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DOCTORAL THESIS

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Structural Graph Theory

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I would like to thank my advisor Martin Loebl for his support throughout this work. I also thank my coauthors Martín Matamala, Maya Stein, José Aliste-Prieto, Ioan Todinca, Mathieu Chapelle, Natacha Astromujoff.

This work it is dedicated to my wife Valeria.

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Strukturální teorie grafů.

Autor: José Zamora

Katedra: Katedra aplikované matematiky

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Abstrakt: V práci studujeme čtyři problémy ze strukturální teorie grafů.

Nejprve se zabýváme strukturou grafů které mají nikde nenulový 5-tok. Podáme charakterizaci takových grafů pomocí existence $(1, 2)-$faktorů.

Ve druhé části zavedeme nový typ dekorace vrcholů grafu, kterému říkáme aditivní barvení. Aditivní barvení je injektivní barvení s omezeními danými grafem. Studujeme strukturu grafů které mají tuto dekoraci, a související algoritmické otázky.

Ve třetí časti studujeme hypotézu kterou formuloval před asi dvaceti lety R. Stanley: je pravda, že U-polynom rozlišuje neizomorfní stromy? Dokážeme tuto hypotézu pro stromy- housenky bez vrcholů stupně dva. O tento výsledek se v minulých letech snažila řada vědců, například S. Noble.

Ve čtvrté části studujeme strukturu nekonečných grafů které mají uplný graf jako minor nebo topologický minor.

Klíčová slova: graf, nikde nenulový tok, faktor grafu, barvení grafů, izomorfismus grafů, strom, U-polynom, minor, topologický minor.

Title: Structural Graph Theory

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Abstract: In this thesis we consider four different problems in structural graph theory:

We start studying the structure of graphs having a nowhere-zero 5-flow. We give a characterization of the graphs that have a nowhere-zero 5-flow in terms of the existence of a $(1,2)$-factor.

For the second problem we introduce a new type of labeling of graphs that we call additive coloring. This coloring is a variation of the injective coloring introduced in [21]. Indeed, it is an injective coloring with arithmetic restrictions determined by the graph. We study the properties and the structure of graphs admitting this type of labeling. Moreover, we study the computational complexity of the problem of computing this labeling for a graph with a fixed number of colors.

In the third problem we study how the structure of caterpillars is encoded by the chromatic symmetric function or, equivalently, the U-polynomial. In [40] Stanley conjectured that the symmetric chromatic polynomial distinguishes non-isomorphic trees. In this thesis we prove that the conjecture is true for proper caterpillars (caterpillars without vertices of degree 2).

Finally, we study the structure of infinite graphs having a complete graph as a minor or as a topological minor. It is known that bounds on the degree of the vertices is not enough to ensure the existence of a complete graph minor in an infinite graph. So, we define a new notion of degree for the ends of an infinite graph. Then, we prove that a condition of minimum degree for the vertices and the ends of the graph ensure the existence of a complete graph as a minor and as a topological minor.

Keywords: nowhere-zero flows, graph labeling, graph polynomial, infinite graphs.

Contents

1. Introduction.

In this work we consider the following problems in structural graph theory:

1. The structure of graphs having a nowhere-zero 5-flow,

2. The structure of graphs having a labeling with arithmetics restrictions.

3. How the structure of proper caterpillars is uniquely encoded in the U-polynomial.

4. The structure of infinite graphs having a complete minor or complete topological minor.

A graph labeling is a function from its vertices or edges to a set of numbers. Graph labeling s were introduced in the late 1960s. In the following years dozens of graph labeling techniques have been studied and have been applied to different problems. For a dynamic survey see [18]. First two parts of my thesis deal with graph labelings.

In chapter 2 we study edge labeling that is called a flow. In this case the edges are oriented and the property of the labeling is flow conservation, i.e., for every vertex the total flow entering is the same as total flow going out. Moreover, if we add the condition that the possible values of the labeling are in the set $\{1, 2, \ldots, k\}$, we get a nowhere-zero k-flow.

The concept of nowhere-zero flow was introduced by Tutte in [48] as a generalization of the face colorings of planar graphs. In this work, Tutte conjectured that every bridgeless graph has a nowhere-zero 5-flow.

This conjecture is still open and the best known result is that every bridgeless graph has a nowhere-zero 6-flow. This result was proved by Seymour in [38].

It is known that it is enough to prove the conjecture for cubic graphs. Indeed, for nowhere-zero 4-flow Jaeger proved in [26] that a cubic graph has a nowhere-zero 4-flow if and only if it has an edge 3-coloring .

In this work we extend this result for nowhere-zero 5-flow. Concretely, we prove that a bridgeless graph has a nowhere-zero 5-flow if there exists a $(1, 2)$-factor satisfying a parity condition.

In chapter 3 we introduce a new type of labeling . Given a graph G and a vertex labeling $\varphi : V(G) \to \mathbb{N}$ we define the induced labeling $\hat{\varphi} : E(G) \to \mathbb{N}$ such that for each edge $uv \in E(G), \hat{\varphi}(uv) = |\varphi(u) - \varphi(v)|$. We define an *additive labeling* as a vertex labeling such that its induced edge labeling is a (proper) edge coloring of the graph. This labeling is inspired by graceful labeling [13] and injective colorings [21].

In a graceful labeling the induced edge labeling must be a bijective function. An injective coloring of a graph is a vertex labeling such that two vertices sharing a common neighbor get different labels.

As usual, we define *additive chromatic index* of a graph G as the smallest integer k such that an additive coloring of a given graph G exists with colors in the set

$\{1, \ldots, k\}$.

We start by proving several upper bounds for the additive chromatic index. On one hand, we prove linear bounds for trees and bipartite graphs. We also prove a quadratic bound in terms of the maximum degree.

On the other hand we study the computational complexity of the problem of deciding whether a graph has additive chromatic index at most some fixed constant k. We prove that this problem is polynomial for graphs with bounded treewidth, but it is NP-complete for $k = 4$, even when restricted to cubic graphs.

In Chapter 4 we study a problem on graph reconstruction. In this kind of problem we will like to know how much information we need to recognize or encode a graph. In particular, we study the U-polynomial introduced in [34]. It is known that there exist two non-isomorphic graphs with the same U-polynomial, but it is an open problem to know if a tree is determined by its U-polynomial. This question is equivalent to long-standing conjecture by Stanley [40], whether the symmetric chromatic polynomial distinguishes non-isomorphic trees. The equivalence was obtained by B. Thatte in [47].

We study the problem for a particular family of trees, the *proper caterpillars*. A caterpillar is a tree such that if we delete its leaves we obtain a path. A caterpillar is *proper* if it has not vertices of degree 2. The key idea is to use the linear structure of caterpillars and connect the problem with a problem in numerical sequences.

Given a caterpillar we associate to it a sequence and a restricted U-polynomial that we call the $\mathcal{L}$-polynomial. This polynomial is equivalent to the notion of the multiset of partition coarsenings defined by Billera, Thomas and van Willigenburg in [2] for sequences. Moreover, they characterize all the sequences with the same multiset of partition coarsenings . Then, we use this characterization to prove that two proper caterpillars with the same $\mathcal{L}$-polynomial must have different U-polynomial. This proves Stanley's conjecture for the class of proper caterpillars.

In Chapter 5 we study an extremal problem in infinite graphs. It is well-known that in finite graphs, large complete minors/topological minors can be forced by assuming a large average degree. More precisely, there is a function f such that every graph of average degree at least $f(k)$ has a K_k-minor/topological minor. These results were proved by Kostochka [29] and by Bollobás and Thomason [4].

Our aim is to extend this fact to infinite graphs. For this, we generalize the notion of the *relative degree* of and end, which had been previously introduced by M. Stein in [42] for locally finite graphs.

In order to give an idea about the relative degree, we need some definitions:

Let G be a locally finite graph and H be a subgraph. The edge-boundary of a subgraph H is the set $E(H, G - H)$. The vertex-boundary of H is the set of all vertices in H that have neighbors in $G - H$. The idea of the relative degree of an end is to calculate the limit of the ratios between the cardinal of edge-boundary and the cardinal of the vertex boundary for certain sequences of subgraphs H_i of G such that these H_i in some sense converge to the end.

In this work, we show that large minimum relative degree at the ends and large minimum degree at the vertices imply the existence of large complete (topological) minors in infinite graphs with countably many ends.

Our main tool is a theorem that permits to find a finite subgraph of large average degree in any (infinite) graph of large minimum (relative) degree at all ends and vertices, and only countably many ends. We believe this result is interesting on its own, and it can be used to extend other finite graphs results to infinite graphs with countably many ends.

2. Nowhere-zero flows and $(1, 2)$-factors

2.1 Introduction

Let $G = (V, E)$ be a bridgeless undirected graph. We say that G has a nowhere-zero k-flow if there exists an orientation $H = (V, A)$ of G and a function $\varphi : A \to \mathbb{Z}$ such that for all $a \in A$, $0 < |\varphi(a)| < k$ and for every $v \in V$, $\sum_{a \in v^+} \varphi(a) = \sum_{a \in v^-} \varphi(a)$, where v^+, v^- are the sets of outgoing and ingoing arcs incident with v, respectively.

The concept of nowhere-zero k-flow was introduced by Tutte [48] as a refinement and a generalization for face coloring problems in planar graphs. The four-color Theorem is equivalent to say that every bridgeless planar graph has a nowhere-zero 4-flow. This result can not be extended to arbitrary bridgeless graph since the Petersen graph has no a nowhere-zero 4-flow. However, in [49] Tutte formulated his famous 5-flow conjecture which still is open:

Conjecture 2.1.1. *Every bridgeless graph admits a nowhere-zero 5-flow.*

Work about Conjecture 2.1.1 have focused on properties of a minimal counterexample (see [28],[27],[9]) and into the study of structural properties of graphs having a nowhere-zero 5-flow (see [41]). In this work we follow the last approach.

The best approximation for Conjecture 2.1.1 is a result of Seymour [38] where he proved that every bridgeless graph has a nowhere-zero 6-flow.

The motivation for studying this conjecture in cubic graphs has two sources. First, it is known that for Conjecture 2.1.1 to be true, it is enough to prove it for cubic graphs. Second, for cubic graphs there exist well known characterizations of the existence of nowhere-zero k-flow for $k = 3, 4$. These characterizations provide some intuition about the structural properties of cubic graphs admitting nowhere-zero $k-$flow. More precisely, Tutte gave the following characterization for nowhere-zero 3-flow:

Theorem 2.1.2. *[48] Let G be a cubic graph. G admits a nowhere-zero 3-flow if and only if G is bipartite.*

This result can be seen as a parity condition that all cycles of a cubic graph must satisfy to admit a nowhere-zero $3-$flow. The parity condition being that each cycle has even length.

For nowhere-zero $4-$flow we only need to check this parity property in the complement of a perfect matching (see, e.g.,[26]).

Theorem 2.1.3. *Let $G = (V, E)$ be a cubic graph. G has a nowhere-zero 4-flow if and only if G has a perfect matching M such that all cycles in $G - M$ have even length.*

Our main result is somehow similar to Theorem 2.1.3. It characterizes the existence of nowhere-zero $5-$flows in a bridgeless cubic graph G in terms of a *parity* condition that must satisfy a family of cycles of G. In our case, the family of cycles $\mathcal{C}$ to be checked is a *basis of cycles* associated to a spanning tree $T = (V, E')$. That is, $\mathcal{C} = \{C_e : e \notin E'\}$, where C_e is the cycle in $T \cup \{e\}$. The role of the perfect matching in Theorem 2.1.3 is played by a subset F of edges such that for each vertex v of G its degree in F, denoted by $d_F(v)$, is 1 or 2. These sets of edges are called $(1,2)$-factors of G. Our main result is the following:

Theorem 2.1.4. *For an undirected cubic graph G the following statements are equivalents.*

1. *The graph G admits a nowhere-zero 5-flow.*
2. *There exists a $(1,2)$-factor F such that, for every cycle C in a basis of cycles associated to a tree, the cardinality of the set*

$$C_F := \{uv \in E(C) : uv \in F \text{ or } d_F(u) = d_F(v)\}$$

is even.

In Figure 2.1 we show a cycle C where edges in C_F are pointed out by an x.

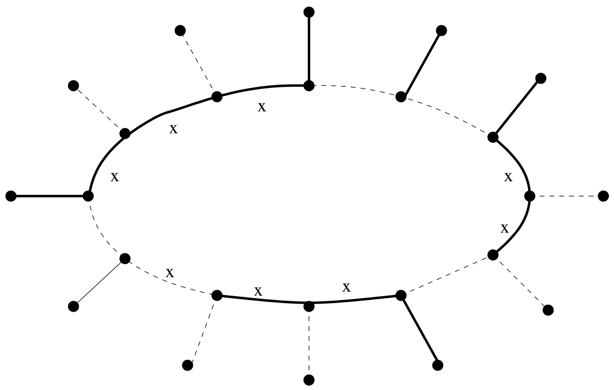

Figure 2.1: Example of C_F. Solid edges are in F. Dashed edges are in $E \setminus F$. Edges in C_F are indicated with a x.

For an Abelian group Γ we have the following analogous definition. A graph G admits a nowhere-zero $\Gamma-$flow if there is an orientation $H = (V, A)$ of G and a function $\varphi : A \to \Gamma$ satisfying the following:

1. For all $a \in A, \varphi(a) \neq 0$.
2. For every $v \in V$, $\sum_{a \in v^+} \varphi(a) = \sum_{a \in v^-} \varphi(a)$.

Arithmetic operations are carried out in Γ. Usually, a function φ satisfying previous conditions is called a nowhere-zero $\Gamma-$flow of G.

It is known by ([49]) that a graph admits a nowhere-zero k-flow if and only if it admits a nowhere-zero Γ-flow, where Γ is any Abelian group of cardinality k. Hence, (i) in Theorem 2.1.4 can be replaced by

(i') The graph G admits a nowhere-zero $\mathbb{Z}_5$-flow.

In the remainder of this text we use $\mathbb{Z}_5$-flow instead 5-flows.

2.2 Preliminaries

Let F be any $(1,2)-$factor of a cubic graph G. Let v be a vertex of G and let u, w, r its three neighbors. We say that the edges uv and vw are $F-$related if $uv, vw \in F$ or $uv, vw \notin F$. We define the $F-$*parity* of the tuple (u, v, w), denoted by $F(u, v, w)$, by 2 when uv and wv are $F-$related. It is 1 when uv and vr are $F-$related; It is 3 when vw and vr are $F-$related. (see Figure 2.2).

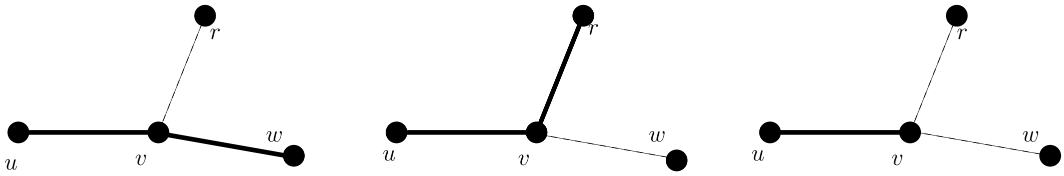

Figure 2.2: Definition of $F(u, v, w)$. Solid edges are in F. Dashed edges are in $E \setminus F$. In the first diagram uv and vw are $F-$related; in the second diagram uv and vr are $F-$related; in the third diagram vr and vw are $F-$related.

We extend the definition of $F-$parity to cycles. Let $C = (u_0, \dots, u_{n-1}, u_n = u_0)$ a cycle of length $n \geq 3$ in G. We define the $F-$parity of C, denoted by $F(C)$, as:

$$F(C) := \sum_{i=0}^{n-2} F(u_i, u_{i+1}, u_{i+2}) + F(u_{n-1}, u_0, u_1) \mod 4$$

Now we prove a technical lemma that will give us the tools to prove Theorem 2.1.4

Lemma 2.2.1. *Let G be a bridgeless cubic graph. For each $(1,2)-$factor F and each cycle C of G we have the following.*

$$F(C) = 2|C_F| \mod 4,$$

where

$$C_F := \{uv \in E(C) : uv \in F \textit{ or } d_F(u) = d_F(v)\}.$$

Proof. Let $C = (u_0, \dots, u_{n-1}, u_0)$. The relation is clear when $E(C) \subseteq F$ since, in this case, $F(C) = 2|E(C)|$. Let Q be a non trivial (at least one edge) connected component of $C \setminus F$. Let us assume that $Q = (u_1, \dots, u_i)$, $i \geq 2$. Let $\alpha(Q)$ be defined by $\alpha(Q) := \sum_{j=1}^{i} F(u_{j-1}, u_j, u_{j+1})$. We prove that $\alpha(Q) = 2|E(Q) \cap C_F| + 2 \mod 4$. Let us consider first the case $i \geq 3$. It is easy to see that $F(u_0, u_1, u_2) + F(u_{i-1}, u_i, u_{i+1}) = 0 \mod 4$ when $u_1u_2, u_{i-1}u_i \in C_F$ or $u_1u_2, u_{i-1}u_i \notin C_F$. Moreover, $F(u_0, u_1, u_2) + F(u_{i-1}, u_i, u_{i+1}) = 2 \mod 4$ when $|\{u_1u_2, u_{i-1}u_i\} \cap C_F| = 1$. Hence,

$$\begin{aligned}
\alpha(Q) &= \sum_{j=1}^{i} F(u_{j-1}, u_j, u_{j+1}) \\
&= F(u_0, u_1, u_2) + 2(i-3) + 2 + F(u_{i-1}, u_i, u_{i+1}) \\
&= 2|E(Q) \cap C_F| + 2 \mod 4.
\end{aligned}$$

For $i = 2$ we have that $F(u_0, u_1, u_2) + F(u_1, u_2, u_3) = 4$ if $u_1 u_2 \in C_F$, and it is 2 otherwise.

Let Q' be a non trivial connected component of $C \cap F$. Let us assume that $Q' = (u_1, \ldots, u_i)$, $i \geq 2$. Let $\alpha(Q')$ be defined as follows. For $i \geq 3$, we set $\alpha(Q') = \sum_{j=2}^{i-1} F(u_{j-1}, u_j, u_{j+1})$. For $i = 2$, we set $\alpha(Q') = 0$. Clearly, with this definition $\alpha(Q') = 2|E(Q') \cap C_F| + 2$, when $i = 2$. When $i \geq 3$ we have

$$\begin{aligned} \sum_{j=2}^{i-1} F(u_{j-1}, u_j, u_{j+1}) &= 2(i-2) \\ &= 2|E(Q') \cap C_F| + 2 \mod 4 \end{aligned}$$

We can now compute $F(C)$ in terms of $\alpha(Q)$, where Q ranges over non trivial connected components of $C \setminus F$ and over non trivial connected components of $C \cap F$. As for each of them $\alpha(Q) = 2|E(Q) \cap C_F| + 2$, and the connected components of $C \setminus F$ and $C \cap F$ alternate, we get the conclusion.

□

When F is an $(1,2)-$factor and C is a cycle such that $F(C) = 0 \mod 4$, we say that F *reduces* C or that C is reduced by F. By using Lemma 2.2.1, our result can be stated as follows: A cubic graph G has a nowhere-zero $\mathbb{Z}_5-$flow if and only if there is a $(1,2)-$factor F reducing every cycle in a basis of cycles of G associated with a tree.

2.3 The proof

We split the proof of Theorem 2.1.4 in two parts associated with the forward and the backward implications.

Proposition 2.3.1. *Let $G = (V, E)$ be an undirected cubic graph. If G admits a nowhere-zero $\mathbb{Z}_5$-flow, then there exists a $(1,2)$-factor F reducing every cycle of G.*

Proof. Let us assume that G has a nowhere-zero $\mathbb{Z}_5$-flow associated with an orientation H and a function φ. Let F be defined as follows.

$$F = F_\varphi := \{uv \in E : \varphi(u, v) \in \{1, 4\} \text{ or } \varphi(v, u) \in \{1, 4\}\}$$

In $\mathbb{Z}_5$, the equation $x + y + z = 0$ has exactly 5 distinct solutions given by $\{\{a, a, 3a\} : a \in \mathbb{Z}_5\}$. Then, for each vertex v, at least one arc incident with v has flow in the set $\{1, 4\}$ and at least one arc incident with v has flow, in the set $\{2, 3\}$. Therefore, F is a $(1,2)-$factor of G.

Let $C = (u_0, \ldots, u_{n-1}, u_0)$ be a cycle of length n in G. To ease the notation, let us define $a_i = (u_i, u_{i+1})$ for $i = 0, \ldots, n-1$. w.l.og we can assume that $a_i \in A$, for $i = 0, \ldots, n-1$. To see this, let $a \in A$ and let $A' = A \setminus \{a\} \cup \{-a\}$ and $\varphi' : A' \to \mathbb{Z}_5$, where $\varphi'(b) = \varphi(b)$, if $b \neq -a$ and $\varphi'(-a) = -\varphi(a)$. Then, $F_{\varphi'} = F$.

Hence, by modifying the orientation assigned for H to any subgraph of G and, accordingly, the associated flow the set F remains the same.

From the choice of F and the definition of $F(u,v,w)$, the reader can chech that $\varphi(v,w) = \varphi(u,v)2^{F(u,v,w)}$, for each path (u,v,w) in H. Hence, for each $i = 0,\dots,n-2$ it holds:

$$\varphi(a_{i+1}) = \varphi(a_i)2^{F(u_i,u_{i+1},u_{i+1})} \tag{2.1}$$

Moreover, $\varphi(a_0) = \varphi(a_{n-1})2^{F(u_{n-1},u_0,u_1)}$. We now prove that $\varphi(a_0) = \varphi(a_0)2^{F(C)}$, which implies that $F(C) = 0 \mod 4$, that is, F reduces C. By starting with a_0, and by iteratively applying 2.1, we have

$$\begin{aligned}
\varphi(a_0) &= \varphi(a_0)\prod_{i=0}^{n-2} 2^{F(u_i,u_{i+1},u_{i+2})}2^{F(u_{n-1},u_0,u_1)} \\
&= \varphi(a_0)2^{\sum_{i=0}^{n-2} F(u_i,u_{i+1},u_{i+2})+F(u_{n-1},u_0,u_1)} \\
&= \varphi(a_0)2^{F(C)}.
\end{aligned}$$

□

Theorem 2.3.2. *Let $G = (V,E)$ be an undirected cubic graph. If there exists a $(1,2)$-factor F reducing every cycle in a basis of cycles associated with a tree T, then G admits a nowhere-zero $\mathbb{Z}_5$-flow*

Proof. Let $\hat{G} = (V,D)$ be the directed graph obtained from G by replacing each edge uv by two arcs (u,v) and (v,u). In $\hat{G}$ the outdegree (resp. indegree) of each vertex is three. A $F-$*valid local solution* for a vertex $v \in \hat{G}$ is any mapping f from v^+ to $\mathbb{Z}_5 \setminus \{0\}$ such that

$$\sum_{a\in v^+} f(a) = 0 \mod 5.$$

and such that $f(v,u) = f(v,w)$ if and only if vu and vw are $F-$related. For each vertex v, there are exactly four $F-$*valid local solutions*, as we show in Figure 2.3

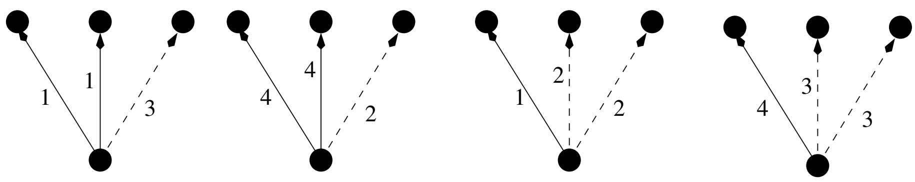

Figure 2.3: The 4 F-valid local solution

Let us root the tree T at a vertex r. We define f_r as any $F-$valid local solution for r. We assign $F-$valid local solutions to the remaining vertices so as, for each edge $uv \in E(T)$, $f_v(v,u) = -f_u(u,v)$. These $F-$valid local solutions are completely determined by F, T and f_r.

Let $\varphi : D \to \mathbb{Z}_5 \setminus \{0\}$ be defined by

$$\varphi(v,u) = f_v(v,u).$$

Clearly, for each arc $a \in D$, $\varphi(a) \neq 0$, and for each vertex v we have that

$$\sum_{a \in v^+} \varphi(a) = \sum_{a \in v^+} f_v(a) = 0 \mod 5.$$

Moreover, for every edge $uv \in E(T)$ it holds that $\varphi(u,v) = -\varphi(v,u)$, and for each edge $vw \in E$

$$\varphi(v,w) = \varphi(u,v)2^{F(u,v,w)}. \tag{2.2}$$

We now prove that equality $\varphi(w,v) = -\varphi(v,w)$ holds for every edge $vw \notin E(T)$. Let $w = u_0, u_1, \ldots, u_n = v$ be the path in T between w and v. By using equation 2.2, it can be proved that

$$\varphi(w,v) = \varphi(v,u_{n-1})2^{F(v,u_{n-1},u_{n-2})+\cdots+F(u_1,w,v)}.$$

Moreover, $\varphi(v,u_{n-1}) = -\varphi(u_{n-1},v)$ and $\varphi(v,w) = \varphi(u_{n-1},v)2^{F(u_{n-1},v,w)}$. Then,

$$\begin{aligned}\varphi(w,v) &= -\varphi(v,w)2^{F(w,v,u_{n-1})+F(v,u_{n-1},u_{n-2})+\cdots+F(u_1,w,v)} \\ &= -\varphi(v,w)2^{F(C)},\end{aligned}$$

where C is the cycle in $T \cup \{vw\}$. As F reduces C, we conclude that $\varphi(w,v) = -\varphi(v,w)$. By choosing any orientation $H = (V,A)$, it follows that the function φ restricted to A is a nowhere-zero $\mathbb{Z}_5-$flow of G.

□

The proof of Theorem 2.1.4 is now easy. The forward direction is included in Proposition 2.3.1, while Theorem 2.3.2 corresponds to the backward implication.

Notice that for a cubic graphs G, if F is a $(1,2)$-factor, then $F^c = E(G) \setminus F$ is a $(1,2)$-factor too. By Lemma 2.2.1, $2|C_F| = F(C) = 2|C_{F^c}|$ we get the following corollary:

Corollary 2.3.3. *Let G be a cubic graph. Then the $(1,2)$-factor F reduces all cycles of G if and only if F^c reduces all cycles of G.*

It is worth to notice the following consequence of Theorems 2.3.1 and 2.3.2.

Corollary 2.3.4. *To decide whether a given $(1,2)-$factor F reduces all cycles of G can be done in polynomial time.*

Proof. Given F and a spanning tree $T = (V,E')$ we first attempt to construct a nowhere-zero $\mathbb{Z}_5-$flow as it was done in the proof of Theorem 2.3.2. If this construction fails, then F does not reduce a cycle C_e, for some $e \notin E'$. Otherwise, a nowhere-zero $\mathbb{Z}_5-$flow φ is constructed such that $F = F_\varphi$. From the proof of Proposition 2.3.1, we conclude that in this case, F reduces every cycle of G.

It is known that one can find a spanning tree in polynomial time. The construction of the flow is in time $O(|E'|)$ and then to check if it is a flow can be done in time $O(|E'|)$ too. □

3. Injective coloring with arithmetic constraints

3.1 Introduction

An *injective coloring* of a graph G is an assignment of colors to the vertices of G such that two vertices sharing a common neighbor get different colors. Injective colorings were introduced in [21]. The smallest integer k such that an injective coloring with k colors exists is called the injective chromatic number of G and it is denoted by $\chi_i(G)$.

In this chapter, we introduce another concept which we call *additive coloring.* An additive coloring c of a graph $G = (V, E)$ is a function assigning positive integers to its vertices such that by assigning to each edge uv the value $|c(u) - c(v)|$, we obtain an edge coloring of G. The smallest integer k such that an additive colorings exists with colors in the set $\{1, \ldots, k\}$ is called the *additive chromatic index* of G. We denote it by $\chi'_a(G)$. Since the set of colors used for coloring the edges is $\{0, \ldots, k-1\}$ whenever colors used in the vertices belong to $\{1, \ldots, k\}$, we get that $\chi'(G) \leq \chi'_a(G)$, where $\chi'(G)$ is the chromatic index of G.

Notice that c is an additive coloring if and only if for every three distinct vertices x, y, z with $xy, yz \in E$, the following two properties hold: $c(x) \neq c(z)$, and $c(x)+c(z) \neq 2c(y)$. Hence, in particular additive colorings are injective colorings. Then, $\Delta(G) \leq \chi_i(G) \leq \chi'_a(G)$, as any additive coloring using colors in $\{1, \ldots, k\}$ is an injective coloring with at most k colors. We shall see throughout this work that these two parameters are closely related and that the additive structure of integers is closely related with the existence of additive colorings.

3.2 Upper bounds

We start by showing a general upper bound of $\chi'_a(G)$ in terms of $\chi_i(G)$.

Proposition 3.2.1. . *Let G be a graph. Then*

$$\chi'_a(G) \leq \chi'_a(K_{\chi_i(G)}),$$

where K_m denotes the complete graph on m vertices.

Proof. Let $m = \chi_i(G)$. Let c be an additive coloring of K_m with maximum value l. Then, given any three distinct vertices i, j, k in K_m, colors $c(i), c(j)$ and $c(k)$ are distinct, and $c(i) + c(j) \neq 2c(k)$.

Let c' be an injective coloring of G with m colors. Then, given any three distinct vertices u, v, w in G with u and w neighbors of v, $c'(u) \neq c'(w)$.

If $c'(v) \in \{c'(u), c'(w)\}$, then $c(c'(u))+c(c'(w)) \neq 2c(c'(v))$. Otherwise, $c'(u), c'(v), c'(w)$ are distinct and $c(c'(u)) + c(c'(w)) \neq 2c(c'(v))$. Hence, by defining $c''(u) :=$

m	$\chi'_a(K_m)$	m	$\chi'_a(K_m)$	m	$\chi'_a(K_m)$	m	$\chi'_a(K_m)$	m	$\chi'_a(K_m)$
1	1	*2*	2	*3*	4	*4*	5	*5*	9
6	11	*7*	13	*8*	14	*9*	20	*10*	24
11	26	*12*	30	*13*	32	*14*	36	*15*	40
16	41	*17*	51	*18*	54	*19*	58	*20*	63
21	71	*22*	74	*23*	82	*24*	84	*25*	92
26	95	*27*	100	*28*	104	*29*	111	*30*	114
31	121	*32*	122	*33*	137	*34*	145	*35*	150
36	157	*37*	163	*38*	165	*39*	169	*40*	174

Table 3.1: $\chi'_a(K_m)$ for $m \leq 40$, [19].

$c(c'(u))$ for every vertex u, we obtain an additive coloring of G with maximum value l. $\square$

Obviously, this bound is tight for complete graphs as $\chi_i(K_m) = m$. We remark that a set of integers defines an additive coloring of K_m if and only if it does not contain arithmetic progressions of length three. Hence, in order to effectively apply the upper bound given in Proposition 3.2.1, we need some information about the smallest integer l such that there is a set of m integers without arithmetic progression of length three and contained in $\{1, \ldots, l\}$. The determination of this value, in our terms $\chi'_a(K_m)$, has been the focus of research for more than 70 years, initiated in [15] where the first upper bound of m in terms of l was given, which was later improved in [35, 36], [46], and [23]. Currently, the best upper bound appears in [5]. On the other hand, the first lower bound was proved in [1] and it was later improved in [14]. These best bounds can be stated as follows: there are constants c_1 and c_2 such that

$$c_1 l \sqrt{\frac{\log^{\frac{1}{8}} l}{2^{4\sqrt{2\log l}}}} \leq m \leq c_2 l \sqrt{\frac{\log\log l}{\log l}}. \tag{3.1}$$

In [19] the exact value of $\chi'_a(K_m)$ [1], for $m \leq 41$, was computed, and lower and upper bounds, for $m \leq 100$, were given (see Table 1).

By doing some standard calculus manipulations, from Proposition 3.2.1 and Inequality 3.1 we get the following super linear upper bound for $\chi'_a(G)$. This bound will later allow us to obtain some (in)approximability results for the computation of the additive chromatic index.

Theorem 3.2.2. . *Let G be a graph. Then,*

$$\chi'_a(G) \leq \chi_i(G) g(\chi_i(G)),$$

where $g(x) = 2^{\sqrt{2\log(x)}}$.

[1] In fact, they studied the dependency of m in terms of l, which they called the Szemeredi number $sz(l)$.

One can see that the injective chromatic number of a graph G is the chromatic number of the graph $G^{(2)}$, obtained from G by adding edges between vertices at distance two and by removing all the original edges. When G is a bipartite graph with independent sets U and W, the graph $G^{(2)}$ is the disjoint union of the graphs $G^U = G^2[U]$ and $G^W = G^2[W]$ induced by the independent sets U and W of G in $G^{(2)}$, respectively. Then, for bipartite graphs we have the following ([21]).

$$\chi_i(G) = \max\{\chi(G^U), \chi(G^W)\}. \tag{3.2}$$

A similar result holds for the additive chromatic index which allow us to get a linear upper bound for $\chi'_a(G)$ in terms of $\chi_i(G)$, for bipartite graphs.

Theorem 3.2.3. . *Let G be a bipartite graph. Then,*

$$\chi'_a(G) \leq 2\chi_i(G) - 1.$$

Proof. Let G be a bipartite graph with independent sets U and W. Let G^U and G^W defined as above. Let $k_U := \chi(G^U)$ and $k_W := \chi(G^W)$. Let c and c' be vertex colorings of G^U and G^W, respectively such that $c(u) \in \{1, \ldots, k_U\}$, for each $u \in G^U$, and $c'(w) \in \{1, \ldots, k_W\}$, for each $w \in G^W$. We define $\varphi : U \cup W \to \mathbb{N}$ as follows. $\varphi(u) = c(u)$ for $u \in U$ and $\varphi(v) = c'(v) + k_U - 1$ for $v \in W$. Then $\max \varphi = k_U + k_W - 1 \leq 2\chi_i(G) - 1$ and it is easy to see that it is an injective coloring. Hence, in order to prove that φ is an additive coloring we prove that $\varphi(u) + \varphi(v) \neq 2\varphi(w)$ for every u, v, w and $w \in N(u) \cap N(v)$. When $u, w \in U$, as c is a coloring of G^U, $\varphi(u) \neq \varphi(v)$ and $\varphi(u), \varphi(v) \leq k_U$. Moreover, $\varphi(w) \geq k_U$. Therefore, $\varphi(u) + \varphi(v) \neq 2\varphi(w)$. We now consider the situation for vertices $u, v \in W$. As c' is a coloring of G^W, $\varphi(u) \neq \varphi(v)$ and $\varphi(u), \varphi(v) \geq k_U$. Moreover, $\varphi(w) \leq k_U$. Therefore, $\varphi(u) + \varphi(v) \neq 2\varphi(w)$. □

In the next result we prove that the previous bound is tight.

Proposition 3.2.4. *Let n be an odd integer, with $n \geq 9$. Then, $\chi'_a(K_{n,n}) = 2n - 1$.*

Proof. Let U and W be the two independent sets of $K_{n,n}$ each of size n. It is clear that both $(K_{n,n})^U$ and $(K_{n,n})^W$ are complete graphs of size n. Hence, $\chi_i(K_{n,n}) = n$.

By coloring U with colors in $\{1, \ldots, n\}$, and coloring B with colors in $\{n, \ldots, 2n-1\}$ we get an additive coloring of $K_{n,n}$. Then, $\chi'_a(K_{n,n}) \leq 2n - 1$.

We now prove the lower bound $\chi'_a(K_{n,n}) \geq 2n - 1$. Let φ be an additive coloring of $K_{n,n}$, and let $A = \varphi(U)$ and $B = \varphi(W)$. Since every vertex in U is adjacent to every vertex in W, the function φ must be injective when restricted to U. Similarly, it must be injective when restricted to W. Hence, $|A| = |B| = n$.

In this situation φ is an additive coloring if and only if $av(A) \cap B = av(B) \cap A = \emptyset$, where $av(C) := \left\{\frac{x+y}{2} \in \mathbb{N} : x, y \in C, x \neq y\right\}$. Therefore,

$$\max\{|av(A)|, |av(B)|\} + n \leq \chi'_a(K_{n,n}).$$

For the sake of contradiction let us assume that $\chi'_a(K_{n,n}) \leq 2n-2$. Then, $|av(A)| \leq n-2$ and $|av(B)| \leq n-2$. We first show that in this situation $av(B) = \{a, a+1, \ldots, a+(n-3)\}$. To this purpose we need the following property.

Claim 3.2.5. *Let D be a set with all its elements with the same parity. For $|D| \geq 2$, it holds that $|av(D)| \geq 2|D|-3$. Moreover, for $|D| \geq 5$, $|av(D)| = 2|D|-3$ if and only if D is an arithmetic progression.*

Proof. Without loss of generality we can assume that all elements of D are even integers. The case $|D| = 2$ is direct. Let $D = \{a_1 < a_2 < \ldots < a_k\}$ be the elements of D and let us assume that $k \geq 3$. Let $b_{i,j} := a_i + a_j$. Then, the following $2k-3$ integers are all distinct and even,

$$b_{1,2} < b_{1,3} < b_{2,3} < \cdots < b_{k-2,k} < b_{k-1,k}.$$

This shows that $|av(D)| \geq 2|D|-3$. Moreover, for $i \geq 4$ the following two sequences have $2i-3$ integers, all distinct and even.

$$b_{1,2} < \cdots < b_{1,i-1} < b_{1,i} < b_{2,i} < \cdots < b_{i-1,i}$$

and

$$b_{1,2} < \cdots < b_{1,i-1} < b_{2,i-1} < b_{2,i} < \cdots < b_{i-1,i}.$$

Since we can continue both previous sequences with another $2k-3-(2i-3)$ terms, we get that if $|av(D)| = 2|D|-3$ then $b_{1,i} = b_{2,i-1}$ for $i \geq 4$. From this equality we get $a_1 + a_i = a_2 + a_{i-1}$ an then $a_2 - a_1 = a_i - a_{i-1}$, for $i \geq 4$. It is easy to see that the equality $a_2 + a_5 = a_3 + a_4$ holds, when $k \geq 5$. Therefore, D is an arithmetic sequence. ☐

We now prove that $av(B) = \{a, a+1, \ldots, a+(n-3)\}$. As $|B| = n$ is odd, it has a subset D with all its elements with the same parity, and such that $2|D| \geq n+1$. From the claim it follows that $|av(D)| \geq 2|D|-3$. Since $av(D) \subseteq av(B)$ and $|av(B)| \leq n-2$ we conclude that $av(D) = av(B)$ and $|av(D)| = 2|D|-3 = n-2$. Since $n \geq 9$, we get that $|D| \geq 5$ which again from the claim implies that D is an arithmetic progression. It is clear that in this case $av(D)$, and then $av(B)$, are arithmetic progression too.

Let $a, b \geq 1$ integers such that $av(B) = \{a, a+b, \ldots, a+b(n-3)\}$. Then, $a \geq 2$ and $a+b(n-3) \leq 2n-3$. Since $n \geq 9$ we obtain that $b \leq 2+1/(n-3) < 3$ which implies $b \leq 2$. If $b = 2$, then elements in $av(B)$ have the same parity. As we are assuming that $av(B) \cap A = \emptyset$ and that $A \subseteq \{1, \ldots, 2n-2\}$, A must contain a set D' of size at least $|av(B)|+1 = n-1$ whose elements have the same parity. Again we apply the claim and we get that $av(A) \geq 2(n-1)-3 = 2n-5 > n-2$ which, for $n \geq 9$, contradicts $\chi'_a(K_{n,n}) \leq 2n-2$. Therefore, we conclude that $b = 1$. That is, $av(B) = \{a, a+1, \ldots, a+(n-3)\}$. Then

$$A = \{1, \ldots, a-1\} \cup \{a+n-2, \ldots, n+n-2\}.$$

Hence, the set $av(A)$ contains the sets $\{2, \ldots, a-2\}$, $\{a+n-1, \ldots, 2n-3\}$, and $\mathbb{N} \cap \{\frac{a+n-1+i}{2} : i = 0, \ldots, n-3\}$. Therefore, $av(A)$ has at least $n-4+\lfloor (n-2)/2 \rfloor$

elements which is larger than $n-2$, for $n \geq 9$ which is a contradiction with $\chi'_a(K_{n,n}) \leq 2n-2$.

□

We can still improve our previous upper bounds when we consider trees. It is easy to see that for trees the injective chromatic number equals the maximum degree. So our previous result can be equivalently formulated as $\chi'_a(T) \leq 2\Delta(T)-1$, for T being a tree. It is also clear that this upper bound reduces to $\chi_i(T) = \Delta(T) = \chi'_a(T)$, when T has radius 1. Similarly, when the radius of T is two, we have that $\chi'_a(T) \leq \lceil 3/2\Delta \rceil - 1$. More generally we have the following.

Proposition 3.2.6. *Let T be a tree. If T has radius at least three, then*

$$\chi'_a(T) \leq \lceil 5\Delta/3 \rceil - 1.$$

Proof. We use a strong inductive hypothesis: an additive coloring exists using colors in the set $\{1,\dots,\Delta+\lceil 2\Delta/3\rceil-1\} \setminus \{\lceil 2\Delta/3 \rceil+1,\dots,\Delta-1\}$. Let v_0 be a vertex such that the distance of each leaf of T to v_0 is at most r. If we remove all the leaves of T we get a tree T' with radius $r-1$. By induction hypothesis there is an additive coloring of T' with maximum value $\Delta+\lceil 2\Delta/3\rceil-1$ and not using colors in $\{\lceil 2\Delta/3\rceil+1,\dots\Delta-1\}$.

Let v be a leaf of T' with color i. When $i \leq \lceil 2\Delta/3 \rceil$, we color its neighbors in $T-T'$ with colors in $\{1,\dots,i\} \cup \{\Delta,\dots,\Delta+\lceil 2\Delta/3\rceil-1\}$, if $2i \geq \lceil 2\Delta/3\rceil$, and with colors in $\{i,\dots,\lceil 2\Delta/3\rceil\} \cup \{\Delta,\dots,\Delta+\lceil 2\Delta/3\rceil-1\}$, otherwise. When $i \geq \Delta$, we color its neighbors in $T-T'$ with colors in $\{\Delta,\dots,i\} \cup \{1,\dots,\lceil 2\Delta/3\rceil\}$, if $2(i-\Delta) \geq \lceil 2\Delta/3\rceil$, and with colors in $\{i,\dots,\Delta+\lceil 2\Delta/3\rceil-1\} \cup \{1,\dots,\lceil 2\Delta/3\rceil\}$, otherwise. □

Previous result leads us to seek an upper bound for $\chi'_a(G)$ in terms of the maximum degree in arbitrary graph. In [21] it was shown that $\chi_i(G) \leq \Delta^2-\Delta+1$ and that this upper bound is attained by the incidence graph of the projective plane of order Δ (P_Δ). We prove a similar result for the additive chromatic index. We first prove that the incidence graph of P_Δ has additive chromatic index $\Delta(\Delta-1)+1$. To this purpose we shall use as set of colors a *perfect differences set* S. It is a set of Δ integers, $s_1,\dots,s_\Delta$, having the property that their $\Delta(\Delta-1)$ differences, $s_i-s_j, i \neq j; i,j = 1,\dots\Delta$, are congruent modulo $\Delta(\Delta-1)+1$, to the integers $1,2,\dots\Delta(\Delta-1)$, in some order. In [39] it was shown that for each $\Delta-1$ being a power of a prime number there is a *perfect differences set* of size Δ.

Notice that if S is a perfect differences set so is the set $S' = \{s-m : s \in S\}$, where m is the minimum element in S. Hence, we shall assume in the sequel that $0 \in S$. Under this assumption for every two elements $s,s' \in S$ it follows that $s+s' \neq 0 \mod n$, where $n = \Delta(\Delta-1)+1$. Otherwise, the two differences $0-s'$ and $s-0$ coincide modulo n.

For each perfect differences set S, with $0 \in S$, and Δ elements, we define the following representation of the incidence graph of P_Δ. Let $G(S) = (U \cup W, E(S))$

be a bipartite graph where U and W are copies of $\{0, 1, \dots n-1\}$ and $E(S) = \{xy | x \in U, y \in W; \exists s \in S : x + s = y \mod n\}$.

Lemma 3.2.7. *The graph $G(S)$ corresponds to the incidence graph of P_Δ. Moreover, $\chi'_a(G(S)) = \Delta(\Delta - 1) + 1$.*

Proof. By its definition, the set of neighbors of a vertex x in U (resp. W) is $\{x + s \bmod n : s \in S\}$ (resp. $\{x - s \bmod n : s \in S\}$), where $n = \Delta(\Delta - 1) + 1$. Hence $G(S)$ is a $\Delta-$regular graph.

In order to see that two vertices in U have exactly one common neighbor in W it is enough to prove that G^U is a complete graph on n vertices. Two vertices $x, x' \in U$ are adjacent in G^U if and only if there are $s, s' \in S$ such that $x + s = x' + s'$. From the definition of a perfect difference set, the set S is such that given $z = x - x' \bmod n \in U$ there are s and s' such that $x - x' = s' - s \bmod n$. Hence, x and x' have a common neighbor in W. By a counting argument we conclude that they have exactly one common neighbor. A similar argument can be applied to prove that two vertices in W have exactly one common neighbor in U. Therefore, $G(S)$ corresponds to the incidence graph of P_Δ.

Moreover, previous analysis shows that $n = \chi(G^U) \leq \chi'_a(G(S))$.

We now need to prove that $\chi'_a(G(S)) = n$. We prove that by assigning to each vertex $i \in U \cup W$ the value i, we obtain an additive coloring. It is clear that this coloring is an injective coloring as $x + s = x + s' \bmod n$ implies $s = s' \bmod n$. By the choice of S this implies that $s = s'$. On the other hand, if there are $x, x' \in U$, $s, s' \in S$ such that $x + s = x' + s' =: y \bmod n$ and $x + x' = 2y$, then we obtain the contradiction $s + s' = 0 \bmod n$. □

Our previous construction shows that the following upper bound is tight up to a constant factor.

Theorem 3.2.8. *Let G be a graph of maximum degree Δ. Then*

$$\chi'_a(G) \leq 2\Delta(\Delta - 1) + 1.$$

Proof. Our first proof of the upper bound was rather elaborated and only gives the result asymptotically. We present here a simpler proof given by B. Reed [**?**]. Let $v_1, \dots, v_n$ be an ordering of the vertices. We construct an additive coloring greedily by following this ordering. When coloring a vertex v_i we have already colored at most $\Delta(\Delta-1)$ vertices at distance two of v_i. Each such vertex forbidds two colors to be used at vertex v. Hence, with $2\Delta(\Delta - 1) + 1$ colors the greedy strategy produces an additive coloring of G with maximum value $2\Delta(\Delta-1)+1$.□

3.3 Lower bounds

We now show some non-trivial lower bounds for the additive index in terms of the minimum degree.

Theorem 3.3.1. *Let $G = (V, E)$ be a graph with minimum degree δ. Then $\chi'_a(G) \geq 5(\delta - 1)/3$.*

Proof. Let $\alpha := \chi'_a(G)$, $l =: \lfloor(\alpha - \delta)/2\rfloor + 1$, and $c := \alpha - \delta - 2(l - 1)$. We shall prove that $2(l - 1) + c \geq (2\delta - 5)/3$, and hence that $\alpha \geq 5(\delta - 1)/3$.

Let φ be an additive coloring of G, with $\varphi : V \to \{1, \ldots, \alpha\}$. Let u be a vertex such that $\varphi(u) < \delta$. Then, at most one color between $\varphi(u) - a$ and $\varphi(u) + a$ can be used to color neighbors of u, for each $a = 0, \ldots, \varphi(u) - 1$. Hence, at least $\delta - \varphi(u)$ colors in $\{2\varphi(u), \ldots, \alpha\}$ are needed to color neighbors of u. This gives $\varphi(u) \leq \alpha - \delta + 1$. Given the definition of c and l, we have that the set of colors that can be used to color neighbors of u is contained in

$$\{1, \ldots, 2l - 1 + c\} \cup \{\delta, \ldots, \delta + 2(l - 1) + c\}.$$

We define four parameters $\theta^-, \theta^+, \rho^-, \rho^+$ as follows.

$$\begin{aligned}
\theta^- &:= \min\{l + c - \varphi(u) : \varphi(u) \leq l + c\}, \\
\theta^+ &:= \min\{\varphi(u) - l - c : l + c \leq \varphi(u) \leq 2l - 1 + c\}, \\
\rho^- &:= \min\{\delta - 1 + l + c - \varphi(u) : \delta \leq \varphi(u) \leq \delta - 1 + l + c\}, \\
\rho^+ &:= \min\{\varphi(u) - l - c - \delta + 1 : \delta - 1 + l + c \leq \varphi(u) \leq \delta - 1 + 2l - 1 + c\}.
\end{aligned}$$

Let u be a vertex such that $\varphi(u) = l + c - \theta^-$. Then, the neighbors of u can use at most $l + c - \theta^-$ colors in $\{1, \ldots, 2(l + c - \theta^-) - 1\}$, at most $2\theta^- - c$ colors in $\{2(l+c-\theta^-), \ldots, 2l+c-1\}$, and at most $2l+c-(\rho^-+\rho^+)$ in $\{\delta, \ldots, \delta+2(l-1)+c\}$. Hence, at most

$$3l + c + \theta^- - \rho^- - \rho^+ \tag{3.3}$$

in total.

By considering a vertex u such that $\varphi(u) = l + c + \theta^+$, we deduce that there are at most

$$3l + 2c + \theta^+ - \rho^- - \rho^+ \tag{3.4}$$

colors available to color neighbors of u.

Similarly, for a vertex u with $\varphi(u) = \delta - 1 + l + c - \rho^-$, there are at most

$$3l + c + \rho^- - \theta^+ - \theta^- \tag{3.5}$$

colors that can be used to color neighbors of u.

Finally, for a vertex with $\varphi(u) = \delta - 1 + l + c + \rho^+$, there are no more than

$$3l + 2c + \rho^+ - \theta^+ - \theta^- - 2 \tag{3.6}$$

colors for coloring neighbors of u.

By adding Equations (3.3), (3.4), (3.5), and (3.6), we get

$$12l + 6c - \theta^- - \theta^+ - \rho^- - \rho^+ - 2 \geq 4\delta,$$

which implies

$$2(l - 1) + c \geq (2\delta - 5)/3.$$

$\square$

When we apply previous bound to Δ-regular graphs we get the following result.

Corollary 3.3.2. *Let $G = (V, E)$ be a Δ-regular graph. Then $\chi'_a(G) \geq 5(\Delta - 1)/3$.*

3.4 Computational complexity of computing $\chi'_a(G)$

Additive coloring are injective coloring with additional constraint on the color allowed in the neighborhood of each vertex. This also holds for $L(p, q)-$labeling, a related concept introduced in [20]. Let $\lambda(p, q)(G)$ be the smallest integer k such that a coloring of G with colors in $\{0, \dots, k\}$ exists with the property that vertices at distance one are assigned with colors at distance at least p while vertices at distance two are assigned with colors at distance at least q. From the computational complexity point of view it is known that for p and q relatively primes the problem of deciding whether $\lambda_{p,q}(G) \leq r$, is NP-complete, even when restricted to trees [17]. On the other hand, in [10], it is shown that $\lambda_{2,1}(G)$ can be computed in polynomial time on trees. More recently, a linear time algorithm for computing $\lambda_{2,1}(G)$ for trees was proved in [22]. The method presented in [10] can be extended to compute $\lambda_{p,1}(G)$ in polynomial time.

Here we show how a similar idea can be used to compute the additive chromatic index of trees. For a tree of maximum degree Δ, we can reduce the computation of its additive chromatic index to the problem of deciding whether it has an additive coloring with maximum value l, for each value $l \in \{\Delta, \dots, 5\Delta/3\}$. We present the result in a framework which includes previous notions.

Let us consider vertex colorings such that when a vertex has color a then its neighbors can only use colors which are not in *conflict* with each other. To model this situation, let $\mathcal{C}$ be a set of colors, and for each color a in $\mathcal{C}$ let $G_a = (V_a, E_a)$ be a graph such that $V_a \subseteq \mathcal{C}$. An edge bc in E_a indicates that no two neighbors u and w of a vertex v can have colors b and c if v has color a.

We call an injective coloring c using colors in $\mathcal{C}$ *feasible* for $(G_a)_{a\in\mathcal{C}}$, if for each vertex u, the set of colors $c(N(u))$ is an independent set in the graph $G_{c(u)}$.

Let $\mathcal{C}$ be a set of colors and for each color a in $\mathcal{C}$ let $G_a = (V_a, E_a)$ be a graph. Given a tree T, a leaf r and a vertex $u \neq r$, we denote by T^u the subtree of T of descendants of vertex u with respect to the partial order defined by r. For each vertex u and each color $a \in \mathcal{C}$ we say color $b \in V_a$ is *compatible* with u and a, if there is an injective coloring of T^u feasible for $(G_a)_{a\in\mathcal{C}}$, and assigning color a to the father of vertex u and color b to u. Let $C(u, a)$ denote the set of colors which are compatible with u and a. It is clear that if we are able to compute sets of compatible colors for all vertices and all colors, then we can decide whether T has a feasible coloring. In fact, if for the child u of the root r there are colors a and b with $b \in C(u, a)$, then a feasible injective coloring exists for T.

Given a tree T, the following dynamic programming algorithm computes for every vertex u and every color a, the set $C(u, a)$.

Compatible Sets

Input: A tree $T = (V, E)$, a set of color $\mathcal{C}$, and a family of graphs $(G_a = (V_a, E_a))_{a \in \mathcal{C}}$.

Output: For each vertex u and each color $a \in \mathcal{C}$, the set of compatible colors $C(u, a)$.

1. For each leaf u, $u \neq r$, for each color a, set $C(u, a) = V_a$; mark vertex u as processed.
2. Iteratively, take a vertex u not yet processed and such that all its children in T are processed. For each color a compute $C(u, a)$ as follows: $b \in C(u, a)$ if and only if for each child v of u, there exists a color $f_v \in C(v, b)$, such that the set $\{f_v : v \text{ a child of } u\} \cup \{a\}$ is independent in G_b. At the end, mark vertex u as processed and continue with unprocessed vertices.

This strategy can be implemented by visiting the tree in time $O(n|\mathcal{C}|^2 K)$, where K is the time needed to determine whether $b \in C(u, a)$, for a given vertex u, and given colors a and b.

As above, given a tree T, a leaf r and a vertex $u \neq r$, we denote by T^u the subtree of T of descendants of vertex u with respect to the partial order defined by r. We also denote by $N'(u)$ the children of u with respect to this partial order. Given two colors a and b in $\mathcal{C}$ and a vertex u of T, we show that to decide whether $b \in C(u, a)$ can be formulated as a maximum matching problem in an auxiliary bipartite graph H, when each connected component of each graph G_a is a complete graph. In this situation an injective coloring c of T is feasible for $(G_a)_{a \in \mathcal{C}}$ if it assigns to each vertex $v \in N(u)$ a color in a different connected component of $G_{c(u)}$. The auxiliary graph H is built as follows.

Let B be the set of connected components of G_b which do not contain color a. Let $G = (N'(u) \cup B, E)$ be a bipartite graph with independent sets $N'(u)$ and B. For each $v \in N'(u)$ and each $s \in B$ we add the edge vs to E whenever $s \cap C(v, b) \neq \emptyset$.

If graph H has a matching saturating $N'(u)$, then we can color $v \in N'(u)$ with any color f_v in $C(v, b) \cap s$, where vs belongs to M. As at most one $v \in N'(u)$ is associated with s we obtain a feasible injective coloring. Conversely, if a feasible injective coloring exists it must assign to each vertex in $N(u)$ a color in a different connected component of G_b. As the father of u has color a, no color in the connected component of G_b containing a can be used for the remaining neighbors of u. Moreover, two distinct vertices in $N'(u)$ are assigned with two different connected components in G_b. Then, a matching exists that saturates $N'(u)$.

The time needed to compute a maximum matching in a bipartite graph whose independent sets are of size $|N(u)| \leq \Delta$ and $|V_a| \leq |\mathcal{C}|$ is $O(\Delta^{1.5}|\mathcal{C}|)$.

Theorem 3.4.1. *In time $O(\Delta^{1.5} n |\mathcal{C}|^3)$ we can decide whether a tree with n vertices and of maximum degree Δ admits an injective coloring feasible for $\mathcal{C}G = (G_a)_{a \in \mathcal{C}}$, when for each color a each connected component of G_a is a complete graph.*

The family $(G_a)_{a \in \mathcal{C}}$ associated to injective colorings is such that set of edges of each graph is empty, while for $\lambda_{p,q}$ coloring for each color a the graph G_a has vertex set $\mathcal{C} \setminus \{a-p, \ldots, a+p\}$ and edge set $\{c \in \mathcal{C} : |c-a| < q\}$. The family $(G_a)_{a\in\mathcal{C}}$ associated to additive colorings is such that the graph G_a is a matching, for each $a \in \mathcal{C}$. From Theorem 3.4.1 applied to this family we get that in time $O(\Delta^{1.5} n l^3)$ we can decide whether a tree T with n vertices and of maximum degree Δ admits an additive coloring with colors in $\{1, \ldots, l\}$. From Proposition 3.2.6 we get the following corollary.

Corollary 3.4.2. *The additive chromatic index can be computed in time $O(\Delta^{4.5} n \log \Delta)$ in a tree with n vertices and of maximum degree Δ.*

We now consider the computation of the additive chromatic index in larger classes of graphs. A natural class to consider is the class of bounded treewidth graphs. For this class it is known that any decision problem admitting a Monadic Second Order Logic formula has a polynomial time algorithm [11]. For k-Additive Coloring, the problem of deciding whether a graph G has additive chromatic index at most k, we have the following for each fixed k.

Lemma 3.4.3. *The problem k-*Additive Coloring *can be expressed by a Monadic Second Order Logic formula of size only depending on k.*

Proof. In the following formula, the sets $X_1, \ldots, X_k$ will correspond to the color sets, and the three vertices x, y, z to any path of length 2. Part (3.8) of the formula states that vertices x, y, z receive colors in $[k] := \{1, \ldots, k\}$, part (3.9) states that vertices x and z receive different colors (same condition as for an injective coloring), and part (3.10) states that the colors given to vertices x, y, z do not create an arithmetic progression.

$$\exists X_1, \ldots, X_k \subseteq V(G) \text{ s.t. } \forall x, y, z \in V(G) : \big(\{x,y\} \in E(G) \wedge \{y,z\} \in E(G)\big) \Rightarrow \tag{3.7}$$

$$\Big[\big(\textstyle\bigvee_{i\in[k]} x \in X_i\big) \wedge \big(\bigvee_{i\in[k]} y \in X_i\big) \wedge \big(\bigvee_{i\in[k]} z \in X_r\big)\Big] \tag{3.8}$$

$$\wedge \qquad \neg\Big[\textstyle\bigvee_{i\in[k]} \big(x \in X_i \wedge z \in X_i\big)\Big] \tag{3.9}$$

$$\wedge \qquad \neg\Big[\textstyle\bigvee_{i,j\in[k]} \big(x \in X_i \wedge y \in X_{i+j} \wedge z \in X_{i+2j}\big)\Big] \tag{3.10}$$

. ∎

From the result in [11] we immediately get the following.

Theorem 3.4.4. *The problem k-*Additive Coloring*, for any fixed k, has a polynomial time algorithm when restricted to classes of graphs of bounded treewidth.*

We do not know whether the problem remains polynomially solvable when k is part of the input, even for serie-parallel graphs, i.e. graphs of treewidth at most two. On the other hand, we prove that the problem is hard for $k = 4$, even when restricted to 3-regular graphs. To this end we show that 4-Additive ColoringReduces 3-Edge Coloring, the problem of deciding whether a graph has a 3-edge coloring, which was proved to be NP-complete in [25], even when restricted to 3-regular graphs.

Theorem 3.4.5. *The problem k-*ADDITIVE COLORING*, for $k = 4$, is* NP*-complete, even when restricted to 3-regular graphs.*

Proof. To see that k-ADDITIVE COLORING belongs to NP, we notice that the additive chromatic index is monotone under subgraphs. Hence an upper bound for its value on the complete graph K_n is an upper bound for its value in any graph on n vertices. As the right hand side of Equation 3.1 is $O(n^2)$, an additive coloring for a graph on n vertices has a description of length polynomial in n.

As previously mentioned, the problem 3-EDGE COLORINGis NP-complete, even when restricted to 3-regular graphs. We build for each 3-regular graph G a 3-regular graph G' such that G is 3-edge-colorable if and only if G' has an additive coloring with colors in $\{1, a, 4\}$, with $a = 2$ or $a = 3$.

The graph G' is obtained from G by replacing each vertex v of G by a copy of the complete graph K_3 which we denote $A(v)$, and for each edge $uv \in G$ we add an edge $u'v'$ in G', where $u' \in A_u$, $v' \in A_v$ such that each vertex in A_u finishes with degree 3 in G'.

The vertices u' and v' in the above construction are called the vertices of G' *associated* to the edge uv in G. It is clear that the construction of G' can be done in polynomial time and that G' is 3-regular.

An edge coloring of G with three colors can be transformed in an additive coloring of G' which uses colors in the set $\{1, 2, 4\}$ as follows. In G' we assign color i to the two vertices u' and v' associated to an edge uv of G with color i, with $i = 1, 2$, and we assing color 4 to those vertices of G' associated with edges of G with color 3.

Conversely, an additive coloring with colors in $\{1, 2, 3, 4\}$ uses either colors $1, 2, 4$ or $1, 3, 4$ in each set A_v. Therefore, colors 1 and 4 are presents in each set A_v. Moreover, the neighbor not in A_v of a vertex v' in A_v has the same color as v'. This allows us to color the edges in G with color 1 or 3 if its associated vertices get color 1 or 4 in G', respectively. The remaining edges are colored with color 2. □

In view of previous results it is interesting to consider whether we can approximate the additive chromatic number in polynomial time. We can use Theorem 3.2.2 to obtain (in)approximability results for the additive chromatic index based on previous results obtained for the injective chromatic numbers. In [24], it was proved that there is a polynomial time approximation algorithm for $\chi_i(G)$ with an approximation factor $n^{1/3}$ when restricted to split graphs. Moreover, they proved that this result is tight in the following sense. They showed that unless $ZPP = NP$, for each $\epsilon > 0$, there is no polynomial time approximation algorithm for $\chi_i(G)$ with a factor $n^{1/3-\epsilon}$, even for the class of split graphs. This result was based on an inapproximability result for the chromatic number obtained in [16]. From a result obtained in [50], we now know that the condition $ZPP = NP$ can be strengthened to $P = NP$.

Let us assume that there are $\epsilon > 0$ and a polynomial time approximation algorithm for $\chi'_a(G)$ which on input G computes an approximation α for the additive chromatic index laying between $\chi'_a(G)$ and $n^{1/3-\epsilon}\chi'_a(G)$.

From Theorem 3.2.2 we have that $\chi'_a(G) \leq \chi_i(G)g(\chi_i(G))$, where function g is $o(n^{\epsilon/2})$, for each $\epsilon > 0$. Moreover, $g(x)$ is a non negative, non decreasing function.

Then, we have $\chi_i(G) \leq \alpha \leq n^{1/3-\epsilon}\chi_i(G)g(n) \leq n^{1/3-\epsilon/2}\chi_i(G)$, for each graph G with n vertices, and n large enough. This allows us to prove the following result.

Theorem 3.4.6. *For each $\epsilon > 0$, unless $P = NP$, there is no polynomial time approximation algorithm with approximation factor $n^{1/3-\epsilon}$ for the additive chromatic index, even when restricted to split graphs.*

On the other hand, if for a graph G on n vertices we can compute in polynomial time a value v such that $\chi_i(G) \leq v \leq \chi_i(G)n^{1/3}$, then we can use $vg(v)$ to get an approximated value for $\chi'_a(G)$. In fact, we know from Theorem 3.2.2 that $\chi'_a(G) \leq \chi_i(G)g(\chi_i(G))$. Hence $\chi'_a(G) \leq vg(v)$ as $xg(x)$ is non-decreasing.

Applying the function $xg(x)$ to $\chi_i(G)n^{1/3}$ we get $\chi_i(G)n^{1/3}g(\chi_i(G)n^{1/3})$. But, $g(\chi_i(G)n^{1/3}) \leq g(n^{4/3})$ and for each value $\epsilon > 0$, $g(n^{4/3})$ is smaller than n^{ϵ}, for n large enough. Therefore,

$$\chi'_a(G) \leq vg(v) \leq \chi'_a(G)n^{1/3+\epsilon}.$$

Theorem 3.4.7. *For each $\epsilon > 0$, there is a polynomial time approximation algorithm with approximation factor $n^{1/3+\epsilon}$ for the additive chromatic index, when restricted to split graphs.*

It is clear that based upon Theorem 3.2.3 we can obtain similar approximation results between the additive chromatic index and the injective chromatic number for bipartite graphs. This linear relation between these two parameters makes interesting the following result.

Theorem 3.4.8. *For each fixed $k \geq 3$, the problem of deciding whether an input graph G has injective chromatic number at most k, is* NP*-complete, even restricted to bipartite graphs of maximum degree k.*

Proof. Let $G = (V, E)$ be a graph a k-regular graph and let $I(G)$ be the *incident graph* of G defined as the bipartite graph with parts V and E such that $ve \in E(I(G))$ whenever, e is incident with v. Notice that if G has maximum degree k, then $I(G)$ also has maximum degree k.

It is not hard to see that for the incidence graph $I(G)$, it holds that $(I(G))^V$ is the original graph G and that $(I(G))^E$ is the line graph of G. Therefore, from Equation 3.2 we get

$$\chi_i(I(G)) = \max\{\chi(G), \chi'(G)\}.$$

From Brooks' Theorem ([6]) we get that $\chi_i(I(G)) = \chi'(G)$, unless G is a complete graph. Therefore, computing the injective chromatic number of $I(G)$ is as hard as computing the chromatic index of G. As it is known that computing the chromatic index of k regular graphs is NP-hard, we obtain the statement of the theorem. $\square$

3.5 Conclusion

We have seen that computing the additive chromatic number on complete graphs as well as in balanced complete bipartite graphs depends on non-trivial additive properties of integers. We think that it is worth to consider the problem in others classes of well structured graphs as products of paths and/or cycles, balanced complete 3-partite graphs, and more generally, balanced complete k-partite graphs. We think that this study may require more sophisticated additive combinatorics tools in order to obtain lower bounds.

4. Proper caterpillars are distinguished by their chromatic symmetric function

4.1 Introduction

The weighted graph polynomial U_G [34] and the chromatic symmetric function X_G [40] of a graph G are powerful invariants. They have been actively studied and have diverse applications as they encode much of the combinatorics of the given graph. In particular, many well-known isomorphism invariants such as the Tutte polynomial and the chromatic polynomial can be obtained as evaluations of them. A natural question about either X_G or U_G is to decide whether they are complete isomorphism invariants. More precisely, do there exist non-isomorphic graphs with the same chromatic symmetric function (respectively the same weighted graph polynomial)? The answer to both questions is affirmative: Examples of non-isomorphic graphs with the same symmetric chromatic function can be found in [40]; on the other hand, one can find non-isomorphic graphs with the same weighted graph polynomial by combining the work in [37, 8]. However, these questions remain open when restricted to trees. In fact, Thatte proved in [47] that they are equivalent for trees. So the question stands: *Do there exist non-isomorphic trees with the same chromatic symmetric function?*. This question is often referred to as *Stanley's question* or *The Stanley conjecture* [40].

Despite the importance of the symmetric function generalization of the chromatic polynomial, not much is known in the literature about this question. We now review some known partial results towards a solution that appeared in [32]. First, we need to recall some definitions. Given a class of trees, we say that X_G distinguishes among this class if trees in the class with the same chromatic symmetric function must be isomorphic. A *caterpillar* is a tree where all the internal edges form a path, which is referred to as the *spine* of the caterpillar. A caterpillar is *proper* if each vertex in the spine is adjacent to a least one leaf. The spine induces a linear structure, which allows us to define the *leaf-sequence* of a caterpillar: To each vertex in the spine, we associate the number of leaves adjacent to it. M. Morin shown that X_G distinguishes among caterpillars with a palindromic leaf sequence and among proper caterpillars having a leaf sequence with all its components being distinct ([33], Theorem 4.3.1) . Thus, determining whether X_G distinguishes among all caterpillars seems to be a natural step towards the solution of Stanley's question. In this chapter, we obtain the following:

Theorem 4.1.1. *The chromatic symmetric function distinguishes among proper caterpillars.*

To this purpose, we give a sufficient condition for caterpillars to have a distinct symmetric function generalization of the chromatic polynomial. Next, we describe a natural embedding of proper caterpillars into the set of integer compositions.

We introduce a polynomial for compositions, which we call the $\mathcal{L}$-polynomial, that mimics the weighted graph polynomial of Noble and Welsh. We also show that the $\mathcal{L}$-polynomial of an integer composition can be computed as an evaluation of the weighted graph polynomial of the corresponding proper caterpillar. Finally, we observe that the $\mathcal{L}$-polynomial is equivalent to the multiset of partition coarsenings defined by Billera, Thomas and van Willigenburg in [2], and then we combine their characterization of integer compositions having the same multiset of partition coarsenings with our sufficient condition to establish our main result.

4.2 Caterpillars versus compositions

4.2.1 Compositions and the $\mathcal{L}$-polynomial

Let $\mathcal{P}$ denote the set of positive integers. Let n be a positive integer. A composition β of n, denoted $\beta \models n$, is a list $\beta_1\beta_2\ldots\beta_k$ of positive integers such that $\sum_i \beta_i = n$. We refer to each of the β_i as components, and say that β has *length* $\ell(\beta) = k$ and *size* $|\beta| = n$. The set of all compositions of n will be denoted by $\mathcal{C}_n$. The set of all compositions is given by

$$\mathcal{C} = \bigcup_{n\in\mathcal{P}} \mathcal{C}_n,$$

and is equal to the set of all non-empty words with alphabet $\mathcal{P}$. The *reverse* of a composition $\beta = \beta_1\beta_2\ldots\beta_k$ is the composition $\beta^* = \beta_k\ldots\beta_2\beta_1$. A composition β is a *palindrome* if and only if $\beta^* = \beta$. We say that $\alpha \sim_* \beta$ if either $\alpha = \beta$ or $\alpha = \beta^*$ and denote by $[\beta]^* := \{\beta, \beta^*\}$ the corresponding *reverse-class.*

Given two compositions $\alpha = \alpha_1\alpha_2\ldots\alpha_k$ and $\beta = \beta_1\beta_2,\ldots\beta_l$, recall that the *concatenation* is given by

$$\alpha_1\alpha_2\cdots\alpha_k \cdot \beta_1\beta_2\cdots\beta_l = \alpha_1\alpha_2\cdots\alpha_k\beta_1\beta_2\cdots\beta_l,$$

and that the *near-concatenation* of α and β is given by

$$\alpha \odot \beta := \alpha_1\alpha_2\ldots\alpha_{k-1}(\alpha_k+\beta_1)\beta_2\beta_3\ldots\beta_l.$$

Next, recall the following partial order on compositions. Given two compositions α and β in $\mathcal{C}$, we say that β is a *coarsening* of α, denoted $\beta \succeq \alpha$ if β can be obtained from α by adding consecutive components from α. That is to say, there exists an increasing finite sequence $1 = j_0 < j_1 < j_2 < ... < j_i < j_{i+1} = \ell(\alpha)+1$ of integer indices such that

$$\beta = \alpha_{j_0}\cdots\alpha_{j_1-1} \odot \alpha_{j_1}\cdots\alpha_{j_2-1} \odot \ldots \odot \alpha_{j_i}\cdots\alpha_{j_{i+1}-1}.$$

For convenience, we will denote $\beta_{i,k} = \beta_i\beta_{i+1}\cdots\beta_k$ to shorten the above notation.

Observe that by a well-known result of MacMahon [30], $(\mathcal{C}_n, \preceq)$ is isomorphic as a poset to the Boolean poset of dimension $n-1$, *i.e.*, the set of all subsets of $\{1, 2, \ldots, n-1\}$ ordered by inclusion.

A *partition* of n is a composition λ of n where the components satisfy $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l$. The *type* of a composition β, denoted by $\lambda(\beta)$, is the partition obtained by reordering the components of β in a weakly decreasing way.

Let $\mathbf{x} = x_1, x_2, \ldots$ be an infinite collection of commuting indeterminates. Given a partition $\lambda = \lambda_1\lambda_2\cdots\lambda_l$ of n, define $\mathbf{x}_\lambda := x_{\lambda_1}x_{\lambda_2}\cdots x_{\lambda_l}$. The *composition-lattice* polynomial of a composition β is defined by

$$\mathcal{L}(\beta, \mathbf{x}) = \sum_{\alpha \succeq \beta} \mathbf{x}_{\lambda(\alpha)}.$$

When will be clear the context, we will use $\mathcal{L}(\beta)$ instead of $\mathcal{L}(\beta, \mathbf{x})$.

If P is any polynomial in $\mathbf{x}$, and λ is a partition, then $[\mathbf{x}_\lambda]P$ will denote the coefficient of $\mathbf{x}_\lambda$ when P is expanded in the standard monomial basis.

4.2.2 The weighted graph polynomial

Let $G = (V, E)$ be a simple graph. If $A \subseteq E$, then $G|_A$ is the graph obtained from G after deleting all the edges in the complement of A from G (but keeping all the vertices). We recall the definition of the weighted graph polynomial (a.k.a. the U-polynomial), originally introduced by Noble and Welsh [34]. Note that we give the definition only for simple graphs (it is possible to define the U-polynomial for graphs with loops and parallel edges but we will not need this generality here).

The rank of A, denoted $r(A)$, is given by

$$r(A) = |V| - k(G|_A),$$

where $k(G|_A)$ denotes the number of connected components of $G|_A$. Let $\lambda(A) = \lambda_1\lambda_2\cdots\lambda_k$ be the partition of $|V|$ induced by the connected components of $G|_A$, that is, $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the cardinalities of the connected components of $G|_A$. The *U-polynomial* of G is defined by:

$$U_G(\mathbf{x}, y) = \sum_{A \subseteq E} \mathbf{x}_{\lambda(A)}(y-1)^{|A|-r(A)}.$$

A graph G is *U-unique* if the U-polynomial of every graph that is not isomorphic to G is different from the U-polynomial of G.

When G is a tree T, it is easy to check that $r(A) = |A|$ for every $A \subseteq E(T)$. Thus, the U-polynomial of T reads

$$U_T(\mathbf{x}) = \sum_{A \subseteq E} \mathbf{x}_{\lambda(A)}.$$

This implies, in particular, that a tree T is U-unique if and only if the U-polynomial of every *tree* that is not isomorphic to T is different from the U-polynomial of T.

Alternatively, by collecting like terms, we get

$$U_T(\mathbf{x}) = \sum_{\lambda \vdash |V|} c_\lambda(T)\mathbf{x}_\lambda,$$

where $c_\lambda(T)$ denotes the the number of subsets $A \subseteq E$ such that $\lambda(A) = \lambda$, and the sum is over all the partitions of $|V|$.

4.2.3 Caterpillars and the U^L-polynomial

Recall that a tree T is a *caterpillar* if the induced subgraph on the internal vertices is a non-trivial path $P(T)$, which is called the *spine* of T. As usual, we will identify $P(T)$ with its set of edges and let $L(T) = E \setminus P(T)$ be the set of leaf edges of T. A caterpillar T is *proper* if every internal vertex of T is adjacent to at least one leaf.

The *restricted weighted polynomial*, or U^L-polynomial, of a caterpillar T is defined by

$$U_T^L(\mathbf{x}) = \sum_{A \subseteq E(T), L(T) \subseteq A} \mathbf{x}_{\lambda(A)}.$$

Proposition 4.2.1. *For every caterpillar T, we have*

$$U_T(x_1 = 0, x_2, x_3, \ldots) = U_T^L(x_1 = 0, x_2, x_3, \ldots). \tag{4.1}$$

Furthermore, if T is proper, then U_T^L does not depend on x_1. In particular, in such a case U_T^L is an evaluation of the U-polynomial of T.

Proof. Let $A \subseteq E$ be such that $L(T)$ is not a subset of A, and pick an edge $e \in L(T)$ that is also in the complement of A. Observe that the leaf adjacent to e is an isolated vertex in $G|_A$. This implies that 1 is a part of $\lambda(A)$, which means that x_1 divides $\mathbf{x}_{\lambda(A)}$. It follows that $\mathbf{x}_{\lambda(A)}|_{x_1=0} = 0$. Thus,

$$U_T(x_1 = 0, x_2, x_3, \ldots) = \sum_{A \subseteq E, L(T) \subseteq A} x(\lambda_A)|_{x_1=0} = U_T^L(x_1 = 0, x_2, x_3, \ldots),$$

which establishes (4.1). To get the conclusion, observe that if T is proper and $A \subseteq E(T)$ contains $L(T)$, then $T|_A$ does not have isolated vertices, which implies that U_T^L is a polynomial that does not depend on x_1. Hence, the last assertion follows from (4.1).

□

4.2.4 Caterpillars versus compositions

Let $\mathcal{T}_+$ be the family of all proper caterpillars and $\mathcal{P}$ be the set of reverse-classes of all integer compositions. There is a natural one to one function of $\mathcal{T}_+$ into $\mathcal{P}$. Indeed, suppose that the internal vertices of T are enumerated as $\{v_1, v_2, \ldots v_k\}$, where v_i is adjacent to v_{i+1} for each $i \in \{1, \ldots, k-1\}$. In other words, the spine of T is the path $P(T) = v_1 v_2 \dots v_k$. Then, for each $i \in \{1, \dots, k\}$, define β_i to be the number of vertices in the connected component of $T|_{L(T)}$ that contains v_i. Finally set

$$\Phi(T) = [\beta_1 \beta_2 \cdots \beta_k]^*.$$

Lemma 4.2.2. *The map $\Phi : \mathcal{T}_+ \to \mathcal{P}$ is one-to-one.*

Proof. Let $\mathcal{P}_+$ be the image of $\mathcal{T}_+$ by Φ. We construct $\Psi : \mathcal{P}_+ \to \mathcal{T}_+$, which is the inverse of Φ, explicitly. Given $[\beta]^*$ in $\mathcal{P}_+$, let $\tilde{T}$ be a path with $\ell(\beta)$ vertices, that is, $\tilde{T} = v_1 \dots v_{\ell(\beta)}$. Next, for every $i \in \{1, \dots, \ell(\beta)\}$, we attach $\beta_i - 1$ leaves to the vertex v_i and denote by $T = \Psi(\beta)$ the caterpillar generated by this process. It is clear that T does not depend on the choice of β in the reverse-class. Moreover, since $[\beta]^*$ belongs to $\mathcal{P}_+$, it is clear that $\beta_i > 1$ for all i, which means that T is proper. Hence, Ψ is well-defined. Finally, it is straightforward to check that Ψ is the inverse of Φ. ☐

Observe that since the $\mathcal{L}$-polynomials of a composition and its reverse coincide, we can define

$$\mathcal{L}(\Phi(T), \mathbf{x}) = \mathcal{L}(\beta, \mathbf{x}), \qquad \beta \in \Phi(T).$$

Proposition 4.2.3. *For every $T \in \mathcal{T}_+$ we have*

$$U_T^L(\mathbf{x}) = \mathcal{L}(\Phi(T), \mathbf{x}).$$

Proof. Fix $\beta \in \Phi(T)$ and an orientation of the spine $P(T) = v_1 \dots v_n$ such that sequence of the number of vertices of the connected components of $T|_{L(T)}$ coincides with β. We will establish a correspondence between compositions $\alpha \succeq \beta$ and sets $A \subseteq E(T)$ containing $L(T)$ that satisfy the relation $\lambda(A) = \lambda(\alpha)$. Indeed, suppose that $A \subseteq E(T)$ contains $L(T)$. Then the edges in $E \setminus A$ are all internal, which means that $E \setminus A = \{v_{j_1} v_{j_1+1}, v_{j_2} v_{j_2+1}, \dots, v_{j_k} v_{j_k+1}\}$ with $j_1 < j_2 < \dots < j_k$. By defining

$$\alpha(A) = |\beta_{1,j_1}||\beta_{j_1+1,j_2}| \dots |\beta_{j_{k-1}+1,j_k}||\beta_{j_k+1,n}|,$$

it is clear that $\alpha(A) \succeq \beta$ and $\lambda(\alpha(A)) = \lambda(A)$. Conversely, if $\alpha \succeq \beta$, then by definition, there exist $1 = j_0 < j_1 < j_2 < \dots < j_i < j_{i+1} = \ell(\beta) + 1$ such that

$$\alpha = \beta_{j_0, j_1 - 1} \odot \beta_{j_1, j_2 - 1} \odot \dots \odot \beta_{j_{i-1}, j_i - 1} \odot \beta_{j_i, j_{i+1} - 1}.$$

By defining

$$A(\alpha) = L(T) \cup \{v_{j_1 - 1} v_{j_1}, v_{j_2 - 1} v_{j_2}, \dots, v_{j_i - 1} v_{j_i}\},$$

we check that $\lambda(A(\alpha)) = \lambda(\alpha)$. It is left to the reader to check that $A(\alpha(A)) = A$ and $\alpha(A(\alpha)) = \alpha$. Finally, by using the last correspondence, we get

$$U_T^L(\mathbf{x}) = \sum_{A \subseteq E(T), L(T) \subseteq A} \mathbf{x}_{\lambda(A)} = \sum_{\alpha \succeq \beta} \mathbf{x}_{\lambda(\alpha)} = \mathcal{L}(\beta, \mathbf{x}).$$

☐

The next corollary follows direct from Proposition 4.2.3 and Proposition 4.2.1.

Corollary 4.2.4. *Let T and T' be two proper caterpillars with the same U-polynomial. Then, $\Phi(T)$ and $\Phi(T')$ have the same $\mathcal{L}$-polynomial.*

We say that $\alpha \sim_{\mathcal{L}} \beta$ if $\mathcal{L}(\alpha, \mathbf{x}) = \mathcal{L}(\beta, \mathbf{x})$ and denote by $[\beta]_{\mathcal{L}} = \{\alpha \in \mathcal{C} \mid \beta \sim_{\mathcal{L}} \alpha\}$ the corresponding *$\mathcal{L}$-class*. A composition β is *$\mathcal{L}$-unique* if and only if $[\beta]_{\mathcal{L}} = [\beta]_*$.

Corollary 4.2.5. *Let T be a proper caterpillar and $\beta \in \Phi(T)$. Suppose that β is $\mathcal{L}$-unique. Then, T is U-unique.*

Proof. Suppose T' is a tree such that $U_T(\mathbf{x}) = U_{T'}(\mathbf{x})$. In (Proposition 10, [32]), it is proved that whether a tree is a caterpillar or not can be recognized from U. Thus, since T and T' have the same U-polynomial, then they have the same degree sequence (Corollary 5 in [32]) and we can recognize that T' must also be a proper caterpillar. Let $\alpha \in \Phi(T')$. It follows from Propositions 4.2.1 and 4.2.3 that $\mathcal{L}(\alpha, \mathbf{x}) = \mathcal{L}(\beta, \mathbf{x})$. The $\mathcal{L}$-uniqueness of β now implies that $\alpha \sim^* \beta$. Thus, T' is isomorphic to T and T is U-unique. □

4.3 Proof of the main result

4.3.1 Description of compositions with the same $\mathcal{L}$-polynomial

It is easy to see that $\mathcal{L}$-polynomial of a composition β is equivalent to the multiset of partitions coarsenings of β

$$\mathcal{M}(\beta) = \{\lambda(\alpha) \mid \alpha \succeq \beta\},$$

introduced in [2]. Since the class of compositions that have the same multiset of partition coarsenings has been completely described in [2], we get a complete description of the $\mathcal{L}$-class of a given composition. We recall now this description. When possible, we follow the notation in [2].

For convenience we write

$$\alpha^{\odot i} := \underbrace{\alpha \odot \alpha \odot \ldots \odot \alpha}_{i \text{ times}},$$

where α is a composition and i is a positive integer. Given $\alpha \models n$ and $\beta \models m$, the composition $\beta \circ \alpha$ is defined by

$$\beta \circ \alpha = \alpha^{\odot \beta_1} \cdot \alpha^{\odot \beta_2} \cdots \alpha^{\odot \beta_k},$$

where $\ell(\beta) = k$. It is clear that $\beta \circ \alpha \models nm$ and since $(\alpha \cdot \beta)^*$ and $(\alpha^{\odot i})^* = (\alpha^*)^{\odot i}$ the the reverse operation commutes with the product $\circ$, it means,

$$(\beta \circ \alpha)^* = (\beta^* \circ \alpha^*). \tag{4.2}$$

If a composition α is written in the form $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$, then we call this a *factorization* of α. A factorization of $\alpha = \beta \circ \gamma$ is called *trivial* if any of the following conditions are satisfied:

1. one of the β,γ is the composition 1,
2. the compositions β and γ both have length 1,
3. the compositions β and γ both have all components equal to 1.

A factorization $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$ is *irreducible* if no $\alpha_i \circ \alpha_{i+1}$ is a trivial factorization, and each α_i admits only trivial factorizations. In this case, each α_i is called an *irreducible factor*.

Theorem 4.3.1 ([2, Theorem 3.6])**.** *Each composition admits a unique irreducible factorization.*

Let $\alpha \models n$ and $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$ be the unique irreducible factorization of α. Let id denote the identity map on $\mathcal{C}$ and R denote the reverse map, that is, $R(\alpha) = \alpha^*$ for all $\alpha \in \mathcal{C}$. The *symmetry-class* of α is defined by

$$\mathrm{Sym}(\alpha) := \big\{T_1(\alpha_1) \circ T_2(\alpha_2) \circ \cdots \circ T_k(\alpha_k) \mid T_i \in \{id, R\} \text{ for all } i \in \{1, \dots, k\}\big\}.$$

For instance, the unique irreducible factorization of 1 2 1 3 2 is 1 2 $\circ$ 1 2. Then

$$\begin{aligned} \mathrm{Sym}(1\ 2\ 1\ 3\ 2) &= \{1\ 2 \circ 1\ 2, 1\ 2 \circ 2\ 1, 2\ 1 \circ 1\ 2, 2\ 1 \circ 2\ 1\} \\ &= \{1\ 2\ 1\ 3\ 2, 2\ 1\ 2\ 3\ 1, 1\ 3\ 2\ 1\ 2, 2\ 3\ 1\ 2\ 1\}. \end{aligned}$$

Theorem 4.3.2 ([2, Corollary 4.2])**.** *For every composition α we have*

$$[\alpha]_{\mathcal{L}} = \mathrm{Sym}(\alpha).$$

4.3.2 Caterpillars with distinct U-polynomials

In this section, we give a very general sufficient condition for two proper caterpillars to have distinct U-polynomials. First, we need some notation.

Recall that α is lexicographically less than β, denoted $\alpha <_L \beta$, if one of the two conditions hold:

1. $\ell(\alpha) < \ell(\beta)$ and $\alpha_i = \beta_i$ for all $i \in \{1, \dots, \ell(\alpha)\}$,
2. There exists $k \in \{1, \dots, \ell(\alpha)\}$ such that $\alpha_k < \beta_k$ and $\alpha_i = \beta_i$ for all $i \in \{1, \dots, k-1\}$.

Given two compositions $\alpha \neq \beta$ of the same length, let

$$k(\alpha, \beta) := \min\{1 \leq k \leq \ell(\alpha) \mid \alpha_k \neq \beta_k\} \tag{4.3}$$

denote the index where the first difference (from left to right) between α and β appears. A composition β is a *prefix* of another composition γ if there exists a composition α such that $\gamma = \beta \cdot \alpha$. Acordingly, β is a suffix of γ if there exists α such that $\gamma = \alpha \cdot \beta$.

Theorem 4.3.3. *Suppose S and T are two proper caterpillars such that $\Phi(S) = [\alpha \circ \gamma]^*$ and $\Phi(T) = [\beta \circ \gamma]^*$, where α, β and γ belong to $\mathcal{C}$, and α and β have the same size. If γ is not a palindrome and $\alpha \neq \beta$, then the U-polynomials of S and T are distinct.*

Proof. To avoid confusion, we sometimes write $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ instead of $\alpha_1\alpha_2\ldots\alpha_k$ for a composition α. Without loss of generality we may assume that $\gamma <_L \gamma^*$ and $\alpha <_L \beta$. Fix $\sigma = \alpha \circ \gamma$, $\tau = \beta \circ \gamma$ and $n = |\alpha| = |\beta|$. Fix also $a = |\alpha_{1,k(\alpha,\beta)}|$ and $b = |\gamma_{1,k(\gamma,\gamma^*)}|$, where $k := k(\alpha, \beta)$ and $l := k(\gamma, \gamma^*)$ are defined by (4.3). Since $\gamma <_L \gamma^*$, it is easy to check that $[\mathbf{x}_{(b,|\gamma|-b)}]\mathcal{L}(\gamma) = 1$ and γ do not has a suffix of size b. Now consider $\delta = \delta_1\delta_2$, where

$$\delta_1 = a|\gamma| + b, \quad \delta_2 = n|\gamma| - \delta_1.$$

We show that

$$[\mathbf{x}_{\lambda(\delta_1,\delta_2)}]\mathcal{L}(\sigma) = 1.$$

Indeed, by the definition of k and l the composition $\rho_1 = (\alpha_{1,k} \circ \gamma) \cdot \gamma_{1,l}$ is a prefix for σ while $\rho_2 = (\alpha_{1,k} \circ \gamma) \odot \gamma_{1,l}$ is a prefix for τ, and both compositions have size δ_1. Hence, $\delta \succeq \sigma$ and $\delta \succeq \tau$.

Let us now suppose that $[\mathbf{x}_{\lambda(\delta_1,\delta_2)}]\mathcal{L}(\sigma) = 2$, that is, there is a suffix ϕ of σ such that $|\phi| = \delta_1$. Moreover, by the definition of σ, this would imply that $\phi = \phi_1 * \phi_2$ where $*$ could be $\cdot$ or $\odot$, $|\phi_1| = b$ and $|\phi_2| = a|\gamma|$, which in turn would yield that ϕ_1 is a suffix of γ, which is a contradiction. Hence, $[\mathbf{x}_{\lambda(\delta_1,\delta_2)}]\mathcal{L}(\sigma) = 1$.

Now we compute $[\mathbf{x}_\lambda]U_S = \sharp\{A \subseteq E(S) \mid \lambda(A) = \lambda\}$ for $\lambda = \lambda(1, \delta_1 - 1, \delta_2)$. Indeed, fix $A \subseteq E(S)$ such that $\lambda(A) = \lambda$. Since S is proper, it follows that $E(S) \setminus A = \{e_1, e_2\}$, where $e_1 = v_iv_{i+1}$ is an internal edge and e_2 is a leaf. This means that $A' = A \cup \{e_2\}$ contains $L(S)$ and either $\lambda(A')$ equals $\{\lambda(\delta_1, \delta_2)\}$ or $\{\lambda(\delta_1 - 1, \delta_2 + 1)\}$. Since A' contains all leaves, it corresponds to the sequence $\zeta := |\sigma_{1,i}||\sigma_{i+1,\ell(\sigma)}|$ and we have $\lambda(\zeta) = \lambda(A')$. Let us see that necessarily $\zeta = \delta$. Indeed, supposing $\zeta = \delta^*$ implies that $(n - b, b) \succeq \gamma$ which is not possible. Next, supposing $\zeta = (\delta_1 - 1, \delta_2 + 1)$ implies, as we already know that $(\delta_1, \delta_2) \succeq \sigma$, that $(\delta_1 - 1, 1, \delta_2) \succeq \sigma$, which is not possible since S is proper. Finally, supposing $(\delta_2 + 1, \delta_1 - 1) \succeq \sigma$ implies that $(n - b + 1, b - 1) \succeq \gamma$. But this is not possible, since by hypothesis, then we have $|(\gamma^*)_{1,l}| > |\gamma_{1,l}| = b > |(\gamma^*)_{1,l-1}|$ and $|(\gamma^*)_{1,l-1}| = |\gamma_{1,l-1}| = b - \gamma_l < b - 1$, where the last inequality follows from the fact that S is proper. Hence we have $\zeta = \delta$. Now, this means that A' is indeed uniquely determined. This implies, in particular, that

$$[\mathbf{x}_\lambda]U_S = \sharp L(\Psi(\rho_1)).$$

with Ψ the function defined in Lemma 4.2.2.

Using a similar argument, it is possible to show that

$$[\mathbf{x}_\lambda]U_T = \sharp L(\Psi(\rho_2)).$$

Thus, to finish the proof, it suffices to compute the number of leaves in $\Psi(\rho_1)$ and $\Psi(\rho_2)$ and show they are different. The following lemma allows us to compute the number of leaves in a proper caterpillar. To simplify notation, we say that a composition $\gamma \in \mathcal{C}$ is *proper* if every element in γ is larger than one.

Lemma 4.3.4. *Suppose $\gamma \in \mathcal{C}$ is proper. Then,*

$$\sharp L(\Psi(\gamma)) = |\gamma| - \ell(\gamma).$$

Proof. Since γ is proper, it is easy to see that $\Psi(\gamma)$ is also proper. Moreover, if $v_1 \dots v_{\ell(\gamma)}$ denotes the spine of $\Psi(\gamma)$, it follow easily from the definition of Ψ that the number of leaves incident to v_i is $\gamma_i - 1$ for every $i \in \{1, \dots, \ell(\gamma)\}$. This implies that $\sharp L(\Psi(\gamma)) = \sum_i (\gamma_i - 1) = |\gamma| - \ell(\gamma)$, which is the desired conclusion. □

Motivated by Lemma 4.3.4, given $\gamma \in \mathcal{C}$, define $N(\gamma) = |\gamma| - \ell(\gamma)$. The following lemma summarises the properties of N.

Lemma 4.3.5. *Suppose γ and α are two proper compositions. Then, the following assertions hold:*

(i) $N(\gamma \cdot \alpha) = N(\gamma) + N(\alpha)$;

(ii) $N(\gamma \odot \alpha) = N(\gamma) + N(\alpha) + 1$;

(iii) $N(\alpha \circ \gamma)) = N(\gamma)|\alpha| + N(\alpha)$.

Proof. *(i)* and *(ii)* are clear from the definition of N. To show *(iii)*, it follows from *(ii)* that $N(\gamma^{\odot \alpha_i}) = \alpha_i N(\gamma) + \alpha_i - 1$ for every $i \in \{1, \dots \ell(\alpha)\}$. Since $\alpha \circ \gamma = \gamma^{\odot \alpha_1} \cdot \gamma^{\odot \alpha_2} \dots \gamma^{\odot \alpha_{\ell(\alpha)}}$ by definition, it follows from *(i)* that

$$N(\alpha \circ \gamma) = \sum_{i=1}^{\ell(\alpha)} (\alpha_i N(\gamma) + \alpha_i - 1) = N(\gamma)|\alpha| + |\alpha| - \ell(\alpha) = N(\gamma)|\alpha| + N(\alpha).$$

□

Now to finish the proof of the theorem, applying Lemma 4.3.5 it is easy to check that

$$N(\rho_2) = N(\rho_1) + 1.$$

Since the sequences ρ_2 and ρ_1 are proper, it follows from Lemma 4.3.4 that $\sharp L(\Psi(\rho_2))$ and $\sharp L(\Psi(\rho_1))$ are distinct. This implies that $[x_\lambda]U_T$ and $[x_\lambda]U_S$ are different and the conclusion now follows. □

4.3.3 Proof of main result

Now we are almost in position to give the proof of our main result. The last result we need is the following:

Proposition 4.3.6. *Suppose that β is a palindrome. Then, β is $\mathcal{L}$-unique.*

Proof. By Theorem 4.3.1, and the fact that the reverse operation commutes with the product $\circ$, it is easy to check that β is a palindrome if and only if all its irreducible factors are palindromes. It is easy to see that the symmetry class of an irreducible factor that is also a palindrome is equal to its reverse-class, which is a singleton. Thus, by Theorem 4.3.2, the $\mathcal{L}$-class of β is equal to $\{\beta\}$, which means that β is $\mathcal{L}$-unique. □

Proof of Main Result. Suppose that T and T' are two proper caterpillars with the same U-polynomial. We assume for contradiction that T and T' are not isomoprhic. By Corollary 4.2.1, it follows that $\alpha \in \Phi(T)$ and $\beta \in \Phi(T')$ have the same $\mathcal{L}$-polynomial and $\alpha \not\sim^* \beta$. By Theorem 4.3.2, we have $\mathrm{Sym}(\alpha) = \mathrm{Sym}(\beta)$. That is, α and β admit irreducible factorizations $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k$ and $\beta = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_k$ such that,

$$\alpha_i \sim_* \beta_i \quad \text{for all } i \in \{1, \ldots, k\}.$$

On the other hand, by Proposition 4.3.6, neither α nor β can be palindromes. Hence, there exists l such that α_l is not a palindrome, and α_i is a palindrome for every $i > l$. Moreover, we may assume that $\alpha_l = \beta_l$ (if $\alpha_l = \beta_l^*$ by Equation (4.2) we can use β^* instead β). By setting $\gamma = \alpha_l \circ \alpha_{l+1} \circ \cdots \circ \alpha_k$, it follows that

$$\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{l-1} \circ \gamma \quad \text{and} \quad \beta = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_{l-1} \circ \gamma.$$

From this, it is direct to check that

$$\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{l-1} \neq \beta_1 \circ \beta_2 \circ \cdots \circ \beta_{l-1}.$$

Hence, by Theorem 4.3.3, the U-polynomials of T and T' are distinct. This gives a contradiction, and thus concludes the proof. $\square$

5. Forcing large complete (topological) minors in infinite graphs

5.1 Introduction

A recurrent question in finite graph theory is how certain substructures, such as specific minors or subgraphs, can be forced by density assumptions. The density assumptions are often expressed via a lower bound on the average or minimum degree of the graph.

The classical example for this type of questions is Turán's theorem, or, in the same vein, the Erdős-Stone theorem. Infinite analogues of these results are not difficult: it is easy to see that if the upper density of an infinite graph G is at least the Turán density for k, i.e. $(k-2)/(k-1)$, then $K^k \subseteq G$, that is, G contains the complete graph of order k as a subgraph (see Bollobás [3], and also [45]). On the other extreme, a result of Mader affirms that an average degree of at least $4k$ ensures the existence of a $(k+1)$-connected subgraph. This result has been extended to infinite graphs by M. Stein [44].

Half-way between the two types of results just discussed lies the question of how to force complete (topological) minors with large average degree. First steps into this direction were taken by Mader [31], we give here two well-known results due to Kostochka [29] and to Bollobás and Thomason [4], respectively, which we sum up in the following theorem.

Theorem 5.1.1. **[12]** *There are constants c_1, c_2 so that for each $k \in \mathbb{N}$, and each graph G the following holds. If G has average degree at least $c_1 k\sqrt{\log k}$ then $K^k \preceq G$ and if G has average degree at least $c_2 k^2$ then $K^k \preceq_{top} G$.*

Our aim is to extend this result to infinite graphs (qualitatively, that is, without necessarily using the same functions $c_1 k\sqrt{\log k}$ and $c_2 k^2$ from Theorem 5.1.1). We call a finite graph H a minor/topological minor of a graph G if it is a minor/topological minor of some finite subgraph of G.[1]

Avoiding the difficulty of defining an average degree for infinite graphs (the upper density mentioned above is too strong for our purposes, cf. [45]), we shall stick to the minimum degree for our extension of Theorem 5.1.1 to infinite graphs. This works fine for rayless graphs: M. Stein showed in [42] that every rayless graph of minimum degree $\geq m \in \mathbb{N}$ has a finite subgraph of minimum degree $\geq m$. Thus, Theorem 5.1.1 can be extended to rayless graphs if we replace the average with the minimum degree.

In general, however, large minimum degree at the vertices alone is not strong

[1]Note that this means all branch-sets of our minor are finite, but as long as H is finite, it clearly makes no difference whether we allow infinite branch-sets or not.

enough to force large complete minors. This is so because of the infinite trees, which may attain any minimum degree condition without containing any interesting substructure. So we need some additional condition that prevents the density from 'escaping to infinity'.

The most natural way to impose such an additional condition is to impose it on the ends[2] of the graph. This approach has also proved successful in other recent work [7, 42, 44]. In this way, that is, defining the degree of an end in appropriate way, the minimum degree, now taken over vertices and ends, can continue to serve as our condition for forcing large complete minors.

In [42], the *relative degree* of an end was introduced for locally finite graphs. In order to explain it, a few notions come in handy.

Let G be a locally finite graph. The *edge-boundary* of a subgraph H of G, denoted by $\partial_e^G H$, or $\partial_e H$ where no confusion is likely, is the set $E(H, G-H)$. The *vertex-boundary* $\partial_v^G H$, or $\partial_v H$, of a subgraph H of G is the set of all vertices in H that have neighbours in $G-H$. Now, the idea of the relative degree is to calculate the ratios of $|\partial_e H_i|/|\partial_v H_i|$ of certain subgraphs H_i of G, and then take the relative degree to be the limit of these ratios as the H_i in some sense converge to ω.

For this, define an ω-region of an end ω of a graph G as an induced connected subgraph which contains some ray of ω and whose vertex-boundary is finite. For $V' \subseteq V(G)$, $\Omega' \subseteq \Omega(G)$, a V'–Ω' separator is a set $S \subseteq V(G)$ such that $V' \nsubseteq S$ and such that no component of $G \setminus S$ contains both a vertex of V' and a ray from ω. We write $\Omega^G(H)$ for the set of all ends of G that have a ray in $H \subseteq G$.

Write $(H_i)_{i\in\mathbb{N}} \to \omega$ if $(H_i)_{i\in\mathbb{N}}$ is an infinite sequence of distinct ω-regions such that $H_{i+1} \subseteq H_i - \partial_v H_i$ and $\partial_v H_{i+1}$ is a minimal $\partial_v H_i$–$\Omega^G(H_{i+1})$ separator in G, for each $i \in \mathbb{N}$. Now, define

$$d_{e/v}(\omega) := \inf_{(H_i)_{i\in\mathbb{N}}\to\omega} \liminf_{i\to\infty} \frac{|\partial_e H_i|}{|\partial_v H_i|}.$$

Note that it does not matter whether we consider the lim inf or the lim sup, because if $(H_i)_{i\in\mathbb{N}} \to \omega$, also all subsequences of $(H_i)_{i\in\mathbb{N}}$ converge to ω. (For the same reason we could restrict our attention only to sequences (H_i) for which $\lim_{i\to\infty} \frac{|\partial_e H_i|}{|\partial_v H_i|}$ exists.)

For more discussion of this notion see Section 5.2. From now on we write $\delta^{V,\Omega}(G)$ for the minimum degree/relative degree, taken over all vertices and ends of G. In [42] M. Stein showed:

Theorem 5.1.2. *Let $m \in \mathbb{N}$ and let G be a locally finite graph. If $\delta^{V,\Omega}(G) \geq m$, then G has a finite subgraph of average degree at least m.*

In particular, this means that if $\delta^{V,\Omega}(G) \geq c_1 k\sqrt{\log k}$ for a locally finite G, then

[2] The *ends* of a graph are the equivalence classes of the rays (i.e. 1-way infinite paths), under the following equivalence relation. Two rays are equivalent if no finite set of vertices separates them. (A set S of vertices is said to *separate* two rays R_1, R_2 if $V(R_1) \setminus S$, and $V(R_2) \setminus S$ lie in different components of $G - S$.) The set of ends of a graph G is denoted by $\Omega(G)$. See the infinite chapter of [12] for more on ends.

$K^k \preceq G$, and if $\delta^{V,\Omega}(G) \geq c_2 k^2$, then $K^k \preceq_{top} G$. (The c_i are the constants from Theorem 5.1.1.)

We extend the notion of the relative degree to arbitrary infinite graphs. For a given end ω of some infinite graph G, let $Dom(\omega)$ denote the set of all vertices that dominate[3] ω. If $G_\omega := G - Dom(\omega)$ does not contain any rays from ω, or if $|Dom(\omega)| \geq \aleph_0$, then we set $d_{e/v}(\omega) := |Dom(\omega)|$. Otherwise, writing $\hat{\omega}$ for the unique[4] end of G_ω that contains rays from ω we define

$$d_{e/v}(\omega) := |Dom(\omega)| + \inf_{(\hat{H}_i)_{i\in\mathbb{N}} \to \hat{\omega}} \liminf_{i\to\infty} \frac{|\partial_e \hat{H}_i|}{|\partial_v \hat{H}_i|}.$$

Note that the $\hat{H}_i$ are $\hat{\omega}$-regions of G_ω. Also note that for locally finite graphs, our definition coincides with the one given earlier. For further discussion of our notion of the relative degree, for examples, and for alternative definitions that do (not) work, see Section 5.2.

Our main result is the following version of Theorem 5.1.2 for graphs with countably many ends.

Theorem 5.1.3. *Let $k \in \mathbb{N}$, let $m \in \mathbb{Q}$, and let G be a graph such that $|\Omega(G)| \leq \aleph_0$ and $\delta^{V,\Omega}(G) > m$. Then $K^k \preceq_{top} G$ or G has a finite subgraph of average degree greater than $m - k + 1$.*

This result, together with Theorem 5.1.1, at once implies the desired extension of Theorem 5.1.1 to graphs with countably many ends.

Theorem 5.1.4. *Let $k \in \mathbb{N}$, and let G be a graph with $|\Omega(G)| \leq \aleph_0$.*

1. *If $\delta^{V,\Omega}(G) \geq c_1 k\sqrt{\log k} + k$, then $K^k \preceq G$.*
2. *If $\delta^{V,\Omega}(G) \geq c_2 k^2 + k$, then $K^k \preceq_{top} G$.*

Clearly, we may take Theorem 5.1.3 as a black box in order to obtain extensions of other extremal finite results to infinite graphs. For instance, as for finite graphs we can use large (relative) degree to force complete graph immersions, many disjoint complete minors or complete bipartite subgraphs/toplogical minors.

We remark that in Theorem 5.1.4, the assumption that G has countably many ends is necessary. To see this, consider the graph we obtain by the following procedure. Let G_1 be a double-ray and let $E_1 := E(G_1)$. Now, for $i \geq 2$, replace each edge $e \in E_{i-1}$ with $\aleph_0$ many paths of length two whose middle points are connected by a (fresh) double-ray D_e. Let E_i be the edges on the D_e. After $\aleph_0$ many steps, we obtain a planar graph G with $\delta^{V,\Omega}(G) = \aleph_0$. Because of its planarity, G has no complete minors of order greater than 4.

Problem 5.1.5. *Is there a 'degree condition' which forces large complete (topological) minors in arbitrary infinite graphs?*

[3]We say a vertex v *dominates* an end ω if v dominates some ray R of ω, that is, if there are infinitely many v–$V(R)$-paths that are disjoint except in v. (It is not difficult to see that then v dominates all rays of ω.)

[4]The uniqueness of $\hat{\omega}$ follows at once from the fact that $|Dom(\omega)| < \aleph_0$.

If one wants to proceed along the same lines as here, it might be useful to resolve the following first.

Problem 5.1.6. *Is there a 'degree condition' which forces a finite subgraph of large average/minimum degree in arbitrary infinite graphs?*

5.2 Discussion of the relative degree

In this section, we will discuss our definition as well as possible alternative definitions of the relative degree of an end. This motivation is not necessary for the understanding of the rest of the chapter and may be skipped at a first reading.

5.2.1 Large vertex-degree is not enough

As we have already seen in the introduction, large minimum degree at the vertices alone is not sufficient for forcing large complete (topological) minors in infinite graphs, because of the trees. A similar example discards the following alternative. The *vertex-/edge-degree* of an end ω was introduced in [44], see also [12], as the supremum of the cardinalities of sets of vertex-/edge-disjoint rays in ω. Clearly, the vertex-degree of an end is always at most its edge-degree, so for our purposes we may restrict our attention to the vertex-degree. It is not difficult to show [43] that an end has vertex-degree $\geq k$ if and only if there is a finite set $S \subseteq V(G)$ so that every S-$\{\omega\}$ separator has order at least k.

Large vertex-degree at the ends together with large degree at the vertices ensures the existence of grid minors [42], and of highly connected subgraphs [44], but it is not strong enough for forcing (topological) minors. This can be seen by inserting the edge set of a spanning path at each level of a large-degree tree, in a way that the obtained graph is still planar.

The reason that the vertex-degree fails to force large complete minors is it only gives information about the sizes of vertex-separators S_i 'converging' to the end in question. But imagine we wish to 'cut off' an end. Then the information we need is not the size of the S_i, but the average amount of edges that the vertices in S_i send 'into' the graph. This idea is made precise in the definition of the relative degree for locally finite graphs as given in the introduction.

We defined the relative degree for locally finite graphs in the introduction as $\inf_{(H_i)} \liminf_i(|\partial_e H_i|/|\partial_v H_i|)$. Let us remark that instead, we might have defined the relative degree as $\inf_{(H_i)} \liminf_i((|\partial_e H_i| + |E(G[\partial_v H_i])|/|\partial_v H_i|)$. This change, and the corresponding change for non-locally finite graphs, would alter the relative degree of the ends (it would make more ends have large relative degree). Thus using such an altered definition, our result would cover a larger class of graphs. In fact it is very easy to check all our proofs do go through in the same way for the altered definition. However, we believe that our definition is more natural.

Let us quickly evaluate a possible alternative definition, that at first sight might seem equally plausible as ours but less complicated. Consider the ratio $d_e(\omega)/d_v(\omega)$

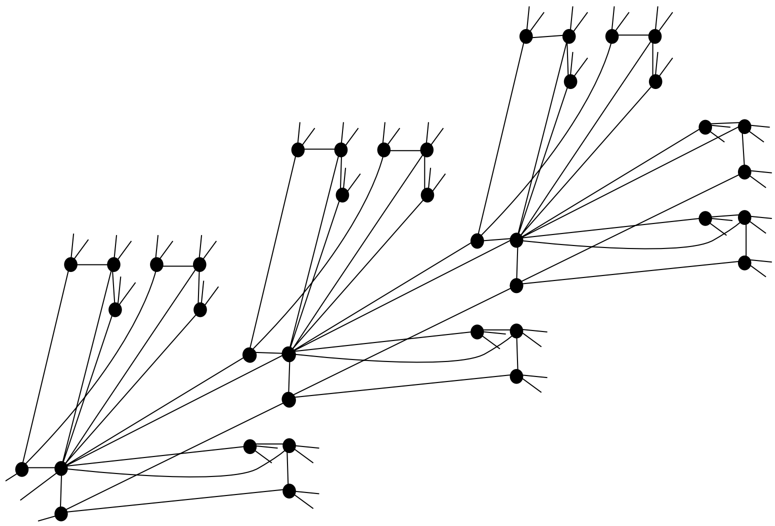

Figure 5.1: The graph from Example 5.2.1, with $r = 5$.

of the edge- and the vertex-degree of some end ω. If we defined this ratio as the degree of ω then large degrees at vertices and ends do not force large complete minors. For an example, see [42].

5.2.2 The role of the separation property

Let us now explain the reason for requiring the vertex-boundaries $\partial_v H_i$ of the graphs H_i to be minimal $\partial_v H_{i-1}$–$\Omega(H_i)$ separators (we shall call this property the *separation property*). First, let us show that a very similar condition, namely requiring the $\partial_v H_i$ to be minimal $\partial_v H_{i-1}$–$\{\omega\}$ separators, is too weak. Even locally finite graphs can satisfy a degree condition thus modified and still not contain any large complete minors. In order to see this, consider the following example.

Example 5.2.1. We start out with the r-regular (infinite) tree T_r with levels L_i. For each of its vertices, we colour half of the edges going to the next level blue, the other half red. Then replace each vertex $v \in V(T_r)$ with a path $x_v y_v z_v$. For each $vw \in E(T_r)$, with v from the ith and w from the $(i+1)$th level of T_r, say, we do the following. If vw is blue, then add the edges $x_v x_w$, $y_v y_w$ and $y_v z_w$, and if vw is red, then add the edges $y_v x_w$, $y_v y_w$ and $z_v z_w$.
The resulting graph Γ_r is locally finite, and it is easy to see that Γ_r does not contain any complete minor of order greater than 4 using the fact that the vertices of any such minor can not be separated by any 2-separator of Γ_r.
Note that each end of Γ_r has relative degree 1. In fact, let $\omega \in \Omega(\Gamma_r)$, and let $\omega' \in \Omega(T_r)$ be the corresponding end of T_r. Suppose $v_1 v_2 v_3 \ldots \in \omega'$ with v_1 being the root of T_r. For $v \in V(T_r)$, let H_v be the subgraph of Γ_r that is induced by all vertices x_w, y_w, z_w such that w lies in the upper closure $\lceil v \rceil$ of v in the tree order of T_r. Then $(H_{v_i})_{i \in \mathbb{N}} \to \omega$, and thus $d_{e/v}(\omega) = 1$.
However, if we would require the $\partial_v H_i$ to be minimal $\partial_v H_{i-1}$–$\{\omega\}$ separators instead of minimal $\partial_v H_{i-1}$–$\Omega^G(H_i)$ separators, then the ends of Γ_r had relative degree $(r+2)/2$. In fact, as the $\partial_v H_{v_i}$ are not minimal $\partial_v H_{i-1}$–$\{\omega\}$ separators, the

sequence $(H_{v_i})_{i\in\mathbb{N}}$ is no longer taken into account when we calculate the relative degree of ω. It is not difficult to see that the relative degree would then be determined by the sequence $(K_{v_i})_{i\in\mathbb{N}}$, which we define now. If v_iv_{i+1} is blue, then K_{v_i} is the subgraph of Γ_r that is induced by $\{x_{v_i}, y_{v_i}\} \cup \{x_w, y_w, z_w : w \in \lceil v \rceil, v \in L_{i+1}, v_iv \text{ is a blue edge}\}$. Otherwise, let K_{v_i} be induced by $\{x_{v_i}, y_{v_i}\} \cup \{x_w, y_w, z_w : w \in \lceil v \rceil, v \in L_{i+1}, v_iv \text{ is a red edge}\}$. As $|\partial_e K_i|/|\partial_v K_i| = (r+2)/2$ for all $i > 1$, the end ω would thus have relative degree $(r+2)/2$.

On the other hand, if we totally gave up the requirement that the $\partial_v H_i$ are minimal separators, then basically every end of every graph would have relative degree 1. To see this, write $(H_i)_{i\in\mathbb{N}} \rightsquigarrow \omega$ if $(H_i)_{i\in\mathbb{N}}$ is an infinite sequence of distinct regions of G with $H_{i+1} \subseteq H_i - \partial_v H_i$ such that ω has a ray in H_i for each $i \in \mathbb{N}$. Let $(H_i)_{i\in\mathbb{N}}$ with $(H_i)_{i\in\mathbb{N}} \rightsquigarrow \omega$, and let $v_i \in \partial_v H_{3i}$ for $i \in \mathbb{N}$. Then the v_i do not have common neighbours. We construct a sequence $(H'_j)_{j\in\mathbb{N}}$ with $H'_0 := H_0$, and, for $j > 0$, we let $H'_j := H_{i_j} - V_j$ where i_j is such[5] that $H_{i_j} \subseteq H'_j - \partial_v H'_j$, and V_j consists of $j|\partial_e H_{i_j}|$ vertices v_i with $i \geq i_j$. Then $(H'_j)_{j\in\mathbb{N}} \rightsquigarrow \omega$, and

$$\liminf_{j\to\infty} \frac{|\partial_e H'_j|}{|\partial_v H'_j|} = \liminf_{j\to\infty} \frac{|\partial_e H_{i_j}| + \sum_{v\in V_j} d(v)}{|\partial_v H_{i_j}| + \sum_{v\in V_j} d(v)} = 1.$$

This shows that the additional condition that $\partial_v H_{i+1}$ is an $\subseteq$-minimal $\partial_v H_i$–$\Omega^G(H_{i+1})$ separator is indeed neccessary for our definition of the relative degree to make sense.

For our notion, every integer, and also $\aleph_0$, appears as the relative degree of an end in some locally finite graph (larger cardinals only appear in non-locally finite graphs). Indeed, let $k \in \mathbb{N}$, and let G be obtained from the disjoint union of $\aleph_0$ many copies K_i of K^k by adding all edges between K_i and K_{i+1}, for all i. Suppose $(H_i)_{i\in\mathbb{N}} \to \omega$ for the unique end ω of G. Then by the separation property, we can conclude inductively that each $\partial_v H_i$ is contained in some K_{i_j}. Thus $d_{e/v}(\omega) = k$. A similar example can be constructed to find an end of relative degree $\aleph_0$.

5.2.3 From locally finite to arbitrary graphs

In arbitrary infinite graphs, we have to face the problem that there might be vertices dominating our end ω, and hence no sequence of subgraphs H_i can satisfy $H_{i+1} \subseteq H_i - \partial_v H_i$. Our way out of this dilemma was to delete the dominating vertices temporarily, find the sequences as above and calculate the corresponding infimum, and then add $|Dom(\omega)|$ to the relative degree.

One might think that alternatively, we might have weakened our requirements on the sequences of ω-regions H_i. For instance we might be satisfied with them obeying $H_{i+1} \subseteq H_i - (\partial_v H_i - Dom(\omega))$. We then should also require that $H_i - Dom(\omega)$ is connected, as otherwise our sequence may 'converge' to more than one end. When no such sequence existed, we would set $\tilde{d}_{e/v}(\omega) := \infty$. This alternative definition would have the advantage that the contribution of a dominating vertex, in terms of outgoing edges, to the edge-boundary of an ω-region is counted.

[5]For instance set $i_j := \max\{dist(v, w) | v \in \partial_v H_0, w \in \partial_v H'_j\} + 1$.

However, the approach does not allow for an extension of Theorem 5.1.1. The problem is vertices that dominate more than one end. Consider an infinite r-regular tree to which we add one vertex that is adjacent to all other vertices. The ends of this graph have infinite degree in the sense just discussed (and relative degree 2), but of course, no K^k-minor for large k, no matter how large r is with respect to k. We can give a similar example for a graph with only two ends.[6]

5.3 Dominating vertices and topological K^k-minors

This section provides some results about dominating vertices that will be needed later on. First, we give a useful characterization of dominating vertices.

Lemma 5.3.1. *In any graph, a vertex v dominates an end ω if and only if there is no finite v–ω separator.*

Proof. For the forward direction, note that every finite set of vertices intersects only a finite number of the infinitely many v–$V(R)$ paths, where $R \in \omega$ is dominated by v. Hence v and ω cannot be finitely separated.

For the backward direction we inductively construct a set of v–$V(R)$ paths, where R is any ray in ω. At each step we use the fact that all paths constructed so far form a finite set which (without v) does not separate v from ω. Hence we can always add a new v–$V(R)$ path, which disjoint from all the others (except in v). $\square$

We now show that for an end ω which is dominated by only finitely many vertices, the graph G_ω contains sequences of subgraphs converging to $\hat{\omega}$.

Lemma 5.3.2. *Let G be a graph, and let $\omega \in \Omega(G)$ with $|Dom(\omega)| < \aleph_0$. Then G_ω contains a sequence $(\hat{H}_i)_{i\in\mathbb{N}} \to \hat{\omega}$.*

Proof. We define a sequence of disjoint finite sets $S_i \subseteq V(G_\omega)$, starting with any finite non-empty set S_1. For $i > 1$ and for each $v \in S_{i-1}$ let S_v^{i-1} be a finite v–$\hat{\omega}$ separator in G_ω (note that such a separator exists by Lemma 5.3.1). Set

$$\tilde{S}_i := \bigcup_{v\in S_{i-1}} S_v^{i-1} \setminus S_{i-1}.$$

Then $\tilde{S}_i$ separates S_{i-1} from $\hat{\omega}$. In fact, otherwise there would be a ray $R \in \hat{\omega}$ with only its first vertex v in S_{i-1}, and disjoint from $\tilde{S}_i$. But R must meet S_v^{i-1}, a contradiction.

Choose $S_i \subseteq \tilde{S}_i$ minimal such that it separates S_{i-1} from $\hat{\omega}$. Now for $i \in \mathbb{N}$, let K_i be the component of $G_\omega - S_i$ that contains a ray of $\hat{\omega}$ (since $S_i \cup Dom(\omega)$ is finite there is a unique such component K_i). Let $\hat{H}_i$ be the subgraph of G_ω that is induced by S_i and K_i. Then, by the choice of S_i, we find that S_i is a minimal S_{i-1}–$\Omega^G(\hat{H}_i)$ separator. Hence, $(\hat{H}_i)_{i\in\mathbb{N}} \to \hat{\omega}$, as desired. $\square$

[6] Take two copies of the r-regular tree add a spanning path in each level of each of the two trees and a new vertex adjacent to all other vertices.

Next, we will see that our desired minor is easy to find whenever there are enough vertices dominating the same end.

Lemma 5.3.3. *Let $k \in \mathbb{N}$, let G be a graph and let $\omega \in \Omega(G)$. If $|Dom(\omega)| \geq k$, then $K^k \preceq_{top} G$.*

Lemma 5.3.3 follows at once from Lemma 5.3.4 below. The *branching vertices* of a subdivision are those vertices that did not arise from subdividing edges.

Lemma 5.3.4. *Let $k \in \mathbb{N}$, let G be a graph, let $\omega \in \Omega(G)$, and let $S \subseteq Dom(\omega)$ with $|S| = k$. Then G contains a subdivision TK^k of K^k whose branching vertices are in S.*

Proof. We use induction on k, the base case $k = 0$ is trivial. So suppose $k \geq 1$. Then, let $S \subseteq Dom(\omega)$ be a set of size k, and let $s \in S$. By the induction hypothesis, $G - s$ contains a subdivision TK^{k-1} of K^{k-1} with branching vertices in $S' := S \setminus \{s\}$.

Successively we define sets $\mathcal{P}_i$ of s–S' paths in G which are disjoint except in s. We start with $\mathcal{P}_0 := \emptyset$. For $i > 0$, suppose there is a vertex $v \in S'$ which is not the endpoint of a path in $\mathcal{P}_{i-1}$. Then, $S_i := S \cup \bigcup_{P \in \mathcal{P}_{i-1}} V(P)$ is finite, and $G - S_i$ has a unique component C_i which contains rays of ω. Since Lemma 5.3.1 implies that neither s nor v can be separated from ω by a finite set of vertices, both s and v have neighbours in C_i. Hence there is an s–v path P_i that is internally disjoint from S_i. Set $\mathcal{P}_i := \mathcal{P}_{i-1} \cup \{P_i\}$. The procedure stops after step $k - 1$, when all vertices of S' are connected to s by a path in $\mathcal{P}_i$. This gives the desired subdivision TK^k. $\square$

With a very similar proof,[7] we also get the following statement (which will not be needed in what follows):

Lemma 5.3.5. *Let G be a graph and let $\omega \in \Omega(G)$. If $|Dom(\omega)| \geq \aleph_0$, then $K^{\aleph_0} \preceq_{top} G$.*

We finish this section with one more basic lemma. This lemma implies that removing a finite part of a graph, or even an infinite part with a finite vertex-boundary, will not alter the relative degree of the remaining ends.

Lemma 5.3.6. *Let G be a graph, let $\omega \in \Omega(G)$, and let G' be an induced subgraph of G such that $\partial_v^G G'$ is finite, and such that G' has an end ω' that contains rays of ω. Then $d_{e/v}(\omega') = d_{e/v}(\omega)$.*

Proof. First of all, observe that since $\partial_v^G G'$ is finite, it follows that $Dom(\omega) \subseteq V(G')$. Hence, we only need to show that $\inf_{(\hat{H}_i)_{i\in\mathbb{N}}} \liminf_i(|\partial_e \hat{H}_i|/|\partial_v \hat{H}_i|)$, is the same for sequences $(\hat{H}_i)_{i\in\mathbb{N}}$ in G_ω and for sequences $(\hat{H}_i)_{i\in\mathbb{N}}$ in $G'_{\omega'}$.

[7] We construct the $TK^{\aleph_0}$ step by step, adding one branching vertex plus the corresponding paths at a time. In each step, the finiteness of the already constructed part ensures the existence of an unused dominating vertex s of ω and enough paths to connect s to the already defined branching vertices.

For this, note that every sequence $(\hat{H}_i)_{i\in\mathbb{N}} \to \hat{\omega}$ in G_ω has a subsequence $(\hat{H}_i)_{i\geq i_0} \to \hat{\omega}'$ in $G'_{\omega'}$ (it suffices to take $i_0 := \max\{dist(v,w) : v \in \partial_v \hat{H}_0, w \in \partial_v G'\} + 1$). Moreover, for every sequence $(\hat{H}'_j)_{j\in\mathbb{N}} \to \hat{\omega}'$ in $G'_{\omega'}$, there is an index j_0 such that $\partial_v \hat{H}'_j \cup N(\hat{H}'_j)$ is the same set in G_ω and in G'_ω (for instance, take $j_0 := \max\{dist(v,w) : v \in \partial_v \hat{H}'_0, w \in \partial_v G'\} + 1$). Hence, the relative degrees of ω and ω' are the same. □

5.4 Proof of Theorem 5.1.3

In this section we prove our main result, Theorem 5.1.3. We start by showing how to find, for a fixed end ω of some graph G, an $\hat{\omega}$-region $\hat{H}$ of G_ω that has an acceptable average degree into $G - \hat{H}$.

Lemma 5.4.1. *Let G be a graph, let $\omega \in \Omega(G)$ with $d_{e/v}(\omega) > m$ for some $m \in \mathbb{Q}$, and let $S \subseteq V(G)$ be finite. If $|Dom(\omega)| < \aleph_0$, then G_ω has a $\hat{\omega}$-region $\hat{H}$ such that*

(a) $S \cap V(\hat{H}) = \emptyset$, *and*

(b) $\frac{|\partial_e \hat{H}|}{|\partial_v \hat{H}|} > m - |Dom(\omega)|$.

Proof. By Lemma 5.3.2, there is a sequence $(\hat{H}_i)_{i\in\mathbb{N}} \to \hat{\omega}$ in G_ω. Let $i_0 = \max\{dist_{G_\omega}(v,w) : v \in S \setminus Dom(\omega), w \in \partial_v \hat{H}_0\}+1$. Then $S \cap V(\hat{H}_i) = \emptyset$ for all $i \geq i_0$. As $d_{e/v}(\omega) > m$, there is a $j_0 \geq i_0$ such that $(|\partial_e \hat{H}_j|/|\partial_v \hat{H}_j|) > m - |Dom(\omega)|$ for all $j \geq j_0$. □

We now apply Lemma 5.4.1 repeatedly to the ends of any suitable fixed countable subset of $\Omega(G)$. Lemma 5.3.6 will ensure that the relative degree of the ends is not disturbed by what has been cut off earlier.

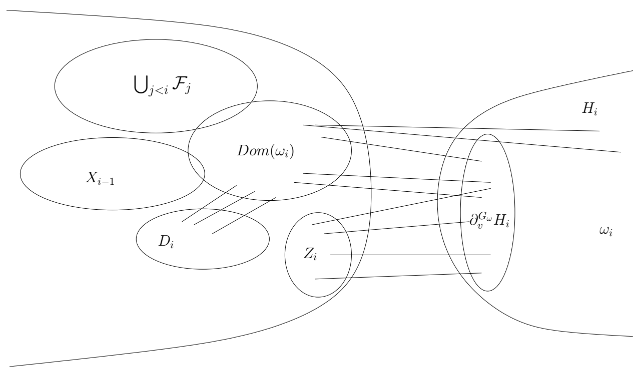

Figure 5.2: Construction of the graph G_i in Lemma 5.4.2.

Lemma 5.4.2. *Let $m \in \mathbb{Q}$, let G be a graph, let $X \subseteq V(G)$ be finite and for $i \in \mathbb{N}$, let $\omega_i \in \Omega(G)$ with $|Dom(\omega_i)| < \aleph_0$. If $\delta^{V,\Omega}(G) > m$, then G contains induced subgraphs G_i, finite sets $X \subseteq X_i \subseteq V(G_i)$, and finite sets $\mathcal{F}_i$ of pairwise disjoint finite subsets of $V(G_i)$ such that for each $i \in \mathbb{N}$*

(A) $G_{i+1} \subseteq G_i$, $X_i \subseteq X_{i+1}$, *and* $\mathcal{F}_{i+1} \supseteq \mathcal{F}_i$,

(B) $X_i \cup \bigcup \mathcal{F}_i \subseteq V(G_i)$,

(C) *there is a family* $\mathcal{H}_i = \{H_F : F \in \mathcal{F}_i\}$ *of disjoint connected subgraphs such that* $V(H_F) \cap V(G_i) = F$ *for each* $F \in \mathcal{F}_i$ *and such that* $G = G_i \cup \bigcup \mathcal{H}_i$,

(D) $\partial_v^G G_i$ *is finite,*

(E) *the average degree of* F *into* $F \cup X_i$ *is* $> m - |Dom(\omega_i)|$, *for each* $F \in \mathcal{F}_i$,

(F) $d_{X_i}(v) > m$ *for all* $v \in \partial_v^G G_i \setminus \bigcup \mathcal{F}_i$, *and*

(G) ω_i *has no rays in* G_i.

Proof. Set $G_0 := G$, $X_0 := X$, and $\mathcal{F}_0 := \emptyset$. Now, for $i \geq 1$ we do the following. If ω_i has no ray in G_{i-1}, then we set $G_i := G_{i-1}$, $X_i := X_{i-1}$ and $\mathcal{F}_i := \mathcal{F}_{i-1}$, which ensures all the desired properties (as they hold for $i-1$).

So suppose ω_i does have a ray in G_{i-1}. Then let D_i be a finite set of vertices of G_{i-1} so that each dominating vertex of ω_i in G_{i-1} has degree greater than m into D_i. (This is possible since by assumption there only finitely many vertices dominating ω_i.)

Observe that because of Equation D we may apply Lemma 5.3.6 to obtain that all ends of G_{i-1} have relative degree $> m$. Hence Lemma 5.4.1 applied to G_{i-1} and the finite set

$$S := X_{i-1} \cup D_i \cup \bigcup \mathcal{F}_{i-1}$$

yields an $\hat{\omega}_i$-region $\hat{H}_i$ of G_ω. We set $\mathcal{F}_i := \mathcal{F}_{i-1} \cup \partial_v \hat{H}_i$ and $G_i := G_{i-1} - (\hat{H}_i - \partial_v \hat{H}_i)$. Choose a finite subset Z_i of $V(G_i)$ such that $\partial_v \hat{H}_i$ has average degree $> m - |Dom(\omega_i)|$ into $Z_i \cup \partial_v \hat{H}_i$ and set $X_i := X_{i-1} \cup D_i \cup Z_i$.

Then, conditions (A), (D) and (G) are clearly satisfied for step i, as they hold for step $i-1$. Conditions (B) and (C) for i follow from Lemma 5.4.1(a), and from (B) and (C) for $i-1$. Condition (E) follows from Lemma 5.4.1(b) and (E) for $i-1$.

Finally, for (F) suppose that $v \in \partial_v^G G_i \setminus \bigcup \mathcal{F}_i$. If $v \in \partial_v^G G_{i-1}$ then (F) for i follows from (F) for $i-1$. Otherwise, v dominates ω_i. Then by construction, v has sufficiently many neighbours in $D_i \subseteq X_i$. $\square$

If G has only countably many ends, then the procedure just described can be used to cut off all ends:

Lemma 5.4.3. *Let* $k \in \mathbb{N}$, *let* $m \in \mathbb{Q}$, *let* G *be a graph with* $|\Omega(G)| \leq \aleph_0$, *and let* $X \subseteq V(G)$ *be finite. Suppose* $|Dom(\omega)| < k$ *for all ends* $\omega \in \Omega(G)$. *If* $\delta^{V,\Omega}(G) > m$, *then* G *has an induced subgraph* G' *and a set* $\mathcal{F}$ *of finite pairwise disjoint vertex sets such that*

(i) $X \cup \bigcup \mathcal{F} \subseteq V(G')$,

(ii) *there is a family* $\{H_F : F \in \mathcal{F}\}$ *of disjoint connected subgraphs of* G *such that* $V(H_F) \cap V(G') = F$ *for each* $F \in \mathcal{F}$,

(iii) *for each vertex* $v \in V(G')$ *of degree* $\leq m$ *there is an* $F \in \mathcal{F}$ *so that* $v \in F$ *and the average degree of* F *in* G' *is* $> m - k + 1$, *and*

(iv) every ray of G has only finitely many vertices in $V(G')$.

Proof. Let $\omega_1, \omega_2, \omega_3, \ldots$ be a (possibly repetitive) enumeration of $\Omega(G)$. Apply Lemma 5.4.2, and then set $G' := \bigcap_{i\in\mathbb{N}} G_i$ and $\mathcal{F} := \bigcup_{i\in\mathbb{N}} \mathcal{F}_i$. We claim that G' and $\mathcal{F}$ are as desired. Indeed, properties (A) and (B) imply property (i), and property (C) together with (A) implies property (ii).

For property (iii) observe that (A) and (B) imply that $X_i \subseteq V(G')$ for all $i \in \mathbb{N}$. Now (iii) follows from (E) and (F) together with the assumption that $\delta^{V,\Omega}(G) > m$.

In order to see (iv), suppose that R is a ray of G that has infinitely many vertices in G'. Say $R \in \omega_j$. Then by (A) for j, the ray R has infinitely many vertices in G_j. So, as $\partial_v^G G_j$ is finite by (D), R has a subray in G_j, a contradiction to (G) for j. $\square$

We are now almost ready to prove our main theorem. We will make use of a standard tool from infinite graph theory, Kőnig's infinity lemma.

Lemma 5.4.4. [12] *Let $V_1, V_2, V_3, \ldots$ be disjoint finite non-empty sets, and let G be a graph on their union. Suppose that for all $i \in \mathbb{N}$, each vertex of V_{i+1} has a neighbour in V_i. Then G has a ray $v_1v_2v_3\ldots$, with $v_i \in V_i$, for each $i \in \mathbb{N}$.*

Let us now prove Theorem 5.1.3.

Proof of Theorem 5.1.3. Suppose that K^k is not a topological minor of G. Then by Lemma 5.3.3, $|Dom(\omega)| < k$ for all $\omega \in \Omega(G)$. Let $u \in V(G)$ and let G' be the subgraph we obtain from Lemma 5.4.3 applied to G and $X := \{u\}$, and let $\mathcal{F}$ be the corresponding set of disjoint finite vertex sets.

For $i \in \mathbb{N}$, we shall successively define finite sets S_i, with $S_i \subseteq S_{i+1}$. We start with setting $S_0 := \emptyset$ and $S_1 := \{u\}$ if $u \notin \bigcup \mathcal{F}$, or $S_1 := F_u$ if there is an $F_u \in \mathcal{F}$ with $u \in F_u$ (by the disjointness of the sets in $\mathcal{F}$, there is at most one such F_u). Note that $S_0 \subseteq S_1 \subseteq V(G')$ because of Lemma 5.4.3 (i).

Our sets S_i will have the following properties for $i \geq 1$:

(I) the average degree of the vertices of S_{i-1} in $G'[S_i]$ is $> m - k + 1$,

(II) the average degree of the vertices of $S_i \setminus S_{i-1}$ in G' is $> m - k + 1$, and

(III) for each $F \in \mathcal{F}$ with $F \cap S_i \neq \emptyset$ there is a $j \leq i$ such that $F \subseteq S_j \setminus S_{j-1}$.

For $i = 1$, property (I) holds trivially, and (II) is satisfied because of Lemma 5.4.3 (iii). By the choice of S_1 and since the $F \in \mathcal{F}$ are disjoint, also (III) holds.

Now, for $i \geq 2$ we choose a finite subset X_i of the neighbourhood of $S_{i-1} \setminus S_{i-2}$ in $G' - S_{i-1}$ so that the average degree of the vertices in S_{i-1} in the graph $G'[S_{i-1} \cup X_i]$ is at least $m - k + 1$. Such a choice is possible by (I) and (II) for $i - 1$.

Let Y_i denote the union of all $F \in \mathcal{F}$ that contain some $v \in X_i$. Note that Y_i is finite since the F are all disjoint and because X_i is finite. Then set $S_i :=$

$S_{i-1} \cup X_i \cup Y_i$. Our choice of the S_i clearly satisfies conditions (I), (II) and (III). This finishes our definition of the sets S_i.

First suppose that $S_i \neq S_{i-1}$ for all $i \in \mathbb{N}$. Then, for each $i \in \mathbb{N}$, let V_i be obtained from $S_i \setminus S_{i-1}$ by collapsing each $F \in \mathcal{F}$ with $F \subseteq S_i \setminus S_{i-1}$ to one vertex v_F (which will be adjacent to all neighbours of F outside F). So, for all $i \in \mathbb{N}$, each vertex of V_{i+1} has a neighbour in V_i, and therefore, we may apply Kőnig's infinity lemma (Lemma 5.4.4) to the sets V_i in order to find a ray R in $\bigcup_{i \in \mathbb{N}} V_i$. We use (III) and Lemma 5.4.3 (ii) to expand R to a ray R' in G. As R' has infinitely many vertices in G', this establishes a contradiction to Lemma 5.4.3 (iv).

So we may assume that there is an $i \in \mathbb{N}$ such that $S_i = S_{i-1}$. Then, by (I), $G'[S_i]$ is a finite graph of average degree $> m - k + 1$, which is as desired. $\square$

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