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BACHELOR THESIS

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Laplaceova transformace na prostorech funkcí

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Děkuji vedoucímu své práce prof. Lubošovi Pickovi za užitečné konzultace a cenné připomínky.

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Název práce: Laplaceova transformace na prostorech funkcí

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Abstrakt: V této práci studujeme chování Laplaceovy transformace na Banachových prostorech funkcí invariantních vůči přerovnání. Náš hlavní cíl je popsat optimální cílový prostor, příslušející zadanému prostoru v kategorii Banachových prostorů funkcí invariantních vůči přerovnání. Nejdříve dokážeme klíčový odhad nerostoucího přerovnání obrazu dané funkce při Laplaceově transformaci. Tento odhad dále použijeme ke konstrukci optimálního cílového prostoru. Tento obecný postup aplikujeme na určení optimálních vztahů mezi Lebesgueovými a Lorentzovými prostory při Laplaceově transformaci.

Klíčová slova: Laplaceova transformace, Lebesgueovy prostory, Lorentzovy prostory, interpolace

Title: Laplace transform on function spaces

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Abstract: In this manuscript we study the action of the Laplace transform on rearrangement-invariant Banach function spaces. Our principal goal is to characterize the optimal range space corresponding to a given domain space within the category of rearrangement-invariant Banach function spaces. We first prove a key pointwise estimate of the non-increasing rearrangement of the image under the Laplace transform of a given function. Then we use this inequality to carry out the construction of the optimal range space. We apply this general result to establish an optimality relation between the Lebesgue and Lorentz spaces under the Laplace transform.

Keywords: the Laplace transform, Lebesgue spaces, Lorentz spaces, interpolation

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Chapter 1

Introduction

The Laplace transform is a well-known classical integral operator defined on real-valued functions on the interval $(0, \infty)$. It is one of the most important integral operators with a wide range of applications throughout mathematical analysis. It is indispensable in certain fields such as the theory of ordinary differential equations and it is very useful in much broader context. It is also clear that one is interested in obtaining the results which are as tight as possible. The ideal situation is when the results are *sharp* or *optimal* in a certain sense.

The principal characteristics of integral operators is how they act on function spaces. Although the Lebesgue spaces L^p, where $p \in [1, \infty]$, play a primary role in many areas of mathematical analysis, there exist other classes of Banach spaces of measurable functions that are also of interest. Some of the well-known larger classes that Lebesgue spaces, such as for instance Orlicz spaces or Lorentz spaces, are of intrinsic importance. The class of the so-called rearrangement-invariant Banach function spaces, which had been built in the first half of the 20th century, mostly through the efforts of Young, Orlicz, Hardy, Littlewood, Pólya, Köthe, Luxemburg, Lorentz, Zaanen and many others, provides a very reasonable and at the same time a fairly wide environment of function spaces. In particular, it constitutes a common roof for all the classes of function spaces mentioned so far, and many more.

Our aim in this thesis is to investigate the action of the Laplace transform on rearrangement-invariant Banach function spaces. The principal goal is to characterize the optimal rearrangement-invariant partner target space when a domain space, also rearrangement-invariant, is given. We shall present a construction of such optimal space by a formula for its norm expressed through the so-called associate space (which is a concept typical for rearrangement-invariant spaces analogous to the dual space in the classical theory of Banach spaces). Even though this construction is fairly explicit, it is not immediately seen what is the optimal partner space for a fixed given domain space, not even in the most simple cases. For this reason we present an example of the optimal partner space in the case when the domain space is L^p with $1 \leq p < \infty$. It turns out that, in this case, the optimal range space is a Lorentz space $L^{p',p}$, where p' is the Lebesgue conjugate index of p.

Our method of finding the optimal range partner is analogous to that which was used for example in Edmunds et al. (2000) or Kerman and Pick (2006) for a different task of Sobolev embeddings. The key background result is an estimate for the non-increasing rearrangement of the Laplace transform of a given function in terms of a generalized Hardy-type integral operator in the spirit of the so-called K-functional used in the interpolation theory. We however do not use the exact definition of the K-functional in order to avoid unnecessary extension of the text.

The thesis is structured as follows. In Chapter 2 we collect all the necessary preliminary material concerning rearrangement-invariant Banach function spaces and concrete function spaces which we will work with throughout the text (this mainly concerns Lebesgue spaces, Lorentz spaces and the space $L^1 + L^\infty$. We also quote most important principles of this theory including the Hölder inequality, the Hardy lemma and the Hardy–Littlewood inequality. Here we study the spaces containing functions defined on a general totally σ-finite measure spaces. We do not insert proofs of the known results, referring the reader for example to the book Bennett and Sharpley (1988). In Chapter 3 we state and prove the auxiliary results that will be useful in the proofs of our main theorems. We start with an analogue of the well-known formula for the K-functional for the pair $(L^1, L^\infty$, quoted below as Theorem 2.21, where the space L^1 is replaced by the two-parameter Lorentz space $L^{1,\infty}$. We then recall known "endpoint" boundedness results of the Laplace transform (on L^1 and L^∞ and then state and prove our key background estimate, Theorem 3.4. Next we note that the Laplace transform is bounded from the Lebesgue space L^p into the Lorentz space $L^{p',p}$ (Theorem 3.6). Finally we present a useful formula for pairs of operators associate with respect to the L^1-pairing (Remark 3.8). Finally, in Chapter 4, we state and prove the main results. More precisely, we present here the general construction of the optimal range partner space to a given domain space (Theorem 4.4) and its application to the concrete example of Lebesgue space (Theorem 4.5). We also show that the functional that determines the optimal space is (under certain mild assumption on the domain space) a rearrangement-invariant Banach function norm (Theorem 4.1).

Chapter 2

Preliminaries

In this section we define basic ingredients of the theory of rearrangement-invariant spaces, fix notation and quote known basic results which will be needed throughout the text. The proofs and further details can be found in Bennett and Sharpley (1988).

We will denote by (R, μ) a totally σ-finite measure space, by $M_0(R, \mu)$ the set of all μ-measurable and a.e. finite functions on R, by $M^+(R, \mu)$ the set of all μ-measurable functions on R whose values lie in $[0, \infty]$ and by $M_0^+(R, \mu)$ the subset of $M_0(R, \mu)$ involving only nonnegative functions.

Definition 2.1. A mapping $\rho : M^+(R, \mu) \to [0, \infty]$ is called a *Banach function norm* or just a *function norm* if, for all $f, g, f_n, (n = 1, 2, ...)$, in $M^+(R, \mu)$, for all constants $a \geq 0$, and for all μ-measurable subsets E of R, the following properties hold:

- (P1): $\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.; $\quad \rho(af) = a\rho(f)$; $\quad \rho(f + g) \leq \rho(f) + \rho(g)$;
- (P2): $0 \leq g \leq f$ μ-a.e. $\Rightarrow \rho(g) \leq \rho(f)$;
- (P3): $0 \leq f_n \nearrow f$ μ-a.e. $\Rightarrow \rho(f_n) \nearrow \rho(f)$;
- (P4): $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$;
- (P5): $\mu(E) < \infty \Rightarrow \int_E f d\mu \leq C_E \rho(f)$ for some constant C_E, $0 < C_E < \infty$, depending on E and ρ but independent of f.

Definition 2.2. Let ρ be a function norm. The collection $X = X(\rho)$ of all functions f in $M(R, \mu)$ for which $\rho(|f|) < \infty$ is called a *Banach function space*. For each $f \in X$, define

$$\|f\|_X = \rho(|f|).$$

An example of Banach function spaces are Lebesgue spaces $L^p = L^p(R, \mu)$.

Definition 2.3. The *distribution function* μ_f of a function f in $M_0(R, \mu)$ is given by

$$\mu_f(\lambda) = \mu\{x \in (0, \infty) : |f(x)| > \lambda\}, \ \lambda \in (0, \infty).$$

Remark 2.4. We note that $\|f\|_X$ is defined for every $f \in M(R, \mu)$ but $f \in X$ if and only if $\|f\|_X < \infty$.

Definition 2.5. Two functions $f \in M_0(R,\mu)$ and $g \in M_0(S,\nu)$ are said to be *equimeasurable* if they have the same distribution function, that is, if $\mu_f(\lambda) = \nu_g(\lambda)$ for all $\lambda \geq 0$.

Definition 2.6. Suppose f belongs to $M_0(R,\mu)$. The *nonincreasing rearrangement* of f is the function f^* defined on $[0,\infty)$ by

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}, \ t \in (0,\infty).$$

Definition 2.7. Let ρ be a function norm over a totally σ-finite measure space (R,μ). Then ρ is said to be *rearrangement-invariant* if $\rho(f) = \rho(g)$ for every pair of equimeasurable functions f and g in $M_0^+(R,\mu)$. In that case, the Banach function space $X = X(\rho)$ generated by ρ is said to be a *rearrangement-invariant space.*

We shall now collect some basic properties of the non-increasing rearrangement of a function.

Proposition 2.8. *Suppose f, g and f_n, $(n = 1, 2, \ldots)$, belong to $M_0(R,\mu)$ and let a be any scalar. The nonincreasing rearrangement f^* is a nonnegative, nonincreasing, right-continuous function on $[0,\infty)$. Furthermore,*

$$|g| \leq |f| \text{ a.e. } \Rightarrow g^* \leq f^*, \tag{2.1}$$

$$(af)^* = |a| f^*, \tag{2.2}$$

$$(f+g)^*(t_1+t_2) \leq f^*(t_1) + g^*(t_2), \tag{2.3}$$

$$|f_n| \nearrow |f| \text{ a.e. } \Rightarrow f_n^* \nearrow f^*. \tag{2.4}$$

We shall introduce another operation involving rearrangements which is sometimes a useful replacemet of f^*. It has certain maximality feature.

Definition 2.9. Let f belong to $M_0(R,\mu)$. Then f^{**} will denote the *maximal function* of f^* defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \ \text{t} > 0.$$

Next theorem shows that certain subadditivity property of the maximal function holds.

Theorem 2.10. *Let f, g belong to $M_0(R,\mu)$, let $t > 0$. Then*

$$(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t).$$

Now we define spaces $L^1 + L^\infty$ and $L^1 \cap L^\infty$. These spaces play a special role in the theory in that they are the largest and the smallest of all rearrangement-invariant spaces.

Definition 2.11. The space $(L^1 + L^\infty)(R, \mu)$ consists of all functions f in $M_0(R, \mu)$ that are representable as a sum $f = g + h$ of functions g in L^1 and h in L^∞. For each f in $L^1 + L^\infty$, let

$$\|f\|_{L^1+L^\infty} = \inf\left\{\|g\|_{L^1} + \|h\|_{L^\infty}\right\},$$

where the infimum is taken over all representations $f = g+h$ of the kind described above.

For each f in the intersection $L^1 \cap L^\infty$ of L^1 and L^∞, let

$$\|f\|_{L^1\cap L^\infty} = \max\left\{\|f\|_{L^1}, \|f\|_{L^\infty}\right\}.$$

We note that the space $L^1 + L^\infty$ was studied in detail in Gould (1959), where he proved among other results that

$$\|f\|_{L^1+L^\infty} = \sup_{|E|\le 1} \int_E |f(x)|\, d\mu = \int_0^1 f^*(t)dt,$$

where the supremum is extended over all μ-measurable subsets of R.

Definition 2.12. If ρ is a function norm, its *associate norm* ρ' is defined on $M^+(R, \mu)$ by

$$\rho'(g) = \sup\left\{\int_R fg d\mu : f \in M^+(R,\mu), \rho(f) \le 1\right\}.$$

Theorem 2.13. *Let ρ be a function norm. Then the associate norm ρ' is itself a function norm.*

Definition 2.14. Let ρ be a function norm and let $X = X(\rho)$ be the Banach function space determined by ρ. Let ρ' be the associate norm of ρ. The Banach function space $X(\rho')$ determined by ρ' is called the *associate space* of X and is denoted by X'.

Remark 2.15. Examples of associate spaces are:

- if $1 \le p \le \infty$ then $(L^p)' = L^{p'}$, where $p' = \begin{cases} \frac{p}{p-1} & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty \\ \infty & \text{if } p = 1 \end{cases}$
- $(L^1 \cap L^\infty)' = L^1 + L^\infty$.

Next theorem shows how else the associate norm can be also expressed.

Theorem 2.16. *Let X be a Banach function space. Then X is rearrangement invariant if and only if the associate space X' is. The norm is given by*

$$\|g\|_{X'} = \sup\left\{\int_0^\infty f^*(s)g^*(s)ds : \|f\|_X \le 1\right\},\ g \in X'.$$

An important property of Banach function spaces is that each Banach function space coincides with the "second associate one". This is of course not true for classical duals of Banach spaces so it is useful to point this fact out explicitly.

Theorem 2.17. *A function f belongs to X if and only if it belongs to X'', and in that case*

$$\|f\|_X = \|f\|_{X''}.$$

The Hölder inequality is well known in the context of Lebesgue spaces. We shall now formulate its general version for Banach function spaces.

Theorem 2.18 (Hölder's inequality). *Let X be a Banach function space with associate space X'. If $f \in X$ and $g \in X'$, then fg is integrable and*

$$\int_R |fg| \, d\mu \le \|f\|_X \|g\|_{X'}.$$

One of the most useful tools in the theory of Banach function spaces is the Hardy lemma and its consequences.

Theorem 2.19 (Hardy's lemma). *Let ξ_1 and ξ_2 be nonnegative measurable functions on $(0, \infty)$ and suppose*

$$\int_0^t \xi_1 (s) \, ds \le \int_0^t \xi_2 (s) \, ds$$

for all $t > 0$. Let η be any nonnegative nonincreasing function on $(0, \infty)$. Then

$$\int_0^\infty \xi_1 (s) \, \eta (s) \, ds \le \int_0^\infty \xi_2 (s) \, \eta (s) \, ds.$$

The following estimate shows certain maximality property of the non-increasing rearrangement concerning integraiton of products of functions.

Theorem 2.20 (Hardy-Littlewood's inequality). *If f and g belong to $M_0^+ (R, \mu)$ then*

$$\int_R |fg| \, d\mu \le \int_0^\infty f^* (s) \, g^* (s) \, ds.$$

We shall now recall the well known formula which states an explicit form of the optimal decomposition of a function with respect to the space L^1, L^∞, and the real parameter t. This formula is a basic fact in the theory of interpolation and it is usually expressed in therm of the so-called K-functional which is avoided here. The theorem can be found for example in (Bennett and Sharpley, 1988, Chapter 2, Theorem 6.2).

Theorem 2.21. *Let (R, μ) be a totally σ-finite measure space and suppose f belongs to $M_0 (R, \mu)$. Then for all $t > 0$*

$$\inf_{f = g + h} \left(\|g\|_{L^1} + t \|h\|_{L^\infty} \right) = \int_0^t f^* (s) \, ds.$$

In the proofs below we shall also need the following classical Hardy inequality.

Theorem 2.22 (Hardy's inequality). *Let $\psi \ge 0$ on $(0, \infty)$, $-\infty < \lambda < 1$ and $1 \le q \le \infty$, then*

$$\left\{ \int_0^\infty \left(t^\lambda \frac{1}{t} \int_0^t \psi (s) \, ds \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}} \le \frac{1}{1 - \lambda} \left\{ \int_0^\infty \left(t^\lambda \psi (t) \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}}.$$

We shall now state a result from Lai (1959) on a weighted inequality for a kernel operator.

Theorem 2.23. *Let* $\phi : (0,\infty) \times (0,\infty) \to (0,\infty)$ *and let*

$$S_{\phi} f(x) = \int_0^{\infty} \phi(x,y) f(y)\, dy; \ \ \Phi(x,r) = \int_0^{r} \phi(x,y)\, dy.$$

If $1 \le q \le p < \infty$ *and for every* $r > 0$ *is*

$$\left(\int_0^r w \right)^{\frac{1}{p}} \le C \left(\int_0^{\infty} \Phi(x,r)^q v \right)^{\frac{1}{q}},$$

then for every f *nonnegative and nonincreasing on* $(0,\infty)$ *we get*

$$\left(\int_0^{\infty} f^p w \right)^{\frac{1}{p}} \le C \left(\int_0^{\infty} (S_{\phi} f)^q v \right)^{\frac{1}{q}}.$$

We shall now introduce the concept of Lorentz spaces, which will be needed throughout the entire text.

Definition 2.24. Let $0 < p, q \le \infty$. Then a *Lorentz space* $L^{p,q} = L^{p,q}(0,\infty)$ is the collection of all measurable functions f on $(0,\infty)$ such that

$$\|f\|_{p,q} = \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,\infty)} < \infty.$$

Proposition 2.25. *The Lorentz space* $L^{p,p}, (0 < p \le \infty)$*, coincides with the Lebesgue space* L^p*, and for* $f \in L^p$

$$\|f\|_{p,p} = \|f\|_p \, .$$

Theorem 2.26. *Suppose* $1 \le q \le p < \infty$ *or* $p = q = \infty$*. Then* $(L^{p,q}, \|\cdot\|_{p,q})$ *is a rearrangement invariant Banach function space.*

Remark 2.27. In cases when $1 < p < q \le \infty$, the Lorentz space $L^{p,q}(R,\mu)$ is not a Banach function space but it is merely *equivalent* to a Banach function space in the following sense: there exists a Banach function space $L^{(p,q)}(R,\mu)$ such that $L^{p,q}(R,\mu) = L^{(p,q)}(R,\mu)$ in the set-theoretical sense and their norms are equivalent, more precisely

$$\|f\|_{L^{p,q}(R,\mu)} \le \|f\|_{L^{(p,q)}(R,\mu)} \le C_{p,q} \|f\|_{L^{p,q}(R,\mu)},$$

where $C_{p,q} \in (0,\infty)$ is an absolute constant independent of f. Namely, the space $L^{(p,q)}(R,\mu)$ is determined by the norm

$$\|g\|_{L^{(p,q)}(R,\mu)} = \left\| t^{\frac{1}{p}-\frac{1}{q}} g^{**}(t) \right\|_{L^q(0,\infty)}, \ g \in M(R,\mu).$$

Thus, we can assume that the Lorentz space $L^{p,q}(R,\mu)$ is a Banach function space if one of the following conditions hold:

- $p = q = 1$
- $1 < p < \infty$ and $1 \le q \le \infty$
- $p = q = \infty$.

Theorem 2.28. *Let* (R,μ) *be a measure space and suppose* $1 < p < \infty$, $1 \le q \le \infty$ *(or* $p = q = 1$ or $p = q = \infty$*). Then the associate space of* $L^{p,q}(R,\mu)$ *is the Lorentz space* $L^{p',q'}(R,\mu)$*.*

Chapter 3

Background results

In this section we shall state and prove our background results. Throughout this section we shall denote by $R := (0, \infty)$ and by μ the one-dimensional Lebesgue measure.

The following result is an analogue of one inequality from Theorem 2.21 for the case when the space L^1 is replaced by the quasinormed space $L^{1,\infty}$. We note that, in view of Remark 2.27, the functional $\|\cdot\|_{L^{1,\infty}}$ is not a norm (it is merely a quasinorm), and the space $L^{1,\infty}$ is therefore not a Banach space.

Theorem 3.1. *For every $t > 0$ and every $f \in M_0(0,\infty)$ we have*

$$\sup_{s\in(0,t)} sf^*(s) \leq \inf_{f=g+h} \left\{ \|g\|_{L^{1,\infty}} + t\|h\|_{L^\infty} \right\},$$

where $\|g\|_{L^{1,\infty}} = \sup_{t\in(0,\infty)} tg^*(t)$.

Proof. We fix f and $t > 0$ and let $\alpha := \inf_{f=g+h} \{\|g\|_{L^{1,\infty}} + t\|h\|_{L^\infty}\}$. Without loss of generality we can assume that $f \in L^{1,\infty} + L^\infty$, therefore $f = g + h$ where $g \in L^{1,\infty}$ and $h \in L^\infty$.

In the next step we use Proposition 2.8, part (2.3) and we get

$$f^*(s) \leq g^*(s) + h^*(0),$$

therefore

$$\begin{aligned} \sup_{s\in(0,t]} sf^*(s) &\leq \sup_{s\in(0,t]} sg^*(s) + \sup_{s\in(0,t]} sh^*(0) \leq \sup_{s\in(0,\infty)} sg^*(s) + th^*(0) \\ &= \|g\|_{L^{1,\infty}} + t\|h\|_{L^\infty}. \end{aligned}$$

This holds for all g and h such that $f = g + h$ so it holds for the infimum as well. ∎

We next recall some well-known "endpoint" results on the boundedness of the operator L.

Proposition 3.2. *The Laplace transform is a bounded operator from $L^1(0,\infty)$ to $L^\infty(0,\infty)$.*

Proof. We want to prove that $\|Lg\|_{\infty} \leq C\|g\|_{1}$ for some constant $C \geq 0$ and for all $g \in L^{1}[0,\infty)$. We have

$$\begin{aligned}
\|Lg\|_{\infty} &= \inf\{\alpha \geq 0 : |Lg| \leq \alpha \text{ a.e. at } [0,\infty)\} \\
&= \inf\left\{\alpha \geq 0 : \left|\int_{0}^{\infty} e^{-st} g(t)\, dt\right| \leq \alpha \text{ a.e. at } [0,\infty)\right\} \\
&\leq \inf\left\{\alpha \geq 0 : \int_{0}^{\infty} |g(t)|\, dt \leq \alpha\right\} \\
&= \inf\{\alpha \geq 0 : \|g\|_{1} \leq \alpha\} = \|g\|_{1}.
\end{aligned}$$

Therefore $C = 1$. □

Proposition 3.3. *The Laplace transform is a bounded operator from $L^{\infty}(0,\infty)$ to $L^{1,\infty}(0,\infty)$.*

Proof. We want to prove that $\|Lg\|_{1,\infty} \leq \|g\|_{\infty}$ for all $g \in L^{\infty}(0,\infty)$. For $s \in (0,\infty)$ we have

$$L(|g|)(s) \leq \int_{0}^{\infty} e^{-st}\|g\|_{\infty}\, dt = \|g\|_{\infty}\left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=\infty} = \frac{\|g\|_{\infty}}{s},$$

therefore

$$sL(|g|)(s) \leq \|g\|_{\infty}.$$

It follows that

$$\sup_{s>0} sL(|g|)(s) \leq \|g\|_{\infty}. \tag{3.1}$$

For $f \geq 0$, $Lf(s)$ is a nonincreasing function, therefore $Lf = (Lf)^{*}$, in our case

$$L|g| = (L|g|)^{*}. \tag{3.2}$$

If $|g| \leq |f|$ a.e., then $g^{*} \leq f^{*}$, therefore

$$|Lg(s)| \leq |L(|g|)(s)| = L(|g|)(s).$$

Thus,

$$(Lg)^{*}(s) \leq (L|g|)^{*}(s). \tag{3.3}$$

Finally, we obtain

$$\begin{aligned}
\|Lg\|_{1,\infty} &= \sup_{s>0} s(Lg)^{*}(s) \leq^{(3.3)} \sup_{s>0} s(L|g|)^{*}(s) \\
&=^{(3.2)} \sup_{s>0} sL(|g|)(s) \leq^{(3.1)} \|g\|_{\infty}.
\end{aligned}$$

□

We shall now present an inequality which will have a key significance in all the estimates for the Laplace transform that will be presented. It gives a pointwise estimate of the nonincreasing rearrangement of the Laplace transform of a given function.

Theorem 3.4. *For every $t > 0$ and every measurable function g on $(0, \infty)$, we have*

$$(Lg)^* (t) \leq \int_0^{1/t} g^* (s) \, ds.$$

Proof. We fix $t > 0$. Then, according to Propositions 3.2 and 3.3 we have

$$\|Lg_1\|_\infty + t \, \|Lg_2\|_{1,\infty} \leq \|g_1\|_1 + t \, \|g_2\|_\infty .$$

Therefore

$$\inf_{g=g_1+g_2} \left(\|Lg_1\|_\infty + t \, \|Lg_2\|_{1,\infty} \right) \leq \inf_{g=g_1+g_2} \left(\|g_1\|_1 + t \, \|g_2\|_\infty \right). \tag{3.4}$$

We also know that L is a linear operator, therefore, according to Theorem 3.1, we have:

$$\begin{aligned}
&\inf_{g=g_1+g_2} \left(\|Lg_1\|_\infty + t \, \|Lg_2\|_{1,\infty} \right) = t \inf_{g=g_1+g_2} \left(\|Lg_2\|_{1,\infty} + \frac{1}{t} \|Lg_1\|_\infty \right) \\
&\geq t \inf_{Lg=Lg_1+Lg_2} \left(\|Lg_2\|_{1,\infty} + \frac{1}{t} \|Lg_1\|_\infty \right) \geq t \inf_{Lg=h_1+h_2} \left(\|h_2\|_{1,\infty} + \frac{1}{t} \|h_1\|_\infty \right) \\
&\geq t \sup_{0<s\leq 1/t} s \, (Lg)^* (s) .
\end{aligned}$$

We used the fact that the infimum over a bigger set is not bigger than the infimum over a smaller set. Using this and (3.4) we get

$$t \sup_{0<s\leq 1/t} s \, (Lg^*) (s) \leq \inf_{g=g_1+g_2} \left(\|g_1\|_1 + t \, \|g_2\|_\infty \right).$$

Now we apply Theorem 2.21 and we get

$$t \sup_{0<s\leq 1/t} s \, (Lg)^* (s) \leq \int_0^t g^* (y) \, dy,$$

which for $s = 1/t$ implies

$$(Lg)^* (1/t) \leq \int_0^t g^* (y) \, dy \quad \text{in other words} \quad (Lg)^* (t) \leq \int_0^{\frac{1}{t}} g^* (y) \, dy.$$

□

The next lemma contains a simple integral inequality.

Lemma 3.5. *Let $p > 1$, then for every $g \in M_0^+(0, \infty)$, we have*

$$\int_0^\infty \left(\frac{1}{t} \int_0^t g (s) \, ds \right)^p dt \leq (p')^p \int_0^\infty g^p (t) \, dt,$$

where p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Let p' be such that $\frac{1}{p}+\frac{1}{p'}=1$. Then using Hölder's inequality in the first step and Fubini's theorem in the fourth step, we get:

$$\int_0^\infty \left(\frac{1}{t}\int_0^t g(s)\, s^{\frac{1}{pp'}} s^{\frac{1}{-pp'}} ds\right)^p dt \le \int_0^\infty t^{-p} \int_0^t g^p(s)\, s^{\frac{1}{p'}} ds \left(\int_0^t s^{-\frac{1}{p}} ds\right)^{\frac{p}{p'}} dt$$
$$= \int_0^\infty t^{-p}\int_0^t g^p(s)\, s^{\frac{1}{p'}} ds \left(\frac{t^{-\frac{1}{p}+1}}{-\frac{1}{p}+1}\right)^{p-1} dt = \int_0^\infty \int_0^t t^{-2+\frac{1}{p}} g(s)^p s^{\frac{1}{p'}} (p')^{p-1} ds dt$$
$$= (p')^{p-1}\int_0^\infty\int_s^\infty t^{-2+\frac{1}{p}} g(s)^p s^{\frac{1}{p'}} dt ds = (p')^{p-1}\int_0^\infty g(s)^p s^{\frac{1}{p'}} \left[\frac{t^{\frac{1}{p}-1}}{\frac{1}{p}-1}\right]_s^\infty ds$$
$$= (p')^p \int_0^\infty g(s)^p s^{\frac{1}{p'}-1+\frac{1}{p}} ds = (p')^p\int_0^\infty g(s)^p ds.$$

□

We shall now focus on the action of the Laplace transform on Lebesgue spaces.

Theorem 3.6. *Let $1<p<\infty$. Then Laplace transform is a bounded operator from $L^p(0,\infty)$ to $L^{p',p}(0,\infty)$.*

Proof. For $f\in M_0(0,\infty)$ we have

$$\|Lf\|_{p',p} = \left\|(Lf)^*(t)\, t^{\frac{1}{p'}-\frac{1}{p}}\right\|_p \le \left\|\int_0^{\frac{1}{t}} f^*(s)\, ds t^{\frac{1}{p'}-\frac{1}{p}}\right\|_p$$
$$= \left(\int_0^\infty \left(\int_0^{\frac{1}{t}} f^*(s)\, ds\right)^p t^{\frac{p}{p'}-1} dt\right)^{\frac{1}{p}} \left(\int_0^\infty \left(\int_0^y f^*(s)\, ds\right)^p \frac{1}{y^{\frac{p}{p'}+1}} dy\right)^{\frac{1}{p}}$$
$$= \left(\int_0^\infty \left(\int_0^y f^*(s)\, ds\right)^p \frac{1}{y^p} dy\right)^{\frac{1}{p}} \left(\int_0^\infty \left(\frac{1}{y}\int_0^y f^*(s)\, ds\right)^p dy\right)^{\frac{1}{p}}.$$

Now we use Lemma 3.5 and we get

$$\|Lf\|_{p',p} \le (p')^p \left(\int_0^\infty f^*(t)^p\, dt\right)^{\frac{1}{p}} = (p')^p\left(\int_0^\infty |f|^p\, dt\right)^{\frac{1}{p}} = (p')^p \|f\|_p.$$

□

We shall need the following well-known simple duality principle.

Lemma 3.7. *$L: X\to Y \Leftrightarrow L: Y'\to X'$ for every pair (X,Y) of rearrangement invariant spaces.*

Proof. For $f,g\ge 0$ we have

$$\int_0^\infty f(x)\, Lg(x)\, dx = \int_0^\infty g(x)\, Lf(x)\, dx.$$

Then from Fubini's theorem we get:

$$\int_0^\infty f(x)\, Lg(x)\, dx = \int_0^\infty f(x) \int_0^\infty g(s)\, e^{-sx} ds dx = \int_0^\infty \int_0^\infty f(x)\, g(s)\, e^{-sx} dx ds$$
$$= \int_0^\infty g(s)\, Lf(s)\, ds.$$

Therefore

$$\begin{aligned} \|L\|_{Y' \to X'} &= \sup_{g \not\equiv 0} \frac{\|Lg\|_{X'}}{\|g\|_{Y'}} = \sup_{g \not\equiv 0} \sup_{f \not\equiv 0} \frac{\left| \int_0^\infty f(x)\, Lg(x) \right| dx}{\|f\|_X \|g\|_{Y'}} \\ &= \sup_{f \not\equiv 0} \frac{1}{\|f\|_X} \sup_{g \not\equiv 0} \frac{\left| \int_0^\infty g(x)\, Lf(x) \right| dx}{\|g\|_{Y'}} \\ &= \sup_{f \not\equiv 0} \frac{1}{\|f\|_X} \|Lf\|_{Y''} = \sup_{f \not\equiv 0} \frac{\|Lf\|_Y}{\|f\|_X} = \|L\|_{X \to Y}. \end{aligned}$$

□

Remark 3.8. We proved that Lemma 3.7 holds for the Laplace transform but in fact the Laplace transform is only a special case of a general result which states that for every pair of operators T and T' such that

$$\int_R T(f) g d\mu = \int_R f T'(g) d\mu \text{ for } f, g \in M(R, \mu).$$

Examples of such operators are, for instance, $T = T' =$ identity, or $Tf(x) = \int_0^x f$ and $T'g(x) = \int_x^\infty g$.

Chapter 4

Main results

In this section we shall state and prove our main results. Our aim now is to construct the optimal range partner space for a given domain space within the category of rearrangement-invariant spaces. We first need to know that certain functional is a rearrangement-invariant norm.

Theorem 4.1. *For a rearrangement invariant space X such that $\frac{1-e^{-x}}{x} \in X'$ we define the functional $F : g \longmapsto \|Lg^*\|_{X'}$ for $g \in M_0^+(0,\infty)$. Then F is a Banach function norm on $(0,\infty)$.*

Proof. We shall verify the axioms (P1)-(P5) of a Banach function norm.

- $(P1)$:

  1. for $\lambda \geq 0$ we have $F(\lambda f) = \lambda F(f)$, which is obvious on using Proposition 2.8 part (2.2).
  2. $F(f) = 0 \Leftrightarrow \|Lf^*\|_{X'} = 0 \Leftrightarrow Lf^* = 0$ a.e. $\Leftrightarrow f^* = 0$ a.e. $\Leftrightarrow f = 0$ a.e.
  3. Given $f,\ g,\ h \in X$ we have

  $$\begin{aligned}\|L(f+g)^*\|_{X'} &= \sup_{\|h\|_X \leq 1} \int_0^\infty h^*(t)\left[L(f+g)^*\right]^*(t)\,dt \\ &= \sup_{\|h\|_X \leq 1} \int_0^\infty h^*(t)\,L(f+g)^*(t)\,dt.\end{aligned}$$

  According to Theorem 2.10 we get

  $$\frac{1}{t}\int_0^t (f+g)^*(s)\,ds \leq \frac{1}{t}\int_0^t f^*(s)\,ds + \frac{1}{t}\int_0^t g^*(s)\,ds,$$

  $$\text{thus } \int_0^t \underbrace{(f+g)^*}_{=:\xi_1}(s)\,ds \leq \int_0^t \underbrace{f^*(s)+g^*(s)}_{=:\xi_2}\,ds.$$

  Now we apply Hardy's lemma (Theorem 2.19) to ξ_1, ξ_2 and $\eta(t) := e^{-xt}$ with $x > 0$ fixed which is non-increasing and nonnegative for every $t > 0$ and for every $x > 0$ and we get

  $$\int_0^\infty (f+g)^*(s)\,e^{-sx}ds \leq \int_0^\infty (f^*(s)+g^*(s))\,e^{-sx}ds,$$

or in other words,

$$L(f+g)^*(x) \leq Lf^*(x) + Lg^*(x) \text{ for every } x > 0.$$

Therefore

$$\begin{aligned}\|L(f+g)^*\|_{X'} &\leq \sup_{\|h\|_X \leq 1} \int_0^\infty h^*(t)\left(Lf^*(t) + Lg^*(t)\right)(t)\,dt \\ &= \|Lf^*\|_{X'} + \|Lg^*\|_{X'}.\end{aligned}$$

- $(P2)$: If $0 \leq g \leq f$ a.e at $(0,\infty)$, then $F(g) \leq F(f)$.

  Now we use Proposition 2.8 part (2.1). Therefore $Lg^*(x) \leq Lf^*(x)$ for every $x \in (0,\infty)$. Then we get:

$$\|Lg^*\|_{X'} = \sup_{\|h\|_X \leq 1} \int_0^\infty h^*(t)\,Lg^*(t)\,dt \leq \sup_{\|h\|_X \leq 1} \int_0^\infty h^*(t)\,Lf^*(t)\,dt = \|Lf^*\|_{X'}.$$

- $(P3)$: If $0 \leq f_n \nearrow f$ a.e. at $(0,\infty)$, then $F(f_n) \nearrow F(f)$.

  We use Proposition 2.8 part (2.4). It follows from Levi's theorem that

$$\lim_{n\to\infty} \int_0^\infty e^{-xs} f_n^*(x)\,dx = \int_0^\infty e^{-xs} f^*(x)\,dx,$$

therefore

$$Lf_n^*(s) \nearrow Lf^*(s) \text{ for every } s \geq 0.$$

Then we get $\|Lf_n^*\|_{X'} \nearrow \|Lf^*\|_{X'}$ because $\|\cdot\|_{X'}$ is a function norm itself, therefore this property holds.

- $(P4)$: If $\mu(E) < \infty$, then $\|L\chi_E^*\|_{X'} < \infty$.

$$L\chi_E^*(x) = L\left(\chi_{(0,\mu(E))}\right)(x) = \int_0^{\mu(E)} e^{-xt}dt = \frac{1-e^{-x\mu(E)}}{x} = \mu(E)\,\frac{1-e^{-x\mu(E)}}{x\mu(E)}$$

It is a fact that for X rearrangement invariant: $u \in X$ if and only if $u(\lambda t) \in X$ (for more details see Bennett and Sharpley (1988), chapter 3, Proposition 5.11). Also $u \in X$ if and only if $\lambda u \in X$. Therefore, by Definition 2.12 and Remark 2.4 we have

$$\begin{aligned}\|L\chi_E^*\|_{X'} &= \sup_{\|h\|_X \leq 1} \int_0^\infty |h(x)|\,L\chi_E^* dx = \sup_{\|h\|_X \leq 1} \int_0^\infty |h(x)|\,\mu(E)\frac{1-e^{-x\mu(E)}}{x\mu(E)}dx \\ &= \mu(E)\left\|\frac{1-e^{-x\mu(E)}}{x\mu(E)}\right\|_{X'} < \infty\end{aligned}$$

Here we used the fact that $\frac{1-e^{-x}}{x} \in X'$.

- $(P5)$: If $\mu(E) < \infty$ then $\int_E g d\mu \leq C_E \|Lg^*\|_{X'}$ for constant $0 < C_E < \infty$ independent of g.

  From Hardy-Littlewood's inequality (Theorem 2.20) we get:

$$\begin{aligned}\int_E g d\mu &\leq \int_E |g|\, d\mu = \int_0^\infty \chi_E(t)\,|g(t)|\, dt \leq \int_0^\infty \chi_E^*(t)\, g^*(t)\, dt \\ &= \int_0^{\mu(E)} g^*(t)\, dt.\end{aligned}$$

  We used the fact that $(\chi_E)^*(t) = \chi_{(0,\mu(E))}(t)$. Because

$$\int_0^{\mu(E)} e^{-xt} dx = \frac{1}{t}\left(1 - e^{-\mu(E)t}\right),$$

  we get

$$\int_E g d\mu \leq \int_0^{\mu(E)} g^*(t) \left(\int_0^{\mu(E)} e^{-xt} dx \right) \frac{t}{1 - e^{-\mu(E)t}} dt.$$

  Now we show that $\frac{t}{1-e^{-\mu(E)t}}$ is bounded on $(0, \mu(E))$. First, we have

$$\lim_{t \to 0} \frac{t}{1 - e^{-\mu(E)t}} = \frac{1}{\mu(E)}.$$

  Let us denote

$$K_E := \max \frac{t}{1 - e^{-\mu(E)t}}.$$

  Therefore, from Fubini's theorem and Hölder's inequality we get:

$$\begin{aligned}\int_E g d\mu &\leq K_E \int_0^{\mu(E)} g^*(t) \int_0^{\mu(E)} e^{-xt} dx dt = K_E \int_0^{\mu(E)} \int_0^{\mu(E)} g^*(t)\, e^{-xt} dt dx \\ &= K_E \int_0^{\mu(E)} \chi_E^* \int_0^{\mu(E)} g^*(t)\, e^{-xt} dt dx \\ &\leq K_E \|\chi_E^*\|_X \left\| \int_0^{\mu(E)} g^*(t)\, e^{-xt} dt \right\|_{X'} \\ &\leq^{(P2)} K_E \|\chi_E^*\|_X \|Lg^*\|_{X'}.\end{aligned}$$

  Since $\|\chi_E^*\|_X < \infty$, we can put $C_E := K_E \|\chi_E^*\|_X$.

□

Remark 4.2. We note that the assumption $\frac{1-e^{-x}}{x} \in X'$ is satisfied, for example, for $X = L^p$ when $p \in [1, \infty)$, but it is not satisfied for instance when $X = L^\infty$. Moreover, instead of assumption $\frac{1-e^{-x}}{x} \in X'$, we can analogically assume $\min\left\{1, \frac{1}{x}\right\} \in X'$.

To achieve our main goal we need to understand the concept of optimality of a function space within certain context. It is reasonable to require of such an optimal space to be a member of the given class of function spaces, to satisfy the property (in this case the boundedness of the operator), and to be the "best" such space, in our case the smallest possible. We shall now formulate a precise definition.

Definition 4.3. Given a rearrangement-invariant space X, an operator T and some class of function spaces W, we say that Y is the *optimal range space for X with respect to T in W* if the following conditions are satisfied:

1. $Y \in W$;
2. $T : X \to Y$;
3. if there exists $Z \in W$ such that $T : X \to Z$ then $Y \hookrightarrow Z$ ($Y \subset Z$ and there exists $c > 0$ such that, for all $g \in M_0^+(0,\infty)$, $\|g\|_Z \leq c\,\|g\|_Y$).

We are now in a position to prove our main result, namely to characterize the general optimal range space coresponding to a given rearrangement invariant space.

Theorem 4.4. *Let X be a rearrangement-invariant space such that $\frac{1-e^{-x}}{x} \in X'$. Define the space Y' by fixing the norm $\|g\|_{Y'} := \|Lg^*\|_{X'}$ as*

$$Y' = \left\{ g \in M_0^+(0,\infty) : \|g\|_{Y'} < \infty \right\}.$$

Then $\|\cdot\|_{Y'}$ is a rearrangement invariant norm and the space Y (obtained via $Y = Y''$) is optimal range space for X with respect to L in the class of rearrangement invariant spaces.

Proof.

1. Y' is rearrangement invariant because $\|g^*\|_{Y'} = \|Lg^*\|_{X'} = \|g\|_{Y'}$ therefore Y is rearrangement invariant.
2. We have (using Lemma 3.7)

$$\begin{aligned} L : X \to Y \Leftrightarrow L : Y' \to X' \Leftrightarrow \|Lg\|_{X'} \leq C\,\|g\|_{Y'} \Leftrightarrow \|Lg\|_{X'} \leq C\,\|g^*\|_{Y'} \\ \Leftrightarrow^{(?)} \|Lg^*\|_{X'} \leq C\,\|g^*\|_{Y'} \end{aligned}$$

   and we know that $\|Lg^*\|_{X'} = \|g\|_{Y'} = \|g^*\|_{Y'}$ so it remains to proof the last equivalence.

   "$\Rightarrow$" is obvious because $\|Lg^*\|_{X'} \leq C\,\|(g^*)^*\|_{Y'} = C\,\|g^*\|_{Y'}$.

   "$\Leftarrow$" We want to show that $\|Lg\|_{X'} \leq \|Lg^*\|_{X'}$. For that we use Hardy-Littlewood's inequality.

   We have $R := (0,\infty)$, e^{-xt} is decreasing for $x \geq 0$, $g \geq 0$ at $(0,\infty)$, therefore, for every $x \geq 0$,

$$Lg(x) = \int_0^\infty e^{-xt} g(t)\,dt \leq \int_0^\infty e^{-xt} g^*(t)\,dt = Lg^*(x),$$

   thus

$$\|Lg\|_{X'} \leq \|Lg^*\|_{X'}.$$

3. Let Z be a rearrangement invariant space such that $L : X \to Z$, then, by Lemma 3.7, $L : Z' \to X'$ therefore $\|Lg\|_{X'} \le K \|g\|_{Z'}$. We also know that $\|g\|_{Z'} = \|g^*\|_{Z'}$. Then also $\|Lg^*\|_{X'} \le K \|g^*\|_{Z'}$ as a special case. Then

$$\|g\|_{Y'} = \|Lg^*\|_{X'} \le K \|g^*\|_{Z'} = K \|g\|_{Z'} .$$

Therefore $Z' \hookrightarrow Y'$ whence $Y \hookrightarrow Z$.

□

Our last theorem is an application of the general result to Lebesgue and Lorentz spaces.

Theorem 4.5. *Let $1 < p < \infty$ then $L^{p',p}(0,\infty)$ is optimal range space for $L^p(0,\infty)$ with respect to L in the class of rearrangement invariant spaces.*

Proof. According to Theorem 4.4 it is enough to show that

$$\|Lg^*\|_{p'} \approx \|g\|_{\left(L^{p',p}\right)'} = \|g\|_{L^{p,p'}} .$$

Therefore we want

$$\underbrace{\int_0^\infty \left(\int_0^\infty e^{-xt} g^*(t)\, dt\right)^{p'} dx}_{\alpha} \approx \underbrace{\int_0^\infty g^*(t)^{p'} t^{p'-2} dt}_{\beta} .$$

1. We shall show that $\alpha \le K_p \beta$ for some constant K_p.

   From Theorem 3.4 we have:

$$\int_0^\infty (Lg^*)^{p'}(t)\, dt \le \int_0^\infty \left(\int_0^{\frac{1}{t}} g^*(s)\, ds\right)^{p'} dt = \int_0^\infty \left(\int_0^y g^*(s)\, ds\right)^{p'} \frac{dy}{y^2}$$
$$= \int_0^\infty \left(\frac{1}{y}\int_0^y g^*(s)\, ds\right)^{p'} \frac{1}{y} y^{p'-1} dy =: (*) .$$

   Now we use Theorem 2.22: $y^{\lambda p} := y^{p'-1}$, then $\lambda = \frac{p'-1}{p'} < 1$; $\psi := g^* \ge 0$ at $(0,\infty)$ and we get

$$(*) \le (p')^{p'} \int_0^\infty g^*(y)^{p'} y^{p'-2} dy.$$

2. Now we shall establish the converse inequality, namely $\alpha \ge C_p \beta$ for some constant C_p.

   We use Theorem 2.23 with $p = q := p'$, $v \equiv 1$, $f := g^*$, $\phi(x,t) := e^{-xt}$, $w(t) := t^{p'-2}$ and we get

$$\left(\int_0^\infty g^*(t)^{p'} t^{p'-2} dt\right)^{\frac{1}{p'}} \le C \left(\int_0^\infty \left(\int_0^\infty e^{-xt} g^*(t)\, dt\right)^{p'}\right)^{\frac{1}{p'}}$$

if for all $r > 0$ the following inequality holds

$$\left(\int_0^r t^{p'-2}dt\right)^{\frac{1}{p'}} \leq C\left(\int_0^\infty \left(\int_0^r e^{-xt}dt\right)^{p'} dx\right)^{\frac{1}{p'}}.$$

However, since

$$\left(\int_0^r t^{p'-2}dt\right)^{\frac{1}{p'}} = \left(\frac{r^{p'-1}}{p'-1}\right)^{\frac{1}{p'}} = C_p' r^{\frac{1}{p}}$$

and

$$\left(\int_0^\infty \left(\int_0^r e^{-xt}dt\right)^{p'} dx\right)^{\frac{1}{p'}} = \left(\int_0^\infty \left(\frac{1-e^{-xr}}{x}\right)^{p'} dx\right)^{\frac{1}{p'}} =: f(r),$$

we finally obtain

$$\begin{aligned}\frac{f(r)}{r} &= \left(\int_0^\infty \left(\frac{1-e^{-xr}}{xr}\right)^{p'} dx\right)^{\frac{1}{p'}} = \left(\int_0^\infty \left(\frac{1-e^{-y}}{y}\right)^{p'} \frac{dy}{y}\right)^{\frac{1}{p'}} \\ &= \left(\frac{1}{r}\right)^{\frac{1}{p'}} f(1).\end{aligned}$$

Therefore $f(r) = r^{\frac{1}{p}} f(1)$; and the desired inequality follows.

□

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