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Pologrupy operátorů a jejich orbity

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Semigroups of operators and its orbits

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Abstrakt: Orbita spojitého lineárního operátoru T na Banachově prostoru je posloupnost $T^n x$, $n = 0, 1, 2, \ldots$, kde x je pevný vektor. Orbity úzce souvisejí především s dynamikou semigrup operátorů a s invariantními podprostory a podmnožinami. Práce studuje vztah operátoru a jeho orbit. Předmětem první části je vztah posloupností $\|T^n x\|$ a $\|T^n\|$, stabilita a orbity v normě rostoucí do nekonečna. Druhá část se zabývá hustými orbitami – hypercyklicitou a příbuznými pojmy. Ve třetí části jsou definovány a studovány orbit-reflexivní operátory, jako analogie reflexivních algeber operátorů. Kromě běžných orbit jsou také zmíněny slabé orbity a orbity C_0-semigrup.

Klíčová slova: operátor, semigrupa, orbita, hypercyklický, orbit-reflexivní

Title: Semigroups of operators and its orbits

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Abstract: The orbit of a bounded linear operator T on a Banach space is a sequence $T^n x$, $n = 0, 1, 2, \ldots$, where x is a fixed vector. The orbits are closely connected to the dynamics of operator semigroups and to the invariant subspaces and subsets. The thesis studies the relation between the operator and its orbits. The subject of the first part is the relation between sequences $\|T^n x\|$ and $\|T^n\|$, stability, and orbits tending to infinity. The second part deals with dense orbits – hypercyclicity and related notions. In the third part, as an analogue of reflexive algebras of operators, orbit reflexive operators are defined and studied. Apart from "normal" orbits of a single operator, the weak orbits and orbits of C_0-semigroups are also mentioned.

Keywords: operator, semigroup, orbit, hypercyclic, orbit reflexive

Contents

I. Introduction

The thesis consists of three topics from functional analysis and operator theory, all connected by the notion of orbit of an operator. An orbit is a sequence $T^n x$ for n running from 0 to infinity, where T is a fixed bounded linear operator on a Banach space, and x is a fixed vector in the Banach space. Orbits have been present in mathematics for a long time, but they were systematically studied first by Beauzamy [15] and received an enormous attention from that time on. The study of orbits is of course related to dynamics of linear systems and semigroup stability in particular, but also to the invariant subspace problem and local spectral theory.

After Part I, which is this introduction, Part II of the thesis deals with regular orbits, that is orbits whose norm tends either to zero (stable orbit) or to infinity. To prepare the ground, in Chapter 1 we begin with the study of the problem how is the asymptotic behaviour of the sequence $\|T^n\|$ related to the asymptotic behaviour of the sequence $\|T^n x\|$, where again T is a bounded linear operator and x a vector. Then, mainly for the sake of completeness, in Chapter 2 we briefly touch the stability theory. Chapter 3 tries to develop a theory of orbits tending to infinity, in particular we use the results from Chapter 1 to obtain some conditions under which an operator has an orbit tending to infinity. Finally, Chapter 4 switches from "normal" orbits to weak orbits, that is the sequences $x^*(T^n x)$ where x^* is a functional.

Part III is titled Irregular orbits and in fact deals with the notion of hypercyclicity and some closely related notions. An operator is called hypercyclic if it has an orbit norm dense in the whole space, and the notion is introduced in Chapter 5. The usual way of showing that a given operator is hypercyclic is the Hypercyclicity criterion, which is studied in Chapter 6, together with a few examples of hypercyclic operators. Next we look at how the set of hypercyclic vectors of a fixed operator looks like – this is done in Chapter 7. Chapter 8 on the other hand studies the set of hypercyclic operators. Finally the topics of Chapters 9 and 10 are notions closely related to hypercyclicity, namely ε-hypercyclicity and weak hypercyclicity.

Finally Part IV is devoted to the concept of orbit reflexivity, which is an orbit-wise analogue of the well studied notion of reflexivity, primarily defined for operator algebras. Orbit reflexivity was introduced some 25 years ago but until very recently it has received almost no attention. Perhaps one of the reasons is that, as we show in Chapter 11, many of the known classes of "nice" operators are orbit reflexive, and only a few examples of non orbit reflexive operators are known. The first constructed is the famous Read's counterexample to the invariant subspace problem on Banach spaces. However, there are some simpler ones even in Hilbert spaces, and we exhibit some of them in Chapter 12.

Some definitions and notation

We will denote the sets of positive integers, non-negative integers, integers, rational numbers, real numbers and complex numbers by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. The letter $\mathbb{T}$ denotes the set of unimodular complex numbers, i.e. $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$.

Let X be a real or complex Banach space, that is, a complete vector space over the field $\mathbb{F}$ of real or complex numbers, with a norm $\|\cdot\|$. If not stated otherwise, the letter X will always denote such a Banach space without explicit mention. We will use the notation $B_X := \{x \in X \colon \|x\| \le 1\}$, and $S_X := \{x \in X \colon \|x\| = 1\}$ to denote the closed unit ball, and the unit sphere in X, respectively. The closure of a set A is written either as $\overline{A}$ or as A^-, and the diameter of A is denoted by $\operatorname{diam} A := \sup\{\|x-y\| \colon x, y \in A\}$. Given $x \in X$ and $x \in X^*$, we sometimes use the notation $\langle x, x^* \rangle := x^*(x)$ to emphasize the duality of the evaluation of x^* at the point x. If $M \subset X^*$ then the pre-annihilator of M is denoted by $M^{\perp} := \{x \in X : \langle x, x^* \rangle = 0 \text{ for every } x^* \in M\}$. The notation $X = Y \oplus Z$ always denotes the direct sum (not the orthogonal sum).

The algebra of the bounded linear operators acting on X will be denoted by $\mathcal{L}(X)$. If not stated otherwise, the letter T will always denote an operator from $\mathcal{L}(X)$ without explicit mention. For $T \in \mathcal{L}(X)$, $\|T\|$ denotes the operator norm, $\|T\|_{ess}$ the essential norm, $\sigma(T)$ the spectrum, $\sigma_p(T)$ the point spectrum, $\sigma_{ess}(T)$ the essential spectrum, $r(T)$ the spectral radius, and $r_{ess}(T)$ the essential spectral radius. The strong operator topology on $\mathcal{L}(X)$ is often abbreviated by SOT, e.g. for $A \subset \mathcal{L}(X)$ the symbol $\overline{A}^{SOT}$ denotes the strong closure of A.

If $T \in \mathcal{L}(X)$ and $x \in X$, the orbit of x under T is the sequence of iterates $\operatorname{Orb}(T, x) := \{T^n x \colon n \in \mathbb{N}_0\}$. We will also use the symbol $\operatorname{Orb}(T) := \{T^n \colon n \in \mathbb{N}_0\}$.

The orbits can behave regularly. For instance, it may happen that:

(i) the orbit is stable: $\|T^n x\| \to 0$,

(ii) the orbit tends to infinity: $\|T^n x\| \to +\infty$,

(iii) the orbit stays inside (or outside) a given ball: there is a point $y \in X$ and $r > 0$ such that for all n we have $\|T^n x - y\| \le r$ (or $\|T^n x - y\| > r$, respectively),

(iv) the orbit is periodic: there is $n \in \mathbb{N}$ such that $T^n x = x$.

On the other hand, the orbits can be also irregular. For instance, it may happen that:

(i) $\inf \|T^n x\| = 0$ and $\sup \|T^n x\| = +\infty$,

(ii) the orbit is dense in the whole space: $\{T^n x\colon\ n \in \mathbb{N}_0\}^- = X$; such a vector is called hypercyclic vector.

Given $T \in \mathcal{L}(X)$, we will examine the sets of vectors in different categories mentioned above, and use the notation $X_s(T)$, $X_\infty(T)$ and $X_h(T)$ to denote the set of points with stable orbit, points with orbit tending to infinity, and hypercyclic points, respectively. We call T hypercyclic, if there is a hypercyclic orbit (in that case there is even a dense set of hypercyclic points, since all iterates $T^n x$ are also hypercyclic).

It is natural to consider a sequence of (possibly independent) operators T_n instead of the sequence of powers of a single operator T^n. All the above notions extend to this setting, so that for instance the sequence $(T_n)_{n\in\mathbb{N}}$ is called hypercyclic if there is a point $x \in X$ such that the set $\{T_n x\colon\ n \in \mathbb{N}\}$ is dense in X. We will use such notions without explicit definition if it cannot be confusing.

Throughout the text, we will sometimes encounter the weighted shift operators. First, let X be one of the sequence spaces ℓ^p, $1 \le p \le \infty$, or c_0, and denote by $(e_k)_{k=0}^\infty$ the standard basis of X. Given a bounded sequence of nonzero complex weights $(w_k)_{k=0}^\infty$, the unilateral backward shift is defined as $Te_k := w_k e_{k-1}$ if $k > 0$ and $Te_0 := 0$, while the unilateral forward shift is defined as $Te_k := w_k e_{k+1}$ if $k \ge 0$. Second, let X be one of the spaces $\ell^p(\mathbb{Z})$, $1 \le p \le \infty$, or $c_0(\mathbb{Z})$, and denote by $(e_k)_{k=-\infty}^{+\infty}$ the standard basis of X. Given a bounded sequence of nonzero complex weights $(w_k)_{k=-\infty}^{+\infty}$, the bilateral backward shift is defined as $Te_k := w_k e_{k-1}$ for all $k \in \mathbb{Z}$, the bilateral forward shift is defined as $Te_k := w_k e_{k+1}$ for all $k \in \mathbb{Z}$. (Note that the difference between the bilateral backward shift and the bilateral forward shift is just formal – there is a one-to-one correspondence between them.)

It is also interesting and useful to study the C_0-semigroups, which are the continuous analogues of the discrete semigroup $(T^n)_{n\in\mathbb{N}}$ of a single operator T. Since we will encounter this notion in the text from time to time, let us briefly summarize the basic notions and theorems.

A C_0-semigroup, or a strongly continuous one-parameter semigroup, is a family $(T(t))_{t\ge 0}$ of bounded linear operators acting on X, indexed by nonnegative real numbers, which satisfies the following three conditions:

(i) $T(0) = I$,

(ii) $T(t)T(s) = T(t+s)$ for any $t, s \ge 0$ (the semigroup property),

(iii) $\lim_{t\to 0+} \|T(t)x - x\| = 0$ for all $x \in X$ (strong continuity).

Note that such a family indeed forms a strongly continuous semigroup. The generator of the semigroup is the linear operator A defined as

$$Ax := \lim_{t\to 0+} \frac{1}{t}(T(t)x - x),$$

for all x in the domain $D(A) \subset X$ consisting of those points where the above limit exists. The generator can be a bounded operator, and then $T(t) = e^{tA}$,

$t \geq 0$, in the Dunford functional calculus sense, but the important case is when $D(A) \neq X$ and the generator is unbounded. The generator is always a closed, densely defined operator, commuting with all $T(t)$, $t \geq 0$, with the spectrum contained in some left half-plane. The reason why the definition is useful lies in the following theorem.

Theorem 0.1. *Operator A with domain $D(A)$ on X is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ if and only if the associated abstract Cauchy problem (ACP)*

$$\begin{aligned} \frac{du}{dt}(t) &= Au(t), \qquad t \geq 0, \\ u(0) &= x, \end{aligned}$$

is well-posed (for each initial value $x \in D(A)$ there is a unique solution u satisfying the equation), A is densely-defined, and there is a continuous dependence on the initial value: if $z_n \to z$ with $z_n, z \in D(A)$, then $u_n(\cdot) \to u(\cdot)$ locally uniformly on $[0, \infty)$, where u and u_n denote the solutions of (ACP) with respect to the initial value $x = z$ and $x = z_n$, respectively.

In this context, the orbits $(T(t)x)_{t\geq 0}$ are called the mild solutions of (ACP) provided $x \in X$, and classical solutions provided $x \in D(A)$.

The asymptotic behaviour of the semigroup $(T(t))_{t\geq 0}$ can be roughly described by its growth bound defined as

$$\omega_0(T) := \inf\{\omega \in \mathbb{R}\colon \text{ there is } C > 0 \text{ such that for all } t \geq 0,\ \|T(t)\| \leq Ce^{\omega t}\}$$

If $\omega_0(T) < 0$ we say that the semigroup is uniformly exponentially stable.

Original versus adopted results

The thesis consists of Parts II, III and IV. Part II and Part IV are more or less based on an original research, Part III is more or less a compilation of known results. A rule of thumb is also the difference between a Theorem, which contains basically an already known statement, and a Proposition, which is basically a new result, either at least partially published in one of the papers [61] and [62] (below denoted just Papers) or unpublished.

In Part II, the statements Proposition 1.3, Proposition 1.5 and Proposition 1.9 are generalisations of results published in the Papers. Example 3.1, Proposition 3.2, Example 3.4, and Example 4.1 appeared in the Papers; Proposition 3.5 and Proposition 4.2 also, but in weaker forms. The simple statements Observation 3.6 and Example 3.8 are new.

Part III consists of a summarisation of known results, with the initial aim of providing an introduction to hypercyclicity, which hasn't appeared in the subject since [38]. However, during the preparations the monography [11] was published and provided a much broader insight into the subject. The only original, unpublished statements are: Proposition 9.2 and Theorem 9.3 which generalise results from [5], Corollary 10.4 which improves the result from [33], and according to my knowledge, condition (iv) in Theorem 5.1.

Part IV is based mainly on the first Paper [61], which already contains roughly partially Theorem 11.3, Proposition 11.4, Corollary 11.5, Example 12.3, and Example 12.5. The trivial yet new proof of part (iv) in Theorem 11.6 was also used independently in [42]. Part (ii) in Observation 11.1 is new and unpublished. (Lemma 11.2 and most of Theorem 11.6 and partially Theorem 11.3 were published already in [44].)

II. Regular orbits

1. Rate of convergence

Let $T \in \mathcal{L}(X)$. The subject of this part will be the question, under what circumstances there is $x \in X$ such that $\|T^n x\| \sim \|T^n\|$ in a sense? For instance, is there always $x \in X$ such that $\|T^n x\| \geq \frac{1}{2}\|T^n\| \cdot \|x\|$ for all $n \in \mathbb{N}$?

We will use the following solution to the affine plank problem asked by T. Bang [9], which was (in case of symmetric bodies) answered positively by K. Ball [8]. The name comes from the definition of plank to be a subset of points between two parallel hyperplanes in a Banach space, that is a set of form

$$\Big\{x \in X\colon \Big|\langle y - x, f\rangle\Big| \leq w/2\Big\}$$

where $f \in X^*$ is a unit functional, $y \in X$ a fixed vector, and $w \geq 0$ the so-called width. We formulate also a dual version of Ball's theorem. For a survey of similar results see Ball's article on the subject [6].

Theorem 1.1. (Ball's plank theorem) *Let X be a Banach space and $f_n \in X^*$, $n \in \mathbb{N}$. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that $a := \sum_{n=1}^{\infty} \alpha_n < 1$; in case X is reflexive, it is sufficient to assume $a \leq 1$. Then for any $y \in X$ there is a point $x \in X$ of norm 1 such that for all $n \in \mathbb{N}$*

$$\Big|\langle y - x, f_n\rangle\Big| \geq \alpha_n \|f_n\|.$$

Theorem 1.2. (dual version) *Let X be a Banach space and $x_n \in X$, $n \in \mathbb{N}$. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that $a := \sum_{n=1}^{\infty} \alpha_n \leq 1$. Then for any $g \in X^*$ there is a functional $f \in X^*$ of norm 1 such that for all $n \in \mathbb{N}$*

$$\Big|\langle x_n, g - f\rangle\Big| \geq \alpha_n \|x_n\|.$$

It is possible to generalize the theorem from functionals to operators, using a proof similar to that in [62].

Proposition 1.3. (plank theorem for operators) *Let X and Y be Banach spaces and $T_n \in \mathcal{L}(X,Y)$, $n \in \mathbb{N}$. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that $a := \sum_{n=1}^{\infty} \alpha_n < 1$. Then for any $y \in X$ there is a point $x \in X$ of norm 1 such that for all $n \in \mathbb{N}$*

$$\|T_n(y-x)\| \geq \alpha_n \|T_n\|.$$

Moreover, in every ball of radius $c > 0$ there is a point $x \in X$ such that for all $n \in \mathbb{N}$ we have $\|T_n x\| \geq c\alpha_n \|T_n\|$. In particular, the set

$$M := \{x \in X \colon \exists c > 0 \quad \forall n \in \mathbb{N} \quad \|T_n x\| \geq c\alpha_n \|T_n\|\}$$

is dense in X.

If the space Y is the scalar field the theorem becomes the ordinary plank theorem for functionals. If the operators T_n are powers of a single operator, that is, if $T_n = T^n$ for given $T \in \mathcal{L}(X)$, the proposition gives an interesting statement about how the growth of $\|T^n\|$ can be realized by a single orbit. Note that there is no dual version of the theorem, but of course it is possible to apply the theorem to the adjoints T_n^* to obtain a statement about the existence of a functional satisfying certain conditions.

Proof of Proposition 1.3. Consider the adjoint operators $T_n^* \in \mathcal{L}(Y^*, X^*)$. Then for any $n \in \mathbb{N}$ there exists $g_n \in Y^*$ such that $\|g_n\| \leq 1$ and $\|T_n^* g_n\| \geq a\|T_n^*\| = a\|T_n\|$.

Let $y \in X$. Applying the plank theorem 1.1 to the functionals $T_n^* g_n$ and coefficients α_n/a, we obtain a point $x \in X$ with $\|x\| = 1$ such that for every $n \in \mathbb{N}$

$$\left|\langle y - x, T_n^* g_n\rangle\right| \geq \frac{\alpha_n}{a}\|T_n^* g_n\| \geq \alpha_n \|T_n\|,$$

and therefore

$$\begin{aligned}
\|T_n(y-x)\| &\geq \|T_n(y-x)\| \cdot \|g_n\| \\
&\geq \left|\langle T_n(y-x), g_n\rangle\right| = \left|\langle y-x, T_n^* g_n\rangle\right| \\
&\geq \alpha_n \|T_n\|.
\end{aligned}$$

For the additional assertion, let $z \in X$ and $c > 0$. There is $x \in X$ with $\|x\| = 1$ satisfying $\|T_n(\frac{z}{c} - x)\| \geq \alpha_n \|T_n\|$ for all $n \in \mathbb{N}$, thus $z - cx$ belongs to M while its distance from z is not more than c.

For complex Hilbert spaces, Ball proved in [7] that a similar assertion holds even if the sum of coefficients is just square summable, with the necessary restriction to the case when $y = 0$.

Theorem 1.4. (complex plank theorem) *Let X be a complex Hilbert space and $f_n \in X^*$, $n \in \mathbb{N}$. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that $a := \sum_{n=1}^{\infty} \alpha_n^2 \leq 1$. Then there is a point $x \in X$ of norm 1 such that for all $n \in \mathbb{N}$*

$$\left|\langle x, f_n\rangle\right| \geq \alpha_n \|f_n\|.$$

(Note that in [62], we wrote $\sum_{n=1}^{\infty} \alpha_n^2 < 1$ instead of the non-strict inequality. The non-strict inequality is used in [7] and is formally stronger, we thus rather write the non-strict inequality here.) It is again possible to extend this complex plank theorem to operators. Although the complex plank theorem holds just if $y = 0$, we are anyway able to obtain the density of the set M, by introducing an additional plank that places the point to any given open set.

Proposition 1.5. (complex plank theorem for operators) *Let X be a complex Hilbert space, Y be a Banach space and $T_n \in \mathcal{L}(X, Y)$, $n \in \mathbb{N}$. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that $a := \sum_{n=1}^{\infty} \alpha_n^2 < 1$. Then there is a point $x \in X$ of norm 1 such that for all $n \in \mathbb{N}$*

$$\|T_n x\| \geq \alpha_n \|T_n\|.$$

Moreover, the set

$$M := \{x \in X\colon\ \exists c > 0 \quad \forall n \in \mathbb{N} \quad \|T_n x\| \geq c\alpha_n \|T_n\|\}$$

is dense in X.

Proof. Consider the adjoint operators $T_n^* \in \mathcal{L}(Y^*, X^*)$. Fix $n \in \mathbb{N}$. As in the previous proposition, there is $g_n \in Y^*$ such that $\|g_n\| = 1$ and $\|T_n^* g_n\| \geq a\|T_n\|$.

Applying the complex plank theorem 1.4 to the functionals $T_n^* g_n$ and coefficients α_n / a, we obtain a point $x \in X$ with $\|x\| = 1$ such that for every $n \in \mathbb{N}$

$$\left|\langle x, T_n^* g_n\rangle\right| \geq \frac{\alpha_n}{a} \|T_n^* g_n\| \geq \alpha_n \|T_n\|,$$

and therefore

$$\begin{aligned}\|T_n x\| &\geq \|T_n x\| \cdot \|g_n\| \\ &\geq \left|\langle T_n x, g_n\rangle\right| = \left|\langle x, T_n^* g_n\rangle\right| \\ &\geq \alpha_n \|T_n\|.\end{aligned}$$

For the density assertion, let $u \in X$ with $\|u\| = 1$ and ε be a real number with $0 < \varepsilon < 1$. By linearity, it is sufficient to prove that there is $x \in X$ and $c > 0$ such that $\|u - x\| \leq \varepsilon$ and $\|T_n x\| \geq c\alpha_n \|T_n\|$ for all $n \in \mathbb{N}$.

Let $\delta := 1 - \frac{\varepsilon^2}{2}$ and $c := \sqrt{1 - \delta^2}$ so that $\delta^2 + \sum_{n=1}^{\infty} (c\alpha_n)^2 \leq 1$. Applying the complex plank theorem to the functionals $\langle \cdot, u\rangle, T_1^* g_1, T_2^* g_2, \ldots$ and coefficients $\delta, c\alpha_1, c\alpha_2, \ldots$, we obtain a point $x' \in X$ with $\|x'\| = 1$ such that

$\left|\langle x', u\rangle\right| \geq \delta$ and $\left|\langle x', T_n^* g_n\rangle\right| \geq c\alpha_n \|T_n^* g_n\|$ for every $n \in \mathbb{N}$. Therefore for all $n \in \mathbb{N}$

$$\begin{aligned}\|T_n x'\| &\geq \|T_n x'\| \cdot \|g_n\| \\ &\geq \left|\langle T_n x', g_n\rangle\right| = \left|\langle x', T_n^* g_n\rangle\right| \\ &\geq c\alpha_n a \|T_n\|.\end{aligned}$$

Moreover, $\left|\langle x', u\rangle\right| \geq \delta$. Therefore, for a special choice of complex number τ of modulus 1, the point $x := \tau x'$ satisfies $\|x - u\| \leq \varepsilon$, while $\|T^n x\| = \|T^n x'\|$ for all $n \in \mathbb{N}$. More precisely, set $\tau := \frac{\overline{\langle x', u\rangle}}{|\langle x', u\rangle|}$, so that

$$\langle \tau x', u\rangle = \frac{\overline{\langle x', u\rangle}}{|\langle x', u\rangle|}\langle x', u\rangle = \left|\langle x', u\rangle\right| \geq \delta,$$

and therefore

$$\begin{aligned}\|x - u\|^2 &= \langle \tau x' - u, \tau x' - u\rangle = \|\tau x'\|^2 - 2\operatorname{Re}\langle \tau x', u\rangle + \|u\|^2 \\ &\leq 1 - 2\delta + 1 = \varepsilon^2.\end{aligned}$$

Given a Banach space X, an interesting question is, what is the maximal possible exponent in the condition on the sequence $(\alpha)_{n=1}^{\infty}$ in Theorem 1.1 and Theorem 1.4. More precisely, let X be a Banach space. The plank number of X is the supremum of all $p > 0$ such that for any unit functionals $f_i \in X^*$, $i \in \mathbb{N}$, and coefficients $\alpha_i \geq 0$, $i \in \mathbb{N}$, satisfying $\sum_{i=1}^{\infty} \alpha_i^p < 1$, there is $x \in X$, $\|x\| \leq 1$, such that $\left|\langle x, f_i\rangle\right| \geq \alpha_i$ for all $i \in \mathbb{N}$. By Theorem 1.1, the plank number of any Banach space is at least 1, and by Theorem 1.4, the plank number of any complex Hilbert space is at least 2.

Example 1.6. ([7]) *The plank number of the real Hilbert space $X := \mathbb{R}^2$ is 1. In other words, Theorem 1.4 is no more valid in real Hilbert spaces.*

Proof. Suppose the plank number is $p > 1$, let $q \in \mathbb{R}$ such that $1 < q < p$ and fix an arbitrary constant $c \in (0;1)$. Let $n \in \mathbb{N}$ be so large that $n^{1/q-1} < c/2$. Let $y_i \in \mathbb{R}^2$ be unit vectors evenly distributed along the circle: $y_i := (\sin\frac{i\pi}{n}, \cos\frac{i\pi}{n})$ for $i = 1\ldots 2n$. Let $\alpha_i := c \cdot n^{-1/q}$.

For a contradiction, suppose that there is $x \in X$, $\|x\| \leq 1$ such that $\left|\langle x, y_i\rangle\right| \geq \alpha_i$ for all $i = 1\ldots 2n$. But by the choice of y_i, there is i for which $\left|\langle x, y_i\rangle\right| \leq \sin\frac{\pi}{2n}$. But this means that for this particular i

$$c \cdot n^{-1/q} = \alpha_i \leq \left|\langle x, y_i\rangle\right| \leq \sin\frac{\pi}{2n} < \frac{2}{n},$$

a contradiction.

An intriguing open problem is to determine the plank number of complex ℓ^p spaces for $p \in (1;\infty)$, $p \neq 2$. Numerical experiments, as well as other evidence (cf. [50], or [11, section 10.1]) suggest that in these complex ℓ^p spaces, the plank number could be strictly bigger that 1, namely it is justified to formulate the following conjecture.

Conjecture 1.7. *Let $X := \ell^p(X)$, $1 \le p \le \infty$, over the field of complex numbers. Then the plank number of X equals* $\min\{p, q\}$ *where* $\frac{1}{p} + \frac{1}{q} = 1$.

In the case of finite number of planks of the same width, we have at least one inequality.

Theorem 1.8. ([74, Proposition 1]) *Let $p > 1$, and let X be a complex $L^p(\mu)$-space, $\frac{1}{p} + \frac{1}{q} = 1$, $m \in \mathbb{N}$, and $f_n \in X^*$ for $n \in \{1, \dots, m\}$. Then there is a point $x \in X$ of norm* 1 *such that for all $n \in \{1, \dots, m\}$*

$$\left|\langle x, f_n\rangle\right| \ge m^{-1/\min\{p,q\}} \|f_n\|.$$

Proof. Since X is reflexive there are points x_n, $n = 1, \dots, m$, of norm 1 such that $\langle x_n, f_n\rangle = \|f_n\|$ for all n.

For isomorphic Banach spaces Y and Z, let $d(Y, Z)$ denote the Banach-Mazur distance

$$d(Y, Z) := \inf\{\|T\| \cdot \|T^{-1}\| \colon\ T\colon\ Y \to Z \text{ is an isomorphism}\}.$$

Let $Z := \operatorname{Span}\{x_1, \dots, x_m\}$ be a subspace of X of dimension $d \le m$ and let $Y := \ell^2(d)$ be the d-dimensional complex Hilbert space. By a result of F. John [49], we have $d(Y, Z) \le d^{|1/2-1/p|}$.

First we show that this means there is an isomorphism $T\colon\ Y \to Z$ of norm 1 with $0 < \|T^{-1}\| \le d^{|1/2-1/p|}$. Indeed, let $C := d^{|1/2-1/p|}$ and consider the set

$$K := \{T \in \mathcal{L}(Y, Z)\colon\ T \text{ is an isomorphism},\ \|T\| = 1,\ \|T^{-1}\| \le C + 1\}.$$

By the above result of F. John, for each $k \in \mathbb{N}$ there is $T_k \in K$ with $\|T^{-1}\| \le C + \frac{1}{k}$. Since K is a compact set in $\mathcal{L}(Y, Z)$, the sequence $(T_k)_{k\in\mathbb{N}}$ has an accumulation point, say T, which is an isomorphism and satisfies $\|T\| = 1$ and $\|T^{-1}\| \le C$.

Now consider the functionals $T^*(f_n|_Z) \in Y^*$, $n \in \{1, \dots, m\}$, and m coefficients $m^{-1/2}$. By the complex plank Theorem 1.4, there is $y \in Y$ of norm 1 such that

$$\begin{aligned}\left|\langle Ty, f_n|_Z\rangle\right| = \left|\langle y, T^*(f_n|_Z)\rangle\right| &\ge m^{-1/2}\|T^*(f_n|_Z)\| \\ &\ge m^{-1/2}\frac{\|f_n|_Z\|}{\|T^{-1}\|} \ge m^{-1/2}\|f_n\| d^{-|1/2-1/p|},\end{aligned}$$

since $\|f_n\| = \|f_n|_Z\|$ by the choice of points x_n. For $x := Ty \in B_Z \subset B_X$ we thus have

$$\begin{aligned}\left|\langle x, f_n\rangle\right| = \left|\langle x, f_n|_Z\rangle\right| &\ge m^{-1/2} d^{-|1/2-1/p|}\|f_n\| \ge m^{-1/2-|1/2-1/p|}\|f_n\| \\ &= m^{-1/\min\{p,q\}}\|f_n\|.\end{aligned}$$

On the other hand, Example 3.1 and Proposition 3.2 will show that the plank number of ℓ^p cannot be greater than p.

It is also possible to formulate a version of the plank theorem for certain continuous families of operators, namely strongly continuous one-parameter semigroups and groups. However, we formulate a more general version. We will use two ad-hoc notions. A function $\phi\colon [0;\infty) \to [0;\infty)$ is said to have uniformly bounded decline (UBD) if there is $\delta > 0$ and $M \geq 1$ such that for all $t, s \in \mathbb{R}$, $0 \leq t < s < \infty$ with $s - t < \delta$ we have $\phi(s) > M^{-1}\phi(t)$. Similarly, it has uniformly bounded growth (UBG) if under the same circumstances we have $\phi(s) < M\phi(t)$. Note that if $(T(t))_{t\geq 0}$ is a C_0-semigroup and $\omega_0(T)$ its growth bound then for any $\omega > \omega_0(T)$ there is $M > 0$ such that for all $t, s \in [0;\infty)$ we have $\|T(t)\| \leq Me^{\omega|t-s|}\|T(s)\|$, so the function $t \mapsto \|T(t)\|$ has UBG. Moreover, for any $x \in X$ the norm of the orbit, i.e. $t \mapsto \|T(t)x\|$ has UBG as well, even uniformly, i.e. the numbers $\delta > 0$ and $M \geq 1$ can be chosen the same independently on $x \in X$.

Proposition 1.9. *Let X and Y be Banach spaces and let $(T(t))_{t\geq 0}$ be a family of operators $T(t) \in \mathcal{L}(X, Y)$ such that the function $\|T(\cdot)\|$ has UBG and the norm of every orbit $\|T(\cdot)x\|$, $x \in X$, has also UBG, all uniformly with the same δ and M. Suppose that the function $\phi\colon [0;\infty) \to [0;\infty)$ has UBD and $\phi \in L^1[0;\infty)$, in case X is a complex Hilbert space just $\phi \in L^2[0;\infty)$. Then for every $\varepsilon > 0$ there is $\delta_0 \in (0;\varepsilon]$ and a dense set of $x \in X$ such that there is $c > 0$ such that $\|T(t-\delta_0)x\| \geq c\phi(t)\|T(t)\|$ for each $t \geq \delta_0$.*

Proof. In case X is a complex Hilbert space, set $p = 2$, otherwise set $p = 1$. By assumption there is a common $\delta > 0$ and $M \geq 1$ from the definitions of uniform UBD and UBG of ϕ, $\|T(\cdot)\|$, $\|T(\cdot)x\|$. Let $\delta_0 := \min\{\varepsilon, \frac{\delta}{2}\}$. Consider the points $t_n := n\delta_0$ for $n \in \mathbb{N}_0$. Since $\phi(s) > M^{-1}\phi(t_n)$ for each $s \in (t_n, t_{n+1})$, $n \in \mathbb{N}_0$, we have

$$\delta_0 \sum_{n=0}^{\infty} \phi(t_n)^p \leq M^p \int_0^{\infty} \phi(s)^p \, ds,$$

so $(\phi(t_n))_{n=0}^{\infty} \in \ell^p$.

Applying the plank theorem for operators 1.3, or 1.5 in case of complex Hilbert space, to the sequence of coefficients $(\phi(t_{n+1}))_{n=0}^{\infty}$ and the sequence of operators $(T(t_n))_{n=0}^{\infty}$, we obtain a dense set of points $x \in X$ for each of which there is $c_0 > 0$ such that for all $n \in \mathbb{N}_0$ we have $\|T(t_n)x\| \geq c_0\phi(t_{n+1})\|T(t_n)\|$. Note that the indices used for the function ϕ and for the family of operators T differ by 1.

Let $t_n \leq t \leq t_{n+1}$, $n \in \mathbb{N}$. That is, $0 \leq t - \delta_0 \leq t_n \leq t \leq t_{n+1}$, and the differences between each two of these subsequent positive numbers are strictly less than δ. We have

$$\begin{aligned}
\|T(t-\delta_0)x\| &> M^{-1}\|T(t_n)x\| \\
&\geq M^{-1}c_0 \cdot \phi(t_{n+1}) \cdot \|T(t_n)\| \\
&> M^{-1}c_0 \cdot M^{-1}\phi(t) \cdot \|T(t_n)\| \\
&> M^{-1}c_0 \cdot M^{-1}\phi(t) \cdot M^{-1}\|T(t)\| \\
&= M^{-3}c_0 \cdot \phi(t)\|T(t)\|.
\end{aligned}$$

Indeed, the first, strict inequality follows from the UBG property of $\|T(\cdot)x\|$: since $t_n - (t-\delta_0) < \delta$ it yields $\|T(t_n)x\| < M\|T(t-\delta_0)x\|$. The second is the inequality obtained from the plank theorem. The third, strict inequality follows from the UBD property of ϕ: since $t_{n+1} - t < \delta$ it yields $\phi(t_{n+1}) > M^{-1}\phi(t)$. Finally the last, strict inequality follows from the UBG property of $\|T(\cdot)\|$: since $t - t_n < \delta$ it yields $\|T(t)\| < M\|T(t_n)\|$.

Unfortunately the proof does not provide the control of the norm of the obtained point (as will be in Corollary 1.14) – the norm of the vectors x depends on δ. Moreover, we have to "skew" the inequality "$\|T(t)x\| \geq C\phi(t)\|T(t)\|$" to "$\|T(t-\delta_0)x\| \geq C\phi(t)\|T(t)\|$" where $\delta_0 > 0$. This is because whereas the C_0-semigroups have a limited growth, they do not have necessarily a limited decline (consider e.g. a nilpotent shift semigroup on $[0;1]$). The formulation is however sufficient for applications to orbits tending to infinity in Chapter 3.

We can also examine other possible definitions of "largeness" of the orbit. Let X be a Banach space and $(\alpha_n)_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that $\lim_{n\to\infty} \alpha_n = 0$. Define

$$M := \{x \in X\colon\ \|T_n x\| \geq \alpha_n \|T_n\| \quad \text{for infinitely many } n\text{'s}\}$$

It is easy to see that the plank theorem for operators implies that M is a dense set. However, using a different technique with Baire theorem as a core, it is possible to prove that M is even a residual set. (Or even a complement of a σ-porous set, as was shown recently in [4], cf. end of Chapter 7.) Recall that a set is called residual if its complement is a first category set or, equivalently, if it contains a G_δ subset dense in the whole space. Again, we formulate the result generally for any sequence of operators $T_n \in \mathcal{L}(X,Y)$.

Theorem 1.10. ([59]) *Let X, Y be Banach spaces, $T_n \in \mathcal{L}(X,Y)$ and $(\alpha_n)_{n=1}^{\infty}$ be a sequence of nonnegative numbers such that $\lim_{n\to\infty} \alpha_n = 0$. Then the set M of those $x \in X$ satisfying $\|T_n x\| \geq \alpha_n \|T_n\|$ for infinitely many n's, is residual.*

Proof. If there are infinitely many n's such that $T_n = 0$ then the assertion is obviously true. Otherwise, fix $k \in \mathbb{N}$ and let

$$M_k := \{x \in X\colon\ \exists n \geq k \quad \|T_n x\| > \alpha_n \|T_n\|\}.$$

Clearly M_k is open. We shall prove it is also dense. Let $z \in X$ and $\varepsilon > 0$ be arbitrary. There exists $n \geq k$ such that $\alpha_n < \varepsilon$ and $T_n \neq 0$. Hence there exists $x \in X$ of norm 1 such that $\|T_n x\| > \alpha_n \varepsilon^{-1} \|T_n\|$. Since

$$2\alpha_n \|T_n\| < \|T_n(2\varepsilon x)\| \leq \|T_n(\varepsilon x + z)\| + \|T_n(\varepsilon x - z)\|,$$

either $\|T_n(z+\varepsilon x)\|$ or $\|T_n(z-\varepsilon x)\|$ is bigger than $\alpha_n\|T_n\|$, so one of the points $z+\varepsilon x$ and $z-\varepsilon x$ is in M_k, while its distance from z is at most ε.

Now by the Baire theorem, the intersection

$$\bigcap_{k=1}^{\infty} M_k \subseteq M$$

is a dense G_δ set.

Corollary 1.11. *The set of $x \in X$ for which the local spectral radius*

$$r_x(T) := \limsup_{n\to\infty} \|T^n x\|^{1/n}$$

is equal to the spectral radius $r(T)$, is residual in X.

If we have a single operator and want to find an orbit which is big with respect to the spectral radii $r(T^n)$ instead of the norms $\|T^n\|$, there is a similar result restricting the sequence of coefficients not to ℓ^1 as in the plank theorems, but again just to c_0.

Theorem 1.12. ([15], [57]) *Let X be a complex Banach space, $\varepsilon > 0$ and $(\alpha_n)_{n=1}^{\infty}$ be a sequence of nonnegative numbers such that $\lim_{n\to\infty} \alpha_n = 0$. Then there is a point $x \in X$, of norm at most $\sup \alpha_n + \varepsilon$, such that $\|T^n x\| \geq \alpha_n r(T^n)$ for all $n \in \mathbb{N}$.*

In case of real Banach spaces, it was proved in [58, Theorem 2.3] using complexification techniques that under the same conditions there is a dense set of $x \in X$ such that for some $c > 0$, for all $n \in \mathbb{N}$ we have $\|T^n x\| \geq c\alpha_n r(T^n)$.

There exists also a version of the above theorem for the C_0-semigroups which offers an alternative to Proposition 1.9. Recall that $\omega_0(T)$ denotes the growth bound of the semigroup. We will just state the version without proof.

Theorem 1.13. ([64, Lemma 3.1.7]) *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with $\omega_0 \geq 0$, $\varepsilon \in (0;1)$ and $\alpha : [0;\infty) \to [0;1]$ be a non-increasing function such that $\lim_{t\to\infty} \alpha(t) = 0$. Then there is a point $x \in X$, of norm 1, such that $\|T(t)x\| \geq (1-\varepsilon)\alpha(t)$ for all $t \geq 0$.*

Corollary 1.14. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup, $\varepsilon > 0$ and $\alpha : [0;\infty) \to [0;\infty)$ be a bounded non-increasing function such that $\lim_{t\to\infty} \alpha(t) = 0$. Then there is a point $x \in X$, of norm at most $\sup_{t\geq 0} \alpha(t) + \varepsilon$, such that $\|T(t)x\| \geq \alpha(t)e^{\omega_0(T)t}$ for all $t \geq 0$.*

Proof of Corollary 1.14. If $\alpha(0) = 0$ then just take $x := 0$. If $\alpha(0) > 0$ we rescale the semigroup, function α and ε to apply Theorem 1.13 in the following way. Define $\tilde{T}(t) := e^{-\omega_0(T)t}T(t)$, $\tilde{\alpha}(t) := \alpha(t)/\alpha(0)$ and $\tilde{\varepsilon} := \frac{\varepsilon}{\varepsilon+\alpha(0)}$. From Theorem 1.13 we obtain $\tilde{x}$ such that $\|\tilde{T}(t)\tilde{x}\| \geq (1-\tilde{\varepsilon})\tilde{\alpha}(t)$ for all $t \geq 0$ so $x := (\alpha(0)+\varepsilon)\tilde{x}$ satisfies $\|T(t)x\| \geq \alpha(t)e^{\omega_0(T)t}$ for all $t \geq 0$.

2. Stable orbits

The operator T is called (strongly) stable if every orbit is (strongly) stable, that is if $\|T^n x\| \to 0$ for all $x \in X$. We will denote $X_s(T) = \{x \in X\colon \lim T^n x = 0\}$.

Stability of both discrete and continuous systems has been thoroughly studied for a long time. It is closely connected not only to the evolutionary differential equations but also ergodic theory, harmonic analysis etc. For some recent results we refer for instance to monographs [29] and [64].

Now let us turn to the Datko-Pazy theorem, first formulated in [26] and [66]. The first implication shows how the semigroup property immediately leads to an integrability condition sufficient for stability. (In fact, instead of the semigroup property, a UBG of the orbits is sufficient.)

Theorem 2.1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup and let $x \in X$. If for some $p \in [1;\infty)$*

$$\int_0^\infty \|T(t)x\|^p\,dt < \infty$$

then $T(t)x \to 0$ as $t \to \infty$.

Proof. To obtain a contradiction, suppose that the orbit of x is not stable. Hence there is $\varepsilon > 0$ and a sequence $(t_n)_{n\in\mathbb{N}_0}$ of real numbers such that $\|T(t_n)x\| \geq \varepsilon$, $t_{n+1} > t_n + 1$ for $n \in \mathbb{N}$, and $t_0 > 1$. Let $M := \sup_{0\leq t\leq 1}\|T(t)\| > 0$. Then for $t \in (t_n - 1, t_n)$, $n \in \mathbb{N}_0$, we have $\|T(t)\| \geq M^{-1}\varepsilon$ by the semigroup property, so

$$\int_0^\infty \|T(t)x\|^p \geq \sum_{n=0}^\infty (M^{-1}\varepsilon)^p = \infty.$$

If the assumption holds for all $x \in X$ with a single $p \in [1;\infty)$, by the closed graph theorem applied to the map $x \to L^p(\mathbb{R}, X)\colon\ x \mapsto T(\cdot)x$, there is a constant $C > 0$ such that for every $x \in X$

$$\int_0^\infty \|T(t)x\|^p\,dt \leq C\|x\|^p.$$

In this case there is a quantitative strengthening of the Datko-Pazy theorem, based on Corollary 1.14.

Theorem 2.2. ([64, Lemma 3.1.8]) *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup. If for some $p \in [1;\infty)$ there is $C > 0$ such that for all $x \in X$*

$$\int_0^\infty \|T(t)x\|^p \, dt \leq C\|x\|^p,$$

then $\omega_0(T) \leq -1/(pC)$.

Proof. Let $\omega := \omega_0(T)$. By Theorem 2.1, every orbit is stable so it follows from the uniform boundedness principle that $\omega \leq 0$. Let $\delta > 0$ and $0 < \varepsilon < 1$ and define $\alpha(t) := e^{-\delta t}$. By Corollary 1.14, there is a point $x_0 \in X$ of norm at most $1 + \varepsilon$ such that

$$\|T(t)x_0\| \geq \alpha(t)e^{\omega t} = e^{(\omega-\delta)t}.$$

Integrating the p-th power of this inequality and using the assumption of the theorem, we obtain

$$C(1+\varepsilon)^p \geq C\|x_0\|^p \geq \int_0^\infty \|T(t)x_0\|^p \, dt \geq -\frac{1}{p(\omega-\delta)}.$$

The numbers $\delta > 0$ and $\varepsilon > 0$ can be chosen arbitrarily small, so $\omega \leq -1/(pC)$.

Corollary 2.3. *A C_0-semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable, if and only if there is $p \geq 1$ such that for every $x \in X$*

$$\int_0^\infty \|T(t)x\|^p \, dt < \infty.$$

Proof. If $\omega_0(T) < 0$ then for some $\omega < 0$ and $C > 0$ we have $\|T(t)\| \leq Ce^{\omega t} \to 0$ exponentially so for every $x \in X$, $\|T(\cdot)x\| \in L^p$ for any $p \geq 1$. The opposite implication follows from the closed graph theorem mentioned above, and Theorem 2.2.

One could ask if it is possible to use Proposition 1.9 instead of Corollary 1.14. In such a case, there is unfortunately no control of the constant M so the method yields nothing apart from the classical, non-quantitative Datko-Pazy theorem.

There are other more general results of Datko-Pazy type, see [79], [76], [63], [65].

3. Orbits tending to infinity

An operator $T \in \mathcal{L}(X)$ is said to be power bounded if there is $C > 0$ such that for all $n \in \mathbb{N}$ we have $\|T^n\| \le C$. By uniform boundedness principle, T is power bounded, if and only if all its orbits are bounded. In other words, the sequence $\|T^n\|$ is unbounded if and only if the sequence $\|T^n x\|$ is unbounded for some $x \in X$.

Analogously, one could ask whether the following equivalence holds: $\|T^n\|$ tends to infinity if and only if $\|T^n x\|$ tends to infinity for some $x \in X$. Of course if there is a point $x \in X$ with orbit tending to infinity then $\|T^n\| \ge \|T^n x\|/\|x\| \to \infty$, so the crucial question lies in the other implication.

If X is finite dimensional then $\|T^n\| \to \infty$ implies $1 < \|T\| = r(T)$ so there is $\lambda \in \sigma_p(T)$ with $|\lambda| > 1$ and the corresponding eigenvector $x \in X$ satisfies $\|T^n x\| = |\lambda|^n \cdot \|x\| \to \infty$. In the infinite dimensional spaces, the implication doesn't hold anymore, as the following example shows. Recall the notation $X_\infty(T) := \{x \in X\colon\ \|T^n x\| \to \infty\}$.

Example 3.1. ([62, Example 4]) *On the space $X = \ell^p$, $1 \le p < \infty$, there is an operator $T \in \mathcal{L}(X)$ satisfying $\|T^n\| = (n+1)^{1/p}$ for all $n \in \mathbb{N}$ but $X_\infty(T) = \emptyset$.*

Proof. Let $(e_k)_{k=1}^\infty$ be the standard basis in the space $X = \ell^p$ (real or complex). Let $T \in \mathcal{L}(X)$ be the weighted backward shift defined by

$$Te_k := \begin{cases} \left(\frac{k}{k-1}\right)^{1/p} e_{k-1} & \text{for } k > 1,\\ 0 & \text{for } k = 1.\end{cases}$$

Hence

$$\|T^n\| = \prod_{k=2}^{n+1}\left(\frac{k}{k-1}\right)^{1/p} = (n+1)^{1/p}$$

for all n. For the contradiction, suppose that there is $x = \sum_{k=1}^\infty c_k e_k \in \ell^p$ such that $\|x\| = \left(\sum_{k=1}^\infty |c_k|^p\right)^{1/p} \le 1$ but $\|T^n x\| \to \infty$ as $n \to \infty$. Consequently,

$$\frac{1}{n}\sum_{j=n}^{2n-1} \|T^j x\|^p \to \infty, \qquad \text{as } n \to \infty.$$

Let us estimate the above arithmetic mean. First we have

$$\begin{aligned}\|T^j x\|^p &= \Big\|\sum_{k=j+1}^{\infty}\left(\frac{k}{k-j}\right)^{1/p} c_k e_{k-j}\Big\|^p\\ &\le \sum_{k=j+1}^{2j} |c_k|^p \frac{k}{k-j} + \sum_{k=2j+1}^{\infty} |c_k|^p \frac{k}{k-j},\end{aligned}$$

where the second sum can be estimated by $2\|x\|^p \leq 2$ since for $k > 2j$ we have $\frac{k}{k-j} < 2$. If we sum up the inequalities we get

$$
\begin{aligned}
\sum_{j=n}^{2n-1} \|T^j x\|^p &\leq 2n + \sum_{j=n}^{2n-1} \sum_{k=j+1}^{2j} |c_k|^p \frac{k}{k-j} \\
&\leq 2n + \sum_{k=n+1}^{4n} |c_k|^p \sum_{i=1}^{k} \frac{k}{i} \\
&\leq 2n + \sum_{k=n+1}^{4n} |c_k|^p \, 4n(1+\log 4n),
\end{aligned}
$$

so that

$$2 + 4(1+\log 4n) \sum_{k=n+1}^{4n} |c_k|^p \geq \frac{1}{n} \sum_{j=n}^{2n-1} \|T^j x\|^p \to \infty.$$

Hence, for all n large enough, the left hand side is greater than 6, i.e., if we write $s_n := \sum_{k=n+1}^{4n} |c_k|^p$ then

$$s_n \geq \frac{1}{1+\log 4n}.$$

But this is a contradiction since for such n we have

$$
\begin{aligned}
1 = \sum_{k=1}^{\infty} |c_k|^p &\geq s_n + s_{4n} + s_{4\cdot 4n} + s_{4\cdot 4\cdot 4n} + \ldots \\
&\geq \sum_{j=1}^{\infty} \frac{1}{1+\log 4^j n} = \sum_{j=1}^{\infty} \frac{1}{1+\log n + j\log 4} = \infty.
\end{aligned}
$$

It is possible to obtain a slightly faster growth: defining T to be the backward shift with weights

$$\left(\frac{(k+1)\log(k+1)}{k\log k}\right)^{1/p} \qquad \text{instead of} \qquad \left(\frac{k}{k-1}\right)^{1/p},$$

a similar proof shows that T has no orbit tending to infinity, but

$$\|T^n\| = \left(\frac{1}{2\log 2}\right)^{1/p} \big((n+2)\log(n+2)\big)^{1/p}.$$

In particular, there is an operator $T \in \mathcal{L}(\ell^1)$ with $\|T^n\| \sim n\log n$ without an orbit tending to infinity.

However, the example is in a sense the best possible, as can be proved using the plank theorem for operators: if the norms of the operator satisfy $\|T^n\| \geq Cn^{1+\varepsilon}$ for some $\varepsilon > 0$, $C > 0$, than there is a dense set of points whose orbit tends to infinity. In the complex Hilbert space the same conclusion holds if $\|T^n\| \geq Cn^{1/2+\varepsilon}$ for some $\varepsilon > 0$, $C > 0$. (The situation in the real Hilbert spaces, or in complex ℓ^p spaces, is not clear as far as we know.) We will formulate the proposition in a more general form, including a technical detail in the conclusion which will be used in Part IV.

Proposition 3.2. *Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X)$ be a sequence of operators satisfying*

$$\sum_{n=1}^{\infty} \frac{1}{\|T_n\|} < \infty, \qquad \text{or when } X \text{ is a complex Hilbert space} \qquad \sum_{n=1}^{\infty} \frac{1}{\|T_n\|^2} < \infty.$$

Then there is a dense set of vectors $x \in X$ such that $\|T_n x\| \to \infty$ and $\inf_{n\in\mathbb{N}} \|T_n x\| > 0$.

Proof. Set $p := 2$ in the case when X is a complex Hilbert space and $p := 1$ otherwise. Let $(\beta_n)_{n=1}^{\infty}$ be a sequence of real numbers tending to infinity such that

$$\sum_{n=1}^{\infty} \frac{\beta_n}{\|T_n\|^p} < \infty$$

so that $(\beta_n^{1/p}/\|T_n\|)_{n=1}^{\infty} \in \ell^p$. By the plank theorem for operators 1.3 or 1.5, respectively, there is a dense set of points $x \in X$ for which there is $c > 0$ such that

$$\|T_n x\| \geq c\frac{\beta_n^{1/p}}{\|T_n\|}\|T_n\| \to \infty.$$

Note that for some particular classes of operators, a slower growth rate could ensure the existence of an orbit tending to infinity, for instance in [67] the following is proved.

Theorem 3.3. *Let*

$$T = \begin{pmatrix} N_1 & M \\ 0 & N_2 \end{pmatrix}$$

be a 2-normal operator, i.e., N_1, M and N_2 are normal, commuting operators. If the polar decompositions can be written as $N_1 = UP_1$ and $N_2 = UP_2$ (U unitary, P_1, P_2 positive) and if there exist $\rho, \delta > 0$ such that $\|T^n\|_{ess} > \rho n^{\delta}$ for each $n \in \mathbb{N}$, then any compact perturbation of T admits an orbit tending to infinity.

J.-M. Augé recently in [4] defined a quantity $q(X)$ as the supremum of all $q > 0$ such that for every non-nilpotent $T \in \mathcal{L}(X)$ there is $x \in X$ such that

$$\sum_{n=1}^{\infty} \left(\frac{\|T^n x\|}{\|T^n\|} \right)^q = \infty.$$

Denote the plank number of X by $\pi(X)$. The proof of Proposition 3.2 in fact shows that $\pi(X) \leq q(X)$. If we consider the complex ℓ^p spaces, Proposition 3.2 together with Example 3.1 proves that $q(\ell^p) \leq p$ if $p \in [1;\infty)$ – and Augé in [4] uses Banach space geometry tools, namely the modulus of asymptotic uniform smoothness, to show the other inequality, so that $q(\ell^p) = p$. His results however doesn't seem to be strong enough to show that $\pi(\ell^p) = p$ for $p \in [1;2]$.

There is also a version of both Example 3.1 and Proposition 3.2 for C_0-semigroups of operators.

Example 3.4. *On the space $X = L^p(1;\infty)$, $1 \leq p < \infty$, there is a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| = (t+1)^{1/p}$ for all $t \geq 0$ but $X_\infty(T) = \emptyset$.*

The semigroup can be constructed as a weighted backward unilateral shift:

$$(T(t)f)(z) := \left(\frac{z+t}{z}\right)^{1/p} f(z+t)$$

for $f \in X = L^p(1;\infty)$, $t \geq 0$, $z \geq 1$. The argument is analogous to that in Example 3.1.

Proposition 3.5. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup satisfying*

$$\int_0^\infty \frac{1}{\|T(t)\|} dt < \infty,$$

or when X is a complex Hilbert space

$$\int_0^\infty \frac{1}{\|T(t)\|^2} dt < \infty.$$

Then there is a dense set of vectors $x \in X$ such that $\|T(t)x\| \to \infty$.

Proof. Let $p := 2$ in case of complex Hilbert space, $p := 1$ otherwise. Let $\beta\colon [0;\infty) \to [0;\infty)$ be a nondecreasing Lebesgue measurable function tending to infinity, such that

$$\int_0^\infty \frac{\beta(t)}{\|T(t)\|^p} dt < \infty$$

so that the function $\phi(t) := \beta(t)^{1/p}/\|T(t)\|$ for $t \geq 0$ satisfies $\phi \in L^p[0;\infty)$. Moreover, ϕ satisfies the UBD condition. Indeed, let ω_0 be the growth bound of T. By the definition there is $M \geq 1$ and $\omega \geq \omega_0$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Without loss of generality, suppose that $\omega > 0$. Set $\delta := \frac{1}{\omega} > 0$. Then for any $s, t \in \mathbb{R}$ such that $0 \leq t < s < \infty$ and $s - t < \delta$, we have $\|T(s)\| \leq Me^{\omega(s-t)}\|T(t)\| < Me^1\|T(t)\|$. Hence

$$\phi(s) = \frac{\beta(s)^{1/p}}{\|T(s)\|} \geq \frac{\beta(t)^{1/p}}{Me\|T(t)\|} = M^{-1}e^{-1}\phi(t).$$

By Proposition 1.9, for $\varepsilon := 1$ there is $\delta_0 \leq \varepsilon = 1$ and a dense set of $x \in X$ such that for some $c > 0$

$$\|T(t-\delta_0)x\| \geq c\phi(t)\|T(t)\| = c\beta(t)^{1/p} \to \infty.$$

An interesting problem is a characterization of the sets which can be written as $X_\infty(T) := \{x \in X\colon \|T^n x\| \to \infty\}$ for some $T \in \mathcal{L}(X)$.

Observation 3.6. *The set $X_\infty(T)$ of $x \in X$ such that $\|T^n x\| \to \infty$, is a homogeneous $F_{\sigma\delta}$ set (in the norm topology). Moreover, $X_\infty(T) \cap S_X$ is either empty or infinite.*

Proof. Clearly if $x \in X_\infty(T)$ then $\lambda x \in X_\infty(T)$ for each $\lambda \in \mathbb{F} \setminus \{0\}$ and

$$X_\infty(T) = \bigcap_{k=1}^\infty \bigcup_{n_0=1}^\infty \bigcap_{n=n_0}^\infty \{x \in X\colon \|T^n x\| \geq k\}$$

where the inner sets are closed. Moreover, suppose that the set $X_\infty(T) \cap S_X$ is nonempty and finite. If $x \in X_\infty(T)$ then $T^n x \in X_\infty(T)$ so there are $n, m \in \mathbb{N}$, $n < m$, such that $T^m x = \lambda T^n x$ for some $\lambda \in \mathbb{F}$. For each $k \in \mathbb{N}$ we have

$$\|T^{n+k(m-n)}\| \cdot \|x\| \geq \|T^{n+k(m-n)}x\| = |\lambda|^k \|T^n x\|.$$

Since $x \in X_\infty(T)$ the middle term tends to infinity as $k \to \infty$ so $|\lambda| > 1$ and the whole sequence $\|T^n\|$ tends to infinity exponentially. Hence by Proposition 3.2, $X_\infty(T)$ is dense in X and we have a contradiction since $X_\infty(T)$ is also homogeneous.

Prăjitură [69] conjectured that if $X_\infty(T) \neq \emptyset$ then $X_\infty(T)$ is even dense in X. However, in all the infinite dimensional separable Banach spaces, the conjecture turned out to be false, as was shown by first by P. Hájek and R. Smith for spaces with symmetric basis and later by J.-M. Augé in general.

Theorem 3.7. ([45], [3]) *Let X be an infinite-dimensional separable Banach space. Then there is $T \in \mathcal{L}(X)$ such that $X_\infty(T)$ is non-empty but not dense, and T can be written as $Id + K$ with K a compact operator. Moreover, if X has a symmetric basis then there is $T \in \mathcal{L}(X)$ such that $X_\infty(T)$ is non-empty and nowhere dense in X.*

The set $X_\infty(T)$ is also far from being linear. Obviously if $x \in X_\infty(T)$ then $-x \in X_\infty(T)$ but $x + (-x) = 0 \notin X_\infty(T)$. However, even $X_\infty(T) \cup X_s(T)$ is not necessarily a linear set, as the following example shows.

Example 3.8. *There is a Hilbert space X, an operator T and vectors $x, y \in X$, such that $x, y \in X_\infty(T)$ but $x + y \notin X_\infty(T) \cup X_s(T)$.*

Proof. Let X be the orthogonal sum of the Hilbert spaces with standard bases $(e_k)_{k\in\mathbb{N}}$, and $(f_k)_{k\in\mathbb{N}}$, respectively. Let the operator T be defined as

$$T(ke_k) := (k+1)e_{k+1} \qquad \text{and} \qquad Tf_k := f_{k+1}, \qquad k \in \mathbb{N}$$

so that for $x := -e_1$, $y := e_1 + f_1$ we have $\|T^n x\| = n+1$ and $\|T^n y\| = \sqrt{(n+1)^2+1}$, while $\|T^n(x+y)\| = 1$. Clearly T is bounded.

We finish by relating the sets X_∞ for different operators, the main part was noted in [67, Proposition 2.2], as a supplement to former Ansari's theorem on hypercyclic operators – see Theorem 8.6.

Theorem 3.9. *Let $m \in \mathbb{N}$ and $\lambda \in \mathbb{T}$. Then $X_\infty(T) = X_\infty(T^m) = X_\infty(\lambda T)$.*

Proof. Suppose that the sequence $\|T^n x\|$ does not tend to infinity so that $M := 1 + \liminf_{n\to\infty} \|T^n x\| < \infty$. Then there is a sequence n_k of positive integers tending to infinity such that $\|T^{n_k} x\| \le M$. Let $m \in \mathbb{N}$ be arbitrary. Then for $n_k > m$ the number n_k can be uniquely written as $n_k = mj_k - i_k$ where $j_k, i_k \in \mathbb{N}_0$ and $0 \le i_k < m$. Note that $\lim_{k\to\infty} j_k = \infty$. Since

$$\|(T^m)^{j_k} x\| = \|T^{i_k+n_k} x\| \le \|T\|^{i_k} \cdot \|T^{n_k} x\| \le \|T\|^m \cdot M$$

the orbit of x under T^m is not tending to infinity. The rest of the statement is easy.

4. Weak orbits

Results analogous to the "normal" orbits $(T^n x)_{n\in\mathbb{N}}$, $x \in X$, can be sometimes also obtained for the weak orbits $(\langle T^n x, x^*\rangle)_{n\in\mathbb{N}}$, $x \in X$, $x^* \in X^*$. For instance, on $\ell^2(\mathbb{N})$ there is an analogy of Example 3.1, only with a linear growth of $\|T^n\|$.

Example 4.1. ([62, Example 8]) *There is a Hilbert space X (real or complex) and an operator $T \in \mathcal{L}(X)$ satisfying $\|T^n\| = n+1$ for each $n \in \mathbb{N}$, such that there is no pair $x, y \in X$ with $\left|\langle T^n x, y\rangle\right| \to \infty$ as $n \to \infty$.*

Proof. Let X be the Hilbert space with orthonormal basis $\{e_{k,j} :\ k \in \mathbb{N}, 1 \le j \le k\}$. Let $T \in \mathcal{L}(X)$ be defined by

$$Te_{k,j} := \begin{cases} \left(\frac{j+1}{j} \cdot \frac{k-j+1}{k-j}\right)^{1/2} e_{k,j+1} & \text{for } j < k, \\ 0 & \text{for } j = k. \end{cases}$$

We have $T^n e_{k,j} = \left(\frac{j+n}{j} \cdot \frac{k-j+1}{k-j+1-n}\right)^{1/2} e_{k,j+n}$ for $j \le k-n$. It is easy to see that $\left(\frac{j+n}{j} \cdot \frac{k-j+1}{k-j+1-n}\right)^{1/2} \le n+1$. Moreover, $T^n e_{n+1,1} = (n+1)e_{n+1,n+1}$, and so $\|T^n\| = n+1$ for each n.

Let $x = \sum_{k,j} \alpha_{k,j} e_{k,j} \in X$, $y = \sum_{k,j} \beta_{k,j} e_{k,j} \in X$ with real or complex coefficients $\alpha_{k,j}, \beta_{k,j}$. Suppose on the contrary that $\left|\langle T^n x, y\rangle\right| \to \infty$. Without loss of generality we may assume that $\|x\| = \|y\| = 1$.

For each n large enough we have

$$\sum_{r=n}^{2n-1} \left|\langle T^r x, y\rangle\right| \ge 7n.$$

On the other hand, we have

$$\begin{aligned} \sum_{r=n}^{2n-1} \left|\langle T^r x, y\rangle\right| &= \sum_{r=n}^{2n-1} \sum_{k=r+1}^{\infty} \sum_{j=1}^{k-r} \Big(\frac{j+r}{j} \cdot \frac{k-j+1}{k-j+1-r}\Big)^{1/2} \left|\alpha_{k,j}\beta_{k,j+r}\right| \\ &\le A+B+C+D, \end{aligned}$$

where

$$A = \sum_{r=n}^{2n-1} \sum_{k=r+1}^{4n} \sum_{j=1}^{k-r} \Big(\frac{j+r}{j} \cdot \frac{k-j+1}{k-j+1-r} \Big)^{1/2} \left| \alpha_{k,j} \beta_{k,j+r} \right| ,$$

$$B = \sum_{k=4n+1}^{\infty} \sum_{r=n}^{2n-1} \sum_{j=n+1}^{k-r-n} \Big(\frac{j+r}{j} \cdot \frac{k-j+1}{k-j+1-r} \Big)^{1/2} \left| \alpha_{k,j} \beta_{k,j+r} \right| ,$$

$$C = \sum_{k=4n+1}^{\infty} \sum_{r=n}^{2n-1} \sum_{j=1}^{n} \Big(\frac{j+r}{j} \cdot \frac{k-j+1}{k-j+1-r} \Big)^{1/2} \left| \alpha_{k,j} \beta_{k,j+r} \right| ,$$

$$D = \sum_{k=4n+1}^{\infty} \sum_{r=n}^{2n-1} \sum_{j=k-r-n+1}^{k-r} \Big(\frac{j+r}{j} \cdot \frac{k-j+1}{k-j+1-r} \Big)^{1/2} \left| \alpha_{k,j} \beta_{k,j+r} \right| .$$

We have:

$$\begin{aligned} A &\le \sum_{k=n+1}^{4n} \sum_{j=1}^{k} \sum_{i=1}^{k} \frac{k}{\sqrt{ij}} \left| \alpha_{k,j} \beta_{k,k-i+1} \right| \\ &\le \frac{4n}{2} \sum_{k=n+1}^{4n} \sum_{i,j=1}^{k} \Big(\frac{\left|\alpha_{k,j}\right|}{\sqrt{i}} \Big)^2 + \Big(\frac{\left|\beta_{k,k-i+1}\right|}{\sqrt{j}} \Big)^2 \\ &\le 2n(1+\ln(4n)) \sum_{k=n+1}^{4n} \sum_{j=1}^{k} (\left|\alpha_{k,j}\right|^2 + \left|\beta_{k,j}\right|^2), \end{aligned}$$

$$\begin{aligned} B &\le \sum_{k=4n+1}^{\infty} \sum_{r=n}^{2n-1} \sum_{j=n+1}^{k-r-n} 3 \left| \alpha_{k,j} \beta_{k,j+r} \right| \\ &\le \frac{3}{2} \sum_{k=4n+1}^{\infty} \sum_{r=n}^{2n-1} \sum_{j=n+1}^{k-r-n} (\left|\alpha_{k,j}\right|^2 + \left|\beta_{k,j+r}\right|^2) \\ &\le \frac{3n}{2} \sum_{k=4n+1}^{\infty} \sum_{j=1}^{k} (\left|\alpha_{k,j}\right|^2 + \left|\beta_{k,j}\right|^2) \le \frac{3n}{2}, \end{aligned}$$

$$
\begin{aligned}
C &\leq \sum_{k=4n+1}^{\infty} \sum_{r=n}^{2n-1} \sum_{j=1}^{n} \Big(\frac{3n-1}{j} \cdot 3\Big)^{1/2} \left|\alpha_{k,j}\beta_{k,j+r}\right| \\
&\leq 3\sqrt{n} \sum_{k=4n+1}^{\infty} \sum_{j=1}^{n} \frac{\left|\alpha_{k,j}\right|}{\sqrt{j}} \sum_{i=n+1}^{3n-1} \left|\beta_{k,i}\right| \\
&\leq 3\sqrt{n} \cdot \sqrt{2n} \sum_{k=4n+1}^{\infty} \sum_{j=1}^{n} \frac{\left|\alpha_{k,j}\right|}{\sqrt{j}} \Big(\sum_{i=n+1}^{3n-1} \left|\beta_{k,i}\right|^2\Big)^{1/2} \\
&\leq \frac{3n}{\sqrt{2}} \sum_{k=4n+1}^{\infty} \sum_{j=1}^{n} \Big(\left|\alpha_{k,j}\right|^2 + \sum_{i=n}^{3n} \frac{\left|\beta_{k,i}\right|^2}{j}\Big) \\
&\leq \frac{3n}{\sqrt{2}} + \frac{3n}{\sqrt{2}}(1+\ln n) \sum_{k=4n+1}^{\infty} \sum_{i=n}^{3n} \left|\beta_{k,i}\right|^2.
\end{aligned}
$$

Since the terms C and D are symmetrical, we have

$$
D \leq \frac{3n}{\sqrt{2}} + \frac{3n}{\sqrt{2}}(1+\ln n) \sum_{k=4n+1}^{\infty} \sum_{i=k-3n}^{k-n} \left|\alpha_{k,i}\right|^2.
$$

Thus for n large enough we have

$$
\begin{aligned}
7n &\leq \sum_{r=n}^{2n-1} \left|\langle T^r x, y\rangle\right| \\
&\leq \frac{3n}{2} + 3n\sqrt{2} + 2n(1+\ln(4n)) \sum_{k=n+1}^{4n} \sum_{j=1}^{k} (\left|\alpha_{k,j}\right|^2 + \left|\beta_{k,j}\right|^2) \\
&\quad + \frac{3n}{\sqrt{2}}(1+\ln n) \sum_{k=4n+1}^{\infty} \sum_{j=n}^{3n} (\left|\beta_{k,j}\right|^2 + \left|\alpha_{k,k-j}\right|^2) \\
&\leq 6n + n(1+\ln(4n))\Big(2 \sum_{k=n+1}^{4n} \sum_{j=1}^{k} (\left|\alpha_{k,j}\right|^2 + \left|\beta_{k,j}\right|^2) \\
&\quad + 3 \sum_{k=4n+1}^{\infty} \sum_{j=n}^{3n} (\left|\beta_{k,j}\right|^2 + \left|\alpha_{k,k-j}\right|^2)\Big).
\end{aligned}
$$

Thus for all $n \geq n_0$ we have

$$
2 \sum_{k=n+1}^{4n} \sum_{j=1}^{k} (\left|\alpha_{k,j}\right|^2 + \left|\beta_{k,j}\right|^2) + 3 \sum_{k=4n+1}^{\infty} \sum_{j=n}^{3n} (\left|\beta_{k,j}\right|^2 + \left|\alpha_{k,k-j}\right|^2) \geq \frac{1}{1+\ln(4n)}.
$$

In particular, for $n = 4^s n_0$, $s = 1, 2, \ldots$, we have

$$\begin{aligned}
10 &= 5\sum_{k=1}^{\infty}\sum_{j=1}^{k}\left(\left|\alpha_{k,j}\right|^2 + \left|\beta_{k,j}\right|^2\right)\\
&\geq 2\sum_{s=1}^{\infty}\sum_{k=4^s n_0+1}^{4^{s+1}n_0}\sum_{j=1}^{k}\left(\left|\alpha_{k,j}\right|^2 + \left|\beta_{k,j}\right|^2\right)\\
&\quad + 3\sum_{s=1}^{\infty}\sum_{k=4^s n_0+1}^{\infty}\sum_{j=4^s n_0}^{3\cdot 4^s n_0}\left(\left|\beta_{k,j}\right|^2 + \left|\alpha_{k,k-j}\right|^2\right)\\
&\geq \sum_{s=1}^{\infty}\frac{1}{1+\ln 4^{s+1}n_0} = \sum_{s=1}^{\infty}\frac{1}{1+\ln n_0 + (s+1)\ln 4} = \infty,
\end{aligned}$$

a contradiction. Hence there are no $x, y \in X$ with $\left|\langle T^n x, y\rangle\right| \to \infty$.

The example is in a sense the best possible.

Proposition 4.2. *Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X)$ be a sequence of operators satisfying*

$$\sum_{n=1}^{\infty}\frac{1}{\|T_n\|^{1/2}} < \infty,$$

or when X is a complex Hilbert space

$$\sum_{n=1}^{\infty}\frac{1}{\|T_n\|} < \infty.$$

Then there is a dense set of pairs $x \in X$, $x^* \in X^*$ such that $|\langle T_n x, x^*\rangle| \to \infty$.

Proof. Set $p := 2$ in the case when X is a complex Hilbert space and $p := 1$ otherwise. Let $(\beta_n)_{n=1}^{\infty}$ be a sequence of real numbers tending to infinity such that

$$\sum_{n=1}^{\infty}\frac{\beta_n}{\|T_n\|^{p/2}} < \infty$$

so that the sequence of coefficients $\alpha_n := \beta_n^{1/p}/\|T_n\|^{1/2}$ belongs to ℓ^p and satisfies the limit condition $\alpha_n^2\|T_n\| \to \infty$.

Now we apply one of the plank theorems for operators 1.3 or 1.5, respectively, to the operators $(T_n^*)_{n\in\mathbb{N}}$ and coefficients $(\alpha_n)_{n\in\mathbb{N}}$. By the theorems, there is a dense set of $x^* \in X^*$ such that there is $c_1 > 0$ such that

$$\|T_n^* x^*\| \geq c_1\alpha_n\|T_n^*\|.$$

Next we use one of the plank theorems 1.1 or 1.4, respectively, to the functionals $T_n^* x^*$ and the coefficients $(\alpha_n)_{n\in\mathbb{N}}$. There is a dense set of vectors $x \in X$ such that there is $c_2 > 0$ such that

$$\left|\langle T_n x, x^*\rangle\right| = \left|\langle x, T_n^* x^*\rangle\right| \geq c_2 \alpha_n \| T_n^* x^* \|.$$

In sum, we have a dense set of pairs $x \in X$, $x \in X^*$ such that for some $c_1, c_2 > 0$

$$\left|\langle T_n x, x^*\rangle\right| \geq c_1 c_2 \alpha_n^2 \| T_n\| \to \infty.$$

Since Example 4.1 works only in Hilbert spaces, the situation for other domains, namely ℓ^p spaces, or the class of Banach spaces in general, is currently not clear.

III. Irregular orbits

5. Hypercyclicity

The notion of hypercyclicity and related topics received a broad attention in the past two decades. However, as noted in a wonderful survey by Grosse-Erdmann [38], a more general concept of universality goes back at least to the year 1914, when Pál Fekete discovered a formal real power series on $[-1;1]$ such that every continuous function g on $[-1;1]$ with $g(0)=0$ is uniformly approximable by partial sums of the series. The first explicit example of a hypercyclic operator on a classical Banach space appeared in an article by Rolewicz [75] – his example is just a multiple of the backward shift on ℓ^p, $1 \le p < \infty$.

Recall that an operator is called hypercyclic if there is a vector with dense orbit; such a vector is then called hypercyclic vector; the term orbital vector also appeared in the literature in the same meaning. Similarly, a sequence $(T_n)_{n\in\mathbb{N}}$ of operators is called hypercyclic if there is $x \in X$ such that $(T_n(x))_{n\in\mathbb{N}}$ is dense. Usually it is supposed that the underlying space is separable – otherwise obviously, there cannot be a hypercyclic vector. There are two other similar notions which are not subject of the thesis but will be briefly touched: an operator is called cyclic if for some $x \in X$ the linear subspace $\operatorname{Span}\operatorname{Orb}(T,x)$ is dense in the whole space, and supercyclic if the cone $\mathbb{F}\cdot\operatorname{Orb}(T,x) = \{\lambda T^n x\colon \lambda \in \mathbb{F}, n \in \mathbb{N}\}$ is dense in the whole space, where $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$ is the underlying field.

Hypercyclicity is connected to many different branches of mathematics and there are many interesting problems concerning hypercyclicity; we will briefly mention some of the connections and some of the problems in this chapter. For a much broader study about the subject we can refer to a recent monograph of Bayart and Mathéron [11].

The key concept in the field is a theorem now attributed to Birkhoff, which states that in the case of separable Banach spaces, hypercyclicity is equivalent to the so-called topological transitivity. The first result of this type appeared already in [20] (thus the name), and later in [51], [35], [19]. In addition to the important statements (i) and (iii), we also state two other, under usual conditions equivalent statements.

Theorem 5.1. (Birkhoff's transitivity theorem) *Let X be a separable complete metric space with metric ϱ. Let $f_n\colon X \to X$, $n \in \mathbb{N}$, be a sequence of continuous mappings. Then the following assertions are equivalent:*

(i) $(f_n)_{n\in\mathbb{N}}$ is topologically transitive, that is, for any non-empty open sets $U, V \subset X$ there is $n \in \mathbb{N}$ such that $f_n(U) \cap V$ is non-empty,

(ii) there is a dense set of hypercyclic points, i.e. of those points $x \in X$ such that the set $\{f_n(x)\colon n \in \mathbb{N}\}$ is dense.

Moreover, if $f_n = f_1^n$ for all $n \in \mathbb{N}$ then the above assertions imply the following assertion (iii). If in addition to that, X is perfect (i.e. without isolated points) then all the four assertions are equivalent.

(iii) f_1 is hypercyclic, that is, there is $x \in X$ such that the set $\{f_1^n(x)\colon n \in \mathbb{N}\}$ is dense,

(iv) there is a dense sequence $(y_i)_{i=1}^{\infty}$ in X such that for each $i, j \in \mathbb{N}$ and $k \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $\varrho(f_1^n(y_i), y_j) < \frac{1}{k}$.

Proof.

(i)⇒(ii) Suppose $(f_n)_{n\in\mathbb{N}}$ is topologically transitive and let $M \subset X$ be any nonempty open set, without loss of generality of $\operatorname{diam} M \le 1$. We will show that M contains a hypercyclic point. Let $(z_k)_{k=1}^{\infty}$ be dense in X and define $V_k := \{y \in X\colon \varrho(z_k, y) < \frac{1}{k}\}$. Set $U_1 := M$. By assumption, there is $n_1 \in \mathbb{N}$ such that $f_{n_1}(U_1) \cap V_1$ is nonempty so that $U_1 \cap f_{n_1}^{-1}(V_1)$ is a nonempty open set. Choose any $s_1 \in U_1 \cap f_{n_1}^{-1}(V_1)$ and define $U_2 := U_1 \cap f_{n_1}^{-1}(V_1) \cap B_{1/2}(s_1)$ so that $\operatorname{diam} U_2 \le 1/2$. Like this, we inductively construct nonempty open sets U_k such that $U_k \supset U_{k+1}$ and $\operatorname{diam} U_k \le 1/k$, so that there is $x \in \bigcap_{k=1}^{\infty} \overline{U_k}$, since X is complete. The point x is hypercyclic. Indeed, let U be any open set. For some $k \in \mathbb{N}$, we have $\overline{V_k} \subset U$ so that

$$f_{n_k}(x) \in f_{n_k}(\overline{U_{k+1}}) \subset f_{n_k}(U_{k+1})^- \subset \overline{V_k} \subset U,$$

while $x \in U_1 \subset M$ and the implication is proved.

(ii)⇒(i) Let U, V be nonempty open sets in X. By assumption, there is a hypercyclic point $x \in U$ so there is $n \in \mathbb{N}$ such that $f_n(x) \in V$, that is $f_n(U) \cap V \neq \emptyset$.

(ii)⇒(iii) Trivial.

(iii)⇒(ii) Suppose that $f_i = f_1^i$ and let x be a hypercyclic point. For any $i \in \mathbb{N}$, the point $f_i(x)$ is hypercyclic since $f_j(f_i(x)) = f_1^{i+j}(x)$ and X is perfect. Hence there is a dense set of hypercyclic points.

(ii)⇒(iv) Suppose that $f_i = f_1^i$ and let x be a hypercyclic point, set $y_i := f_i(x)$. Since X is perfect, each y_i is hypercyclic and thus its orbit comes arbitrarily close to any given y_j.

(iv)⇒(i) If $U, V \subset X$ are nonempty open sets then $y_i \in U$, $y_j \in V$ for some $i, j \in \mathbb{N}$. There is $k \in \mathbb{N}$ so that the ball with diameter $\frac{1}{k}$ and center y_j is also contained in V. By assumption, there is $n \in \mathbb{N}$ such that $\varrho(f_n(y_i), y_j) < \frac{1}{k}$ so $f_n(y_i) \in V$. Thus $(f_n)_{n\in\mathbb{N}}$ is topologically transitive.

In particular, if X is a Banach space and $f_n = T^n$ for a single bounded linear operator T, we have the following equivalence.

Corollary 5.2. *Let $T \in \mathcal{L}(X)$. Then the following are equivalent:*

(i) T is topologically transitive, i.e. for any non-empty open sets $U, V \subset X$ there is $n \in \mathbb{N}$ such that $T^n(U) \cap V$ is non-empty,

(ii) T is hypercyclic,

(iii) T has a dense set of hypercyclic points.

6. The Hypercyclicity criterion

A first version of a powerful sufficient condition for hypercyclicity which is now known as the Hypercyclicity criterion was first proposed by C. Kitai [51]. We rephrase a generalized version of it as follows [19], [39]: $T \in \mathcal{L}(X)$ satisfies the Hypercyclicity criterion iff there are dense sets $D, D' \subset X$ and a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that:

(i) $T^{n_k} z \to 0$ for all $z \in D$,

(ii) for each $z' \in D'$ there is a sequence $(x_k)_{k\in\mathbb{N}} \subset X$ such that $x_k \to 0$ and $T^{n_k} x_k \to z'$.

In such a case we will also say that T satisfies the Hypercyclicity criterion with respect to the sequence $(n_k)_{k\in\mathbb{N}}$. The original condition of Kitai is now usually referred to as Kitai's criterion, and it requires the sequence to be $n_k := k$, and $D = D'$.

It is not difficult to prove that if T satisfies the Hypercyclicity criterion then it is hypercyclic – see the part "(i)⇒(ii)" of the next theorem. A natural question is the opposite implication: whether all the hypercyclic operators satisfy the Hypercyclicity criterion. Independently in 1992, Herrero [48] asked another question: whether $T \oplus T$ is hypercyclic whenever T is hypercyclic. In fact, the two questions turned out to be equivalent.

Theorem 6.1. ([19]) *The following are equivalent:*

(i) T satisfies the Hypercyclicity criterion,

(ii) T is hereditarily hypercyclic,

(iii) T is (topologically) weakly mixing, that is, the direct sum $T \oplus T$ acting on $X \oplus X$ is topologically transitive.

Here T is called hereditarily hypercyclic with respect to a sequence of non-negative integers $(n_k)_{k\in\mathbb{N}}$ if for all subsequences $(n_{k_j})_{j\in\mathbb{N}}$, the sequence of operators $(T^{n_{k_j}})_{j\in\mathbb{N}}$ is hypercyclic. T is called hereditarily hypercyclic if it is hereditarily hypercyclic with respect to some sequence $(n_k)_{k\in\mathbb{N}}$.

Proof.

(i)⇒(ii) First we will prove that T is topologically transitive. Let U and V be nonempty open sets in X. Let D, D' be the dense sets from the Hypercyclicity criterion so there are $z \in U \cap D$ and $z' \in V \cap D'$. Let $\varepsilon > 0$ be such that $B(z,\varepsilon) \subset U$ and $B(z', 2\varepsilon) \subset V$ and let (x_k) be the sequence from the Hypercyclicity criterion.

There is $k \in \mathbb{N}$ such that

$$\begin{aligned} \|T^{n_k} z\| &< \varepsilon \\ \|x_k\| &< \varepsilon \\ \|T^{n_k} x_k - z'\| &< \varepsilon \end{aligned}$$

and therefore

$$\begin{aligned} \|(z + x_k) - z\| &< \varepsilon \\ \|T^{n_k}(z + x_k) - z'\| &< 2\varepsilon. \end{aligned}$$

This means that $z + x_k \in U$ and $T^{n_k}(z + x_k) \in V$, so $T^{n_k} U \cap V \neq \emptyset$.

Furthermore, it is easy to see that if T satisfies the Hypercyclicity criterion with respect to (n_k) then it is satisfied also for any subsequence of (n_k) and we can proceed as above. Therefore T is hereditarily hypercyclic with respect to (n_k).

(ii)⇒(iii) Suppose that T is hereditarily hypercyclic with respect to the sequence $(n_k)_{k\in\mathbb{N}}$, we will prove that $T \oplus T$ is topologically transitive. Let U_1, V_1, U_2, V_2 be nonempty open sets in X. The sequence (T^{n_k}) is hypercyclic, so there is a dense set of hypercyclic points. By Theorem 5.1, there is $n_{k_1} \in \mathbb{N}$ such that $T^{n_{k_1}}(U_1) \cap V_1 \neq \emptyset$. However, the sequence $T^{n_{k_1+1}}, T^{n_{k_1+2}} \ldots$ is also hypercyclic, so by again, there is $n_{k_2} \in \mathbb{N}$ such that $T^{n_{k_2}}(U_1) \cap V_1 \neq \emptyset$. Proceeding by induction, we obtain a subsequence $(n_{k_j})_{j\in\mathbb{N}}$ of (n_k) such that

$$T^{n_{k_j}}(U_1) \cap V_1 \neq \emptyset \qquad \text{for each } j \in \mathbb{N}.$$

Finally since $(T^{n_{k_j}})$ still forms a hypercyclic sequence of operators we can apply Theorem 5.1 once more again, so there is $m \in (n_{k_j})_{j\in\mathbb{N}}$ such that $T^m(U_2) \cap V_2 \neq \emptyset$. So $T \oplus T$ is topologically transitive and thus hypercyclic.

(iii)⇒(i) Let (x, y) be a hypercyclic vector for $T \oplus T$, let $D = D' := \{T^n x : n \in \mathbb{N}\}$. Fix $k \in \mathbb{N}$ for a moment. We will prove that the vector $(x, T^k y)$ is also hypercyclic for $T \oplus T$. First note that $\operatorname{Rng} T^k$ is dense. Second, given any $(u, v) \in T \oplus T$, $v \in \operatorname{Rng} T^k$ and $\varepsilon > 0$, let $\varepsilon_0 := (1 + \|T^k\|)\varepsilon$ and take any $n \in \mathbb{N}$ larger than k such that $\|(T^n x, T^n y) - (u, T^{-k} v)\| < \varepsilon_0$. Then

$$\begin{aligned} \|(T^n x, T^n T^k y) - (u, v)\| &\leq \|T^n x - u\| + \|T^n T^k y - v\| \\ &\leq \varepsilon_0 + \|T^k\| \|T^n y - T^{-k} v\| \leq \varepsilon. \end{aligned}$$

Hence $(x, T^k y)$ is hypercyclic for $T \oplus T$, so that there is a dense set M of vectors $u \in X$ such that (x, u) is hypercyclic for $T \oplus T$. In particular, given $k \in \mathbb{N}$ there is a vector $u_k \in M$ of norm at most $1/k$, so that $\|T^{n_k}(x, u_k) - (0, x)\| < 1/k$ for certain exponent $n_k \in \mathbb{N}$. In other words, for each $k \in \mathbb{N}$ there is $u_k \in X$ and $n_k \in \mathbb{N}$ such that

$$\begin{gathered} \|u_k\| < 1/k, \\ \|T^{n_k} x\| < 1/k, \\ \|T^{n_k} u_k - x\| < 1/k. \end{gathered}$$

First, this means that for any $z = T^m x \in D$, $m \in \mathbb{N}$, we have $T^{n_k} z = T^m T^{n_k} x \to 0$ as $k \to \infty$. Second, let $z' = T^m x \in D'$, $m \in \mathbb{N}$. Then for $x_k = T^m u_k$ we have $x_k \to 0$ and $T^{n_k} x_k = T^m T^{n_k} u_k \to T^m x = z'$ as $k \to \infty$.

If we restrict the Hypercyclicity criterion to the case when $n_k = k$, we can obtain an analogous result regarding stronger notions. The proof is similar.

Theorem 6.2. *The following are equivalent:*

(i) T satisfies the Hypercyclicity criterion with respect to $n_k = k$,

(ii) T is hereditarily hypercyclic with respect to $n_k = k$,

(iii) T is (topologically) mixing, that is, for any nonempty open subsets $U, V \subset X$ there exists $m \in \mathbb{N}$ such that for all $n \geq m$ the set $T^n(U)$ intersects V.

The natural question whether every hypercyclic operator satisfies the Hypercyclicity criterion (or equivalently, whether for all T hypercyclic, $T \oplus T$ is hypercyclic as well), turned out to be one of the most important problems on the notion of hypercyclicity. In 1991, it was proved by Salas [77] and Herrero [47] that even in the Hilbert space setting, there are hypercyclic operators which do not satisfy the Hypercyclicity criterion with respect to the sequence $n_k = k$. However, the general case remained open until 2006 when De La Rosa and Read constructed a counterexample.

Example 6.3. ([27]) *There is a Banach space X and a hypercyclic operator $T \in \mathcal{L}(X)$ such that $T \oplus T$ is not hypercyclic, that is, T does not satisfy the Hypercyclicity criterion.*

In the sequel, Bayart and Mathéron [10] showed that such an operator can be constructed in every Banach space with an unconditional basis such that the forward shift corresponding to the basis is continuous, which includes the spaces c_0 and ℓ_p, $1 \le p < \infty$.

In the opposite direction, in some broad classes of operators the Hypercyclicity criterion is equivalent to hypercyclicity. In particular, the equivalence holds for operators T with dense generalized kernel $\bigcup_{n\ge1} \operatorname{Ker}(T^n)$, for instance generalized backward shifts.

Theorem 6.4. ([19]) *Let T be hypercyclic with dense generalized kernel, i.e.*

$$\bigcup_{n\ge1} \operatorname{Ker}(T^n)^- = X.$$

Then T satisfies the Hypercyclicity criterion.

Proof. Let x be any hypercyclic vector. Since T is topologically transitive, for each $k \in \mathbb{N}$ there is $n_k \in \mathbb{N}$ and $y_k \in X$ such that $\|y_k\| < 1/k$ and $\|T^{n_k} y_k - x\| < 1/k$. Obviously we can suppose that $(n_k)_{k\in\mathbb{N}}$ is increasing. The set $D := \bigcup_{n\ge1} \operatorname{Ker} T^n$ is dense in X; for each $z \in D$ we have $T^{n_k} z = 0$ for sufficiently big k so point (i) of the Hypercyclicity criterion holds. Let $D' := \{T^n x\colon\ n \in \mathbb{N}\}$. If $z' = T^m x \in D'$ then $x_k := T^m y_k$ satisfies $\|x_k\| < \|T^m\| \cdot 1/k \to 0$ as $k \to \infty$, as well as $T^{n_k} x_k = T^m T^{n_k} y_k \to T^m x = z'$ as $k \to \infty$. Thus the point (ii) of the Hypercyclicity criterion also holds.

Using the Hypercyclicity criterion, it is not hard to prove that several operators are hypercyclic, for instance it allows us to reprove the classical result of Rolewicz in just a few lines.

Example 6.5. ([75]) *Let $S\colon\ \ell^p \to \ell^p$, $1 \le p < \infty$, be the backward shift operator, and $\lambda \in \mathbb{C}$ satisfy $|\lambda| > 1$. Then the operator $T := \lambda S$ satisfies the Hypercyclicity criterion and is thus hypercyclic.*

Proof. Take $n_k := k$ and $D = D' := c_{00}(\mathbb{N})$ be the subspace of all finitely supported sequences. Let $z \in D$, then $T^k z \to 0$. Set $x_k := \lambda^{-k} S'^k z$ where S' is the forward shift operator, so that $\|x_k\| = |\lambda|^{-k} \|z\| \to 0$ and

$$T^k x_k = \lambda^k S^k(\lambda^{-k} S'^k z) = z.$$

7. Set of hypercyclic vectors

If x is a hypercyclic vector then for any $n \in \mathbb{N}$ the points $T^n x$ are also hypercyclic, that is, there is a dense set of hypercyclic points. Moreover, it is easy to determine the topological complexity of the set, and the linear structure allows us to prove even more.

Theorem 7.1. *Let T be a hypercyclic operator. The set X_h of hypercyclic points under T is a dense G_δ set. If $x \in X_h$ then $p(T)x \in X_h$ for any nonzero polynomial p, so that $X_h \cup \{0\}$ contains a dense linear manifold and X_h is a connected set.*

The assertion about the linear manifold was proven in [47, Proposition 4.1] and independently [21], for real Banach spaces the proof was given by Bès [17]. The proofs gave a surprising answer to a question whether there is an operator whose set of hypercyclic points contains a nontrivial vector space: in fact every hypercyclic operator has such a property. We show the proof only for complex Banach spaces.

Proof. Let $(x_j)_{j=1}^\infty$ be dense in X. Observe that

$$X_h = \bigcap_{j\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \{y \in X\colon \; \|T^n y - x_j\| < \frac{1}{k}\}.$$

The inner sets are open so the whole set is a G_δ.

Let T be a hypercyclic operator. We will first prove, that the adjoint T^* has no eigenvalue, so that for each $\lambda \in \mathbb{C}$, the operator $T - \lambda I$ has dense range. Suppose on the contrary that $T^* x^* = \lambda x^*$ for some $\lambda \in \mathbb{C}$ and $x^* \in X^*$, $x^* \neq 0$. Let $x \in X$ be a hypercyclic vector. Then

$$\mathbb{C} = \{\langle T^n x, x^*\rangle\colon \; n \in \mathbb{N}\}^- = \langle x, x^*\rangle\{\bar{\lambda}^n\colon \; n \in \mathbb{N}\}^-,$$

but the sequence $\{\bar{\lambda}^n\colon \; n \in \mathbb{N}\}$ cannot be dense in $\mathbb{C}$ (it is monotone in modulus) so we have a contradiction.

Now let $x \in X_h$ and p be a polynomial. Then

$$\{T^n p(T)x\colon \; n \in \mathbb{N}\} = \alpha(T - \beta_1 I)(T - \beta_2 I)\cdots(T - \beta_k I)\{T^n x\colon \; n \in \mathbb{N}\}$$

where $k \in \mathbb{N}_0$, $\alpha, \beta_1, \beta_2, \ldots, \beta_k \in \mathbb{C}$, $\alpha \neq 0$, are fixed. The set $\{T^n x\colon \; n \in \mathbb{N}\}$ is dense and all the operators $T - \beta_i I$ have dense ranges, so the set $\{T^n p(T)x : n \in \mathbb{N}\}$ is also dense.

Note that in the proof we showed an assertion which is interesting in itself: if T is hypercyclic then $\lambda_p(T^*) = \emptyset$. In particular, on a finite dimensional space there is no hypercyclic operator.

Montes-Rodríguez [56] found conditions, sometimes called the Strong Hypercyclicity Criterion, ensuring the existence of even infinite-dimensional closed subspace consisting of hypercyclic points.

Theorem 7.2. ([56]) *Let X be a separable Banach space and let T satisfy the Hypercyclicity criterion. Moreover suppose, that there is a closed infinite-dimensional subspace $X_0 \subset X$ consisting of stable vectors. Then there is a closed infinite-dimensional subspace consisting of vectors hypercyclic under T.*

The famous invariant subspace problem asks, whether each bounded linear operator has a nontrivial closed invariant subspace. The question can be reformulated in terms of closed linear span of the orbits: if T is a bounded linear operator, is there a nonzero orbit, whose linear span is not dense in the whole space? Similarly, the invariant *subset* problem asks, whether each bounded linear operator has a nontrivial closed invariant *subset*, that is, if T is a bounded linear operator, is there a nonzero orbit which is not dense in the whole space?

In the case of Banach spaces, the questions have been solved negatively by Enflo (solved in the 1970s, published in [30], see also Beauzamy [13]) and Read ([71], [72] for ℓ_1, [73] for c_0). However, in reflexive spaces, especially Hilbert spaces, the questions are still open. In particular, it is unknown whether there is a hypertransitive operator on a Hilbert space, i.e. an operator with all nonzero points hypercyclic, i.e. without a nontrivial closed invariant subset. There is also the following open conjecture: if T is a bounded linear operator on a Banach space then its adjoint T^* has a nontrivial closed invariant subset.

In the past twenty years there have been several attempts to construct an operator on a separable Hilbert space with a big set of hypercyclic points, generally using techniques similar to that of Read. Beauzamy in [16] (see also [14]) constructed an operator with hypercyclic point x, such that for any nonzero polynomial p, the point $p(T)x$ is also hypercyclic. Later it was discovered that such a property is satisfied for every hypercyclic operator – see Theorem 7.1. Grivaux and Roginskaya [37] constructed an operator, such that the set of non-hypercyclic points is in many senses "small", but the operator has also other interesting properties:

(i) $X \setminus X_h(T)$ is Gauss-null, i.e. for every non-degenerate Gaussian measure μ on X, we have $\mu(X \setminus X_h(T)) = 0$,

(ii) in particular, $X \setminus X_h(T)$ is Haar null, i.e. there exists a Borel probability measure such that any translate of $X \setminus X_h(T)$ has measure 0,

(iii) $X \setminus X_h(T)$ is σ-porous, i.e. it is a countable union of porous sets; a set A is called porous if there is $\lambda \in (0;1)$ such that for any $x \in A$ and $\varepsilon > 0$ there exists $y \in X \setminus A$, with $\|y - x\| < \varepsilon$, such that $A \cap B(y, \lambda\|y - x\|) = \emptyset$,

(iv) each orbit forms a linear manifold,

(v) T is orbit-unicellular, i.e. the family of the closures of all orbits is totally ordered with respect to the inclusion relation,

(vi) T is not orbit reflexive, i.e. there is an operator A such that it is not in the closure of $\{T^n : n \in \mathbb{N}\}$ in the strong operator topology, but $Ax \in \{T^n x : n \in \mathbb{N}\}^-$ for each $x \in X$; the notion is thoroughly studied in Part IV.

8. Set of hypercyclic operators

Fact 8.1. *The set HC(X) of all hypercyclic operators on X is a G_δ set in the operator norm topology.*

Proof. Hypercyclicity is equivalent to the topological transitivity. Therefore, given any sequence $(y_i)_{i\in\mathbb{N}}$ dense in X, an operator T is hypercyclic iff for any $i, j \in \mathbb{N}$ and $k \in \mathbb{N}$ there is $n \in \mathbb{N}$ and $u \in X$ such that $\|u - y_i\| < 1/k$ and $\|T^n u - y_j\| < 1/k$. Hence

$$HC(X) = \bigcap_{i\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcup_{\substack{u\in X \\ \|u-y_i\|<1/k}} \{T \in \mathcal{L}(X) \colon \ \|T^n u - y_j\| < 1/k\}.$$

The inner sets are obviously open in the operator norm topology, while all the intersections are countable, so $HC(X)$ is a G_δ set.

It is a long time ago since the first explicit hypercyclic operators were constructed, however, until recently it was not clear whether hypercyclic operators can be found in all Banach spaces.

Theorem 8.2. ([2], [53], [46], [36]) *On any infinite dimensional separable Banach space, there is an operator of the form "identity plus compact" which has a closed infinite dimensional subspace of hypercyclic vectors. Moreover, given any sequence $(v_n)_{n\in\mathbb{N}}$ of linearly independent vectors, there is an operator T such that the set v_n precisely coincides with orbit of some vector (not necessarily in the same order).*

It became clear that being hypercyclic is definitely not a rare and obscure phenomenon: on the contrary, the set $HC(X)$ is big. However, the set of hypercyclic operators cannot be dense in the norm topology, since a hypercyclic operator must obviously have norm strictly greater than 1. On the other hand, the following holds.

Theorem 8.3. ([23]) *Let X be a separable infinite dimensional Hilbert space. Then the closed linear span of the set of hypercyclic operators is dense in the norm topology.*

On a separable complex Hilbert space, the norm-closure of the set of hypercyclic operators was characterized already by Herrero in [47] in spectral terms, and later Müller proved the following generalization. Note that the full statement does not hold in a general Banach space setting.

Theorem 8.4. ([60]) *Let X be a separable complex infinite-dimensional Hilbert space. Then the norm-closures of the following sets coincide: hypercyclic operators, weakly mixing operators, mixing operators, chaotic operators, frequently hypercyclic operators and finally the operators which are at the same time mixing, chaotic and frequently hypercyclic.*

Here T is called frequently hypercyclic if there exists $x \in X$ such that for any non-empty $U \subset X$ the set $M := \{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density, i.e.

$$\liminf_{n\to\infty} \frac{|M \cap \{1, \dots, n\}|}{n} > 0.$$

For the notion of chaotic operators see Chapter .

If one turns to the strong operator topology (SOT), for "nice" Banach spaces the set of hypercyclic operators is dense – in particular, perhaps surprisingly, the zero operator is an SOT-limit of hypercyclic operators. The theorem was first proved by K. Chan in [23] for the Hilbert spaces, and then simplified in [18] and [68], using the result from [43].

Theorem 8.5. *Let T be an operator such that for any $n \in \mathbb{N}$ there are vectors $x_1, \dots, x_n$ such that all the vectors $x_1, \dots, x_n, T(x_1), \dots, T(x_n)$ are linearly independent; e.g., let T be a hypercyclic operator. Then the similarity orbit $\{JTJ^{-1} : J \in \mathcal{L}(X)$ invertible$\}$ of T is SOT-dense in $\mathcal{L}(X)$. In particular, if X is a separable infinite dimensional Banach space then $HC(X)$ is SOT-dense in $\mathcal{L}(X)$.*

Proof. Suppose T satisfies the assumption. Let $A \in \mathcal{L}(X)$ be arbitrary operator and let

$$U := \{S \in \mathcal{L}(X) : \|Se_1 - Ae_1\| < \varepsilon, \dots, \|Se_n - Ae_n\| < \varepsilon\}$$

be its SOT-neighborhood, where $e_1, \dots, e_n \in X$ are linearly independent. The space is clearly infinite dimensional, so there are $f_1, \dots, f_n \in X$ such that $\|f_i - Ae_i\| < \varepsilon$ for all $i = 1, \dots, n$, and all the vectors $e_1, \dots, e_n, f_1, \dots, f_n$ are linearly independent. By assumption, there are $x_1, \dots, x_n$ such that $x_1, \dots, x_n, Tx_1, \dots, Tx_n$ are linearly independent. Therefore, there is an invertible operator $J \in \mathcal{L}(X)$ mapping each x_i to e_i and each Tx_i to f_i. Thus $JTJ^{-1}e_i = f_i$ so $JTJ^{-1} \in U$. This proves that the similarity orbit of T is SOT-dense in $\mathcal{L}(X)$.

If T is hypercyclic, then for any hypercyclic x, the points $(T^{2k+1}x)_{k=0}^{\infty}$ and its images $(T^{2k+2}x)_{k=0}^{\infty}$ are linearly independent, so T satisfies the assumption of the theorem. By Theorem 8.2, every separable infinite dimensional Banach space admits a hypercyclic operator T; the hypercyclicity is preserved under similarity, so the similarity orbit of T is dense in $\mathcal{L}(X)$ and consists purely of hypercyclic operators.

An important result is the Ansari theorem relating the hypercyclicity of T and T^n for any fixed $n \in \mathbb{N}$. The Ansari theorem also follows from the result of

Bourdon and Feldman, which states that a somewhere dense orbit is necessarily everywhere dense, i.e. that if the closed orbit has a nonempty interior then it is already dense in the whole space, see [22, Theorem 2.4]. We however show the original argument.

Theorem 8.6. ([1]) *Let $x \in X$ be a hypercyclic vector for operator $T \in \mathcal{L}(X)$. Then for any $n \in \mathbb{N}$, x is hypercyclic for T^n.*

Proof. Denote Y the dense linear manifold $\{p(T)x : \ p \text{ is a polynomial}\}$, so that Y is invariant under T and every nonzero element of Y is hypercyclic for T, by Theorem 7.1. In the rest of the proof all the sets lie in Y, and closures are taken also relatively to Y. Fix $n \in \mathbb{N}$.

Let $S := \{x, T^n x, T^{2n}x, \ldots\}$. We are going to prove that $\overline{S} = Y$. For any k, $1 \le k \le n$, define

$$S_k := \bigcup_{i_1,\ldots,i_k} \overline{T^{i_1}S} \cap \ldots \cap \overline{T^{i_k}S}$$

where the union runs over all k-tuples satisfying $0 \le i_1 < i_2 < \ldots < i_k < n$. In particular, we have $S_1 = \bigcup_{i=0}^{n-1} \overline{T^i S} = Y$ and $S_n = \bigcap_{i=0}^{n-1} \overline{T^i S}$. In general, every S_k is closed and we have $S_n \subset S_{n-1} \subset \ldots \subset S_2 \subset S_1 = \mathrm{Orb}(T,x)^- = Y$.

Every S_k is invariant under T. To see this, let $0 \le i_1 < \ldots < i_k < n$, then for each $j \in \{1,\ldots,k\}$ we have $T(\overline{T^{i_j}S}) \subset \overline{T^{i_j+1}S}$ if $i_j < n-1$, or $T(\overline{T^{i_j}S}) \subset \overline{T^n S} \subset \overline{T^0 S} = \overline{S}$ if $i_j = n-1$. In any case, $T(\overline{T^{i_1}S}) \cap \ldots \cap T(\overline{T^{i_k}S}) \subset S_k$ so $TS_k \subset S_k$.

Note that $0 \in S_n$. Indeed, $0 \in Y = S_1$ so that $0 \in \overline{T^i S}$ for some i and thus $0 \in T^{n-i}\overline{T^i S} \subset \overline{T^n S} \subset \overline{S}$. Since $0 \in S$ also $0 \in S_n$.

Now suppose that for some k, $S_k = Y$ but $S_{k+1} \neq Y$. We will show that this leads to a contradiction. Since S_{k+1} is invariant under T, and every nonzero vector in Y is hypercyclic under T, necessarily $S_{k+1} = \{0\}$. This means that for any $l > k$ and any l-tuple $(i_1,\ldots,i_l)$ we have $\overline{T^{i_1}S} \cap \ldots \cap \overline{T^{i_l}S} = 0$. Therefore, the sets in the union

$$Y \setminus \{0\} = S_k \setminus \{0\} = \bigcup_{i_1,\ldots,i_k} \overline{T^{i_1}S} \cap \ldots \cap \overline{T^{i_k}S} \setminus \{0\}$$

are pairwise disjoint: if $(i_1,\ldots,i_k) \neq (j_1,\ldots,j_k)$ then

$$(\overline{T^{i_1}S} \cap \ldots \cap \overline{T^{i_k}S}) \cap (\overline{T^{j_1}S} \cap \ldots \cap \overline{T^{j_k}S}) \subset \{0\}.$$

But $Y \setminus \{0\}$ is a connected set so just one of the sets mentioned in the union is nonempty, say it is the one with indices $i_1,\ldots,i_k$. Then, arguing as above, for some $j_1,\ldots,j_k$ we have

$$T(Y) \setminus \{0\} = T(\overline{T^{i_1}S}) \cap \ldots \cap T(\overline{T^{i_k}S}) \setminus \{0\} \subset (\overline{T^{j_1}S}) \cap \ldots \cap T(\overline{T^{j_k}S}) \setminus \{0\}$$

and the latter set is empty by the connectedness argument above. But $T(Y)$ clearly can't be nor empty set nor $\{0\}$, and we have a contradiction.

Corollary 8.7. *If T satisfies the Hypercyclicity criterion then its any power does as well.*

Proof. If $T \oplus T$ is hypercyclic then $(T \oplus T)^n = T^n \oplus T^n$ is also hypercyclic by Theorem 8.6. So by Theorem 6.1, T^n satisfies the Hypercyclicity criterion as well.

Observation 8.8. *If T is hypercyclic and invertible, then T^{-1} is hypercyclic as well.*

Proof. Let U, V be any nonempty open sets in X. Then there is $n \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$ so $U \cap T^{-n} V = T^{-n}(T^n U \cap V) \neq \emptyset$ and T^{-1} is thus topologically transitive. Now use Corollary 5.2.

Note that Theorem 8.6 implies that for any rational $\lambda \in \mathbb{Q}$, the operators T and $e^{2\pi i\lambda} T$ share the same set of hypercyclic points, i.e. hypercyclicity is preserved under rational rotations. León-Saavedra and Müller proved that the statement in fact holds for any rotations.

Theorem 8.9. ([54]) *Let $x \in X$ be a hypercyclic vector for T and let $\lambda \in \mathbb{T}$. Then x is a hypercyclic vector for λT as well.*

9. ε-hypercyclic operators

Let $\varepsilon > 0$ and $p \geq 0$. Vector x is called ε-hypercyclic with exponent p iff for any $y \in X$ there is $n \in \mathbb{N}$ such that $\|T^n x - y\| < \varepsilon \|y\|^p$; if p is not mentioned $p = 1$ is assumed. An operator is called ε-hypercyclic if it admits an ε-hypercyclic point. Feldman [32] showed that if an operator is ε-hypercyclic with exponent 0 for some $\varepsilon > 0$, then the operator is already hypercyclic. Later Badea, Grivaux and Müller [5] showed that a natural setting for the notion of ε-hypercyclicity would be rather $p = 1$: they proved that in ℓ^1, Feldman's theorem is no longer true for $p = 1$, but that an operator is hypercyclic if it is ε-hypercyclic for all $\varepsilon > 0$. We generalize the arguments in the following three theorems.

Theorem 9.1. *Let $p \geq 0$ and $\varepsilon > 0$. Moreover, if $p \leq 1$ then suppose that $\varepsilon < 1$, if $p > 1$ then suppose that $\varepsilon < 5^{-p}$. Then if T is ε-hypercyclic with exponent p then $\sigma_p(T^*) = \emptyset$.*

Proof. Suppose that $T^* x^* = \lambda x^*$ for some $\lambda \in \mathbb{C}$ and $x^* \in X^*$, $\|x^*\| = 1$. Let $x \in X$ be ε-hypercyclic under T with exponent p. We distinguish three possibilities.

(a) Consider the case when $\langle x, x^* \rangle \neq 0$, $|\lambda| > 1$ and $p = 0$. Let $m \in \mathbb{N}$ be so big that $|\lambda|^m (|\lambda| - 1) > 2\varepsilon / |\langle x, x^* \rangle|$ and put $\alpha := \frac{1}{2}(|\lambda|^m + |\lambda|^{m+1})$, so that

$$\alpha - |\lambda|^m > \frac{\varepsilon}{|\langle x, x^* \rangle|} \qquad \text{and} \qquad |\lambda|^{m+1} - \alpha > \frac{\varepsilon}{|\langle x, x^* \rangle|}.$$

Since $(|\lambda|^n)_{n\in\mathbb{N}}$ is an increasing sequence, for any $n \in \mathbb{N}$ we have

$$\Big||\lambda|^n - \alpha\Big| > \frac{\varepsilon}{|\langle x, x^*\rangle|}.$$

Now let $y = \alpha x$. The point x is ε-hypercyclic with exponent 0 so there is $n \in \mathbb{N}$ such that

$$\begin{aligned}\varepsilon > \|T^n x - y\| &\geq \Big|\langle T^n x, x^*\rangle - \langle y, x^*\rangle\Big| \\ &= \Big|\langle x, T^{*n}x^*\rangle - \alpha\langle x, x^*\rangle\Big| = \Big|\langle x, x^*\rangle\Big| \cdot \Big||\lambda|^n - \alpha\Big|\end{aligned}$$

and we have a contradiction.

(b) Next suppose that still $\langle x, x^*\rangle \neq 0$ and $|\lambda| > 1$ but $p > 0$. Then choose $y \in X$ so small that $\|y\| < 1$, $\|y\| < |\langle x, x^*\rangle| / (1+\varepsilon)$ and $\|y\|^p < |\langle x, x^*\rangle| / (1+\varepsilon)$. Since x is ε-hypercyclic vector with exponent p, there is $n \in \mathbb{N}$ such that

$$\begin{aligned}\varepsilon\|y\|^p &> \Big|\langle T^n x - y, x^*\rangle\Big| = \Big|\langle x, T^{*n}x^*\rangle - \langle y, x^*\rangle\Big| \\ &\geq \Big|\lambda^n\Big|\Big|\langle x, x^*\rangle\Big| - \Big|\langle y, x^*\rangle\Big| \geq \Big|\langle x, x^*\rangle\Big| - \|y\|\|x^*\|.\end{aligned}$$

If $0 < p < 1$ then $\|y\| \leq \|y\|^p$ so we have $\varepsilon\|y\|^p > |\langle x, x^*\rangle| - \|y\|^p$ and thus $\|y\|^p > |\langle x, x^*\rangle| / (1+\varepsilon)$, a contradiction. If $p \geq 1$ then $\|y\| \geq \|y\|^p$ so we have $\varepsilon\|y\|^p \geq \varepsilon\|y\| > |\langle x, x^*\rangle| - \|y\|$ and thus $\|y\| > |\langle x, x^*\rangle| / (1+\varepsilon)$, also a contradiction.

(c) The remaining possibility is when either $|\lambda| \leq 1$ or $\langle x, x^*\rangle = 0$. In case $p \leq 1$ we choose $\alpha > 0$ so that $\alpha > (\|x\| + 1)/(1-\varepsilon)$, in case $p > 1$ we choose $\alpha > 0$ so that $\alpha > 2(\|x\| + 1)$. Let $y \in X$ satisfy $\langle y, x^*\rangle > 1 - \varepsilon/\alpha$ and $\|y\| = 1$. Since x is ε-hypercyclic with exponent p, there is $n \in \mathbb{N}$ such that

$$\begin{aligned}\varepsilon\alpha^p = \varepsilon\|\alpha y\|^p &> \|T^n x - \alpha y\| = \|T^n x - \alpha y\| \cdot \|x^*\| \\ &\geq \Big|\langle \alpha y, x^*\rangle - \langle x, T^{*n}x^*\rangle\Big| \geq \alpha - \varepsilon - |\lambda|^n \Big|\langle x, x^*\rangle\Big| \\ &\geq \alpha - 1 - \|x\|.\end{aligned}$$

Therefore $\alpha - \varepsilon\alpha^p < 1 + \|x\|$. If $p \leq 1$ then α was chosen so that $1 + \|x\| < \alpha - \varepsilon\alpha \leq \alpha - \varepsilon\alpha^p$, and we have a contradiction. If $p > 1$ then α was chosen so that $1 + \|x\| < \alpha/2$, so $\varepsilon\alpha^p > \alpha/2$, but by assumption $\varepsilon < 5^{-p} \leq \frac{1}{2}\alpha^{1-p}$, and we have also a contradiction.

Proposition 9.2. *Let $p \geq 0$. If T is ε-hypercyclic with exponent p for every $\varepsilon > 0$, then T is hypercyclic.*

Proof. By Theorem 9.1, X is infinite dimensional. Now observe that if U is a non-empty open set then there is $\varepsilon > 0$ such that inside U, there is an infinite number of pairwise disjoint open balls, all with radius $\varepsilon > 0$. Indeed, let $B \subset U$ be a closed ball of radius ε_0 and let $0 < \varepsilon < \varepsilon_0/2$. Suppose that inside B, there are disjoint open balls $B(u_1, \varepsilon)$, ..., $B(u_n, \varepsilon)$ such that any other open ball of radius ε inside B would intersect one of them. This means that $B(u_1, 2\varepsilon)$, ..., $B(u_n, 2\varepsilon)$ form a finite open covering of B with balls of radius $2\varepsilon < \varepsilon_0$ – but this is impossible in an infinite dimensional space.

We will prove that T is topologically transitive. Let U and V be any non-empty disjoint open sets in X. Without loss of generality, we can suppose that

$$\alpha := \max\{\sup_{u\in U} \|u\|, \sup_{v\in V} \|v\|\} < \infty.$$

By the previous observation, there are pairwise disjoint open balls of diameter ε, infinite number of them in U, with central points denoted $(u_i)_{i\in\mathbb{N}}$, and infinite number of them in V, with central points denoted $(v_i)_{i\in\mathbb{N}}$. Since x is $(\varepsilon \cdot \alpha^{-p})$-hypercyclic with exponent p, for each $i \in \mathbb{N}$ there are exponents $n_i, m_i \in \mathbb{N}$ such that

$$\|T^{n_i}x - u_i\| < \varepsilon\alpha^{-p}\|u_i\|^p \le \varepsilon\|u_i\|^{-p}\|u_i\|^p = \varepsilon,$$
$$\|T^{m_i}x - v_i\| < \varepsilon\alpha^{-p}\|v_i\|^p \le \varepsilon\|v_i\|^{-p}\|v_i\|^p = \varepsilon.$$

This shows that the orbit of x intersects both U and V infinitely many times, so in particular, there is $k, l \in \mathbb{N}$, $k < l$, such that $T^k x \in U$ and $T^l x \in V$. Thus $T^{l-k}(U) \cap V \neq \emptyset$.

Theorem 9.3. *Let $p \ge 0$, $p \neq 1$. Suppose that for a single $\varepsilon > 0$, T is ε-hypercyclic with exponent p. Then it is ε-hypercyclic with exponent p for all $\varepsilon > 0$, and thus hypercyclic.*

Proof. Suppose that T is δ-hypercyclic with exponent p. Let $\varepsilon > 0$ be arbitrary and set

$$c := (\varepsilon/\delta)^{\frac{1}{p-1}}.$$

Let $y \in X$. By assumption applied to $cy \in X$ instead of y, there exists a point $x \in X$ and $n \in \mathbb{N}$ such that $\|T^n x - cy\| < \delta\|cy\|^p$. Now if we divide the inequality by c, we have

$$\|T^n(\tfrac{1}{c}x) - y\| < \delta c^{p-1}\|y\|^p = \varepsilon\|y\|^p,$$

so $\frac{1}{c}x$ is an ε-hypercyclic point with exponent p.

Note that the theorem does not assert that any ε-hypercyclic vector with exponent p is hypercyclic – in fact it's not true in general. The above results show that the only interesting case is when $p = 1$. In such a case the notion really differs from the notion of hypercyclicity – see the following result which was first obtained for ℓ^1 in [5], and consequently by Bayart for Hilbert spaces in [12].

Theorem 9.4. *There is a Hilbert space such that for any $\varepsilon > 0$ there exists an operator which is ε-hypercyclic but not hypercyclic.*

10. Weakly hypercyclic operators

Up to now we dealt with orbits dense with respect to the norm topology. In [24], the notion of weak hypercyclicity was broadly studied for the first time. An operator T is said to be weakly hypercyclic if it admits an orbit which is dense in X with respect to the weak topology. Weakly hypercyclic operators share some of the properties with hypercyclic operators, there are however many pitfalls: only one of two implications of the Birkhoff transitivity theorem 5.1 holds, there is no known version of the Hypercyclicity Criterion valid for the weak topology etc.

Clearly, if $x \in X$ is a (norm) hypercyclic vector then it is weakly hypercyclic, but the converse is not true. However, if x is weakly hypercyclic then $\operatorname{Span}\operatorname{Orb}(T, x)$ is weakly dense so, as a convex set, it is norm dense. In other words, every weakly hypercyclic vector is weakly cyclic and thus cyclic.

Theorem 10.1. ([24, Corollary 3.3]) *Let $2 \le p < \infty$. There exists a bilateral weighted shift on $\ell^p(\mathbb{Z})$, that is weakly hypercyclic, but bounded from below by 1 and thus not hypercyclic.*

In fact, the constructed shift operator admits a strictly norm increasing, weakly dense orbit. In particular, in the space $\ell^p(\mathbb{Z})$, there is a weakly dense sequence which is strictly norm increasing. However, S. Shkarin showed that the growth of such a sequence has certain limits.

Theorem 10.2. ([78, Proposition 5.2 and 5.4]) *Let $(x_n)_{n=1}^{\infty}$ be a sequence in X satisfying*

$$C := \sum_{n=1}^{\infty} \|x_n\|^{-p} < \infty$$

where $p = 2$ if X is a Hilbert space, or $p = 1$ in general. Then (x_n) is weakly closed.

Proof. Let $z \in X$ be such that for no $n \in \mathbb{N}$ is $z = x_n$. Observe that the sequence $x_n - z$ satisfies the same growth assumption as x_n.

Applying Theorem 1.2, or Theorem 1.4 in the complex Hilbert space case, to the sequence $(x_n - z)/\|x_n - z\|$ and to the coefficients $(C^{1/p}\|x_n - z\|)^{-1}$, we get a functional $f \in X^*$ such that

$$\left|\langle \frac{x_n - z}{\|x_n - z\|}, f\rangle\right| \ge \frac{1}{C^{1/p}\|x_n - z\|}$$

and so $\left|\langle x_n - z, f\rangle\right| \ge C^{-1/p}$. Therefore z is not in the weak closure of $(x_n)_{n=1}^{\infty}$.

For a real Hilbert space, consider its complexification $X_{\mathbb{C}} = X \oplus iX$. Proceeding as above for the complex Hilbert space $X_{\mathbb{C}}$, we get a functional $f_{\mathbb{C}} = f \oplus if' \in X_{\mathbb{C}}^*$, such that

$$C^{-1/p} \le \left|\langle x_n \oplus ix_n - z \oplus iz, f_{\mathbb{C}}\rangle\right| = \left|\langle x_n - z, f\rangle\right| + \left|\langle ix_n - iz, f'\rangle\right|$$

and thus $\max(\left|\langle x_n - z, f\rangle\right|, \left|\langle x_n - z, if'\rangle\right|) \geq C^{-1/p}/2$ for each $n \in \mathbb{N}$. Therefore z is not in the weak closure of $(x_n)_{n=1}^{\infty}$.

In [11], an intermediate result is obtained for Banach spaces with the dual space of (Rademacher) type p – even though an intermediate plank theorem is unknown for ℓ^p spaces.

The following corollary is a strengthening of a result of Kitai [51]. It immediately implies that no compact operator can be weakly hypercyclic (and the more so hypercyclic) since the infinite-dimensional compact operators have always one component of the spectrum consisting of a single point with modulus less than 1.

Corollary 10.3. ([28]) *If T is weakly hypercyclic then every component of its spectrum intersects the unit circle.*

Proof. Let $\sigma \subset \mathbb{D}$ be a non-empty component of the spectrum of T, P_σ the corresponding spectral projection onto the spectral subspace X_σ and $T_\sigma = T|_{X_\sigma}$. If $x \in X$ is a weakly hypercyclic vector for T then $y = P_\sigma x$ is a weakly hypercyclic vector for T_σ, since the orbit of y is an image of weakly dense orbit of x under the weakly continuous projection P_σ.

Without loss of generality, $\|y\| = 1$. Clearly, σ cannot be a subset of $\{\lambda : |\lambda| < 1\}$: in such a case $r(T_\sigma) < 1$ and $T_\sigma^n y \to 0$. On the other hand, if $\sigma \subset \{\lambda : |\lambda| > 1\}$ then T_σ is invertible and by the spectral mapping theorem $r(T_\sigma^{-1}) < 1$. Hence for some $a < 1$ we have $\|T_\sigma^{-n}\|/a^n \to 0$, so that $\|T_\sigma^{-n}\| < a^n$ for all sufficiently large $n \in \mathbb{N}$. This yields

$$1 = \|y\| = \|T_\sigma^{-n} T_\sigma^n y\| \leq a^n \|T_\sigma^n y\|,$$

so $\|T_\sigma^n y\| \geq a^n$ for large $n \in \mathbb{N}$. By Theorem 10.2, $\mathrm{Orb}(T_\sigma, y)$ is weakly closed – a contradiction.

We finish the chapter with the following corollary, which is a slight improvement of [33, Theorem 4.1].

Corollary 10.4. *Let T be a weakly hypertransitive operator, i.e., an operator which has no nontrivial weakly closed invariant subset. Then the sum*

$$S := \sum_{n=1}^{\infty} \|T^n\|^{-1/2}$$

diverges. When X is a complex Hilbert space, even the sum

$$S' := \sum_{n=1}^{\infty} \|T^n\|^{-1}$$

diverges. In particular, $r(T) = 1$.

Proof. In the general Banach space case, putting $\alpha_n = S^{-1}\|T^n\|^{-1/2}$ we obtain by Proposition 1.3 a vector $x \in X$ such that $\|T^n x\| \geq \alpha_n \|T^n\|$ so

$$\sum_{n=1}^{\infty} \|T^n x\|^{-1} \leq S^{-1} \sum_{n=1}^{\infty} (\|T^n\|^{-1/2} \|T^n\|)^{-1} = S^{-1} \sum_{n=1}^{\infty} \|T^n\|^{-1/2} = S^2.$$

Therefore by Theorem 10.2, the orbit of x is weakly closed in X, so T is not weakly hypertransitive. In the complex Hilbert space case, we proceed similarly, using Proposition 1.5 instead of Proposition 1.3.

In particular every weakly hypertransitive operator satisfies $r(T) \leq 1$. However, if $r(T) < 1$ then for each $x \in X$ we have $\|T^n x\| \to 0$, so the operator is even not weakly hypercyclic.

IV. Orbit reflexivity

11. Introduction and some conditions

The notion of orbit reflexivity is a natural analogue of the well-known notion of reflexivity. It is connected to the invariant subset problem in the same way as is the reflexivity connected to the invariant subspace problem (both problems were already mentioned in the end of Chapter 7). The notion was introduced in [44] and studied e.g. in [55] and [42]; for a wider context see [40], and for another similar notions see [41] and again [42].

Recall that we say that $T \in \mathcal{L}(X)$ is reflexive if every $A \in \mathcal{L}(X)$ belongs to the closure of the set $\{p(T) : \ p \text{ polynomial}\}$ in the strong operator topology, whenever for each $u \in X$, Au belongs to the closure of the set $\{p(T)u : \ p \text{ polynomial}\}$. This is the same as saying that the only operators, which leave invariant all the closed subspaces invariant under T, are those in the SOT closure of the span of the powers of T. In fact the term comes from operator algebras: an operator algebra $\mathcal{A}$ is called reflexive if it is equal to the algebra of operators, which leave invariant all the subspaces invariant under all the operators from $\mathcal{A}$. The definitions coincide: operator T is reflexive iff the strongly closed algebra generated by I and T is reflexive.

An analogous notion exists for the invariant subsets in place of invariant subspaces. We say that T is orbit reflexive if every $A \in \mathcal{L}(X)$ belongs to the closure of $\mathrm{Orb}(T)$ in the strong operator topology, whenever for each $u \in X$, Au belongs to the closure of $\mathrm{Orb}(T, u)$. Again, in human language this means that the only operators, which leave invariant all the subsets invariant under T, are those in the SOT closure of the powers of T. Note that the term "hyperreflexivity" is already reserved for a different notion derived from reflexivity.

To shed some light on the definition, consider the following simple example. Let $X = \mathbb{R}^2$ and T be the diagonal matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, $a, b > 1$. In this case for $u \in X$ and a matrix A, the statement $Au \in \{T^n u \colon \ n \in \mathbb{N}_0\}$ means $Au = T^k u$ for some $k \in \mathbb{N}_0$ dependent on $u \in X$. Therefore T is orbit reflexive just if the exponent k is independent on $u \in X$. With the help of linearity it is thus sufficient to prove that the exponent is the same for a vector $u = \left(\begin{smallmatrix} c \\ 0 \end{smallmatrix}\right)$ and a vector $v = \left(\begin{smallmatrix} 0 \\ d \end{smallmatrix}\right)$, where $c, d \neq 0$. To prove this statement, suppose that $Au = T^n u = a^n u$ and

$Av = T^m v = b^m v$ for some $n, m \in \mathbb{N}_0$; moreover there is $k \in \mathbb{N}$ such that $A(u+v) = T^k(u+v) = a^k u + b^k v$. But

$$a^n u + b^m v = Au + Av = A(u+v) = a^k u + b^k v,$$

while u and v are linearly independent. Hence $a^n = a^k$ and $b^m = b^k$, so $n = k$ and $m = k$. In other words, there is a common exponent $k \in \mathbb{N}$ such that for any $u \in X$, we have $Au = T^k u$, so $A = T^k$ and T is orbit reflexive.

A few similar notions were also considered, for instance recently in [42], the notion of null-orbit reflexivity was defined: T is null-orbit reflexive iff every A belongs to the closure of $\{0\} \cup \mathrm{Orb}(T)$ in SOT, whenever Au belongs to the closure of $\{0\} \cup \mathrm{Orb}(T, x)$ for each $u \in X$.

However, let us focus on the orbit reflexivity itself.

Observation 11.1.

(i) If T is orbit reflexive and S is invertible then STS^{-1} is also orbit reflexive.

(ii) If T is orbit reflexive then $T \oplus T$ is also orbit reflexive.

Proof.

(i) Similarity preserves both the norm topology on X and the strong operator topology on $\mathcal{L}(X)$. Hence "$SAS^{-1}x \in \mathrm{Orb}(STS^{-1}, x)^-$ for all $x \in X$" is equivalent to "$Ax \in \mathrm{Orb}(T, x)^-$ for all $x \in X$", which is by assumption equivalent to "$A \in \mathrm{Orb}(T)^{-SOT}$", which is equivalent to "$SAS^{-1} \in \mathrm{Orb}(STS^{-1})^{-SOT}$".

(ii) Let $A \in \mathcal{L}(X \oplus X)$ and suppose that $Au \in \mathrm{Orb}(T \oplus T, u)^-$ for all $u \in X \oplus X$. In particular, for any $x \in X$ we have

$$A(x, 0) \in \mathrm{Orb}(T \oplus T, (x, 0))^- \subset X \times \{0\}$$

and similarly for the second component, so that both copies of X are A-invariant. Hence A can be written as $B \oplus C$ where $B, C \in \mathcal{L}(X)$.

Moreover, if $x \in X$ then $(Bx, Cx) = A(x, x) \in \mathrm{Orb}(T \oplus T, (x, x))^- \subset \{(y, y) : y \in X\}$, so $B = C$. Now since T is orbit reflexive and $Bx \in \mathrm{Orb}(T, x)^-$ for all $x \in X$, we have $B \in \mathrm{Orb}(T)^{-SOT}$, and therefore $A = B \oplus B \in \mathrm{Orb}(T \oplus T)^{-SOT}$.

However, orbit reflexivity is in general not preserved under scaling: we will see in Corollary 11.5 that if T is not orbit reflexive and $c \in \mathbb{C}$ with $|c| \neq 1$, then cT is orbit reflexive.

Many known classes of operators turn out to be orbit reflexive. Most of the results were obtained for Hilbert spaces in [44] and in fact the original proof applies to the Banach space setting as well, as was shown in [61]. We'll need the following Baire category argument.

Lemma 11.2. *Let $A, T_1, T_2, \ldots \in \mathcal{L}(X)$. If the set of vectors $u \in X$ for which $Au \in \{T_1 u, T_2 u, \ldots\}$ is of second category, then $A \in \{T_1, T_2, \ldots\}$.*

Proof. Denote the above mentioned set of second category by U, so $U \subset \bigcup_{n=1}^{\infty} \operatorname{Ker}(A - T_n)$. By Baire category theorem, there exists $m \in \mathbb{N}$ such that $\operatorname{Ker}(A - T_m)$ has a nonempty interior. Since $\operatorname{Ker}(A - T_m)$ is a linear subspace, we have $\operatorname{Ker}(A - T_m) = X$, and so $A = T_m$.

Theorem 11.3. *T is orbit reflexive in any of the following cases:*

(i) there is a nonempty open subset $U \subset X$ such that for each $x \in U$, the orbit $\operatorname{Orb}(T, x)$ is closed,

(ii) there is a nonempty open subset $U \subset X$ such that for each $x \in U$, $\|T^n x\| \to \infty$,

(iii) for each $x \in X$, $\|T^n x\| \to 0$,

(iv) the set $\mathcal{T} := \operatorname{Orb}(T)^{-SOT}$ is countable and strongly compact,

(v) $\sigma(T) \cap \mathbb{T} = \emptyset$.

Proof.

(i) Follows from Lemma 11.2.

(ii) Follows from (i).

(iii) Apply Lemma 11.2 to operators $0 = \lim_{n\to\infty} T^n$, and $I, T, T^2, T^3, \ldots$.

(iv) Let $u \in X$ and suppose $Au = \lim T^{n_k} u$ for some sequence $(n_k)_{k\in\mathbb{N}}$ of non-negative integers. Since $\mathcal{T}$ is strongly compact, the sequence $(T^{n_k})_{k\in\mathbb{N}}$ has a strongly convergent subsequence. Therefore $Au \in \mathcal{T}u$ for each $u \in X$, and we can use Lemma 11.2.

(v) By the Riesz functional calculus, we can decompose $X = Y \oplus Z$, $T = T_Y \oplus T_Z$ such that $r(T_Y) < 1$ and the spectrum of T_Z lies outside the unit disc. If $Z \neq \{0\}$ then there is a nonempty open set $U \subset X$ with $U \cap Y = \emptyset$ such that the orbit of every point in U tends to infinity and by (ii), T is orbit reflexive. On the other hand, if $Z = \{0\}$ then $\|T^n\| \to 0$ so by (iii), T is orbit reflexive.

Proposition 11.4. ([61]) *T is orbit reflexive if*

$$\sum_{n=1}^{\infty} 1/\|T^n\| < \infty,$$

or when X is a complex Hilbert space

$$\sum_{n=1}^{\infty} 1/\|T^n\|^2 < \infty.$$

Proof. Let $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty$. Let $A \in \mathcal{L}(X)$ be such that $Au \in \mathrm{Orb}(T,u)^-$ for each $u \in X$. Suppose that $A \neq T^n$ for all $n \in \mathbb{N}_0$, otherwise there is nothing to prove. Observe that

$$\sum_{n=1}^{\infty} \frac{1}{\|T^n - A\|} < \infty.$$

Indeed, since $\|T^n\| \to \infty$ we have $\|T^n - A\| \geq \|T^n\| - \|A\| \geq \frac{1}{2}\|T^n\|$ for all n large enough. So for certain $n_0 \in \mathbb{N}$ we have

$$\sum_{n=n_0}^{\infty} \frac{1}{\|T^n - A\|} \leq \sum_{n=n_0}^{\infty} \frac{1}{\|T^n\| - \|A\|} \leq \sum_{n=n_0}^{\infty} \frac{2}{\|T^n\|} < \infty.$$

Therefore, the operators $S_n := T^n - A$ satisfy the conditions in Proposition 3.2. So there exists (in fact a dense set of points) $x \in X$ with $\|(T^n - A)x\| > 0$ for all $n \in \mathbb{N}_0$ and $\|(T^n - A)x\| \to \infty$. Thus there is a constant $C > 0$ such that $\inf_n \|(T^n - A)x\| \geq C > 0$ and we have a contradiction with the assumption that $Ax \in \mathrm{Orb}(T,x)^-$.

The Hilbert space case can be proved similarly.

Corollary 11.5. *Let $r(T) \neq 1$. Then T is orbit reflexive.*

Proof. Either $r(T) < 1$ so $\lim_{n\to\infty} \|T^n\| = 0$ and we can apply Theorem 11.3 (iii), or $r(T) > 1$ so $\|T^n\| > n^2$ for all $n \in \mathbb{N}$ large enough since otherwise we would have $r(T) = \inf_{n\to\infty} \|T^n\|^{1/n} \leq \lim_{n\to\infty}(n^2)^{1/n} \leq 1$, and we can apply Proposition 11.4.

For Hilbert spaces, it was proved in [44] that many classes of operators are orbit reflexive. The crucial tool for the following statements is the spectral theorem for normal operators, so the method does not extend to Banach spaces in general.

We also obtain an easy proof that any operator with $r(T) = \|T\|$ is orbit reflexive. This applies to subnormal operators (for which orbit reflexivity was shown recently in [41, Theorem 8] using a different proof), and compact operators (for which orbit reflexivity was shown in [44] also using a different proof). The same easy proof was independently used in [42] to prove null-orbit reflexivity of operators T satisfying $r(T) = \|T\|$. Recall that an operator is called weakly stable if every weak orbit tends to zero.

Theorem 11.6. ([44]) *Let X be a Hilbert space. Then T is orbit reflexive in any of the following cases:*

(i) T is normal,

(ii) T is a direct sum of a normal operator and a weakly stable operator,

(iii) $\|T\| \le 1$,

(iv) $r(T) = \|T\|$, in particular, T is subnormal or compact or algebraic (that is, $p(T) = 0$ for some non-zero polynomial p),

(v) T is a unilateral or bilateral weighted shift with positive weights.

Proof.

(i) Let T be normal and suppose that $S \in \mathcal{L}(X)$ satisfies $Sx \in \mathrm{Orb}(T,x)^-$ for all $x \in X$. Let $y_1, y_2, \dots, y_n \in X$ and $\varepsilon > 0$. Since S leaves invariant all the invariant subspaces of T, it is in the von Neumann algebra $\mathcal{A}$ generated by T. Without loss of generality, X can be identified with the smallest reducing subspace of $\mathcal{A}$ containing $y_1, y_2, \dots, y_n$.

In particular, X is separable so by the spectral theorem (cf. e.g. [70, Theorem 1.6]), there is a finite measure space (X,μ) such that all the operators in $\mathcal{A}$ are unitarily equivalent to a multiplication operator by a function from $L^\infty(\mu)$, acting on $L^2(\mu)$. Let $\tilde{y}_k$, $k = 1\dots n$, be the representations of y_k in $L^2(\mu)$ and let $\tilde{x} := \left|\tilde{y}_1\right| + \dots + \left|\tilde{y}_n\right| \in L^2(\mu)$ be the representation of certain $x \in X$. By assumption, there is $j \in \mathbb{N}_0$ such that $\|(S - T^j)x\|^2 < \varepsilon$.

If S and T^j are represented by multiplication by ϕ and $\psi_j \in L^\infty(\mu)$, respectively, we have for $k = 1\dots n$

$$\|(S - T^j)y_k\|^2 = \int \left|(\phi - \psi_j)\tilde{y}_k\right|^2 d\mu \le \int \left|(\phi - \psi_j)\tilde{x}\right|^2 d\mu = \|(S - T^j)x\|^2 < \varepsilon.$$

Therefore $S \in \mathrm{Orb}(T)^-$ so T is orbit reflexive.

(ii) Let the direct sum $T = T_Y \oplus T_Z$ correspond to the decomposition $X = Y \oplus Z$, where T_Y is normal and T_Z is weakly stable. Suppose that $Ax \in \mathrm{Orb}(T,x)^-$ for all $x \in X$. As in Observation 11.1 (ii), we see that A can be also written as a direct sum $A_Y \oplus A_Z$ corresponding to the decomposition $X = Y \oplus Z$. Since $T_Z^n \to 0$ weakly, for any $z \in Z$ we have $\mathrm{Orb}(T_Z, z)^- \subset \{0, z, T_Z z, T_Z^2 z, \dots\}$ so by Lemma 11.2, $A_Z \in \{0, I, T_Z, T_Z^2, \dots\}$.

By assumption, $\langle T_Z^n z, z^* \rangle \to 0$ for all $z \in Z$ and $z^* \in Z^*$, so T_Z is power bounded by the uniform boundedness principle applied twice. Now let $z \in Z$ satisfy $0 \in \mathrm{Orb}(T_Z, z)^-$. Then $\lim_{n\to\infty} T_Z^n z = 0$. Indeed, let $M := \sup_{n\in\mathbb{N}_0} \|T_Z^n\|$ and let $\varepsilon > 0$. Then there is $m \in \mathbb{N}$ such that $\|T_Z^m z\| < \varepsilon/M$, so for any $n \ge m$

$$\|T_Z^n z\| \le \|T_Z^{n-m}\| \cdot \|T_Z^m z\| \le M\|T_Z^m z\| < \varepsilon.$$

Now let us turn to the subspace Y. As in (i) we can suppose that X is separable and use the spectral theorem to find a finite measure μ and represent

T_Y and A_Y, on $L^2(\mu)$ by multiplication by ϕ and $\psi \in L^\infty(\mu)$, respectively. Let $h \in Y$ be the vector which is represented by the constant 1 function on $L^2(\mu)$ and suppose that $A_Y h = \lim_{k\to\infty} T_Y^{n_k} h$ for some sequence n_k of nonnegative integers. Either n_k is bounded, or unbounded. In the former case there is $p \in \mathbb{N}_0$ such that $n_k = p$ for infinitely many k's, so $A_Y h = T_Y^p h$ which means $\psi 1 = \phi^p 1$ a.e., so $A_Y = T_Y^p$. In the latter case there is a subsequence of n_k tending to infinity – denote this subsequence again n_k. We have $A_Y h = \lim T_Y^{n_k} h$ which means $\psi = \lim \phi^{n_k}$ in $L^2(\mu)$, so there is a subsequence, denote it again n_k, such that $\psi = \lim \phi^{n_k}$ almost everywhere. In particular $\|\phi\|_\infty \le 1$. Therefore we can use the dominated convergence theorem: if $y \in Y$ is represented by $\tilde{y} \in L^2(\mu)$ then

$$\|A_Y y - T_Y^{n_k} y\|^2 = \int \left|\psi\tilde{y} - \phi^{n_k}\tilde{y}\right|^2 d\mu \le \int \left|\tilde{y}\right|^2 d\mu \cdot \int \left|\psi - \phi^{n_k}\right| d\mu \to 0$$

so $A_Y = \lim T_Y^{n_k}$ in SOT.

We distinguish three possibilities.

(a) Suppose that $A_Z = 0$ and $A_Y \notin \operatorname{Orb}(T_Y)$. By (i), there is a sequence n_k tending to infinity such that $T_Y^{n_k} \to A_Y$ strongly. Since $A_Z = 0$ we have $0 \in \operatorname{Orb}(T_Z, z)^-$ for each $z \in Z$, so as was shown in the first paragraph, $T_Z^n \to 0 = A_Z$ strongly. In sum, $T_Y^{n_k} \oplus T_Z^{n_k} \to A_Y \oplus A_Z$ strongly.

(b) Suppose that $A_Z = 0$ and $A_Y = T_Y^m$ for some $m \ge 0$. Either $T_Z = 0$ so $A = T^m$, or $T_Z \ne 0$ so that we can choose $z \in Z$ such that $T_Z^m z \ne 0$. Let h be the vector mentioned above. By assumption, $A(h,z) \in \operatorname{Orb}(T,(h,z))^-$ so there is a sequence n_k of nonnegative integers such that $\lim_{k\to\infty} T_Z^{n_k} z = A_Z z = 0$, and $\lim_{k\to\infty} T_Y^{n_k} h = A_Y h$. We already know that the latter means that for some subsequence (n_{k_j}) we have $T_Y^{n_{k_j}} \to A_Y$ strongly. Now there are two possibilities: either $\limsup n_{k_j} = \infty$ so for some subsequence $(n_{k_{j_i}})$ we have $T_Z^{n_{k_{j_i}}} \to 0 = A_Z$ by the above power boundedness argument, and we are done. Or $\limsup n_{k_j} < \infty$, so there is $s \in \mathbb{N}$ such that $T_Y^{m+s} = T_Y^m$ so $T_Y^{m+ks} = T_Y^m \to T_Y^m = A_Y$ strongly as $k \to \infty$, while $T_Z^{m+ks} \to 0 = A_Z$ strongly as $k \to \infty$.

(c) Finally if $A_Z \ne 0$ then $A_Z = T_Z^m$ for some $m \ge 0$. Choose $z \in Z$ such that $T_Z^m z \ne 0$ and let h be again the vector mentioned above. Similarly as in (b), since $A(h,z) \in \operatorname{Orb}(T,(h,z))^-$ there is a sequence n_k of integers such that $\lim_{k\to\infty} T_Z^{n_k} z = A_Z z = T_Z^m z \ne 0$, and $\lim_{k\to\infty} T_Y^{n_k} h = A_Y h$ and thus $T_Y^{n_{k_j}} \to A_Y$ strongly for some subsequence (n_{k_j}). But by assumption $T_Z^n z \to 0$ as $n \to \infty$ so the sequence (n_{k_j}) is necessarily bounded, i.e. there is $p \in \mathbb{N}$ such that $n_{k_j} = p$ for infinitely many j's. Hence $T_Y^p = A_Y$ and $T_Z^p z = A_Z z = T_Z^m z$. This implies that $m = p$: if not then $T_Z^{|m-p|} z = z$ but this contradicts that $T_Z \to 0$ weakly. Hence $A = T^m$.

(iii) By a classical theorem by Langer [52] and Sz.-Nagy and Foiaş [34], T can be written as a direct sum of a unitary operator A, and a completely non-unitary contraction B which is weakly stable (c.n.u. means that any restriction to a reducing non-null subspace is not a unitary operator). Hence we can apply (ii).

(iv) Let $r(T) = \|T\|$. Either $r(T) > 1$, so T is orbit reflexive by Corollary 11.5. Or $\|T\| \le 1$, so T is orbit reflexive by (iii). The subnormal operators satisfy $\|T\| = r(T)$, cf. [25, Theorem 30.12], as do the compact, and algebraic operators.

(v) Let either $X := \ell^2(\mathbb{N})$ and T be a unilateral weighted shift on X, or $\ell^2(\mathbb{Z})$ and T be a bilateral weighted shift on X. Suppose that $A \in \mathcal{L}(X)$ satisfies $Ax \in \mathrm{Orb}(T, x)^-$ for all $x \in X$. For all types of shifts we first show that

$$A \in \{0, I, T, T^2, \ldots\}, \tag{1}$$

and afterwards we eliminate the possibility that $A = 0$.

First, if T is a unilateral forward shift then fix any $x \in X$. The set $\{y \in X : \langle T^n x, y\rangle \to 0\}$ is dense in X so $Ax \in \{0, x, Tx, T^2x, \ldots\}$ for all $x \in X$. Now (1) follows from Lemma 11.2. Assume that $A = 0$. Let T be given by $Te_i := \alpha_i e_{i+1}$, $i \in \mathbb{N}$. Since the weights $(\alpha_i)_{i\in\mathbb{N}}$ are nonzero, $0 = Ae_i \in \mathrm{Orb}(T, e_i)^-$ happens exactly when the sequence of products $(\alpha_i \cdot \alpha_{i+1} \cdot \ldots \cdot \alpha_{i+n-1})_{n\in\mathbb{N}}$ has 0 among its accumulation points. Thus the commuting family $(S_n)_{n\in\mathbb{N}_0}$ of normal operators, defined by

$$S_n e_i := \alpha_i \alpha_{i+1} \ldots \alpha_{i+n-1} e_i, \qquad n \in \mathbb{N}_0, i \in \mathbb{N},$$

satisfies $0 = Ax \in \{S_n x : n \in \mathbb{N}_0\}^-$ for each $x \in X$. Now a proof similar to the case (i) of a single normal operator yields that $A \in \{S_n : n \in \mathbb{N}_0\}^-$, so $A \in \mathrm{Orb}(T)^-$.

Second, if T is a bilateral shift then there is an increasing sequence of invariant subspaces $Y_1 \subset Y_2 \subset \ldots \subset X$ whose union Z is dense in X and such that for each $j \in \mathbb{N}$ the restriction $T|_{Y_j}$ is a unilateral forward shift. For each $j \in \mathbb{N}$ we thus already know that $A|_{Y_j} \in \{0, I|_{Y_j}, T|_{Y_j}, T|^2_{Y_j} \ldots\}$. Since the sequence of the subspaces is increasing we have $A|_Z \in \{0, I|_Z, T|_Z, T|^2_Z \ldots\}$ and thus (1) holds on $X = \overline{Z}$ as well. The rest now follows from almost the same proof as in the case of the unilateral forward shift, only the normal operators S_n are indexed by $\mathbb{Z}$ instead of $\mathbb{N}_0$.

Finally if T is a unilateral backward shift then there is an increasing sequence of invariant subspaces $Y_j := \mathrm{Ker}\, T^j$, $j \in \mathbb{N}$, whose union Z is dense in X. Fix $j \in \mathbb{N}$ and consider the restriction $T|_{Y_j}$. As in the unilateral forward shift case, the set $\{y \in Y_j : \langle T^n x, y\rangle \to 0\}$ is dense in Y_j so by Lemma 11.2 we have $A|_{Y_j} \in \{0, I|_{Y_j}, T|_{Y_j}, T|^2_{Y_j} \ldots\}$. As in the bilateral shift case, the sequence of the subspaces is increasing so we have $A|_Z \in \{0, I|_Z, T|_Z, T|^2_Z \ldots\}$ and thus (1) holds for $X = \overline{Z}$ as well. The rest is similar as in the first two cases.

12. The operators that are not orbit reflexive

Whereas there are many operators known not to be reflexive, until recently the only non-orbit reflexive operators discovered were those without a nontrivial closed invariant subset, i.e. basically the Read operator constructed on ℓ^1 in [72]. In particular, no such operator was known in the setting of reflexive Banach spaces.

Observation 12.1. *Let X be an infinite dimensional separable Banach space, and $T \in \mathcal{L}(X)$ a hypertransitive operator, that is an operator without a non-trivial closed invariant subset. Then T is not orbit reflexive.*

Proof. Suppose that T is orbit reflexive, we will show that this leads to a contradiction. Any bounded linear operator A leaves invariant all the closed subsets invariant under T, so that necessarily $\mathrm{Orb}(T)^- = \mathcal{L}(X)$. But $\mathrm{Orb}(T)^-$ is contained in the commutant of T, hence even all the one dimensional projections must commute with T, so T must leave invariant all the one dimensional subspaces. This means T is a scalar multiple of identity – and a scalar multiple of identity on at least two dimensional space obviously has a non-trivial closed invariant subset.

Recently, Grivaux and Roginskaya [37] modified the construction of Read to obtain an operator with a very small set of vectors which are not hypercyclic; the properties of the operator were mentioned in the end of Chapter 7. They proved that a slight modification of their example leads to a non-orbit reflexive operator. Consequently, Esterle [31] showed that the above mentioned modification is not necessary since every "Read type operator" on a Hilbert space is non-orbit reflexive itself.

Theorem 12.2. ([31]) *Let X be a separable Hilbert space. Then an operator $T \neq 0$ is not orbit reflexive if it is Read type, i.e. it satisfies the following two conditions:*

(i) for every $x \in X$, the closed orbit $\mathrm{Orb}(T,x)^-$ is a closed linear subspace,

(ii) the closed orbits are totally ordered, i.e. if $x, y \in X$ then

$$\textit{either} \quad \mathrm{Orb}(T,x)^- \subset \mathrm{Orb}(T,y)^- \quad \textit{or} \quad \mathrm{Orb}(T,y)^- \subset \mathrm{Orb}(T,x)^- .$$

Proof. For a contradiction, suppose that T is orbit reflexive and satisfies both conditions. If for any $x \in X \setminus \{0\}$ the orbit is dense then T has no nontrivial closed invariant subset and T is not orbit reflexive by Observation 12.1. Hence suppose there is a nonzero $y \in X$ such that $Y := \mathrm{Orb}(T,y)^- \neq X$. Let P be a projection to Y; we will show that $PT \neq TP$. Let $x \in \mathrm{Ker}\, P \setminus \{0\}$. Since $x \notin \mathrm{Orb}(T,y)^-$ and the closed orbits are totally ordered, we know that $y \in \mathrm{Orb}(T,x)^-$. Hence $y = \lim_{k\to\infty} T^{n_k}x$ for some sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers, so $0 \neq y = Py = \lim_{k\to\infty} PT^{n_k}x$. However, $T^nPx = 0$ for each

$n \in \mathbb{N}_0$ by the choice of $x \in X$, so $PT \neq TP$. But the closure of $\mathrm{Orb}(T)$ in the strong operator topology is contained in the commutant of T so we have a contradiction.

Another, simpler construction than that of Grivaux and Roginskaya was shown by V. Müller and the author in [61]. It is perhaps of interest, that the operators obtained by this construction are null-orbit reflexive, whereas the operator of Grivaux and Roginskaya is not null-orbit reflexive, cf. [42].

The first example is an operator on a Hilbert space which is not orbit reflexive.

Example 12.3. ([61]) *There exists a Hilbert space X and an operator $T \in \mathcal{L}(X)$ such that*

(i) for all $x \in X$, we have $\inf_{n\in\mathbb{N}} \|T^n x\| = 0$,

(ii) there are vectors $e_0, f_0 \in X$ such that

$$\inf_{n\in\mathbb{N}} \max\{\|T^n e_0\|, \|T^n f_0\|\} > 0.$$

Thus T is not orbit reflexive.

In the proof we use the following approximation lemma. Denote by m the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Denote by $\|\cdot\|_2$ and $\|\cdot\|_\infty$ the norms in the Hardy spaces $H^2(m)$ and $H^\infty(m)$, respectively.

Lemma 12.4. *Let p, q be complex polynomials, $\|p\|_2 \le 1$, $\|q\|_2 \le 1$ and let $0 < \varepsilon < 1/3$. Then there exist polynomials r, s such that $\|rp + sq\|_2 < \varepsilon$, $\|r\|_\infty \le 1$, $\|s\|_\infty \le 1$ and $\max\{\|r\|_2, \|s\|_2\} \ge 1/3$.*

Proof. Let $M_1 := \{z \in \mathbb{T} : \left|p(z)\right| \ge \left|q(z)\right|\}$, $M_2 = \mathbb{T} \setminus M_1$. Without loss of generality we can assume that $m(M_1) \ge 1/2$. Define functions $g, h \colon \mathbb{T} \to \mathbb{C}$ by

$$h(z) := \begin{cases} -1 & (z \in M_1) \\ 0 & (z \in M_2) \end{cases}$$

$$g(z) := \begin{cases} \frac{q(z)}{p(z)} & (z \in M_1) \\ 0 & (z \in M_2) \end{cases}$$

(if $p(z) = q(z) = 0$ then set $g(z) := 0$). Note that $\|g\|_\infty \le 1$, $\|h\|_\infty \le 1$ and $pg + qh = 0$.

Let $K = \max\{1, \|p\|_\infty, \|q\|_\infty\}$. There exist continuous functions $g_1, h_1 : \mathbb{T} \to \mathbb{C}$ such that $\|g_1 - g\|_2 < \frac{\varepsilon}{4K}$ and $\|h_1 - h\|_2 < \frac{\varepsilon}{4K}$. Define $g_2, h_2 \colon \mathbb{T} \to \mathbb{C}$ by

$$g_2(z) := \frac{g_1(z)}{\max\{1, \left|g_1(z)\right|\}},$$

$$h_2(z) := \frac{h_1(z)}{\max\{1, |h_1(z)|\}}.$$

Clearly g_2, h_2 are continuous, $\|g_2\|_\infty \le 1$, $\|h_2\|_\infty \le 1$, $\|g_2 - g\|_2 < \frac{\varepsilon}{4K}$ and $\|h_2 - h\|_2 < \frac{\varepsilon}{4K}$.

There exist trigonometric polynomials g_3, h_3 such that $\|g_3 - g_2\|_\infty < \varepsilon/4K$, $\|h_3 - h_2\|_\infty < \varepsilon/4K$. Moreover, we may assume that $\|g_3\|_\infty \le 1$, $\|h_3\|_\infty \le 1$.

Choose $l \in \mathbb{N}$ such that $r := z^l g_3$ and $s := z^l h_3$ are polynomials. Then $\|r\|_\infty \le 1$, $\|s\|_\infty \le 1$ and

$$\begin{aligned}
\|rp + qs\|_2 &= \|z^l g_3 p + z^l h_3 q\|_2 \\
&\le \|z^l gp + z^l hq\|_2 + \|z^l(g_3 - g)p\|_2 + \|z^l(h_3 - h)q\|_2 \\
&\le K\|g_3 - g\|_2 + K\|h_3 - h\|_2 \\
&\le K(\|g_3 - g_2\|_2 + \|g_2 - g\|_2) + K(\|h_3 - h_2\|_2 + \|h_2 - h\|_2) \\
&< \varepsilon.
\end{aligned}$$

Finally,

$$\|s\|_2 = \|h_3\|_2 \ge \|h\|_2 - \|h_3 - h\|_2 \ge \frac{1}{2} - \frac{\varepsilon}{2K} \ge \frac{1}{3}.$$

If $m(M_1) < 1/2$ then $m(M_2) \ge 1/2$ and we can proceed similarly. At the end we obtain $\|r\|_2 \ge 1/3$.

Proof of Example 12.3. For $N = 1, 2, 3\dots$ let $\varepsilon_N := N^{-1/3}$.

The underlying Hilbert space will be

$$X = Z \oplus \bigoplus_{k=1}^{\infty} Y_k,$$

where Z is the Hilbert space with an orthonormal basis $\{e_j, f_j :\ j = 0, 1, 2\dots\}$ and Y_k are finite-dimensional Hilbert spaces which will be determined in the construction.

We construct inductively integers k_N, $N = 0, 1, 2\dots$, integers a_k, spaces Y_k and elements $w_k \in Z$, $k = 1, 2, 3\dots$, in the following way. Set formally $k_0 := 0$ and $a_0 := 0$. Let $N \ge 1$ and suppose that the integers k_{N-1}, a_k, spaces Y_k and elements $w_k \in Z$ have already been defined for $1 \le k \le k_{N-1}$. Write for short $b_{N-1} := a_{k_{N-1}}$. Let $Z_N := \operatorname{Span}\{e_j, f_j :\ j = 0, \dots, b_{N-1}\}$ and let $w_{k_{N-1}+1}, \dots, w_{k_N}$ be an ε_N^2-net in the closed unit ball of Z_N.

For $k = k_{N-1} + 1, \dots, k_N$ we can write $w_k = \sum_{i=0}^{b_{N-1}} (\alpha_i^{(k)} e_i + \beta_i^{(k)} f_i)$ with complex coefficients $\alpha_i^{(k)}$, $\beta_i^{(k)}$. We define numbers μ_i, $0 \le i \le b_{N-1}$ in the following way. If $1 \le M \le N - 1$, $k_{M-1} < l < k_M$ and $a_l < i \le 2a_l$ then set $\mu_i = \varepsilon_M^{-1}$. If $2a_l < i < 3a_l$ then $\mu_i = \varepsilon_M^{-(3a_l - i)/a_l}$. Set $\mu_i = 1$ otherwise.

Consider the polynomials p_k, q_k defined by $p_k(z) := \sum_{i=0}^{b_{N-1}} \mu_i \alpha_i^{(k)} z^i$ and $q_k(z) := \sum_{i=0}^{b_{N-1}} \mu_i \beta_i^{(k)} z^i$. We have $\|p_k\|_2 \le \varepsilon_{N-1}^{-1}$ and $\|q_k\|_2 \le \varepsilon_{N-1}^{-1}$.

By Lemma 12.4 for the polynomials $\varepsilon_{N-1} p_k$, $\varepsilon_{N-1} q_k$, there exist $m_k \in \mathbb{N}$ and polynomials $r_k(z) = \sum_{i=0}^{m_k} \gamma_i^{(k)} z^i$, $s_k(z) = \sum_{i=0}^{m_k} \delta_i^{(k)} z^i$ such that $\|r_k\|_\infty \le 1$, $\|s_k\|_\infty \le 1$, $\max\{\|r_k\|_2, \|s_k\|_2\} \ge 1/3$ and $\|r_k p_k + s_k q_k\|_2 < \varepsilon_N$.

Choose numbers a_k, $k_{N-1}+1 \le k \le k_N$, such that $a_{j+1} > a_j^2 + 3a_j + m_j$, $j = k_{N-1}, \dots, k_N - 1$. Let Y_k be the finite-dimensional Hilbert space with an orthonormal basis $u_{k,j}$, $j = 0, \dots, m_k + 2a_k - 1$.

Using induction, we continue the construction as described above.

Now we define the operator $T \in \mathcal{L}(X)$ by:

$$
\begin{aligned}
&Tu_{k,i} := u_{k,i+1} \qquad (k \in \mathbb{N}, 0 \le i \le m_k + 2a_k - 2),\\
&Tu_{k,m_k+2a_k-1} := 0,\\
&Te_{a_k} := \varepsilon_N e_{a_k+1} + \sum_{i=0}^{m_k} \gamma_i^{(k)} u_{k,i} \qquad (k_{N-1} < k \le k_N),\\
&Tf_{a_k} := \varepsilon_N f_{a_k+1} + \sum_{i=0}^{m_k} \delta_i^{(k)} u_{k,i} \qquad (k_{N-1} < k \le k_N),\\
&Te_j := \varepsilon_N^{-1/a_k} e_{j+1} \qquad (k_{N-1} < k \le k_N, 2a_k \le j < 3a_k),\\
&Tf_j := \varepsilon_N^{-1/a_k} f_{j+1} \qquad (k_{N-1} < k \le k_N, 2a_k \le j < 3a_k),\\
&Te_j := e_{j+1} \quad \text{and} \quad Tf_j = f_{j+1} \qquad \text{otherwise.}
\end{aligned}
$$

That is, T acts on the standard basis of Z as a pair of weighted shifts, up to the points of the form e_{a_k} and f_{a_k}. It is easy to see that T defines a bounded linear operator on X. It is easy to check that $\|T\| \le 2$. Note also that for each $k \in \mathbb{N}$, we have $T^{a_k - a_{k-1}} e_{a_{k-1}} = e_{a_k}$ and $T^{a_k - a_{k-1}} f_{a_{k-1}} = f_{a_k}$.

Let $E := \operatorname{Span}\{e_i : i = 0, 1, \dots\}$, $F := \operatorname{Span}\{f_i : i = 0, 1, \dots\}$ and $Y := \bigoplus_{k=1}^{\infty} Y_k$. For a closed subspace $M \subset X$ denote by P_M the orthogonal projection onto M.

To prove (ii), let $j \in \mathbb{N}$. If $j \notin \bigcup_{k=1}^{\infty} \{a_k + 1, \dots, 3a_k\}$ then $\|T^j e_0\| \ge \|P_Z T^j e_0\| = \|e_j\| = 1$. So we may assume that $a_k + 1 \le j \le 3a_k$ for some k. Then

$$
\begin{aligned}
\max&\{\|T^j e_0\|, \|T^j f_0\|\}\\
&\ge \max\{\|P_{Y_k} T^j e_0\|, \|P_{Y_k} T^j f_0\|\}\\
&= \max\{\|P_{Y_k} T^{j-a_k} e_{a_k}\|\}, \|P_{Y_k} T^{j-a_k} f_{a_k}\|\}\\
&= \max\{\|P_{Y_k} Te_{a_k}\|, \|P_{Y_k} Tf_{a_k}\|\}\\
&= \max\{\|\sum_{i=0}^{m_k} \gamma_i^{(k)} u_{k,i}\|, \|\sum_{i=0}^{m_k} \delta_i^{(k)} u_{k,i}\|\}\\
&= \max\{\|r_k\|_2, \|s_k\|_2\} \ge 1/3.
\end{aligned}
$$

So $\max\{\|T^j e_0\|, \|T^j f_0\|\} \ge 1/3$ for all j.

To prove (i), suppose that $x \in X$ is of norm 1 and $0 < \varepsilon < \frac{1}{2}$.

There exists $M \geq 1$ such that $\|(P_Z - P_{Z_M})x\| < \frac{\varepsilon}{18}$. There exists $N > M$ such that

$$\varepsilon_N^{1/2} < \frac{\varepsilon \cdot \varepsilon_M}{9},$$

$$\left\| \sum_{k'=k_{N-1}+1}^{\infty} P_{Y_{k'}} x \right\| < \frac{\varepsilon}{9},$$

$$\|P_{Z_{N+1}} x - P_{Z_N} x\| < \varepsilon_N^{3/2}.$$

Indeed, the first two conditions are satisfied for all N sufficiently large. Suppose on the contrary that $\|P_{Z_{N+1}} x - P_{Z_N} x\| \geq \varepsilon_N^{3/2}$ for all $N \geq N_0$. Then

$$1 = \|x\|^2 \geq \sum_{N=N_0}^{\infty} \|P_{Z_{N+1}} x - P_{Z_N} x\|^2 \geq \sum_{N=N_0}^{\infty} \varepsilon_N^3 = \infty,$$

a contradiction. Fix N with these properties.

Find k, $k_{N-1} < k \leq k_N$ such that $\|P_{Z_N} x - w_k\| \leq \varepsilon_N^2$. Set $j = 2a_k + 1$. We have

$$\begin{aligned}
\|T^j x\| \leq & \left\| \sum_{k'=1}^{k_{N-1}} T^j P_{Y_{k'}} x \right\| + \left\| \sum_{k'=k_{N-1}+1}^{\infty} T^j P_{Y_{k'}} x \right\| + \|P_Z T^j P_{Z_M} x\| \\
& + \|P_Z T^j (P_{Z_N} - P_{Z_M}) x\| + \|P_Z T^j (P_{Z_{N+1}} - P_{Z_N}) x\| + \|P_Z T^j (P_Z - P_{Z_{N+1}}) x\| \\
& + \|P_Y T^j (P_Z - P_{Z_{N+1}}) x\| + \|P_Y T^j (P_{Z_{N+1}} - P_{Z_N}) x\| \\
& + \|P_Y T^j (P_{Z_N} x - w_k)\| + \|P_Y T^j w_k\|.
\end{aligned}$$

We estimate all the terms in the previous formula.

Since $k > k_{N-1}$ and $j > a_k > 2a_{k_{N-1}} + m_{k_{N-1}}$, we have $\sum_{k'=1}^{k_{N-1}} T^j P_{Y_{k'}} x = 0$.

For $k' > k_{N-1}$ we have $\|T^j|_{Y_{k'}}\| \leq 1$, and so

$$\left\| \sum_{k'=k_{N-1}+1}^{\infty} T^j P_{Y_{k'}} x \right\| \leq \left\| \sum_{k'=k_{N-1}+1}^{\infty} P_{Y_{k'}} x \right\| < \frac{\varepsilon}{9}.$$

It is easy to see that

$$\|P_Z T^j P_{Z_M}\| = \sup\{\|P_Z T^j e_i\| \colon\ i \leq b_{M-1}\} \leq \varepsilon_M^{-1} \varepsilon_N \varepsilon_N^{-i/a_k} < \varepsilon_M^{-1} \varepsilon_N^{1/2} < \frac{\varepsilon}{9},$$

and so $\|P_Z T^j P_{Z_M} x\| \leq \frac{\varepsilon}{9} \|P_{Z_M} x\| \leq \frac{\varepsilon}{9}$.

Similarly,

$$\begin{aligned}
\|P_Z T^j (P_{Z_N} - P_{Z_M})\| &= \max\{\|P_Z T^j e_i\| \colon\ b_{M-1} < i \leq b_{N-1}\} \leq 2, \\
\|P_Z T^j (P_{Z_{N+1}} - P_{Z_N})\| &= \max\{\|P_Z T^j e_i\| \colon\ b_{N-1} < i \leq b_N\} \leq \varepsilon_N^{-1}
\end{aligned}$$

and

$$\|P_Z T^j (P_Z - P_{Z_{N+1}})\| = \max\{\|P_Z T^j e_i\| \colon\ b_N < i\} \le 2.$$

Thus

$$\|P_Z T^j (P_{Z_N} - P_{Z_M})x\| \le 2\|(P_{Z_N} - P_{Z_M})x\| < \frac{\varepsilon}{9},$$
$$\|P_Z T^j (P_{Z_{N+1}} - P_{Z_N})x\| \le \varepsilon_N^{-1} \varepsilon_N^{3/2} = \varepsilon_N^{1/2} < \frac{\varepsilon}{9}$$

and

$$\|P_Z T^j (P_Z - P_{Z_{N+1}})x\| \le 2\|(P_Z - P_{Z_{N+1}})x\| < \frac{\varepsilon}{9}.$$

We show that $\|P_{Y_k} T^j P_Z\| \le 2\varepsilon_N^{-1}$. Clearly $\|P_{Y_k} T^j P_E\| = \|P_{Y_k} T^j P_{E_k}\|$ where $E_k = \operatorname{Span}\{e_0, \dots, e_{a_k}\}$. Let $y = \sum_{i=0}^{a_k} \lambda_i e_i$, $\|y\| = 1$. Note that the numbers μ_i mentioned in the construction satisfy $0 < \mu_i \le \varepsilon_N^{-1}$, $0 \le i \le a_k$ and $T^{a_k - i} e_i = \mu_i e_{a_k}$. We have

$$\|P_{Y_k} T^j y\| = \left\| r_k(z) \cdot \sum_{i=0}^{m_k} \lambda_i \mu_i z^i \right\|_2 \le \|r_k\|_\infty \cdot \left\| \sum_{i=0}^{m_k} \lambda_i \mu_i z^i \right\|_2$$
$$\le \Bigl(\sum_{i=0}^{m_k} \bigl|\lambda_i \mu_i\bigr|^2 \Bigr)^{1/2} \le \varepsilon_N^{-1} \Bigl(\sum_{i=0}^{m_k} |\lambda_i|^2 \Bigr)^{1/2} = \varepsilon_N^{-1}.$$

So $\|P_{Y_k} T^j P_E\| \le \varepsilon_N^{-1}$ and similarly, $\|P_{Y_k} T^j P_F\| \le \varepsilon_N^{-1}$. Hence

$$\|P_{Y_k} T^j P_Z\| \le \|P_{Y_k} T^j P_E\| + \|P_{Y_k} T^j P_F\| \le 2\varepsilon_N^{-1}.$$

It is easy to show that for $k' > k$ we have $\|P_{Y_{k'}} T^j P_Z\| \le 2$, and so $\|P_Y T^j P_Z\| = \sup_{k' \ge 1} \|P_{Y_{k'}} T^j P_Z\| \le 2\varepsilon_N^{-1}$. Furthermore,

$$\|P_Y T^j (P_Z - P_{Z_{N+1}})\| = \sup_{k' > k_N} \|P_{Y_{k'}} T^j (P_Z - P_{Z_{N+1}})\| \le 2.$$

So

$$\|P_Y T^j (P_Z - P_{Z_{N+1}})x\| \le 2\|(P_Z - P_{Z_{N+1}})x\| \le \frac{\varepsilon}{9},$$
$$\|P_Y T^j (P_{Z_{N+1}} - P_{Z_N})x\| \le 2\varepsilon_N^{-1} \|(P_{Z_{N+1}} - P_{Z_N})x\| < 2\varepsilon_N^{-1} \varepsilon_N^{3/2} = 2\varepsilon_N^{1/2} < \frac{\varepsilon}{9}$$

and

$$\|P_Y T^j (P_{Z_N} x - w_k)\| \le 2\varepsilon_N^{-1} \|P_{Z_N} x - w_n\| \le 2\varepsilon_N^{-1} \varepsilon_N^{3/2} = 2\varepsilon_N^{1/2} < \frac{\varepsilon}{9}.$$

Finally,

$$\|P_Y T^j w_k\| = \|r_k p_k + s_k q_k\|_2 \le \varepsilon_N < \frac{\varepsilon}{9}.$$

Hence $\|T^j x\| < \varepsilon$.

Consequently, T is not orbit reflexive since the zero operator is not in the strong operator topology closure of polynomials of T but $0 \in \mathrm{Orb}(T, x)^-$ for each $x \in X$.

Moreover on ℓ^1, there is an operator which is reflexive but not orbit reflexive. The existence of a Hilbert space operator with such properties seems to be an open question, in particular it is not clear whether the operator constructed above is reflexive.

Note that there are many operators which are orbit reflexive but not reflexive, for instance the Volterra operator V on the Hilbert space $L^2[0;1]$, defined as $(Vf)(x) := \int_0^x f(t)dt$. Indeed, it is well-known that V is compact, and hence it is orbit reflexive, and that the lattice of invariant subspaces of V consists of subspaces of the form $L^2[\alpha;1]$ where $\alpha \in [0;1]$, cf. [25, §28]. The multiplication operator $(Tf)(x) := xf(x)$ leaves all these subspaces invariant, but does not commute with V since for $f := 1$ we have $(VTf)(x) = x^2/2 \neq x^2 = (TVf)(x)$. The Volterra operator is thus not reflexive.

Example 12.5. ([61]) *There is an operator T on ℓ^1 which is reflexive but not orbit reflexive.*

Proof. For $N = 1, 2, 3\dots$ let $\varepsilon_N := 1/\sqrt{N}$. Let a_k, $k = 1, 2, 3\dots$, be an increasing sequence of positive integers such that $a_{k+1} > 6a_k^2$.

The underlying space will be the ℓ_1-direct sum

$$X = Z \oplus \bigoplus_{k=1}^{\infty} Y_k$$

where Z is the ℓ_1 space with standard basis $\{e_j, f_j : j = 0, 1, 2\dots\}$ and Y_k are the ℓ_1 spaces with standard bases $\{u_{k,i}, v_{k,i} : i = 1, 2, \dots, 5a_k^2\}$.

We construct inductively integers k_N, $N = 0, 1, 2\dots$, and elements $w_k \in Z$, $k = 1, 2, 3\dots$, in the following way. Set formally $k_0 := 0$ and $a_0 := 0$. Write for short $b_N := a_{k_N}$. Let $N \ge 1$ and suppose that the integer k_{N-1} and elements $w_1, \dots, w_{k_{N-1}}$ have already been defined. Let $Z_N := \mathrm{Span}\{e_j, f_j : j = 0, \dots, b_{N-1}\}$ and let $w_{k_{N-1}+1}, \dots, w_{k_N}$ be an ε_N^2-net in the closed unit ball of Z_N.

Using induction, we continue the construction in the above described way.

Now we define the operator $T \in \mathcal{L}(X)$ by:

$$Te_{a_k} := e_{a_k+1} + \tfrac{1}{a_k^2}\textstyle\sum_{i=1}^{a_k^2} u_{k,i}, \qquad Tf_{a_k} := f_{a_k+1} + \tfrac{1}{a_k^2}\textstyle\sum_{i=1}^{a_k^2} v_{k,i},$$

$$Te_{a_k+3a_k^2} := \varepsilon_N e_{a_k+3a_k^2+1}, \qquad Tf_{a_k+3a_k^2} := \varepsilon_N f_{a_k+3a_k^2+1}$$

if $(k_{N-1} < k \leq k_N)$,

$$Te_j := \varepsilon_N^{-1/a_k^2} e_{j+1}, \qquad Tf_j := \varepsilon_N^{-1/a_k^2} f_{j+1}$$

if $(k_{N-1} < k \leq k_N,\ a_k + 3a_k^2 < j \leq a_k + 4a_k^2)$,

$$Te_j := e_{j+1}, \qquad Tf_j := f_{j+1}$$

otherwise.

Thus T acts on the standard basis of Z as a pair of weighted shifts, up to the points of the form e_{a_k} and f_{a_k}.

Further, let

$$\begin{aligned} &Tu_{k,5a_k^2} := 0, &&Tv_{k,5a_k^2} := 0, \\ &Tu_{k,i} := u_{k,i+1}, &&Tv_{k,i} := v_{k,i+1} \qquad (1 \leq i < 2a_k^2 \text{ or } 2a_k^2 < i < 5a_k^2). \end{aligned}$$

It remains to define T on $\operatorname{Span}\{u_{k,2a_k^2}, v_{k,2a_k^2}\}$. Since $w_k \in Z_N$ for $k_{N-1} < k \leq k_N$, we have $w_k = \sum_{i=0}^{b_{N-1}}(\alpha_i^{(k)} e_i + \beta_i^{(k)} f_i)$ for some complex coefficients $\alpha_i^{(k)}, \beta_i^{(k)}$. For $i = 0, \ldots, b_{N-1}$ we have $T^{a_k - i} e_i = \mu_i e_{a_k}$ and $T^{a_k-i} f_i = \mu_i f_{a_k}$ for some $\mu_i \in \mathbb{C}$ satisfying $\left|\mu_i\right| \leq \varepsilon_N^{-1}$. Set $\alpha^{(k)} = \sum_{i=0}^{b_{N-1}} \mu_i \alpha_i^{(k)}$ and $\beta^{(k)} = \sum_{i=0}^{b_{N-1}} \mu_i \beta_i^{(k)}$. Without loss of generality we may assume that $\left|\alpha^{(k)}\right| \neq \left|\beta^{(k)}\right|$.

If $\left|\alpha^{(k)}\right| < \left|\beta^{(k)}\right|$ then set $Tu_{k,2a_k^2} := u_{k,2a_k^2+1}$ and $Tv_{k,2a_k^2} := -\frac{\alpha^{(k)}}{\beta^{(k)}} u_{k,2a_k^2+1}$. If $\left|\alpha^{(k)}\right| > \left|\beta^{(k)}\right|$ then set $Tv_{k,2a_k^2} := v_{k,2a_k^2+1}$ and $Tu_{k,2a_k^2} := -\frac{\beta^{(k)}}{\alpha^{(k)}} v_{k,2a_k^2+1}$. Note that in both cases we have $T(\alpha^{(k)} u_{k,2a_k^2} + \beta^{(k)} v_{k,2a_k^2}) = 0$.

Let $Y = \bigoplus_{k=1}^{\infty} Y_k$. Denote by P_Z, P_Y, P_{Z_N} and P_{Y_k} the natural projections onto the corresponding subspace of X.

It is easy to check that $\|T\| \leq 2$. Note also that for each $k \in \mathbb{N}$, we have the identities $T^{a_k - a_{k-1}} e_{a_{k-1}} = e_{a_k}$ and $T^{a_k - a_{k-1}} f_{a_{k-1}} = f_{a_k}$.

We prove that

$$\max\{\|T^n e_0\|, \|T^n f_0\|\} \geq 1$$

for all $n = 0, 1, 2 \ldots$, and for each $x \in X$ and $\varepsilon > 0$ there is a $j \in \mathbb{N}$ such that $\|T^j x\| < \varepsilon$. As in Example 12.3, this gives automatically that T is not orbit reflexive.

To prove the first statement, let $n \in \mathbb{N}$. If $n \notin \bigcup_{k=1}^{\infty}\{a_k + 3a_k^2 + 1, \ldots, a_k + 4a_k^2\}$ then $P_Z T^n e_0 = e_n$, and so $\max\{\|T^n e_0\|, \|T^n f_0\|\} \geq \|P_Z T^n e_0\| = 1$.

Let $a_k + 3a_k^2 < n \leq a_k + 4a_k^2$ for some k. Recall that $w_k = \sum_{i=0}^{b_{N-1}}(\alpha_i^{(k)} e_i + \beta_i^{(k)} f_i)$, $\alpha^{(k)} = \sum_{i=0}^{b_{N-1}} \mu_i \alpha_i^{(k)}$ and $\beta^{(k)} = \sum_{i=0}^{b_{N-1}} \mu_i \beta_i^{(k)}$, where $T^{a_k - i} e_i = \mu_i e_{a_k}$

and $T^{a_k-i}f_i=\mu_i f_{a_k}$. First suppose that $\left|\alpha^{(k)}\right|<\left|\beta^{(k)}\right|$ so that T is a shift on $u_{k,i}$. It is then easy to show that

$$P_{Y_k}T^n e_0=\frac{1}{a_k^2}\sum_{i=n-a_k}^{n-a_k+a_k^2-1}u_{k,i},$$

and so $\|T^n e_0\|\geq 1$. If $\left|\alpha^{(k)}\right|>\left|\beta^{(k)}\right|$, then we obtain in the same way that $\|T^n f_0\|\geq 1$. Hence $\max\{\|T^n e_0\|,\|T^n f_0\|\}\geq 1$ for all $n\in\mathbb{N}$.

To prove the second statement, suppose that $x\in X$ is of norm 1 and $0<\varepsilon<1$.

There exists $M\geq 2$ such that $\|(P_Z-P_{Z_M})x\|<\frac{\varepsilon}{18}$. There exists $N>M$ such that

$$\begin{aligned}\varepsilon_N^{1/2}&<\frac{\varepsilon\cdot\varepsilon_M}{9},\\ b_{N-1}\varepsilon_N&>\frac{18}{\varepsilon},\\ \sum_{k'=k_{N-1}+1}^{\infty}\|P_{Y_{k'}}x\|&<\frac{\varepsilon}{9},\\ \|P_{Z_{N+1}}x-P_{Z_N}x\|&<\varepsilon_N^2.\end{aligned}\tag{2}$$

Indeed, the first three conditions of (2) are satisfied for all N sufficiently large. Suppose on the contrary that $\|P_{Z_{N+1}}x-P_{Z_N}x\|\geq\varepsilon_N^2$ for all $N\geq N_0$. Then

$$1=\|x\|\geq\sum_{N=N_0}^{\infty}\|P_{Z_{N+1}}x-P_{Z_N}x\|\geq\sum_{N=N_0}^{\infty}\varepsilon_N^2=\infty,$$

a contradiction. Fix N with properties (2).

Find k, $k_{N-1}<k\leq k_N$ such that $\|P_{Z_N}x-w_k\|\leq\varepsilon_N^2$. Set $j=a_k+3a_k^2+1$. We have

$$\begin{aligned}\|T^j x\|\leq{}&\left\|\sum_{k'=1}^{k_{N-1}}T^jP_{Y_{k'}}x\right\|+\left\|\sum_{k'=k_{N-1}+1}^{\infty}T^jP_{Y_{k'}}x\right\|+\|P_ZT^jP_{Z_M}x\|\\ &+\|P_ZT^j(P_{Z_N}-P_{Z_M})x\|+\|P_ZT^j(P_{Z_{N+1}}-P_{Z_N})x\|+\|P_ZT^j(P_Z-P_{Z_{N+1}})x\|\\ &+\|P_YT^j(P_Z-P_{Z_{N+1}})x\|+\|P_YT^j(P_{Z_{N+1}}-P_{Z_N})x\|\\ &+\|P_YT^j(P_{Z_N}x-w_k)\|+\|P_YT^jw_k\|.\end{aligned}$$

Since $k > k_{N-1}$ and $j > a_k > 5a_{k_{N-1}}^2$, we have $\sum_{k'=1}^{k_{N-1}} T^j P_{Y_{k'}} x = 0$. For $k' > k_{N-1}$ we have $\|T^j|_{Y_{k'}}\| \le 1$, and so

$$\Big\|\sum_{k'=k_{N-1}+1}^{\infty} T^j P_{Y_{k'}} x\Big\| \le \Big\|\sum_{k'=k_{N-1}+1}^{\infty} P_{Y_{k'}} x\Big\| < \frac{\varepsilon}{9}.$$

The following four terms can be estimated by $\varepsilon/9$ similarly as in the Hilbert space case. We omit the details.

We have

$$\begin{aligned}
&\|P_Y T^j (P_Z - P_{Z_{N+1}})\| = \max\{\|P_Y T^j e_i\|, \|P_Y T^j f_i\| \colon\ i > b_N\}\\
&\le \max\{\|P_Z T^{j'} e_i\|, \|P_Z T^{j'} f_i\| \colon\ j' \le j, i > b_N\} \le \varepsilon_{N+1}^{-j/a_{k_{N+1}}^2} \le 2
\end{aligned}$$

and similarly

$$\|P_Y T^j P_{Z_{N+1}}\| \le \max\{\|P_Z T^{j'} e_i\|, \|P_Z T^{j'} f_i\| \colon\ j' \le j, i \le b_N\} \le \varepsilon_N^{-1}.$$

Thus

$$\begin{aligned}
\|P_Y T^j (P_Z - P_{Z_{N+1}})x\| &\le 2\|(P_Z - P_{Z_{N+1}})x\| \le \frac{\varepsilon}{9},\\
\|P_Y T^j (P_{Z_{N+1}} - P_{Z_N})x\| &\le \varepsilon_N^{-1}\|(P_{Z_{N+1}} - P_{Z_N})x\| < \varepsilon_N^{-1}\varepsilon_N^2 = \varepsilon_N < \frac{\varepsilon}{9}
\end{aligned}$$

and

$$\|P_Y T^j (P_{Z_N} x - w_k)\| \le \varepsilon_N^{-1}\|P_{Z_N} x - w_k\| \le \varepsilon_N^{-1}\varepsilon_N^2 < \frac{\varepsilon}{9}.$$

It remains to estimate $\|P_Y T^j w_k\|$. We have

$$\begin{aligned}
&\|P_Y T^j w_k\| = \|P_{Y_k} T^j w_k\|\\
&= \Big\|T^{3a_k^2} \sum_{i=0}^{b_{N-1}} \Big(\frac{\mu_i \alpha_i^{(k)}}{a_k^2} \sum_{i'=1}^{a_k^2} u_{k,i+i'} + \frac{\mu_i \beta_i^{(k)}}{a_k^2} \sum_{i'=1}^{a_k^2} v_{k,i+i'}\Big)\Big\|\\
&= \frac{1}{a_k^2}\Big\|T^{3a_k^2}\Big(\mu_0\alpha_0^{(k)} u_{k,1} + \mu_0\beta_0^{(k)} v_{k,1} + (\mu_0\alpha_0^{(k)} + \mu_1\alpha_1^{(k)}) u_{k,2}\\
&\quad + (\mu_0\beta_0^{(k)} + \mu_1\beta_1^{(k)}) v_{k,2} + \cdots + \sum_{s=b_{N-1}+1}^{a_k^2} (\alpha^{(k)} u_{k,s} + \beta^{(k)} v_{k,s}) + \cdots\\
&\quad \cdots + \mu_{b_{N-1}}\alpha_{b_{N-1}}^{(k)} u_{k,a_k^2+b_{N-1}} + \mu_{b_{N-1}}\beta_{b_{N-1}}^{(k)} v_{k,a_k^2+b_{N-1}}\Big)\Big\|\\
&\le \frac{1}{a_k^2}\cdot 2\varepsilon_N^{-1}(b_{N-1}+1)\|w_k\| \le \frac{2}{\varepsilon_N a_k} \le \frac{2}{\varepsilon_N b_{N-1}} < \frac{\varepsilon}{9}.
\end{aligned}$$

Hence $\|T^j x\| < \varepsilon$. This implies that T is not orbit reflexive.

We show now that T is reflexive. Suppose that an operator $A \in \mathcal{L}(X)$ leaves invariant all the closed subspaces which are invariant for T. Without loss of generality we may assume that $\|A\| = 1$. We have to show that A is a limit of polynomials of T in the strong operator topology.

Let $k \in \mathbb{N}$ and let $y \in Y_k$, $y \neq 0$. Let s satisfy $T^s y \neq 0$ and $T^{s+1} y = 0$. Since $\mathrm{Span}\{y, Ty, \dots, T^s y\}$ is invariant for A, there are numbers $\lambda_0, \dots, \lambda_s \in \mathbb{C}$ such that $Ay = \sum_{i=0}^{s} \lambda_i T^i y$.

Fix any natural numbers $l > k$ such that $\left|\alpha^{(l)}\right| < \left|\beta^{(l)}\right|$ (so that T is a shift on $u_{l,i}$; such a number certainly exists) and consider the spaces invariant for T generated by the vectors $u_{l,1}$ and $y + u_{l,1}$, respectively. Since these subspaces are invariant for A, there are complex numbers ξ_i and η_i such that

$$Au_{l,1} = \sum_{i=0}^{5a_l^2-1} \xi_i T^i u_{l,1}$$

and

$$A(y + u_{l,1}) = \sum_{i=0}^{5a_l^2-1} \eta_i T^i (y + u_{l,1}).$$

Thus

$$\sum_{i=0}^{s} \eta_i T^i y + \sum_{i=0}^{s} \eta_i T^i u_{l,1} + \sum_{i=s+1}^{5a_l^2-1} \eta_i T^i u_{l,1}$$
$$= \sum_{i=0}^{s} \lambda_i T^i y + \sum_{i=0}^{s} \xi_i T^i u_{l,1} + \sum_{i=s+1}^{5a_l^2-1} \xi_i T^i u_{l,1}.$$

Since the vectors $T^i y$, $0 \le i \le s$, and $T^i u_{l,1}$, $0 \le i \le 5a_l^2 - 1$, are linearly independent, we have $\lambda_i = \xi_i = \eta_i$, $0 \le i \le s$, and $Ay = \sum_{i=0}^{5a_k^2-1} \xi_i T^i y$. Note that this equality does not depend on $y \in Y_k$. Note also that $\sum_{i=0}^{5a_k^2-1} |\xi_i| \le \left\|\sum_{i=0}^{5a_k^2-1} \xi_i T^i u_{l,1}\right\| \le \|Au_{l,1}\| \le \|A\| = 1$. Moreover, if $Ay = \sum_{i=0}^{5a_l^2-1} \xi_i' T^j y$ for all $y \in Y_l$ then $\xi_i = \xi_i'$, $0 \le i \le 5a_k^2 - 1$.

Thus there are numbers $\xi_0, \xi_1, \dots$ such that $\sum_{i=0}^{\infty} |\xi_i| \le 1$ and $Ay = \sum_{i=0}^{5a_j^2-1} \xi_i T^i y$ for all $j \in \mathbb{N}$ and $y \in Y_j$.

For $k \in \mathbb{N}$ let $p_k(z) := \sum_{i=0}^{5a_k^2-1} \xi_i z^i$. Then $\|p_k(T)|_Y\| \le 1$, and so we have $Ay = \lim_{k\to\infty} p_k(T) y$ for all $y \in Y$.

Let $E := \mathrm{Span}\{e_j : j \ge 0\}$ and $F := \mathrm{Span}\{f_j : j \ge 0\}$. Let $x_1, \dots, x_n \in E$ and $x_{n+1}, \dots, x_m \in F$ be unit vectors, $q \in \mathbb{N}$ and let $0 < \varepsilon < 1$. It is sufficient to show that there is a $k \geq q$ such that $\|p_k(T)x_i - Ax_i\| < \varepsilon$, $i = 1, \dots, m$. This will show that A belongs to the closure of polynomials of T in the strong operator topology.

As above, it is possible to show that there is an N such that

$$\tag{3} \begin{gathered} \varepsilon_N < \frac{\varepsilon}{8}, \\ \sum_{j=k_N+1}^{k_{N+1}} \left|\xi_j\right| < \varepsilon_N^2, \\ \|(I - P_{Z_{N+1}})x_i\| < \frac{\varepsilon}{16} \qquad (i = 1, \dots, m), \\ \|(P_{Z_{N+1}} - P_{Z_N})x_i\| < \varepsilon_N^2 \qquad (i = 1, \dots, m), \\ \left\|\Big(I - P_{Z_N} - \sum_{k'=1}^{k_N} P_{Y_{k'}}\Big) Ax_i\right\| < \frac{\varepsilon}{4} \qquad (i = 1, \dots, m). \end{gathered}$$

Set $k = k_N$. Fix $i \in \{1, \dots, n\}$ (for $n + 1 \leq i \leq m$ the proof will be similar). Let $x_i = \sum_{j=j_0}^{\infty} \gamma_j e_j$ with $\gamma_{j_0} \neq 0$. Clearly $j_0 \leq b_{N-1}$. Let $s = 5a_k^2 + a_k - j_0$. Let Q be the natural projection onto the space $\mathrm{Span}\{e_0, \dots, e_{5a_k^2+a_k}, Y_{k'} \quad (k' \leq k), v_{k+1,1}, \dots, v_{k+1,s+1}\}$.

Consider the vectors x_i, $v_{k+1,1}$ and $x_i + v_{k+1,1}$. We have

$$QAv_{k+1,1} = \sum_{j=0}^{s} \xi_j T^j v_{k+1,1}$$

and there are complex numbers ν_j, η_j such that

$$QAx_i = Q\sum_{j=0}^{s} \nu_j T^j x_i$$

and

$$QA(x_i + v_{k+1,1}) = Q\sum_{j=0}^{s} \eta_j T^j (x_i + v_{k+1,1}).$$

As above, we have $\nu_j = \xi_j = \eta_j$, $0 \leq j \leq s$. So $QAx_i = Q\sum_{j=0}^{s} \xi_j T^j x_i$.

We have

$$\|(A - p_k(T))x_i\| \leq \|(I - Q)Ax_i\| + \|Q(A - p_k(T))x_i\| + \|(I - Q)p_k(T)x_i\|.$$

By (3), $\|(I-Q)Ax_i\| < \varepsilon/4$ and

$$\|Q(A-p_k(T))x_i\| = \left\|Q\sum_{j=5a_k^2}^{s}\xi_j T^j x_i\right\| \le \left\|\sum_{j=5a_k^2}^{s}\xi_j T^j x_i\right\|$$

$$\le \sum_{j=5a_k^2}^{s}\left|\xi_j\right| \cdot \max\{\|T^j\|\colon\ 5a_k^2 \le j \le s\} \le \varepsilon_N^2 \cdot 2\varepsilon_N^{-1} = 2\varepsilon_N < \varepsilon/4.$$

Furthermore, since $(I-Q)p_k(T)P_{Z_N}x_i = 0$, we have

$$\begin{aligned}&\|(I-Q)p_k(T)x_i\|\\ &\le \|(I-Q)p_k(T)(I-P_{Z_{N+1}})x_i\| + \|(I-Q)p_k(T)(P_{Z_{N+1}}-P_{Z_N})x_i\|\\ &\le \|p_k(T)(I-P_{Z_{N+1}})x_i\| + \|p_k(T)(P_{Z_{N+1}}-P_{Z_N})x_i\|,\end{aligned}$$

where

$$\|p_k(T)(I-P_{Z_{N+1}})x_i\| = \left\|\sum_{j=0}^{5a_k^2-1}\xi_j T^j(I-P_{Z_{N+1}})x_i\right\|$$

$$\le \Big(\sum_{j=0}^{5a_k^2-1}\left|\xi_j\right|\Big)\max\{\|T^j(I-P_{Z_{N+1}})\|\colon\ 0 \le j \le 5a_k^2-1\} \cdot \|(I-P_{Z_{N+1}})x_i\|$$

$$\le \frac{4\varepsilon}{16} = \frac{\varepsilon}{4}$$

and

$$\begin{aligned}&\|p_k(T)(P_{Z_{N+1}}-P_{Z_N})x_i\| \le \|p_k(T)\| \cdot \|(P_{Z_{N+1}}-P_{Z_N})x_i\|\\ &\le \max\{\|T^j\|\colon\ 0 \le j \le 5a_k^2-1\} \cdot \varepsilon_N^2 \le 2\varepsilon_N^{-1}\varepsilon_N^2 = 2\varepsilon_N < \varepsilon/4.\end{aligned}$$

Hence $\|(A-p_k(T))x_i\| < \varepsilon$ for each i, $1 \le i \le n$, and similarly, for $n+1 \le i \le m$. This implies that A is a limit of polynomials of T in the strong operator topology and hence, T is reflexive.

It was suggested in [44] that it's possible to extend the definition of orbit reflexivity from Banach spaces to arbitrary topological spaces, passing from operator theory to topological dynamics. Let X be a topological space and let $f\colon\ X \to X$ be a continuous map. Then f is called orbit reflexive if the only maps that leave invariant all the closed subsets invariant under f, are those in the pointwise closure of $\mathrm{Orb}(f) = \{f^n\colon\ n \in \mathbb{N}_0\}$.

In such a setting, it is very easy to find a map which is not orbit reflexive. For instance let $X := \{a, b, c\}$ with a discrete topology and let f be defined as $a \mapsto b, b \mapsto c, c \mapsto a$. Then $f^2 = f^{-1}$ and $f^3 = Id$, so there is no nontrivial invariant subset. But g defined by $a \mapsto a, b \mapsto c, c \mapsto b$ is not in the pointwise closure of $\mathrm{Orb}(f)$. Hence f is not orbit reflexive. Analogously if $X = \mathbb{T}$ and

$\lambda \in \mathbb{R} \setminus \mathbb{Q}$ then the map $x \mapsto e^{i\lambda} x$ has no nontrivial closed invariant subset and is not orbit reflexive.

We finish the chapter by posing several open questions related to orbit reflexivity:

(i) Does the orbit reflexivity of T imply orbit reflexivity of T^n? Or perhaps conversely?

(ii) Let T be orbit reflexive and $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Is then λT orbit reflexive?

(iii) Is any Banach space contraction orbit reflexive? Is every mean ergodic operator orbit reflexive?

(iv) Is "identity plus a quasinilpotent" always orbit reflexive? Does there exist space where all operators are orbit reflexive?

(v) Is there a Hilbert space operator, which is not orbit reflexive, but is reflexive? In particular, is the operator from Example 12.3 reflexive?

(vi) Are there semigroups which are not orbit reflexive?

V. References

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