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MASTER THESIS

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Nekonečné matroidy

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Nekonečné matroidy

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Abstrakt: Práce prezentuje aktuální pokroky v oblasti teorie nekonečných matroidů. V práci jsou zadefinovány a dokázány základní vlastnosti nekonečných matroidů a předvedeny známé třídy těchto struktur. Práce se zaměřuje na problematiku souvislosti nekonečných matroidů a poukazuje na vztahy některých matroidových operací se souvislostí. Hlavní výsledek práce ukazuje existenci nekonečných matroidů libovolné konečné souvislosti se speciálními vlastnostmi – bez konečných kružnic a kokružnic.

Klíčová slova: matroidy, nekonečno, souvislost, kružnice, kokružnice

Title: Infinite matroids

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Abstract: We summarize and present recent results in the field of infinite matroid theory. We define and prove basic properties of infinite matroids and we discuss known classes of examples of these structures. We focus on the topic of connectivity of infinite matroids and we link some matroid properties to connectivity. The main result of this work is the proof of existence of infinite matroids with arbitrary finite connectivity, but without finite circuits or cocircuits.

Keywords: matroids, infinite, connectivity, circuit, cocircuit

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1. Preface

1.1 Short history of infinite matroids

The notion of a finite matroid has been introduced in the 1930s by Whitney [Whi35] as a generalization of linear dependence and graph structures, and the idea of extending the notion of a matroid into infinite sets arrived not much later.

The original definition of an infinite matroid made it a *finitary* structure, that is, with all circuits to be finite. These matroids are not even closed on duality, which is another essential property of the finite matroid theory.

Therefore, in 1966, Rado [Rad66] asked for a structure which extended these finitary matroids and was as general as possible while still preserving duality. The first (and ultimately the best) answer to this question was a structure called a *B-matroid* proposed by Higgs [Hig69c] and worked on by others, with some results from that time summarized in Oxley's 1978 paper on infinite matroids [Oxl78].

While the B-matroid definition was general enough and was closed on duality, its axiomatization used both the *independence axioms* and *basis axioms* of the finite matroid theory, plus one new axiom describing maximality called (IM). This may have been one of the reasons why there was little progress in the area of infinite matroids until the early 2010s.

Starting in 2010, a Hamburg research group consisting of Bruhn, Diestel, Kriesell, Pendavingh, Wollan and others ([BDK$^+$13], [BC12b], [BD11], [AHCF11a] and more) made a substantial breakthrough in the axiomatization of infinite matroids. They have proven that the axiom (IM) can be added to most finite matroid axiomatization, with the resulting class being identical to the class of B-matroids.

Since then, several papers have extended on their results, trying to translate concept from finite matroid theory into infinite matroid theory. It turns out that for infinite matroids, the concept of *rank* is not as useful as for their finite counterparts. This means that a lot of the new proofs for infinite matroids provide also new insights into the structure of finite matroids.

1.2 Thesis overview

The aim of the thesis is to summarize the current breakthroughs in the field of infinite matroid theory, extend the set of known infinite matroids with new classes, and provide new insight into the connectivity of some infinite matroids.

In Chapter 2, we introduce the axiomatics and basic properties of the infinite matroid theory, such as duality, minors and extending bases. We also show already known examples of matroids.

In Chapter 3, we present the concept of connectivity of an infinite matroid, explain its basic properties with relation to duality and minors and finally show connectivity bounds for some of the infinite matroid classes.

In Chapter 4, we present the plus operation that extends independent sets of a matroid and discuss its previous use in the literature. We then present the major new result of this thesis – that the plus operation increases connectivity in a class of matroids not containing any finite circuit or cocircuit.

A corollary of this result is the fact that there exists an infinite matroid without finite circuits and cocircuits with finite but arbitrary large connectivity. The question of existence of infinite matroids with infinite connectivity remains open.

In Chapter 5, we show recent results on the intersection size of a circuit and a cocircuit in an infinite matroid, and show their relation to the class of graphs without a finite circuit or cocircuit. We also briefly discuss intersections of pairs of circuits.

1.2.1 Further links

The thesis focuses on the topic of matroid connectivity; therefore, some less related recent results on infinite matroids (such as the concept of infinite matroid union, intersection and packing/covering) are not covered. For these topics, we refer the reader to the paper of Bowler and Carmesin [BC12a] and the papers of Aigner-Horev, Carmesin and Fröhlich [AHCF11a], [AHCF11b].

If the reader wishes to keep up with the recent progress in infinite matroid theory, the Hamburg research group on infinite matroids collects their published and unpublished articles on the website [HPo].

2. Basic definitions

2.1 Notation

Matroid theory abounds with sets, set systems, unions and intersections. For consistency, we standardize on the following notation:

Notation 1. We will always denote a single element by a lowercase letter, such as x, y and e. Sets of elements, for example an independent set or a circuit, will be denoted with uppercase characters, such as X, Y and K. Set systems, such as the system of all bases, will be denoted with caligraphic characters, such as $\mathcal{B}$ or $\mathcal{C}$. Finally, we will denote individual matroids using the blackboard bold script, such as $\mathbb{M}_1$ and $\mathbb{M}_2$.

As for addition and removal of single elements, we will use the following notation:

Notation 2. $I + x$ will stand for $I \cup \{x\}$, similarly $I - x$ will mean $I \setminus \{x\}$.

For general unions and intersections between two sets, we use $X \cup Y$ and $X \setminus Y$ as usual.

Notation 3. When discussing duality on a ground set E and $X \subseteq E$, we use $\overline{X}$ for $E \setminus X$ while for dual structures we prefer the star notation: $\mathcal{B}^* = \{\overline{B} | B \in \mathcal{B}\}$.

Notation 4. We employ $\downarrow\mathcal{B}$ for a down-closure of the set system $\mathcal{B}$, that is, an extension of $\mathcal{B}$ with all subsets of sets in $\mathcal{B}$.

2.2 The notion of an infinite matroid

We assume that the reader is already knowledgeable in the basics concepts of finite matroid theory. For a good introduction see e.g. the textbook of Oxley [Oxl11].

Definition 1. A finite matroid is a tuple $(E, \mathcal{I})$ such that E is finite, $\mathcal{I}$ (a family of *independent sets*) is a set system of E and fulfills the following three conditions:

(I1) An empty set is in $\mathcal{I}$.

(I2) $\mathcal{I}$ is closed on taking subsets.

(I3) If $I_1, I_2 \in \mathcal{I}$, $|I_1| = k$ and $|I_2| > k$, then $\exists a \in I_2 \setminus I_1$ such that $I_1 + a \in \mathcal{I}$.

This definition of a finite matroid is standard in the literature, but it uses the sizes of the independent sets I_1 and I_2, which is not very relevant in the infinite setting. A very small modification will be of more use to us. We first prove the equivalence of the small modification with the standard axiomatics for finite matroids:

Observation 1. *For finite matroids, the conditions (I1),(I2),(I3) in the finite matroid definition are equivalent to (I1),(I2) and the following:*

(I3') If $I_1, I_2 \in \mathcal{I}$, I_2 is maximal in $\mathcal{I}$ inclusion-wise but I_1 is not, then $\exists x \in I_2 \setminus I_1$ such that $I_1 + a \in \mathcal{I}$.

Proof. (I1),(I2),(I3) imply (I1),(I2),(I3'). This follows immediately, as any set maximal in $\mathcal{I}$ can be used as I_2 in (I3).

(I1), (I2), (I3') imply (I1), (I2), (I3). If I_2 is not maximal, we extend it by a greedy process from elements in $E \setminus I_2$ until we arrive at a maximal set I_2'. We iteratively apply (I3') until we arrive at a maximal set I_1' that extends I_1 with elements from I_2'.

Since at the start we had that $|I_1| < |I_2|$ and we cannot apply (I3') anymore, it must hold that $|I_1'| = |I_2'|$ and we see that at least one element from I_2 had to have been added to I_1 during this process. This element can play the role of a in the rule (I3). □

Now that we have an axiomatics for finite matroids based not on relative sizes, but on maximality, we can present the definition of an infinite matroid, as stated by Bruhn, Diestel et al.:

Definition 2. [BDK+13] An *infinite matroid* $(X, \mathcal{I})$, is a structure with a finite or infinite set X fulfilling the requirements (I1), (I2), (I3') for finite matroids, and also the following condition:

(IM) For any set $S \subseteq X$ and any $I \in \mathcal{I}, I \subseteq S$, there exists an inclusion-wise maximal independent subset of S that contains I.

Besides independent sets, the most important notions in matroid theory are those of bases, circuits and cocircuits.

Definition 3. In a matroid $\mathbb{M}$, a *basis* is any inclusion-wise maximal independent subset of E. A *circuit* is any inclusion-wise minimal dependent subset of E, and a *cocircuit* is an inclusion-wise minimal set such that it intersects every basis of $\mathbb{M}$.

It is well-known that we can define matroids using axiomatics that talk about bases and circuits. The same can be done for infinite matroids, only we have to be more careful in the statements and we also need to usually include the axiom (IM) in some form. No axiomatics that does not employ (IM) has yet been proposed.

We state the basis and circuit definitions for completeness:

Claim 1. *A matroid is a structure* $\mathbb{M} = (E, \mathcal{B})$ *for which the following three axioms hold:*

(B1) $\mathcal{B} \neq \emptyset$.

(B2) $\forall B_1 \neq B_2 \in \mathcal{B}, \forall x \in B_2 \setminus B_1 \exists y \in B_1 \setminus B_2$ such that $B_1 - y + x \in \mathcal{B}$.

(BM) The set system $\downarrow\mathcal{B}$ fulfills the axiom (IM).

Claim 2. *A matroid is a structure* $\mathbb{M} = (E, \mathcal{C})$ *for which the following four axioms hold:*

(C1) $\emptyset \notin \mathcal{C}$.

(C2) $\forall C_1 \neq C_2 \in \mathcal{C}$, C_1 is not a subset of C_2.

(C3) (Infinite circuit elimination) Given a circuit $C \in \mathcal{C}$, a subset of elements $X \subseteq C$, a set of circuits $\mathcal{D}$ indexed by elements of X ($\mathcal{D} = \{C_x | x \in C, x \in C_x\}$) such that $\forall C_x, C_y \in \mathcal{D} : y \in C_x \implies x = y$, and an element $z \in C \setminus (\bigcup_{C_x \in \mathcal{D}} C_x)$, there exists a circuit C' such that $z \in C' \subseteq (C \cup \bigcup_{C_x \in \mathcal{D}} C_x) \setminus X$.

(CM) If we define a set system $\mathcal{I}$ such that $I \in \mathcal{I} \equiv \forall C \in \mathcal{C} : C \nsubseteq I$, then $\mathcal{I}$ fulfills the axiom (IM).

The equivalence was proven in the seminal paper of Bruhn, Diestel et al:

Theorem 1. *[BDK+13] A structure satisfies the conditions (B1),(B2),(BM) if and only if it satisfies (I1),(I2),(I3') and (IM). Similarly, a structure satisfies (C1),(C2),(C3) and (CM) if and only if it satisfies (I1),(I2),(I3') and (IM).*

It is a natural question to ask at this point about the *rank function*, which is invaluable for many of the proofs of finite matroid theory and the submodularity of which has sparked research into submodular functions in general.

The biggest problem of the rank function is that $r(X)$ only calculates the size of the largest independent set of X. However, $r(X) = \omega$ is a common phenomenon in infinite matroids, and it actually does not tell us very much. Specifically, whenever we add an element e to a set X which already has $r(X) \geq \omega$, we have no way to verify whether we extended the largest independent set or not.

This issue has been sidestepped in the paper [BDK+13] by showing that a similar function, a *relative rank function*, can be used to axiomatize matroids (along with the ever-present axiom (IM)). The relative rank function calculates the maximum independent size difference between two sets Y and $X, X \subseteq Y$.

In the papers following [BDK+13], however, the relative rank function has only been used sparingly; suggesting that it may be easier to prove theorems by arguing about the set structure of the matroid directly.

It is easy to see that the axiom (I3') together with (IM) does not guarantee only finitely many extensions of an independent set, but that we can extend any set to a maximal set. This fact is stated precisely in the following lemma:

Lemma 1. *For a matroid* $\mathbb{M}$*, given an indepedent set* X *and a basis* B*, we can find a basis* B_X *that contains only elements from* $X \cup B$ *such that* X *is fully contained in* B_X*.*

Proof. Applying (IM) we find a maximal independent superset of X inside the set $X \cup B$. Denote it B_X. This B_X must also be maximal, because if not, we can extend it by a single element using (I3'), which is a contradiction with B_X being inclusion-maximal by (IM). □

Notation 5. We apply Lemma 1 frequently, and so whenever we apply it, we say we *extend* the set X to the set B_X with elements from a basis B.

The most important axiom of the circuit and basis definitions is arguably the circuit elimination axiom, as it produces a nice tool for generating further circuits and arguing about connectivity. We use it several times in later chapters, and so we propose the following notation:

Notation 6. When applying the axiom (C3) as a tool, we call the circuit C the *main circuit*, the set of circuits $\mathcal{D}$ the *intersecting circuits*, the set X will be denoted as the *indexing set* and $z \in C, z \notin \bigcup_{C_x \in \mathcal{D}} C_x$ a *guaranteed element.*

An important fact about basis size was already observed by Higgs in 1969:

Claim 3. *[Hig69a] Assuming the generalized continuum hypothesis, all bases of a matroid* $\mathbb{M}$ *are of the same cardinality.*

A weaker lemma on bases specifies the exchange property without assuming Claim 3:

Lemma 2. *If* B_1*,* B_2 *are bases of a matroid with* $|B_1 \setminus B_2|$ *finite, then* $|B_2 \setminus B_1| = |B_1 \setminus B_2|$*.*

Proof. We start with basis B_2 and for every item that is in $B_2 \setminus B_1$, we apply the basis exchange axiom (B2). Since $|B_1 \setminus B_2|$ was finite, we can only employ it $|B_1 \setminus B_2|$ times until we end with the basis B_1. Since we have switched a single element for a single element each time, we have performed $|B_2 \setminus B_1|$ operations but also $|B_1 \setminus B_2|$ operations, and the numbers must be equal. □

Using the definition of bases, we may define duality of matroids:

Definition 4. Given a set of all bases $\mathcal{B}$ of a matroid $\mathbb{M}$, the set $\downarrow(\mathcal{B}^*)$ is the independent set of a matroid $\mathbb{M}^*$ called the *dual* of $\mathbb{M}$.

Of course, immediately after this definition we need to verify the correctness of the definition. We will employ the following useful observation:

Observation 2. *Given a matroid* $\mathbb{M}$ *and a dual structure* $\mathbb{M}^*$ *as defined above, a set* X *will be in* $\mathcal{I}^*$ *if and only if the set* $\overline{X}$ *is spanning, that is, there exists a basis of* $\mathbb{M}$ *inside* $\overline{X}$*.*

Claim 4. *Given a set of all bases* $\mathcal{B}$ *of an infinite matroid* $\mathbb{M}$*, the set* $\downarrow(\mathcal{B}^*)$ *fulfills axioms (I1), (I2), (I3') and (IM).*

Proof. Axioms (I1) are (I2) are inherited immediately from the down-closure operator.

For (I3') for the dual matroid, suppose we have a (co)independent set $I \in \mathcal{I}^*$ non-maximal and a maximal element $B \in \mathcal{B}^*$. Since I is non-maximal in the dual, the complement $\overline{I}$ is dependent. Pick a $\mathbb{M}$-basis B inside $\overline{I}$ and remove one element from $\overline{I}$ that is not inside B. In the dual, we have added an element to I, creating I', but as $\overline{I'}$ contains a basis, therefore I' is still independent.

Finally we verify the axiom (IM) for the dual structure. Suppose that we want to find a $\mathbb{M}^*$-maximal independent subset of a coindependent set I^* within a set

X. We apply (IM) for $\mathbb{M}$ to find a maximal independent subset of $\overline{X}$, denoting it as J.

From our observation above we know that since I^* is independent in the dual, the set $E \setminus I^*$ is spanning, and we can use the axiom (IM) to extend J using some set $J' \subseteq X \setminus I^*$ to a basis of $\mathbb{M}$.

The set $I' = X \setminus J'$ contains the set I because of its construction. We show that it is both independent and maximal, thus satisfying (IM).

- Independence: Since $J' \cup J$ is a basis of $\mathbb{M}$, its complement is a maximal element of $\mathcal{B}^*$, and I' is a subset of such a complement.

- Maximality: Suppose we can extend I' within X to a larger maximal set I'' of $\downarrow(\mathcal{B}^*)$, and we denote $X \setminus I''$ as J''. As I'' is an extension of I', J'' is a restriction (subset) of J', and therefore it is independent.

  As $J \cup J'$ is a basis, $J \cup J''$ is its proper subset, and therefore non-maximal. We can therefore use (I3') for $\mathbb{M}$ and extend it either from the set X or the set $\overline{X}$.

  If we are able to extend it from X, it contradicts the membership of I'' within $\downarrow(\mathcal{B}^*)$, but if we are able to extend it from $\overline{X}$, it contradicts our original application of (IM) which guaranteed that J was a maximal independent subset of $\overline{X}$. In either case, we arrive at a contradiction.

□

We have mentioned in Chapter 1 that the original definitions of a matroid were defined as *finitary* structures. Instead of the axiom (IM), they had a condition on a set being independent if and only if all its finite subsets were independent.

To explain what *finitary* means in the context of infinite matroids, we present the following definition:

Definition 5. A matroid is called *finitary* if all its circuits are of finite size. A matroid is called *cofinitary* if all its cocircuits have finite size; i.e. if its dual is finitary.

While the notion of the cocircuit was defined in Definition 3, we can also define it as follows:

Definition 6. For a matroid M, a cocircuit is a subset of elements such that it forms a circuit of the dual matroid $\mathbb{M}^*$.

Observation 3. *The two definitions of cocircuits (Definition 3 and Definition 6) are equivalent for all matroids.*

Proof. Consider a circuit C of $\mathbb{M}^*$ (a cocircuit of the primal). This is a minimal dependent set in $\mathbb{M}^*$. If there was a $\mathbb{M}$-basis B avoiding C, then the complement $\overline{B}$ would be a $\mathbb{M}^*$-basis, which is impossible, as $\overline{B}$ is a superset of C, which is a $\mathbb{M}^*$-circuit and thus $\mathbb{M}^*$-dependent.

The same argument said in reverse shows that every set that interesects every basis of $\mathbb{M}$ must be $\mathbb{M}^*$-dependent.

It is easy to see that the minimality of one implies minimality of the other, thus both conditions are preserved and the definitions are equivalent. □

Claim 5. *[BDK⁺13] In an infinite matroid $\mathbb{M}$, no circuit and cocircuit can meet in exactly one element.*

Proof. Suppose that the circuit C and the cocircuit D meet in one element e.

$D - e$ is not a cocircuit, and so there exists a basis B_1 of $\mathbb{M}$ not containing $D - e$. Necessarily, this basis contains e.

$C - e$ is independent, and can be extended using (IM) to a basis B_2 by extending $C - e$ with elements from B_1. The complement $\overline{B_2}$ is a cobasis by the duality condition, but it also contains both $D - e$ (because no element from $D - e$ was in B_1 and by extension B_2) and e (because e could not have been added to B_2).

The cobasis $\overline{B_2}$ therefore contains a cocircuit, which is a contradiction. □

Historical note. The above definition of the infinite matroid (Definition 2) was not the first definition that has been used in matroid theory. The important goal was to find a definition as broad possible (for example one that does not restrict only to locally finite matroids) while preserving the key notion of duality. The first definition to fulfill these requirements was the *B-matroid* definition by Higgs in the 1960s [Hig69c]:

Definition 7. We call a structure $(E, \mathcal{I})$ a B-matroid if all following requirements are met:

(I1) $\emptyset \in \mathcal{I}$.

(I2) $\mathcal{I}$ is down-closed.

(IM) $\forall X \subseteq E, \forall I \subseteq X, I \in \mathcal{I}$, there exists an inclusion-maximal subset of X in $\mathcal{I}$ extending I.

(B2) $\forall B_1, B_2$, which are two inclusion-maximal sets of $\mathcal{I}$, and $\forall x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $B_1 - x + y$ is also an inclusion-maximal set in $\mathcal{I}$.

The main advantage of Definition 2 is that it uses the standard axioms (I1),(I2),(I3) with only a small variation on (I3). Definition 2 can also be easily shown to be equivalent with the definition of a B-matroid:

Observation 4. *A structure $\mathbb{M}$ satisfies the conditions of the B-matroid if and only if it satisfies the conditions (I1),(I2),(I3') and (IM).*

Proof. An infinite matroid is a B-matroid. The condifitions (I1), (I2) and (IM) are common to both. As for (B2), we can apply (I3') to a non-maximal independent set $B_1 - x$ and extend it to a maximal independent set by an element from B_2 to get the required claim.

B-matroid is an infinite matroid. We show that the set of maximal elements of a B-matroid satisfies (B1), (B2) and (IM). (B2) and (IM) are immediate, as they are common to both definitions, and (B1) follows from (I1) and a simple application of (IM) on the ground set E. □

2.3 Properties of infinite matroids

2.3.1 Minors

Since we established duality of infinite matroids, to show validity of all minor operations we only need to show that restriction produces a matroid.

Lemma 3. *[BDK+13] Given a matroid* $\mathbb{M}$*, a basis* $B \in \mathcal{B}$ *and* $F \subseteq E$*,* $B \cap F$ *is maximal in* $\mathcal{I} \cap 2^F$ *if and only if* $\overline{B} \cap \overline{F}$ *is maximal in* $\mathcal{I}^* \cap 2^{\overline{F}}$*.*

Proof. $\Rightarrow$: If $B \cap F$ is maximal but the implication is not true, then we extend $\overline{B} \cap \overline{F}$ with $Z \neq \emptyset$ to a maximal independent superset $\overline{B'}$ of $\overline{F}$ and extend this $\overline{B'}$ to a cobasis $\overline{B''}$ of $\mathbb{M}$ using elements from $\overline{B}$.

The complement B'' is therefore a basis of $\mathbb{M}$.

Now, B'' contains $B \cap F$ (because we extended from $\overline{F}$ and $\overline{B}$ only) and so either $B'' \subseteq B$ (which is a contradiction with B'' being a basis) or $B \cap F \subset B'' \cap F$, contradicting the maximality of $B \cap F$ within $\mathcal{I} \cap 2^F$.

$\Leftarrow$: Apply the first implication in the matroid $\mathbb{M}^*$. □

Claim 6. *[BDK+13] Given an infinite matroid* $\mathbb{M}$ *and a subset of edges* $F \subseteq E$*, the restriction of* $\mathbb{M}$ *to* F *(denoted* $\mathbb{M}|F$*) will also constitute a matroid.*

Proof. (I1),(I2) and (IM) hold automatically. To show (I3'), we are given I and I_m, where I_m is maximal in $\mathbb{M}|F$ and I is independent non-maximal in $\mathbb{M}|F$. We extend I_m into a $\mathbb{M}$-basis B_m, then extend I into an $\mathbb{M}$-basis B using elements from B_m.

Since $B \cap \overline{F} \subseteq B_m \cap \overline{F}$, taking complements means $\overline{B_m} \cap \overline{F} \subseteq \overline{B} \cap \overline{F}$, and as $\overline{B_m} \cap \overline{F}$ is maximal in $\mathbb{M}^*|\overline{F}$, so is $\overline{B} \cap \overline{F}$. Applying Lemma 3 now, we know that $B \cap F$ is maximal also, and so $B \cap F$ is the desired extension of I with elements from I_m. □

Notation 7. Given a set $X \subseteq E$, a minor of $\mathbb{M}$ created by removing X from the ground set will be denoted as $\mathbb{M} \setminus X$ and a minor of $\mathbb{M}$ created by removing the set $Y \subseteq E$ in the dual (contraction) will be denoted $\mathbb{M}/Y$.

We will denote restriction on a specific set Z, i.e. removing the elements $\mathbb{M} \setminus (E \setminus Z)$, as $\mathbb{M}|Z$ and contracting all but a specific set W ($\mathbb{M}/(E \setminus W)$) as $\mathbb{M}.W$.

2.3.2 Properties of minors

Lemma 4. *[BDK+13] Given a matroid* $\mathbb{M}$ *and sets* $I \subseteq W \subseteq E$*, the following conditions are equivalent:*

1. *I is a basis of* $\mathbb{M}.W$*.*
2. *There exists a basis* I' *of* $\mathbb{M} \setminus W$ *such that* $I \cup I' \in \mathcal{B}(\mathbb{M})$*.*
3. *For every basis* I'' *of* $\mathbb{M} \setminus W$ *it holds that* $I \cup I'' \in \mathcal{B}(\mathbb{M})$*.*

Proof.
(1) ⇒ (2): If I is a basis of $\mathbb{M}.W$, then $W \setminus I$ is the basis of $\mathbb{M}^*|W$. We extend the set $W \setminus I$ to a basis D of $\mathbb{M}^*$. This means that $W \setminus I = D \cap W$ is maximal in $\mathbb{M}^*|W$, and by Lemma 3, we have that $\overline{D} \cap \overline{W}$ is a basis of $\mathbb{M} \setminus W$. $\overline{D} \cap \overline{W}$ is a superset of I, and the independent set $I' \equiv ((\overline{D} \cap \overline{W}) \setminus I)$ is the desired set from the condition (2).

(2) ⇒ (1): This argument follows by restating the previous argument in reverse.

(3) ⇒ (2): Immediate, as there necessarily exists a basis of $\mathbb{M} \setminus W$.

(2) ⇒ (3): Having proven the equivalence of (1) and (2), we can assume that we have a basis I of $\mathbb{M}.W$ such that there exists a basis I' of $\mathbb{M} \setminus W$ for which $I \cup I' \in \mathcal{B}(\mathbb{M})$. Given any basis I'' of $\mathbb{M} \setminus W$, we extend it with elements from $I \cup I'$ to a basis B of $\mathcal{B}(\mathbb{M})$. As I'' was maximal in $\mathbb{M} \setminus W$ and I' also, the only elements that could have been added are from I.

We also see that B is a basis of $\mathbb{M}$, and so $\overline{B}$ is a basis of $\mathbb{M}^*$ and more specifically $\overline{B} \cap \overline{W}$ is maximal, and by applying Lemma 3 as in (1), $B \cap W$ is maximal in $\mathbb{M}.W$, which implies that $B \cap W = I$ and $I \cup I'' \in \mathcal{B}(\mathbb{M})$. □

Corollary 1. *A set I is independent in $\mathbb{M}.W$ if and only if $I \cup I' \in \mathcal{I}(\mathbb{M})$ for all I' which are independent in $\mathbb{M} \setminus W$.*

Proof. ⇒: If I is independent in $\mathbb{M}.W$, extend it to a basis of $\mathbb{M}.W$. Using Point (3) of Lemma 4, we know that for every basis I'' of $\mathbb{M} \setminus W$ $B_W \cup I'' \in \mathcal{B}\mathbb{M}$, and therefore for all independent sets J of $\mathbb{M} \setminus W$ (which are subsets of bases, $I \cup J \in \mathcal{I}(\mathbb{M})$.

⇐: Choose a specific I' to be a basis of $\mathbb{M} \setminus W$. Extend $I \cup I'$ by a set X to a basis of $\mathbb{M}$; since I' was already a basis of $\mathbb{M} \setminus W$, we have only extended it by elements from W. Due to Point (2) in Lemma 4, we know that $I' \cup X$ is a basis of $\mathbb{M}.W$, and I' is therefore independent. □

Lemma 5. *Given a matroid $\mathbb{M}$ and its contracted minor $\mathbb{M}/X$, we can extend any circuit $C \in \mathcal{C}(\mathbb{M}/X)$ into a circuit of $\mathbb{M}$ by a subset of X.*

Proof. Pick any $M|X$-basis B_X. $C \cup B_X$ is necessarily $\mathbb{M}$-dependent, because C was $\mathbb{M}/X$ dependent. Therefore $C \cup B_X$ contains an $\mathbb{M}$-circuit C' which fulfills the equation $C' \setminus X = C$. □

Lemma 6. *If C is a circuit of $\mathbb{M}$ and $X \subset C$, then $C \setminus X$ is a circuit of $\mathbb{M}/X$.*

Proof. Since C is $\mathbb{M}$-dependent, $C \setminus X$ is $\mathbb{M}/X$-dependent and it is minimal with such property because of the minimality of C. □

Lemma 7. *Any element not included in X that lies in a circuit C of $\mathbb{M}$ also lies in a circuit of $\mathbb{M}/X$.*

Proof. Suppose it does not. Then $\{e\}$ is a one-element $\mathbb{M}/X$-cocircuit which intersects the $\mathbb{M}/X$-circuit $C \setminus X$ in exactly one element. □

2.4 Examples of infinite matroids

The study of the class of finite matroids has been motivated by the fact that matroids arise both as a generalization of the class of graphs and as a generalization of the class of vector sets with independence. We list some major classes of infinite matroids that can be generalized from the finite setting and we also give examples of those that cannot.

2.4.1 Cycle and bond matroids

The traditional approach to creating matroids from graphs is to base it on its cycle set or a bond set (edge cut-set). Therefore, the following matroids were suggested:

Finite-cycle and finite-bond matroids. The finite cycle and finite bond matroids arise from infinite graphs by considering all finite cuts and finite cycles. Note that these two classes are not dual to each other, as the dual of the finite-cycle matroid of a graph is the (both finite and infinite) bond matroid of such a graph.

Cycle and bond matroids. While cycles of infinite size are hard to define for infinite graphs, as it is not easy to see how to define them, it is easy to imagine infinite bonds in a graph (for example a vertex of infinite degree).

From these sets of both finite and infinite bonds, the general bond matroids were defined. The key theorem for the cathegorization of graph matroids was the following:

Theorem 2. *[BD11] Let G be any graph.*

- *The bonds of G, finite or infinite, are the circuits of a matroid $\mathbb{M}_B(G)$.*
- *This matroid is the dual of the finite-cycle matroid $\mathbb{M}_{FC}(G)$ of G.*

In keeping with the idea of Theorem 2, the general cycle matroid of a graph G was therefore defined as the dual of a finite bond matroid of the given graph G.

2.4.2 Algebraic cycle matroids

Given an infinite double ray (an infinite graph that is a path without an endpoint), should we consider this double ray a cycle or not? It arises as a path structure but shares many properties with cycles; for example, its embedding in the plane cuts it in two sections.

One construction which includes these double rays as circuits of the resulting matroid is called the *algebraic cycle matroid*. Here, we consider a circuit to be every inclusion-minimal subset of edges of the given graph, where the subset induces degree 2 or 0 in all the vertices of the graph.

For the further chapters, the most important example of an algebraic cycle graph is the *omega tree* matroid denoted as T_ω. To create this matroid, we construct an infinite rooted tree with all degrees equal to ω, and build an algebraic cycle matroid of this graph.

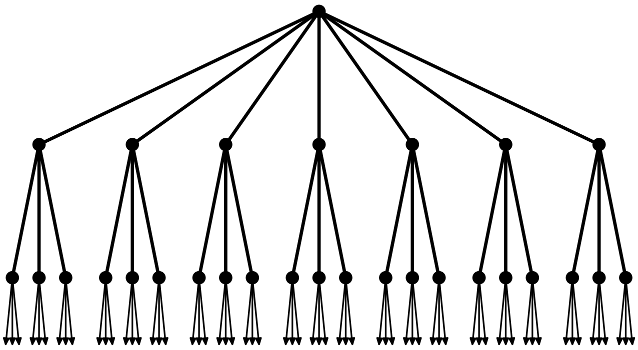

Figure 2.1: A visualisation of an omega tree – in reality, all vertices have degree ω.

The other important example of the algebraic cycle construction is the fact that not all graphs can be used to create an algebraic cycle matroid. To show this, we consider the Bean graph, created as a double ray with an *additional vertex* that is connected to vertices of only one ray, as shown below:

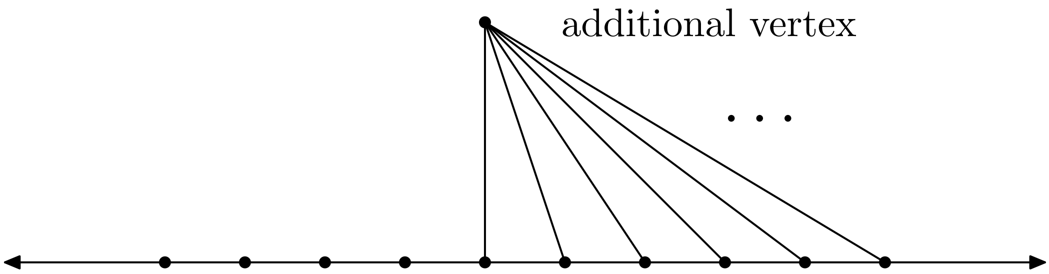

Figure 2.2: The Bean graph BG.

Observation 5. *An algebraic cycle structure formed from the Bean graph BG does not fulfill the circuit axiomatics of an infinite matroid.*

Proof. The double ray spans a circuit $C \in \mathcal{C}$. Consider a set of consecutive triangles $T_1, T_2, T_3 \ldots$ which is formed from the additional vertex of BG. These triangles are also members of $\mathcal{C}$. Each of these triangles has exactly one edge common with the double ray, and the union of these edges will form the set X.

We now apply circuit elimination for the circuit C, the circuit system $\{T_i\}$ with indexing edge set X and a guaranteed edge z arbitrary from $C \setminus X$. From (C3) we learn that there must exist a circuit C' containing z and avoiding the entirety of $\bigcup T_i$. As C' induces a subgraph in the Bean graph of degrees either 2 or 0, the additional vertex must also satisfy this. Therefore there exists a finite number j such that the last edge in the circuit C' that intersects $\bigcup T_i$ belongs to the triangle T_j. Therefore, one vertex of this T_j has degree 1 and C' cannot be an algebraic cycle of BG. $\square$

However, the Bean graph is essentially the only graph that does not form a matroid through algebraic circuits:

Claim 7. *[Hig69b] The elementary algebraic cycles of an infinite graph G are the circuits of the matroid on the edge set $E(G)$ if and only if G does not contain any subdivision of the Bean graph.*

2.4.3 Thin sums matroids

The class of matroids defined by linear independence conditions are necessarily finitary, so by introducing infinite matroids we lose an important set of examples. Therefore, in [BDK$^+$13], a class of matroids inspired by representability was introduced:

Definition 8. Let F be a field, and A a set. A set of functions $\{\varphi_i A \to F\}$ is called *thin* if $\forall a \in A$ there are only finitely many indices i such that $\varphi_i(a) \neq 0$.

Definition 9. We say that a family Ψ of functions from A to F is *thinly independent* if for every thin subfamily $\Psi' \subseteq \Psi$ the linear independence condition holds, i.e. for every set of coefficients $\{\lambda_i | i \in I\}$ we have $\sum_{i \in I} \lambda_i \varphi_i$ is a zero function ($f(a) = 0$ for all $a \in A$) if and only if all λ_i are set to be zero.

We call a matroid $\mathbb{M}$ *thinly represented* over F if it can be represented with a family of functions with independence coinciding with thin independence.

The thin sums matroids have been studied rather extensively, for example in the papers of [BC12b] and [AB13]. The main question was whether all thin sums systems form a matroid, and it was answered negatively using the following definition and claim:

Definition 10. Given a graph G, an *algebraic cycle system* $\mathrm{ac}(G)$ is the system specified by each subset of its edges classified as dependent or independent. A subset is dependent if and only if it contains an algebraic cycle of G.

Claim 8. *[AB13] For every (infinite) graph G, the algebraic cycle system* $\mathrm{ac}(G)$ *forms a thin sums system over any field F of three or more elements.*

Proof. First, fix an arbitrary orientation of G. Our system will be consisting of functions $f_e : V \to -1, 0, 1$. For every edge e, define a function $f_e(v) = 1$ if v is the source of the edge e, -1 if it is the target, and 0 otherwise.

We show that $\mathrm{ac}(G)$ is dependent if and only if this thin-sum representation f_e is thinly dependent.

If we have a dependent set S in $\mathrm{ac}(G)$, then it contains a cycle or a double ray, so we take only the edges of the cycle or the double ray. Assume an arbitrary orientation of the double ray or circuit, and then assign $c(e) = 1$ if the chosen orientation agrees with f_e, $c(e) = -1$ if the orientation disagrees and $c(e) = 0$ if the edge is not present in the double ray or circuit.

Clearly, after this setting, $\sum_{v \in V(C)} c(e) f_e(v) = 0$ but c is non-trivial assignment, and so the the set of f_e is thinly dependent.

On the other hand, if we have a thinly dependent set in the thin representation we defined, then every vertex of the dependent set has to be present at least two edges. Therefore, we have a subgraph that contains a 2-regular subgraph and therefore it contains a double ray or a cycle.

□

Combining Observation 5 with Claim 8 we get the following:

Corollary 2. *[AB13] The thinly representable graph BG does not form a matroid.*

A class of thin sums systems which are all proven to be matroids are the thin sums systems where the ground set itself is thin. Unfortunately, in this case, the systems have also a very predictable structure:

Theorem 3. *[AB13] A matroid arises as a thin sums system over a thin family for some field F if and only if it is a dual of a representable matroid.*

2.4.4 Structures not forming a matroid

While the previous subsections listed analogues of finite matroids in infinite matroid theory, it is important to realize that even though infinite matroids are closed on duality and taking minors, not all generalizations of finite matroids yield an infinite matroid.

The first observation shows the limit of the axiom (IM):

Observation 6. *The uniform structure of the type $U_{\alpha,\beta}$, that is, a structure with a ground set of size $|E| = \beta > \alpha \geq \omega$ with all sets of cardinality less than or equal to (or only strictly less than) α in $\mathcal{I}$ does not constitute a matroid.*

Proof. The system does not satisfy the axiom (IM). If we are given an independent set I, suppose that we can extend it to a maximal independent set I_m. As $\beta > \alpha$ there is an element e outside I_m. Then $I_m + e$ is of size less than or equal to α and therefore independent, contradicting maximality of I_m.

□

The non-existence of infinite uniform structures seems like a minor one. A more serious example of a non-matroid arises from the *closure operator*. The closure axiomatics is one of the more useful ones for finite matroids and it can also be translated into an axiomatics for infinite matroids:

Claim 9. *[BDK+13] A structure $\mathbb{M}$ on the ground set E is a matroid if and only if there exists a function $\mathrm{cl} : 2^E \to 2^E$ satisfying the following properties:*

(CL1) *For all $X \subseteq E$ it holds that $X \subseteq \mathrm{cl}(X)$.*

(CL2) *For all $X \subseteq Y \subseteq E$ it holds that $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$.*

(CL3) cl *is idempotent; that is* $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$.

(CL4) *For all $X \subseteq E$ and $a, b \in E$, if $a \in (\mathrm{cl}(X + b) \setminus \mathrm{cl}(X))$ then $b \in \mathrm{cl}(X + a)$.*

(CLM) *If we define the set system $\mathcal{I}$ as containing exactly the sets I that satisfy $\forall x \in I : \mathrm{cl}(I - i) \neq I$, then $\mathcal{I}$ satisfies (IM).*

Using the closure operator in finite matroids, we can define *closed sets* as those satisfying $\mathrm{cl}(X) = X$ and *open sets* as their complements, and we can work with these sets topologically.

However, the paper [BDK+13] showed the following:

Observation 7. *Given the topological closure operator* cl *on the set of real numbers* $\mathbb{R}$*,* cl *satisfies axioms (CL1) - (CL4), but not (CLM).*

Proof. The first four axioms follow from general properties of open and closed sets. However, because of the condition $\forall x \in I : \text{cl}(I - i) \neq I$, all independent sets must be discrete – that is, consisting of isolated points.

However, there is no maximal discrete set in $\mathbb{R}$ (we can always add another point) and so (CLM) fails. □

Losing the examples from topology is unfortunate (but necessary) because topologies abound with various non-standard structures, while the other classes of infinite matroids we presented (graph matroids, thin sums matroids) are often finitary or cofinitary.

3. Connectivity in infinite matroids

3.1 Original definition of connectivity

The standard definition of connectivity for finite matroids, usually called Tutte connectivity, starts with the notion of equivalence based on membership in a circuit. For higher connectivity, we employ minimization of the separation function, which is usually defined through the rank operation.

Notation 8. For a finite matroid $\mathbb{M}$, the set $X \subseteq E$ identifies a partition of its elements into disjoint parts $(X, \overline{X})$. While this partition is defined only by the subset X, sometimes it is useful to denote the two sides (X, Y) so that it is clear at all times where the elements x_i or y_i belong.

We will therefore use (X, Y) or $(X, \overline{X})$ depending on the situation.

Definition 11. Given a finite matroid $\mathbb{M}$, a matroid is *connected* (or *simply connected*) when every pair of elements lies on a common circuit.

Definition 12. Given a finite matroid $\mathbb{M}$, a matroid is Tutte k-connected when the following equality holds:

$$\min_{X \subseteq E,\ |X| \geq k, |\overline{X}| \geq k} \kappa(X) + 1 = k,$$

where the connectivity function κ is defined by calculating the rank of both sides of the partition:

$$\kappa(X) = r(X) + r(\overline{X}) - r(E).$$

It seems intuitive that we can use the definition of the simple connectivity for infinite matroids verbatim, while the definition of higher connectivity might be salvaged by interpreting the rank function directly in the infinite matroid. This is indeed the case. The notions of connectivity in infinite matroids, as well as most proofs in this sections, were introduced in the 2012 paper of Bruhn and Wollan [BW12].

3.2 Simple connectivity

As we suggested in the previous section, we will verify that the definition of simple connectivity for finite matroids is still salvageable for infinite matroids:

Definition 13. Given an infinite matroid $\mathbb{M}$, $\mathbb{M}$ is connected (simply connected) when every pair of elements lies on a common circuit.

While the definition seems plausible at a first glance, we need to verify that it behaves properly. The key to this is the standard lemma on connectivity:

Claim 10. *[BW12] Given a matroid $\mathbb{M}$, the relation: "$(x, y) \in R \equiv x$ and y lie on a common circuit" forms an equivalence.*

To prove this we need three simple lemmas:

Lemma 8. *Given a matroid $\mathbb{M}$ and its contracted minor $\mathbb{M}/X$, we can extend any circuit $C \in \mathcal{C}(\mathbb{M}/X)$ into a circuit of $\mathbb{M}$ by a subset of X.*

Proof. Pick any $M|X$-basis B_X. $C \cup B_X$ is necessarily $\mathbb{M}$-dependent, because C was $\mathbb{M}/X$ dependent. Therefore $C \cup B_X$ contains an $\mathbb{M}$-circuit C' which fulfills the equation $C' \setminus X = C$. □

Lemma 9. *If C is a circuit of $\mathbb{M}$ and $X \subset C$, then $C \setminus X$ is a circuit of $\mathbb{M}/X$.*

Proof. Since C is $\mathbb{M}$-dependent, $C \setminus X$ is $\mathbb{M}/X$-dependent and it is minimal with such property because of the minimality of C. □

Lemma 10. *Any element not included in X that lies in a circuit C of $\mathbb{M}$ also lies in a circuit of $\mathbb{M}/X$.*

Proof. Suppose it does not. Then $\{e\}$ is a one-element $\mathbb{M}/X$-cocircuit which intersects the $\mathbb{M}/X$-circuit $C \setminus X$ in exactly one element. □

Proof. (Proof of Claim 10.)

Reflexivity and symmetry of the relation is straightforward.

For transitivity, assume that e, f lie on a common circuit C_1 and f, g lie on C_2. Our goal is to show that there is a circuit containing elements e, g.

We cannot assume any elements outside $C_1 \cup C_2$, so we restrict ourselves on this set. We also contract any elements inside $C_1 \cap C_2$ besides f, as we cannot assume any of those elements either. It is an allowed operation as we know from previous lemmas that we can extend a circuit from a minor to the original matroid. We will denote the resulting matroid $\mathbb{M}'$.

If the circuit C_2 was a 2-element one, we would be finished with the proof, as we could use (C3) with C_1 as the main circuit, e as the guaranteed element and $\{f\}$ as the indexing set, generating a circuit containing both e and g.

We therefore try to contract $\mathbb{M}'$ using the set $P_1 = C_2 - f - g$. The set $\{f, g\}$ becomes a circuit because of Lemma 9, but it may happen that C_1 ceased to be a circuit and became a dependent set containing a circuit C_3'. We can extend the $\mathbb{M}'/P_1$-circuit C_3' to a $\mathbb{M}'$-circuit C_3 applying Lemma 8.

Instead of $P_1 = C_2 - f - g$, we now contract $\mathbb{M}'$ by the set $P_2 = C_2 - f - g \setminus C_3$. We claim that C_3' remains a circuit after this contraction. If it was not, C_3' would be a dependent set, and applying the same argument as before, it would contain a subset C_4' that is a $\mathbb{M}'/P_2$-circuit, but because we contracted less elements in P_2 than in P_1, it must have been a $\mathbb{M}/P_1$ circuit as well, which is a contradiction of (C2).

We arrive at the situation where we have a circuit C_3 that contains e and a circuit $C_2' = C_3 \cap C_2 + f + g$ that contains g and differs from C_3 by two elements. In our argument, we have tried contracting elements from C_2. Mirroring the same argument now but contracting elements from C_3 while preserving the element e, we obtain circuits C_4 and C_5 where $e \in C_4$, $g \in C_5$ and $|C_4 \setminus C_5| = |C_5 \setminus C_4| = 2$.

We now contract the intersection $C_4 \cap C_5$ to one element and get a finite matroid on 5 elements, for which our claim holds immediately. We finish the proof by decontracting the resulting circuit using Lemma 8.

□

3.3 Higher connectivity

As we stated at the start of this chapter, it is crucial to interpret the connectivity function κ in a non-rank manner, since otherwise we would end up with equations of type $\omega + \omega - \omega$, which have an undefined result.

Looking at its definition again:

$$\kappa(X) = r(X) + r(\overline{X}) - r(E)$$

we see that once we split the matroid $\mathbb{M}$ into two parts, we sum the size of the maximal independent set in each half, we get $r(X) + r(\overline{X})$. From this, we substract the size of the basis of $\mathbb{M}$, which is $r(E)$.

Instead of thinking about sizes, we interpret the rank function literally. In the first part, we therefore pick two maximal independent sets, one of X and one of $\overline{X}$. Let us denote them by B_X and $B_{\overline{X}}$. From the addition/union of these two maximal independent sets, we want to subtract the size of the basis. The result of this subtraction corresponds to the minimum amount of elements that we remove from $B_X \cup B_{\overline{X}}$, until we arrive at an independent set – this independent set must naturally be a basis.

Formally, we define a function $\text{del}_{\mathbb{M}}(X)$ (we usually drop the subscript when the matroid is clear from context) that will output one result of such a calculation as described above:

Definition 14. [BW12] For a finite matroid $\mathbb{M}$ and any subset $X \subseteq E$, $\text{del}(X) \equiv |K|$ for a finite set K that satisfies the condition:

$$B_X \cup B_{\overline{X}} \setminus K \in \mathcal{B} \tag{3.1}$$

where B_X is an arbitrary maximal independent subset of X and $B_{\overline{X}}$ an arbitrary maximal independent subset of $\overline{X}$.

If no such finite set K exists, we set $\text{del}(X) = \infty$.

This definition seems very fragile at a first glance; it could happen that two sets K_1, K_2 satisfy Equation 3.1 yet have different cardinality, or it could happen that the choice of the bases B_X and $B_{\overline{X}}$ changes the cardinality of K. The following claim proves that neither is the case. It also shows a much stronger condition: that we may choose a specific part from which we are eliminating elements and the cardinality stays the same.

Claim 11 (Properties of the del function, [BW12])**.** *Let $\mathbb{M}$ be a matroid, $X \subseteq E(\mathbb{M})$. Then:*

1. *For a given $B_X \in \mathcal{B}(\mathbb{M}|X)$ and $B_{\overline{X}} \in \mathcal{B}(\mathbb{M}|\overline{X})$, the cardinality $|K|$ is the same for any finite set $K \subseteq B_X \cup B_Y$ that satisfies Equation 3.1.*

2. *For a given subset X, the cardinality $|K|$ of a finite set fulfilling Equation 3.1 is independent on the choice of the bases $B_X \in \mathcal{B}(\mathbb{M}|X)$ and $B_{\overline{X}} \in \mathcal{B}(\mathbb{M}|\overline{X})$.*

3. $\mathrm{del}(X) = |K|$ *for any finite set K such that $(B_X \setminus K) \cup B_{\overline{X}}$ is a basis.*

Proof. (1): Let us make sure that for two *specific* bases B_X and $B_{\overline{X}}$, the number of the elements is the same for any two sets K_1, K_2 such that $B_X \cup B_{\overline{X}} \setminus K_i \in \mathcal{B}$. If this were not the case, then we would have two bases B_1, B_2 that would have finite difference $|B_1 \setminus B_2| \neq |B_2 \setminus B_1|$, which is a contradiction because of Lemma 2 in Chapter 2.

We now state a weaker version of point (3):

Observation 8. *For a given $B_X \in \mathcal{B}(\mathbb{M}|X)$ and $B_{\overline{X}} \in \mathcal{B}(\mathbb{M}|\overline{X})$, if a given set $K \subseteq B_X$ satisfies Equation 3.1, then it is equicardinal to any set $K \subseteq B_X \cup B_Y$ that also satisfies 3.1.*

Proof of Observation 8. Any set $K \subseteq B_X$ can also be thought as being from $B_X \cup B_{\overline{X}}$, and the statement follows from (1). □

Note that if $|K|$ is finite, we can always remove edges only from X: simply remove edges arbitrarily until $(B_X \setminus K) \cup B_{\overline{X}}$ ceases to be dependent. At that moment, the resulting system will be maximal, and necessarily a basis of $\mathbb{M}$.

(2): We are given two pairs of bases $(B^1_X, B^1_{\overline{X}})$ and $(B^2_X, B^2_{\overline{X}})$, and we suppose that the set K satisfying $B^1_X \cup B^1_{\overline{X}}) \setminus K \in \mathcal{B}$ is of size k (either finite or inf).

We now use Observation 8, choosing elements of K only from B^1_X to move from a pair $(B^1_X, B^1_{\overline{X}})$ to a pair $(B^1_X, B^2_{\overline{X}})$ with the same k, using the rule (1) we can now switch to removing elements of $B^2_{\overline{X}}$ without changing k, and finally applying Observation 8 again we end up with the pair $(B^2_X, B^2_{\overline{X}})$ with k unchanged throughout all of the operations.

(3): Follows by combining Observation 8 with point (2). □

Definition 15. We say that an infinite matroid $\mathbb{M}$ is k-connected and write $\kappa(\mathbb{M}) = k$ if and only if

$$\min_{X \subseteq E, |X| \geq k, |\overline{X}| \geq k} \mathrm{del}(X) + 1 = k.$$

If no such finite k exists, we say that $\mathbb{M}$ is infinitely connected.

Notation 9. Our $\mathrm{del}(X)$ function now plays essentially the same role as the $\kappa(X)$ function, and indeed they behave the same way on finite matroids. The authors of [BW12] prefer to switch back to the original notation $\kappa(X)$, while we do not use this notation except for computing global connectivity.

The following simple observation is very handy when arguing about the behavior of circuits in $B_X \cup B_{\overline{X}}$ that get removed during computation of $\mathrm{del}(X)$:

Lemma 11. *For an infinite matroid with a given partition* (X, Y) *and bases* (B_X, B_Y) *thereof with finite separation* k*, every element of a separating edge set* K *of* (B_X, B_Y) *lies on a circuit* C *(which we call an identifying circuit) such that no other element of* K *lies on this circuit.*

Proof. Suppose that for an element $p \in K$, no such identifying circuit exists. Then every circuit of $B_X \cup B_Y$ is already covered by a different element in K and removing K creates a smaller separation, which is a contradiction. $\square$

The following lemma verifies that the del function preserves the submodularity of the connectivity function in finite matroids:

Lemma 12. *[BW12] For all sets* $X, Y \subseteq E$ *of a matroid* $\mathbb{M}$*:*

$$\mathrm{del}(X) + \mathrm{del}(Y) \geq \mathrm{del}(X \cup Y) + \mathrm{del}(X \cap Y).$$

Proof. The idea of the proof is to construct very similar bases for all 4 partitions.

First, define the following two bases:

- $B_\cap \equiv$ a basis of $X \cap Y$.
- $B_{\overline{\cup}} \equiv$ a basis of $\overline{X \cup Y}$.

Next, we set $F_{\cap \& \overline{\cup}}$ to be a set that needs to be deleted from $B_\cap \cup B_{\overline{\cup}}$ to get a basis of the matroid on the union of their ground sets (which is $\mathbb{M}|(X \cap Y) \cup (\overline{X \cup Y})$).

We denote the resulting independent set as $B_{\cap \& \overline{\cup}}$. We extend this set to a basis B of $\mathbb{M}$. The elements we have added to $B_{\cap \& \overline{\cup}}$ are from $X \setminus Y$ and $Y \setminus X$; therefore, B is of the form $B = B_{\cap \& \overline{\cup}} \cup I_{X \setminus Y} \cup I_{Y \setminus X}$.

Now, imagine that we went back to $B_\cap$ and before merging it with $B_{\overline{\cup}}$, we tried to merge $B_\cap$ instead with $I_{X \setminus Y}$ and $I_{Y \setminus X}$. We claim that this union $B_\cap \cup I_{X \setminus Y} \cup I_{Y \setminus X}$ cannot form a circuit.

For the contradiction, suppose that it does contain a circuit C. This circuit intersects $B_\cap$ and therefore also F, as C is not a circuit in B.

Select all identifying circuits of $F \cap C$ as described in Lemma 11 and apply circuit elimination (the guaranteed element can be any element of $I_{X \setminus Y}$). We thus get a circuit avoiding all of F, which is a contradiction.

Thus, the set $B_\cap \cup I_{X \setminus Y} \cup I_{Y \setminus X}$ is independent in $\mathbb{M}$. Symmetrically, the set $B_{\overline{\cup}} \cup I_{X \setminus Y} \cup I_{Y \setminus X}$ is also independent. We will denote these two sets as $I_\cup$ and $I_{\overline{\cap}}$, as they are independent set in the union or in the complement of intersection, respectively.

Finally, since we have an independent set of the union $I_\cup$ which may not be a basis, we extend it to a basis $B'_\cup$ of $X \cup Y$. The elements that we extended $I_\cup$ with come from either $(X \setminus Y) \setminus I_\cup$ or $(Y \setminus X) \setminus I_\cup$. We denote the former as $F_\cup^X$ and the latter as $F_\cup^Y$. We also define the $F_{\overline{\cap}}^X$ and $F_{\overline{\cap}}^Y$ in a similar fashion.

The last pieces of notation will be I_X, which will consist of the part of the basis $B'_\cup$ that lies on the side of X, and I_Y similarly. Analogously, we define $I_{\overline{X}}$ and $I_{\overline{Y}}$.

We now summarize all the definitions we have:

- $B_\cap$, a basis of $X \cap Y$.
- $B_{\overline{\cup}}$, a basis of $\overline{X \cup Y}$.
- $B_{\cap\&\overline{\cup}}$, a basis of $(X \cap Y) \cup (\overline{X \cup Y})$. Extends $B_\cap$,$B_{\overline{\cup}}$.
- $F_{\cap\&\overline{\cup}}$, a set that we removed when creating $B_{\cap\&\overline{\cup}}$.
- B, a basis of $\mathbb{M}$. Extends $B_{\cup\&\overline{\cap}}$.
- $I_{X\setminus Y}$ and $I_{Y\setminus X}$, the independent sets used in the extension to B.
- $I_\cup$, an independent set of $X \cup Y$. Union of $B_\cap$, $I_{X\setminus Y}$ and $I_{Y\setminus X}$.
- $I_{\overline{\cap}}$, an independent set of $\overline{X \cap Y}$. Union of $B_{\overline{\cup}}$, $I_{X\setminus Y}$ and $I_{Y\setminus X}$.
- $B'_\cup$, a basis of $X \cup Y$. Extends $I_\cup$.
- $B'_{\overline{\cap}}$, a basis of $\overline{X \cap Y}$. Extends $I_{\overline{\cap}}$.
- $F^X_\cup, F^Y_\cup$, sets that were used in the extension of $I_\cup$ to $B'_\cup$.
- $F^X_{\overline{\cap}}, F^Y_{\overline{\cap}}$, ditto for $B'_{\overline{\cap}}$.
- I_X and I_Y, restrictions of $B'_\cup$ on X and Y respectively.
- $I_{\overline{X}}$ and $I_{\overline{Y}}$, restrictions of $B'_\cap$ on $\overline{X}$ and $\overline{Y}$ respectively.

We now prove the submodularity by making a lower bound on the left side which will match an upper bound on the right side. First we prove the lower bound on $\mathrm{del}(X)$. I_X and $I_{\overline{X}}$ are independent sets of X and $\overline{X}$, we can choose them as the starting points for bases of $(X, \overline{X})$ and so removing all circuits of $I_X \cup I_{\overline{X}}$ is a good lower bound on removing all circuits of $(X, \overline{X})$.

Now, observe that $I_X \cup I_{\overline{X}}$ is composed of the following sets:

$$B_{\cap\&\overline{\cup}} \cup F_{\cap\&\overline{\cup}} \cup I_{X\setminus Y} \cup I_{Y\setminus X} \cup F^X_\cup \cup F^Y_{\overline{\cap}}.$$

We see that in this union there are two bases – B and $B'_\cup$ – and any element of the F-sets creates a basis in at least one of those bases. Therefore, we need to remove at least as many elements as the sizes of the F-sets. We now have that $\mathrm{del}(X) \geq |F_{\cap\&\overline{\cup}}| + |F^X_\cup| + |F^Y_{\overline{\cap}}|$.

Applying the same argument for Y, we get that

$$\mathrm{del}(X) + \mathrm{del}(Y) \geq 2|F_{\cap\&\overline{\cup}}| + |F^X_\cup| + |F^Y_{\overline{\cap}}| + |F^Y_\cup| + |F^X_{\overline{\cap}}|. \tag{3.2}$$

Now we prove the upper bound on $\mathrm{del}(X \cup Y) + \mathrm{del}(X \cap Y)$. We start with $\mathrm{del}(X \cap Y)$, $\mathrm{del}(X \cup Y)$ can be proved symmetrically. We have the basis $B_\cap$ of $X \cap Y$ and we have the basis $B'_{\overline{\cap}}$ of $\overline{X \cap Y}$. $\mathrm{del}(X \cap Y)$ will be upper bounded by the size of the set needed to make $B_\cap \cup B_{\overline{\cap}}$ circuit-free.

First we remove $F_{\cap\&\overline{\cup}}$, which means that from $B_\cap$ and $B_{\overline{\cup}}$ (contained in $B_{\overline{\cap}}$ only an independent set $B_{\cap\&\overline{\cap}}$ will remain. After removing $F^Y_{\overline{\cap}}$ and $F^X_{\overline{\cap}}$ we are left with subsets of $I_{X\setminus Y}$ and $I_{Y\setminus X}$ which together with $B_{\cap\&\overline{\cap}}$ form a subset of the basis B.

Summing up $\mathrm{del}(X \cap Y)$ and $\mathrm{del}(X \cup Y)$, we have

$$2|F_{\cap\&\overline{\cup}}| + |F^X_\cup| + |F^Y_\cup| + |F^X_{\overline{\cap}}| + |F^Y_{\overline{\cap}}| \geq \mathrm{del}(X \cap Y) + \mathrm{del}(X \cup Y). \tag{3.3}$$

Equations 3.3 and 3.2 together give us the desired submodular inequality.

□

Another key lemma is that duality preserves connectivity of any partition:

Lemma 13. *[BW12] For any $\mathbb{M}$ and partition $(X, \overline{X})$ it holds that* $\mathrm{del}_{\mathbb{M}}(X) = \mathrm{del}_{\mathbb{M}^*}(X)$ *and therefore it also holds that* $\kappa(\mathbb{M}) = \kappa(\mathbb{M}^*)$.

Proof. Let B_X be the basis of X and $B_{\overline{X}}$ the basis of $\overline{X}$. Suppose that K_X is the minimal separation when we remove elements only from B_X, and define $K_{\overline{X}}$ in a similar fashion.

We remember from Lemma 4 in Chapter 2, a set I is a base of $\mathbb{M}.X$ if and only if there exists a base of $\overline{X}$ that extends I to a base of $\mathbb{M}$. That is clearly true for $B_X \setminus K_X$ and $B_{\overline{X}} \setminus K_{\overline{X}}$, and so they are bases of $\mathbb{M}.X$ and $\mathbb{M}.\overline{X}$, respectively.

Moving to the dual, $B^*_X \equiv (X \setminus B_X) \cup K_X$ is an $\mathbb{M}^*|X$-basis and $B^*_{\overline{X}}$, defined symmetrically, is a $\mathbb{M}^*|X$-basis.

We can also see that $(B^*_X \setminus K_X) \cup B^*_{\overline{X}} = E(\mathbb{M}) \setminus (B_X \cup (B_{\overline{X}} \setminus K_{\overline{X}}))$, and so $(B^*_X \setminus K_X) \cup B^*_{\overline{X}}$ is a basis of $\mathbb{M}^*$ and we have removed exactly the same elements K_X in the process. The lemma follows. □

Claim 12. *The matroid $\mathbb{M}$ is 2-connected if and only if it is simply connected.*

Proof. Suppose that the matroid is 2-connected but not simply connected. Therefore there exist elements x, y that are not on the same circuit. Because of Claim 10 we know that the relation "being on the same circuit" forms an equivalence, and so the equivalence classes of x and y are disjoint.

Suppose X is the equivalence class of x. We consider the partition $(X, \overline{X})$ and claim that $\mathrm{del}(X) = 0$. If not, then choosing B_X and $B_{\overline{X}}$ forms a circuit that needs to be disconnected by the set K, but then we have a circuit going outside the equivalence class of x, which is a contradiction.

For the other direction of the equivalence, suppose that $\mathbb{M}$ is simply connected but not 2-connected. Therefore $\mathrm{del}(X) = 0$ for some set X where $|X| \geq 1, |\overline{X}| \geq 1$. Since $K = \emptyset$ this means that there was no circuit present in $B_X \cup B_{\overline{X}}$ for any choice of B_X and $B_{\overline{X}}$.

This is however a contradiction because we can find such B_X and $B_{\overline{X}}$ which contain a circuit: Pick an element $x \in X$ and $y \in \overline{X}$ and consider a common circuit C for these two elements. As this circuit has elements in both sides of the partition $(X, \overline{X})$, $C \cap X$ and $C \cap \overline{X}$ are both independent sets. Extending $C \cap X$ and $C \cap \overline{X}$ into bases of B_X and $B_{\overline{X}}$ respectively yields the desired contradiction.

□

Observation 9. *Finite circuit or cocircuit limits connectivity. Given a matroid $\mathbb{M}$ that contains a finite circuit C or a finite cocircuit D of sizes at most c, the connectivity of this matroid is at most c.*

Proof. Using duality of separations we can assume without loss of generality that there exists a cocircuit D of size c. If we set $X = D$ and Y for the remaining elements, we can see that with any choice of B_Y, we can disconnect all circuits in $B_Y \cup X$ using $c-1$ elements from X, as D was a cocircuit and thus there existed an element from D which extended the basis B_Y. □

The previous observation motivated Bruhn and Wollan to state the following problem:

Open problem 1. *[BW12] Give an example of an infinite infinitely connected matroid.*

Note that there are finite uniform matroids of the form $U_{k/2,k}$ which have infinite connectivity even in the finite matroid setting, but the infinite connectivity of these matroids is generally considered a fluke of the definition.

3.4 Tutte's Linking Theorem

Similarly as in the finite matroid theory, we can extend our new connectivity function del not just on partitions, but on any two disjoint subsets:

Definition 16. Given two disjoint subset X, Y of the ground set of $\mathbb{M}$, the function $\mathrm{del}(X, Y)$ is defined as follows:

$$\mathrm{del}(X, Y) = \min_{X \subseteq X' \subseteq E \setminus Y} \mathrm{del}(X').$$

This extension of the del function allows us to compare connectivity of a disjoint pair of sets in $\mathbb{M}$ with its connectivity in a minor of $\mathbb{M}$. A simple lemma states connectivity is decreasing:

Lemma 14. *[BW12] Given a mutually disjoint quadruple of sets X, Y, C, D, the connectivity function satisfies $\mathrm{del}_{\mathbb{M}}(X, Y) \geq \mathrm{del}_{\mathbb{M}/C \setminus D}(X, Y)$.*

Proof. From Lemma 13 we know that computing connectivity in the primal is the same as computing it in the dual, and so we can assume that $C = \emptyset$, as it would follow the same argument as we apply for $D \neq \emptyset$, but in the dual matroid.

Assume we have a partition $(X', Y'), X \subseteq X', Y \subseteq Y'$ of the matroid $\mathbb{M}$ that certifies the minimality of Definition 16, i.e. $\mathrm{del}(X') = \mathrm{del}(X, Y)$. Assume that $B_{DX'}$ is the maximal independent set of $D \cap X'$. and $B_{DY'}$ the maximal independent set of $D \cap Y'$. We extend $B_{DX'}$ to a basis of X' and $B_{DY'}$ to a basis of Y'.

In the minor $\mathbb{M} \setminus D$, we can see that $B_{DX'} \setminus D$ and $B_{DY'} \setminus D$ are bases of $X' \setminus D$ and $Y' \setminus D$, respectively.

Now, assume that there is a set K such that $B_{DX'} \cup B_{DY'} \setminus K$ is a basis of $\mathbb{M}$. Looking at the matroid $\mathbb{M} \setminus D$, the only change to the partition is that we may have removed additional elements from the bases $B_{DX'}$ and $B_{DY'}$. Therefore, since K removed all circuits in $B_{DX'} \cup B_{DY'}$, it also removes all circuits in $B_{DX'} \cup B_{DY'} \setminus D$.

Therefore:

$$\mathrm{del}_{\mathbb{M} \setminus D}(X, Y) = \mathrm{del}_{\mathbb{M} \setminus D}(X' \setminus D, Y' \setminus D) \leq \mathrm{del}_{\mathbb{M}}(X', Y') = \mathrm{del}_{\mathbb{M}}(X, Y).$$

$\square$

A rather well-known result of Tutte states that whenever we compute $\mathrm{del}(X, Y)$ for some subsets X, Y that are not partition of $\mathbb{M}$, we can create a minor of $E \setminus (X \cup Y)$ which has the same value of $\mathrm{del}(X, Y)$.

The paper of Bruhn and Wollan shows that this is the case also for finitary and cofinitary matroids:

Theorem 4 (Tutte's Linking Theorem for finitary matroids)**.** *Let* $\mathbb{M}$ *be a finitary or a cofinitary matroid. Then for any disjoint pair* X, Y *there exist disjoint sets* $C, D \subseteq E \setminus (X \cup Y)$ *such that* $\mathrm{del}_{\mathbb{M}}(X, Y) = \mathrm{del}_{\mathbb{M}/C \setminus D}(X, Y)$.

However, the proof of this theorem uses finitarity in a very essential manner – simplistically said, the proof iteratively generates a finite union Z of finite circuits linking X to Y, and concludes that there are still many more circuits to be chosen from $E \setminus Z$ unless the connectivity of $\mathbb{M}|Z$ is high enough.

Therefore, the following still remains open:

Open problem 2. *Does Tutte's linking theorem (Theorem 4) hold for general infinite matroids?*

3.5 Connectivity of known matroids

The following series of claims shows the limits of current examples of matroids, at least for potential candidates for infinite connectivity.

3.5.1 Cycle and bond matroids

Claim 13. *Given an infinite graph* G*, any cycle matroid or bond matroid, finite or infinite, is either finitary or cofinitary.*

This claim follows from the theorem of Bruhn and Diestel:

Theorem 5. *[BD11] Given an infinite graph* G*, a cycle matroid of* G *is a dual of the finite bond matroid of* G*, and the bond matroid of* G *is a dual of the finite cycle matroid of* G*.*

Because of this theorem, any cycle or bond matroid contains a finite circuit or cocircuit and the connectivity is bounded by the length of such a circuit.

3.5.2 Algebraic cycle matroids

The algebraic cycle matroids contain matroids without finite circuits and cocircuits – the omega tree T_ω was given as an example in the previous section.

However, T_ω is essentially the only such matroid in its class:

Claim 14. *Given an algebraic cycle matroid with degree* $\leq \omega$*, it either is isomorphic to* T_ω *or it contains a finite circuit or a cocircuit.*

Proof. Consider any graph G. G must be graph-connected, otherwise it would have a cocircuit of size 2. Pick a vertex arbitrarily and create its spanning tree using a BFS-like traversal. If we arrive at a vertex with a finite number of neighbors, we know that there is a finite cocircuit in the algebraic cycle matroid. Therefore, all the vertices need to be of the degree ω.

If at any point we arrive at an already traversed vertex, we necessarily get a finite circuit. Therefore, at no finite point will the BFS procedure halt, and we get a spanning tree that spans the entire graph G, and the graph is an omega tree. □

We give a proof in the later section of this chapter that the omega tree has connectivity two. Therefore, all algebraic cycle matroids have finite connectivity.

3.5.3 Thin sums matroids

As we have learned by Corollary 2, not all thin sums structures are matroids. We also learned that thin sums matroids are not closed on duality.

Unfortunately, when we focus on the class of thin sums matroids that are closed on duality – the set of thin sums matroids on thin function sets – we must concede that Theorem 3 applies and all such matroids are cofinitary, which limits their usefulness for connectivity in infinite matroids.

3.6 Infinitary matroids

In the search for the infinite matroid with infinite connectivity, due to Observation 9 we restrict ourselves only to matroids which have no finite circuit or a cocircuit. Since we refer to them frequently, we employ the term *infinitary matroid* for such matroids.

In an infinitary matroid, any finite set is automatically independent and coindependent. For any partition (X, Y) where X is of finite size, we immediately know that there exists a basis of M completely within Y and such a partition is not relevant for the connectivity of M.

The following lemma also establishes that some non-trivial infinite partitions are irrelevant as well:

Lemma 15. *Given a partition (X, Y) of an infinitary matroid M such that both parts are infinite and one of them is either independent or coindependent,* $\mathrm{del}(X, Y) \geq \omega$.

Proof. We already know that $\mathrm{del}_M(X, Y) = \mathrm{del}_{M^*}(X, Y)$, so without loss of generality the set X is coindependent. This means that the set X does not contain a cocircuit, which is equivalent to the fact that there exists a basis B of M that avoids a set X. This basis B lies in M, so we pick this B as B_Y and the largest independent set B_X (which is of size at least ω) in X.

Since $B_Y = B$ is a basis of M, any element of X automatically creates a circuit. If we choose to remove elements only on the side of X, in order to disconnect all circuits, we have to remove all elements of B_X and therefore $\mathrm{del}(X, Y) \geq \omega$. □

3.6.1 Connectivity and properties of the omega tree

The only infinitary matroid presented in the literature to date was the omega tree T_ω, defined in Chapter 2. Since this matroid is of interest to us in the later chapters, we list its properties and calculate its connectivity.

Observation 10. *Any basis B of T_ω has the following structure: from the root leads exactly one infinite ray and all the other paths are finite. At the end of any finite path, there is one edge $e = (\alpha, \beta)$ not in B. The vertex β is a root of another basis B' of T_ω.*

Proof. The proof is straightforward. There must exist at least one ray from the root of the omega tree, otherwise we could add one more edge and preserve independence. Clearly, two infinite paths would constitute a circuit, and so all the other paths end after a finite number of edges. □

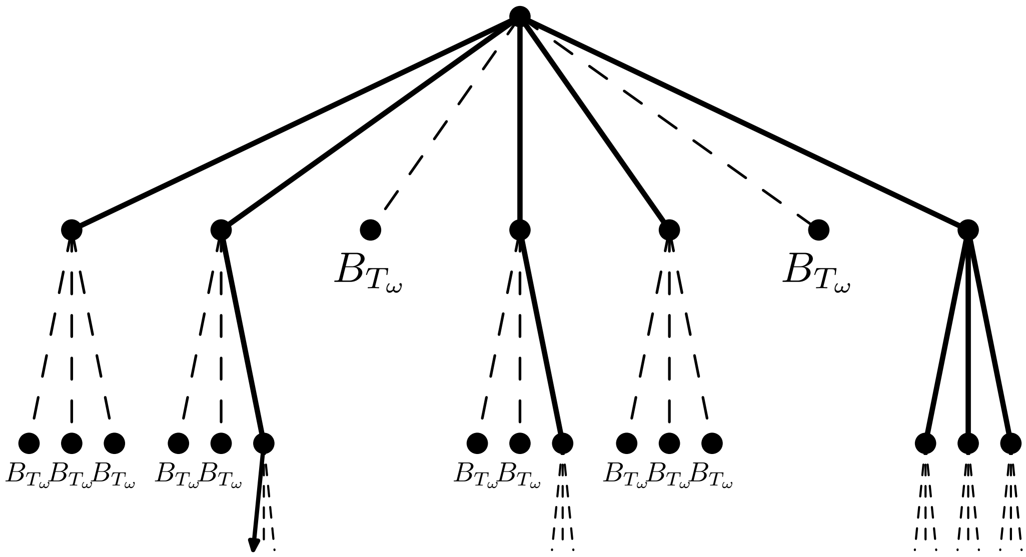

Figure 3.1: A recursive example of a basis of T_ω. Dashed lines are not within the basis. Every B_{T_ω} pictured may be a different basis from $\mathcal{B}(T_\omega)$.

Observation 11. *The omega tree matroid is not self-dual.*

Proof. If T_ω was self dual, then all cocircuits of T_ω can be mapped to circuits of T_ω bijectively.

First we note that if we take a vertex v of the graph representation of T_ω, then $\delta(V)$ (the edges adjacent to v) form a cocircuit. This is clear because if there was a basis B avoiding $\delta(v)$, then adding a single edge of $\delta(v)$ cannot create a double ray, as this edge would have one endpoint of degree 1. Therefore $\delta(v)$ intersects all bases of T_ω.

We can also see that $\delta(v)$ is minimal with the property of avoiding all bases, as we can form a basis that uses any single edge to v and avoids all others by using Observation 10.

Similarly, we can observe that if we are given a finite subtree S of T_ω that contains the root, the edge set $\delta(S)$ that exit the subtree S form a cocircuit. This can be proven by contracting the subtree to the root.

Finally, suppose that the vertex v is contained in one such finite subtree S and v is the leaf of S. Then all the edges of $\delta(v)$ except one are contained in $\delta(S)$. If T_ω was self-dual, then $\delta(v)$ forms a double ray, we have removed one edge of such a double ray, added infinitely many others, and we arrive at another double ray. This is not possible in T_ω, we have to remove infinitely many elements to move from a double ray to another double ray. □

Corollary 3. *The dual of the matroid T_ω is not an algebraic cycle matroid.*

Proof. From Observation 11 we know that ${T_\omega}^*$ is not self-dual but from Observation 14 we know that any infinitary matroid that is also an algebraic cycle matroid is isomorphic to T_ω.

□

Claim 15. *The connectivity of the matroid T_ω is 2.*

Proof. The matroid is immediately 2-connected, because every element lies on a cycle. To see that the connectivity is not higher, we create a partition (X, Y) by choosing an arbitrary edge $e \in T_\omega$ and putting all edges of one graph-connected component of $T_\omega - e$ plus e itself into X and all the edges of the other graph-connected component into Y.

From Observation 10 on bases of T_ω we know that given a basis B_X of X and a basis B_Y of Y, all the arising circuits of $B_X \cup B_Y$ must use the edge e. Removing it therefore creates an independent set, thus $\mathrm{del}(X, Y) = 1$. □

4. The plus operation

4.1 Basic properties

The *plus operation* is a unary operation on matroids that was introduced in [AHCF11a] and applied in [BC12b] in order to settle a question about a circuit and cocircuit meeting in infinitely many points (See Chapter 4 for the application).

Definition 17. Given a finite or an infinite matroid $\mathbb{M}$ where E is not independent, the matroid $\mathbb{M}^+$ is created by extending bases by one element not present in them. Formally:

$$\mathcal{B}(\mathbb{M}^+) = \{B + x | B \in \mathcal{B}(\mathbb{M}), x \notin B\}.$$

Definition 18. The minus operation forms a matroid $\mathbb{M}^-$ (provided that $\mathbb{M}$ has a nonempty basis) by shrinking bases by an element. Formally:

$$\mathcal{B}(\mathbb{M}^-) = \{B - x | B \in \mathcal{B}(\mathbb{M}), x \in B\}.$$

The duality of the plus and minus operations follows from the duality of the bases.

Observation 12.

$$(\mathbb{M}^*)^+ = (\mathbb{M}^-)^*.$$

We now verify that the operations create a valid matroid from a valid matroid:

Observation 13. *The plus and minus operation forms a matroid for every matroid* $\mathbb{M}$*.*

Proof. Because of duality it suffices to prove this for $\mathbb{M}^+$ only. The independent sets of $\mathbb{M}^+$ are either of the form $\{I | I \in \mathcal{I}(\mathbb{M})\}$ or of the form $\{I + x | I \in \mathcal{I}(\mathbb{M}), x \notin I\}$. Any indepenedent set of the second form is an extension of the old independent set, while the old independent sets remain indepenedent by definition. Thus (I1) and (I2) are fulfilled.

(I3') holds because we can extend any independent set either the same way as in $\mathbb{M}$, or (if the first option is not possible) with any single element that remains.

As for (IM), we can select a maximal independent subset of any set simply by finding such a set for $\mathbb{M}$, then extending it by one element that is not present, if such an element exists. □

While the operations are defined similarly for bases, it is notable to see that much changes in the family of independent sets. For those, the plus operation extends every independent set by any element that would have made it dependent in $\mathbb{M}$ (and old independent sets remain independent), the minus operation changes no independent set except for the bases, which all disappear.

There is a useful description of circuits of $\mathbb{M}^+$ and $\mathbb{M}^-$ as well:

Observation 14. *The circuits of* $\mathbb{M}^-$ *are either bases of* $\mathbb{M}$ *or circuits of* $\mathbb{M}$*, unless there was a circuit of* $\mathbb{M}$ *that contained a basis – these circuits are not present in* $\mathbb{M}^-$*. The circuits of* $\mathbb{M}^+$*, on the other hand, arise as minimal sets* C *with the following property: "erasing any one element* e *from the set* C *will produce an* $\mathbb{M}$*-dependent set* $C - e$ *that contains exactly one* $\mathbb{M}$*-circuit."*

We can also think of the circuits in $\mathbb{M}^+$ as subsets of the unions of circuits $C_1 \cup C_2$ based on dependence of the subsets of C_2 in $\mathbb{M}/C_1$. This notion is made precise in Lemma 18.

4.2 Applying plus on the omega tree

We first investigate applying the plus operation on the infinite matroid T_ω, defined in Section 2.

The bases are of the form B_1+x+B_2, where B_1 and B_2 are bases of two copies of T_ω that can be thought of as the "components" of $T_\omega - x$. This means that the new bases will always contain double rays. It turns out that as the number of rays rises, so does connectivity:

Claim 16. *Applying* $+$ *on* T_ω *increases its connectivity by one.*

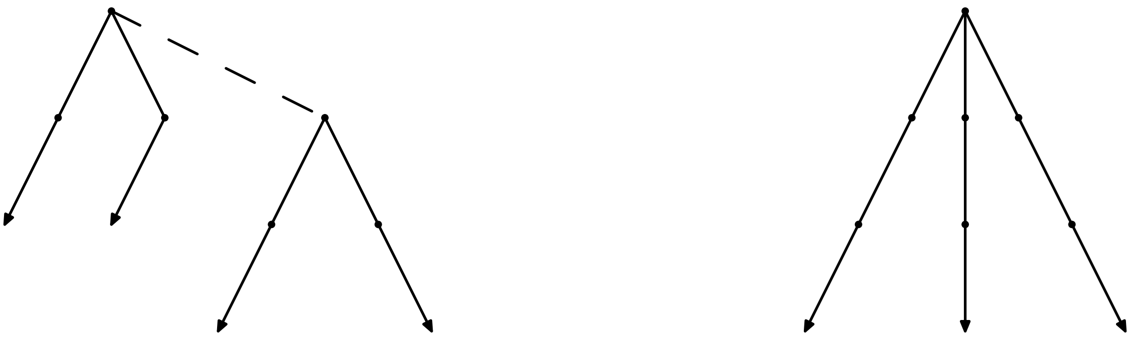

Figure 4.1: Two types of circuits in the matroid ${T_\omega}^+$. On the left is a pair of double rays, on the right a triple ray. All circuits of ${T_\omega}^+$ are of these types.

Proof. The upper bound on the increase of connectivity is easy to observe, as the connectivity cannot be increased by more than 1 since we can apply the same argument for connectivity of T_ω to the connectivity of ${T_\omega}^+$, while removing 2 edges.

As for the lower bound, we consider a partition (X, Y). Start from the root of T_ω and include all encountered edges into B_X or B_Y. If there are at least two infinite rays from the root with edges only in X, we include those two rays in B_X and take only finite sections of all the remaining rays.

We know that after finitely many steps from the root, we encounter edges in Y. We include these edges in B_Y in the same manner (if the entire subtree is in Y, include two infinite rays). For all the remaining subtrees we encounter, we preserve the independence of B_Y and B_X – by including only one infinite ray in the subtree whenever we avoid an edge of X and the same for Y.

Regardless of the separation we end up with the pair (B_X, B_Y) that are independent and contain at least four rays. We know that any basis of $\mathbb{M}^+$ contains exactly two infinite rays in one connected subtree and exactly one infinite ray in all the other connected subtrees, therefore we need at least two edges removed from B_X, B_Y to arrive at a basis of $\mathbb{M}^+$. □

It is easy to see that this argument can be easily extended to higher iterations of the plus operation. We have therefore produced a sequence of infinitary matroids which are not isomorphic to each other. This also implies the following corollary:

Corollary 4. *For any $2 \leq k < \omega$ there exists a matroid without finite circuits and cocircuits with connectivity k.*

4.3 Decomposition of T_ω using 2-sums

The main advantage of working with T_ω is that despite not being a graph matroid itself, it has a nice graph-theoretical description.

We have shown in Chapter 3 that T_ω has connectivity 2. Finite matroids can be decomposed using the 2-sum operation into three-connected components, for reference see e.g. Chapter 7 in the book of Oxley [Oxl11].

The 2-sum operation is not dependent on the finiteness of the matroid $\mathbb{M}$ and can be restated for infinite matroids:

Definition 19. Given a matroid $\mathbb{M}_1$ and a matroid $\mathbb{M}_2$ for which holds that $E(\mathbb{M}_1) \cap E(\mathbb{M}_2) = \{e\}$, then the *2-sum* $\mathbb{M}_1 \oplus_2 \mathbb{M}_2$ is a matroid defined by its circuit space, which is

$$\mathcal{C}(\mathbb{M}_1 \oplus_2 \mathbb{M}_2) = \{C | e \notin C, C \in \mathcal{C}(\mathbb{M}_1)\} \cup \{C | e \notin C, C \in \mathcal{C}(\mathbb{M}_2)\}$$

$$\cup \{(C_1 - e) \cup (C_2 - e) | e \in C_1 \cap C_2, C_1 \in \mathcal{C}(\mathbb{M}_1), C_2 \in \mathcal{C}(\mathbb{M}_2)\}.$$

The validity of such a definition was shown first by [AHDP11]:

Lemma 16. *[AHDP11] For two infinite matroids that have a single element in common, $\mathbb{M}_1 \oplus_2 \mathbb{M}_2$ produces a matroid.*

The main result of [AHDP11] enables us to decompose 2-connected matroids which are not 3-connected using 2-sums into smaller objects:

Theorem 6. *[AHDP11] Every connected matroid with at least three elements can be decomposed into a unique, irreducible tree where nodes are either 3-connected matroids, circuits, or cocircuits, and two nodes are connected by an edge exactly when the two nodes can be joined by a 2-sum operation. Summing the nodes as indicated by the edges produces the original matroid $\mathbb{M}$.*

As the tree T_ω is a 2-connected matroid which is not 3-connected, Theorem 6 guarantees a tree decomposition int 2-sums. In order to understand the structure of T_ω better, we show this decomposition.

Since T_ω has a tree-like structure, it is natural to try splitting it into 2-sums at its vertices. If we contract everything else except one vertex and its edges, each edge will represent a ray, and no two rays are contained within an independent set. Therefore, we obtain a copy of the uniform matroid $U_{1,\omega}$, where all elements (pictured as arrows on Figure 4.2) are edges that will disappear after the 2-sum is applied. Note that $U_{1,\omega}$ is a cocircuit of size ω.

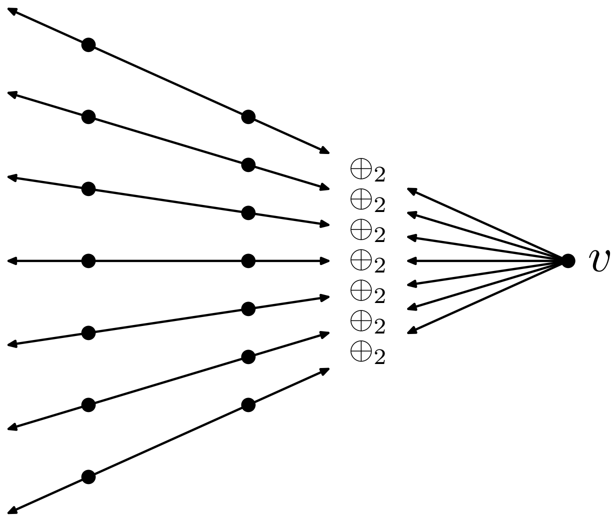

Figure 4.2: The decomposition of one vertex v of the graph T_ω. Note that none of the edges of the matroid $U_{1,\infty}$ associated with v remains in the final matroid T_ω.

We see that every element e of the matroid T_ω is adjacent to two vertices and therefore it needs at least two additional edges in its node. These edges are depicted on Figure 4.3 as arrows. The resulting node is a circuit of size 3 (a triangle).

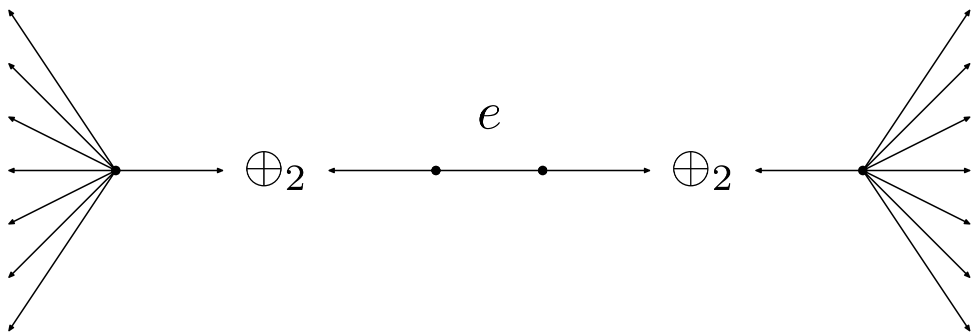

Figure 4.3: The decomposition of an edge e of T_ω in the 2-sum. The edge itself is the undirectional one, the other edges will be used as common edges in the 2-sum. The matroids $U_{1,\omega}$ representing vertices are on the left and on the right of the edge e.

4.4 Applying plus on algebraic cycle matroids

We have already shown the circuit structure of ${T_\omega}^+$, but we can similarly describe all circuits of $\mathbb{M}^+$ for a given algebraic cycle matroid $\mathbb{M}$. This description has first appeared in [BC12b].

Before presenting the description of the circuits of $\mathbb{M}^+$ for $\mathbb{M}$ algebraic, we show a necessary and a sufficient condition for being a circuit of $\mathbb{M}^+$ for any matroid, not only an algebraic cycle matroid.

Lemma 17. *[BC12b] Let C be a circuit of $\mathbb{M}$ and I a disjoint independent set. Then, $C \cup I$ is independent in $\mathbb{M}^+$ if and only if I is independent in $\mathbb{M}/C$.*

Proof. $\Rightarrow$: If $C \cup I$ is independent in $\mathbb{M}^+$, then we can extend it to a basis B of $\mathbb{M}^+$. This basis is of the form $B' + e$ for some basis B' of $\mathbb{M}$ and one element e not present.

Since the fundamental circuit of $B' + e$ must be C, we know that for any $f \in C$, $(C - f) \cup I$ is independent in $\mathbb{M}$. Applying Corollary 1 in Chapter 2, we get I being independent in $\mathbb{M}/C$.

$\Leftarrow$: Suppose that I is independent in $\mathbb{M}/C$. Using Corollary 1, we know that $I \cup I' \in \mathcal{I}(\mathbb{M})$ for all $I' \in \mathcal{I}(\mathbb{M}|C)$. This especially means that $I \cup (C - f) \in \mathcal{I}(\mathbb{M})$. Adding f to $I \cup (C - f)$ gives us an independent set of the form $I' + f$ and such a set is therefore also $\mathbb{M}^+$-independent.

□

Lemma 18. *[BC12b] Let C_1 be a circuit of $\mathbb{M}$ and C_2 a circuit of $\mathbb{M}/C_1$. Then $C_1 \cup C_2$ is a circuit of $\mathbb{M}^+$. Furthermore, every circuit in $\mathbb{M}^+$ arises in this way.*

Proof. Since C_1 is a circuit of $\mathbb{M}$ and C_2 is a dependent set, by Lemma 17 we have that $C_1 \cup C_2$ is dependent in $\mathbb{M}^+$.

We now show that it is minimal with such property. Clearly, removing any element from C_2 will form an independent set in $\mathbb{M}/C_1$ and we will reach an independent set.

Consider therefore removing an element f from C_1 and assume that $C_1 \cup C_2 - f$ is still dependent. $C_1 - f$ is now independent. We select any $g \in C_2$, which means $C_2 - g$ is $\mathbb{M}/C_1$-independent, and extend $C_2 - g$ to a basis B of $\mathbb{M}/C_1$. Applying Lemma 4, we get that $B \cup (C_1 - f)$ is independent in $\mathbb{M}$. Therefore, adding g to $B \cup C_1 - f$ makes it of the form $I + g = C_1 \cup C_2 - f$ and therefore $\mathbb{M}^+$-independent, which is a contradiction.

□

Corollary 5. *[BC12b] Every union of two circuits of $\mathbb{M}$ forms a dependent set of $\mathbb{M}^+$.*

Claim 17. *[BC12b] Given an algebraic cycle matroid $\mathbb{M}$, all circuits of $\mathbb{M}^+$ are subdivisions of the form (a) - (h) on Figure 4.4.*

Proof. From Lemma 18, we know that every circuit of $\mathbb{M}^+$ where $\mathbb{M}$ is an algebraic cycle matroid must be of the form $C_1 \cup C_2$, where C_2 is a circuit of a contraction of $\mathbb{M}/C_1$.

From Corollary 5 we know that any circuit of $\mathbb{M}^+$ arises as a subgraph of the union of two circuits in $\mathbb{M}$. We list the possible situations and categorize them.

Consider two circuits C_1 and C_2 in the matroid $\mathbb{M}$ and let us see what happens when we contract C_1 to a single element. If C_2 was finite and shared a vertex

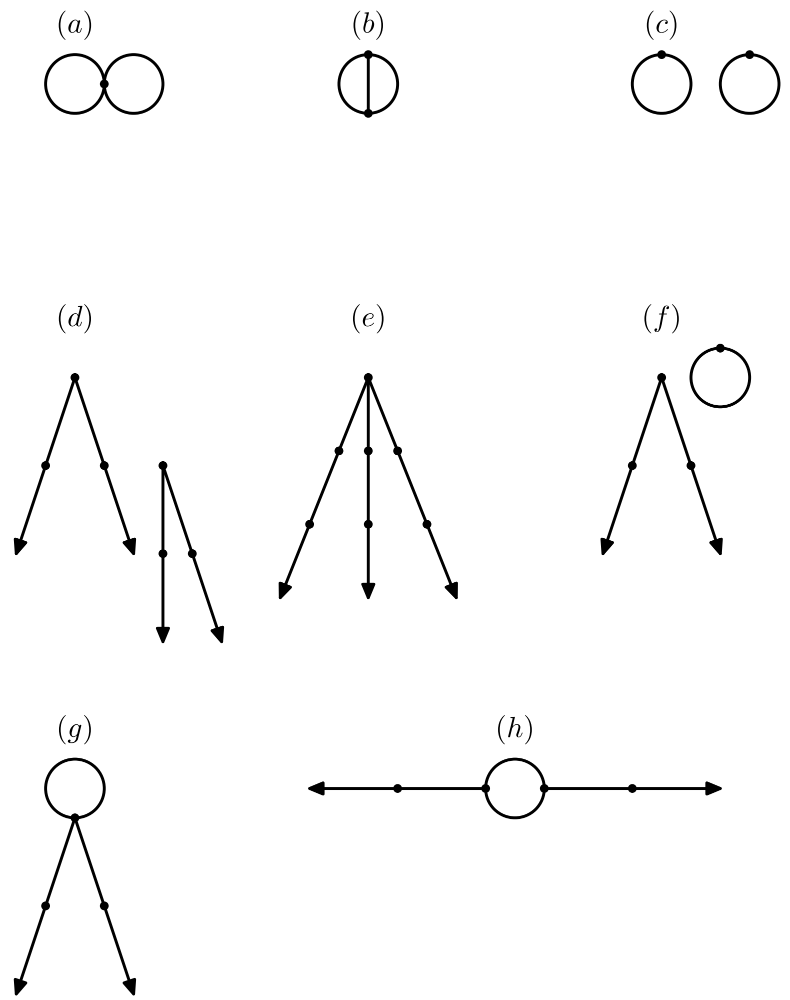

Figure 4.4: The only possible contractions of a circuit of a $\mathbb{M}^+$ when $\mathbb{M}$ is an algebraic cycle matroid.

with C_1 or was completely disjoint from C_1, we get circuits is $\mathbb{M}^+$ of the type (a) or (c). If the circuit C_2 had an intersection with C_1, we get a circuit of type (b).

Suppose C_1 and C_2 are both infinite circuits (double rays). If the double rays do not share an edge or a vertex, contracting one again does not affect the other, and we get a circuit of the case (d).

If C_1 and C_2 share an edge or a vertex, then we note that a circuit of $\mathbb{M}/C_1$ is any inclusion-minimal set that can be extended by elements from C_1 to a circuit of $\mathbb{M}$. From this we get that any single ray of C_2 that is adjacent to the circuit of C_1 will form a circuit of $\mathbb{M}/C_1$, and we get a circuit of the form (e).

The circuits (f), (g) and (h) are created by the same arguments, only in these three cases one of the circuit is finite and one is infinite. □

4.5 Applying plus on infinitary matroids

The proof of our claim about increasing connectivity in T_ω was linked to the structure of the tree, however, the plus operation does increase connectivity in general matroids without finite circuits and cocircuits. This is expressed by the following theorem:

Theorem 7. *For every infinitary matroid $\mathbb{M}$ with connectivity $2 \leq k < \omega$, the matroid $\mathbb{M}^+$ has connectivity $k+1$.*

Proof. Suppose that there is a k-separation of $\mathbb{M}^+$. This means that we have a partition (X, Y) such that $I = B_X \cup B_Y \setminus K \in \mathcal{I}(\mathbb{M}^+)$, where K is a set of k elements. We can assume that I is a basis of $\mathbb{M}^+$. As such, it is of the form $B + e, B \in \mathcal{B}(\mathbb{M})$. Our goal is to show that this partition (X, Y) also admits a $(k-1)$-separation of $\mathbb{M}$.

Case 1. Suppose that neither side was independent in M, that is, B_X is of the form $B_X^M + e_X$ and B_Y of the form $B_Y^M + e_Y$.

Observation 15. *Without loss of generality, we can assume that $B_X \cup B_Y \setminus K = B + e_Y$ and $K \subseteq B_X$.*

Proof of Observation 15. From Claim 11 in Chapter 3, we know that the size of the cut set is the same if we remove elements only in the part X. Since the maximum independent set on the side of Y is of the form $B_Y^M + e_Y$, we know that there is a unique $\mathbb{M}$-circuit in $B_Y^M + e_Y$ that contains e_Y. This circuit must also be the unique circuit contained in $B_X \cup B_Y \setminus K$, and so e_Y can be the extra element present in the final independent set. □

Case 1a. Suppose that e_X was removed in the process – it is included in the removed set K. We look at bases $B_X^{\mathbb{M}}$ and $B_Y^{\mathbb{M}}$. Since we do not need to remove e_X in $\mathbb{M}$ (e_X is not part of the basis B_X) and $B = B + e_Y - e_Y$ is a basis of $\mathbb{M}$, we see that (B_X, B_Y) are bases witnessing the $k-1$ separation of (X, Y) in $\mathbb{M}$.

Case 1b. Suppose that e_X is not present in K. Looking at the $\mathbb{M}$-separation, we have removed K elements plus e_X, which was not present in the original $B_X^{\mathbb{M}}$. We need to show that at least one of the elements of K can be returned to $B_X \cup B_Y$ while it remains $\mathbb{M}$-independent.

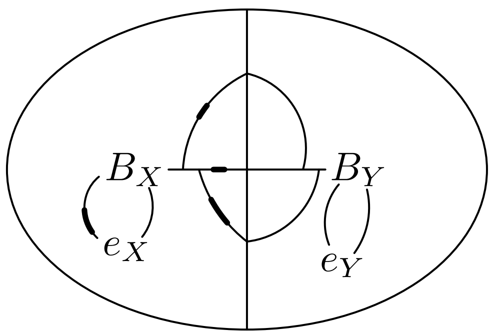

Figure 4.5: A visual representation of the situation in the proof of Theorem 11. The bold segments represent the elements of the set K, which is only in the part X. Our goal is to show e_X can play the role of one element of $C_X \cap K$.

From our last observation we know that in the $\mathbb{M}^+$-basis $B + e_Y$, the only $\mathbb{M}$-circuit that is present is fully contained in the ($\mathbb{M}$-dependent, $\mathbb{M}^+$-independent) set $B_Y + e_Y$. Therefore, the single fundamental $\mathbb{M}$-circuit C_X of $B_X + e_X$ must have been intersected by at least one element from K.

We now return all elements of K back, therefore working with the ground set of $B_X + e_X \cup B_Y + e_Y$. We will now employ circuit elimination, along with the Lemma 11 from Chapter 2:

We now apply circuit elimination (C3) on the system as follows:

- Our main circuit will be the fundamental circuit C_X containing e_X.
- The guaranteed element will be e_X.
- The indexing family will be $K \cap C_X$.
- The circuits indexed by $K \cap C_X$ will be the $\mathbb{M}$-circuits of $B_X + e_X \cup B_Y + e_Y$ that are identifying for the respective elements of K. We also require that e_X does not lie on the identifying circuits of K.

We are not guaranteed that the circuit elimination will succeed, therefore we consider the failure of the procedure also.

If the circuit elimination procedure succeeds, this means that we have found a circuit in $B_X + e_X \cup B_Y + e_Y$ that contains e_X but no element of K. This circuit could not have been contained in the $\mathbb{M}^+$-basis $B + e$ because only a single $\mathbb{M}$-circuit is contained in this basis and such circuit does not contain any elements of X, and so we reach a contradiction.

If the circuit elimination could not have been realized, this means that one of the conditions was not met. If there is more than one element of K on C_X, then the identfying circuits exist and are distinct from C_X.

The remaining option is then that there is one element $p \in K \cap C$ on C_X and all identifying circuits of p contain e_X. We can therefore remove p out of K and add e_X to it. As all identifying circuits of p contained e_X also, we are guaranteed that all the $\mathbb{M}$-circuits of $B_X + e_X \cup B_Y + e_Y \setminus K$ are removed except for the single circuit contained in $B_Y + e_Y$ (which does not get removed or play any role as we only remove elements from the part X).

The new set $K' = K - p' + e_X$ is therefore a separating set and we can use Case 1a to reach a contradiction again.

Case 2. Suppose that $B_X^{M^+}$ is of the form B_X^M, that is, the independent set is directly inherited from M. This means that the partition X was independent in M. However, Lemma 15 from Chapter 3 gives us that any independent set on one side automatically results in a separation of ∞ in an infintary matroid, and so this holds for M^+ also.

□

From Theorem 11 and the fact that every infinitary matroid is also a dual of an infinitary matroid, we get the following:

Corollary 6. *For every infinitary matroid $\mathbb{M}$ with connectivity $2 \leq k < \omega$, the matroid $\mathbb{M}^-$ has connectivity $k + 1$.*

Remark. While the results of this chapter (primarily Corollary 8) on the systematic increase of connectivity for infinitary matroids could make us hopeful that we can find an infinitely connected infinitary matroid through taking some limit of a sequence of matroids with increasing connectivity, this does not seem realistic.

For example, looking at the sequence of increasing plus iterations of the omega tree, we see that with every iteration, the bases of $(T_\omega)^{k+}$ are those of the form $B+S$, where $|S|=k$ and $B \in \mathcal{B}(T_\omega)$.

However, considering a matroid of the form $B+S$, where $B \in \mathcal{B}(T_\omega)$ and $|S| = \omega$, we notice the same problem which leads to the structure $U_{\omega,2^\omega}$ not forming a matroid – the form $B+S$ would allow us to create ω rays from the root, but adding one more would again constitute a structure of the form $B+S$, thus failing (IM).

We also present the following small observation related to the previous remark:

Observation 16. *If there exists an infinite matroid* $\mathbb{M}$ *such that it has connectivity* ∞*, then* $\mathbb{M}^+$ *and* $\mathbb{M}^-$ *are matroids with the same property.*

This observation makes it probable that once a matroid $\mathbb{M}$ with infinite connectivity were to be found, we could generate many with such property.

5. Intersections of circuits

5.1 Circuit-cocircuit intersection

The very useful Claim 5 of Chapter 2, proven for infinite matroids in [BDK$^+$13] and likely observed for B-matroid even prior to this paper, proves that circuits and cocircuits of a matroid cannot intersect in one element.

It is not difficult to construct a matroid $\mathbb{M}$ where a circuit and a cocircuit intersects in k elements with k finite. However, infinite circuit-cocircuit intersection was not present in most examples of infinite matroids. The following definition was proposed for such matroids that have this intersection:

Definition 20. A matroid $\mathbb{M}$ is called *wild* if it contains a circuit-cocircuit intersection of size at least ω. A matroid is called *tame* if all such circuit-cocircuit intersections are finite.

In [BDK$^+$13], the question was posed whether infinite matroids can have a circuit-cocircuit intersection of infinite size. The existence of such intersection was considered to be false by some authors (for instance by Dress [Dre86]), and potential axiomatics of infinite matroids were built on the presumption that circuit-cocircuit intersection is always finite.

This presumption of Dress was however false, as infinite matroids often contain such an intersection, and a simple example was shown by Bowler and Carmesin in in [BC12b]:

Theorem 8. *[BC12b] There exists a wild matroid.*

Proof. Consider an algebraic cycle structure $\mathbb{N}$ such that it arises as a disjoint loop l along with a union of two single rays with vertices $i_1, i_2, i_3, \ldots$ and $j_1, j_2, j_3 \ldots$ with additional edges $\{i_k, j_k\}$ for all $k \in \omega$. The graph in Figure 5.1 does not contain a subdivision of the Bean graph and so $\mathbb{N}$ is a matroid by Claim 7.

The matroid $\mathbb{N}$ is depicted in Figure 5.1

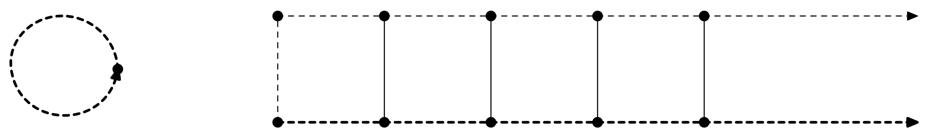

Figure 5.1: The matroid $\mathbb{N}^+$ used in the proof of Bowler and Carmesin. The circuit C is dashed, the cocircuit D is bold.

We see that the set $R = \{\{i_1, j_1\}\} \cup \bigcup_k \{i_k, i_{k+1}\} \cup \bigcup_k \{j_k, j_{k+1}\}$ is a double ray and therefore forms a circuit in $\mathbb{N}$, and so $C \equiv R + l$ forms a circuit in $\mathbb{N}^+$.

To find a cocircuit of $\mathbb{N}^+$, we look for a circuit of $(\mathbb{N}^*)^-$. Using Observation 14 in Chapter 4, we see that some circuits of a matroid $\mathbb{N}^-$ are bases of $\mathbb{N}$. In our case, we look for a basis of $\mathbb{N}^*$, which means a complement of a basis of $\mathbb{N}$. From this argument we have that $D = \{\bigcup_k \{j_k, j_{k+1}\}\} + l$ forms a circuit of $(\mathbb{N}^*)^-$ and a cocircuit in $\mathbb{N}^+$.

The intersection $C \cap D$ is the entire single ray $\bigcup_k \{j_k, j_{k+1}\} + l$ and it is therefore infinite.

□

Note. The application of the plus operation on $\mathbb{N}$ was necessary, because all the cocircuits of $\mathbb{N}$ are finite otherwise, in fact every cocircuit of $\mathbb{N}$ is of cardinality two or three.

The paper [BC12b] also shows that almost every matroid $\mathbb{M}^+$ is a wild matroid. We will employ Lemma 17 and Lemma 18 from Chapter 4.

Theorem 9. *Let $\mathbb{M}$ be any matroid that fulfills these conditions:*

1. *$\mathbb{M}$ contains at least two circuits.*
2. *There exists a circuit of $\mathbb{M}$ and for this circuit there exists a basis B such that $C \setminus B$ is infinite.*

Then, $\mathbb{M}^+$ is a wild matroid.

Proof. The proof is a generalization of the construction in Theorem 8.

Suppose that $\mathbb{M}$ contains a circuit C_1 and a circuit C_2 and that C_1 satisfies the condition (2) with a basis B. C_2 is dependent in $\mathbb{M}/C_1$ because of Corollary 1 in Chapter 2, and contains a circuit C_2' of $\mathbb{M}/C_1$. Then, by Lemma 18, we know that $C_1 \cup C_2'$ forms a circuit in $\mathbb{M}^+$.

We already know that for C_1 and B holds that $C_1 \setminus B$ is infinite, therefore it also holds for the set $C_1 \cup C_2'$. The cardinality of the difference $(C_1 \cup C_2') \setminus B$ is the same as the cardinality of the intersection $(C_1 \cup C_2') \cap \overline{B}$, with a cobasis $\overline{B}$ of $\mathbb{M}$ which becomes a cocircuit of $\mathbb{M}^+$, and the matroid $\mathbb{M}^+$ is wild.

□

Corollary 7. *$\mathbb{M}^+$ is a wild matroid for every infinitary matroid $\mathbb{M}$.*

Proof. We verify both conditions of Theorem 9.

1. If a matroid has an infinite cocircuit D, then $D - e$ is codependent and there exists a basis B that avoids $D - e$. Any element f from the still infinite set $D - e$ forms a unique fundamental circuit within $B + f$. Therefore, the condition (1) is met.
2. Consider a circuit C. The size of the circuit is infinite. Consider the set C in the dual matroid $\mathbb{M}^*$. Apply (IM) to find a maximum coindependent set A in the set C. Since the matroid is infinitary, the set A is infinite. Since we applied it in the dual, the set A is coindependent, and so there exists a basis of $\mathbb{M}$ that avoids it completely. For this basis B, $C \setminus B$ is infinite and the condition (2) holds also.

□

The existence of the wild matroid was open for a time primarily because such matroids are not common among the standard examples of infinite graph matroids, as our next claim illustrates:

Claim 18. *Every matroid $\mathbb{M}$ formed as an algebraic cycle matroid (and also every matroid formed as a graph cycle matroid) is tame.*

To prove this claim, it is useful to show the following description of the cocircuit space of the algebraic matroids:

Definition 21. Given an algebraic cycle graph G, a *skew cut* F is an edge cut in the graph G if and only if at least one part of the graph G separated by the skew cut contains no ray and F is inclusion-minimal with this property.

Observation 17. *[BD11] The cocircuits of any algebraic cycle matroid* $\mathbb{M}$ *formed from the graph* G *are precisely the skew cuts of* G.

Proof. Suppose we are given a skew cut F of G. Suppose that F is not a cocircuit, then F avoids a basis B. Adding any element of $e \in F$ to B forms a circuit, which has to be either a finite circuit or a double ray. However, we also know that e connects to a part of the skew cut without any ray. Adding one edge cannot form neither a finite circuit (because F is a cut) nor a double ray, and we have a contradiction.

Now, suppose we have a cocircuit F of $\mathbb{M}$. Suppose that both sides of the cut contain a ray. Consider an edge $e \in F$. If both endpoints of e are in the same side of the cut, we have a circuit and a cocircuit meeting in one element, violating Claim 5.

We can now suppose that the endpoints are in different sides of the cut. As both sides contained a ray (which is independent by itself), we can form a basis B that contains the rays on both sides, and adding an edge e to this basis, we form a double ray that meets exactly in one element with F, again violating Claim 5.

Therefore, we know that one side of the cut contains no ray. We also know F is minimal with this property because the cocircuit F itself is also inclusion-minimal and from the previous paragraph we know that the properties are equivalent. □

Proof of Claim 18. We can again assume without loss of generality that the graph G is graph-connected, as otherwise a cocircuit would only be in one connected component.

Consider a skew cut F, which splits the graph G into a part $V(S)$ without no ray and the rest. Suppose that a circuit C has an infinite intersection with it. Then such a circuit must automatically be a double ray.

After each transition from the part $V(S)$ to the other part, the circuit C has to include exactly one edge of F and it visits every vertex of $V(S)$ at most once.

We have assumed that the circuit C has infinite intersection with F, and so it must visit infinitely many disjoint vertices in $V(S)$. However, this means that the side $V(S)$ has infinitely many vertices and it is connected, and so it contains a ray, which is a contradiction. □

While many matroids arising from the "natural" constructions from graphs and thin sums contain a finite circuit or a cocircuit, we have already shown many examples of infinitary matroids, where no such structures exist. Motivated by this lack of finitarity in matroids, we could pose the question whether there exists a matroid such that all circuit-cocircuit intersections are infinite.

However, this question is rather easy to answer negatively using the following lemma, found for example in the PhD thesis of Robin Christian [Chr10]:

Lemma 19. *[Chr10] Given an (infinite) matroid* $\mathbb{M}$ *without loops and coloops, a dependent set* C *is a circuit if and only if for every two distinct elements* $e, f \in C$ *there exists a cocircuit* D *such that* $C \cap D = \{e, f\}$*.*

Proof.
$\Rightarrow$: Pick any element $e \in C$. $C - e$ is independent, so we can extend it to a basis B. Looking at the cobasis $\overline{B}$, we see that it contains only the element e from C.

Consider any other element $f \in C - e$. The set $\overline{B} + f$ is coindependent, and so it contains a fundamental cocircuit D . This cocircuit D contains f but because of Claim 5, the intersection $C \cap D$ cannot be of size 1, but the only other element $C \cap D$ can contain is e, and so $C \cap D = \{e, f\}$.

$\Leftarrow$: Suppose that the condition holds but C is dependent and not a circuit. Therefore, there exists a strict subset $C' \subseteq C$ such that C' is a circuit.

Choosing the element e from C' and the element $f \in C \setminus C'$, we find a cocircuit that intersects C in $\{e, f\}$, but this cocircuit intersects C' only in the element e, which is a contradiction with Claim 5.

□

The authors of [BC12b] believe that wild matroids serve as a good counterexample to many claims about infinite matroids. To back this assumption, they provide the following theorem, which we state without proof:

Theorem 10. *[BC12b] Consider the algebraic cycle matroid* $\mathbb{N}$ *used in the proof of Theorem 8 and depicted on Figure 5.1. Then, matroid* $\mathbb{N}$ *is a thin-sums matroid as is* $\mathbb{N}^+$*, but* $(\mathbb{N}^+)^* = (\mathbb{N}^*)^-$ *is not a thin sums matroid.*

5.2 Circuit-circuit intersection

We now briefly consider the behaviour of intersections of two circuits (or dually, two cocircuits). The question whether there exists a matroid with infinite circuit-circuit intersection is not very interesting, as already an algebraic cycle matroid of a triple ray satisfies this condition.

In the still ongoing search for an infinitary matroid with infinite connectivity, it is expected that if such a matroid exists, it has its circuits interlocked very densely, as that is the general understanding of connectivity.

Since we know there are many infinitary wild matroids, we can ask whether there is an infinitary matroid where not only all the circuits and cocircuits pairwise intersect in an infinite number of elements.

Such a condition sounds too strict, and it is indeed the case:

Observation 18. *In an infinitary matroid* $\mathbb{M}$*, there is either a pair of circuits or a pair of cocircuits such that their intersection is of size at most two.*

Proof. Choose an arbitrary circuit C. We now apply Lemma 19 to get a cocircuit D that intersects C in at most two elements.

Now, apply (IM) to find the maximum independent set B_D in D. If that set is not the entire set D, then adding an element from $D \setminus B_D$ would create a circuit with the desired property.

If the set B_D is the entire set D, then the cocircuit D is also independent, which means (looking at the matroid $\mathbb{M}^*$) that there is a cobasis B' in the set $\overline{D}$. Adding an element of D to this basis forms a cocircuit, and this cocircuit has exactly one element common with D. $\square$

If we look at the structure of the circuits of the matroid T_ω, we see a much larger variety of circuit-circuit intersections – for example a large set of mutually disjoint circuits. The examples strongly suggest that there are much stronger lemmas about the structure of infinitary matroids and their circuit spaces. We therefore believe Observation 18 can be strengthened into a much stronger one. Motivated by this, we ask the following:

Open problem 3. *How many disjoint circuits or cocircuits can be found in any infinitary matroid* $\mathbb{M}$*?*

6. Conclusion

6.1 Summary

We have presented most of the core of the infinite matroid theory throughout the thesis. Having presented the notion of infinite matroid connectivity and its basic properties in Chapter 3, we have used it in conjunction with the plus function from Chapter 4 to study the omega tree matroid, its dual, the as of yet still unexplored class of infinitary matroids and other matroids. We have also briefly investigated the structure of the intersections between circuits and cocircuits.

We restate our main results here for convenience:

Theorem 11. *For every infinitary matroid* $\mathbb{M}$ *with connectivity* $2 \leq k < \omega$*, the matroid* $\mathbb{M}^+$ *has connectivity* $k+1$*.*

Corollary 8. *For any* $2 \leq k < \omega$ *there exists a matroid without finite circuits and cocircuits with connectivity* k*.*

We would also like to note that before this thesis, the only referenced example of an infinitary matroid in the literature was the algebraic cycle matroid T_ω. We have employed an existing method (the plus function) to generate many more.

6.2 Future work and final notes

We believe our future work will focus on the Open problem 1 (search for infinitely connected infinite matroid), as it is the most essential open problem of the infinite matroid theory of today. The key to solving this problem is finding stronger theorems on the properties of infinitary matroids, as any infinitely connected matroid must be infinitary. Open problem 3 may be a good starting mark.

The constructive approach to finding the infinitely connected matroid would likely involve discovering an entirely new class of infinitary matroids, as the current classes of infinite matroids are often either finitary or simply with a mix of finite and infinite circuits.

However, finding a large class of new matroid examples, perhaps one that has its base in geometry or set theory, would be very useful on its own, as infinite matroids (in our opinion) still suffer from lack of examples.

Besides Open problem 1, there are many other open question in infinite matroid theory, for example relating to different axiomatics, the class of thin sum structures which form a matroid, or related to the Tutte Linking Theorem (Open problem 2.)

It would be also very notable if we could apply relative rank function or some other new property of infinite matroids to re-prove some of the theorems already existing in this field. We would like to note Lemma 12 and Claim 10, which are seemingly stating a very simple fact but their current proofs are rather technical and involved.

We hope that we have been able to convince you that the theory of infinite matroids is interesting, active, yet still full with undiscovered examples and theorems.

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