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Rapport sur la thèse de doctorat
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Titre : Degenerate Parabolic Stochastic Partial Differential Equations
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Stochastic Partial Differential Equations constitute a new and active domain of present mathematical analysis. This PhD contributes to a new aspect, dealing with degenerate parabolic equations, kinetic equations and their relaxation toward a hyperbolic scalar conservation law with stochastic forcing. To my knowledge Debussche et Vovelle are the only authors to have touched this topics and their method and ideas allow to go much further than whatever existed before. Notice that a chapiter proposes to come back on an older subject, namely the construction of Ito's solutions to Stochastic Differential Equations, when coefficients are merely continous.

It is also noticeable that several parts of the PhD manuscript have already been published or are accepted for 5 publications in international journals.

The core of the work on SPDE deals with problems as

$$du(t) + \mathrm{div}B(u)dt - \mathrm{div}A(x)\nabla u dt = \Phi(u)dW(t), \tag{1}$$

and the construction goes through the kinetic version of it

$$\partial_t f + b(\xi)\nabla f - \mathrm{div}A(x)\nabla f = \delta(u-\xi)\Phi(\xi)\dot{W} + \partial_\xi[m - \frac{1}{2}G^2(x,\xi)\delta(u-\xi)],$$

where $W(t)$ is a brownian motion. The main difficulty is degeneracy of the matrix A that can possibly vanish completely. Hence the difficulties of hyperbolic conservation laws are underlying the theory developed in the thesis : a correct weak formulation is needed as well as a refined construction process.

Chapter 2 gives the first step of this construction when the coeficients are smooth enough and non-degenerate. Existence of a unique solution is achieved in Sobolev spaces with integrability $p > 2$ and has as many derivatives as possible from the coeficients regularity. The steps here use a technical chain rule in the appropriate spaces, a construction through Picard iterations, stochastic integration in L^p spaces and refined estimates on stochastic integrals in Sobolev spaces,

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Chapter 3 addresses the well posedness for (1) when the matrix A can degenerate together with a contraction principle stated with expectancy of L^1 norms in space. In order to handle the degenerate parabolic term, it takes a different route than Debussche and Vovelle by proving directly the strong convergence of solutions built in Chapter 2. The approach is based on the definition of a convenient notion of kinetic formulation which keeps the trace of the parabolic dissipation in the nondegeneracy zone as introduced in Chen and Perthame. To analyze it, a new kind of martingale solution to SPDE, which is also related to the last chapter of the thesis, is introduced which is the heart of the approach. It has to be combined with refined technical estimates that are deep enough to deal with the weak regularity available. Other ingredients are Young measures, a deep pathwise uniqueness (contraction) result for kinetic solutions. Existence is obtained by regularization (and reduction to the result of the previous chapter) ; then uniform bounds are proved and compactness is obtained thanks to a Sobolev bound with low exponent for their law that allows the authors to derive that the limit solve the appropriate martingale problem and then is a pathwise solution.

Chapter 4 introduces another regularization method towards the weak solution in the fully degenerate (purely hyperbolic) case. The method is the BGK equation introduced in [65] and the stochastic forcing lead to modify the right hand side of the BGK equation so as to take into account the new 'random dissipation' in an explicit form so as to recover Ito's type of corrections. The existence of a solution to this new problem is proved and the relaxation limit toward the stochastic conservation law à la 'Debussche-Vovelle' are studied in this chapter. At first glance this might seem a direct extension of the deterministic case but this is not true ; several difficulties arise. Among them the characteristic equation, needed to define solutions, has completely different properties because it becomes a SDE after reformulation in a Stratonovich form. As a consequence the 'hyperbolic' property of finite speed of propagation is lost and technical assumptions are needed. This chapter is really remarkable.

Chapter 5 treats of a new construction of SDE with coefficients that are only continuous with a growth control at infinity.The unity with the previous chapters stems from the similar methods handled here which are inspired from SPDEs. This construction is of purely probabilistic interest and my only comment is that the question is very subtile, related to solutions in law or a.e. (that is the selection of a correct measurable representative), according to seminal papers of Ito and Skorokhod.

Not only the subject is timely, but also all the results are new and require difficult methods and innovative concepts. The manuscript is written with all the details and there is no doubt that M. HOFMANOVÁ has a global view of the methods, techniques and of the subject.

There is no doubt that the PhD can be defenced.

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