal{A}\).

Let us remark that \(\|a^*a\| = \|a\|^2\) and \(\|ab\| \leq \|a\| \|b\|\) imply \(\|a^*\| = \|a\|\). This means that the involution on a \(C^*\)-algebra is norm preserving and therefore it is continuous in the norm topology.

**Definition 2.1.4.** A unit \(1_\mathcal{A}\) of an algebra \(\mathcal{A}\) is an element of \(\mathcal{A}\) satisfying

\[
a1_\mathcal{A} = 1_\mathcal{A}a = a
\]

for every \(a \in \mathcal{A}\). When no confusion can arise we shall write \(1\) instead of \(1_\mathcal{A}\). An algebra containing the unit is said to be unital.

It is easy to see that an algebra can have at most one unit. If \(\mathcal{A}\) is a unital \(C^*\)-algebra, then it may be shown that \(\|1\| = 1\).

**Definition 2.1.5.** Let \(\mathcal{A}\) be a unital \(*\)-algebra. An element \(a \in \mathcal{A}\) is said to be invertible if there exists an element \(a^{-1} \in \mathcal{A}\) such that

\[
aa^{-1} = a^{-1}a = 1.
\]

The element \(a^{-1}\) is called the inverse of \(a\).

If an algebra \(\mathcal{A}\) over a field \(\mathbb{F}\) is not unital, then \(\mathcal{A}\) can be embedded into a unital algebra \(\mathcal{A}_f = \mathcal{A} \oplus \mathbb{F}\) called unitization of \(\mathcal{A}\). Moreover, if \(\mathcal{A}\) is a \(C^*\)-algebra without the unit, then it can be shown that there is a unique norm which makes \(\mathcal{A}_f\) a unital \(C^*\)-algebra.

---

1. The symbol \(\mathcal{A} \oplus \mathbb{F}\) means the direct sum of linear spaces \(\mathcal{A}\) and \(\mathbb{F}\) endowed with the multiplication defined by \((a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda \mu)\) for every \((a, \lambda), (b, \mu) \in \mathcal{A} \oplus \mathbb{F}\).
Definition 2.1.6. Let \( \mathcal{A} \) be a \( C^* \)-algebra and let \( a \in \mathcal{A} \). Set \( \tilde{\mathcal{A}} = \mathcal{A} \) if \( \mathcal{A} \) is unital, and \( \tilde{\mathcal{A}} = \mathcal{A}_I \), where the \( C^* \)-algebra \( \mathcal{A}_I \) is the unitization of \( \mathcal{A} \), if \( \mathcal{A} \) is not unital. The **spectrum** of \( a \) in \( \mathcal{A} \) is the set

\[
\sigma_{\mathcal{A}}(a) = \{ \lambda \in \mathbb{C} \mid (a - \lambda 1) \text{ is not invertible in } \tilde{\mathcal{A}} \}.
\]

The **spectral radius** of \( a \) is the number

\[
r_{\mathcal{A}}(a) = \sup \{ |\lambda| \mid \lambda \in \sigma_{\mathcal{A}}(a) \}.
\]

We introduce the following convention. When no confusion can arise, we shall write \( \sigma(a) \) and \( r(a) \) in place of \( \sigma_{\mathcal{A}}(a) \) and \( r_{\mathcal{A}}(a) \), respectively.

The next result describes the set \( \sigma(a) \) for a \( C^* \)-algebra \( \mathcal{A} \). It is a generalization of the similar theorem known from the theory of bounded operators on a Hilbert space.

**Theorem 2.1.7.** Let \( a \) be an element of a \( C^* \)-algebra \( \mathcal{A} \). Then \( \sigma(a) \) is a non-empty closed subset of the closed disk in \( \mathbb{C} \) with center 0 and radius \( ||a|| \).

If \( a \) is an element of a \( C^* \)-subalgebra \( \mathcal{B} \) of a unital \( C^* \)-algebra \( \mathcal{A} \) containing the unit of \( \mathcal{A} \), then it is clear that \( \sigma_{\mathcal{A}}(a) \subseteq \sigma_{\mathcal{B}}(a) \). The following important result says that the reverse inclusion also holds and so we cannot lose any inverses if we restrict our attention to \( \mathcal{B} \).

**Theorem 2.1.8.** Let \( \mathcal{B} \) be a \( C^* \)-subalgebra of a unital \( C^* \)-algebra \( \mathcal{A} \) containing the unit of \( \mathcal{A} \). Suppose that \( a \in \mathcal{B} \). Then \( \sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a) \).

Let us describe so-called *-homomorphisms and *-isomorphisms between two *-algebras. These maps are defined in such a way that they preserve the algebraic structure of the *-algebras.

**Definition 2.1.9.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be *-algebras. We say that a map \( \pi \) from \( \mathcal{A} \) into \( \mathcal{B} \) is a *-*homomorphism if

(i) \( \pi(\lambda a + b) = \lambda \pi(a) + \pi(b) \),

(ii) \( \pi(ab) = \pi(a)\pi(b) \),

(iii) \( \pi(a^*) = \pi(a)^* \)

for any \( a, b \in \mathcal{A} \) and \( \lambda \in \mathbb{C} \). We say that \( \pi \) is unital if \( \mathcal{A} \) and \( \mathcal{B} \) are unital and \( \pi(1_\mathcal{A}) = 1_\mathcal{B} \). A bijective *-homomorphism is called a *-*isomorphism.

Now we shall give an important theorem concerning *-homomorphisms between \( C^* \)-algebras \( \mathcal{A} \) and \( \mathcal{B} \). We shall see that each *-homomorphism from \( \mathcal{A} \) into \( \mathcal{B} \) is a continuous map whose range is a \( C^* \)-subalgebra of \( \mathcal{B} \).
Chapter 2. Operator Algebras

Theorem 2.1.10. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-algebras and $\pi: \mathcal{A} \to \mathcal{B}$ is a *-homomorphism. Then

(i) $\|\pi(a)\| \leq \|a\|$ for every $a \in \mathcal{A}$.

(ii) If $\pi$ is injective, then $\|\pi(a)\| = \|a\|$ for each $a \in \mathcal{A}$.

(iii) $\pi(\mathcal{A})$ is a $C^*$-subalgebra of $\mathcal{B}$.

Definition 2.1.11. Let $\mathcal{A}$ be an abelian (i.e., commutative) $C^*$-algebra. The spectrum $\hat{\mathcal{A}}$ of $\mathcal{A}$ is the set of all non-zero homomorphisms from $\mathcal{A}$ to $\mathbb{C}$.

Note that the spectrum $\hat{\mathcal{A}}$ is a locally compact Hausdorff topological space in the weak* topology.

Let $X$ be a locally compact Hausdorff topological space. It can be shown that the algebra $C_0(X)$ of all continuous complex-valued functions on $X$ vanishing at infinity forms an abelian $C^*$-algebra. The following significant theorem says that every abelian $C^*$-algebra can be viewed as $C_0(X)$.

Theorem 2.1.12 (Gelfand). Let $\mathcal{A}$ be an abelian $C^*$-algebra. Then there is *-isomorphism from $\mathcal{A}$ onto $C_0(\hat{\mathcal{A}})$.

In the case of $\mathcal{B}(\mathcal{H})$, special classes of operators (for example, self-adjoint operators, projections, etc.) play an important role. Let us now define these special types of elements for a general *-algebra.

Definition 2.1.13. Let $\mathcal{A}$ be a *-algebra. An element $a \in \mathcal{A}$ is called

(i) normal if $a^*a = aa^*$;

(ii) self-adjoint if $a = a^*$;

(iii) positive if there is an element $b \in \mathcal{A}$ such that $a = b^*b$;

(iv) a projection if $a = a^* = a^2$;

(v) a partial isometry if $a = aa^*a$.

If $\mathcal{A}$ is unital, then an element $a \in \mathcal{A}$ is unitary whenever $a^*a = aa^* = 1$.\footnote{The algebra $C_0(X)$ is endowed with the supremum norm and the involution defined by $f^*(x) = f(x)$ for all $x \in X$.}\footnote{Let us note that $C_0(X)$ is unital if and only if $X$ is compact. In this case, $C_0(X) = C(X)$, where $C(X)$ is the algebra of all continuous complex-valued functions on $X$.}
In the sequel, the sets of all normal, self-adjoint, and positive elements of a \( *\)-algebra \( \mathcal{A} \) will be denoted by \( \mathcal{A}_n \), \( \mathcal{A}_{sa} \), and \( \mathcal{A}_+ \), respectively. The set of all projections in \( \mathcal{A} \) will be denoted by \( P(\mathcal{A}) \).

The spectrum of an element of a \( C^* \)-algebra is a pure algebraic concept. However, there is a connection between the spectral radius and the norm.

**Proposition 2.1.14.** Let \( \mathcal{A} \) be a \( C^* \)-algebra.

(i) For each \( a \in \mathcal{A} \), \( r(a) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}} \).

(ii) If \( a \in \mathcal{A} \) is normal, then \( \|a\| = r(a) \).

The previous proposition, together with the condition \( \|a^*a\| = \|a\|^2 \), ensures that \( \|a\| = \sqrt{r(a^*a)} \) for any \( a \in \mathcal{A} \). This means that the norm is determined by the algebraic structure of the \( C^* \)-algebra. This has an important corollary.

**Corollary 2.1.15.** There is at most one norm on a \( * \)-algebra making it a \( C^* \)-algebra.

Now let us describe the structure of the set \( \mathcal{A}_+ \) of all positive elements in a \( C^* \)-algebra.

**Theorem 2.1.16.** Suppose that \( \mathcal{A} \) is a \( C^* \)-algebra.

(i) \( \mathcal{A}_+ \) is closed in \( \mathcal{A} \).

(ii) \( \lambda a \in \mathcal{A}_+ \) if \( a \in \mathcal{A}_+ \) and \( \lambda \in [0, \infty) \).

(iii) \( a + b \in \mathcal{A}_+ \) if \( a, b \in \mathcal{A}_+ \).

(iv) \( ab \in \mathcal{A}_+ \) if \( a, b \in \mathcal{A}_+ \) and \( ab = ba \).

(v) If \( a \in \mathcal{A}_+ \) and \( -a \in \mathcal{A}_+ \), then \( a = 0 \).

The foregoing theorem shows that \( \mathcal{A}_+ \) is a (closed) positive cone in the \( C^* \)-algebra \( \mathcal{A} \). Consequently, we can introduce a partial order \( \leq \) on the space \( \mathcal{A}_{sa} \) as follows: \( a \leq b \) if \( b - a \in \mathcal{A}_+ \).

The following result gives a useful characterization of positive elements in \( C^* \)-algebras. Note that the condition (iii) determines the positive square root of a positive element.

**Theorem 2.1.17.** For a self-adjoint element \( a \) of a \( C^* \)-algebra \( \mathcal{A} \), the following conditions are equivalent:

(i) \( a \in \mathcal{A}_+ \).
(ii) \( \sigma(a) \subseteq [0, \infty) \).

(iii) \( a = h^2 \) for some \( h \in \mathcal{A}_+ \).

When these conditions are satisfied, the element \( h \) occurring in (iii) is unique.

One of the powerful results of the theory of \( C^* \)-algebras is the continuous functional calculus for normal elements. We formulate this functional calculus in Theorem 2.1.18. In Theorem 2.1.20, we then specify the spectrum of normal elements arising in this functional calculus.

**Theorem 2.1.18.** If \( a \) is a normal element of a unital \( C^* \)-algebra \( \mathcal{A} \), \( C(\sigma(a)) \) is the abelian \( C^* \)-algebra of all continuous complex-valued functions on \( \sigma(a) \), and \( \iota \) in \( C(\sigma(a)) \) is defined by \( \iota : t \mapsto t \) (\( t \in \sigma(a) \)), then there is a unique unital injective *-homomorphism \( \Phi : C(\sigma(a)) \to \mathcal{A} \) such that \( \Phi(\iota) = a \). For each \( f \) in \( C(\sigma(a)) \), \( \Phi(f) \) is normal, and is the limit of a sequence of polynomials in \( 1, a, \) and \( a^* \). The set

\[ \{ \Phi(f) \mid f \in C(\sigma(a)) \} \]

is an abelian \( C^* \)-algebra, and is the smallest \( C^* \)-subalgebra of \( \mathcal{A} \) that contains the element \( a \).

The *-homomorphism \( \Phi : C(\sigma(a)) \to \mathcal{A} \) described in the previous theorem is called the continuous functional calculus for the normal element \( a \) of the \( C^* \)-algebra \( \mathcal{A} \). We usually denote by \( f(a) \) the element \( \Phi(f) \) of \( \mathcal{A} \).

**Corollary 2.1.19.** Let \( \mathcal{A} \) be a unital \( C^* \)-algebra and let \( a \in \mathcal{A} \). Then

(i) \( a \) is self-adjoint if and only if \( \sigma(a) \subseteq \mathbb{R} \).

(ii) \( a \) is unitary if and only if \( \sigma(a) \subseteq \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \} \).

(iii) \( a \) is a projection if and only if \( \sigma(a) \subseteq \{0, 1\} \).

**Theorem 2.1.20** (Spectral mapping theorem). If \( a \) is a normal element of a unital \( C^* \)-algebra \( \mathcal{A} \) and \( f \in C(\sigma(a)) \), then

\[ \sigma(f(a)) = \{ f(t) \mid t \in \sigma(a) \} \]

Moreover, if \( g \in C(\sigma(f(a))) \), then \( (g \circ f)(a) = g(f(a)) \), where \( g(f(a)) \) denotes the element of \( \mathcal{A} \) that corresponds to \( g \) in the continuous functional calculus for the normal element \( f(a) \).

**Proposition 2.1.21.** Let \( \mathcal{A} \) be a unital \( C^* \)-algebra.
(i) If $a \in \mathcal{A}_{sa}$, then $-\|a\|_1 \leq a \leq \|a\|_1$.

(ii) If $a, b \in \mathcal{A}_{sa}$ and $-b \leq a \leq b$, then $\|a\| \leq \|b\|$.

Now we formulate a useful result saying that a unital $*$-homomorphism commutes with the continuous functional calculus.

**Proposition 2.1.22.** Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital $C^*$-algebras and $\varphi$ is a unital $*$-homomorphism from $\mathcal{A}$ into $\mathcal{B}$. If $a$ is a normal element of $\mathcal{A}$ and $f \in C(\sigma(a))$, then $\varphi(a)$ is a normal element of $\mathcal{B}$, and $f(\varphi(a)) = \varphi(f(a))$.

The following theorem gives useful information about elements which commute with a normal element of a $C^*$-algebra.

**Theorem 2.1.23** (Fuglede). Let $a$ be a normal element of a $C^*$-algebra $\mathcal{A}$. If $b \in \mathcal{A}$ commutes with $a$, then $b^*$ commutes with $a$.

Extreme points of the closed unit ball play an important role in many aspects of the theory of Banach spaces. The next result describes extreme points of the closed unit ball of a $C^*$-algebra.

**Theorem 2.1.24.** Let $\mathcal{A}$ be a $C^*$-algebra. Then the closed unit ball of $\mathcal{A}$ has an extreme point if and only if $\mathcal{A}$ is unital. In this case, extreme points in the closed unit ball of $\mathcal{A}$ are precisely the partial isometries $a$ such that $(1 - aa^*)\mathcal{A}(1 - a^*a) = 0$.

### 2.2 States and representations

Positive linear functionals, especially so-called states, play an important role in the theory of operator algebras. In particular, we shall see that they are fundamental tools for a construction of representations of a $C^*$-algebra. Moreover, the states are also important in physics, where they describe physical states of a system.

**Definition 2.2.1.** A linear functional $\varphi$ on a $*$-algebra $\mathcal{A}$ is said to be hermitian if $\varphi(a^*) = \overline{\varphi(a)}$ for all $a \in \mathcal{A}$. A linear functional $\varphi$ on $\mathcal{A}$ is said to be positive, written $0 \leq \varphi$, if $0 \leq \varphi(a^*a)$ for every $a \in \mathcal{A}$. For two hermitian linear functionals $\varphi$ and $\psi$, we write $\varphi \leq \psi$ if $0 \leq \psi - \varphi$.

**Definition 2.2.2.** A positive linear functional $\varphi$ on a unital $*$-algebra $\mathcal{A}$ is called a state if $\varphi(1) = 1$. A state $\varphi$ on $\mathcal{A}$ is called faithful if $\varphi(a^*a) = 0$ implies $a = 0$. A state $\varphi$ on $\mathcal{A}$ is said to be tracial if $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$. 
Proposition 2.2.3. Let \( \varphi \) be a positive linear functional on a \(*\)-algebra \( \mathcal{A} \). Then
\[
(i) \quad \varphi(a^*b) = \overline{\varphi(b^*a)}, \\
(ii) \quad |\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b)
\]
for all \( a, b \in \mathcal{A} \).

It follows immediately from (i) that each positive linear functional on a unital \(*\)-algebra is hermitian. Note also that the inequality (ii) is called the Cauchy-Schwarz inequality.

Theorem 2.2.4. Let \( \mathcal{A} \) be a \( \mathcal{C}^* \)-algebra.
\[
(i) \quad \text{If } \varphi \text{ is a positive linear functional on } \mathcal{A}, \text{ then } \varphi \text{ is bounded.} \\
(ii) \quad \text{If } \mathcal{A} \text{ is unital, then a linear functional } \varphi \text{ on } \mathcal{A} \text{ is positive if and only if} \\
\quad \varphi \text{ is bounded and } \|\varphi\| = \varphi(1).
\]

The previous theorem implies that each state on a unital \( \mathcal{C}^* \)-algebra \( \mathcal{A} \) is a bounded (and so continuous) linear functional and its norm is equal to one. Thus the states are elements from the unit sphere of a continuous dual space of \( \mathcal{A} \). This observation allows us to generalize the notion of state to a (not necessarily unital) \( \mathcal{C}^* \)-algebra as follows.

Definition 2.2.5. A state on a \( \mathcal{C}^* \)-algebra \( \mathcal{A} \) is a positive linear functional on \( \mathcal{A} \) of norm one. The set \( \mathcal{S}(\mathcal{A}) \) of all states on \( \mathcal{A} \), with relative weak* topology, is called the state space of \( \mathcal{A} \). An extreme point of \( \mathcal{S}(\mathcal{A}) \) is called a pure state. The set of all pure states on \( \mathcal{A} \) is denoted by \( \mathcal{P}(\mathcal{A}) \).

The next proposition implies that the set \( \mathcal{S}(\mathcal{A}) \) is non-empty.

Proposition 2.2.6. Suppose that \( \mathcal{A} \) is a \( \mathcal{C}^* \)-algebra and \( a \in \mathcal{A} \).
\[
(i) \quad \text{If } \varphi(a) = 0 \text{ for every } \varphi \in \mathcal{S}(\mathcal{A}), \text{ then } a = 0. \\
(ii) \quad a \in \mathcal{A}_{sa} \text{ if and only if } \varphi(a) \in \mathbb{R} \text{ for every } \varphi \in \mathcal{S}(\mathcal{A}).
\]

\[\text{4}\text{The set } \mathcal{S}(\mathcal{A}) \text{ is weak* compact if and only if } \mathcal{A} \text{ is a unital } \mathcal{C}^* \text{-algebra.}\]
(iii) $a \in \mathcal{A}_+$ if and only if $\varphi(a) \geq 0$ for every $\varphi \in \mathcal{S}(\mathcal{A})$.

In the following definitions, we introduce basic notions concerning representations of $C^*$-algebras.

**Definition 2.2.8.** A representation of a $C^*$-algebra $\mathcal{A}$ is a pair $(\pi, \mathcal{H})$ consisting of a Hilbert space $\mathcal{H}$ and a *-homomorphism $\pi$ from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. If there is a vector $\xi$ in $\mathcal{H}$ such that the linear subspace $\pi(\mathcal{A})\xi = \{\pi(a)\xi \mid a \in \mathcal{A}\}$ is dense in $\mathcal{H}$, then $(\pi, \mathcal{H})$ is called a cyclic representation and $\xi$ is called a cyclic vector (or generating vector) for $(\pi, \mathcal{H})$.

Note that, by Theorem 2.1.10, $\pi(\mathcal{A})$ is a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

**Definition 2.2.9.** Representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of a $C^*$-algebra $\mathcal{A}$ are said to be unitary equivalent if there exists an isometry $u$ of $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $u\pi_1(a)u^* = \pi_2(a)$ for any $a \in \mathcal{A}$.

**Definition 2.2.10.** A representation $(\pi, \mathcal{H})$ of a $C^*$-algebra $\mathcal{A}$ is said to be faithful if $\pi$ is an injective *-homomorphism.

We can use a state on a $C^*$-algebra to construct a representation. This important construction is called Gelfand-Neumark-Segal construction (or GNS construction) and it is used in the proof of the following theorem (see, for example, [46]). This theorem asserts that every state can be realized by a vector state in an appropriate representation.

**Theorem 2.2.11** (GNS representation). If $\varphi$ is a state on a $C^*$-algebra $\mathcal{A}$, there is a cyclic representation $(\pi_\varphi, \mathcal{H}_\varphi)$ of $\mathcal{A}$, and a unit cyclic vector $\xi_\varphi$ for $(\pi_\varphi, \mathcal{H}_\varphi)$ such that $\varphi(a) = (\pi_\varphi(a)\xi_\varphi, \xi_\varphi)$ for all $a \in \mathcal{A}$. Moreover, the representation is unique up to unitary equivalence.

A connection between an abstract $C^*$-algebra and a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ is given by the next fundamental result.

**Theorem 2.2.12** (Gelfand-Neumark). Each $C^*$-algebra has a faithful representation.

---

5Let $\xi$ be a unit vector of a Hilbert space $\mathcal{H}$. The relation $\varphi_\xi(a) = (a\xi, \xi)$, $a \in \mathcal{B}(\mathcal{H})$, defines the state $\varphi_\xi$ on $\mathcal{B}(\mathcal{H})$ which is called a vector state.
Definition 2.2.13. Let $S$ be a subset of $\mathcal{B}(\mathcal{H})$. The set
\[ S' = \{ a \in \mathcal{B}(\mathcal{H}) \mid ab = ba \text{ for every } b \in S \} \]
is the commutant of $S$. The double commutant $S''$ of $S$ is $(S')'$.

The following result is a version of Radon-Nikodym theorem for states on $C^*$-algebras.

Theorem 2.2.14 (Radon-Nikodym). Let $\varphi$ be a state on a $C^*$-algebra $A$. Suppose that $(\pi_\varphi, \mathcal{H}_\varphi)$ is a GNS representation associated to $\varphi$ and $\xi_\varphi \in \mathcal{H}_\varphi$ is a unit cyclic vector for $(\pi_\varphi, \mathcal{H}_\varphi)$. Then, for each positive functional $\psi$ on $A$ with $\psi \leq \varphi$, there is a unique element $\tilde{b}$ in the commutant $\pi_\varphi(A)'$ of $\pi_\varphi(A)$ with $0 \leq \tilde{b} \leq 1_{\mathcal{B}(\mathcal{H}_\varphi)}$ such that
\[ \psi(a) = (\pi_\varphi(a)\tilde{b}\xi_\varphi, \xi_\varphi) \]
for all $a \in A$.

2.3 Von Neumann algebras

A number of important locally convex topologies can be introduced on the $C^*$-algebra $\mathcal{B}(\mathcal{H})$. We discuss here two of them that play a fundamental role in the theory of von Neumann algebras.

Definition 2.3.1. The strong operator topology on $\mathcal{B}(\mathcal{H})$ is a locally convex topology generated by the family of semi-norms of the form
\[ p_\xi : a \mapsto \|a\xi\|, \quad a \in \mathcal{B}(\mathcal{H}), \xi \in \mathcal{H}. \]
The weak operator topology on $\mathcal{B}(\mathcal{H})$ is a locally convex topology generated by the family of semi-norms of the form
\[ p_{\xi,\eta} : a \mapsto |(a\xi, \eta)|, \quad a \in \mathcal{B}(\mathcal{H}), \xi, \eta \in \mathcal{H}. \]

Note that a net $(a_\alpha)_{\alpha \in \Lambda}$ in $\mathcal{B}(\mathcal{H})$ converges to an operator $a \in \mathcal{B}(\mathcal{H})$ in the strong operator topology if and only if $\|a_\alpha \xi - a\xi\| \to 0$ for every $\xi \in \mathcal{H}$. A net $(a_\alpha)_{\alpha \in \Lambda}$ in $\mathcal{B}(\mathcal{H})$ converges to an operator $a \in \mathcal{B}(\mathcal{H})$ in the weak operator topology if and only if $(a_\alpha \xi, \eta) \to (a\xi, \eta)$ for every $\xi, \eta \in \mathcal{H}$.

The multiplication is separately continuous (i.e., for fixed $b \in \mathcal{B}(\mathcal{H})$, the maps $a \in \mathcal{B}(\mathcal{H}) \mapsto ab \in \mathcal{B}(\mathcal{H})$ and $a \in \mathcal{B}(\mathcal{H}) \mapsto ba \in \mathcal{B}(\mathcal{H})$ are continuous) in the strong operator topology. However, it is not jointly continuous (i.e., the map $(a, b) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \mapsto ab \in \mathcal{B}(\mathcal{H})$ is not continuous) in the strong operator topology if $\mathcal{H}$ is infinite dimensional.\footnote{It can be shown that $(a, b) \in B \times \mathcal{B}(\mathcal{H}) \mapsto ab \in \mathcal{B}(\mathcal{H})$, where $B$ is a bounded subset of $\mathcal{B}(\mathcal{H})$, is continuous in the strong operator topology.}

Theorem 2.3.2. Let $S$ be a convex subset of $\mathcal{B}(\mathcal{H})$. Then the strong operator closure of $S$ coincides with the weak operator closure of $S$.

Theorem 2.3.3. The closed unit ball $\mathcal{B}(\mathcal{H})_1$ is compact in the weak operator topology.

It is well known that monotone bounded sequences of real numbers are convergent. The following useful theorem generalizes this result to $\mathcal{B}(\mathcal{H})$.

Theorem 2.3.4 (Vigier). Let $(a_\alpha)_{\alpha \in \Lambda}$ be a net of bounded self-adjoint operators on a Hilbert space $\mathcal{H}$. If $(a_\alpha)_{\alpha \in \Lambda}$ is increasing and bounded above, then it is strongly operator convergent to an element $a \in \mathcal{B}(\mathcal{H})_{sa}$, and $a$ is the supremum of $(a_\alpha)_{\alpha \in \Lambda}$.

Corollary 2.3.5. Suppose that $(p_\alpha)_{\alpha \in \Lambda}$ is a net of projections on a Hilbert space $\mathcal{H}$.

(i) If $(p_\alpha)_{\alpha \in \Lambda}$ is increasing, then it is strongly operator convergent to the projection of $\mathcal{H}$ onto the closure of $\bigcup_{\alpha \in \Lambda} p_\alpha(\mathcal{H})$.

(ii) If $(p_\alpha)_{\alpha \in \Lambda}$ is decreasing, then it is strongly operator convergent to the projection of $\mathcal{H}$ onto $\bigcap_{\alpha \in \Lambda} p_\alpha(\mathcal{H})$.

We introduce von Neumann algebras as certain $C^*$-subalgebras of $\mathcal{B}(\mathcal{H})$. Note that von Neumann algebras can also be defined in an abstract way, without referring to the concrete $C^*$-algebra $\mathcal{B}(\mathcal{H})$, by using the notion of predual (see, for example, [69]).

Definition 2.3.6. A $C^*$-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is called von Neumann algebra if $\mathcal{A}$ is strongly operator closed.

The following important result, known as double commutant theorem, shows that double commutant of a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ is closely associated with the strong operator closure of such algebra.

Theorem 2.3.7 (Double commutant). Let $\mathcal{A}$ be a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ and let $1_{\mathcal{B}(\mathcal{H})} \in \mathcal{A}$. Then $\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$.

Proposition 2.3.8. If $\mathcal{A}$ is a von Neumann algebra, then it is unital.

The unit of a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ need not be the identity operator on $\mathcal{H}$. However, it can be easily proved that it is necessarily a projection $p$. If we restrict our attention to the smaller Hilbert space $\mathcal{K} = p\mathcal{H}$, then $\mathcal{A}$ can be considered as $C^*$-subalgebra of $\mathcal{B}(\mathcal{K})$ containing the identity operator on $\mathcal{K}$ and so, by double commutant theorem, $\mathcal{A} = \mathcal{A}''$.

A significant feature of von Neumann algebras is that they have abundance of projections in contrast to general $C^*$-algebras.
Theorem 2.3.9. Let $\mathcal{A}$ be a von Neumann algebra. Then $\mathcal{A}$ is a closed linear span of its projections.

It is well known that the set of all projections in $\mathcal{B}(\mathcal{H})$ forms a complete lattice (i.e., every subset has an infimum and a supremum). The following theorem generalizes this result to von Neumann algebras.

Theorem 2.3.10. If $\mathcal{A}$ is a von Neumann algebra, then the set $P(\mathcal{A})$ of all projections in $\mathcal{A}$ is a complete lattice (under the usual operator order $\leq$).

Definition 2.3.11. If $a \in \mathcal{B}(\mathcal{H})$, then its range projection $p_a$ is the projection of $\mathcal{H}$ on the closure of $a\mathcal{H}$.

Theorem 2.3.12. If $\mathcal{A}$ is a von Neumann algebra, then it contains the range projections of all of its elements.

Every operator $a \in \mathcal{B}(\mathcal{H})$ has the decomposition of the form $a = u|a|$, where $|a| = \sqrt{a^*a}$ and $u \in \mathcal{B}(\mathcal{H})$ is a partial isometry such that $u^*u = p_{|a|}$. If $\mathcal{A}$ is a von Neumann algebra and $a \in \mathcal{A}$, then, by the functional calculus, $|a| \in \mathcal{A}$. It is natural to ask whether the partial isometry $u$ is also an element of $\mathcal{A}$.

Theorem 2.3.13 (Polar decomposition). For each element $a$ in a von Neumann algebra $\mathcal{A}$ there is a unique partial isometry $u \in \mathcal{A}$ such that $u^*u = p_{|a|}$ and $a = u|a|$.

The following Kaplansky theorem is a powerful tool in the theory of von Neumann algebras.

Theorem 2.3.14 (Kaplansky). Let $\mathcal{A}$ be a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ with strong operator closure $\mathcal{B}$. Then

(i) $\mathcal{A}_{sa}$ is strongly operator dense in $\mathcal{B}_{sa}$.

(ii) The closed unit ball of $\mathcal{A}_{sa}$ is strongly operator dense in the closed unit ball of $\mathcal{B}_{sa}$.

(iii) The closed unit ball of $\mathcal{A}$ is strongly operator dense in the closed unit ball of $\mathcal{B}$.

(iv) If $1_{\mathcal{B}(\mathcal{H})} \in \mathcal{A}$, then the unitaries of $\mathcal{A}$ are strongly operator dense in the unitaries of $\mathcal{B}$.
It follows from Theorem 2.1.12 that every abelian von Neumann algebra is *-isomorphic to the $C^*$-algebra $C(X)$, where $X$ is a compact Hausdorff topological space. Since von Neumann algebras contain a lot of projections, the space $X$ has a very specific topological structure. We describe this fact in Theorem 2.3.17.

**Definition 2.3.15.** A positive linear functional $\varphi$ on $C(X)$ is said to be **normal** if $\varphi(f_\alpha) \to \varphi(f)$ whenever $(f_\alpha)_{\alpha \in \Lambda}$ is a bounded increasing net of real functions on $X$ converging to $f$.

**Definition 2.3.16.** A Hausdorff topological space is called **extremely disconnected** if closure of every open set is open. A compact extremely disconnected space $X$ is said to be **hyperstonean** if for each positive nonzero function $f \in C(X)$ there is a positive normal linear functional $\varphi$ with $\varphi(f) \neq 0$.

**Theorem 2.3.17.** Let $\mathcal{A}$ be an abelian $C^*$-algebra. Then its spectrum is a hyperstonean Hausdorff topological space if and only if $\mathcal{A}$ is *-isomorphic to an abelian von Neumann algebra.

There is also a measure theoretic viewpoint on abelian von Neumann algebras. It can be shown that $L^\infty(\Gamma, \mu)$, where $\Gamma$ is a locally compact space and $\mu$ is a positive Radon measure, is *-isomorphic to an abelian von Neumann algebra. The following theorem says that every abelian von Neumann algebra can be regarded as a $C^*$-algebra $L^\infty(\Gamma, \mu)$.

**Theorem 2.3.18.** Let $\mathcal{A}$ be an abelian von Neumann algebra. Then $\mathcal{A}$ is *-isomorphic to a $C^*$-algebra $L^\infty(\Gamma, \mu)$, where $\Gamma$ is a locally compact space and $\mu$ is a positive Radon measure.

### 2.4 Types of $C^*$-algebras and von Neumann algebras

We have seen that projections are essential elements of von Neumann algebras. This fact motivates the deeper study of these elements.

**Definition 2.4.1.** Let $\mathcal{A}$ be a *-algebra. Projections $e, f \in \mathcal{A}$ are said to be **equivalent**, written $e \sim f$, if there is a partial isometry $u \in \mathcal{A}$ such that $u^*u = e$ and $uu^* = f$. In this case, the projections $e$ and $f$ are called **initial** and **final** projections of $u$, respectively.

It can be easily verified that the relation $\sim$ is an equivalence relation on the set of all projections. This allows us to introduce the concepts of finite and infinite projections.
Definition 2.4.2. Let $p$ be a projection in a $C^*$-algebra $\mathcal{A}$.

(i) We say that $p$ is \emph{finite} if $p \sim p_0 \leq p$ implies $p = p_0$.

(ii) We say that $p$ is \emph{infinite} if it is not finite.

(iii) We say that $p$ is \emph{abelian} if $pAp$ is abelian.

(iv) We say that $p$ is \emph{central} if it commutes with each element of $\mathcal{A}$.

Let us now define various types of unital $C^*$-algebras and von Neumann algebras.

Definition 2.4.3. A unital $C^*$-algebra $\mathcal{A}$ is called \emph{finite} (resp. \emph{infinite}) if the unit $1$ of $\mathcal{A}$ is finite (resp. infinite).

Definition 2.4.4. Let $\mathcal{A}$ be a von Neumann algebra.

(i) $\mathcal{A}$ is said to be of \emph{Type I} if every nonzero central projection in $\mathcal{A}$ majorizes a nonzero abelian projection in $\mathcal{A}$.

(ii) $\mathcal{A}$ is said to be of \emph{Type II} if it has no nonzero abelian projection and every nonzero central projection majorizes a nonzero finite projection.

(iii) $\mathcal{A}$ is said to be of \emph{Type III} if it has no nonzero finite projection.

(iv) If $\mathcal{A}$ is of Type II and finite, then it is said to be of \emph{Type II}_1.

(v) If $\mathcal{A}$ is of Type II and has no nonzero finite central projection, then it is said to be of \emph{Type II}_\infty.

The following theorem shows that an arbitrary von Neumann algebra is built up from the types mentioned above.

Theorem 2.4.5. \emph{Every von Neumann algebra is uniquely decomposable into a direct sum of von Neumann algebras of Type I, Type II}_1, Type II_\infty, and Type III.\end{quote}

Definition 2.4.6. A von Neumann algebra $\mathcal{A}$ is called a \emph{factor} if $\mathcal{A} \cap \mathcal{A}' = \mathbb{C}1$.

Note that if a von Neumann algebra is a factor, then it is exactly one of the Types I, II_1, II_\infty, and III.

We shall now take a closer look at the structure of von Neumann algebras of Type I.
**Definition 2.4.7.** A von Neumann algebra $\mathcal{A}$ is said to be of *Type $I_n$* if the unit $1$ of $\mathcal{A}$ is the sum of $n$ equivalent abelian projections.

**Theorem 2.4.8.** A von Neumann algebra of Type $I$ is uniquely decomposable into a direct sum of von Neumann algebras of Type $I_n$, where $n \in \Lambda$ and $\Lambda$ is a family of mutually distinct cardinal numbers.

A factor of Type I is necessarily of Type $I_n$ for some cardinal number $n$. Consequently, the following theorem gives a complete description of factors of Type I.

**Theorem 2.4.9.** If $\mathcal{A}$ is a factor of Type $I_n$, then $\mathcal{A}$ is $*$-isomorphic to $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ has dimension $n$.

### 2.5 Jordan algebras

The set $\mathcal{B}(\mathcal{H})_{sa}$ of all self-adjoint elements of $\mathcal{B}(\mathcal{H})$ forms a real vector space. If we endow this space with the product given by $a \circ b = \frac{1}{2}(ab + ba)$, we obtain the commutative real algebra which is nonassociative in general. This example leads to the concepts of Jordan algebras and JB algebras.

**Definition 2.5.1.** A real (not necessarily associative) algebra $\mathcal{A}$ with the product $(a, b) \mapsto a \circ b$ is called a Jordan algebra if

(i) $a \circ b = b \circ a$,

(ii) $a \circ (b \circ a^2) = (a \circ b) \circ a^2$,

for all $a, b \in \mathcal{A}$.

**Proposition 2.5.2.** Let $a$ be an element of a Jordan algebra. Then, for any $m, n \in \mathbb{N} \setminus \{0\}$,

$$a^{m+n} = a^m \circ a^n.$$ 

**Definition 2.5.3.** A Jordan algebra $\mathcal{A}$ equipped with a norm $\|\cdot\|$ is called a JB algebra if $\mathcal{A}$ is complete with respect to the norm $\|\cdot\|$ and this norm satisfies

(i) $\|a \circ b\| \leq \|a\| \|b\|$,

(ii) $\|a^2\| = \|a\|^2$,

(iii) $\|a^2\| \leq \|a^2 + b^2\|$.

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7This algebra is clearly nonassociative whenever $\dim \mathcal{H} \geq 2$. 
for all \( a, b \in \mathcal{A} \).

**Definition 2.5.4.** We say that two elements \( a, b \) in a Jordan algebra \( \mathcal{A} \) operator commute if \( a \circ (b \circ c) = b \circ (a \circ c) \) for all \( c \in \mathcal{A} \). Two subalgebras \( \mathcal{B} \) and \( \mathcal{C} \) of a Jordan algebra \( \mathcal{A} \) operator commute if every element of \( \mathcal{B} \) operator commutes with all elements of \( \mathcal{C} \).

In the following definitions, we introduce positive elements and states in a similar manner as for \(*\)-algebras.

**Definition 2.5.5.** Let \( \mathcal{A} \) be a Jordan algebra. An element \( a \in \mathcal{A} \) is positive, written \( 0 \leq a \), if there is \( b \in \mathcal{A} \) such that \( a = b^2 \). We write \( a \leq b \) if \( 0 \leq b - a \).

**Definition 2.5.6.** A linear functional \( \varphi \) on a unital Jordan algebra \( \mathcal{A} \) is called a state if \( \varphi(1) = 1 \) and \( \varphi(a^2) \geq 0 \) for every \( a \in \mathcal{A} \). A state \( \varphi \) is called faithful if \( \varphi(a^2) = 0 \) implies \( a = 0 \). A state \( \varphi \) is said to be tracial if

\[
\varphi(a \circ (b \circ c)) = \varphi((a \circ b) \circ c)
\]

for all \( a, b, c \in \mathcal{A} \).

**Proposition 2.5.7.** Let \( \mathcal{A} \) be a unital Jordan algebra and let \( \varphi \) be a state on \( \mathcal{A} \). If \( a, b \in \mathcal{A} \), then

\[
\varphi(a \circ b)^2 \leq \varphi(a^2) \varphi(b^2).
\]

Let us note that the preceding proposition is a version of Cauchy-Schwarz inequality for the states on Jordan algebras.

**Definition 2.5.8.** Two elements \( a \) and \( b \) in a Jordan algebra \( \mathcal{A} \) are called orthogonal if \( a \circ b = 0 \). An element \( s \) of a unital Jordan algebra is called a symmetry, if \( s^2 = 1 \).

**Definition 2.5.9.** Let \( \mathcal{A} \) be a unital Jordan algebra. A spin system in \( \mathcal{A} \) is a collection \( \mathcal{P} \) of at least two symmetries different from \( \pm 1 \) such that \( s \circ t = 0 \) whenever \( s, t \in \mathcal{P} \) and \( s \neq t \).

**Definition 2.5.10.** A unital JB algebra generated as a JB algebra by a spin system is called a spin factor.

A spin factor can be constructed as follows. Given spin system \( \mathcal{P} \) in a unital Jordan algebra \( \mathcal{A} \), let \( \mathcal{H}_0 \) denote the (real) linear span of \( \mathcal{P} \). It is not difficult to show that there is an inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{H}_0 \) such that \( a \circ b = \langle a, b \rangle 1 \). Therefore, \( \mathcal{H}_0 \) is a real pre-Hilbert space. As a consequence of the following proposition, we obtain that \( \mathcal{H} \oplus \mathbb{R}1 \), where \( \mathcal{H} \) is a completion of \( \mathcal{H}_0 \), is a spin factor.
Proposition 2.5.11. Let $\mathcal{H}$ be a real Hilbert space of dimension at least two. Suppose that $A = \mathcal{H} \oplus \mathbb{R}1$ has the norm $\|a + \lambda 1\| = \|a\| + |\lambda|$, where $a \in \mathcal{H}$ and $\lambda \in \mathbb{R}$. Define a product $\circ$ in $A$ by

$$(a + \lambda 1) \circ (b + \mu 1) = (\mu a + \lambda b) + ((a, b) + \lambda \mu)1,$$

where $(\cdot, \cdot)$ is an inner product on $\mathcal{H}$, $a, b \in \mathcal{H}$, and $\lambda, \mu \in \mathbb{R}$. Then $A$ is a unital JB algebra.

It may seem that the spin factors constructed above are very special. However, the next result says that every spin factor arises in this way.

Proposition 2.5.12. For each cardinal number $n \geq 2$ there is, up to isomorphism, a unique spin factor generated by a spin system of cardinality $n$.

If the cardinality of spin system $\mathcal{P}$ is $n < \infty$, then the (real) linear span $\mathcal{H}_0$ of $\mathcal{P}$ is a Hilbert space of dimension $n$ and $\mathcal{H}_0 \oplus \mathbb{R}1$ is a finite dimensional spin factor. We shall denote this spin factor by $V_n$.

Let us look at a concrete realization of the spin factor $V_2$. It is given by the algebra $H_2(\mathbb{R})$ of all real hermitian $2 \times 2$ matrices with the product defined by $a \circ b = \frac{1}{2}(ab + ba)$, for all $a, b \in H_2(\mathbb{R})$. In this case, the spin system is formed, for example, by the Pauli spin matrices

$$s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
Chapter 3

Bell inequalities

In this chapter, we consider the Cirel’son inequality in a general setting of (real and complex) linear spaces. The main goal of this chapter is the study of elements responsible for maximal violation of the (CHSH version of) Bell inequality in the framework of *-algebras and Jordan algebras. In particular, we show that these elements are closely associated with Pauli spin matrices. This fact has some interesting structural consequences which demonstrate that the maximal violation requires a very “nonclassical” position of the subsystems.

Note that the results of Sections 3.2, 3.3, and 3.4 were originally published in the papers [15, 16].

3.1 Introduction

The formulation of so-called Bell inequality was an important milestone in the discussion on hidden variables. Bell showed [10] that correlations between measurements performed on two systems (which are no longer interacting but have interacted in past) have to satisfy a certain inequality if the global system can be described by a local hidden-variable theory. Moreover, he proved that this inequality is violated in the case of quantum theory. Thus the conflict between quantum theory and local hidden-variable theories can be decided by an experiment. A number of experiments were made (see, for example, [5, 6, 34]) and their results agree with the predictions of quantum theory.

Bell inequality can be formulated as follows. Let $X_1$, $X_2$, $Y_1$, and $Y_2$ be random variables on a probability space $(\Omega, \Sigma, \mu)$ taking values in the interval $[-1, 1]$. It is not difficult to show that the expectation value of the “combined” observable $X_1(Y_1 + Y_2) + X_2(Y_1 - Y_2)$ is in the interval $[-2, 2]$. 

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In other words,
\[
\frac{1}{2} \sup |E(X_1(Y_1 + Y_2) + X_2(Y_1 - Y_2))| \leq 1,
\]
where \(E\) denotes the expectation operator and the supremum is taken over all random variables \(X_1, X_2, Y_1,\) and \(Y_2\) on a probability space \((\Omega, \Sigma, \mu)\) taking values in the interval \([-1, 1]\). This is a prototype of a Bell inequality. Note that this is not the original form of the Bell inequality but the form sometimes called the CHSH version of Bell inequality (or CHSH inequality) [26].

Let \(A\) be a unital *-algebra. In the sequel, we shall use the following convention: we write \(a \leq b\) for elements \(a, b \in A\) if \(b - a \in A_+\). In order to generalize the inequality (3.1) to unital *-algebras, we have to replace the expectation operator by a state and the random variables by elements of the set
\[
\{a \in A_{sa} | a^2 \leq 1\}.
\]
This leads naturally to the following definitions.

**Definition 3.1.1.** Let \(A\) and \(B\) be *-subalgebras of a unital *-algebra \(C\). We say that \(x \in C\) is a Bell operator (with respect to the ordered pair \((A, B)\)) if
\[
x = \frac{1}{2} (a_1 (b_1 + b_2) + a_2 (b_1 - b_2)),
\]
where \(a_i \in A\) and \(b_i \in B\) (\(i = 1, 2\)) are self-adjoint elements with \(a_i^2, b_i^2 \leq 1\).
The set of all Bell operators with respect to the ordered pair \((A, B)\) is denoted by \(F(A, B)\).

**Definition 3.1.2.** Let \(A\) and \(B\) be *-subalgebras of a unital *-algebra \(C\) and let \(\varphi\) be a state on \(C\). We say that \(\varphi\) satisfies the Bell inequality (with respect to the ordered pair \((A, B)\)) if
\[
\sup_{x \in F(A, B)} |\varphi(x)| \leq 1.
\]
The inequality (3.2) is called the Bell inequality. The number \(|\varphi(x)|\), where \(x \in F(A, B)\), is called a Bell correlation (in the state \(\varphi\)).

Let us note that, for every self-adjoint element \(a\) of a unital \(C^*\)-algebra, the condition \(a^2 \leq 1\) is equivalent to \(-1 \leq a \leq 1\)
\footnote{Indeed, if \(a^2 \leq 1\), then \(\|a\|^2 = \|a^2\| \leq 1\| \) and so \(-1 \leq -\|a\| \leq a \leq \|a\| \leq 1\). Conversely, if \(-1 \leq a \leq 1\), then \(0 \leq (1 + a)(1 - a) = 1 - a^2\).}. Therefore, if the global algebra \(C\) is an abelian \(C^*\)-algebra, then (3.2) amounts to (3.1).
It is well known that a state $\varphi$ satisfies the Bell inequality in the case of $C^*$-algebras provided that $A$ and $B$ commute and at least one of them is abelian (see, for example, [7]). On the other hand, it was proved [25, 49, 75] that Bell correlations cannot exceed $\sqrt{2}$ whenever the local $C^*$-algebras $A$ and $B$ commute.

**Theorem 3.1.3.** Let $A$ and $B$ be mutually commuting $C^*$-subalgebras of a unital $C^*$-algebra $C$. Then, for any state $\varphi$ on $C$, we have

$$\sup_{x \in F(\mathcal{A}, \mathcal{B})} |\varphi(x)| \leq \sqrt{2}. \quad (3.3)$$

**Proof.** If $\|\varphi|_A\| = 0$, then the required result follows immediately from the Cauchy-Schwarz inequality.

Let $\|\varphi|_A\|$ be nonzero. Without loss of generality, we may assume that $\varphi|_A$ is a state on $A$. For each $b \in B$, the equation $\varphi_b(a) = \varphi(ab)$ defines a linear functional $\varphi_b$ on $A$. Since a product of commuting positive elements is positive, we have $-a \leq ab \leq a$ whenever $a \in A_+$ and $b \in B$ with $-1 \leq b \leq 1$. This ensures that $-\varphi|_A \leq \varphi_b \leq \varphi|_A$ for every $b \in B$ with $-1 \leq b \leq 1$. Let $(\pi_{\varphi}, \mathcal{H}_{\varphi})$ be the GNS representation of $A$ associated to the state $\varphi|_A$ and let $\xi_{\varphi}$ be a unit cyclic vector for $(\pi_{\varphi}, \mathcal{H}_{\varphi})$. It follows from Theorem 2.2.14 that, for each $b \in B$ with $-1 \leq b \leq 1$, there exists a unique element $\tilde{b} \in \pi_{\varphi}(A)'$ with $-\mathbb{1}_{\mathcal{H}_{\varphi}} \leq \tilde{b} \leq \mathbb{1}_{\mathcal{H}_{\varphi}}$ such that $\varphi_b(a) = (\pi_{\varphi}(a)\tilde{b}\xi_{\varphi}, \xi_{\varphi})$ for all $a \in A$.

Suppose that

$$x = \frac{1}{2}(a_1(b_1 + b_2) + a_2(b_1 - b_2)), \quad (\text{where } a_i \in A \text{ and } b_i \in B \ (i = 1, 2) \text{ are self-adjoint elements with } a_i^2, b_i^2 \leq 1.)$$

Let $\tilde{b}_1$ and $\tilde{b}_2$ be elements of $\pi_{\varphi}(A)'$ corresponding to the elements $b_1$ and $b_2$, respectively. Put

$$A = \frac{1}{2} \pi_{\varphi}(a_1 + ia_2) \quad \text{and} \quad B = \frac{1}{2\sqrt{2}}(\tilde{b}_1 + \tilde{b}_2 + i(\tilde{b}_1 - \tilde{b}_2)).$$

Then a straightforward calculation shows that

$$A^*A + AA^* = \frac{1}{2} \pi_{\varphi}(a_1^2 + a_2^2) \leq \pi_{\varphi}(1) \leq \|\varphi|_A\| \mathbb{1}_{\mathcal{H}_{\varphi}} \leq \mathbb{1}_{\mathcal{H}_{\varphi}},$$

$$B^*B + BB^* = \frac{1}{2} (\tilde{b}_1^2 + \tilde{b}_2^2) \leq \mathbb{1}_{\mathcal{H}_{\varphi}}.$$

---

2If $\varphi|_A$ is not a state on $A$, then we consider $\frac{1}{\|\varphi|_A\|} \varphi$ instead of $\varphi$. 

As
\[
\begin{align*}
(A^*B \xi_\varphi, \xi_\varphi) &= \frac{1}{4\sqrt{2}} \left[ (\pi_\varphi(a_1 + a_2) \tilde{b}_1 \xi_\varphi, \xi_\varphi) + (\pi_\varphi(a_1 - a_2) \tilde{b}_2 \xi_\varphi, \xi_\varphi) \right] \\
&\quad + \frac{i}{4\sqrt{2}} \left[ (\pi_\varphi(a_1 - a_2) \tilde{b}_1 \xi_\varphi, \xi_\varphi) - (\pi_\varphi(a_1 + a_2) \tilde{b}_2 \xi_\varphi, \xi_\varphi) \right] \\
&= \frac{1}{4\sqrt{2}} \left[ \varphi((a_1 + a_2)b_1) + \varphi((a_1 - a_2)b_2) \right] \\
&\quad + \frac{i}{4\sqrt{2}} \left[ \varphi((a_1 - a_2)b_1) - \varphi((a_1 + a_2)b_2) \right],
\end{align*}
\]
we obtain
\[
\sqrt{2} |\varphi(x)| = 4 \text{Re} (A^*B \xi_\varphi, \xi_\varphi).
\]
We may assume without loss of generality that \( \text{Re} (B \xi_\varphi, A \xi_\varphi) \geq 0 \). Since \( A \) commutes with \( B \), \( A^*A + AA^* \leq 1_{\varphi(\mathcal{H})} \), and \( B^*B + BB^* \leq 1_{\varphi(\mathcal{H})} \), we have
\[
\sqrt{2} |\varphi(x)| = 4 \text{Re} (A^*B \xi_\varphi, \xi_\varphi) = 2 \text{Re} (B \xi_\varphi, A \xi_\varphi) + 2 \text{Re} (A^* \xi_\varphi, B^* \xi_\varphi)
\]
\[
= \|A \xi_\varphi\|^2 + \|B \xi_\varphi\|^2 - \|(B - A) \xi_\varphi\|^2 + \|A^* \xi_\varphi\|^2 + \|B^* \xi_\varphi\|^2 - \|(A^* - B^*) \xi_\varphi\|^2
\]
\[
\leq ((A^*A + AA^* + B^*B + BB^*) \xi_\varphi, \xi_\varphi) \leq 2.
\]
(3.4)

The previous theorem can also be proved by elementary computation (see [25]). We have presented more complicated proof published in [75] because the analysis of the inequality (3.4) will be used to prove Theorem 5.1.6.

Note that the inequality (3.3) is sometimes called Cirel’son inequality. This inequality gives the upper bound for the violation of Bell inequality in the case of mutually commuting \( C^* \)-subalgebras of a unital \( C^* \)-algebra. We shall see later that this upper bound is valid also in the more general case which leads to the following definition.

**Definition 3.1.4.** Let \( A \) and \( B \) be \( * \)-subalgebras of a unital \( * \)-algebra \( C \) and let \( \varphi \) be a state on \( C \). We say that the Bell inequality is *maximally violated in the state \( \varphi \) if there are elements \( a_i \in \mathcal{A}_{sa} \) and \( b_i \in \mathcal{B}_{sa} \) \( (i = 1, 2) \) with \( a_i^2, b_i^2 \leq 1 \) such that
\[
\frac{1}{2} |\varphi(a_1(b_1 + b_2) + a_2(b_1 - b_2))| = \sqrt{2}.
\]
The elements \( a_1, a_2, b_1, b_2 \) are called *maximal violators* of the Bell inequality in the state \( \varphi \).

The Cirel’son inequality, as well as the Bell inequality, belongs to the set of inequalities which are called *Bell-type inequalities* (or simply *Bell inequalities*).
Let us recall that the Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$ are defined by
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (3.5)

The matrices $\sigma_1, \sigma_2, \sigma_3$ are self-adjoint and satisfy the following canonical relations for all $j, k = 1, 2, 3$:
\[
\sigma_j^2 = 1, \\
\sigma_j \sigma_k + \sigma_k \sigma_j = 0, \quad j \neq k.
\]

On the other hand, it is well known that the identities above characterize the Pauli spin matrices. In general, let $\mathcal{A}$ be a unital $*$-algebra and let $a_1, a_2 \in \mathcal{A}$ be the self-adjoint elements such that
\[
a_1^2 = a_2^2 = 1, \\
a_1 a_2 + a_2 a_1 = 0.
\] (3.6)

It can be shown that the $*$-subalgebra of $\mathcal{A}$ generated by $a_1, a_2$ is $*$-isomorphic to the algebra $M_2(\mathbb{C})$ of two by two complex matrices in such a way that $a_1, a_2, \text{ and } a_3 := -\frac{1}{2}(a_1 a_2 - a_2 a_1)$ correspond to the Pauli spin matrices $\sigma_1, \sigma_2,$ and $\sigma_3$.

**Definition 3.1.5.** Self-adjoint elements $a_1, a_2$ of a unital $*$-algebra are called a realization of Pauli spin matrices if they satisfy (3.6).

The following theorem was proved by Summers and Werner [75]. It explains maximal violation of the Bell inequality for commuting $C^*$-subalgebras of a unital $C^*$-algebra $\mathcal{C}$ containing the unit of $\mathcal{C}$. Let us note that the assumption of commutativity of partial algebras is crucial in the proof.

**Theorem 3.1.6.** Let $\mathcal{A}$ and $\mathcal{B}$ be mutually commuting $C^*$-subalgebras of a unital $C^*$-algebra $\mathcal{C}$ containing the unit of $\mathcal{C}$. Suppose that $\varphi$ is a state on $\mathcal{C}$ which restricts to a faithful state on both $\mathcal{A}$ and $\mathcal{B}$. Let $a_i \in \mathcal{A}$, $b_i \in \mathcal{B}$ $(i = 1, 2)$ be maximal violators of the Bell inequality in the state $\varphi$. Then $a_1, a_2, \text{ as well as } b_1, b_2,$ are a realization of Pauli spin matrices. Moreover, $\varphi$ restricts to a tracial state on the $C^*$-subalgebras generated by $\{a_1, a_2\}$ and $\{b_1, b_2\}$, respectively.

**Proof.** The assumption $\frac{1}{2}|\varphi(a_1(b_1 + b_2) + a_2(b_1 - b_2))| = \sqrt{2}$ implies that the equality occurs in (3.4). Therefore, we have
\[
\|(B - A)\xi_\varphi\| = \|((A^* - B^*)\xi_\varphi\| = 0, \\
((A^* A + A A^* + B^* B + B B^*)\xi_\varphi, \xi_\varphi) = 2.
\]
This ensures that $A\xi_\varphi = B\xi_\varphi$ and $A^*\xi_\varphi = B^*\xi_\varphi$. Moreover, since
\[
(B^*B\xi_\varphi, \xi_\varphi) = (B\xi_\varphi, B\xi_\varphi) = (A\xi_\varphi, A\xi_\varphi) = (A^*A\xi_\varphi, \xi_\varphi)
\]
and similarly
\[
(BB^*\xi_\varphi, \xi_\varphi) = (AA^*\xi_\varphi, \xi_\varphi),
\]
we have
\[
1 = ((A^* + AA^*)\xi_\varphi, \xi_\varphi) = \frac{1}{2}(\pi_\varphi(a_1^2 + a_2^2)\xi_\varphi, \xi_\varphi) = \frac{1}{2}(\pi_\varphi(a_1^2)\xi_\varphi, \xi_\varphi)
\]
\[
+ \frac{1}{2}(\pi_\varphi(a_2^2)\xi_\varphi, \xi_\varphi) \leq \frac{1}{2}\|\pi_\varphi(a_1^2)\xi_\varphi\| + \frac{1}{2}\|\pi_\varphi(a_2^2)\xi_\varphi\| \leq 1,
\]
where we have used the Cauchy-Schwarz inequality and the facts that $a_1^2 \leq 1$, $a_2^2 \leq 1$, and $\|\xi_\varphi\| = 1$. Accordingly, $\pi_\varphi(a_1^2)\xi_\varphi = \alpha_1\xi_\varphi$ and $\pi_\varphi(a_2^2)\xi_\varphi = \alpha_2\xi_\varphi$, where $\alpha_1, \alpha_2 \geq 0$ ($\alpha_1, \alpha_2 \geq 0$ because $(\pi_\varphi(a_j^2)\xi_\varphi, \xi_\varphi) \geq 0$ for $j = 1, 2$). As $\|\pi_\varphi(a_j^2)\xi_\varphi\| = 1$, we infer $\alpha_1 = \alpha_2 = 1$ and so
\[
\pi_\varphi(a_1^2)\xi_\varphi = \pi_\varphi(a_2^2)\xi_\varphi = \xi_\varphi.
\]
Hence, for any $a \in \mathcal{A}$,
\[
\varphi(aa_j^2) = (\pi_\varphi(a)\pi_\varphi(a_j^2)\xi_\varphi, \xi_\varphi) = (\pi_\varphi(a)\xi_\varphi, \xi_\varphi) = \varphi(a), \quad j = 1, 2.
\]
In particular, if $a = 1$, then $\varphi(a_j^2) = \varphi(a_j^2) = \varphi(1)$. Using the faithfulness of $\varphi$ on $\mathcal{A}$, we see that $a_1^2 = a_2^2 = 1$.

The straightforward computation gives $A^2 - A^{*2} = \frac{i}{2}\pi_\varphi(a_1a_2 + a_2a_1)$ and $B^2 - B^{*2} = \frac{i}{2}\pi_\varphi(a_1a_2 - a_2a_1)$. By the same reasoning as for $\pi_\varphi(a_j^2)\xi_\varphi = \pi_\varphi(a_j^2)\xi_\varphi = \xi_\varphi$, we get $\tilde{b}_1^2\xi_\varphi = \tilde{b}_2^2\xi_\varphi = \xi_\varphi$. Hence, as $A\xi_\varphi = B\xi_\varphi$, $A^*\xi_\varphi = B^*\xi_\varphi$, and $A$ commutes with $B$,
\[
\pi_\varphi(a_1a_2 + a_2a_1)\xi_\varphi = 2\frac{i}{2}(A^2 - A^{*2})\xi_\varphi = 2\frac{i}{2}(B^2 - B^{*2})\xi_\varphi = (\tilde{b}_1^2 - \tilde{b}_2^2)\xi_\varphi = 0.
\]
This implies that
\[
\varphi(a(a_1a_2 + a_2a_1)) = 0
\]
for all $a \in \mathcal{A}$. If we put $a = a_1a_2 + a_2a_1$, then $\varphi((a_1a_2 + a_2a_1)^2) = 0$. By the faithfulness of $\varphi$ on $\mathcal{A}$, $a_1a_2 + a_2a_1 = 0$.

We have proved that the equations (3.6) hold for $a_1$ and $a_2$. Therefore, the elements $a_1$ and $a_2$ are a realization of Pauli spin matrices.

Let us show that the restriction of $\varphi$ is a tracial state on the C*-subalgebra generated by $a_1$ and $a_2$. For every $a \in \mathcal{A}$,
\[
\varphi(a(a_1 + ia_2)) = 2(\pi_\varphi(a)A\xi_\varphi, \xi_\varphi) = 2(\pi_\varphi(a)B\xi_\varphi, \xi_\varphi) = 2(\pi_\varphi(a)\xi_\varphi, B^*\xi_\varphi)
\]
\[
= 2(\pi_\varphi(a)\xi_\varphi, A^*\xi_\varphi) = 2(A\pi_\varphi(a)\xi_\varphi, \xi_\varphi) = \varphi((a_1 + ia_2)a)
\]
and similarly

\[ \varphi(a(a_1 - ia_2)) = \varphi((a_1 - ia_2)a). \]

Thus, \( \varphi(aa_1 - a_1a) = \varphi(aa_2 - a_2a) = 0 \) for all \( a \in A \). This ensures that a restriction of \( \varphi \) is a tracial state on the \( C^* \)-subalgebra generated by \( a_1 \) and \( a_2 \).

By the symmetry\(^4\), the same relations as for \( a_1 \) and \( a_2 \) hold for \( b_1 \) and \( b_2 \).

### 3.2 Bell inequalities and linear spaces

As we have seen in Theorem 3.1.3, the upper bound for violation of the Bell inequality is \( \sqrt{2} \) in the case of mutually commuting \( C^* \)-algebras. In this section, we generalize Theorem 3.1.3 to (real and complex) linear spaces with a pseudo inner product and show that the upper bound \( \sqrt{2} \) is valid even in this more general situation. It is also investigated in which cases the bound \( \sqrt{2} \) can be attained.

We shall now recall a few concepts and fix the notation. In the sequel, the symbol \( \mathbb{F} \) denotes either the real field \( \mathbb{R} \) or the complex field \( \mathbb{C} \).

**Definition 3.2.1.** Let \( X \) be a linear space over \( \mathbb{F} \). A **pseudo inner product** on \( X \) is a sesquilinear form \( Q : X \times X \to \mathbb{F} \) such that, for all \( x, y \in X \),

1. \( Q(x, y) = \overline{Q(y, x)} \),
2. \( Q(x, x) \geq 0 \).

A pseudo inner product \( Q : X \times X \to \mathbb{F} \) induces a pseudonorm \( \| \cdot \|_Q \) given by \( \| x \|_Q = \sqrt{Q(x, x)} \), \( x \in X \). The set of all elements \( x \in X \) with \( \| x \|_Q = 0 \) will be denoted by \( N_Q \).

Let us note that the set \( N_Q \) is a linear subspace of \( X \) and it is equal to

\[ X^\perp = \{ x \in X \mid Q(x, y) = 0 \text{ for all } y \in X \}. \]

Indeed, this follows from the Cauchy-Schwarz inequality for a pseudo inner product which states that

\[ |Q(x, y)| \leq \| x \|_Q \| y \|_Q \]

\(^4\varphi(a_1(b_1 + b_2) + a_2(b_1 - b_2)) = \varphi(b_1(a_1 + a_2) + b_2(a_1 - a_2)) \) because \( A \) and \( B \) are mutually commuting \( C^* \)-subalgebra.
for all \( x, y \in X \). Next, if \( \mathbb{F} = \mathbb{R} \), then a pseudo inner product is a symmetric bilinear form such that \( Q(x, x) \geq 0 \) for all \( x \in X \). If \( \mathbb{F} = \mathbb{C} \), then, by the polarization identity, the assumption \( Q(x, x) \geq 0 \) for all \( x \in X \) implies \( Q(x, y) = Q(y, x) \) for all \( x, y \in X \).

**Lemma 3.2.2.** Let \( X \) be a linear space over \( \mathbb{F} \) equipped with a pseudo inner product \( Q \). Then
\[
\|u + v\|_Q + \|u - v\|_Q \leq 2\sqrt{2},
\]
whenever \( \|u\|_Q, \|v\|_Q \leq 1 \). Moreover, the equality in the above inequality occurs if and only if
\[
Q(u, v) + Q(v, u) = 0 \quad \text{and} \quad \|u\|_Q = \|v\|_Q = 1.
\]

**Proof.** Suppose that \( u, v \in X \) with \( \|u\|_Q, \|v\|_Q \leq 1 \). Set
\[
t = Q(u, v) + Q(v, u).
\]
Using the obvious identity
\[
\|u + v\|_Q = \sqrt{Q(u + v, u + v)} = \sqrt{\|u\|_Q^2 + t + \|v\|_Q^2},
\]
we obtain
\[
\left( \|u + v\|_Q + \|u - v\|_Q \right)^2 = \left( \sqrt{\|u\|_Q^2 + t + \|v\|_Q^2} + \sqrt{\|u\|_Q^2 - t + \|v\|_Q^2} \right)^2 \leq \left( \sqrt{2 + t + \sqrt{2 - t}} \right)^2 = 2 \left( 2 + \sqrt{4 - t^2} \right) \leq 8.
\]
From the preceding inequality, we immediately have
\[
\|u + v\|_Q + \|u - v\|_Q \leq 2\sqrt{2}.
\]

It is clear from the above discussion that \( \|u + v\|_Q + \|u - v\|_Q = 2\sqrt{2} \) if and only if \( Q(u, v) + Q(v, u) = 0 \) and \( \|u\|_Q = \|v\|_Q = 1 \).

Let us remark that the previous lemma abstracts the well known geometric fact that the parallelogram, whose sides are less than or equal to one, has the sum of diagonals at most \( 2\sqrt{2} \). The maximum is attained exactly when the parallelogram is the unit square.

---

5This means \( Q(x, y) = Q(y, x) \) for all \( x, y \in X \).
Theorem 3.2.3. Let $X$ be a linear space over $\mathbb{F}$ equipped with a pseudo inner product $Q$. Then
\[
\frac{1}{2} \sup |Q(a_1, b_1 + b_2) + Q(a_2, b_1 - b_2)| \leq \sqrt{2},
\]
where the supremum is taken over elements $a_i, b_i \in X$ ($i = 1, 2$) such that $\|a_i\|_Q, \|b_i\|_Q \leq 1$.

Proof. Employing the Cauchy-Schwarz inequality, we infer that
\[
\begin{align*}
|Q(a_1, b_1 + b_2)| &\leq \|a_1\|_Q \|b_1 + b_2\|_Q \leq \|b_1 + b_2\|_Q, \\
|Q(a_2, b_1 - b_2)| &\leq \|a_2\|_Q \|b_1 - b_2\|_Q \leq \|b_1 - b_2\|_Q.
\end{align*}
\]
According to Lemma 3.2.2, we have
\[
\frac{1}{2} |Q(a_1, b_1 + b_2) + Q(a_2, b_1 - b_2)| \leq \frac{1}{2} \left( \|b_1 + b_2\|_Q + \|b_1 - b_2\|_Q \right) \leq \sqrt{2}.
\]
The proof is completed.

It is apparent that every state $\varphi$ on a unital $*$-algebra $\mathcal{A}$ defines a pseudo inner product $Q_\varphi$ on $\mathcal{A}$ given by $Q_\varphi(a, b) = \varphi(b^*a)$, $a, b \in \mathcal{A}$. Moreover, if $a \in \mathcal{A}$ is self-adjoint with $a^2 \leq 1$, then $\|a\|_{Q_\varphi} = \sqrt{\varphi(a^2)} \leq 1$. Therefore, the inequality (3.7) is stronger than (3.3) and so the previous theorem can be regarded as a generalization of Theorem 3.1.3 to linear spaces over $\mathbb{F}$ equipped with a pseudo inner product.

The next result has some interesting consequences. For example, any maximal violation of the Bell inequality inevitably leads to linear relations between maximal violators. Besides, it turns out that maximal violators have to be nearly orthogonal which reveals the geometric content of Bell inequalities.

Theorem 3.2.4. Suppose that $X$ is a complex linear space endowed with a pseudo inner product $Q$. Let $a_i, b_i$ ($i = 1, 2$) be elements of $X$ with pseudonorms $\|a_i\|_Q, \|b_i\|_Q \leq 1$. If
\[
\frac{1}{2} |Q(a_1, b_1 + b_2) + Q(a_2, b_1 - b_2)| = \sqrt{2},
\]
then the following holds:

(i) $\|a_i\|_Q = \|b_i\|_Q = 1$.

(ii) $\text{Re} Q(b_1, b_2) = 0$. 
(iii) \( \text{Re} Q(a_1, a_2) = 0. \)

(iv) There is a complex unit \( \gamma \) and elements \( n_1, n_2 \in N_Q \) such that
\[
a_1 = \frac{\gamma}{\sqrt{2}}(b_1 + b_2) + n_1,
\]
\[
a_2 = \frac{\gamma}{\sqrt{2}}(b_1 - b_2) + n_2.
\]

**Proof.** Suppose that (3.8) holds. By inspecting the inequalities in the proof of Theorem 3.2.3, we can conclude that \( \|a_1\|_Q = \|a_2\|_Q = 1 \) and
\[
\left| Q(a_1, b_1 + b_2) + Q(a_2, b_1 - b_2) \right| = \left| Q(a_1, b_1 + b_2) \right| + \left| Q(a_2, b_1 - b_2) \right| = \|b_1 + b_2\|_Q + \|b_1 - b_2\|_Q = 2\sqrt{2}.
\] (3.9)

Applying Lemma 3.2.2, we see that \( \|b_1\|_Q = \|b_2\|_Q = 1 \) and \( \text{Re} Q(b_1, b_2) = 0. \)

As
\[
\left| Q(a_1, b_1 + b_2) \right| = \|a_1\|_Q \|b_1 + b_2\|_Q,
\]
we conclude that the vectors \( a_1 \) and \( b_1 + b_2 \) must be collinear in the quotient space \( X/N_Q \). Taking into account that \( a_1 \) and \( b_1 + b_2 \) have nonzero pseudonorms, we induce that there is \( \alpha \in \mathbb{C} \) and \( n_1 \in N_Q \) such that
\[
a_1 = \alpha(b_1 + b_2) + n_1.
\]
Similarly, there exists \( \beta \in \mathbb{C} \) and \( n_2 \in N_Q \) with
\[
a_2 = \beta(b_1 - b_2) + n_2.
\]
The fact that \( \|a_1\|_Q = 1 \) implies
\[
1 = Q(\alpha(b_1 + b_2) + n_1, \alpha(b_1 + b_2) + n_1) = \alpha \overline{\alpha} Q(b_1 + b_2, b_1 + b_2)
\]
\[
= |\alpha|^2 (\|b_1\|_Q^2 + 2\text{Re} Q(b_1, b_2) + \|b_2\|_Q^2) = 2|\alpha|^2.
\]
So \( |\alpha| = \frac{1}{\sqrt{2}} \). In the same way, we can prove that \( |\beta| = \frac{1}{\sqrt{2}} \). Let us write \( \alpha = \frac{\gamma_1}{\sqrt{2}}, \beta = \frac{\gamma_2}{\sqrt{2}} \), where \( \gamma_1 \) and \( \gamma_2 \) are complex units. It is clear that
\[
Q(a_1, b_1 + b_2) = Q(a_1, \sqrt{2\gamma_1}a_1) = \sqrt{2\gamma_1} \|a_1\|_Q^2 = \sqrt{2\gamma_1}.
\]
In the same way,
\[
Q(a_2, b_1 - b_2) = \sqrt{2\gamma_2}.
\]
By (3.9) and previous computations, we have
\[
|\gamma_1 + \gamma_2| = 2.
\]
Easy geometric argument shows that the foregoing equation holds if and only if \( \gamma_1 = \gamma_2. \)

It remains to show (iii). But this follows readily from (i), (ii), and (iv). \( \square \)
Now, we formulate a real version of the previous result. The proof is based on the same arguments as the proof of Theorem 3.2.4 and therefore it is omitted.

**Theorem 3.2.5.** Let $X$ be a real linear space with the pseudo inner product $Q$. Suppose that $a_i, b_i \in X$, $i = 1, 2$, are such that $\|a_i\|_Q, \|b_i\|_Q \leq 1$. If

$$\frac{1}{2} |Q(a_1, b_1 + b_2) + Q(a_2, b_1 - b_2)| = \sqrt{2},$$

then the following statements hold:

(i) $\|a_i\|_Q = \|b_i\|_Q = 1$,

(ii) $Q(a_1, a_2) = Q(b_1, b_2) = 0$,

(iii) there are elements $n_1, n_2 \in N_Q$ such that

$$a_1 = \frac{\alpha}{\sqrt{2}} (b_1 + b_2) + n_1,$$

$$a_2 = \frac{\alpha}{\sqrt{2}} (b_1 - b_2) + n_2,$$

where $\alpha \in \{-1, 1\}$.

### 3.3 Maximal violation in $\ast$-algebras

Applying results from the preceding section to $\ast$-algebras, we obtain interesting assertions concerning maximal violation of the Bell inequality. In particular, we show that maximal violators are closely related to realizations of Pauli spin matrices.

Let us recall that any state $\varphi$ on a unital $\ast$-algebra $\mathcal{C}$ induces a pseudo inner product $Q_{\varphi}$ on $\mathcal{C}$ given by

$$Q_{\varphi}(a, b) = \varphi(b^*a), \quad a, b \in \mathcal{C}.$$

This pseudo inner product amounts to an inner product exactly when $\varphi$ is faithful.

We shall concentrate on the question of when the upper bound $\sqrt{2}$ of Bell correlations is attained in the setting of $\ast$-algebras. The following examples show that the Bell inequality is maximally violated in the case of a Bell operator constructed from realizations of Pauli spin matrices.
Example 3.3.1. Suppose that $a_1$ and $a_2$ are a realization of Pauli spin matrices in a $\ast$-algebra $\mathcal{A}$. Put

$$
\begin{align*}
  b_1 &= \frac{1}{\sqrt{2}}(a_1 + a_2), \\
  b_2 &= \frac{1}{\sqrt{2}}(a_1 - a_2).
\end{align*}
$$

Then $b_1$ and $b_2$ are again a realization of Pauli spin matrices and

$$
\frac{1}{2} (a_1 (b_1 + b_2) + a_2 (b_1 - b_2)) = \frac{1}{2} \left( \sqrt{2}a_1^2 + \sqrt{2}a_2^2 \right) = \sqrt{2} 1.
$$

Consequently,

$$
\frac{1}{2} |\varphi (a_1 (b_1 + b_2) + a_2 (b_1 - b_2))| = \sqrt{2}
$$

for each state $\varphi$ on $\mathcal{A}$ and thus the elements $a_1, a_2, b_1, b_2$ maximally violate the Bell inequality in every state.

Theorem 3.3.2. Let $\varphi$ be a faithful state on a unital $\ast$-algebra $\mathcal{C}$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are $\ast$-subalgebras of $\mathcal{C}$. Let $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ ($i = 1, 2$) be such that $a_i^\ast a_i, b_i^\ast b_i \leq 1$. Assume that

$$
\frac{1}{2} |\varphi ((b_1 + b_2)^* a_1 + (b_1 - b_2)^* a_2)| = \sqrt{2}.
$$

Then the following relations hold:

(i) $a_i^* a_i = b_i^* b_i = 1$,

(ii) $a_i^* a_2 + a_2^* a_1 = 0$,

(iii) $b_i^* b_2 + b_2^* b_1 = 0$,

(iv) there is a complex unit $\alpha$ such that

$$
\begin{align*}
  a_1 &= \alpha \frac{1}{\sqrt{2}}(b_1 + b_2), \\
  a_2 &= \alpha \frac{1}{\sqrt{2}}(b_1 - b_2).
\end{align*}
$$

Proof. As $a_i^* a_i, b_i^* b_i \leq 1$, we have $\|a_i\|_{Q_\varphi}, \|b_i\|_{Q_\varphi} \leq 1$. By Theorem 3.2.4, $\|a_i\|_{Q_\varphi} = \|b_i\|_{Q_\varphi} = 1$. From this, $\varphi(1 - a_i^* a_i) = \varphi(1 - b_i^* b_i) = 0$, which, by faithfulness of $\varphi$, means that $a_i^* a_i = b_i^* b_i = 1$.

Applying Theorem 3.2.4 for the inner product $Q_\varphi$ we immediately see (iv).
Using relations (iv), we can compute
\[ a_1^* a_2 + a_2^* a_1 = \frac{1}{2} \left[ (b_1 + b_2)^* (b_1 - b_2) + (b_1 - b_2)^* (b_1 + b_2) \right] = b_1^* b_1 - b_2^* b_2 = 1 - 1 = 0. \]
The same identity for \( b_i \) follows from relations (iv).

Note that in Theorem 3.3.2 we can rewrite the assumption
\[ \frac{1}{2} |\varphi((b_1 + b_2)^* a_1 + (b_1 - b_2)^* a_2)| = \sqrt{2} \]
into the form
\[ \frac{1}{2} |\varphi(a_1^* (b_1 + b_2) + a_2^* (b_1 - b_2))| = \sqrt{2}. \]

When we restrict to self-adjoint elements, then we obtain that maximal violation of the Bell inequality can occur only in the case of a Bell operator constructed from realizations of Pauli spin matrices.

**Corollary 3.3.3.** Let \( A \) and \( B \) be \(*\)-subalgebras of a unital \(*\)-algebra \( C \). Let \( \varphi \) be a faithful state on \( C \). Suppose that \( a_i \in A \) and \( b_i \in B \) \((i = 1, 2)\) are maximal violators of the Bell inequality in the state \( \varphi \). Then

(i) \( a_i^2 = b_i^2 = 1 \),

(ii) \( a_1 a_2 + a_2 a_1 = b_1 b_2 + b_2 b_1 = 0 \).

Moreover, there is \( \alpha \in \{-1, 1\} \) such that
\[ a_1 = \frac{\alpha}{\sqrt{2}} (b_1 + b_2), \]
\[ a_2 = \frac{\alpha}{\sqrt{2}} (b_1 - b_2). \]

**Proof.** The proof follows immediately from the previous theorem and the remark following Theorem 3.3.2.

It follows from Corollary 3.3.3 and Example 3.3.1 that the Bell inequality is maximally violated in a faithful state if and only if the violators are realizations of Pauli spin matrices from Example 3.3.1.

We shall see later that if the subalgebras commute, one cannot construct maximal violators at any faithful state. The following classical example exhibits the maximal violation of the Bell inequality for mutually commuting copies of the two by two matrix algebra in a (necessarily) nonfaithful state.
Example 3.3.4. Let \( \{e_1, e_2\} \) be the canonical orthonormal basis of \( \mathbb{C}^2 \). We shall consider the matrix algebra \( C = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \) together with two subalgebras \( \mathcal{A} = M_2(\mathbb{C}) \otimes 1 \) and \( \mathcal{B} = 1 \otimes M_2(\mathbb{C}) \). Obviously, \( \mathcal{A} \) and \( \mathcal{B} \) commute. Put

\[
\xi = \frac{e_1 \otimes e_2}{\sqrt{2}} - \frac{e_2 \otimes e_1}{\sqrt{2}}.
\]

Then \( \xi \) is a unit vector which induces a vector state \( \varphi_\xi \) on \( \mathbb{C}(i.e., \varphi_\xi(a) = (a, \xi, \xi) \) for all \( a \in \mathcal{C} \). It can be verified easily that the state \( \varphi_\xi \) restricts to a tracial state on both \( \mathcal{A} \) and \( \mathcal{B} \). Let us set

\[
a_1 = \sigma_1 \otimes 1, \quad b_1 = 1 \otimes \frac{\sigma_1 + i \sigma_2}{\sqrt{2}},
\]

\[
a_2 = \sigma_3 \otimes 1, \quad b_2 = 1 \otimes \frac{\sigma_1 - i \sigma_2}{\sqrt{2}},
\]

where \( \sigma_1 \) and \( \sigma_3 \) are Pauli matrices defined by (3.5). A straightforward computation shows that the maximal violation of the Bell inequality occurs for \( a_1, a_2 \in \mathcal{A} \) and \( b_1, b_2 \in \mathcal{B} \) in the state \( \varphi_\xi \):

\[
\frac{1}{2} |\varphi_\xi(a_1(b_1 + b_2) + a_2(b_1 - b_2))| = \sqrt{2}.
\]

Let us remark that the foregoing example goes back to the famous Bohm’s version of EPR paradox [18, 32]. This construction is also an idealized description of the well known Aspect’s experiment [5, 6] that found the maximal violation of the Bell inequality in nature.

Theorem 3.3.2 has several surprising consequences saying that, in general, the maximal violation of the Bell inequality implies that the subsystems described by \( \mathcal{A} \) and \( \mathcal{B} \) must be in a very nonclassical position. For example, there is no product state across \( \mathcal{A} \) and \( \mathcal{B} \), the subalgebras must have a non-trivial intersection, etc. We summarize the consequences of this kind in Theorem 3.3.7. Before doing so we recall some terminology.

Definition 3.3.5. Let \( \mathcal{A} \) and \( \mathcal{B} \) be \(*\)-subalgebras of a unital \(*\)-algebra \( \mathcal{C} \). A state \( \varphi \) on \( \mathcal{C} \) is called a product state across \( \mathcal{A} \) and \( \mathcal{B} \) if \( \varphi(ab) = \varphi(a)\varphi(b) \) for all \( a \in \mathcal{A} \) and \( b \in \mathcal{B} \). We say that \( \mathcal{A} \) and \( \mathcal{B} \) are strongly maximally correlated in a state \( \varphi \) on \( \mathcal{C} \) if there is an element \( x \in \mathcal{F}(\mathcal{A}, \mathcal{B}) \) such that \( |\varphi(x)| = \sqrt{2} \).

Proposition 3.3.6. Let \( \mathcal{A} \) and \( \mathcal{B} \) be \(*\)-subalgebras of a unital \(*\)-algebra \( \mathcal{C} \). Then every product state across \( \mathcal{A} \) and \( \mathcal{B} \) satisfies the Bell inequality.

Proof. Suppose that \( a_i \in \mathcal{A}_{sa} \) and \( b_i \in \mathcal{B}_{sa} \) (\( i = 1, 2 \)) are such that \( a_i^2, b_i^2 \leq 1 \). Let \( \varphi \) be a product state across \( \mathcal{A} \) and \( \mathcal{B} \). Then

\[
[\varphi(a_1(b_1 + b_2) + a_2(b_1 - b_2))]^2 = \varphi(a_1)^2\varphi(b_1 + b_2)^2 + \varphi(a_2)^2\varphi(b_1 - b_2)^2 + 2\varphi(a_1)\varphi(a_2)\varphi(b_1 + b_2)\varphi(b_1 - b_2).
\]
By Cauchy-Schwarz inequality,
\[ \varphi(a_i)^2 = \varphi(a_i^2)\varphi(1) \leq 1, \]
and similarly \( \varphi(b_i)^2 \leq 1. \) Therefore,
\[
\left[ \varphi(a_1(b_1 + b_2) + a_2(b_1 - b_2)) \right]^2 \leq \varphi(b_1 + b_2)^2 + \varphi(b_1 - b_2)^2 \\
+ 2\varphi(a_1)\varphi(a_2) \left[ \varphi(b_1)^2 - \varphi(b_2)^2 \right] \\
= 2\varphi(b_1)^2 + 2\varphi(b_2)^2 \\
+ 2\varphi(a_1)\varphi(a_2) \left[ \varphi(b_1)^2 - \varphi(b_2)^2 \right] \\
= 2\varphi(b_1)^2 [1 + \varphi(a_1)\varphi(a_2)] \\
+ 2\varphi(b_2)^2 [1 - \varphi(a_1)\varphi(a_2)].
\]
As \( |\varphi(a_1)\varphi(a_2)| \leq 1, \)
\[
\left[ \varphi(a_1(b_1 + b_2) + a_2(b_1 - b_2)) \right]^2 \leq 2 [1 + \varphi(a_1)\varphi(a_2) + 1 - \varphi(a_1)\varphi(a_2)] = 4.
\]
Hence
\[
\frac{1}{2} |\varphi(a_1(b_1 + b_2) + a_2(b_1 - b_2))| \leq 1.
\]

\[\square\]

**Theorem 3.3.7.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be \(*\)-subalgebras of a unital \(*\)-algebra \( \mathcal{C} \). Suppose that \( \mathcal{C} \) admits a faithful state. The following conditions are equivalent:

(i) \( \mathcal{A} \) and \( \mathcal{B} \) are strongly maximally correlated in some faithful state \( \varphi \) on \( \mathcal{C} \).

(ii) \( \mathcal{A} \) and \( \mathcal{B} \) are strongly maximally correlated in every state \( \varphi \) on \( \mathcal{C} \).

(iii) \( \sqrt{2} \mathbf{1} \in \mathcal{F}(\mathcal{A}, \mathcal{B}) \).

(iv) \( \mathcal{A} \cap \mathcal{B} \) contains a unital subalgebra \(*\)-isomorphic to \( M_2(\mathbb{C}) \).

Moreover, if any one of the previous equivalent conditions holds, then the following statements are true:

(v) The intersection of \( \mathcal{A} \) and \( \mathcal{B} \) is nontrivial.

(vi) \( \mathcal{A} \) and \( \mathcal{B} \) cannot commute.

(vii) There is no product state across \( \mathcal{A} \) and \( \mathcal{B} \).
Proof. Let us show that (i) implies (iv). If (i) is true then, by Corollary 3.3.3, there are realizations of Pauli spin matrices $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$ such that $a_1 = \frac{1}{\sqrt{2}}(b_1 + b_2)$ and $a_2 = \frac{1}{\sqrt{2}}(b_1 - b_2)$. Therefore, the elements $a_1, a_2$ generate a unital *-subalgebra in $\mathcal{A} \cap \mathcal{B}$ which is *-isomorphic to $M_2(\mathbb{C})$.

(iv) $\Rightarrow$ (iii). If the intersection of algebras $\mathcal{A}$ and $\mathcal{B}$ contains as a unital subalgebra a copy of $M_2(\mathbb{C})$ we can show easily (see Example 3.3.1) that $\sqrt{2}1$ is in $\mathcal{F}(\mathcal{A}, \mathcal{B})$.

The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious.

Further, (v) and (vi) are obvious consequences of (iv).

The statement (vii) follows immediately from (iii) and Proposition 3.3.6. 

Let us note that the assumption on the existence of a faithful state is satisfied if $\mathcal{C}$ is a $C^*$-algebra acting on a separable Hilbert space.

As we have seen, the maximal violation of the Bell inequality in some faithful state is not compatible with the mutual commutativity of the individual subalgebras. Thereby in the case when the algebras commute the global state maximally violating the Bell inequality cannot be faithful. However, if we assume the faithfulness only on the partial algebras, then we can again obtain realizations of Pauli spin matrices. Moreover, as a general fact we show that the restriction of a global state has to act as a tracial state on the subalgebras generated by the realizations of Pauli spin matrices. This was shown in the context of $C^*$-algebras by Summers and Werner [75] (see Theorem 3.1.6). In order to generalize this result to *-algebras only, we shall need the following definition.

Definition 3.3.8. Let $(\mathcal{A}, \mathcal{B})$ be a pair of *-subalgebras of a unital *-algebra $\mathcal{C}$. A state $\varphi$ on $\mathcal{C}$ is said to be weakly uncoupled across $(\mathcal{A}, \mathcal{B})$ if, for all $a \in \mathcal{A}$ and $b, c \in \mathcal{B}$, we have

$$\varphi(abc) = \varphi(bac).$$

(3.10)

It should be pointed out that the definition of weakly uncoupled state across $(\mathcal{A}, \mathcal{B})$ is not symmetric under interchange of partial *-subalgebras $\mathcal{A}$ and $\mathcal{B}$. It is also clear that if $\mathcal{B}$ contains the unit of $\mathcal{C}$, then, for any weakly uncoupled state across $(\mathcal{A}, \mathcal{B})$, we automatically have $\varphi(ab) = \varphi(ba)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Let us remark that if the algebras $\mathcal{A}$ and $\mathcal{B}$ commute then any state on $\mathcal{C}$ is weakly uncoupled across $(\mathcal{A}, \mathcal{B})$.

In the paper [15], we defined the weakly uncoupled state across $(\mathcal{A}, \mathcal{B})$ by the condition

$$\varphi(abc) = \varphi(cab),$$

(3.11)
where \( a \in \mathcal{A} \) and \( b, c \in \mathcal{B} \), and the condition (3.10) was given only as an alternative to (3.11). However, the condition (3.11) clearly implies that the state \( \varphi \) has to be a tracial state on the \(*\)-subalgebra \( \mathcal{B} \) whenever the algebra \( \mathcal{A} \) contains the unit of \( \mathcal{C} \). Therefore, in the Definition 3.3.8, we use the condition (3.10) which does not imply such restrictive consequences.

**Lemma 3.3.9.** Let \( \varphi \) be a state on a unital \(*\)-algebra \( \mathcal{C} \). Suppose that \( n \) is a self-adjoint element of \( N_{Q_\varphi} = \{ n \in \mathcal{C} \mid \varphi(n^*n) = 0 \} \). Then

\[
\varphi(na) = \varphi(an) = 0
\]

for all \( a \in \mathcal{C} \). Moreover, if \( m \in N_{Q_\varphi} \), then \( m^*m \in N_{Q_\varphi} \).

**Proof.** From the Cauchy-Schwarz inequality, we have

\[
0 \leq |\varphi(na)|^2 = |\varphi(n^*a)|^2 \leq \varphi(n^*n)\varphi(a^*a) = 0.
\]

Since \( |\varphi(an)| = |\varphi(n^*a^*)| = |\varphi(na^*)| \), we obtain \( \varphi(an) = 0 \).

If \( m \in N_{Q_\varphi} \), then, by Cauchy-Schwarz inequality,

\[
0 \leq \varphi(m^*mm^*m)^2 \leq \varphi(m^*m)\varphi(m^*mm^*mm^*m) = 0
\]

and so \( m^*m \in N_{Q_\varphi} \).

**Theorem 3.3.10.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be \(*\)-subalgebras of a unital \(*\)-algebra \( \mathcal{C} \) containing the unit of \( \mathcal{C} \). Let \( \varphi \) be a weakly uncoupled state on \( \mathcal{C} \) across \((\mathcal{A}, \mathcal{B})\). If \( a_i \in \mathcal{A} \) and \( b_i \in \mathcal{B} \) (\( i = 1, 2 \)) are maximal violators of the Bell inequality in the state \( \varphi \), then

(i) \( \varphi(a_1^2 c) = \varphi(c) \),

(ii) \( \varphi(b_1^2 c) = \varphi(c) \),

(iii) \( \varphi((a_1 a_2 + a_2 a_1)a) = \varphi((b_1 b_2 + b_2 b_1)b) = 0 \)

for all \( a \in \mathcal{A} \), \( b \in \mathcal{B} \), and \( c \in \mathcal{C} \). Moreover, \( \varphi \) restricts to a tracial state on the unital \(*\)-subalgebras generated by \( \{1, a_1, a_2\} \) and \( \{1, b_1, b_2\} \), respectively.

**Proof.** As, by Theorem 3.2.4 \( \|a_i\|_{Q_\varphi} = 1 \), we have \( 1 - a_i^2 = x_i^*x_i \), where \( i = 1, 2 \) and \( x_i \in N_{Q_\varphi} \). Thus \( \varphi(a_i^2 c) = \varphi(c - x_i^*x_i c) \) and so, applying Lemma 3.3.9, we obtain \( \varphi(a_i^2 c) = \varphi(c) \). Obviously, the same relation holds for \( b_i \).

From Theorem 3.2.4 we have

\[
a_1 = \frac{\gamma}{\sqrt{2}}(b_1 + b_2) + n_1,
\]

\[
a_2 = \frac{\gamma}{\sqrt{2}}(b_1 - b_2) + n_2,
\]

(3.12)
where \( n_1, n_2 \in N_{Q_\varphi} \) and \( \gamma \) is a complex unit. Let us show that \( n_1 \) and \( n_2 \) are self-adjoint elements. Since \( \varphi \) is weakly uncoupled state across \( (\mathcal{A}, \mathcal{B}) \) and \( n_1 = a_1 - \frac{\gamma}{\sqrt{2}} (b_1 + b_2) \),

\[
\varphi(n_1 n_1^*) = \varphi(a_1^2) + 2 \varphi((b_1 + b_2)^2) - \frac{\gamma}{\sqrt{2}} \varphi(a_1 (b_1 + b_2)) - \frac{\gamma}{\sqrt{2}} \varphi((b_1 + b_2)a_1) \\
= \varphi(a_1^2) + 2 \varphi((b_1 + b_2)^2) - \frac{\gamma}{\sqrt{2}} \varphi((b_1 + b_2)a_1) - \frac{\gamma}{\sqrt{2}} \varphi(a_1 (b_1 + b_2)) \\
= \varphi(n_1^* n_1) = 0
\]

and so \( n_1^* \in N_{Q_\varphi} \). Since \( N_{Q_\varphi} \) is a linear subspace of \( \mathcal{C} \), \( n_1 = m_1 + im_2 \), where \( m_1, m_2 \in \mathcal{C}_{sa} \cap N_{Q_\varphi} \). As \( a_1 \) is self-adjoint,

\[
a_1 = \frac{\gamma - \sqrt{2}}{\sqrt{2}} (b_1 + b_2) + m_1 + im_2 = \frac{\gamma}{\sqrt{2}} (b_1 + b_2) + m_1 - im_2.
\]

Hence

\[
\frac{\gamma - \sqrt{2}}{\sqrt{2}} (b_1 + b_2) = -2im_2.
\]

If \( \gamma - \sqrt{2} \neq 0 \), then \( b_1 + b_2 \in N_{Q_\varphi} \) which is a contradiction with \( \varphi(a_1^* a_1) = 1 \). Therefore, \( \gamma = \sqrt{2} \) and so \( n_1 \) and \( n_2 \) are self-adjoint elements. Using relations (3.12), we can compute

\[
(a_1 a_2 + a_2 a_1) a = (b_1^2 - b_2^2) a + \frac{\gamma}{\sqrt{2}} [(b_1 + b_2) n_2 + (b_1 - b_2) n_1] a \\
+ \frac{\gamma}{\sqrt{2}} [n_1 (b_1 - b_2) + n_2 (b_1 + b_2)] a + n_1 n_2 a + n_2 n_1 a.
\]

Lemma 3.3.9 clearly implies

\[
\varphi(n_1 (b_1 - b_2) a + n_2 (b_1 + b_2) a) = \varphi(n_1 n_2 a + n_2 n_1 a) = 0,
\]

\[
\varphi((b_1^2 - b_2^2) a) = \varphi((y_1^* y_1 - y_2^* y_2) a) = 0,
\]

where we have used the fact that, by \( \|b_1\|_{Q_\varphi} = 1 \), we have \( 1 - b_1^2 = y_1^* y_1 \) for some \( y_1 \in N_{Q_\varphi} \). Moreover,

\[
\varphi((b_1 + b_2) a_2 a + (b_1 - b_2) a_1 a) = \varphi((b_1 + b_2) n_2 a + (b_1 - b_2) n_1 a) \\
+ \frac{\gamma}{\sqrt{2}} \varphi((b_1 + b_2) (b_1 - b_2) a + (b_1 - b_2) (b_1 + b_2) a) \\
= \varphi((b_1 + b_2) n_2 a + (b_1 - b_2) n_1 a),
\]
and, as $\varphi$ is weakly uncoupled state across $(\mathcal{A}, \mathcal{B})$,

$$\varphi((b_1 + b_2)a_2a + (b_1 - b_2)a_1a) = \varphi(a_2a(b_1 + b_2) + a_1a(b_1 - b_2))$$

$$= \frac{\gamma}{\sqrt{2}}\varphi((b_1 - b_2)a(b_1 + b_2) + (b_1 + b_2)a(b_1 - b_2))$$

$$= \gamma\sqrt{2}\varphi(a(b_1^2 - b_2^2)) = \gamma\sqrt{2}\varphi((b_1^2 - b_2^2)a) = 0.$$ 

Hence $\varphi((a_1a_2 + a_2a_1)a) = 0$ for all $a \in \mathcal{A}$. For similar reasons, we obtain $\varphi((b_1b_2 + b_2b_1)b) = 0$ for all $b \in \mathcal{B}$.

Since $\varphi$ is a weakly uncoupled across $(\mathcal{A}, \mathcal{B})$, we have

$$\varphi(a_1a) = \frac{\gamma}{\sqrt{2}}\varphi((b_1 + b_2)a) + \varphi(n_1a) = \frac{\gamma}{\sqrt{2}}\varphi(a(b_1 + b_2)) = \varphi(aa_1) - \varphi(an_1)$$

$$= \varphi(aa_1).$$

In the same way, we can prove that $\varphi(a_2a) = \varphi(aa_2)$. Therefore, $\varphi$ is a tracial state on the unital *-subalgebra generated by $\{1, a_1, a_2\}$ and for the same reason we obtain that $\varphi$ is a tracial state on the unital *-subalgebra generated by $\{1, b_1, b_2\}$.

**Corollary 3.3.11.** Let $\mathcal{A}$ and $\mathcal{B}$ be *-subalgebras of a unital *-algebra $\mathcal{C}$ containing the unit of $\mathcal{C}$. Let $\varphi$ be a weakly uncoupled state on $\mathcal{C}$ across $(\mathcal{A}, \mathcal{B})$ which restricts to a faithful state on $\mathcal{A}$ and $\mathcal{B}$. If $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ $(i = 1, 2)$ are maximal violators of the Bell inequality in the state $\varphi$, then

(i) $a_1^2 = b_1^2 = 1$,

(ii) $a_1a_2 + a_2a_1 = b_1b_2 + b_2b_1 = 0$.

Moreover, $\varphi$ restricts to a tracial state on the unital *-subalgebras generated by $\{a_1, a_2\}$ and $\{b_1, b_2\}$, respectively.

**Proof.** The proof follows immediately from the previous theorem and faithfulness of $\varphi$ on $\mathcal{A}$ and $\mathcal{B}$. 

3.4 Maximal violation in Jordan algebras

In this section, we formulate and study the Bell inequality and its maximal violation in the context of Jordan algebras. We show that maximal Bell correlations are naturally connected with the Jordan structure and prove that spin factors are responsible for maximal correlations of the subsystems.

The Bell inequality is defined for Jordan algebras in a similar manner as for *-algebras.
Definition 3.4.1. Let \(\mathcal{A}\) and \(\mathcal{B}\) be Jordan subalgebras of a unital Jordan algebra \(\mathcal{C}\). We say that \(x \in \mathcal{C}\) is a Bell operator (with respect to \(\mathcal{A}\) and \(\mathcal{B}\)) if
\[
x = \frac{1}{2} (a_1 \circ (b_1 + b_2) + a_2 \circ (b_1 - b_2)),
\]
where \(a_i \in \mathcal{A}\) and \(b_i \in \mathcal{B}\) \((i = 1, 2)\) satisfy \(a_i^2, b_i^2 \leq 1\). The set of all Bell operators with respect to \(\mathcal{A}\) and \(\mathcal{B}\) is denoted by \(\mathcal{F}(\mathcal{A}, \mathcal{B})\).

Since the Jordan product \(\circ\) is commutative, we have \(\mathcal{F}(\mathcal{A}, \mathcal{B}) = \mathcal{F}(\mathcal{B}, \mathcal{A})\). Furthermore, in the case of unital JB algebras, the condition \(a^2 \leq 1\) is equivalent to the inequality \(-1 \leq a \leq 1\).

Definition 3.4.2. Let \(\mathcal{A}\) and \(\mathcal{B}\) be Jordan subalgebras of a unital Jordan algebra \(\mathcal{C}\) and let \(\varphi\) be a state on \(\mathcal{C}\). We say that \(\varphi\) satisfies Bell inequality (with respect to \(\mathcal{A}\) and \(\mathcal{B}\)) if
\[
\sup_{x \in \mathcal{F}(\mathcal{A}, \mathcal{B})} |\varphi(x)| \leq 1.
\]
(3.13)
The inequality (3.13) is called the Bell inequality. The number \(|\varphi(x)|\), where \(x \in \mathcal{F}(\mathcal{A}, \mathcal{B})\), is called a Bell correlation (in the state \(\varphi\)).

Suppose for a moment that \(\mathcal{C}\) is an associative JB algebra. Then \(\mathcal{C}\) is the self-adjoint part of an abelian \(C^*\)-algebra and as it is well known the Bell inequality is satisfied for all states on \(\mathcal{C}\). We have showed that, for a pair of nonabelian \(C^*\)-subalgebras of a unital \(C^*\)-algebra, there is the upper bound \(\sqrt{2}\) of Bell correlations which can be attained. Since any state \(\varphi\) on a unital Jordan algebra induces naturally a pseudo inner product \(Q_\varphi\) on this algebra given by
\[
Q_\varphi(a, b) = \varphi(a \circ b),
\]
we have, by Theorem 3.2.3, that \(\sqrt{2}\) is the upper bound for Bell correlations also in the case of Jordan algebras.

Definition 3.4.3. Let \(\mathcal{A}\) and \(\mathcal{B}\) be Jordan subalgebras of a unital Jordan algebra \(\mathcal{C}\) and let \(\varphi\) be a state on \(\mathcal{C}\). We say that the Bell inequality is maximally violated in the state \(\varphi\) if there are elements \(a_i \in \mathcal{A}\) and \(b_i \in \mathcal{B}\) \((i = 1, 2)\) with \(a_i^2, b_i^2 \leq 1\) such that
\[
\frac{1}{2} |\varphi(a_1 \circ (b_1 + b_2) + a_2 \circ (b_1 - b_2))| = \sqrt{2}.
\]
The elements \(a_1, a_2, b_1, b_2\) are called maximal violators of the Bell inequality in the state \(\varphi\).
The next example shows typical maximal violation of the Bell inequality in the Jordan case.

**Example 3.4.4.** Let \( \{s_1, s_2\} \) be the spin system in the spin factor \( V_2 \). Let
\[
  u_1 = \frac{1}{\sqrt{2}} (s_1 + s_2),
\]
\[
  u_2 = \frac{1}{\sqrt{2}} (s_1 - s_2).
\]
It is straightforward to check that
\[
  s_1 \circ (u_1 + u_2) + s_2 \circ (u_1 - u_2) = 2\sqrt{2} \mathbf{1}.
\]
Therefore, for any state \( \varphi \) on \( V_2 \), we have
\[
  \frac{1}{2} \varphi (s_1 \circ (u_1 + u_2) + s_2 \circ (u_1 - u_2)) = \sqrt{2}.
\]
So the maximal violation occurs in every state on \( V_2 \).

Employing Theorem 3.2.5, we can describe precisely maximal violators of the Bell inequality in the context of Jordan algebras.

**Theorem 3.4.5.** Let \( A \) and \( B \) be Jordan subalgebras of a unital Jordan algebra \( C \). Let \( \varphi \) be a faithful state on \( C \). Suppose that \( a_i \in A \) and \( b_i \in B \) \((i = 1, 2)\) are maximal violators of the Bell inequality in the state \( \varphi \). Then \( a_1, a_2 \) and \( b_1, b_2 \) are orthogonal symmetries. Moreover, there is \( \alpha \in \{-1, 1\} \) such that
\[
  a_1 = \frac{\alpha}{\sqrt{2}} (b_1 + b_2),
\]
\[
  a_2 = \frac{\alpha}{\sqrt{2}} (b_1 - b_2).
\]

**Proof.** According to Theorem 3.2.5, we see that \( \varphi (a_i^2) = \varphi (b_i^2) = 1 \), which, by faithfulness of \( \varphi \), implies that \( a_i^2 = b_i^2 = \mathbf{1} \). Further, as \( N_{Q_\varphi} = \{0\} \),
\[
  a_1 = \frac{\alpha}{\sqrt{2}} (b_1 + b_2),
\]
\[
  a_2 = \frac{\alpha}{\sqrt{2}} (b_1 - b_2),
\]
where \( \alpha \in \{-1, 1\} \). By this,
\[
  a_1 \circ a_2 = \frac{1}{2} (b_1^2 - b_2^2) = 0.
\]
By symmetry, \( b_1 \circ b_2 = 0 \) as well. \( \square \)
Definition 3.4.6. We say that Jordan subalgebras \( A \) and \( B \) of a unital Jordan algebra \( C \) are strongly maximally correlated in the state \( \varphi \) if there are maximal violators \( a_i \in A, b_i \in B, i = 1, 2 \), of the Bell inequality in the state \( \varphi \).

In the next theorem, we shall identify the spin factor \( V_2 \) as the algebra responsible for the maximal violation of the Bell inequality in a faithful state.

**Theorem 3.4.7.** Let \( A \) and \( B \) be Jordan subalgebras of a unital Jordan algebra \( C \) containing the unit of \( C \). Suppose that \( C \) admits a faithful state. The following conditions are equivalent:

(i) \( A \) and \( B \) are strongly maximally correlated in some faithful state.

(ii) \( A \) and \( B \) are strongly maximally correlated in every state.

(iii) \( \sqrt{2} \mathbf{1} \in F(A, B) \).

(iv) \( A \cap B \) contains a Jordan subalgebra isomorphic to \( V_2 \).

**Proof.** Let us show that (i) implies (iv). By Theorem 3.4.5, \( A \cap B \) contains the spin system of cardinality two and so it contains a Jordan subalgebra isomorphic to the spin factor \( V_2 \).

The statement (iii) follows from (iv) directly by Example 3.4.4.

The remaining implications (iii) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (i) are trivial.

One of the consequences of the previous theorem is that subalgebras \( A \) and \( B \) which are strongly maximally correlated cannot operator commute for otherwise \( A \cap B \) would be an associative algebra which cannot contain a Jordan subalgebra isomorphic to \( V_2 \). The description of maximal correlation for operator commuting subalgebras will be given below. We start with the model example of maximal violation in this situation which is much more difficult than Example 3.4.4.

**Example 3.4.8.** Let \( H_2(\mathbb{R}) \) be the algebra of all real hermitian \( 2 \times 2 \) matrices with the product defined by \( a \circ b = \frac{1}{2} (ab + ba) \), for all \( a, b \in H_2(\mathbb{R}) \). Let \( C = H_2(\mathbb{R}) \otimes H_2(\mathbb{R}) \) be the tensor product algebra. Put \( A = H_2(\mathbb{R}) \otimes \mathbf{1} \) and \( B = \mathbf{1} \otimes H_2(\mathbb{R}) \). Consider the canonical spin matrices

\[
\begin{align*}
s_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
s_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{align*}
\]

and set

\[
\begin{align*}
a_1 &= s_1 \otimes \mathbf{1}, \\
b_1 &= \mathbf{1} \otimes \frac{s_1 + s_2}{\sqrt{2}}, \\
a_2 &= s_2 \otimes \mathbf{1}, \\
b_2 &= \mathbf{1} \otimes \frac{s_1 - s_2}{\sqrt{2}}.
\end{align*}
\]
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Let $\varphi_\xi$ be a vector state on $\mathcal{C}$ induced by the vector

$$\xi = \frac{(1,0) \otimes (0,1)}{\sqrt{2}} - \frac{(0,1) \otimes (1,0)}{\sqrt{2}}.$$  

It is straightforward to show that

$$\frac{1}{2} |\varphi_\xi (a_1 \circ (b_1 + b_2) + a_2 \circ (b_1 - b_2))| = \sqrt{2}.$$

**Lemma 3.4.9.** Let $\varphi$ be a state on a unital Jordan algebra $\mathcal{A}$. Then $N_{Q_\varphi}$ is a Jordan subalgebra of $\mathcal{A}$.

**Proof.** As

$$N_{Q_\varphi} = \{ n \in \mathcal{A} \mid \varphi (n \circ a) = 0 \text{ for all } a \in \mathcal{A} \},$$

we see that $N_{Q_\varphi}$ is a linear subspace of $\mathcal{A}$. For proving that $N_{Q_\varphi}$ is a subalgebra, we have to show that $N_{Q_\varphi}$ is closed under multiplication. Since

$$a \circ b = \frac{1}{2}((a + b)^2 - a^2 - b^2)$$

for all $a,b \in \mathcal{A}$, it is enough to prove that $n \in N_{Q_\varphi}$ implies $n^2 \in N_{Q_\varphi}$. However,

$$\varphi (n^4) = \varphi (n \circ n^3) = 0$$

by power associativity (see Proposition 2.5.2). Hence $n^2 \in N_{Q_\varphi}$. \hfill \qed

Now, we would like to define so-called uncorrelated state. We are motivated by the following (natural) requirement: if subalgebras $\mathcal{A}$ and $\mathcal{B}$ of a unital Jordan algebra $\mathcal{C}$ operator commute then any state on $\mathcal{C}$ should be uncorrelated with respect to $\mathcal{A}$ and $\mathcal{B}$. Let us note that, from the physical point of view, the operator commutation of $\mathcal{A}$ and $\mathcal{B}$ embodies the fact that any (bounded) observable from the system $\mathcal{A}$ can be measured simultaneously with any (bounded) observable from the system $\mathcal{B}$. Bearing in mind this motivation we define uncorrelated state as follows.

**Definition 3.4.10.** Let $\mathcal{A}$ and $\mathcal{B}$ be Jordan subalgebras of a unital Jordan algebra $\mathcal{C}$. We say that a state $\varphi$ on $\mathcal{C}$ is **uncorrelated across $\mathcal{A}$ and $\mathcal{B}$** if

$$\varphi (a \circ (b \circ c)) = \varphi (b \circ (a \circ c)),$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $c \in \mathcal{A} \cup \mathcal{B}$.

Of course, if $\mathcal{A}$ and $\mathcal{B}$ operator commute, then any state on $\mathcal{C}$ is uncorrelated across $\mathcal{A}$ and $\mathcal{B}$ as we require. Moreover, any tracial state on $\mathcal{C}$ is uncorrelated across any pair of subalgebras. Let us also note that this definition is symmetric under exchanging of subalgebras $\mathcal{A}$ and $\mathcal{B}$. 
Lemma 3.4.11. Let $A$ and $B$ be Jordan subalgebras of a unital Jordan algebra $C$ and let $\varphi$ be a state on $C$ uncorrelated across $A$ and $B$. Then, for all $a, \tilde{a} \in A$ and $b, \tilde{b} \in B$,

(i) $\varphi(a \circ (\tilde{a} \circ b)) = \varphi(\tilde{a} \circ (a \circ b))$,

(ii) $\varphi(b \circ (\tilde{b} \circ a)) = \varphi(\tilde{b} \circ (b \circ a))$.

Proof. Since $\varphi$ is an uncorrelated state on $C$ across $A$ and $B$, we have

$$
\varphi(a \circ (\tilde{a} \circ b)) = \varphi(b \circ (\tilde{a} \circ a)) = \varphi(\tilde{a} \circ (a \circ b)).
$$

In the same way, we obtain (ii). 

Theorem 3.4.12. Let $A$ and $B$ be Jordan subalgebras of a unital Jordan algebra $C$. Let $\varphi$ be a state on $C$ uncorrelated across $A$ and $B$. Suppose that $a_i \in A$ and $b_i \in B$ ($i = 1, 2$) are maximal violators of the Bell inequality in the state $\varphi$. Then, for all $a \in A$, $b \in B$, and $c \in C$,

(i) $\varphi(a_i^2 \circ c) = \varphi(c)$,

(ii) $\varphi(b_i^2 \circ c) = \varphi(c)$,

(iii) $\varphi((a_1 \circ a_2) \circ a) = \varphi((b_1 \circ b_2) \circ b) = 0$,

(iv) $\varphi(a_i) = \varphi(b_i) = 0$.

Proof. By Theorem 3.2.5 we have $(1 - a_i^2) = x_i^2$, where $x_i \in N_{\varphi}$. By Lemma 3.4.9 $x_i^2 \in N_{\varphi}$ and so

$$
0 = \varphi(x_i^2 \circ c) = \varphi((1 - a_i^2) \circ c)
$$

for all $c \in C$. This implies (i). In a similar way, we prove the statement (ii).

Again by Theorem 3.2.5, we may assume, without loss of generality, that

$$
\begin{align*}
    a_1 &= \frac{1}{\sqrt{2}}(b_1 + b_2) + n_1, \\
    a_2 &= \frac{1}{\sqrt{2}}(b_1 - b_2) + n_2,
\end{align*}
$$

where $n_1, n_2 \in N_{\varphi}$. Let us compute the term $\varphi((a_1 \circ a_2) \circ a)$, where $a \in A$. Employing the previous relations, we obtain

$$
\begin{align*}
    \varphi((a_1 \circ a_2) \circ a) &= \frac{1}{2} \varphi((b_1^2 - b_2^2) \circ a) + \frac{1}{\sqrt{2}} \varphi((b_1 + b_2) \circ n_2) \circ a) \\
    &\quad + \frac{1}{\sqrt{2}} \varphi((n_1 \circ (b_1 - b_2)) \circ a) + \varphi((n_1 \circ n_2) \circ a).
\end{align*}
$$
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The first summand on the right hand side of the foregoing equation must be zero by (ii). By virtue of Lemma 3.4.9, \( \varphi((n_1 \circ n_2) \circ a) = 0 \). Using (3.14), (ii), Lemma 3.4.11, and the fact that \( \varphi \) is an uncorrelated state across \( A \) and \( B \), we have

\[
\varphi(((b_1 + b_2) \circ n_2) \circ a) = \varphi(((b_1 + b_2) \circ a_2) \circ a) - \frac{1}{\sqrt{2}} \varphi((b_1^2 - b_2^2) \circ a) \\
= \varphi(a \circ (a_2 \circ (b_1 + b_2))) = \varphi(a_2 \circ (a \circ (b_1 + b_2))) \\
= \frac{1}{\sqrt{2}} \varphi((b_1 - b_2) \circ (a \circ (b_1 + b_2))) \\
= \frac{1}{\sqrt{2}} \varphi(a \circ (b_1^2 - b_2^2)) = 0. \tag{3.15}
\]

Similarly,

\[
\varphi((n_1 \circ (b_1 - b_2)) \circ a) = 0.
\]

Thus

\[
\varphi((a_1 \circ a_2) \circ a) = 0. \tag{3.16}
\]

It follows from (i) and (3.14) that

\[
\varphi(a_1) = \varphi(a_1 \circ (a_2 \circ a_2)) = \frac{1}{\sqrt{2}} \varphi(a_1 \circ ((b_1 - b_2) \circ a_2)) + \varphi(a_1 \circ (n_2 \circ a_2)) \\
= \frac{1}{\sqrt{2}} \varphi((b_1 - b_2) \circ (a_1 \circ a_2)) + \varphi(a_1 \circ (n_2 \circ a_2)).
\]

Since \( \varphi \) is an uncorrelated state across \( A \) and \( B \), let us observe that, thanks to distributivity,

\[
\varphi(a \circ (b \circ x)) = \varphi(b \circ (a \circ x))
\]

whenever \( x \) is in the linear span of \( A \cup B \), \( a \in A \), and \( b \in B \). Accordingly,

\[
\varphi((b_1 + b_2) \circ (a_2 \circ n_2)) = \varphi(a_2 \circ ((b_1 + b_2) \circ n_2)).
\]

By this and (3.15), we clearly obtain \( \varphi(a_1 \circ (a_2 \circ n_2)) = 0 \). It is apparent from (3.16) that \( \varphi((b_1 - b_2) \circ (a_1 \circ a_2)) = 0 \). Therefore, \( \varphi(a_1) = 0 \). Analogously, we obtain \( \varphi(a_2) = 0 \).

By the symmetry, the same relations as for \( a_1 \) hold for \( b_1 \).

**Corollary 3.4.13.** Let \( A \) and \( B \) be operator commuting Jordan subalgebras of a unital Jordan algebra \( C \) containing the unit of \( C \). Let \( \varphi \) be a state on \( C \) which restricts to a faithful state on \( A \) and \( B \). Suppose that \( a_i \in A \), \( b_i \in B \), \( i = 1, 2 \), are maximal violators of the Bell inequality in the state \( \varphi \). Then \( \{a_1, a_2\} \) and \( \{b_1, b_2\} \) are spin systems and \( \varphi \) restricts to the tracial state on Jordan subalgebras generated by \( \{a_1, a_2\} \) and \( \{b_1, b_2\} \), respectively.
Proof. It follows directly from Theorem 3.4.12 and faithfulness of \( \varphi \) on local algebras that \( \{a_1, a_2\} \) and \( \{b_1, b_2\} \) are spin systems. Since \( \{a_1, a_2\} \) and \( \{b_1, b_2\} \) generate the spin factors \( V_2 \) and \( \varphi(a_i) = \varphi(b_i) = 0 \), we have (see [40, Prop. 6.1.7] and [65, Theorem]) that \( \varphi \) restricts to a tracial state on algebras generated by \( \{a_1, a_2\} \) and \( \{b_1, b_2\} \).

The preceding corollary generalizes the result of Summers and Werner [75]. Indeed, if we specify \( A \) and \( B \) to be self-adjoint parts of mutually commuting \( C^* \)-subalgebras of a unital \( C^* \)-algebra, then Corollary 3.4.13 amounts to Theorem 3.1.6.

Let us remark that the conclusion of Corollary 3.4.13 holds also under the assumption that the state \( \varphi \) is an uncorrelated state across \( A \) and \( B \). This condition is weaker than mutual operator commutation of partial algebras. Indeed, consider a (faithful) tracial state \( \tau \) on the spin factor \( V_n \). Then \( \tau \) is uncorrelated across any pair of subalgebras \( A \) and \( B \) of \( V_n \). But it is easy to choose \( A \) and \( B \) not operator commuting.
Chapter 4

Star order

This chapter is concerned with a partial order, called star order, on operator and function algebras. We briefly explore the basic properties of this order. After that, we investigate the star order on partial isometries. The connection between the star order on partial isometries and infinite projections in unital $C^*$-algebras is given. It leads to a new characterization of infinite $C^*$-algebras. The main results concern the infimum and supremum problem (i.e., the question of the existence of the infimum and the supremum) for the star order on the algebra of all continuous complex-valued functions $C(X)$, where $X$ is a Hausdorff topological space. In particular, we show that if $X$ is locally connected or extremely disconnected, then any upper bounded set in $C(X)$ has the infimum and the supremum in the star order.

Note that the results of Sections 4.2, 4.3, 4.4, and 4.5 were originally published in the paper [14].

4.1 Introduction

It is well known that the bounded observables of quantum systems are represented by elements of the set $B(H)_{sa}$ of all bounded self-adjoint operators on a Hilbert space $H$. The set $B(H)_{sa}$ can be endowed with different useful partial orders. Besides the usual order defined by the positive cone and spectral order [39, 52, 62], the next useful partial order is Gudder order [36] defined as follows.

Definition 4.1.1. We say that $a \in B(H)_{sa}$ is less than or equal to $b \in B(H)_{sa}$ in the Gudder order, written $a \leq_G b$, if there exists $c \in B(H)_{sa}$ such that $ac = 0$ and $b = a + c$.

We shall see later (Proposition 4.2.2) that the Gudder order is in fact the
restriction of the star order from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})_{sa}$ which was pointed out by Pulmannová and Vinceková in [68].

Now we introduce a special class of $*$-algebras on which we shall define the star order.

**Definition 4.1.2.** A $*$-algebra $\mathcal{A}$ is said to be proper if $a^*a = 0$ implies $a = 0$ for any $a \in \mathcal{A}$.

Important examples of proper $*$-algebras are a $C^*$-algebra and a $*$-algebra $C(X)$ of all continuous complex-valued functions on a Hausdorff topological space $X$.

The star order was introduced by Drazin [30] in a general context of so-called proper $*$-semigroups. Since a proper $*$-algebra $\mathcal{A}$ carries the multiplicative structure of a proper $*$-semigroup, we can define, following Drazin, a partial order on $\mathcal{A}$ as follows.

**Definition 4.1.3.** Let $\mathcal{A}$ be a proper $*$-algebra. We say that $a \in \mathcal{A}$ is less than or equal to $b \in \mathcal{A}$ in the star order, written $a \preceq b$, if

$$a^*a = a^*b \quad \text{and} \quad aa^* = ba^*.$$ 

We write $a \prec b$ if $a \preceq b$ and $a \neq b$.

Now, let us verify that the star order is really a partial order on proper $*$-algebras.

**Proposition 4.1.4.** The binary relation $\preceq$ on a proper $*$-algebra $\mathcal{A}$ is a partial order.

**Proof.** It is clear that $\preceq$ is a reflexive relation on $\mathcal{A}$.

Let $a, b$ be elements of $\mathcal{A}$ such that $a \preceq b$ and $b \preceq a$. Then $a^*a = a^*b = b^*b$. Hence

$$(a - b)^*(a - b) = a^*a - a^*b - b^*a + b^*b = 0.$$ 

Using the fact that $\mathcal{A}$ is a proper $*$-algebra, we obtain that $a = b$.

Let $a, b, c$ be elements of $\mathcal{A}$ such that $a \preceq b$ and $b \preceq c$. Then a straightforward calculation shows that

$$aa^*a = aa^*b = ab^*b = ab^*c = a^*a,$$

$$a^*aa^* = a^*ab^* = a^*bb^* = a^*bc^* = a^*ac^*.$$ 

Therefore,

$$(a^*a - a^*c)(a^*a - a^*c) = a^*aa^*a - a^*aa^*c - c^*aa^*a + c^*aa^*c = 0,$$

$$(aa^* - ac^*)(aa^* - ac^*) = aa^*aa^* - aa^*ac^* - ca^*aa^* + ca^*ac^* = 0.$$ 

Since $\mathcal{A}$ is a proper $*$-algebra, we have $a^*a = a^*c$ and $aa^* = ca^*$. Thus, we have proved that $a \preceq c$. \qed
4.2 Basic properties

In this section, we summarize the basic properties of the star order. A number of them generalize the well known results for $B(H)$ and matrix algebras. We observe that the star order is preserved by a $*$-homomorphism. This simple observation is especially useful in the case of an abelian $C^*$-algebra $C$ because we can consider a $C^*$-algebra $C_0(X)$, where $X$ is the spectrum of $C$, instead of $C$. This motivates the investigation of the star order on function algebras in last two sections of this chapter. Furthermore, we show that the star order on $C^*$-algebras is well behaved with respect to the continuous functional calculus. A similar result was proved for the star order on $B(H)$ in [3] and also for the Gudder order in [36].

The star order is closely related to the concept of the $*$-orthogonality which was first discovered by Hestenes [42] in the context of matrix algebras.

Definition 4.2.1. Let $A$ be a $*$-algebra. We say that two elements $a, b \in A$ are $*$-orthogonal, written $a \perp b$, if $a^*b = ba^* = 0$.

It is easy to see that the $*$-orthogonality is a symmetric binary relation which, for self-adjoint elements, coincides with the usual notion of orthogonality.

Proposition 4.2.2. Let $A$ be a proper $*$-algebra. If $a, b$ are elements of $A$, then the following conditions are equivalent:

(i) $a \preceq b$.

(ii) There is an element $c$ of $A$ such that $a \perp c$ and $b = c + a$.

Proof. (i) $\Rightarrow$ (ii). If $c = b - a$, then, using $a^*a = a^*b$ and $aa^* = ba^*$, it is clear that $ca^* = a^*c = 0$. Furthermore, $b = a + (b - a) = a + c$.

(ii) $\Rightarrow$ (i). If there is $c$ such that $ca^* = a^*c = 0$ and $b = a + c$, then, multiplying $b = a + c$ by $a^*$ from the left, we obtain $a^*b = a^*a + a^*c = a^*a$. In the same way, we can prove that $aa^* = ba^*$.

Let us note that the equivalence of (i) and (ii) in the previous theorem was first proved by Hestenes [42] for matrix algebras. Further, the condition (ii) is a natural extension of the definition of the Gudder order. The condition (ii) also provides the useful insight into the star order. Loosely speaking, $a \preceq b$ means that $a$ is an orthogonal part of $b$. Moreover, we can see that the second orthogonal part $c = b - a$ of $b$ also satisfies $c \preceq b$ because the $*$-orthogonality is a symmetric relation.

Let $\mathcal{C}$ be a $C^*$-algebra acting on a Hilbert space $\mathcal{H}$. We shall denote the range of operator $a \in \mathcal{C}$ by $\mathcal{R}(a)$ and the corresponding projection onto
the closure $\overline{R(a)}$ by $p_a$. Clearly, the projection onto $\overline{R(a)}$ is an element of the second commutant $C''$ of the $C^*$-algebra $C$. The null space of $a$ will be denoted by $N(a)$. The following proposition is a minor modification of the results known for the Gudder order \cite{Gudder} and the star order on $B(H)$ \cite{Blackadar}.

**Proposition 4.2.3.** Let $C$ be a $C^*$-algebra acting on a Hilbert space $H$ and let $a,b \in C$. Then the following conditions are equivalent:

(i) $a \preceq b$.

(ii) $a\xi = b\xi$ for any $\xi \in \overline{R(a^*)}$ and $a^*\zeta = b^*\zeta$ for any $\zeta \in \overline{R(a)}$.

(iii) $a = bp_a$ and $a^* = b^*p_a$.

**Proof.** (i) $\Rightarrow$ (ii). If $a \preceq b$, then $aa^*\eta = ba^*\eta$ for any $\eta \in H$. Therefore, $a\xi = b\xi$ for all $\xi \in \overline{R(a^*)}$. Analogously, $a^*\zeta = b^*\zeta$ for all $\zeta \in \overline{R(a)}$.

(ii) $\Rightarrow$ (iii). By (ii), we immediately have $ap_a\xi = bp_a\xi$ and $a^*p_a\zeta = b^*p_a\zeta$ for all $\xi \in H$. Further, $ap_a = (p_a^*a)^* = a$ and $a^*p_a = (p_a^*a)^* = a^*$ and so (iii) holds.

(iii) $\Rightarrow$ (i). If (iii) holds, then

\[
a^*b = b^*p_a b = b^*p_a p_a b = a^* a, \quad ba^* = bp_a b^* = bp_a p_a b^* = aa^*.\]

The next proposition describes the elementary properties of the star order. Some of them were published in the case of $A$ being a matrix algebra in \cite{Blackadar}. The proof follows directly from the definition of the star order and will be omitted.

**Proposition 4.2.4.** Let $a,b$ be elements of a proper $*$-algebra $A$.

(i) $a \preceq b$ if and only if $a^* \preceq b^*$.

(ii) $a \preceq b$ if and only if $\lambda a \preceq \lambda b$ for any $\lambda \in \mathbb{C} \setminus \{0\}$.

(iii) If $a \preceq b$ and an element $x \in A$ commutes with $a$ and $b$, then $ax \preceq bx$.

(iv) If $u,v \in A$ are unitary elements, then $a \preceq b$ if and only if $uv \preceq ubv$.

(v) If $a$ is a normal element and $a \preceq b$, then $a^*b = ba^*$. If, in addition, at least one of elements $a$ and $b$ is self-adjoint, then $a$ commutes with $b$.

(vi) If $\Phi : A \to B$ is a $*$-homomorphism between proper $*$-algebras and $a \preceq b$, then $\Phi(a) \preceq \Phi(b)$. 
Further, simple features of the star order will be discussed in a series of the following propositions. In particular, we show that the set of all upper bounds of a given element is convex. Moreover, we discuss the behavior of the star order with respect to the tensor product and the strong operator limit.

Proposition 4.2.5. Let \( a, b, a_1, b_2 \) be elements of a proper \(*\)-algebra \( \mathcal{A} \). If \( a \preceq b_1 \) and \( a \preceq b_2 \), then \( a \preceq \lambda b_1 + (1 - \lambda) b_2 \) for every \( \lambda \in \mathbb{C} \).

Proof. If \( a \preceq b_1 \) and \( a \preceq b_2 \), then
\[
a^*(\lambda b_1 + (1 - \lambda) b_2) = \lambda a^*b_1 + (1 - \lambda)a^*b_2 = \lambda a^*a + (1 - \lambda)a^*a = a^*a.
\]
Similarly, \((\lambda b_1 + (1 - \lambda) b_2)a^* = aa^*\).

Proposition 4.2.6. Let \( C \) be a \( C^\ast \)-algebra. If \( a_i, b_i \in C \) (\( i = 1, 2 \)) and \( a_i \preceq b_i \), then \( a_1 \otimes a_2 \preceq b_1 \otimes b_2 \) in \( C \otimes C \).

Proof. From \( a_i \preceq b_i \) and elementary properties of the tensor product, it follows that
\[
(a_1 \otimes a_2)^*(b_1 \otimes b_2) = (a_1^*b_1) \otimes (a_2^*b_2) = (a_1^*a_1) \otimes (a_2^*a_2) = (a_1 \otimes a_2)^*(a_1 \otimes a_2).
\]
Similarly, \((b_1 \otimes b_2)(a_1 \otimes a_2)^* = (a_1 \otimes a_2)(a_1 \otimes a_2)^*\).

Proposition 4.2.7. Let \( C \) be a \( C^\ast \)-algebra acting on a Hilbert space. Suppose that \((b_\alpha)_{\alpha \in \Lambda} \) is a net of elements from \( C \) whose limit in the strong operator topology is \( b \in C \). If \( a \preceq b_\alpha \) for all \( \alpha \in \Lambda \), then \( a \preceq b \).

Proof. If \( a \preceq b_\alpha \), then \( a^*a = a^*b_\alpha \) and \( aa^* = b_\alpha a^* \). Since the multiplication is separately continuous in the strong operator topology, \( a^*a = a^*b \) and \( aa^* = ba^* \).

Let \( \mathcal{A} \) be a proper \(*\)-algebra. We have seen in Proposition [4.2.4(v)] that if \( a \preceq b \) (\( a, b \in \mathcal{A} \)) and \( a \) is self-adjoint, then \( a \) commutes with \( b \). If we restrict our attention to \( C^\ast \)-algebras, we can strengthen this result even for normal elements. Note that the following proposition and theorem were proved for the case of \( \mathcal{B}(\mathcal{H}) \) in [3]. The extension to \( C^\ast \)-algebras is straightforward. Nevertheless, we give alternative proofs based on \( C^\ast \)-algebraic viewpoint.

Proposition 4.2.8. Let \( a \) and \( b \) be elements of a \( C^\ast \)-algebra \( C \). If \( a \) is normal and \( a \preceq b \), then \( a \) commutes with \( b \).
Proposition 4.2.4(vi) that $\pi$ using this and Proposition 4.2.3(iii), we obtain

Consequently, $a$

**Theorem 4.2.9.** Let $C$ be a unital $C^*$-algebra and let $a$ and $b$ be normal elements of $C$ such that $a \leq b$. Suppose that $f : \sigma(b) \cup \{0\} \to \mathbb{C}$, where $\sigma(b)$ is the spectrum of $b$, is a continuous function satisfying $f(0) = 0$. Then $f(a) \leq f(b)$.

**Proof.** Since $a$ is normal and $a \leq b$, it follows from Proposition 4.2.8 that $a$ and $b$ commute. Therefore $a$, $b$, and the unit element $1$ of $C$ generate a unital abelian $C^*$-algebra which is $*$-isomorphic to an algebra $C(X)$ of all continuous complex-valued functions on a compact Hausdorff topological space $X$. Let $\Phi$ be a corresponding unital $*$-isomorphism from the subalgebra of $C$ onto $C(X)$. Since the spectrum of a continuous function is the range of the function, we obtain from simple properties of the star order on functions (see, for example, Proposition 4.4.1 below) that $\sigma(a) \subseteq \sigma(b) \cup \{0\}$. Further, Proposition 4.2.4(vi) implies that $\Phi(a) \leq \Phi(b)$. In addition, it can be proved (see Proposition 4.4.1) that $\Phi(a) \leq \Phi(b)$ if and only if $\Phi(a) = \chi_{\text{Supp}(\Phi(a))} \Phi(b)$, where $\chi_{\text{Supp}(\Phi(a))}$ is the characteristic function of the set

$$\text{Supp}(\Phi(a)) = \{x \in X|\Phi(a)(x) \neq 0\}.$$  

Using the assumption $f(0) = 0$, we obtain

$$f\big|_{\sigma(a)} \circ \Phi(a) = f\big|_{\sigma(a)} \circ (\chi_{\text{Supp}(\Phi(a))} \Phi(b)) = (f\big|_{\sigma(b)} \circ \Phi(b)) \chi_{\text{Supp}(\Phi(a))}$$

which implies

$$f\big|_{\sigma(a)} \circ \Phi(a) \leq f\big|_{\sigma(b)} \circ \Phi(b).$$

Since $f\big|_{\sigma(a)} \circ \Phi(a) = \Phi(f\big|_{\sigma(a)}(a))$ and $f\big|_{\sigma(b)} \circ \Phi(b) = \Phi(f\big|_{\sigma(b)}(b))$ (see Proposition 2.1.22), we have $\Phi(f\big|_{\sigma(a)}(a)) \leq \Phi(f\big|_{\sigma(b)}(b))$. From this, it follows, by Proposition 4.2.4(vi), that $f(a) \leq f(b)$. 

\[ \square \]
As a direct consequence of the preceding theorem, it can be shown that the map \( a \mapsto |a| = \sqrt{a^*a} \) preserves the star order on a unital \( \mathcal{C}^* \)-algebra \( \mathcal{C} \). The proof is based on the same arguments which have been applied in the case \( \mathcal{B}(\mathcal{H}) \) (see [3]).

**Corollary 4.2.10.** Let \( \mathcal{C} \) be a unital \( \mathcal{C}^* \)-algebra and \( a, b \) be elements of \( \mathcal{C} \). If \( a \preceq b \), then \( |a| \preceq |b| \).

**Proof.** It is easy to see that \( a \preceq b \) implies \( a^*a \preceq b^*b \). By Theorem 4.2.9, we obtain that \( \sqrt{a^*a} \preceq \sqrt{b^*b} \).

---

**4.3 Star order and partial isometries**

Partial isometries play a significant role in the geometry of Hilbert spaces and in the theory of operator algebras. It is well known, for example, that the set of extreme points of the closed unit ball of a unital \( \mathcal{C}^* \)-algebra consists of partial isometries (see Theorem 21.24). Further, partial isometries are important ingredients in the polar decomposition of operators. Using the polar decomposition, it was shown in [3] that the star order on \( \mathcal{B}(\mathcal{H}) \) can be investigated on the set of partial isometries and on the set of positive elements separately. So the star order on partial isometries is an important component of the star order on general elements of \( \mathcal{B}(\mathcal{H}) \). These facts motivate the study of the star order on partial isometries in general proper \( \mathcal{C}^* \)-algebras. The main goal of this section is to give a new characterization of infinite \( \mathcal{C}^* \)-algebras via the star order on partial isometries.

**Proposition 4.3.1.** Let \( u \) be an element of a proper \( \mathcal{C}^* \)-algebra \( \mathcal{A} \). Then the following conditions are equivalent:

(i) \( u \) is a partial isometry.

(ii) \( uu^* \) is a projection.

(iii) \( u^*u \) is a projection.

**Proof.** The implication (i) \( \Rightarrow \) (ii) is obvious.

(ii) \( \Rightarrow \) (iii). Suppose that \( uu^* \) is a projection. Then

\[
(u^*u - u^*uu^*u)(u^*u - u^*uu^*u) = u^*uu^*u - 2u^*(uu^*)^2u + u^*(uu^*)^3u = 0.
\]

Since \( \mathcal{A} \) is a proper \( \mathcal{C}^* \)-algebra, we have \( u^*u = u^*uu^*u \).

(iii) \( \Rightarrow \) (i). Suppose that \( u^*u \) is a projection. Then

\[
(u - uu^*u)(u - uu^*u) = u^*u - 2(u^*u)^2 + (u^*u)^3 = 0.
\]

Hence \( u = uu^*u \). \( \square \)
CHAPTER 4. STAR ORDER

The following statement, first pointed out by Drazin [30], says that any element less than or equal to a partial isometry has to be a partial isometry.

**Proposition 4.3.2.** Let $u, v$ be elements of a proper $*$-algebra. If $u \preceq v$ and $v$ is a partial isometry, then $u$ is also partial isometry.

*Proof.* Since $v$ is a partial isometry, $v = vv^*v$. By $u \preceq v$, we have

$$u^*u = u^*v = u^*v v^*v = u^*uu^*u.$$  

From Proposition 4.3.1 we see that $u$ is a partial isometry. □

We can interpret the next proposition as an analogue of Proposition 4.2.3 for partial isometries in a proper $*$-algebra where the range projections are replaced by initial and final projections.

**Proposition 4.3.3.** Let $u, v$ be partial isometries of a proper $*$-algebra. Then $u \preceq v$ if and only if $u = fv = ve$, where $e = u^*u$ and $f = uu^*$.

*Proof.* If $u \preceq v$, then

$$u = uu^*u = uu^*v = fv,$$

$$u = uu^*u = vv^*u = ve.$$  

Conversely, if $u = fv = ve$, then

$$u^*u = u^*fv = u^*uu^*v = u^*v,$$

$$uu^* = veu^* = vu^*uu^* = vu^*.$$  □

Let us recall that the standard partial order $\leq_P$ on the set $P(A)$ of all projection of a $*$-algebra $A$ is defined by setting $e \leq_P f$ whenever $ef = e$. If $A$ is a proper $*$-algebra, then it follows immediately from the definition of the star order that $\preceq$ and $\leq_P$ coincide on $P(A)$.

**Theorem 4.3.4.** Let $A$ be a proper $*$-algebra. Suppose that $u_i \in A$ ($i = 1, 2$) are partial isometries, $e_i = u_i^*u_i$, and $f_i = u_iu_i^*$. If $u_1 \preceq u_2$, then

$$e_1 \preceq e_2, \quad f_1 \preceq f_2, \quad \text{and} \quad e_2 - e_1 \sim f_2 - f_1.$$  

\(^1\)In the case of $C^*$-algebras, $\leq_P$ and the order $\leq$ given by positive cone coincide on the set of all projections.
**Proof.** From $u_1 \leq u_2$, it follows that $u_1^*u_1 = u_1^*u_2$ and $u_1u_1^* = u_2u_2^*$. By this and Proposition 4.3.3 we have
\[
e_1e_2 = e_1u_2^*u_2 = u_1^*u_2 = u_1^*u_1 = e_1,
\]
\[
f_1f_2 = f_1u_2^2u_2^* = u_1u_2^* = u_1u_1^* = f_1.
\]
Therefore, $e_1 \leq e_2$ and $f_1 \leq f_2$. Using Proposition 4.2.2 and Proposition 4.3.2, we see that $u = u_2 - u_1$ is a partial isometry satisfying $u \leq u_2$.

Now we show that $u^*u = e_2 - e_1$ and $uu^* = f_2 - f_1$. We have
\[
u^*u = (u_2 - u_1)^*(u_2 - u_1) = u_2^*u_2 - u_2^*u_1 - u_1^*u_2 + u_1^*u_1 = u_2^*u_2 - u_1^*u_1 = e_2 - e_1,
\]
\[
u u^* = (u_2 - u_1)(u_2^* - u_1^*) = u_2^*u_2 - u_2u_2^* - u_1u_2^* + u_1^*u_1 = u_2^*u_2 - u_1u_2^* = f_2 - f_1.
\]
Thus $e_2 - e_1 \sim f_2 - f_1$.

In the following theorem, partial isometries which are above a given partial isometry are characterized in terms of initial and final projections. A similar result for partial isometries which are below a fixed partial isometry is proved in Theorem 4.3.6.

**Theorem 4.3.5.** Let $\mathcal{A}$ be a proper $*$-algebra and let $e_i, f_i \in \mathcal{A}$ ($i = 1, 2$) be projections. Suppose that $u_1$ is a partial isometry such that $u_1^*u_1 = e_1$ and $u_1u_1^* = f_1$. Then the following conditions are equivalent:

(i) There is a partial isometry $u_2$ such that $u_2^*u_2 = e_2$, $u_2u_2^* = f_2$, and $u_1 \leq u_2$.

(ii) $e_1 \leq e_2$, $f_1 \leq f_2$, and $e_2 - e_1 \sim f_2 - f_1$.

**Proof.** (i) $\Rightarrow$ (ii) follows immediately from Theorem 4.3.4.

(ii) $\Rightarrow$ (i). If $e_2 - e_1 \sim f_2 - f_1$, then there is a partial isometry $v$ such that $v^*v = e_2 - e_1$ and $vv^* = f_2 - f_1$. Since
\[
(u_1^*v)(u_1^*v) = v^*u_1^*u_1v = v^*f_1v = v^*f_1vv^*v = v^*f_1(v_2 - f_1)v = v^*(f_1 - f_1)v = 0,
\]
\[
(vu_1^*)(vu_1^*) = u_1v^*vu_1^* = u_1(e_2 - e_1)u_1^* = u_1(e_2 - e_1)u_1^* = u_1(e_2 - e_1)u_1^* = 0,
\]
we have $u_2^*v = v u_1^* = 0$. Now we set $u_2 = u_1 + v$. It is easily seen (e.g., by Proposition 4.2.2) that $u_1 \leq u_2$. Moreover,
\[
u_2^*u_2 = (u_1 + v)^*(u_1 + v) = u_1^*u_1 + u_1^*v + v^*u_1 + v^*v = e_1 + e_2 - e_1 = e_2,
\]
\[
u_2u_2^* = (u_1 + v)(u_1 + v)^* = u_1u_1^* + u_1^*v + vu_1^* + vv^* = f_1 + f_2 - f_1 = f_2.
\]
Thanks to this $u_2$ is a partial isometry with required properties. \qed
Theorem 4.3.6. Let $\mathcal{A}$ be a proper $*$-algebra and let $e_i, f_i \in \mathcal{A}$ ($i = 1, 2$) be projections. Suppose that $u_2$ is a partial isometry such that $u_2^*u_2 = e_2$ and $u_2u_2^* = f_2$. Then the following conditions are equivalent:

(i) There is a partial isometry $u_1$ such that $u_1^*u_1 = e_1$, $u_1u_1^* = f_1$, and $u_1 \preceq u_2$.

(ii) $e_1 \preceq e_2$, $f_1 \preceq f_2$, and $u_2e_1 = f_1u_2$.

Proof. (i) $\Rightarrow$ (ii). It follows from Theorem 4.3.4 that $e_1 \preceq e_2$ and $f_1 \preceq f_2$. Since $u_1 \preceq u_2$, we obtain, by Proposition 4.3.3, that $f_1u_2 = u_2e_1$.

(ii) $\Rightarrow$ (i). Let us set $u_1 = u_2e_1$. Then

\[
\begin{align*}
    u_1^*u_1 &= e_1u_2^*u_2e_1 = e_1e_2e_1 = e_1, \\
    u_1u_1^* &= u_2e_1u_2^* = u_2e_1u_2^* = f_1u_2u_2^* = f_1f_2 = f_1.
\end{align*}
\]

Thus $u_1$ is a partial isometry and, by Proposition 4.3.3, $u_1 \preceq u_2$. \hfill \Box

Let us remark that, in the light of Proposition 4.3.2, the previous theorem describes the set of all elements underneath a given partial isometry.

If we apply the following theorem to unital $C^*$-algebras, we obtain the connection between infinite projections and the star order on partial isometries.

Theorem 4.3.7. Let $\mathcal{A}$ be a unital proper $*$-algebra and let $f_i \in \mathcal{A}$ ($i = 1, 2$) be projections. Then the following conditions are equivalent:

(i) $f_2 \sim f_1 < f_2$.

(ii) There are partial isometries $u_i \in \mathcal{A}$ ($i = 1, 2$) such that $u_1 < u_2$, \
\[u_1u_1^* = u_2^*u_2 = f_1, \quad \text{and} \quad u_2u_2^* = f_2.\]

Proof. (i) $\Rightarrow$ (ii). By $f_2 \sim f_1$, there is a partial isometry $u_2$ such that $u_2^*u_2 = f_1$ and $u_2u_2^* = f_2$. Put $e = u_2^*f_1u_2$. Now we show that $e$ is a projection. Indeed, $e^* = e$. Furthermore,

\[
e^2 = u_2^*f_1u_2u_2^*f_1u_2 = u_2^*f_1f_2f_1u_2 = u_2^*f_1u_2.
\]

Thus $e$ is a projection. Further, we show that $e < f_1$. Since

\[ef_1 = u_2^*f_1u_2u_2^*u_2 = u_2^*f_1u_2 = e,
\]

we have $e \preceq f_1$. Now suppose that $f_1 = e$. Then

\[0 = u_2(f_1 - e)u_2^* = u_2u_2^*(1 - f_1)u_2u_2^* = f_2(1 - f_1)f_2 = f_2 - f_1,
\]
which is a contradiction with \( f_1 \neq f_2 \). Therefore, \( f_1 \neq e \) and so \( e \prec f_1 \). Now let us put \( u_1 = u_2e \). It remains to show that \( u_1 \) is a partial isometry with the required properties. We observe that

\[
u_1 = u_2u_2^*f_1 u_2 = f_2 f_1 u_2 = f_1 u_2.
\]

Further,

\[
u_1^*u_1 = eu_2^* u_2 e = ef_1 e = e,
\]

\[
u_1 u_1^* = f_1 u_2 u_2^* f_1 = f_1 f_2 f_1 = f_1.
\]

Hence \( u_1 \) is a partial isometry. Using Proposition \[4.3.3\] we obtain that \( u_1 \preceq u_2 \). Since \( f_1 \prec f_2 \), we have \( u_1 \neq u_2 \).

Now we show that \((ii) \Rightarrow (i)\). From \( u_1 \prec u_2 \) we have, applying Theorem \[4.3.3\] \( f_1 \preceq f_2 \). If we suppose that \( f_1 = f_2 \), then, using Proposition \[4.3.3\] we can compute

\[
u_1 = f_1 u_2 = f_2 u_2 = u_2,
\]

which is a contradiction. Thus \( f_1 \prec f_2 \).

Our result developed in the general context of proper *-algebras may be applied to the comparison theory of \( C^* \)-algebras and von Neumann algebras. This enables us to characterize infiniteness of \( C^* \)-algebras in Murray-von Neumann comparison theory. Let us recall that an element \( u \) of a unital \( C^* \)-algebra is called coisometry if \( uu^* = 1 \). In other words, an element \( u \) of a unital \( C^* \)-algebra is coisometry, if \( u^* \) is isometry.

**Corollary 4.3.8.** A unital \( C^* \)-algebra \( C \) is infinite if and only if there are a partial isometry \( u_1 \in C \) and a coisometry \( u_2 \in C \) such that \( u_1 \prec u_2 \) and \( u_1 u_1^* = u_2^* u_2 \).

Let us note that all isometries and coisometries, in a unital \( C^* \)-algebra \( A \) are extreme points of the closed unit ball \( A_1 \) of \( A \) (see Theorem \[2.1.24\]). Moreover, they are maximal elements of the set of all partial isometries of \( A \) with respect to the star order. \[^2\] A simple observation shows that every extreme point of \( A_1 \) is a maximal element of \( A \). \[^3\] Indeed, let \( u \) be an extreme

\[^2\]This follows immediately from Proposition \[4.3.3\]. By a minor modification of the first part of the proof of Proposition \[4.3.3\] one can prove that they are also maximal elements of \( A \) with respect to the star order.

\[^3\]The reverse implication is not true. It is easy to see that \( 21 \) is a maximal element of \( A \) but it is not an extreme point of \( A_1 \) because \( 21 \) is not a partial isometry. However, it is an open question whether every maximal element of the set of all partial isometries of \( A \) with respect to the star order is an extreme point of \( A_1 \).
point of \( \mathcal{A}_1 \) (and therefore partial isometry) and let \( a \) be an element of \( \mathcal{A} \) such that \( u \preceq a \). Then
\[
(1 - uu^*)a(1 - u^*u) = (a - uu^*)a(1 - u^*u) = (a - u)(1 - u^*u) = (a - au^*u - u + uu^*u) = a - u.
\]
Since \( u \) is an extreme point, we see, by Theorem 2.1.24, that \( u = a \).

### 4.4 Infimum problem for function algebras

The infimum problem (i.e., the question of the existence of an infimum) was solved in the affirmative for the Gudder order in [36, 68]. For example, it was shown that the infimum of two elements in the Gudder order always exists. Similar results concerning infimum were also proved for the star order on \( \mathcal{B}(\mathcal{H}) \) and matrix algebras in [3, 41]. In this section, we investigate the infimum problem for the star order on a *-algebra \( C(X) \) of all continuous complex-valued functions on a Hausdorff topological space \( X \). It is shown that the infimum problem has the positive answer for the *-algebra \( C(X) \) whenever \( X \) is a locally connected or extremely disconnected Hausdorff topological space.

In the sequel, \( C(X) \) denotes a *-algebra of all continuous complex-valued functions on a Hausdorff topological space \( X \). If \( f \in C(X) \), we set
\[
\text{Supp}(f) = \{ x \in X | f(x) \neq 0 \},
\]
\[
\text{Null}(f) = \{ x \in X | f(x) = 0 \}.
\]
The characteristic function of a set \( M \) is denoted by \( \chi_M \). Let us remark that, in the case of \( C(X) \), the definition of \( f \preceq g \ (f, g \in C(X)) \) is reduced to only one equation \( \overline{f}f = \overline{f}g \), where \( \overline{f} \) is the complex conjugate of \( f \). This definition can be expressed in useful equivalent ways which are summarized in the following proposition analogous to Proposition 4.2.2 and Proposition 4.2.3.

A similar result for random variables can be found in [36].

**Proposition 4.4.1.** If \( f, g \in C(X) \), then the following conditions are equivalent:

(i) \( f \preceq g \).

(ii) \( f(x) = g(x) \) for all \( x \in \text{Supp}(f) \).

(iii) \( f = g\chi_{\text{Supp}(f)} \).

(iv) There is a function \( h \in C(X) \) such that \( \overline{f}h = 0 \) and \( g = f + h \).
Proof. (i) ⇔ (iv) is a special case of Proposition 4.2.2.

(i) ⇒ (ii). If \( f \preceq g \), then \( f(x)f(x) = f(x)g(x) \) for any \( x \in X \). Consequently, \( f(x) = g(x) \) for \( x \in \text{Supp}(f) \).

(ii) ⇒ (iii). If \( x \in \text{Supp}(f) \), then
\[
f(x) = g(x) = g(x)\chi_{\text{Supp}(f)}(x).
\]

If \( x \in \text{Null}(f) \), then
\[
f(x) = 0 = g(x)\chi_{\text{Supp}(f)}(x).
\]

Since \( X = \text{Supp}(f) \cup \text{Null}(f) \), we obtain \( f = g\chi_{\text{Supp}(f)} \).

(iii) ⇒ (i). Suppose that \( f = g\chi_{\text{Supp}(f)} \). Then
\[
\overline{f}f = \overline{f}g\chi_{\text{Supp}(f)} = \overline{f}\chi_{\text{Supp}(f)}g = \overline{f}g.
\]

Motivated by the condition (iii) in the preceding proposition, let us concentrate on the question of when the function \( \chi_M f \), where \( f \in C(X) \) and \( M \) is an open subset of \( X \), is continuous.

**Proposition 4.4.2.** Let \( X \) and \( Y \) be topological spaces. If \( X = A \cup B \), where \( A \) and \( B \) are both open (or both closed) in \( X \), and if \( f : X \to Y \) is a function such that both \( f|_A \) and \( f|_B \) are continuous, then \( f \) is continuous.

**Proof.** The proof of this standard topological result can be found, for example, in [84, p. 45].

**Proposition 4.4.3.** Let \( X \) be a Hausdorff topological space and let \( M \subseteq X \) be open. Assume that \( f : X \to \mathbb{C} \) is continuous. Then \( \chi_M f \) is a continuous function if and only if \( f \) vanishes on \( \partial M \).

**Proof.** Suppose that \( \chi_M f \) is a continuous function. Let \( z \) be an element of \( \partial M \). Then there is a net \( (x_\alpha) \) such that \( x_\alpha \in M \) and \( x_\alpha \to z \). Moreover, \( z \notin M \) because \( M \) is an open set. Since \( \chi_M f \) is a continuous function, we have
\[
f(x_\alpha) = (\chi_M f)(x_\alpha) \to (\chi_M f)(z) = 0.
\]

As \( f \) is a continuous function, \( f(x_\alpha) \to f(z) \). Hence \( f(z) = 0 \).

For the converse implication suppose now that \( f \) vanishes on \( \partial M \). Let us put \( g = \chi_M^{-1}f \). It is easy to see that \( g|_M \) and \( g|_{X\setminus M} \) are continuous functions. Proposition 4.4.2 implies that the function \( g = \chi_M^{-1}f = \chi_M f \) is continuous. \(\square\)
Consider the functions \( f_1, f_2 \in C(X) \). Now we would like to describe the set of all elements \( g \in C(X) \) satisfying \( g \preceq f_1 \) and \( g \preceq f_2 \) since the infimum \( f_1 \land f_2 \) is the maximal element of this set. Denote

\[
\begin{align*}
\Omega &= \{ x \in X | f_1(x) = f_2(x), f_1(x) \neq 0 \}, \\
\mathcal{M} &= \{ M | M \subseteq \Omega, \chi_M f_1 \in C(X) \}.
\end{align*}
\]

(4.1)

It is clear, by Proposition 4.4.1, that \( g \) has the form \( g = \chi_M f_1 \), where \( M \in \mathcal{M} \). Conversely, any element \( M \) of \( \mathcal{M} \) uniquely determines a function \( g \) satisfying \( g \preceq f_1 \) and \( g \preceq f_2 \). Therefore we can investigate the set \( \mathcal{M} \) instead of the set of all common lower bounds for \( f_1 \) and \( f_2 \). Note that \( \mathcal{M} \) is a nonempty set because \( \emptyset \in \mathcal{M} \).

Lemma 4.4.4. If \( M \) is an element of \( \mathcal{M} \), then \( M \) is an open set.

Proof. If \( M \) is an element of \( \mathcal{M} \), then \( \chi_M f_1 \) is a continuous function and therefore \( \text{Null}(\chi_M f_1) \) is closed. Moreover, \( \text{Supp}(\chi_M f_1) = M \). From this it follows that \( M = X \setminus \text{Null}(\chi_M f_1) \) is an open set. \( \square \)

Proposition 4.4.5. If \( M, N \) are elements of \( \mathcal{M} \), then \( M \cup N \) is an element of \( \mathcal{M} \).

Proof. As \( M \) and \( N \) are open, it is clear that \( \partial(M \cup N) \subseteq \partial M \cup \partial N \). By Proposition 4.4.3 we have that \( f_1(x) = 0 \) for any \( x \in \partial(M \cup N) \) and therefore \( \chi_{M \cup N} f_1 \) is continuous. Clearly, \( M \cup N \subseteq \Omega \). Hence \( M \cup N \in \mathcal{M} \). \( \square \)

Let us recall some topological concepts. A more detailed treatment can be found, for example, in [47, 84].

Definition 4.4.6. We say that a family of neighborhoods of \( x \) in a topological space \( X \) is a local base at \( x \) if every neighborhood of \( x \) contains a member of the family. A topological space \( X \) is called locally connected if each point of \( X \) has a local base consisting of connected open sets.

Note that a topological space is locally connected if and only if components of each open set are open sets.

We have seen in Proposition 4.4.5 that the family \( \mathcal{M} \) is closed under forming finite unions. This implies that the infimum \( f_1 \land f_2 \) exists in the case when the family \( \mathcal{M} \) has finitely many elements. However, the situation is more complicated if the family \( \mathcal{M} \) is infinite. In the sequel, we shall prove that the infimum \( f_1 \land f_2 \) exists in locally connected spaces. Before doing this, we need some auxiliary topological results.
Lemma 4.4.7. Let $S$ be an open set in a topological space $X$ and let $U$ be an open connected subset of $X$ such that $S \cap U \neq \emptyset$ and $U \setminus S \neq \emptyset$. Then there is $z \in \partial S$ such that $z \in U$.

Proof. For a contradiction, suppose that there is no $z \in \partial S$ such that $z \in U$. This implies that

$$\emptyset \neq S \cap U = \overline{S} \cap U.$$ 

Thus $\overline{S} \cap U$ and $U \cap (X \setminus \overline{S}) = U \setminus \overline{S}$ are nonempty open disjoint sets with union $U$. This is a contradiction with connectedness of $U$. \qed

Lemma 4.4.8. Let $X$ be a locally connected Hausdorff topological space and let $\mathcal{N}$ be a family of open sets in $X$. Assume that $(x_\alpha)$ is a net of elements of $\bigcup_{N \in \mathcal{N}} N$ satisfying $x_\alpha \to x$, where $x \in X \setminus \bigcup_{N \in \mathcal{N}} N$. Then there is a net $(y_\beta)$ such that $y_\beta \in \bigcup_{N \in \mathcal{N}} \partial N$ and $y_\beta \to x$.

Proof. Let $\mathcal{B}_x$ be a local base at $x$ consisting of connected open sets. If $x_\alpha \to x$, then, for any set $U \in \mathcal{B}_x$, there is $\alpha_0$ such that $x_\alpha$ is an element of $U$ for any $\alpha$ satisfying $\alpha_0 \leq \alpha$. For the given set $U \in \mathcal{B}_x$, there exists $N \in \mathcal{N}$ such that $U \cap N \neq \emptyset$ and $U \cap N' \neq \emptyset$ since $x_\alpha \in \bigcup_{M \in \mathcal{N}} M$ and $x \in X \setminus \bigcup_{M \in \mathcal{N}} M$. By Lemma 4.4.7, there is a point $y_U \in \partial N$ such that $y_U \in U$. In this way, we can construct the net $(y_U)$, indexed by the elements of $\mathcal{B}_x$, such that $y_U \in \bigcup_{N \in \mathcal{N}} \partial N$ and $y_U \to x$. \qed

Using the previous result, we prove the following theorem which plays a key role in further discussion. A simple consequence of this theorem is Corollary 4.4.10 which implies that $\bigcup_{M \in \mathcal{M}} M$ is an element of $\mathcal{M}$ in the case of a locally connected topological space. It ensures the existence of the infimum of two continuous functions on a locally connected space (see Theorem 4.4.11). In particular, there is the infimum in the important case of the algebra $C([0, 1])$.

Theorem 4.4.9. Let $X$ be a locally connected Hausdorff topological space and let $f : X \to \mathbb{C}$ be a continuous function. If $\mathcal{N}$ is a family of open sets such that $\chi_N f$ is continuous for every $N \in \mathcal{N}$, then $\chi_M f$, where $M = \bigcup_{N \in \mathcal{N}} N$, is continuous.

Proof. Let $M = \bigcup_{N \in \mathcal{N}} N$. Using Lemma 4.4.8, we obtain that for $z \in \partial M$ there exists a net $(y_\alpha)$ such that $y_\alpha \in \bigcup_{N \in \mathcal{N}} \partial N$ and $y_\alpha \to z$. Since $\chi_N f$ is continuous for any $N \in \mathcal{N}$, we have, by Proposition 4.4.3, that $f$ vanishes on $\bigcup_{N \in \mathcal{N}} \partial N$. Thus $0 = f(y_\alpha) \to f(z)$ and so $f$ vanishes on $\partial M$. It follows from Proposition 4.4.3 that $\chi_M f$ is a continuous function. \qed

Corollary 4.4.10. If $X$ is a locally connected Hausdorff topological space and $\mathcal{N} \subseteq \mathcal{M}$, then $\bigcup_{N \in \mathcal{N}} N \in \mathcal{M}$. 

Proof. Denote $M = \bigcup_{N \in \mathcal{N}} N$. From Theorem 4.4.9 it follows that $\chi_M f_1$ is a continuous function. Moreover, it is clear that $M \subseteq \Omega$. Hence $M \in \mathcal{M}$. □

Theorem 4.4.11. Suppose that $f_1, f_2 \in C(X)$, where $X$ is a locally connected Hausdorff topological space. Then $f_1 \wedge f_2$ exists in $C(X)$. Moreover, $f_1 \wedge f_2 = \chi_N f_1$, where $N = \bigcup_{M \in \mathcal{M}} M$.

Proof. Let us put $N = \bigcup_{M \in \mathcal{M}} M$. It follows from Corollary 4.4.10 that $N \in \mathcal{M}$ and so $g = \chi_N f_1$ satisfies $g \preceq f_1, f_2$. Now we prove that if $h \in C(X)$ and $h \preceq f_1, f_2$, then $h \preceq g$. Clearly, $h = \chi_A f_1$ where $A \in \mathcal{M}$. However, since $N = \bigcup_{M \in \mathcal{M}} M$, $A \subseteq N$ which implies, by Proposition 4.4.1, that $h \preceq g$. □

Applying the previous theorem, we show that the infimum of two elements exists in abelian $C^*$-algebras whose spectrum is locally connected.

Corollary 4.4.12. Suppose that $A$ is an abelian $C^*$-algebra whose spectrum is locally connected. Let $a$ and $b$ be elements of $A$. Then $a \wedge b$ exists.

Proof. By Theorem 2.1.12 and Proposition 4.2.4(vi), we may assume, without loss of generality, that $A = C_0(X)$, where $X$ is a locally connected and locally compact Hausdorff topological space. The algebra $C_0(X)$ can be considered as a *-subalgebra of the *-algebra $C(X)$. Moreover, if $f \in C(X)$, $g \in C_0(X)$, and $f \preceq g$ in $C(X)$, then it follows from Proposition 4.4.1(iii) that $f \in C_0(X)$. Therefore, by Theorem 4.4.11, the infimum $f_1 \wedge f_2$ exists for all $f_1, f_2 \in C_0(X)$. □

We have seen that the infimum problem for two elements in $C(X)$ has the positive answer whenever $X$ is locally connected. Next, we show that there is a positive answer also in the case of extremely disconnected spaces.

Theorem 4.4.13. Let $X$ be an extremely disconnected Hausdorff topological space and $f_1, f_2 \in C(X)$. Then $f_1 \wedge f_2$ exists. Moreover, $f_1 \wedge f_2 = \chi_N f_1$, where $N = \bigcup_{M \in \mathcal{M}} M$.

Proof. Denote $N = \bigcup_{M \in \mathcal{M}} M$. Since $X$ is an extremely disconnected Hausdorff topological space, $\overline{N}$ is a clopen set. Thanks to this, $\chi_{\overline{N}}$ is a continuous function and therefore $\chi_{\overline{N}} f_1$ is continuous. Put $S = \{x \in \overline{N} \mid f_1(x) \neq 0\}$. It is clear that $\chi_{\overline{N}} f_1 = \chi_S f_1$ and $S \subseteq \Omega$. Hence $S \in \mathcal{M}$ and so $S \subseteq N$. It can be easily verified that $N \subseteq S$. Therefore $N = S$ which implies $\chi_{\overline{N}} f_1 = \chi_N f_1$. The same arguments as in the proof of Theorem 4.4.11 show that $\chi_N f_1$ is the infimum $f_1 \wedge f_2$. □
The next consequence of Theorem 4.4.13 follows immediately from the fact that an abelian $AW^*$-algebra is *-isomorphic to $C(X)$, where $X$ is an extremely disconnected compact Hausdorff topological space (see [11]).

**Corollary 4.4.14.** Let $A$ be an abelian $AW^*$-algebra and $a, b \in A$. Then $a \land b$ exists.

Note that, in the special case of abelian von Neumann algebras, the preceding result can be proved without using Theorem 4.4.13. It is well known that an abelian von Neumann algebra is *-isomorphic to $L^\infty(\Gamma, \mu)$ for which the proof is straightforward.

The following example shows that infimum need not exist in the case of a Hausdorff topological space which is neither locally connected nor extremely disconnected.

**Example 4.4.15.** Consider $X = [0, 1]$ endowed with topology whose base is

$$\mathcal{B} = \{(a, b) \cap [0, 1] | -\infty < a < b < \infty\} \cup \left\{\left\{\frac{1}{n}\right\} \mid n \in \mathbb{N} \setminus \{0\}\right\}.$$ 

Put $A = \left\{\frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\}\right\}$ and $F = X \setminus A$. Clearly, the set $A$ is open but not closed. Moreover, $\overline{A} = A \cup \{0\}$ is not open and so $X$ is not extremely disconnected. Since the component of 0 is $\{0\}$ and $\{0\}$ is not open, the space $X$ is not locally connected. Denote $M_n = [\frac{1}{n}, 1] \setminus A$, where $n \in \mathbb{N} \setminus \{0\}$. The set $M_n$ is a clopen set for each $n \in \mathbb{N} \setminus \{0, 1\}$, 

$$M_n = \bigcup_{k=1}^{n-1} \left(\frac{1}{k+1}, \frac{1}{k}\right) = \bigcup_{k=1}^{n-1} X \setminus U_k,$$

where $U_k = [0, \frac{1}{k+1}) \cup \left\{\frac{1}{k+1}\right\} \cup \left\{\frac{1}{k}\right\} \cup \left(\frac{1}{k}, 1\right]$. Let $f_1 : X \to \mathbb{C}$ be the function given by

$$f_1 : x \mapsto \begin{cases} 1 & \text{if } x \in F, \\ 1 - x & \text{if } x \in A. \end{cases}$$

It is apparent that $f_1 \in C(X)$.

Consider the function $f_2 \in C(X)$ such that $f_2 : x \mapsto 1$. We observe that $M_n \subseteq M_{n+1}$ and $M_n \in \mathcal{M}$ for all $n \in \mathbb{N} \setminus \{0\}$,

---

4For the definition of the $AW^*$-algebra, we refer the reader to [11][64].

5The set $A$ is not closed because $\frac{1}{n} \to 0 \notin A$.

6Indeed, let $\mathcal{O} \subseteq \mathbb{C}$ be an open ball. If $1 \notin \mathcal{O}$, then $f_1^{-1}(\mathcal{O}) \subseteq A$ and so $f_1^{-1}(\mathcal{O})$ is open. If $1 \in \mathcal{O}$, then there is $n_0 \in \mathbb{N} \setminus \{0\}$ such that $f_1^{-1}(\mathcal{O}) = F \cup \left\{\frac{1}{n} \mid n \geq n_0\right\} = [0, \frac{1}{n_0}) \cup \left\{\frac{1}{n_0}\right\} \cup M_{n_0}$, and therefore $f_1^{-1}(\mathcal{O})$ is an open set.
where $\mathcal{M}$ is defined in (4.1). In order to show that $f_1 \wedge f_2$ does not exist, it is sufficient to prove that there is no set $N \in \mathcal{M}$ such that $\bigcup_{n=1}^{\infty} M_n \subseteq N \subseteq F$. Since $\bigcup_{n=1}^{\infty} M_n = F \setminus \{0\}$, the only subsets of $F$ containing $\bigcup_{n=1}^{\infty} M_n$ are $F \setminus \{0\}$ and $F$. It is easily seen that the sets

$$(\chi_{F \setminus \{0\}} f_1)^{-1} (\mathbb{C} \setminus \{1\}) = X \setminus (F \setminus \{0\}) = A \cup \{0\},$$

$$(\chi_{F} f_1)^{-1} (\mathbb{C} \setminus \{0\}) = F$$

are not open and so $F \setminus \{0\} \notin \mathcal{M}$ and $F \notin \mathcal{M}$. Therefore, $f_1 \wedge f_2$ does not exist.

The results of this section, concerning the infimum problem for two functions, can be easily generalized to the infimum problem for an arbitrary subset of $C(X)$. Indeed, let $(f_\alpha)_{\alpha \in \Lambda}$ be a family of elements of $C(X)$. For fixed $\beta \in \Lambda$, we can denote

$$\Omega = \{ x \in X | f_\beta(x) = f_\alpha(x) \text{ for all } \alpha \in \Lambda \setminus \{\beta\}, f_\beta(x) \neq 0 \},$$

$$\mathcal{M} = \{ M | M \subseteq \Omega, \chi_M f_\beta \in C(X) \}$$

and repeat the discussion given in this section. From this, we conclude that the following theorem holds.

**Theorem 4.4.16.** Let $(f_\alpha)_{\alpha \in \Lambda}$ be a family of elements of $C(X)$. Then the infimum $\bigwedge_{\alpha \in \Lambda} f_\alpha$ exists whenever $X$ is locally connected or extremely disconnected.

### 4.5 Supremum problem for function algebras

It was shown in [36] that the supremum of two elements in the Gudder order exists if and only if there is a common upper bound for these two elements. In this section, we show that this situation occurs also for the star order on the $\ast$-algebra $C(X)$ of all continuous complex-valued functions on a Hausdorff topological space $X$. Moreover, we investigate the existence of the supremum of an arbitrary subset of $C(X)$. It is proved that the supremum of a bounded (with respect to the star order) subset of $C(X)$ exists whenever the Hausdorff topological space $X$ is locally connected or extremely disconnected.

Let us look at the simple example showing that there are functions for which the supremum does not exist.

**Example 4.5.1.** Let $X$ be a Hausdorff topological space. Consider the continuous functions $f_1, f_2 : X \to \mathbb{C}$ given by $f_1 : x \mapsto 1$ and $f_2 : x \mapsto 2$. Since $f_1(x) \neq f_2(x)$ for all $x \in X$, functions $f_1$ and $f_2$ have no common upper bound with respect to the star order and so the supremum $f_1 \vee f_2$ does not exist.
The important aspect in the previous example has been the absence of a common upper bound. The following theorem shows that this is the only situation in which the supremum of two elements does not exist.

**Theorem 4.5.2.** Let \( f_1 \) and \( f_2 \) be elements of \( C(X) \) where \( X \) is a Hausdorff topological space. The supremum \( f_1 \vee f_2 \) exists if and only if there is \( h \in C(X) \) such that \( f_1, f_2 \preceq h \).

**Proof.** If \( f_1 \vee f_2 \) exists, then \( f_1, f_2 \preceq f_1 \vee f_2 \).

Let us prove the reverse implication. Denote \( M_1 = \text{Supp}(f_1) \) and \( M_2 = \text{Supp}(f_2) \). Obviously, \( M_1 \) and \( M_2 \) are open. If there is \( h \in C(X) \) such that \( f_1, f_2 \preceq h \), then \( f_1 = \chi_{M_1}h \) and \( f_2 = \chi_{M_2}h \). By Proposition 4.4.3, \( h \) vanishes on \( \partial M_1 \cup \partial M_2 \) and so the inclusion \( \partial (M_1 \cup M_2) \subseteq \partial M_1 \cup \partial M_2 \) implies that \( g = \chi_{M_1 \cup M_2}h \) is continuous. Now we prove that \( g = f_1 \vee f_2 \). It is clear that \( f_1, f_2 \preceq g \). If \( \tilde{g} \in C(X) \) is such that \( f_1, f_2 \preceq \tilde{g} \), then \( h(x) = f_1(x) = \tilde{g}(x) \) for \( x \in M_1 \) and \( h(x) = f_2(x) = \tilde{g}(x) \) for \( x \in M_2 \). Therefore, \( \tilde{g}(x) = h(x) \) for \( x \in M_1 \cup M_2 \). Hence \( g = \chi_{M_1 \cup M_2}h = \chi_{M_1 \cup M_2}\tilde{g} \). Thus \( g = \tilde{g} \).

We have seen in the previous theorem that, unlike the infimum problem, there is no restriction on the topological space \( X \) in the case of the supremum problem of two (and so finitely many) functions. In the sequel, we shall see that the supremum problem for an arbitrary bounded subset of \( C(X) \) has a positive answer if \( X \) is locally connected or extremely disconnected.

**Theorem 4.5.3.** Let \( X \) be a locally connected Hausdorff topological space. Suppose that \( (f_\alpha)_{\alpha \in \Lambda} \) is a family of elements of \( C(X) \). Then the following conditions are equivalent:

(i) There exists \( \bigvee_{\alpha \in \Lambda} f_\alpha \).

(ii) There is \( h \in C(X) \) such that \( f_\alpha \preceq h \) for any \( \alpha \in \Lambda \).

**Proof.** (i) \( \Rightarrow \) (ii) is clear.

(ii) \( \Rightarrow \) (i). Let us put 

\[ \mathcal{N} = \{ N \mid N = \text{Supp}(f_\alpha), \alpha \in \Lambda \} \]

and \( M = \bigcup_{N \in \mathcal{N}} N \). Since \( f_\alpha \preceq h \) for each \( \alpha \in \Lambda \), we have \( \chi_N h \) is continuous for every \( N \in \mathcal{N} \). By Theorem 4.4.9, the function \( g = \chi_M h \) is continuous. Similarly to the proof of Theorem 4.5.2, we can verify that the function \( g \) is the supremum of the family \( (f_\alpha)_{\alpha \in \Lambda} \).

**Corollary 4.5.4.** Suppose that \( \mathcal{A} \) is an abelian \( C^* \)-algebra with a locally connected spectrum and \( (a_\alpha)_{\alpha \in \Lambda} \) is a family of elements of \( \mathcal{A} \). Then the following conditions are equivalent:
(i) There exists $\bigvee_{\alpha \in \Lambda} a_\alpha$.

(ii) There is $b \in A$ such that $a_\alpha \preceq b$ for any $\alpha \in \Lambda$.

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (i). By Theorem 2.1.12 and Proposition 4.2.4(vi), we may assume, without loss of generality, that $A = C_0(X)$, where $X$ is a locally connected and locally compact Hausdorff topological space. The algebra $C_0(X)$ can be considered as a $*$-subalgebra of the $*$-algebra $C(X)$. Let $(f_\alpha)_{\alpha \in \Lambda}$ be a family of elements of $C_0(X)$. Suppose that there is an element $h \in C_0(X)$ such that $f_\alpha \preceq h$ for all $\alpha \in \Lambda$. It follows from Theorem 4.5.3 that $\bigvee_{\alpha \in \Lambda} f_\alpha$ exists in $C(X)$. Since $\bigvee_{\alpha \in \Lambda} f_\alpha \preceq h$ and $h \in C_0(X)$, we have, by Proposition 4.4.1(iii), that $\bigvee_{\alpha \in \Lambda} f_\alpha \in C_0(X)$.

Let us prove the analogue of Theorem 4.5.3 for extremely disconnected topological spaces.

Theorem 4.5.5. Let $X$ be an extremely disconnected Hausdorff topological space. Suppose that $(f_\alpha)_{\alpha \in \Lambda}$ is a family of elements of $C(X)$. Then the following conditions are equivalent:

(i) There exists $\bigvee_{\alpha \in \Lambda} f_\alpha$.

(ii) There is $h \in C(X)$ such that $f_\alpha \preceq h$ for all $\alpha \in \Lambda$.

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (i). Put $M_\alpha = \text{Supp}(f_\alpha)$, $M = \bigcup_{\alpha \in \Lambda} M_\alpha$, and $f = \chi_M h$. Since $M$ is clopen, $f \in C(X)$. Let us show that $f = \bigvee_{\alpha \in \Lambda} f_\alpha$. It follows immediately from Proposition 4.4.1(iii) that $f_\alpha \preceq f$ for all $\alpha \in \Lambda$. If $g \in C(X)$ is such that $f_\alpha \preceq g$ for all $\alpha \in \Lambda$, then $f_\alpha = \chi_{M_\alpha} g$ for all $\alpha \in \Lambda$. Moreover, (ii) implies that $f_\alpha = \chi_{M_\alpha} h$ for every $\alpha \in \Lambda$. Consequently, we have $g = h$ on $M$. By continuity, $g = h$ on $\overline{M}$. Therefore,

$$f g = f \chi_M g = f \chi_M h = f$$

and so $f \preceq g$.

Corollary 4.5.6. Let $A$ be an abelian $AW^*$-algebra. Suppose that $(a_\alpha)_{\alpha \in \Lambda}$ is a family of elements of $A$. Then the following conditions are equivalent:

(i) There exists $\bigvee_{\alpha \in \Lambda} a_\alpha$.

(ii) There is $b \in A$ such that $a_\alpha \preceq b$ for all $\alpha \in \Lambda$. 
We say that a poset $L$ is *boundedly complete* if any bounded subset of $L$ has the infimum and the supremum. Since a lower bound (with respect to the star order) of any subset of $C(X)$ is a function identically equal to zero, every subset, which has an upper bound, is bounded. Thus, combining results of this section and Section 4.4, we obtain that the poset $C(X)$, where $X$ is a locally connected or an extremely disconnected Hausdorff topological space, is boundedly complete. On the other hand, it is well known (see [46, 72]) that $C(X)$ endowed with the usual order is boundedly complete if and only if $X$ is an extremely disconnected topological space. We can conclude that $C(X)$ endowed with the star order forms a boundedly complete poset under a quite different condition than in the case of the usual order.
Chapter 5

Preservers of the star order

In this chapter, we investigate maps preserving the star order. In particular, our attention is focused on continuous star order isomorphisms between normal parts of von Neumann algebras without Type I$_2$ direct summand. We show that if these preservers satisfy only a mild condition generalizing the linearity with respect to the scales (multiples of identity), then they have a nice compact form whose essential components are the continuous functional calculus and a Jordan *-isomorphism.

Note that the results of Section 5.3 can be found in the paper [17].

5.1 Jordan *-homomorphisms

Jordan *-homomorphisms arise in many circumstances. For example, the interesting Kadison’s result [45] shows that Jordan *-isomorphisms are basic components of surjective linear isometries between $C^\ast$-algebras. The Jordan *-isomorphisms are also important in the generalization of famous Wigner’s unitary-antiunitary theorem [83] given by Dye [31]. In this section, we recall some well known results concerning the Jordan *-homomorphisms between $C^\ast$-algebras which can also be found in [46], [63]. After that, we prove that Jordan *-isomorphisms preserve the star order in both direction.

Definition 5.1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be a $C^\ast$-algebras. We say that $\varphi : \mathcal{A} \to \mathcal{B}$ is a Jordan *-homomorphism if $\varphi$ is a linear map satisfying

(i) $\varphi(a^\ast) = \varphi(a)^\ast$ for all $a \in \mathcal{A}$,

(ii) $\varphi(a^2) = \varphi(a)^2$ for all $a \in \mathcal{A}_{sa}$.

A bijective Jordan *-homomorphism is called Jordan *-isomorphism.
In the following lemma, we summarize some basic identities for Jordan $^*$-homomorphisms. Recall that the commutator $[a, b]$ of elements $a$ and $b$ of a $C^*$-algebra is defined by $[a, b] = ab - ba$.

**Lemma 5.1.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a Jordan $^*$-homomorphism. Then

(i) $\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a)$ for all $a, b \in \mathcal{A}$;

(ii) $\varphi(a^n) = \varphi(a)^n$ for all $a \in \mathcal{A}$ and $n \in \mathbb{N} \setminus \{0\}$;

(iii) $\varphi(aba) = \varphi(a)\varphi(b)\varphi(a)$ for all $a, b \in \mathcal{A}$;

(iv) $\varphi(abc + cba) = \varphi(a)\varphi(b)\varphi(c) + \varphi(c)\varphi(b)\varphi(a)$ for all $a, b, c \in \mathcal{A}$;

(v) $\varphi([a, b], c) = [[\varphi(a), \varphi(b)], \varphi(c)]$ for all $a, b, c \in \mathcal{A}$;

(vi) $\varphi([a, b]^2) = [\varphi(a), \varphi(b)]^2$ for all $a, b \in \mathcal{A}$.

**Proof.** (i) First suppose that $a, b \in \mathcal{A}_{sa}$. Since $ab + ba = (a + b)^2 - a^2 - b^2$, we obtain

$$\varphi(ab + ba) = \varphi(a + b)^2 - \varphi(a)^2 - \varphi(b)^2 = (\varphi(a) + \varphi(b))^2 - \varphi(a)^2 - \varphi(b)^2 = \varphi(a)\varphi(b) + \varphi(b)\varphi(a).$$

Now suppose that $a$ and $b$ are arbitrary elements of $\mathcal{A}$. Thus $a = x_1 + iy_1$ and $b = x_2 + iy_2$, where $x_1, x_2, y_1, y_2 \in \mathcal{A}_{sa}$. A simple calculation yields

$$ab + ba = (x_1 + x_2)^2 - x_1^2 - x_2^2 + i((x_1 + x_2)(y_1 + y_2) + (y_1 + y_2)(x_1 + x_2))$$

$$-i(x_1y_1 + y_1x_1) - i(x_2y_2 + y_2x_2) - (y_1 + y_2)^2 + y_1^2 + y_2^2.$$

Hence $\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a)$.

(ii) We prove the statement by induction. The formula $\varphi(a^n) = \varphi(a)^n$ is obvious if $n = 1$. Assume that the formula holds for $n$. Then, by (i), we have

$$\varphi(a^{n+1}) = \frac{1}{2}\varphi(aa^n + a^n a) = \frac{1}{2}(\varphi(a)\varphi(a)^n + \varphi(a)^n \varphi(a)) = \varphi(a)^{n+1}.$$ (iii) The assertion follows immediately from (i), (ii), and the identity

$$2aba = (ab + ba)a + a(ab + ba) - ba^2 - a^2b.$$

(iv) Since

$$abc + cba = (a + c)b(a + c) - aba - cbc,$$

we obtain from (iii) that (iv) holds.
(v) We observe that
\[
[[a, b], c] = abc + cba - (bac + cab).
\]
From this and (iv), we get the statement (v).

(vi) The identity
\[
[a, b]^2 = a(bab) + (bab)a - ab^2a - ba^2b,
\]
together with (i), (ii), and (iii), implies (vi).

**Proposition 5.1.3.** Let \( A \) and \( B \) be \( C^* \)-algebras and let \( \varphi : A \to B \) be a Jordan \( * \)-homomorphism. If \( a, b \in A \) and \( [a, b] = 0 \), then
\[
\varphi(ab) = \varphi(a)\varphi(b) = \varphi(b)\varphi(a).
\]

**Proof.** Consider the smallest \( C^* \)-subalgebra \( C \) of \( B \) that contains \( \varphi(A) \). As \( [a, b] = 0 \), it follows from Lemma 5.1.2(v) that
\[
0 = \varphi([a, b], c)) = [[\varphi(a), \varphi(b)], \varphi(c)]
\]
for all \( c \in A \). This shows that \( [\varphi(a), \varphi(b)] \) commutes with all element of \( C \). In particular, \( [\varphi(a), \varphi(b)] \) is a normal element of \( C \). Moreover, it follows from Lemma 5.1.2(vi) that
\[
[\varphi(a), \varphi(b)]^2 = \varphi([a, b]^2) = 0.
\]
Since \( [\varphi(a), \varphi(b)] \) is normal,
\[
\|[[\varphi(a), \varphi(b)]\| = r([\varphi(a), \varphi(b)]) = \lim_{n \to \infty} \|[\varphi(a), \varphi(b)]^n\|^\frac{1}{n} = 0
\]
and so \( [\varphi(a), \varphi(b)] = 0 \). Furthermore, by Lemma 5.1.2(i),
\[
\varphi(ab) = \frac{1}{2}(\varphi(a)\varphi(b) + \varphi(b)\varphi(a)) = \varphi(a)\varphi(b).
\]

**Corollary 5.1.4.** Let \( \varphi : A \to B \) be a surjective Jordan \( * \)-homomorphism from a unital \( C^* \)-algebra \( A \) onto a unital \( C^* \)-algebra \( B \). Then \( \varphi(1) = 1 \).

**Proof.** It follows from Proposition 5.1.3 that
\[
\varphi(a) = \varphi(a1) = \varphi(a)\varphi(1) = \varphi(1)\varphi(a).
\]
for all \( a \in A \). By surjectivity of \( \varphi \), \( \varphi(1) = 1 \).
The following proposition is a trivial consequence of continuity of so-called positive maps\footnote{A linear map $\varphi$ from a $C^*$-algebra $A$ into a $C^*$-algebra $B$ is called positive if $\varphi(A_+) \subseteq B_+$. Note that if $B = \mathbb{C}$, then positive maps are nothing but positive functionals on $A$.} between $C^*$-algebras.

**Proposition 5.1.5.** Let $A$ and $B$ be $C^*$-algebras and let $\varphi : A \to B$ be a Jordan $*$-homomorphism. Then $\varphi$ is continuous.

**Proof.** If $a \in A_+$, then $\varphi(a) = \varphi(\sqrt{a})^2 \geq 0$. Therefore, $\varphi(A_+) \subseteq B_+$. We shall show that the graph of $\varphi$ is closed. Suppose that $a_n \to 0$ in $A$ and $\varphi(a_n) \to b$ in $B$. For any positive functional $\psi$ on $B$, $\psi \circ \varphi$ is a positive linear functional on $A$. By Theorem 2.2.4(i), $\psi \circ \varphi$ is continuous and so $(\psi \circ \varphi)(a_n) \to 0$. Accordingly, $\psi(b) = 0$ for all positive linear functionals on $B$. Applying Proposition 2.2.7(i), we get $b = 0$ and thus the graph of $\varphi$ is closed. Hence, by the closed graph theorem, $\varphi$ is continuous. \hfill $\Box$

The next proposition shows that Jordan $*$-isomorphisms preserve the star order in both directions.

**Proposition 5.1.6.** Let $A$ and $B$ be $C^*$-algebras. If $\varphi : A \to B$ is a Jordan $*$-isomorphism, then

$$a \preceq b \iff \varphi(a) \preceq \varphi(b)$$

for all $a, b \in A$.

**Proof.** By Proposition 4.2.2, it is sufficient to prove that, for all $a, b \in A$, $a \perp b$ if and only if $\varphi(a) \perp \varphi(b)$. This follows from Proposition 5.1.3 and the fact that $\varphi^{-1}$ is a Jordan $*$-isomorphism. \hfill $\Box$

## 5.2 Star order isomorphisms

In this section, we concentrate our attention on the study of star order isomorphisms which are defined as follows.

**Definition 5.2.1.** Let $A$ and $B$ be $C^*$-algebras. Let $M$ and $N$ be subsets of $A$ and $B$, respectively. We say that $\varphi : M \to N$ is a star order isomorphism if $\varphi$ is a bijection such that

$$a \preceq b \iff \varphi(a) \preceq \varphi(b)$$

for all $a, b \in M$. A star order isomorphism $\varphi : M \to N$ is called unital if $A$ and $B$ are unital, $1_A \in M$, $1_B \in N$, and $\varphi(1_A) = 1_B$. 
It is clear that if $\varphi : M \to N$ is a star order isomorphism, then the inverse map $\varphi^{-1} : N \to M$ is also a star order isomorphism. Furthermore, it follows from Proposition 5.1.6 that every Jordan $*$-isomorphism between $C^*$-algebras is a (linear continuous) star order isomorphism.

Let us briefly examine linear unital star order isomorphisms from a unital $C^*$-algebra $A$ onto a unital $C^*$-algebra $B$. Since, by Proposition 4.2.2, a linear map $\varphi : A \to B$ is a star order isomorphism if and only if it preserves $*$-orthogonality in both directions, we can use a number of existing results concerning the preservers of the $*$-orthogonality [23, 24, 85, 86]. For example, Wong proved the following result [86].

**Theorem 5.2.2.** Let $A$ and $B$ be unital $C^*$-algebras and let $\varphi : A \to B$ be a bounded linear map. Then the following conditions are equivalent:

(i) $\varphi$ is a triple homomorphism.

(ii) $\varphi(1)$ is a partial isometry and $a \perp b$ implies $\varphi(a) \perp \varphi(b)$ for every $a, b \in A$.

Note that every Jordan $*$-isomorphism is a triple isomorphism. Indeed, this fact follows directly from Lemma 5.1.2(iv).

**Corollary 5.2.3.** Let $A$ and $B$ be unital $C^*$-algebras and let $\varphi : A \to B$ be a linear bounded unital star order isomorphism. Then the map $\varphi$ is a Jordan $*$-isomorphism.

**Proof.** Since $\varphi$ is linear star order isomorphism, $a \perp b$ implies $\varphi(a) \perp \varphi(b)$ for all $a, b \in A$. By Theorem 5.2.2, $\varphi$ is a triple isomorphism (i.e., bijective triple homomorphism) and so

$$\varphi(ab^*c + cb^*a) = \varphi(a)\varphi(b)^*\varphi(c) + \varphi(c)\varphi(b)^*\varphi(a)$$

for all $a, b, c \in A$. If we set $a = c = 1$ in (5.1), then $\varphi(b^*) = \varphi(b)^*$ for every $b \in A$. Furthermore, if we set $a = c$ and $b = 1$ in (5.1), then $\varphi(a^2) = \varphi(a)^2$ for every $a \in A$. Therefore, $\varphi$ is a Jordan $*$-isomorphism.

By the following theorem proved in [24], the assumption of boundedness of $\varphi$ can be omitted in the preceding corollary provided that $A$ and $B$ are von Neumann algebras.

---

2This means that $a \perp b \Leftrightarrow \varphi(a) \perp \varphi(b)$ for all $a, b \in A$.

3By a triple homomorphism we mean a linear map $\varphi$ from a $C^*$-algebra $A$ into a $C^*$-algebra $B$ such that

$$\varphi(ab^*c + cb^*a) = \varphi(a)\varphi(b)^*\varphi(c) + \varphi(c)\varphi(b)^*\varphi(a)$$

for all $a, b, c \in A$. 
**Theorem 5.2.4.** Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras. Suppose that $\varphi: \mathcal{A} \to \mathcal{B}$ is linear surjection such that $a \perp b \iff \varphi(a) \perp \varphi(b)$ holds for all $a, b \in \mathcal{A}$. Then $\varphi$ is continuous.

At the end of this section, we consider continuous star order isomorphisms of $\mathcal{B}(\mathcal{H})_{sa}$ which are nonlinear in general. Since the Gudder order is a restriction of the star order on $\mathcal{B}(\mathcal{H})$ to the self-adjoint part of $\mathcal{B}(\mathcal{H})$, the structure of continuous star order isomorphisms $\varphi: \mathcal{B}(\mathcal{H})_{sa} \to \mathcal{B}(\mathcal{H})_{sa}$, where $\dim \mathcal{H} \geq 3$, is completely described by the nice result of Dolinar and Molnár [29] which can be reformulated as follows.

**Theorem 5.2.5.** Let $\mathcal{H}$ be a complex Hilbert space with $\dim \mathcal{H} \geq 3$. Let $\varphi: \mathcal{B}(\mathcal{H})_{sa} \to \mathcal{B}(\mathcal{H})_{sa}$ be a continuous star order isomorphism. Then there exist a continuous bijective function $f: \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and either a unitary or an antiunitary operator $u$ on $\mathcal{H}$ such that

$$\varphi(a) = uf(a)u^*$$

for all $a \in \mathcal{B}(\mathcal{H})_{sa}$.

It should be noted that, by Proposition [4.2.4(4)] and Theorem [4.2.9] every map of the form (5.2) is star order isomorphism on $\mathcal{B}(\mathcal{H})_{sa}$. Moreover, the assumption $\dim \mathcal{H} \geq 3$ is crucial. It was shown in [29] that if $\dim \mathcal{H} = 2$, then there is a continuous star order isomorphism which is not of the form (5.2).

### 5.3 Star order isomorphisms and von Neumann algebras

In the light of the fact that any Jordan *-isomorphism on a Type I factor is implemented by either a unitary or an antiunitary operator (see, for example, [30] Section 7.5]), one can reformulate Theorem 5.2.5 by saying that any continuous star order isomorphism of the set $\mathcal{B}(\mathcal{H})_{sa}$ of all self-adjoint operators, where $\dim \mathcal{H} \geq 3$, is given by the continuous functional calculus followed by a Jordan *-isomorphism. The aim of this section is to extend this result in a few direction, mainly to general von Neumann algebras. The important assumption in Theorems [5.2.5] that $\dim \mathcal{H} \geq 3$ indicates the necessity to exclude Type I$_2$ direct summand from our consideration.

To generalize Theorem 5.2.5 we shall need some auxiliary results. First, we characterize multiples of projections in terms of the star order. Recall that the set of all projection in a $C^*$-algebra $\mathcal{A}$ is denoted by $P(\mathcal{A})$. 
Proposition 5.3.1. Let $A$ be a unital $C^*$-algebra and let $a \in A$. Then $a \preceq \lambda 1$, for $\lambda \in \mathbb{C}$, if and only if there is a projection $p \in A$ such that $a = \lambda p$.

Proof. Suppose first that $a = \lambda p$, where $p \in P(A)$ and $\lambda \in \mathbb{C}$. Then

\[
a^*a = |\lambda|^2 p = a^* \lambda 1,
\]
\[
aa^* = |\lambda|^2 p = \lambda 1 a^* ,
\]
and so $a \preceq \lambda 1$.

For the reverse implication assume that $a \preceq \lambda 1$. Then $a^*a = aa^* = \lambda a^*$. Write $a$ in the form $a = \lambda x$, where $x \in A$. Then

\[
|\lambda|^2 x^* x = |\lambda|^2 xx^* = |\lambda|^2 x^*
\]
and so, for $\lambda \neq 0$,

\[x^* x = xx^* = x^* .\]

The last equation says that $x$ is a self-adjoint idempotent and thereby a projection. (The case $\lambda = 0$ is trivial.)

Proposition 5.3.2. Let $A$ be a $C^*$-algebra and let $p, q \in P(A)$. Then the following conditions are equivalent:

(i) For each $\lambda \in \mathbb{C} \setminus \{0, 1\}$, there is an element $a \in A_n$ such that

\[p, \lambda q \preceq a .\]

(ii) There is $\lambda \in \mathbb{C} \setminus \{0, 1\}$ and $a \in A$ such that

\[p, \lambda q \preceq a .\]

(iii) The projections $p$ and $q$ are mutually orthogonal.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). The relation $p \preceq a$ means that $p = pa = ap$, while $\lambda q \preceq a$ means that $|\lambda|^2 q = \lambda qa = \lambda q a$. Putting this together, we see that

\[pq = paq = \lambda pq ,\]

which means that $pq = 0$.

(iii) $\Rightarrow$ (i). If $p$ and $q$ are mutually orthogonal, then it can be easily verified that, for all $\lambda \in \mathbb{C} \setminus \{0, 1\}$, $a = p + \lambda q$ is a normal element majorazing both $p$ and $\lambda q$. \qed
We shall now formulate standing assumptions valid in the following text. In the sequel, let $A$ and $B$ be von Neumann algebras and let $\varphi : A_n \to B_n$ be a map between normal parts of $A$ and $B$ satisfying the following conditions:

- $\varphi$ is a continuous star order isomorphism.
- There is an invertible central element $c \in B$ and a function $f : \mathbb{C} \to \mathbb{C}$ such that
  $$\varphi(\lambda 1) = f(\lambda)c$$
  for all $\lambda \in \mathbb{C}$.

In the next lemma, we derive a few elementary properties of $\varphi$ and $f$ that follow from the assumptions given above.

**Lemma 5.3.3.**

(i) $\varphi(0) = 0$.

(ii) $f(0) = 0$ and $f(1) \neq 0$.

(iii) $f$ is continuous.

(iv) $f$ is injective.

**Proof.** 
(i) It is clear that 0 is the infimum of the set of all normal operators of a von Neumann algebra in the star order. Since $\varphi$ is star order isomorphism, $\varphi(0) = 0$.

(ii) $0 = \varphi(0 \cdot 1) = f(0)c$ and so $f(0) = 0$. Further, $\varphi(1 \cdot 1) = f(1)c$, hence $f(1) \neq 0$, for otherwise $\varphi(1) = \varphi(0) = 0$, contradicting the injectivity of $\varphi$.

(iii) It is a consequence of continuity of $\varphi$ because
  $$f(\lambda)1 = \varphi(\lambda 1)c^{-1}.$$

(iv) As $\varphi(\lambda 1) = f(\lambda)c$, the injectivity of $f$ is a consequence of the injectivity of $\varphi$. \qed

Consider the map $\varrho : A_n \to B_n$ given by
  $$\varrho(a) = \frac{1}{f(1)}\varphi(a)c^{-1}, \quad a \in A_n.$$  

It is apparent that we can investigate, without loss of generality, $\varrho$ instead of $\varphi$.

---

4An element of a von Neumann algebra $A$ is said to be central if it commutes with all elements of $A$. 
Proposition 5.3.4. The following statements hold:

(i) There is an injective continuous function \( g : \mathbb{C} \to \mathbb{C} \) such that \( g(0) = 0, g(1) = 1, \) and
\[
g(\lambda 1) = g(\lambda) 1.
\]

(ii) \( \phi \) is a continuous unital star order isomorphism.

Proof. (i) Set
\[
g(\lambda) = \frac{f(\lambda)}{f(1)}.
\]
Injectivity, continuity, and the facts that \( g(0) = 0 \) and \( g(1) = 1 \) follow immediately from Lemma 5.3.3. Moreover,
\[
g(\lambda 1) = \frac{1}{f(1)} \varphi(\lambda 1)c^{-1} = \frac{1}{f(1)} f(\lambda) cc^{-1} = g(\lambda) 1.
\]

(ii) This is obvious from (i), Proposition 4.2.4(iii), and the corresponding properties of \( \varphi \).

In the sequel, by the function \( g \) we shall always mean the function from the previous proposition.

Proposition 5.3.5. Let \( p \in P(A) \) and \( \lambda \in \mathbb{C} \). There is a projection \( q \in B \) such that
\[
\phi(\lambda p) = g(\lambda) q.
\]

Proof. We have that \( \lambda p \preceq \lambda 1 \) and so
\[
\phi(\lambda p) \preceq g(\lambda 1) = g(\lambda) 1.
\]
By Proposition 5.3.1 we infer
\[
\phi(\lambda p) = g(\lambda) q,
\]
where \( q \) is a certain projection in \( B \).

Proposition 5.3.6. \( \phi \) maps \( P(A) \) onto \( P(B) \).

Proof. As \( g(1) = 1 \), it follows from Proposition 5.3.5 that \( \phi \) maps \( P(A) \) into \( P(B) \). If \( e \in P(B) \), then there is \( a \in A_n \) such that \( \phi(a) = e \). Since \( \phi^{-1} : B_n \to A_n \) is a unital star order isomorphism and \( e \preceq 1 \), we see that
\[
a = \phi^{-1}(\phi(a)) \preceq \phi^{-1}(1) = 1.
\]
Thus, by Proposition 5.3.1 \( a \in P(A) \).
Recall that the set of all projections in a von Neumann algebra forms a complete lattice under the usual operator order \( \leq \). In the following proposition, we denote the supremum of two projections \( p \) and \( q \) in this lattice by \( p \vee q \).

**Proposition 5.3.7.** Let \( p \) and \( q \) be projections in \( A \). Then

(i) \( \varphi(p \vee q) = \varphi(p) \vee \varphi(q) \).

(ii) If \( p \) and \( q \) are mutually orthogonal, then \( \varphi(p) \) and \( \varphi(q) \) are mutually orthogonal projections in \( B \).

**Proof.** (i) Since \( \varphi \) is a star order isomorphism such that \( \varphi(P(A)) = P(B) \) and the star order restricts to usual order on sets \( P(A) \) and \( P(B) \), it is easily verified that

\[
\varphi(p \vee q) = \varphi(p) \vee \varphi(q)
\]

for all \( p, q \in P(A) \).

(ii) Let us assume that \( p \) and \( q \) are mutually orthogonal projections. By Proposition 5.3.2, for each \( \lambda \in \mathbb{C} \setminus \{0,1\} \), there is \( a_\lambda \in A_n \) such that

\[
p, \lambda q \preceq a_\lambda.
\]

Hence, for any \( \lambda \in \mathbb{C} \setminus \{0,1\} \), there is a \( b_\lambda \in B_n \) such that \( \varphi(p), \varphi(\lambda q) \preceq b_\lambda \). Moreover, by Proposition 5.3.5, \( \varphi(\lambda q) = \varphi(\lambda)e_\lambda \) for some projection \( e_\lambda \) in \( P(A) \). Applying Proposition 5.3.2 again, we see that

\[
\varphi(p)\varphi(\lambda q) = 0.
\]

(5.3)

Using continuity of \( \varphi \) and making the limit \( \lambda \to 1 \) in (5.3), we obtain

\[
\varphi(p)\varphi(q) = 0.
\]

\( \square \)

An important ingredient of the next discussion is the following result that is an interesting application of generalized Gleason theorem. Its proof can be found in [38].

**Theorem 5.3.8.** Let \( A \) be a von Neumann algebra without Type \( I_2 \) direct summand and let \( C \) be a \( C^* \)-algebra. Suppose that \( \omega : P(A) \to P(C) \) is a map satisfying

\[
\omega(p + q) = \omega(p) + \omega(q)
\]

whenever \( p, q \in P(A) \) are mutually orthogonal projections. Then there is a unique Jordan *-homomorphism \( \psi : A \to C \) such that \( \omega = \psi|_{P(A)} \).
Proposition 5.3.9. If $\mathcal{A}$ has no direct summand of Type $I_2$, then there is a (unique) Jordan $^*$-isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that $\varrho |_{P(\mathcal{A})} = \psi |_{P(\mathcal{A})}$.

Proof. It follows from Proposition 5.3.7 and Theorem 5.3.8 that $\varrho |_{P(\mathcal{A})}$ extends to a Jordan $^*$-homomorphism $\psi : \mathcal{A} \to \mathcal{B}$. It remains to show that $\psi$ is a bijection.

From Proposition 5.3.6, we have $\psi(P(\mathcal{A})) = g(P(\mathcal{A})) = P(\mathcal{B})$. Since $\mathcal{B}$ is the closed linear span of $P(\mathcal{B})$ and $\psi$ is a continuous linear map, $\psi$ is surjective.

Using the same arguments as in the proof of [38, Theorem 8.1.2], we show that $\psi$ is injective. The first step is to prove that if $a \in \mathcal{A}_+$ and $\psi(a) = 0$, then $a = 0$. We can assume without loss of generality that $0 \leq a \leq 1$. It follows from [53, Theorem 4.1.13] that there is a sequence $(e_n)_{n=1}^{\infty}$ of projections in $\mathcal{A}$ such that $a = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n$. We conclude from this that

$$\frac{1}{2^n} \psi(e_n) \leq \psi(a) = 0$$

for all $n$. Accordingly, $0 = \psi(e_n) = g(e_n)$ for all $n$ and so, by injectivity of $\varrho$, $e_n = 0$ for all $n$. Hence $a = 0$.

Assume now that $a \in \mathcal{A}_{sa}$ and $\psi(a) = 0$. Then $0 = \psi(a)^2 = \psi(a^2)$. Since $a^2 \in \mathcal{A}_+$, we have $a^2 = 0$. Thus $\|a\|^2 = \|a^2\| = 0$ which implies that $a = 0$.

Finally, suppose that $a \in \mathcal{A}$ and $\psi(a) = 0$. As $a = x + iy$, where $x, y \in \mathcal{A}_{sa}$, we have

$$0 = \psi(a) = \psi(x) + i\psi(y).$$

Therefore, $\psi(x) = \psi(y) = 0$ and so $x = y = 0$. Consequently, $a = 0$. This shows that $\psi$ is injective.

In the sequel, we shall assume that the von Neumann algebra $\mathcal{A}$ is without direct summand of Type $I_2$. Moreover, we shall denote

$$\theta = \psi^{-1} \circ \varrho,$$

where $\psi$ is a Jordan $^*$-isomorphism specified in Proposition 5.3.9 above. In the following result, we summarize some properties of the map $\theta$.

Proposition 5.3.10. The following statements hold:

(i) The map $\theta : \mathcal{A}_n \to \mathcal{A}_n$ is a continuous unital star order isomorphism.

(ii) $\theta(\lambda 1) = g(\lambda) 1$ for all $\lambda \in \mathbb{C}$.
(iii) For each $p \in P(\mathcal{A})$ and $\lambda \in \mathbb{C}$ there is a projection $e_\lambda \in P(\mathcal{A})$ such that $\theta(\lambda p) = g(\lambda) e_\lambda$.

(iv) $\theta(p) = p$ for all $p \in P(\mathcal{A})$.

Proof. (i) It is clear from Proposition 5.1.3 that $\psi^{-1}(\mathcal{B}_n) = \mathcal{A}_n$. Moreover, the maps $\psi^{-1}: \mathcal{B} \to \mathcal{A}$ and $\varrho: \mathcal{A}_n \to \mathcal{B}_n$ are continuous unital star order isomorphisms. This shows that $\theta = \psi^{-1} \circ \varrho$ is a continuous unital star order isomorphism from $\mathcal{A}_n$ onto $\mathcal{A}_n$.

(ii) This follows immediately from Proposition 5.3.4(i) and the fact that $\psi^{-1}$ is a Jordan $*$-isomorphism.

(iii) This statement is a direct consequence of Proposition 5.3.5.

(iv) Since Jordan $*$-isomorphism $\psi$ extends $\varrho|_{P(\mathcal{A})}$,

$$p = \psi^{-1}(\psi(p)) = \psi^{-1}(\varrho(p)) = \theta(p)$$

for all $p \in P(\mathcal{A})$. □

Lemma 5.3.11. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and $p \in P(\mathcal{A})$. Suppose that $e_\lambda$ and $f_\lambda$ are projections in $\mathcal{A}$ such that $\theta(\lambda p) = g(\lambda) e_\lambda$ and $\theta(\lambda (1-p)) = g(\lambda) f_\lambda$. Then $e_\lambda \perp f_\lambda$.

Proof. The case $\lambda = 1$ is straightforward.

Let $\lambda \in \mathbb{C} \setminus \{0,1\}$. As $p \perp 1-p$, we have, by Proposition 5.3.2, that there are elements $a,b \in \mathcal{A}_n$ such that

$$\lambda p, 1-p \preceq a \quad \text{and} \quad p, \lambda (1-p) \preceq b.$$

Applying $\theta$ to these relations and using its properties, we obtain

$$g(\lambda)e_\lambda, 1-p \preceq \theta(a) \quad \text{and} \quad p, g(\lambda)f_\lambda \preceq \theta(b).$$

It follows from Proposition 5.3.2 that $1-p \perp e_\lambda$ and $p \perp f_\lambda$. Therefore, $e_\lambda \perp f_\lambda$. □

In the next lemma, we denote the supremum of elements $a,b \in \mathcal{A}$ (with respect to the star order) by $a \vee b$.

Lemma 5.3.12. Let $p,q \in P(\mathcal{A})$ and $p \perp q$. Then, for each $\lambda \in \mathbb{C}$,

$$(\lambda p) \vee (\lambda q) = \lambda (p+q).$$
Proof. It is clear that $\lambda p, \lambda q \preceq \lambda(p + q)$.

If $a \in A$ is such that $\lambda p, \lambda q \preceq a$, then

$$|\lambda|^2p = \overline{\lambda pa} = \overline{\lambda ap},$$

$$|\lambda|^2q = \overline{\lambda qa} = \overline{\lambda aq}.$$

This implies that

$$|\lambda|^2(p + q) = \overline{\lambda(p + q)a} = \overline{\lambda a(p + q)}.$$

Therefore, $\lambda(p + q) \preceq a$. \hfill $\square$

Lemma 5.3.13. For every $p \in P(A)$ and $\lambda \in \mathbb{C}$ we have

$$\theta(\lambda(1 - p)) = \theta(\lambda 1) - \theta(\lambda p).$$

Proof. The case $\lambda = 0$ is obvious.

Let $\lambda \neq 0$. It follows from Lemma 5.3.12 that $(\lambda p) \lor \lambda(1 - p) = \lambda 1$. Since $\theta$ is a star order isomorphism, we have

$$\theta(\lambda 1) = \theta((\lambda p) \lor \lambda(1 - p)) = \theta(\lambda p) \lor \theta(\lambda(1 - p)).$$

We know that $\theta(\lambda p) = g(\lambda)e_\lambda$ and $\theta(\lambda(1 - p)) = g(\lambda)f_\lambda$, where $e_\lambda, f_\lambda \in P(A)$. By Lemma 5.3.11 and Lemma 5.3.12, we infer

$$\theta(\lambda p) \lor \theta(\lambda(1 - p)) = (g(\lambda)e_\lambda) \lor (g(\lambda)f_\lambda) = g(\lambda)(e_\lambda + f_\lambda) = \theta(\lambda p) + \theta(\lambda(1 - p)).$$

Hence

$$\theta(\lambda(1 - p)) = \theta(\lambda 1) - \theta(\lambda p).$$

\hfill $\square$

Proposition 5.3.14. For each $p \in P(A)$ and $\lambda \in \mathbb{C}$ we have

$$\theta(\lambda p) = g(\lambda)p.$$

Proof. The cases $\lambda = 0, 1$ are straightforward.

Let $\lambda \in \mathbb{C} \setminus \{0, 1\}$. We know that $\theta(\lambda p) = g(\lambda)e_\lambda$, where $e_\lambda$ is an element of $P(A)$. Moreover, by Proposition 5.3.10(ii) and Lemma 5.3.13, we have $\theta(\lambda(1 - p)) = g(\lambda)(1 - e_\lambda)$. Let us prove that $e_\lambda = p$. Since $p \perp 1 - p$, it follows from Proposition 5.3.2 that there are $a, b \in A_n$ such that

$$\lambda p, 1 - p \preceq a \quad \text{and} \quad p, \lambda(1 - p) \preceq b.$$

Therefore,

$$g(\lambda)e_\lambda, 1 - p \preceq \theta(a) \quad \text{and} \quad p, g(\lambda)(1 - e_\lambda) \preceq \theta(b).$$

By Proposition 5.3.2, $1 - p \perp e_\lambda$ and $p \perp 1 - e_\lambda$. This means that $e_\lambda \preceq p$, $p \preceq e_\lambda$ and so $e_\lambda = p$. \hfill $\square$
CHAPTER 5. PRESERVERS OF THE STAR ORDER

Now we have arrived at the main result of this section.

**Theorem 5.3.15.** Let $\mathcal{A}$ be a von Neumann algebra without Type $I_2$ direct summand and let $\mathcal{B}$ be a von Neumann algebra. Let $\varphi : \mathcal{A}_n \to \mathcal{B}_n$ be a continuous star order isomorphism. Suppose that there is an invertible central element $c \in \mathcal{B}$ and a function $f : \mathbb{C} \to \mathbb{C}$ such that

$$\varphi(\lambda 1) = f(\lambda) c$$

for all $\lambda \in \mathbb{C}$. Then $f$ is a continuous bijection with $f(0) = 0$ and there is a unique Jordan $^*$-isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that

$$\varphi(a) = \psi(f(a))c \quad (5.4)$$

for all $a \in \mathcal{A}_n$.

**Proof.** Following notation introduced in the text we shall prove at first that

$$\theta(a) = g(a)$$

for all $a \in \mathcal{A}_n$. Let us remark that this part of the proof is similar to the corresponding part of the proof of Theorem 5.2.5 (see [29]). We already know that

$$\theta(\lambda p) = g(\lambda)p$$

for all $\lambda \in \mathbb{C}$ and $p \in P(\mathcal{A})$. Suppose now that $p_1, \ldots, p_n$ are pairwise orthogonal projections and $\lambda_i \in \mathbb{C} \setminus \{0\}$, where $i = 1, \ldots, n$. We have

$$\lambda_i p_i \preceq \sum_{j=1}^n \lambda_j p_j$$

and so

$$g(\lambda_i) p_i = \theta(\lambda_i p_i) \preceq \theta\left(\sum_{j=1}^n \lambda_j p_j\right)$$

for all $i = 1, \ldots, n$. This implies

$$\left(\sum_{i=1}^n g(\lambda_i) p_i\right)^* \theta\left(\sum_{j=1}^n \lambda_j p_j\right) = \left(\sum_{i=1}^n g(\lambda_i) p_i \theta\left(\sum_{j=1}^n \lambda_j p_j\right)\right) =$$

$$= \sum_{i=1}^n |g(\lambda_i)|^2 p_i = \left(\sum_{i=1}^n g(\lambda_i) p_i\right)^* \left(\sum_{j=1}^n g(\lambda_j) p_j\right).$$
Similarly, one can establish
\[ \theta \left( \sum_{j=1}^{n} \lambda_j p_j \right) \left( \sum_{i=1}^{n} g(\lambda_i) p_i \right)^* = \left( \sum_{j=1}^{n} g(\lambda_j) p_j \right) \left( \sum_{i=1}^{n} g(\lambda_i) p_i \right)^*. \]

In other words,
\[ \sum_{j=1}^{n} \theta(\lambda_j p_j) = \sum_{j=1}^{n} g(\lambda_j) p_j \preceq \theta \left( \sum_{j=1}^{n} \lambda_j p_j \right). \tag{5.5} \]

As \( \theta^{-1} \) has similar properties as \( \theta \), we can analogically derive, considering linear combinations \( \sum_{j=1}^{n} g(\lambda_j) p_j \), that
\[ \sum_{j=1}^{n} \lambda_j p_j = \sum_{j=1}^{n} \theta^{-1}(g(\lambda_j) p_j) \preceq \theta^{-1} \left( \sum_{j=1}^{n} g(\lambda_j) p_j \right). \]

Therefore,
\[ \theta \left( \sum_{j=1}^{n} \lambda_j p_j \right) \preceq \sum_{j=1}^{n} g(\lambda_j) p_j. \tag{5.6} \]

Combining (5.5) with (5.6), we have
\[ \sum_{j=1}^{n} g(\lambda_j) p_j = \theta \left( \sum_{j=1}^{n} \lambda_j p_j \right). \]

In other words, \( \theta(a) = g(a) \) for any normal element of \( a \) with finite spectrum. Now density of such elements in the set \( \mathcal{A}_n \) and continuity of \( \theta \) imply that \( \theta(a) = g(a) \) for all \( a \in \mathcal{A}_n \).

From the discussion in this section, we see that
\[ \varphi(a) = f(1)\psi(\theta(a))c = f(1)\psi(g(a))c = \psi(f(1)g(a))c = \psi(f(a))c \]
for all \( a \in \mathcal{A}_n \).

It remains to prove that \( f \) is necessarily surjective. For this, it is sufficient to show that \( g \) is surjective. By the surjectivity of \( \theta \), for every \( \lambda \in \mathbb{C} \), there exists \( a \in \mathcal{A}_n \) such that \( \theta(a) = \lambda 1 \). Let \( \sigma(\theta(a)) \) denote the spectrum of \( \theta(a) \). Since
\[ \{\lambda\} = \sigma(\theta(a)) = \sigma(g(a)) = g(\sigma(a)), \]
we have the following: for every \( \lambda \in \mathbb{C} \), there is \( \mu \in \mathbb{C} \) such that \( g(\mu) = \lambda \). \( \square \)
Note that every map of the form (5.4) is a continuous star order isomorphism $\varphi : A_n \rightarrow B_n$. This follows immediately from Proposition 4.2.4(iii), Theorem 4.2.9, and Proposition 5.1.6.

It is perhaps interesting to mention that the star order isomorphism (5.4) has the nonlinear part (given by the functional calculus $a \mapsto f(a)$) isolated on the algebra $A$.

The following corollary describing continuous star order isomorphisms between abelian von Neumann algebras is a direct consequence of the preceding theorem.

**Corollary 5.3.16.** Let $A$ and $B$ be abelian von Neumann algebras. Let $\varphi : A \rightarrow B$ be a continuous star order isomorphism. Suppose that there is an invertible element $c \in B$ and a function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\varphi(\lambda 1) = f(\lambda)c$$

for all $\lambda \in \mathbb{C}$. Then $f$ is a continuous bijection with $f(0) = 0$ and there is a unique $*$-isomorphism $\psi : A \rightarrow B$ such that

$$\varphi(a) = \psi(f(a))c$$

for all $a \in A$.

**Proof.** Using Proposition 5.1.3 we see that every Jordan $*$-isomorphism from $A$ onto $B$ is $*$-isomorphism. Moreover, abelian von Neumann algebras are of Type $I_1$. Therefore, the desired conclusion follows from Theorem 5.3.15. \(\square\)

The next theorem is a version of Theorem 5.3.15 for star order isomorphisms between self-adjoint parts of a von Neumann algebras. This result can be proved in the same way as Theorem 5.3.15 and so the proof will be omitted.

**Theorem 5.3.17.** Let $A$ be a von Neumann algebra without Type $I_2$ direct summand and let $B$ be a von Neumann algebra. Let $\varphi : A_{sa} \rightarrow B_{sa}$ be a continuous star order isomorphism. Suppose that there is an invertible central self-adjoint element $c \in B$ and a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\varphi(\lambda 1) = f(\lambda)c$$

(5.7)

for all $\lambda \in \mathbb{R}$. Then $f$ is a continuous bijection with $f(0) = 0$ and there is a unique Jordan $*$-isomorphism $\psi : A \rightarrow B$ such that

$$\varphi(a) = \psi(f(a))c$$

(5.8)

for all $a \in A_{sa}$. 
Let \( \varphi : A_{sa} \to B_{sa} \) be a star order isomorphism satisfying the assumptions of Theorem 5.3.17. It seems to be interesting that the nonlinearity of \( \varphi \) on the multiples of identity determines the nonlinearity of \( \varphi \) on all elements of \( A_{sa} \). More precisely, the function \( f \) appearing in (5.8) is uniquely given by the condition (5.7). In particular, if the map \( \varphi \) is unital and linear only on the one-dimensional subspace generated by the unit, then \( \varphi \) must be automatically linear on \( A_{sa} \). This interesting phenomenon is formulated in the following corollary.

**Corollary 5.3.18.** Let \( A \) be a von Neumann algebra without Type \( I_2 \) direct summand and let \( B \) be a von Neumann algebra. Let \( \varphi : A_{sa} \to B_{sa} \) be a continuous star order isomorphism. If

\[
\varphi(\lambda 1) = \lambda 1
\]

for all \( \lambda \in \mathbb{R} \), then \( \varphi \) is the restriction of a Jordan *-isomorphism \( \psi : A \to B \) to \( A_{sa} \).

Let us now discuss the role of the condition (5.7) in Theorem 5.3.17. By a simple example, we show that not all continuous star order isomorphisms between self-adjoint parts of von Neumann algebras (without Type \( I_2 \) direct summand) satisfy (5.7). This justifies our additional assumption \( (5.7) \) and the similar assumption in Theorem 5.3.15.

**Example 5.3.19.** Consider \( A = \mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H}) \), where \( \dim \mathcal{H} = \infty \). Put \( z_1 = (1, 0) \) and \( z_2 = (0, 1) \), where \( 1 \) is a unit of \( \mathcal{B}(\mathcal{H}) \). Let \( f_1, f_2 \) be two continuous bijections of \( \mathbb{R} \) with \( f_1(0) = f_2(0) = 0 \) and \( f_1(1) = f_2(1) = 1 \) that are linearly independent. Define the map \( \psi : A_{sa} \to A_{sa} \) as follows:

\[
\psi(a) = f_1(az_1) + f_2(az_2), \quad a \in A_{sa}.
\]

It is easily verified by Theorem 4.2.9 that this map is a continuous unital star order isomorphism. For \( \lambda \in \mathbb{R} \), we have

\[
\psi(\lambda 1) = f_1(\lambda z_1) + f_2(\lambda z_2) = f_1(\lambda)z_1 + f_2(\lambda)z_2.
\]

We show that \( \psi \) does not act on scalars multiples of identity in the sense of (5.7). Suppose the contrary and try to reach a contradiction. Let there be a real function \( h \) such that

\[
\psi(\lambda 1) = h(\lambda)z
\]

5The similar conclusion can be stated for star order isomorphisms appearing in Theorem 5.3.14.
for all $\lambda \in \mathbb{R}$, where $z$ is an invertible central self-adjoint element of $\mathcal{A}$. This central element can be written as

$$z = \alpha z_1 + \beta z_2,$$

where $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. From (5.9), we have

$$f_1(\lambda)z_1 + f_2(\lambda)z_2 = h(\lambda)\alpha z_1 + h(\lambda)\beta z_2.$$

This means that $f_1 = \alpha h$ and $f_2 = \beta h$. But this contradicts the fact that $f_1$ and $f_2$ are linearly independent.

The following result is another version of Theorem 5.3.15. It can be proved in the same way as Theorem 5.3.15 and therefore the proof will be omitted.

**Theorem 5.3.20.** Let $\mathcal{A}$ be a von Neumann algebra without Type $I_2$ direct summand and let $\mathcal{B}$ be a von Neumann algebra. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a continuous star order isomorphism. Suppose that there is an invertible central element $c \in \mathcal{B}$ and a function $f : \mathbb{C} \to \mathbb{C}$ such that

$$\varphi(\lambda 1) = f(\lambda)c$$

for all $\lambda \in \mathbb{C}$. Then $\varphi(\mathcal{A}_n) \subseteq \mathcal{B}_n$, $f$ is an injective function with $f(0) = 0$, and there is a unique Jordan $^*$-isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that

$$\varphi(a) = \psi(f(a))c$$

for all $a \in \mathcal{A}_n$.

As a simple consequence of the previous theorem, we obtain a result concerning the automatic linearity of certain star order isomorphisms between von Neumann algebras. This result is an analogue of Corollary 5.3.18 and provides a new characterization of Jordan $^*$-isomorphisms in which the condition of linearity seems to be very relaxed. Note that we do not assume that $\varphi$ preserves the self-adjoint elements.

**Corollary 5.3.21.** Let $\mathcal{A}$ be a von Neumann algebra without Type $I_2$ direct summand and let $\mathcal{B}$ be a von Neumann algebra. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a continuous star order isomorphism such that

$$\varphi(\lambda 1) = \lambda 1$$

for all $\lambda \in \mathbb{C}$. Suppose that

$$\varphi(a + ib) = \varphi(a) + i\varphi(b)$$

for all $a, b \in \mathcal{A}_{sa}$. Then $\varphi$ is a Jordan $^*$-isomorphism.
Proof. By Theorem 5.3.20, there is a Jordan *-isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(a) = \psi(a)$ for all $a \in \mathcal{A}_s$. If $a \in \mathcal{A}$ is arbitrary, then there are $x_1, x_2 \in \mathcal{A}_{sa}$ such that $a = x_1 + ix_2$. Therefore,

$$\varphi(a) = \varphi(x_1) + i\varphi(x_2) = \psi(x_1) + i\psi(x_2) = \psi(x_1 + ix_2) = \psi(a).$$

$\square$
Chapter 6

Conclusion

This thesis has dealt with Bell inequalities and the star order from the viewpoint of operator algebras. The results obtained in the thesis can be briefly summarized as follows.

In Chapter 3 we have studied the Cirel’son inequality and the (CHSH version of) Bell inequality. In Section 3.2, the Cirel’son inequality has been generalized to real and complex linear spaces endowed with a pseudo inner product. The investigation on this abstract level has led to interesting consequences for the maximal violation of the Bell inequality. In Section 3.3, we have proved that the Bell inequality is maximally violated in some faithful state on a unital *-algebra if and only if the maximal violators form realizations of Pauli spin matrices. As a consequence, the intersection of *-subalgebras describing local systems has to be nontrivial. Therefore, the maximal violation of the Bell inequality in the faithful state is not compatible with mutual commutativity of local subalgebras. This has motivated the definition of the weakly uncoupled state across unital *-algebras. We have shown that if the Bell inequality is maximally violated in a weakly uncoupled state that is faithful on local subalgebras, then maximal violators are again realizations of Pauli spin matrices and given uncoupled state must act as a tracial state on local algebras generated by maximal violators. In Section 3.4, similar statements have been proved for maximal violators of the Bell inequality in purely algebraic context of nonassociative structures given by Jordan algebras. The results of Sections 3.2, 3.3, and 3.4 were published in the papers [15, 16] and are far-reaching generalizations of the result in [75].

Chapter 4 has been concerned with the star order on operator and function algebras. In Section 4.2, some elementary properties known from matrix algebras and the algebra of all bounded operator on a Hilbert space have been generalized. In particular, we have described the behavior of the star order
with respect to the continuous function calculus. Section 4.3 has been devoted to the star order on partial isometries. The connection between the star order on partial isometries and relations of their initial and final projections has been determined. Moreover, we have provided quite new characterization infinite $C^*$-algebras in terms of the star order. The infimum and supremum problem for the star order on the algebra $C(X)$ has been investigated in Section 4.4 and Section 4.5. We have shown that every upper bounded subset of $C(X)$ has the infimum and the supremum whenever $X$ is a locally connected or an extremely disconnected Hausdorff topological space. Applying this result to abelian $C^*$-algebras, we have obtained that the infimum and supremum of any upper bounded subset exists in the case of abelian $C^*$-algebras with locally connected spectrum and abelian von Neumann algebras. Moreover, we have constructed the example showing that the solution of infimum problem has negative answer in general and so our restriction to locally connected and extremely disconnected topological spaces is well-justified. The results of Chapter 4 were published in the paper [14].

Chapter 5 has been devoted to star order isomorphisms. In Section 5.3 we have completely described the structure of certain continuous star order isomorphisms from the normal part of a von Neumann algebra without Type I$_2$ direct summand onto the normal part of another von Neumann algebra which are nonlinear in general. We have also discussed several modifications of this result. It should be noted that a characterization of star order isomorphisms was known only for continuous star order isomorphisms between self-adjoint part of Type I factors before our investigation [29]. As a consequence of our analysis we have provided a new characterization of Jordan *-isomorphisms among not necessary linear maps preserving the star order. The results of Section 5.3 can be found in the paper [17].

The area of quantum structures provides many possibilities for further research. Therefore, we outline here only directions related to the themes presented in this thesis. The results concerning Bell inequalities could be extended especially by the investigation of multipartite Bell inequalities or Bell inequalities for more than two observables per site. This research can be inspired by the nice results which have been recently obtained by using operator space theory [43, 141, 66, 67].

The star order could be generalized to JBW algebras which are nonassociative counterparts of von Neumann algebras. Another line of the research of the star order could be the study of the order topology induced by this order and its relation to standard locally convex topologies on operator algebras.
Bibliography


