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ZOBECNĚNÉ BOOLEOVSKÉ MODELY A KLASICKÁ PREDIKÁTOVÁ LOGIKA

GENERALIZED BOOLEAN MODELS AND CLASSICAL PREDICATE LOGIC

Bakalářská práce

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Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl všechny použité prameny a literaturu.

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Abstract

This bachelor thesis is dealing with complete Boolean algebras and its use in semantics of first-order predicate logic. This thesis has two main goals, at first it is to show that every Boolean algebra can be extended to a complete Boolean algebra such that the former Boolean algebra is its dense subalgebra. This theorem is proved using topological construction. Afterwards, in the second part, we define semantics for first-order predicate logic with respect to complete Boolean algebras, which includes introduction of the Boolean-valued model. Then we prove completeness theorem with respect to all complete Boolean algebras. The theorem is proven using ultrafilters on Boolean algebras.

Keywords: Boolean algebras, complete Boolean algebras, classical logic.

Abstract

Tato bakalářská práce pojednává o úplných Booleových algebrách a o jejich užití v semantice prvořádové predikátové logiky. Práce má dva hlavní cíle, v první řadě dokázat, že každá Booleova algebra může být rozšířena na úplnou Booleovu algebru tak, že původní algebra je její hustá podalgebra. Toto tvrzení je dokázáno pomocí topologické kontrukce. Následně, ve druhé části, definujeme sémantiku prvořádové predikátové logiky s ohledem na úplné Booleovy algebry, současně také zavedeme pojem Booleovsky- ohodnoceného modelu. Poté dokážeme větu o úplnosti s ohledem na všechny úplné Booleovy algebry. To je dokázáno pomocí ultrafiltrů na Booleových algebrách.

Klíčová slova: Booleovy algebry, úplné Booleovy algebry, klasická logika.

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1 Introduction

In this thesis we will prove that every Boolean algebra can be extended to a complete Boolean algebra, we will demand for this complete Boolean algebra to satisfy some properties (see Definition 2.33). Then, in the second part, we we will speak of satisfaction in complete Boolean algebras.

The motivation of this work can be described as follows. Standard semantics for the first-order predicate logic is actually a semantics with respect to the Boolean algebra $\{0, 1\}$, we can speak of a so called algebraic semantics. Our main goal is to generalize this notion to all complete Boolean algebras, in other words to prove that the first-order predicate logic is complete with respect to the class of all complete Boolean algebras.

In Section 2, we prove that for every Boolean algebra B, there is a unique complete Boolean algebra, we denote it $\mathrm{cm}(B)$, such that B is a dense subalgebra of $\mathrm{cm}(B)$. We prove this using a topological construction by Balcar and Štěpánek ([1]). At the beginning of this section, after defining basic terms, we speak of regular open sets. We define the system of all regular open sets of a topological space, $\mathrm{RO}(X)$, and show that with properly defined operations it is a complete Boolean algebra, which we will denote as $\mathrm{B}(\mathrm{RO}(X))$ (Theorem 2.13). Then we define the notion of a separated ordering and show that every partially ordered set can be factorized to a separative partially ordered set (Theorem 2.21). In the next subsection, we concentrate on dense subsets. We show that every element b in Boolean algebra B can be expressed by certain subset of a dense subset of B (Lemma 2.26), moreover we show that two complete Boolean algebras with isomorphic dense subsets are also isomorphic (Theorem 2.30), which is the key statement to prove the uniqueness of the completion, $\mathrm{cm}(B)$. In the next subsection we speak of the topology of lower subsets. In the proof of the completion theorem we use this important fact: Let (Q, τ) be a topology of lower subsets based on the separative partially ordered set Q, then for every q in Q the smallest lower subset containing q, $(\leftarrow, q]$, is in $\mathrm{RO}(Q)$. And finally we prove completion Theorems 2.34 and 2.36. In these theorems we use the fact that $(B^+, \leq)$ is a separative partial order. However, we also mention in Corollary 2.35 a weaker version for orderings, which are not separated.

In Section 3, we prove completeness theorem with respect to all complete Boolean algebras. First we define Boolean-valued models following [4]. Later we define full Boolean-valued models and we show that every Boolean-valued model can be extended to a full Boolean-valued model, which satisfies some important properties, see Theorem 3.15. In the next subsection we discuss ultrafilters. For a full Boolean-valued model M^B and ultrafilter G on B, we show how to construct the quotient M/G, a two-valued model (Theorem 3.21). In Theorem 3.24 we prove the completeness. In this theorem we use the notion of a quotient model of a full Boolean-valued model, which enables us to reduce the completeness to the completeness theorem for standard two-valued predicate logic, which we suppose as a fact.

2 Completion theorem for BAs (Boolean algebras)

In this section we prove completion theorem for BAs. The greatest part of this section is inspired by Balcar and Štěpánek, [1].

2.1 Introduction to BAs

Definition 2.1. A structure $(B, \vee, \wedge, -, 0, 1)$ with binary functions $\vee, \wedge$, which we denote as join and meet, and unary function $-$, which we call complement, and constants 0 a 1 is called *Boolean algebra* if following axioms are satisfied:

(i) *Associativity* $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, $x \vee (y \vee z) = (x \vee y) \vee z$

(ii) *Commutativity* $x \wedge y = y \wedge x$, $x \vee y = y \vee x$

(iii) *Absorption* $x \wedge (x \vee y) = x, x \vee (x \wedge y) = x$

(iv) *Distributivity* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(v) *Complement* $x \wedge (-x) = 0, x \vee (-x) = 1$

We say that Boolean algebra B is *complete* if for every $S \subseteq B$ there exist $\bigvee S = \sup$[1](S)[2].

Definition 2.2. Let B be a boolean algebra, we define *canonical ordering* $\leq$ on B as follows: For every $x, y \in B$: $x \leq y \leftrightarrow_{\text{def}} x \wedge y = x$.

Definition 2.3. Let B be a Boolean algebra. We say that the elements $x, y \in B$ are *disjoint* and write $x \perp y$ if $x \wedge y = 0$.

For the purpose of this work, we mention only one more property of BAs, which will be widely used. For more detailed information on BAs see [1] or [5].

Let B be a BA, then for every $x, y \in B$:

$$x \leq y \leftrightarrow x \wedge -y = 0 \tag{1}$$

[1]For the definition of the supremum see definition 2.19 on page 14.

[2]It can be proved that if for a given set S the supremum exists then also the infimum exists (and vice versa), thus every subset of a complete BA B has both supremum and infimum.

2.2 Regular open sets

Definition 2.4. Let X be a nonempty set and τ a subset of $P(X)$. Assume (X, τ) satisfies the following conditions:

(i) $\emptyset, X \in \tau$

(ii) $A, B \in \tau$ then $A \cap B \in \tau$

(iii) let I be a set and $\{A_i \in X \mid i \in I\}$ be a family of sets in τ, then the union $\cup_{i \in I} A_i$ is also in τ.

Then we call the pair (X, τ) a *topological space* and the system τ a *topology* on X. We call the set A open, if $A \in \tau$. If A is open, then its complement $X \setminus A$ is called closed.

Definition 2.5. Let (X, τ) be a topological space and let $A \subseteq \tau$ be given. We define:

(i) *Closure* of A as the smallest closed superset of A and we denote it $\mathrm{cl}(A)$, i.e. $\mathrm{cl}(A)$ is the intersection of all closed sets containing A.

(ii) *Interior* of A as the greatest open subset of A and we denote it $\mathrm{int}(A)$, i.e. $\mathrm{int}(A)$ is the union of all open sets contained in A.

(iii) *Regularization* of A as $\mathrm{r}(A) = \mathrm{int}(\mathrm{cl}(A))$.

Fact 2.6. Properties of interior and closure

(i) Int and cl are monotonous functions.

(ii) $\mathrm{int}(A) = \mathrm{int}(\mathrm{int}(A)), \mathrm{cl}(A) = \mathrm{cl}(\mathrm{cl}(A))$.

(iii) A is closed(open), if and only if $A = \mathrm{cl}(A)(A = \mathrm{int}(A))$.

(iv) $A \subseteq \mathrm{cl}(A)$, $\mathrm{int}(A) \subseteq A$.

(v) $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$, $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

Definition 2.7. We call a set A *regular open set* if $\mathrm{r}(A) = A$. We denote $\mathrm{RO}(X)$ the system of all regular open sets of a topological space (X, τ).

We can imagine regularization as a function that "removes holes" from an open set. As an example let us have a topological space of real numbers $(\mathbb{R}, \tau_{\mathbb{R}})$ and fix an open set $A = (1, 3) \cup (3, 5)$. Then $\mathrm{cl}(A) = [1, 5]$, thus $\mathrm{r}(A) = (1, 5)$. We can view number 3 as a hole in the open set A. Regular open sets are then open sets without such a holes.

Observation 2.8. Operation regularization of topological space (X, τ) is *monotonous*, i.e. $A \subseteq B \subseteq X$ then $\mathrm{r}(A) \subseteq \mathrm{r}(B)$

Proof. Easy consequence of the monotonicity of operations interior and closure. □

Definition 2.9. Let (X, τ) be a topological space and $b \in X$. We say a set $V \subseteq X$ is a *neighbourhood* of b if there is an open set $U \in \tau$ such $U \subseteq V$ and $b \in U$. Moreover we say V is an *open neighbourhood* if V is open.

Lemma 2.10. Let (X, τ) be a topological space and $A \in \tau$, then for every $b \in X$: $b \in \mathrm{cl}(A)$ if and only if for every open neighbourhood V of b: $V \cap A \neq \emptyset$.

Proof. ad $\rightarrow$: Let V be such an open neighbourhood of b, so that $V \cap A = \emptyset$. Obviously the set $X \setminus V$ is closed, $A \subseteq X \setminus V$ and moreover $b \notin X \setminus V$. It easily follows that $b \notin \mathrm{cl}(A)$.

ad $\leftarrow$: Let us have $b \notin \mathrm{cl}(A)$ and define the set $B = X \setminus \mathrm{cl}(A)$. Because $\mathrm{cl}(A)$ is closed, the set B is open. Moreover it holds that B is an open neighbourhood of b and $B \cap A = \emptyset$.

□

Lemma 2.11. Set $A \subseteq X$ of a topological space (X, τ) is regular open if and only if A is open and for every $p \in X$: if there is an open neighbourhood V of p such that $V \subseteq \mathrm{cl}(A)$, then $p \in A$.

Proof. ad $\rightarrow$: Let us have $A \subseteq X$ regular open. A is obviously open. Now consider $p \in X$ with an open neighbourhood V, which satisfies the condition $V \subseteq \mathrm{cl}(A)$. If $p \notin A$, then $A \subsetneq A \cup V \subseteq \mathrm{cl}(A)$. And because the set $A \cup V$ is open, we have a contradiction with the fact that $A = \mathrm{int}(\mathrm{cl}(A))$.

ad $\leftarrow$: We will show that $A = \mathrm{int}(\mathrm{cl}(A))$. $\subseteq$: Obvious, because A is open and $A \subseteq \mathrm{cl}(A)$. $\supseteq$: if $b \in \mathrm{int}(\mathrm{cl}(A))$, then $\mathrm{int}(\mathrm{cl}(A))$ is an open neighbourhood of b and moreover $\mathrm{int}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$ and hence $b \in A$.

□

Lemma 2.12. Let A, B be two open sets then $\mathrm{r}(A \cap B) = \mathrm{r}(A) \cap \mathrm{r}(B)$.

Proof. First we show that for an open set A and for an arbitrary set Q of a topological space holds:

$$A \cap \mathrm{cl}(Q) \subseteq \mathrm{cl}(A \cap Q) \tag{2}$$

To see this let us have an element $b \in A \cap \mathrm{cl}(Q)$, by Lemma 2.10 we want to prove that every open neighbourhood V of b satisfies: $V \cap A \cap Q \neq \emptyset$. So let V be an open neighbourhood of b, because b is in A and also in V, we get $A \cap V \neq \emptyset$ and because A is open, $A \cap V$ is also open and because it contains b, it follows that $A \cap V$ is an open neighbourhood of b and hence again by Lemma 2.10 $V \cap A \cap Q \neq \emptyset$.

ad $\subseteq$: follows immediately by monotonicity of regularization.

ad $\supseteq$: first, by (2) we get:

$$A \cap \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cap B)$$

By Fact 2.6 (v) and (iv):

$$A \cap \mathrm{r}(B) \subseteq \mathrm{r}(A \cap B) \subseteq \mathrm{cl}(A \cap B)$$

Now we again apply (2) and we get: $\mathrm{cl}(A) \cap \mathrm{r}(B) \subseteq \mathrm{cl}(A \cap \mathrm{r}(B)) \subseteq \mathrm{cl}(A \cap B)$, where the last relation follows from the previous equation using monotonicity of closure. We again apply Fact 2.6 (v) and we have:

$$\mathrm{r}(A) \cap \mathrm{r}(B) \subseteq \mathrm{r}(A \cap B)$$

□

Theorem 2.13. The system $\mathrm{RO}(X)$ of a not empty topological space (X, τ) with operations:

$$A \wedge B = A \cap B,\ A \vee B = \mathrm{r}(A \cup B),\ -A = \mathrm{int}(X \setminus A)$$

and constants $0 = \emptyset$ and $1 = X$ makes a complete Boolean algebra. Moreover if $S \subseteq \mathrm{RO}(X)$ then

$$\bigwedge S = \mathrm{r}(\bigcap S) \text{ and } \bigvee S = \mathrm{r}(\bigcup S)$$

We denote this complete Boolean algebra $\mathrm{B}(\mathrm{RO}(X))$.

Proof. First we need to show that the system $\mathrm{RO}(X)$ is closed under operations. The cases of join and complement are obvious by the definition and meet follows by Lemma 2.12.

Commutativity and associativity follows by commutativity and associativity of the operations of the set theoretical functions union and intersection.

To see that distributivity holds let us have $A, B, C \in \mathrm{RO}(X)$, we known that for the set theoretical operations holds:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Now apply regularization on both sides of the equation. And by the definition and Lemma 2.12 we get for the left side:

$$\mathrm{r}(A \cap (B \cup C) = \mathrm{r}(A) \cap \mathrm{r}(B \cup C) = A \wedge (B \vee C)$$

and for the right side:

$$\mathrm{r}((A \cap B) \cup (A \cap C)) = (A \wedge B) \vee (A \wedge C)$$

thus we can conclude:

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

The second case is similar.

Absorption is easy to derive using distributivity and the set theoretical equivalent of absorption.

Now we show that the axiom of complement holds, let us have $A \in \mathrm{RO}(X)$

$$A \wedge (-A) = A \cap \mathrm{int}(X \setminus A) = \emptyset \tag{3}$$

$$A \vee (-A) = \mathrm{r}(A \cup \mathrm{int}(X \setminus A)) = X \tag{4}$$

ad (3) Obvious (for every set A: $\mathrm{int}(A) \subseteq A$).
ad (4) It is enough to show that $X = \mathrm{cl}(A \cup \mathrm{int}(X \setminus A))$. For contradiction suppose that $\mathrm{cl}(A \cup \mathrm{int}(X \setminus A)) \neq X$. So there is a closed set C which satisfies $(A \cup \mathrm{int}(X \setminus A)) \subseteq C$ and $C \neq X$. Thus the complement of C is open and not empty subset of $(X \setminus A)$[3] and moreover $\mathrm{int}(X \setminus A) \cap (X \setminus C) = \emptyset$. So the set $\mathrm{int}(X \setminus A) \cup (X \setminus C)$ contradicts the fact that $\mathrm{int}(X \setminus A)$ is the greatest open subset of $(X \setminus A)$.

So far we have shown that the so defined system on $\mathrm{RO}(X)$ is a Boolean algebra. The rest to prove is the completeness. So let us have a set S such that $S \subseteq \mathrm{RO}(X)$ and put $A = \mathrm{r}(\bigcap S)$, $A \in \mathrm{RO}(X)$ and for every B, $B \in S$: $A \subseteq \mathrm{r}(B) = B$, thus we have shown that A is a lower bound of S, now we show it is the greatest lower bound. To see that consider a lower bound $C \in \mathrm{RO}(X)$, such a C satisfies $C \subseteq \bigcap S$ and by Observation 2.8 $C = \mathrm{r}(C) \subseteq \mathrm{r}(\bigcap S) = A$. The second case is similar. □

Observation 2.14. The canonical ordering on $B = \mathrm{B}(\mathrm{RO}(X))$ is in fact the set theoretical inclusion $\subseteq$.

[3]Follows by fact that $A \subseteq B \leftrightarrow (X \setminus B) \subseteq (X \setminus A)$.

2.3 Separated ordering

Definition 2.15. We say a set X to by *partially ordered* by binary relation $\leq$, if for every $x, y, z \in X$ holds:

(i) *reflexivity* $x \leq x$

(ii) *transitivity* $(x \leq y \land y \leq z) \to x \leq z$

(iii) *weak antisymmetry* $(x \leq y \land y \leq x) \to x = y$

Observation 2.16. Canonical ordering on every BA is a partial ordering.

Definition 2.17. Let $\leq$ be an ordering on X, we say the ordering is *linear* if for every $x, y \in X$: $x \leq$ y or $y \leq x$ or $x = y$.

Definition 2.18. Let $(X, \leq)$ be a partial ordering. We say that the element $x \in X$:

(i) is *maximal* if for every $y \in X$, $y \neq x$: $x \nleq y$.

(ii) is *minimal* if for every $y \in X$, $y \neq x$: $y \nleq x$.

(iii) is *the greatest* if for every $y \in X$: $y \leq x$.

(iv) is *the least* if for every $y \in X$: $x \leq y$.

Definition 2.19. Let $(X, \leq)$ be partial ordering and $P \subseteq X$. We say that the element $x \in X$:

(i) is *an upper bound of* P if for every $y \in P$: $y \leq x$.

(ii) is *an lower bound of* P if for every $y \in P$: $x \leq y$.

(iii) is *the supremum of* P if x is the least upper bound of P.

(iv) is *the infimum of* P if x is the greatest lower bound of P.

It is easy to see that the supremum and the infimum of a set P is unique element if it exists. We denote it $\sup(P)$ and $\inf(P)$.

Definition 2.20. let $(X, \leq)$ be a partial ordering.

(i) We say elements $x, y \in X$ to be *disjoint* and write $x \perp_o y$, if there is no element $z \in X$ such that $z \leq x$ and $z \leq y$. Otherwise we say x, y are *compatible*.

(ii) The ordering is called *separated* on the set X if for every x, y ∈ X the following property holds.

$$x \not\leq y \rightarrow (\exists z \in X)(z \leq x \wedge z \perp_o y) \tag{5}$$

We have just introduced a second definition for disjoint elements, this time for partial ordering. To ease the reading we use atypical notation $\perp_o$ instead of $\perp$. Realize that if we have a partially ordered set with the least element, then there are no disjoint elements, hence for every BA there are no disjoint elements in sense of $\perp_o$ and therefore no BA is ordered separately. However, if for a given BA B we consider a set $B^+ = B - \{0\}$, then for every $x, y \in B^+$:

$$x \perp y \text{ in } B \leftrightarrow x \perp_o y \text{ in } B^+. \tag{6}$$

Theorem 2.21. [4] Let $(P, \leq)$ be a partially ordered set, then there is a separative partially ordered set $(Q, \preceq)$ and a mapping $h : P \rightarrow Q$ such that for every $x, y \in P$:

(i) if $x \leq y$, then $h(x) \preceq h(y)$

(ii) x and y are compatible in P if and only if $h(x)$ and $h(y)$ are compatible in Q.

Proof. First we define following equivalence relation on P:

$x \sim y$ if and only if $\forall z(z$ is compatible with $x \leftrightarrow z$ is compatible with $y)$.

Relation $\sim$ is obviously an equivalence. So let Q be a quotient set of P by $\sim$, i.e. $Q = P/\sim = \{[x] \,|\, x \in P\}$, where $[x] = \{y \in P \,|\, y \sim x\}$ is an equivalence class of x.

Definition of the ordering $\preceq$ on Q:

$$[x] \preceq [y] \leftrightarrow (\forall z \leq x)[z \text{ and } y \text{ are compatible}].$$

Easily, by mere rewriting of definitions, it can be verified that the set Q with ordering $\preceq$ is a separative partially ordered set.

Definition of the mapping $h : P \rightarrow Q$: $(\forall x \in P)(h(x) = [x])$.

Ad (i): Let $x \leq y$. By definition we show that every $z \leq x$ is compatible with y. Because $x \leq y$: $z \leq y$ and moreover $z \leq z$, therefore z, y are compatible.

[4] This theorem can be found in [2], p. 205.

Ad (ii): $\rightarrow$: If x, y are compatible, then there is $z \in P$ such that $z \leq x$, $z \leq y$, therefore by (i) above: $[z] \preceq [x]$ and $[z] \preceq [y]$. $\leftarrow$: Let us have such $[x]$, $[y]$, so they are compatible. There is $[z] \in Q$: $[z] \preceq [x]$, $[z] \preceq [y]$. Fix arbitrary $k \leq z$, by definition k and x are compatible. So there is $l \in P$: $l \leq k$, $l \leq x$, because $l \leq z$, it follows that l is compatible with y and thus there is $m \in P$: $m \leq l \leq x$ and $m \leq y$.

□

2.4 Dense sets

Definition 2.22. Let B be a Boolean algebra. We say a set $D \subseteq B$ is *dense* in B if $0 \notin D$ and for every nonzero $b \in B$ there exists $x \in D$ such that $x \leq b$.

Examples:

(i) For every BA B the set $B - \{0\}$ is dense in B.

(ii) [5] Boolean algebra B is atomic iff the set of all its atoms is dense in B.

The ordering on every dense subset D of a Boolean algebra B, which is restriction of the canonical ordering on B to the set D, is separated (it is easy to prove this, use (1) and Lemma 2.26).

Definition 2.23. Let B be a BA and $b \in B$. We say a set $X \subseteq B$ is an *antichain* if for every $x, y \in X$: $x \neq y \rightarrow x \perp y$.

Definition 2.24. Let B be a BA and $b \in B$. We say a set $P \subseteq B$ is the *partition of* b if following holds:

(i) $0 \notin P$.

(ii) P is an antichain.

(iii) $b = \bigvee P$.

Definition 2.25 (Principle of maximality, PM). Let $(X \leq)$ be a partial ordering, then for every $x \in X$ there is maximal element over x if the following condition is satisfied (chain condition): Every linearly ordered subset S of X has an upper bound.

Lemma 2.26. If a set D is dense in Boolean algebra B, then for every element $b \in B$ following holds:

[5] recall that an nonzero element $a \in B$ is called an *atom* if there is no b such that $0 < b < a$. And BA B is called *atomic* if under every nonzero $b \in B$ there is an atom.

(i) $b = \bigvee\{x \in D \,|\, x \leq b\}$.

(ii) there exists a partition P of b, which only consist of elements of D.

Proof. Ad (i): Let b be given. We define $X = \{x \in D \,|\, x \leq b\}$. We want to show that b is the supremum of X. If $b = 0$ then X is empty and thus $\sup(X) = 0$. Now suppose that $b \neq 0$, it implies that $X \neq \emptyset$. The element b is obviously an upper bound of X, to see it is also the lowest upper bound let us have an upper bound c, we will show that $b \leq c$. For contradiction suppose $b \nleq c$ and by (1) we get a nonzero element $d \in B$ such that $d \leq b$, $d \leq -c$. Because D is dense in B and $d \leq b$, we get a nonzero element $e \in X$ such that $e \leq d$, however $e \nleq c$, contradiction.

Ad (ii): We omit the case, where $b = 0$, so let $b \neq 0$ be given. We define $X = \{x \in D \,|\, x \leq b\}$ and $P = \{Y \subseteq X \,|\, Y \text{ is antichain}\}$, it is easy to verify that P ordered with inclusion $\subseteq$ satisfies the chain condition of PM and because X is not empty, P is also not empty, therefore we can choose an arbitrary element $a \in P$ and by PM we get maximal element in P over a, we denote it M. We claim that M is a partition of b. It is obvious that M is an antichain and $0 \notin M$. b is obviously an upper bound of M, to see it is also the lowest upper bound let $c \in B$ be upper bound of M and suppose for contradiction that $b \nleq c$. We obtain a nonzero element $e \in X$ the same way as in (i). Because M is maximal in P there must be element $f \in M$ such that $f \wedge e \neq 0$, however $f \nleq c$ (by *equation* (1): $f \wedge -c = (f \wedge b) \wedge -c = f \wedge d \neq 0$), contradiction.

□

Lemma 2.27. If B is BA and $\emptyset \neq P \subseteq B$ and $c = \bigvee P$ then for every $0 \neq a \leq c$, there is an element $p \in P$: $p \wedge a \neq 0$.

Proof. For contradiction suppose that for every $p \in P$: $p \wedge a = 0$. We will show that the element $c \wedge -a$ contradicts the fact that c is the supremum of P. First we show $c \wedge -a \lneq c$. Obviously $c \wedge -a \leq c$ and $c \wedge -a \neq c$ because otherwise:

$$0 = a \wedge (c \wedge -a) = c \wedge a = a \neq 0$$

For every $p \in P$: $p \wedge a = 0$ therefore by (1): $p \leq -a$, moreover $p \leq c$ and thus $p \leq c \wedge -a$. Contradiction, c is not the least upper bound of P. □

Definition 2.28. Two structures $\mathfrak{A}$, $\mathfrak{B}$ are *isomorphic*, we write $\mathfrak{A} \cong \mathfrak{B}$, if there is a function $e : A \to B$ satisfying following conditions:

(i) e is *bijection*, i.e. : Satisfies two following properties.

(a) is *injection (1-1)*: $x \neq_{\mathfrak{A}} y \to e(x) \neq_{\mathfrak{B}} e(y)$.

(b) is *surjection (onto)*: $\{b \in B \mid \exists(a \in A)(e(a) = b)\} = B$.

(ii) For every constant c holds: $e(c_{\mathfrak{A}}) = c_{\mathfrak{B}}$.

(iii) For every n-ary function symbol F holds:

$$e(F_{\mathfrak{A}}(a_1, .., a_n)) = F_{\mathfrak{B}}(e(a_1), ..., e(a_n)).$$

(iv) For every n-ary predicate symbol P holds:

$$(a_1, ..., a_n) \in P_{\mathfrak{A}} \leftrightarrow (e(a_1), ..., e(a_1)) \in P_{\mathfrak{B}}.$$

Fact 2.29. Boolean algebras $(B_0, \wedge_0, \vee_0, -_0, 0_0, 1_0)$ and $(B_1, \wedge_1, \vee_1, -_1, 0_1, 1_1)$ are isomorphic iff they are isomorphic with regard to their canonical orderings, i.e. if $(B_0, \leq_0) \cong (B_1, \leq_1)$.

Proof. Proof can be found for example in [1], p. 10. □

Theorem 2.30. Let us have two complete Boolean algebras B_1, B_2, such that some dense subset $D_1 \subseteq B_1$ is isomorphic with some dense subset $D_2 \subseteq B_2$ with regard to the canonical ordering, then algebras B_1, B_2 are isomorphic.

Proof. Let $j : D_1 \to D_2$ be an isomorphism between dense subsets of BAs B_1 and B_2 with regard to their canonical orderings $\leq_1$ and $\leq_2$. We define the mapping $J : B_1 \to B_2$ as follows: For every x in B_1:

$$J(x) = \bigvee_2 \{j(y) \mid y \in D_1, y \leq_1 x\} \tag{7}$$

Because both algebras are complete it follows that J is mapping from B_1 to B_2. First we show that J extends mapping j. For every $x \in D_1$ we have $J(x) = \bigvee_2\{z \in D_2 \mid z \leq_2 j(x)\}$ and by Lemma 2.26 (i): $J(x) = j(x)$.

Now we prove that J is onto. So let us have $z \in B_2$ and define $x = \bigvee_1\{y \in D_1 \mid j(y) \leq_2 z\}$. Let us denote $P_1 = \{x \in D_2 \mid x \leq_2 z\}$ and $P_2 = \{j(y) \mid y \in D_1, y \leq_1 x\}$. We will show that $J(x) = z$, to see this is enough to show that $P_1 = P_2$, because by Lemma 2.26 (i): $z = \bigvee_2 P_1$, and by (7): $J(x) = \bigvee_2 P_2$. Ad $\subseteq$: let $p \in P_1$ then there is $p_0 \in D_1$: $j(p_0) = p$ and because $j(p_0) \leq_2 z$ by the definition of x: $p_0 \leq_1 x$ and thus $j(p_0) = p \in P_2$. Ad $\supseteq$: Let us have $p \in P_2$, then by definition of P_2 there is $p_0 \in D_1$, $p_0 \leq_1 x$: $j(p_0) = p$. $j(p_0)$ is obviously in D_2, so for contradiction suppose that $j(p_0) \not\leq_2 z$. Because the ordering $\leq_2$ on D_2 is separated, we get by (5) an element $b \in D_2$: $b \leq_2 j(p_0)$ and $b \perp_o z$ on D_2. Because $j : D_1 \to D_2$ is isomorphism, there is an element $a \in D_1$, $j(a) = b$, and because $b = j(a) \leq_2 j(p_0)$: $a \leq_1 p_0$, therefore $a \leq_1 x$ and hence by definition of x and by Lemma 2.27 there must be $y \in D_1$: $y \wedge a \neq 0$, so by density of D_1 we have $0 \neq c \in D_1$, $c \leq_1 y \wedge a$. Obviously $0 \neq j(c) \leq_2 j(a) = b$ and by the definition of x it follows that $j(c) \leq_2 j(y) \leq_2 z$, contradiction with the fact that z, b are disjoint.

J preservers the canonical ordering i.e. $x \leq_1 y \leftrightarrow J(x) \leq_2 J(y)$: Ad $\to$: Obvious, $J(y)$ is an upper bound of $\{j(y) \,|\, y \in D_1, y \leq_1 x\}$. Ad $\leftarrow$: Suppose $x \not\leq_1 y$. We will apply (5) considering separate ordering on B_1^+. However we first need to cover cases, where $x = 0$ (but it is not possible, because $\forall x(0 \leq x)$) and $y = 0$, but if $y = 0$ then $J(y) = 0$. Now suppose $x \neq 0$ and $y \neq 0$ and apply (5). We get a nonzero $c \in B_1$, $c \leq x$, $c \perp_o y$ in B_1^+ and by density of D_1, we have $b \in D_1$, $b \leq_1 c$. By definition of J: $j(b) \leq_2 J(x)$, for contradiction suppose $j(b) \leq_2 J(y)$ then by the definition of J and by Lemma 2.27 there is $p \in D_1$, $p \leq_1 y$ and $j(b) \wedge j(p) \neq 0$. And by density of D_1 it follows that there must be an element $a \in D_1$, $a \leq_1 b \leq_1 c$ and $a \leq_1 p \leq_1 y$, but $c \perp_o y$ in B_1^+, contradiction, hence $J(x) \not\leq_2 J(y)$.

J is 1-1: let $x \neq_1 y$, by weak antisymmetry $x \not\leq_1 y$ or $y \not\leq_1 x$. Without loss of generality suppose that $x \not\leq_1 y$ then because J preserves orderings: $J(x) \not\leq_2 J(y)$. If $J(x) =_2 J(y)$ then $J(x) \not\leq_2 J(x)$, contradiction with reflexivity, hence $J(x) \neq_2 J(y)$. □

2.5 Topology of lower subsets

We already know that from a given topological space we can obtain a complete Boolean algebra B(RO(X)). In this section we describe topology of lower subsets. This topology enables us to to get for a given Boolean algebra a topological space, but we will proceed more generally and define this topology for nonempty ordered set $(Q, \leq)$.

Definition 2.31. Let $(Q, \leq)$ be a nonempty ordered set. We call a set $X \subseteq Q$ a *lower subset of* Q if for every $p, q \in Q$ is satisfied:

$$(p \leq q \wedge q \in X) \to p \in X$$

If $(Q, \leq)$ is nonempty ordered set then both $\emptyset$ and Q are lower subsets of Q, the intersection of two lower subsets is also a lower subset of Q, and for a system S of lower subsets of Q, its union is a lower subset.

It implies that the system of all lower subsets of Q makes a topological space. We call it a *topology of lower subsets*.

Now we introduce an important formula describing regular open sets of a topology of lower subsets. We will show that a subset X of Q is regular open if and only if X is open and following formula holds:

$$(\forall p \in Q)[p \in X \leftrightarrow (\forall q \leq p)(X \cap (\leftarrow, q]^{6} \neq \emptyset)] \tag{8}$$

Proof. First realize that $(\leftarrow, p]$ is the least open neighbourhood of p, i.e.

$$\text{if } V \text{ is an open neighbourhood of } p \text{ then } (\leftarrow, p] \subseteq V. \tag{9}$$

Ad $\rightarrow$: Let X be a regular open then X is open, we will show that (8) holds. Direction from left to right is obvious, because if $p \in X$ then for every $q \leq p$: $(\leftarrow, q) \subseteq X$ (because X is open, i.e. X is a lower subset of Q). To show the other direction let us have $p \in Q$ and $(\forall q \leq p)(X \cap (\leftarrow, q] \neq \emptyset)$ holds. By Lemma 2.10 and (9): $\forall(q \leq p)(q \in \mathrm{cl}(X))$ which can be written as $(\leftarrow, p] \subseteq \mathrm{cl}(X)$ and thus by Lemma 2.11: $p \in X$.

ad $\leftarrow$: As in Lemma 2.11 we show that $X = \mathrm{int}(\mathrm{cl}(X))$. $\subseteq$: The same as Lemma 2.11. $\supseteq$: Let us have $p \in \mathrm{int}(\mathrm{cl}(X))$ then because X is open: $\forall(q \leq p)(q \in \mathrm{cl}(X))$ and therefore by Lemma 2.10: $(\forall q \leq p)(X \cap (\leftarrow, q] \neq \emptyset)$. □

2.6 Completion theorem

Definition 2.32. Let B be a Boolean algebra. We say that $A \subseteq B$ is a *subalgebra of* B if A is closed under operations in B.

Definition 2.33. We say that a complete Boolean algebra B is *completion* of a Boolean algebra A and we write $B = \mathrm{cm}(A)$ if A is a dense subalgebra of B.

Theorem 2.34. Let Q be a nonempty separative partially ordered set then there exists a complete Boolean algebra B and function $j : Q \rightarrow B$ which satisfies:

(i) $j[Q] = \{b \in B \mid \exists q \in Q(j(q) = b)\}$ is dense in B.

(ii) j preserves ordering, i.e. if $p \leq q$ in Q if and only if $j(p) \leq j(q)$ in B.

[6] $(\leftarrow, q] = \{p \in Q | p \leq q\}$

(iii) j preserves disjunction, i.e. if $p \perp_o q$ in Q if and only if $j(p) \perp j(q)$ in B.

(iv) j is 1-1 function (which means that j is in fact an isomorphism from Q onto $j[Q]$).

(v) algebra B is defined uniquely (up to isomorphism).

Proof. Let us consider a Boolean algebra $B = \mathrm{B}(\mathrm{RO}(Q))$, where Q stands for a topology of lower subsets based on Q, and define $j(q) = (\leftarrow, q]$. We show that j is our desired function and B our desired BA.

First we show that for every $q \in Q$, $j(q)$ is a regular open set and thus $j(q) \in B$ (i.e. the function j is properly defined). $j(q)$ is obviously open thus it is enough to show that (8) for $j(q)$ holds. Direction $\rightarrow$ is easy. Direction $\leftarrow$: Suppose $p \notin j(p)$ thus $p \nleq q$ and because ordering on Q is separative, there is $z \leq p$ such that $z \perp_o q$ and therefore $j(q) = (\leftarrow, q] \cap (\leftarrow, z] = \emptyset$.

Ad (i): For all $q \in Q$: $j(q) \neq 0$. Let us have $X \in B \neq 0$ then there is some $p \in X$. The result follows by (9), which says that for every $p \in X$: $j(p) \subseteq X$.

Ad (ii): Obvious.

Ad (iii): $p \perp_o q$ means by definition that there is no element z in Q such that $z \leq p$ and $z \leq q \leftrightarrow (\leftarrow, p] \cap (\leftarrow, q] = \emptyset \leftrightarrow j(p) \wedge j(q) = 0$ which by definition means $j(p) \perp j(q)$.

Ad (iv): Easy consequence of (ii).

Ad (v): Let C be an arbitrary Boolean algebra and mapping $k : Q \rightarrow C$ satisfies conditions (i)-(iii). We show that $(j[Q], \leq_B) \cong (k[Q], \leq_C)$. We define mapping $m : j[Q] \rightarrow k[Q]$ as follows: for all $p \in j[Q]$: $m(p) = k(j^{-1}(p))$. It is easy to verify that m is an isomorphism between dense subset of BA B and dense subset of BA C and therefore by Theorem 2.30 BAs B and C are isomorphic. □

Corollary 2.35. [7] For every partially ordered set $(P, \leq)$ there is a complete Boolean algebra B and mapping $j : P \rightarrow B$ such that:

(i) j[P] is dense in B.

(ii) if $p \leq q$ in P then $j(p) \leq j(q)$ in B.

(iii) $p \perp_o q$ in P if and only if $j(p) \perp j(q)$ in B.

[7] This corollary can be found in [2],p. 206.

(iv) B is unique up to isomorphism.

Proof. Consequence of Theorems 2.21 and 2.34. □

Theorem 2.36. For every BA A there is a BA B such that $B = \mathrm{cm}(A)$. This algebra B is defined uniquely (up to isomorphism).

Proof. Apply Theorem 2.34 on A^+. We have obtained BA B such that A^+ is dense in B. We only need to verify that A is a subalgebra of B, i.e. that A is closed under operations: Because for BA holds that ($z \leq x$ and $z \leq y$) $\leftrightarrow$ $z \leq x \wedge y$ and ($z \leq x$ or $z \leq y$) $\leftrightarrow$ $z \leq x \vee y$ we get $j(p) \wedge j(q) = j(p \wedge q)$, $j(p) \vee j(q) = j(p \vee q)$.

To see that $-j(p) = j(-p)$, we need to show that $\mathrm{int}(A^+ - (\leftarrow, p]) = (\leftarrow, -p]$. $\supseteq$: Follows by fact that $(\leftarrow, -p]$ is an open subset of $(A^+ - (\leftarrow, p])$. $\subseteq$: Let us have $q \in \mathrm{int}(A^+ - (\leftarrow, p])$ then it follows that $q \not\leq p$. By (5) we get $k \in A^+$ such that $k \leq q$ and $k \perp_o p$ and therefore by (6) $k \perp p$ and by (1) $k \leq -p$. □

3 Completeness theorem for Boolean valued predicate logic

Most of the statements in this section are from Handbook of Boolean algebras, Volume 3, [4]. The general idea of the proofs in this section can be found in [4], however we have decided to be more detailed with proofs, which sometimes causes difficulties due to the complexity of the proofs.

3.1 Infinite operations on BAs

To go further we need some more information on Boolean algebras. We state without proof some important properties concerning infinite subsets of BAs. Fore more information and for proofs in this subsection see [1], chapter IV § 1.
Notation: If $\langle a_i | \imath \in I \rangle$ is a set of elements of BA B then:

$$\bigvee_{i \in I} a_i \text{ stands for } \bigvee \{a_i \,|\, i \in I\} \qquad \text{and}$$
$$\bigwedge_{i \in I} a_i \text{ stands for } \bigwedge \{a_i \,|\, i \in I\}$$

Fact 3.1 (**Infinite distributive laws**). If $\bigvee_{i \in I} a_i$, $\bigwedge_{i \in I} a_i$ and $\bigvee_{i \in J} b_i$, $\bigwedge_{i \in J} b_i$ exists, then for every $c \in B$:

(i) $c \wedge \bigvee a_i = \bigvee \{c \wedge a_1 \,|\, i \in I\}$

(ii) $c \vee \bigwedge a_i = \bigwedge \{c \vee a_1 \,|\, i \in I\}$

(iii) $\bigvee_{i \in I} a_i \wedge \bigvee_{j \in J} b_j = \bigvee \{a_i \wedge b_j \,|\, i \in I, j \in J\}$

(iv) $\bigwedge_{i \in I} a_i \vee \bigwedge_{j \in J} b_j = \bigwedge \{a_i \vee b_j \,|\, i \in I, j \in J\}$

Fact 3.2 (**De Morgan laws**). For a subset S of a Boolean algebra B:

$$-\bigvee S = \bigwedge \{-a \,|\, a \in S\}$$
$$-\bigwedge S = \bigvee \{-a \,|\, a \in S\}$$

3.2 Partition refinement

Moreover we again without proof introduce some properties of partitions on BAs. Proofs can be found in [1], chapter IV § 2.

Definition 3.3. Let P and P' be two partitions of some element b in Boolean algebra B. We say P' is a *refinement* of P (or P' refines P) if for every $p' \in P'$ there is $p \in P$ such that $p' \leq p$.

Realize that if P' is a refinement of P then for every $p' \in P'$, there is a unique element $p \in P$, which satisfies $p' \leq p$, and moreover for every $p \in P$:

$$\bigvee\{p' \in P' \,|\, p' \leq p\} = p. \tag{10}$$

We speak of a *common refinement* if P refines the same time more than one refinement. For every finite system of refinements there always exsists a common refinement (This statement doesn't hold for every infinite system).

3.3 Boolean valued models

In this section we introduce the notion of Boolean-valued models (from now on we write only BV-model). To ease the reading we will use notation $+$ to denote $\vee$ and $\cdot$ to denote $\wedge$ in BAs, so we could easier distinguish between operations on BAs and operations of predicate calculus.

So let $\mathcal{L}$ be first-order language, B be a complete Boolean algebra and M be a set (universe of the model). We consider a function from $M \times M$ into B, we denote this function $\|x = y\|$.

Now we describe several condition we want to be satisfied in BV-model.

Definition 3.4. The function $\|x = y\|$ has to satisfy for every $a, b, c \in M$ following:

$$\begin{aligned} &\|a = a\| = 1 \\ &\|a = b\| = \|b = a\| \\ &\|a = b\| \cdot \|b = c\| \leq \|a = c\| \end{aligned} \tag{A}$$

For every n-ary predicate symbol $R(x_1, ..., x_n)$ of language $\mathcal{L}$ let $\|R(x_1, ..., x_n)\|$ be an n-ary function from M^n into B satisfying for each $a_i \in M$, where $i = 1, ..., n$, and every $b \in M$:

$$\|a_i = b\| \cdot \|R(..., a_i, ...)\| \leq \|R(..., a_{i-1}, b, a_{i+1}, ...)\| \tag{B}$$

For every n-ary function symbol $F(x_1, ..., x_n)$ of $\mathcal{L}$ we have a function $F{:}M^n \to M$ such that for each $a_i \in M$, where $i = 1, ..., n$, and every $b \in M$:

$$\|a_i = b\| \leq \|F(..., a_i, ...) = F(..., a_{i-1}, b, a_{i+1}, ...)\| \tag{C}$$

From (A)-(C) it follows that the binary relation $\|x = y\| = 1$ is a congruence on M with respect to functions $\|R\|$ and F. Thus we postulate:

$$\text{if } \|a = b\| = 1, \text{ then } a = b \tag{D}$$

Definition 3.5. A *Boolean-valued model* for $\mathcal{L}$ is

$$M^B = \langle M, \|x = y\|, \|R(x_1, ...x_n)\|, ..., F, ..., c, ...\rangle$$

satisfying (A)-(D).

3.4 Boolean-valued semantics

Definition 3.6. Let M^B be a BV-model and $e : VAR \to M$ an evaluation function. We define the *value of a term* in model M^B, $t^M[e] \in M$, as follows:

(i) if $t = x$, where $x \in VAR$[8], then $t^M[e] = e(x)$.

(ii) if $t_1, ..., t_n \in TERM$[9] and $t = F(t_1, ..., t_n)$, then $t^M[e] = F(t_1, ..., t_n)^M[e] = F(t_1^M[e], ..., t_n^M[e])$.

Definition 3.7. Let M^B be a BV-model and $e : VAR \to M$ an evaluation function. We define *Boolean-value of a formula* $\varphi(x_1, ...x_n)$ in model M^B under evaluation e, we write $\|\varphi(x_1, ..., x_n)\|[e]$, as follows:

(i) If φ is an atomic formula and $t_1, ..., t_n \in TERM$:

$$\|t_1 = t_2\|[e] = \|t_1^M[e] = t_2^M[e]\|$$
$$\|R(t_1, ..., t_n)\|[e] = \|R(t_1^M[e], ..., t_n^M[e])\|$$

(ii) Boolean value of the logical connectives we define by:

$$\|\neg\varphi\|[e] = -\|\varphi\|[e]$$
$$\|\varphi \wedge \psi\|[e] = \|\varphi\|[e] \cdot \|\psi\|[e]$$
$$\|\varphi \vee \psi\|[e] = \|\varphi\|[e] + \|\psi\|[e]$$

(iii) And for quantifiers:

$$\|\exists x\varphi\|[e] = \bigvee_{a \in M} \|\varphi(x)\|[e^x/_a]$$
$$\|\forall x\varphi\|[e] = \bigwedge_{a \in M} \|\varphi(x)\|[e^x/_a]$$

[8] set of all variables.
[9] set of all terms.

We say a formula φ is satisfied in M^B under evaluation e and we write $M^B, e \models \varphi$ if $\|\varphi\|[e] = 1$ in M^B (if necessary we write $\|\varphi\|_M[e] = 1$). Moreover we say M^B satisfies φ and write $M^B \models \varphi$ if $\forall e(M^B, e \models \varphi)$.

Lemma 3.8. If e is an evaluation on BV-model M^B then for every formula φ and $a, b \in M$:

$$\|a = b\| \cdot \|\varphi\|[e^x/_a] \leq \|\varphi\|[e^x/_b]$$

Proof. First by induction on a term t we show that for every $t \in TERM$:

$$\|a = b\| \leq \|t[e^x/_a] = t[e^x/_b]\|$$

(i) If $t = x$ then clearly:

$$\|a = b\| \leq \|t[e^x/_a] = t[e^x/_b]\|$$

If $t = z$ then:

$$\|a = b\| \leq \|t[e^x/_a] = t[e^x/_b]\| = \|e(z) = e(z)\| = 1$$

(ii) if $t = F(t_1, ..., t_n)$ then:

$\|a = b\| \leq \|t_1[e^x/_a] = t_1[e^x/_b]\|$ induction assumption
$\|t_1[e^x/_a] = t_1[e^x/_b]\| \leq$
$\|F(.., t_i[e^x/_a], ..) = F(t_1[e^x/_b], .., t_i[e^x/_a], ..)\|$ by (C) in Definition 3.4

After applying this procedure n-times we have:

$$\|a = b\| \leq \|F(t_1[e^x/_a], ..., t_n[e^x/_a]) = F(t_1[e^x/_b], ..., t_n[e^x/_b])\| = \|t[e^x/_a] = t[e^x/_b]\|$$

Now we use induction on the complexity of the formula φ

(i) if $\varphi = R(t_1, ..., t_n)$
We have shown:

$$\|a = b\| \leq \|t_1[e^x/_a] = t_1[e^x/_b]\|$$

and thus by (B):

$$\|a = b\| \cdot \|R(.., t_i[e^x/_a], ..)\| \leq \|R(t_1[e^x/_b], .., t_i[e^x/_a], ..)\|$$

After applying this procedure n-times we get the result.

(ii) for the connectives:

¬: By induction assumption, we have:

$$\|a = b\| \cdot \|\varphi\|[e^x/_b] \leq \|\varphi\|[e^x/_a] \leftrightarrow$$
$$\|a = b\| \cdot \|\varphi\|[e^x/_b] \cdot \|\varphi\|[e^x/_a] = \|a = b\| \cdot \|\varphi\|[e^x/_b]$$

And thus we can easily argue that:

$$\|a = b\| \cdot -\|\varphi\|[e^x/_a] \cdot \|\varphi\|[e^x/_b] = 0$$

Which is by (1), what we wanted.

∧: By induction assumption we have:

$$\|a = b\| \cdot \|\varphi\|[e^x/_a] \leq \|\varphi\|[e^x/_b]$$
$$\|a = b\| \cdot \|\psi\|[e^x/_a] \leq \|\psi\|[e^x/_b]$$

The result is then obtained using monotonicity[10] of ∧ in BAs.

∨: Similar, only uses monotonicity of ∨.

(iii) for the quantifiers:

$\varphi = \exists z \psi(z)$ and $z \neq x$ by induction assumption for all $c \in M$:

$$\|a = b\| \cdot \|\psi(z)\|[e^z/_c, ^x/_a] \leq \|\psi(z)\|[e^z/_c, ^x/_b]$$

The result follows easily by Fact 3.1.

□

3.5 Full Boolean-valued models

Definition 3.9. We say the BV-model M^B is *full* if for every partition P of 1 in B and every function $f{:}P \to M$ there is an element $a \in M$ such that for all $p \in P$: $p \leq \|a = f(p)\|$.

This element is unique. Suppose there are two such elements a and a', then for all $p \in P$: $p \leq \|a = f(p)\| \cdot \|a' = f(p)\| \leq \|a = a'\|$ and therefore $\|a = a'\|$ is an upper bound of P and hence $\|a = a'\| = 1$.

We shall use formal notation:

$$a = \bigvee_{p \in P} f(p) \cdot p \tag{11}$$

[10]See [1], p. 329.

Proposition 3.10. If M^B is full then for every formula $\varphi(x, x_1, ...x_n)$ and every evaluation e there exists $a \in M$ such that:

$$\|\varphi(x, x_1, ...x_n)\|[e^x/a] = \|\exists z\varphi(z, x_1, ...x_n)\|[e] \tag{12}$$

Proof. Obviously $\|\varphi(x, x_1, ...x_n)\|[e^x/a] \leq \|\exists z\varphi(z, x_1, ...x_n)\|[e]$, we now show the other inequality.

We define function $f{:}B^+ \to M$ such that $\forall p \in B^+$:

$f(p) =$	some $b \in M$ such that $p \leq \|\varphi(x, x_1, ...x_n)\|[e^x/_b]$	if such a b exists.
$f(p)$ is undefined.		otherwise.

Now we consider arbitrary maximal antichain P on $Dom(f)$, $Dom(f)$ is empty only if $\|\exists z\varphi(z, x_1, ...x_n)\|[e] = 0$ and in this case every $a \in M$ will work. In the other case by maximality principle such an antichain always exists.

We show that P is partition of $\|\exists z\varphi(z, x_1, ...x_n)\|[e]$. By definition P is an antichain and $0 \notin P$. We only need to verify that $\bigvee P = \|\exists z\varphi(z, x_1, ...x_n)\|[e]$. For every $p \in P$: $p \leq \|\varphi(x, x_1, ...x_n)\|[e^x/_{f(p)}] \leq \|\exists z\varphi(z, x_1, ...x_n)\|[e]$ and thus $\|\exists z\varphi(z, x_1, ...x_n)\|[e]$ is an upper bound of P. To see it is also the least upper bound let us have $A \in B$, which is an upper bound of P. For contradiction suppose $\|\exists z\varphi(z, x_1, ...x_n)\|[e] \not\leq A$ then by (1) there is $q \in B$:

$$\|\exists z\varphi(z, x_1, ...x_n)\|[e] \cdot -A = q \text{ and } q \neq 0 \tag{13}$$

By Lemma 2.27 , by definition of Boolean value and because $0 \neq q \leq \|\exists z\varphi(z, x_1, ...x_n)\|[e]$, there is $b \in M$ and $r \in Dom(f)$:

$$\|\varphi(x, x_1, ...x_n)\|[e^x/_b] \cdot q = r \text{ and } r \neq 0 \tag{14}$$

Now we argue that $r \leq \bigvee P$ a thus we show that $r \leq A$, which causes contradiction because by (13) and (14) $r \leq q \leq -A$, however $r \neq 0$. So let us suppose that $r \not\leq \bigvee P$, by (5) considering B^+ we get $s \in B$: $s \leq r$ and $s \perp_o \bigvee P$. It follows that for every $p \in P$: $s \cdot p = 0$ and because $s \in Dom(f)$ we have contradiction with maximality of P.

Because M^B is full we can fix $a = \bigvee_{p \in P} f(p) \cdot p$ and show that (12) holds. Direction $\leq$ is obvious. Ad $\geq$: For all $p \in P$:

$$p \leq \|\varphi(x, x_1, ...x_n)\|[e^x/_{f(p)}]$$
$$p \leq \|f(p) = a\|$$

and thus

$$p \leq \|\varphi(x, x_1, ...x_n)\|[e^x/_{f(p)}] \cdot \|f(p) = a\|$$

and therefore by Lemma 3.8

$$p \leq \|\varphi(x, x_1, ...x_n)\|[e^x/_a]$$

$\|\varphi(x, x_1, ...x_n)\|[e^x/_a]$ is therefore an upper bound of P and hence $\|\exists z\varphi(z, x_1, ...x_n)\|[e] \leq \|\varphi(x, x_1, ...x_n)\|[e^x/_a]$. However we didn't use the property of a full model correctly, but realize that with every partition P' such that $P \subseteq P'$ and $\bigvee P' = 1$ (for example $P' = P \cup \{1 - \bigvee P\}$) and with arbitrary expansion of function f, the proof proceeds the same way.

□

Let M^B be a BV-model, we will now describe a construction of BV-model N^B based on M^B. N^B will be full and will satisfy other important properties, of which we will speak later.

Definition 3.11. Definition of the structure N^B:

(N) We define N as a set of all formal expressions:

$$a = \bigvee_{p \in P} p \cdot f(p),$$

where P is a partition of 1 in B and $f : P \to M$ is a function.

(R) For every n-ary predicate symbol $R(x_1, ..., x_n)$, and every $a_1, ..., a_n \in N$ we define:

$$\|R(a_1, ..., a_n)\|_N =$$
$$\bigvee\{\|R(f_1(p_1), ..., f_n(p_n))\|_M \cdot p_1 \cdot ... \cdot p_n \,|\, p_1 \in P_1, ..., p_n \in P_n\},$$

where every a_i is of a formal form $\bigvee_{p_i \in P_i} p_i \cdot f_i(p_i)$. This definition also covers the definition of $\|x = y\|_N$.

(F) For every n-ary function symbol $F(x_1, ..., x_n)$, and every $a_1, ..., a_n \in N$ we define:
Let P be a common refinement of all partition on which are $a_1, ...a_n$ based. We extend each f_i so that $P = dom(f_i)$ as follows: for all $p \in P$:

$f_i(p) = f_i(b)$ for the unique $b \in P_i$, such that $p \leq b$. Then we define $f : P \to M$: $f(p) = F(f_1(p), ..., f_n(p))_M$ and finally:

$$F(a_1, ..., a_n)_N = a = \bigvee_{p \in P} p \cdot f(p).$$

For a constant c we define $c_N = \bigvee_{p \in P} p \cdot f(p)$, where $P = \{1\}$ and $f(1) = c$.

We will write $\underline{p_i \in P_i}$ as a shortcut for $p_1 \in P_1, ..., p_n \in P_n$; $\bar{a}_i$ as a shortcut for $a_1, ..., a_n$; $\overline{f_i(p_i)}$ as a shortcut for $f_1(p_1), ..., f_n(p_n)$ and $\bigwedge p_i$ as a shortcut for $p_1 \cdot p_2 \cdot ... \cdot p_n$.

Lemma 3.12. N^B is BV-model.

Proof. we need to verify conditions (A)-(C) from the definition 3.4. So let us have $a, a_1, ..., a_n \in N$.

(A) (a) $\|a_1 = a_1\|_N =$
$\bigvee\{\|f_1(p_1) = f_1(p_1')\|_M \cdot p_1 \cdot p_1' \,|\, p_1 \in P_1, p_1' \in P_1\} =^*$
$\bigvee\{\|f_1(p) = f_1(p)\|_M \cdot p \,|\, p \in P_1\} = \bigvee\{p \,|\, p \in P_1\} = 1$

ad (*) if $p_1 \neq p_1'$ then $p_1 \wedge p_1' = 0$

(b) $\|a_1 = a_2\|_N \cdot \|a_2 = a_3\|_N =^*$
$\bigvee\{\|f_1(p_1) = f_2(p_2)\|_M \cdot \|f_2(p_2) = f_3(p_3)\|_M \cdot p_1 \cdot p_2 \cdot p_3 \,|\, p_1 \in P_1 ...\} \leq^{*1} \bigvee\{\|f_1(p_1) = f_3(p_3)\|_M \cdot p_1 \cdot p_3 \,|\, p_1 \in P_1, p_3 \in P_3\}$

ad (*) by Fact 3.1 (iii) and the reason why we didn't use p_2' is the same as in (a).
ad (*1) it is obvious that the later expression is an upper bound of the previous one.

(c) $\|a_1 = a_2\|_N = \|a_2 = a_1\|_N$ similar.

(B) $\|a = a_1\|_N \cdot \|R(\bar{a}_i)\|_N =^*$
$\bigvee\{\|f(p) = f(p_1')\|_M \cdot \|R(\overline{f_i(p_i)})\|_M \cdot p \cdot p_1' \cdot \bigwedge p_i \,|\, p \in P, p_1' \in P_1, p_i \in P_i\} \leq^{*1}$
$\bigvee\{\|R(f(p), \overline{f_i(p_i)})\|_M \cdot p \cdot \bigwedge_{2 \leq i \leq n} p_i \,|\, p \in P, p_i \in P_i \text{ for } 2 \leq i \leq n\} =$
$\|R(a, a_2, \ldots, a_n)\|_N$

ad (*) by Fact 3.1 (iii)
ad (*1) The later expression is an upper bound of the previous.

(C) We want to show that $\|a = a_1\|_N \leq \|F(a, a_2, \ldots, a_n) = F(\bar{a}_i)\|_N$, thus by definition 3.11 (F):

$F(a, a_2, \ldots, a_n)_N = \bigvee_{p' \in P'} p' \cdot f'(p')$ and
$F(\bar{a}_i)_N = \bigvee_{p'' \in P''} p'' \cdot f''(p'')$,
where $f'(p') = F(f(p'), f_2(p'), ..., f_n(p'))$ and $f''(p'') = F(f_1(p''), ..., f_n(p''))$ and P' and P'' are partitions by definition.
By definition of $\|x = y\|_N$:

$\|a = a_1\|_N = \bigvee\{\|f(p) = f_1(p_1)\|_M \cdot p \cdot p_1 \,|\, p \in P, p_1 \in P_1\}$ and
$\|F(a, a_2, \ldots, a_n) = F(\bar{a}_i)\|_N =$
$\bigvee\{\|f'(p') = f''(p'')\|_M \cdot p' \cdot p'' \,|\, p' \in P', p'' \in P''\}$.

We show that for every $p \in P$ and $p_1 \in P_1$:
$\|f(p) = f_1(p_1)\|_M \cdot p \cdot p_1 \leq \|F(a, a_2, \ldots, a_n) = F(\bar{a}_i)\|_N$.

By definition for every $p' \leq p$: $f(p') = f(p)$ and for every $p'' \leq p_1$: $f_1(p'') = f_1(p_1)$ and thus by definition 3.4 (C):

$$F(f(p'), f_2(p'), ..., f_n(p')) = F(f(p), f_2(p'), ..., f_n(p'))$$
$$F(f_1(p''), ..., f_n(p'')) = F(f_1(p_1), ..., f_n(p''))$$

for every $p' \leq p$ and $p'' \leq p_1$ such that $p' \bot p'' \neq 0$:

$$\|f(p) = f_1(p_1)\|_M \leq$$
$$\|F(f(p), f_2(p'), ..., f_n(p')) = F(f_1(p_1), ..., f_n(p'')))\|_M =$$
$$\|f'(p') = f''(p'')\|_M$$

Realize that the condition $p' \bot p'' \neq 0$ causes that every $f_i(p') = f_i(p'')$.

Moreover by (10) because P' is a refinement of P and P'' of P_1:
$\bigvee\{p' \in P' | p' \leq p\} = p$ and the same for P_1. And therefore by Fact 3.1 we can conclude:

$\|f(p) = f(p_1)\|_M \cdot p \cdot p_1 \leq$
$\bigvee\{\|f'(p') = f''(p'')\|_M \cdot p' \cdot p'' \,|\, p' \in P', p'' \in P'', p' \leq p, p'' \leq p_1\} \leq$
$\|F(a, a_2, \ldots, a_n) = F(\bar{a}_i)\|_N$

□

Lemma 3.13. N^B is full.

Proof. Let us consider an arbitrary partition P of 1 in B and an arbitrary function $f{:}P \to N$. We will find $a \in N = \bigvee_{p' \in P'} p' \cdot f'(p')$ such that for each $p \in P$: $p \leq \|a = f(p)\|_N$.

We denote the partition of every formal expression $f(p)$ as $P_{f(p)}$ and define $P' = \{p \cdot q \neq 0 \,|\, p \in P, q \in P_{f(p)}\}$.

Obviously $0 \notin P'$ and for every $p_1, p_2 \in P'$ such that $p_1 \neq p_2$ holds $p_1 \cdot p_2 = 0$. Moreover for all $p \in P$ holds:

$$\bigvee\{p \cdot q \,|\, q \in P_{f(p)}\} = p, \tag{15}$$

because by Fact 3.1 and by (10): $\bigvee\{p \cdot q \,|\, q \in P_{f(p)}\} = p \cdot \bigvee\{q \,|\, q \in P_{f(p)}\} = p \cdot 1 = p$. This means that $\bigvee P'$ is an upper bound of P and thus $\bigvee P' = 1$. We have shown that P' is a partition of 1 in B.

We define $f'{:}P' \to M$, but first we introduce notation: for every $p \in P$ the value of $f(p)$ we will denote as $\bigvee_{q \in P_{f(p)}} q \cdot f_{f(p)}(q)$. It is easy to verify that for each $p' \in P'$ there is a unique $p \in P$ such that $p' \leq p$ and a unique $q \in P_{f(p)}$ such that $p' \leq q$, thus we define $f'(p') = f_{f(p)}(q)$.

To see that $a = \bigvee_{p' \in P'} p' \cdot f'(p')$ is our desired element let us have $p \in P$. From the definition we have:

$$\|a = f(p)\|_N = \bigvee\{\|f'(p') = f_{f(p)}(q)\|_M \cdot p' \cdot q \,|\, p' \in P', q \in P_{f(p)})\}$$

If we consider p' such that $p' \leq p$ and $q \in P_{f(p)}$ such that $p' \leq q$ then $f'(p') = f_{f(p)}(q)$ and $p' \cdot q = p'$, therefore $\|f'(p') = f_{f(p)}(q)\|_M \cdot p' \cdot q = p'$. The set of all such a p's is equal with the set in (15), thus we can conclude that $p \leq \|a = f(p)\|_N$. □

(R) in definition 3.11 can extended to all formulas in following way:

Lemma 3.14. For every formula φ with free variables among $x_1, ..., x_n$ and for every $a_1, ..., a_n \in N$ holds:

$$\|\varphi\|_N[e\ {}^{\bar{x}_i}/_{\bar{a}_i}] = \bigvee\{\|\varphi\|_M[e\ {}^{\bar{x}_i}/_{\overline{f_i(p_i)}}] \cdot \bigwedge p_i \,|\, p_i \in P_i\}, \tag{16}$$

where $a_i = \bigvee_{p_i \in P_i} p_i \cdot f_i(p_i)$.

Proof. To ease the reading, without loss of generality, we consider only 2-ary predicate a function symbols.

First we show that for every term t with free variables among $x_1, ..., x_n$ and for every elements $a_1, ..., a_n \in N$ holds:

$$\text{If } t^N[e\ {}^{\bar{x}_i}/_{\bar{a}_i}] = \bigvee_{p \in P} p \cdot f(p), \text{ then for all } p \in P:$$
$$f(p) = t^M[e\ {}^{\bar{x}_i}/_{\overline{f_i(p_i)}}], \tag{17}$$

where for all i such that x_i is free in t: p_i is the unique element in P_i such that $p \leq p_i$ (such a p_i always exists, because P is a refinement of each P_i, by definition 3.11 (F)), the others p_i's are arbitrary. We will verify this using induction on the complexity of term t.

(i) Let $t = x_1$ then $t^N[e\ {}^{\bar{x}_i}/_{\bar{a}_i}] = a_1 = \bigvee_{p_1 \in P_1} p_1 \cdot f_1(p_1)$ and trivially $f(p_1) = t^M[e\ {}^{\bar{x}_i}/_{\overline{f_i(p_i)}}]$.
The case $t = c$, where c is a constant, is obvious by definition 3.11 (F).

(ii) If $t = F(t_1, t_2)$ and $t^N[e\ {}^{\bar{x}_i}/_{\bar{a}_i}] = \bigvee_{p \in P} p \cdot f(p)$
Let us have $p \in P$ then (each $p_i \in P_i$ is the unique element such that $p \leq p_i$)

$$\begin{aligned}
&f(p) =^* F(f_{t_1}(p), f_{t_2}(p)) =^{*1} \\
&F(f_{t_1}(p_{t_1}), f_{t_2}(p_{t_2})) =^{*2} \\
&F(t_1^M[e\ {}^{\bar{x}_i}/_{\overline{f_i(p_i)}}], t_2^M[e\ {}^{\bar{x}_i}/_{\overline{f_i(p_i)}}]) = \\
&t^M[e\ {}^{\bar{x}_i}/_{\overline{f_i(p_i)}}]
\end{aligned}$$

ad (*) $t_i^N[e\ {}^{\bar{x}_i}/_{\bar{a}_i}] = \bigvee_{p_{t_i} \in P_{t_i}} p_{t_i} \cdot f_{t_i}(p_{t_i})$, thus the result follows by definition 3.11 (F).
ad (*1)By definition 3.11 (C), there are elements $p_{t_1} \in P_{t_1}$ and $p_{t_2} \in P_{t_2}$ such that $p \leq p_{t_2}$ and $p \leq p_{t_2}$, for these elements holds: $f_{t_1}(p) = f_{t_1}(p_{t_1})$ and $f_{t_2}(p) = f_{t_2}(p_{t_2})$.
ad (*2) By induction assumption.

Now we verify (16) using induction on the complexity of formula φ.

(i) $\varphi = R(t, s)$
Let us denote the value of the term t, $t^N[e\ {}^{\bar{x}_i}/_{\bar{a}_i}]$, as $\bigvee_{p_t \in P_t} p_t \cdot f_t(p_t)$ and the value of the term s, $s^N[e\ {}^{\bar{x}_i}/_{\bar{a}_i}]$, as $\bigvee_{p_s \in P_s} p_s \cdot f_s(p_s)$, then by definition 3.11 (R):

$$\|R(t, s)\|_N[e\ {}^{\bar{x}_i}/_{\bar{a}_i}] = \bigvee \{\|R(f_t(p_t), f_s(p_s))\|_M \cdot p_t \cdot p_s \,|\, p_t \in P_t, p_s \in P_s\}$$

We want to show that:
$\bigvee\{\|R(f_t(p_t), f_s(p_s))\|_M \cdot p_t \cdot p_s \,|\, p_t \in P_t, p_s \in P_s\} =$
$\bigvee\{\|R(t,s)\|_M[e\ ^{\bar{x}_i}/_{\overline{f_i(p_i)}}] \cdot \bigwedge p_i \,|\, p_i \in P_i\}$.

$\leq$: by (17) for every p_t and p_s:

$$f_t(p_t) = t^M[e\ ^{\bar{x}_i}/_{\overline{f_i(p_i)}}] \tag{18}$$
$$f_s(p_s) = s^M[e\ ^{\bar{x}_i}/_{\overline{f_i(p_i)}}], \tag{19}$$

where for every i such that x_i is in t: p_i is the unique element in P_i (partition of a_i) such that $p_t \leq p_i$ (other p_i's are arbitrary) and the same holds for term s.

Realize that the set of p_i's in (18) and (19) can be different. So let us use notation:$(p_i)_t$ and $(p_i)_s$, however we will now show that in important cases ($p_t \cdot p_s \neq 0$) they can be considered the same. Suppose that $(p_i)_t \neq (p_i)_s$, this can happen only in three cases. First: x_i is not free in one of terms s, t, without loss of generality, suppose it is not free in s, then $(p_i)_s$ is arbitrary and thus we can choose it to be $(p_i)_t$. Second: x_i is not free in t and also not free in s, thus we can choose arbitrary $p_i \in P_i$ and say $p_i = (p_i)_s = (p_i)_t$. Third: x_i is free in both terms, and $(p_i)_t \neq (p_i)_s$, however this is only possible when $p_t \cdot p_s = 0$. Therefore if $p_t \cdot p_s \neq 0$, we can get common set of p_i's as in (18) and (19).

If $p_t \cdot p_s \neq 0$, then by (18) and (19):

$$\|R(f_t(p_t), f_s(p_s))\|_M = \|R(t,s)\|_M[e\ ^{\bar{x}_i}/_{\overline{f_i(p_i)}}].$$

Moreover $p_t \cdot p_s \leq \bigwedge p_i$[11], however this holds only when we count those p_i's such that x_i is either in t or s, but if x_i is not in either of these terms, then x_i is not free in R(t,s) and the value $\|R(t,s)\|_M[e\ ^{\bar{x}_i}/_{\overline{f_i(p_i)}}]$ does not depend on p_i we choose thus by (10) we can conclude that:

$$\|R(f_t(p_t), f_s(p_s))\|_M \cdot p_t \cdot p_s \leq \bigvee\{\|R(t,s)\|_M[e\ ^{\bar{x}_i}/_{\overline{f_i(p_i)}}] \cdot \bigwedge p_i \,|\, p_i \in P_i\}.$$

$\geq$: The other inequality is very similar.

(ii) $\varphi = \neg\psi$

$\|\varphi\|_N[e\ ^{\bar{x}_i}/_{\bar{a}_i}] = -\|\psi\|_N[e\ ^{\bar{x}_i}/_{\bar{a}_i}] =$	by definition
$-\bigvee\{\|\psi\|_M[e\ ^{\bar{x}_i}/_{\overline{f_i(p_i)}}] \cdot \bigwedge p_i \,\|\, p_i \in P_i\}$	by induction assumption

[11] Follows by monotonicity of $\wedge$, see [1], p. 329.

We want: $\|\varphi\|_N[e\ {}^{\bar{x}_i}/_{\bar{a}_i}] = \bigvee\{\|\neg\psi\|_M[e\ {}^{\bar{x}_i}/_{\overline{f_i(p_i)}}] \cdot \bigwedge p_i \,|\, p_i \in P_i\}$

Let us denote $A = \bigvee\{\|\psi\|_M[e\ {}^{\bar{x}_i}/_{\overline{f_i(p_i)}}] \cdot \bigwedge p_i \,|\, p_i \in P_i\}$ and
$B = \bigvee\{\|\neg\psi\|_M[e\ {}^{\bar{x}_i}/_{\overline{f_i(p_i)}}] \cdot \bigwedge p_i \,|\, p_i \in P_i\}$.
Because $A \cdot B = 0$ and $A + B = 1$, we can conclude that B is a complement of A, i.e. $-A = B$.[12]

$\wedge$: follows easily by Fact 3.1.
$\vee$: follows by infinite associativity laws.[13]

(iii) $\varphi = \exists x\psi$

$$\|\exists x\psi\|_N[e\ {}^{\bar{x}_i}/_{\bar{a}_i}] =^*$$
$$\bigvee\{\|\psi\|_N[e\ {}^{\bar{x}_i}/_{\bar{a}_i}, {}^x/_a] \,|\, a \in N\} =^{*1}$$
$$\bigvee_{a\in N} \bigvee\{\|\psi\|_M[e\ {}^{\bar{x}_i}/_{\overline{f_i(p_i)}}, {}^x/_{f(p_a)}] \cdot \bigwedge p_i \cdot p_a \,|\, p_i \in P_i, p_a \in P_a\} =^{*2}$$
$$\bigvee\{\|\psi\|_M[e\ {}^{\bar{x}_i}/_{\overline{f_i(p_i)}}, {}^x/_a] \cdot \bigwedge p_i \,|\, p_i \in P_i, a \in M\} =^{*3}$$
$$\bigvee\{(\bigvee_{a\in M} \|\psi\|_M[e\ {}^{\bar{x}_i}/_{\overline{f_i(p_i)}}, {}^x/_a]) \cdot \bigwedge p_i \,|\, p_i \in P_i\} =^*$$
$$\bigvee\{\|\exists x\psi\|_M[e\ {}^{\bar{x}_i}/_{\overline{f_i(p_i)}}] \cdot \bigwedge p_i \,|\, p_i \in P_i\}$$

(*) By definition.
(*1) By induction assumption.
(*2) Obvious.
(*3) By Fact 3.1.

□

Theorem 3.15. Every BV-model M^B can be embedded in a full BV-model N^B such that for all φ and all evaluation e on M^B, there is an evaluation $\tilde{e}$ on N^B such that:

$$\|\varphi\|_M[e] = \|\varphi\|_N[\tilde{e}], \tag{20}$$

moreover for every φ:

$$\text{if } M^B \models \varphi \text{ then } N^B \models \varphi. \tag{21}$$

[12] See [1], p. 328.
[13] See [1], p. 330.

Proof. Let us consider model N^B, described by (N), (R) and (F) in definition 3.11, by Lemmas 3.12, 3.13 we know that N^B is a full BV-model.

Definition of evaluation $\tilde{e}$: $\tilde{e}$ is an evaluation on N^B corresponding to evaluation e, i.e. for every variable x if $e(x) = a$ then $\tilde{e}(x) = a_N$, where $a_N = \bigvee_{p \in P} p \cdot f(p)$ and $P = \{1\}$ and $f(1) = a$.

First we show by induction on the complexity of term t that for every term t and evaluation e on M^B:

$$\text{if } t^M[e] = a \text{ then } t^N[\tilde{e}] = a_N \tag{22}$$

(i) $t = x$ then $t^M[e] = e(x) = a$ and $t^N[\tilde{e}] = \tilde{e}(x) = a_N$.
$t = c$, c is a constant, then $t^M[e] = c$ and by definition 3.11 (F) $t^N[\tilde{e}] = c_N$.

(ii) $t = F(t_1, ..., t_n)$ then for all i: $t_i^M[e] = b_i$ and by induction assumption $t_i^N[\tilde{e}] = b_{iN} = \bigvee_{p_i \in \{1\}} p_i \cdot f_i(p_i)$, where $f_i(1) = b_i$. Therefore it follows that $t^M[e] = F(b_1, ..., b_n)$. By definition (F) $t^N[\tilde{e}] = \bigvee_{p \in P} p \cdot f(p)$, where $P = \{1\}$ and $f(1) = F(f_1(1), ..., f_n(1)) = F(b_1, ..., b_n)$, therefore $t^N[\tilde{e}] = F(b_1, ..., b_n)_N$.

(20) we prove by induction on the complexity of formula φ.

(i) $\varphi = R(t_1, ..., t_n)$. Let us have an evaluation e on M^B, then
$\|R(t_1, ..., t_n)\|_M[e] = \|R(t_1^M[e], ..., t_n^M[e])\|_M = \|R(a_1, ..., a_n)\|_M =^*$
$\|R(f_1(1), ...f_n(1))\|_M = \|R(t_1, ..., t_n)\|_N[\tilde{e}]$
ad (*) consequence of (22) and that $t_i^M[e] = a_i$.

(ii) For connectives:

$\neg$: $\|\neg\psi\|_M[e] = -\|\psi\|_M[e] =^* -\|\psi\|_N[e] = \|\neg\psi\|_N[\tilde{e}]$.
(*) by induction assumption.

$\wedge$: $\|\psi_1 \wedge \psi_2\|_M[e] = \|\psi_1\|_M[e] \cdot \|\psi_2\|_M[e] =^* \|\psi_1\|_N[\tilde{e}] \cdot \|\psi_2\|_N[\tilde{e}] = \|\psi_1 \wedge \psi_2\|_N[\tilde{e}]$
(*) by induction assumption.

$\vee$: similar.

(iii) For quantifiers:
$\varphi = \exists x \psi$, obviously by induction assumption:
$\bigvee\{\|\psi\|_M[e\ ^x/_a] \mid a \in M\} = \|\varphi\|_M[e] \leq \|\varphi\|_N[\tilde{e}] =$
$\bigvee\{\|\psi\|_N[\tilde{e}\ ^x/_a] \mid a \in N\}$

The other inequality is an easy consequence of Lemma 3.14. Realize that Lemma 3.14 says:

$$\|\psi\|_N[\tilde{e}\ {}^{x}/_{a}] = \bigvee\{\|\psi\|_M[e\ {}^{x}/_{f(p)}] \cdot p \mid p \in P\}$$

(21) follows easily by (20) and by Lemma 3.14.

□

3.6 Ultrafilters

Definition 3.16. Let B be a Boolean algebra. We say $F \subseteq B$ is *filter* on B if it satisfies for all $a, b \in B$:

(i) $1 \in F$.

(ii) If $a \in F$ and $a \leq b$, then $b \in F$.

(iii) If $a, b \in F$, then $a \cdot b \in F$.

Moreover we say it is a *proper filter* if $0 \notin X$.

Definition 3.17. We say a subset X of Boolean algebra B has the *finite intersection property, FIP* , if for every finite $S \subseteq X$: $\bigwedge S \neq 0$.

Definition 3.18. We call the proper filter F on BA B:

(i) *maximal* if every for filter F' such that $F \subsetneq F'$ holds that $0 \in F'$.

(ii) *an ultrafilter* if for each $a \in B$ either $a \in F$, or $-a \in F$.

(iii) *prime* if every $a, b \in B$: $a + b \in F \leftrightarrow a \in F$ or $b \in F$.

Fact 3.19. For every proper filter F on Boolean algebra B the following is equivalent:

(i) F is maximal.

(ii) F is an ultrafilter.

(iii) F is prime.

Theorem 3.20 (**Boolean prime ideal theorem, BPI**)**.** For every $X \subseteq B$ with FIP there is an ultrafilter F such that $X \subseteq F$.

Proof. Omitted, see for example [5]. □

With the notion of ultrafilters we can for a given BV-model M^B and for an ultrafilter G on B construct the *quotient* M/G, a two-valued model. The universe of M/G is the quotient of M by equivalence relation $\|x = y\| \in G$

Functions are interpreted as

$$F([a_1], ..., [a_n])_{M/G} = [F(a_1, ..., a_n)_M]$$

and for the predicates

$$R([a_1], ..., [a_n]) \text{ iff } \|R(a_1, ..., a_n)\| \in G. \tag{23}$$

Theorem 3.21. Let M^B be a full BV-model and let G be an ultrafilter on B. For every formula φ and every evaluation e on M^B and its corresponding evaluation e' on M/G holds:

$$M/G \models \varphi[e'] \text{ iff } \|\varphi\|[e] \in G. \tag{24}$$

Proof. It is easy to verify that for for every term t:

$$t^{M/G}[e'] = [t^M[e]] \tag{25}$$

Now again by induction on the complexity of formula φ we show (24)

(i) $\varphi = R(t_1, ..., t_n)$.

$$\begin{aligned} M/G \models \varphi[e'] &\text{ iff } R(t_1^{M/G}[e'], ..., t_n^{M/G}[e']) \\ &\text{ iff } R([t_1^M[e]], ..., [t_n^M[e]]) && \text{by (25)} \\ &\text{ iff } \|R(t_1^M[e], ..., t_n^M[e])\| \in G && \text{by (23)} \\ &\text{ iff } \|R(t_1, ..., t_n)\|[e] \in G. \end{aligned}$$

(ii) $\varphi = \neg\psi$

$$\begin{aligned} M/G \models \varphi[e'] &\text{ iff } M/G \not\models \psi[e'] \\ &\text{ iff } \|\psi\|[e] \notin G && \text{induction assumption} \\ &\text{ iff } \|\varphi\|[e] \in G && \text{property of an ultrafilter} \end{aligned}$$

$\varphi = \psi_1 \wedge \psi_2$

$$\begin{aligned} M/G \models \varphi[e'] &\text{ iff } M/G \models \psi_1[e'] \text{ and } M/G \models \psi_2[e'] \\ &\text{ iff } \|\psi_1\|[e] \in G \text{ and } \|\psi_2\|[e] \in G && \text{ind. assumption} \\ &\text{ iff } \|\psi_1\|[e] \cdot \|\psi_2\|[e] \in G && \text{property of a filter} \\ &\text{ iff } \|\psi_1 \wedge \psi_2\|[e] \end{aligned}$$

$\varphi = \psi_1 \vee \psi_2$

Similar (uses the property of a prime filter).

(iii) $\varphi = \exists x \psi(x)$

$$\begin{aligned} M/G \models \varphi[e'] & \text{ iff } \exists a \in M \ M/G \models \psi(x)[e'^x/_{[a]}] & \\ & \text{ iff } \exists a \in M \ \|\psi(x)\|[e^x/a] \in G & \text{induction assumption} \\ & \text{ iff } \|\exists x \psi(x)\|[e] \in G & \text{because } M^B \text{ is full} \end{aligned}$$

□

Corollary 3.22. If $M^B \models \varphi$ then for every ultrafilter G on B:

$$M/G \models \varphi.$$

Proof. Easy consequence of Theorem 3.21. □

3.7 Completeness Theorem

Definition 3.23. Let B be a complete Boolean algebra and Γ be a set of sentences in language $\mathcal{L}$ and φ formula in $\mathcal{L}$. We say φ is a *consequence* of Γ (or Γ *implies* φ) and write $\Gamma \models_B \varphi$ if

$$\forall M^B (\forall \gamma \in \Gamma (M^B \models \gamma) \rightarrow M^B \models \varphi)$$

i.e. if every BV-model over BA B, which satisfies every formula in Γ, also satisfies φ.

Theorem 3.24. Let B be a complete Boolean algebra. Let Γ be a set of sentences in language $\mathcal{L}$ and φ formula in $\mathcal{L}$, then

$$\Gamma \models_B \varphi \leftrightarrow \Gamma \vdash \varphi$$

Proof. $\leftarrow$: $\Gamma \vdash \varphi$ and for contradiction suppose $\Gamma \not\models_B \varphi$, then by definition there is a model M^B such that $\forall \gamma \in \Gamma (M^B \models \gamma)$ and there is an evaluation e such that $M^B, e \not\models \varphi$.

By Theorem 3.15 there is a full BV-model N^B such that $\forall \gamma \in \Gamma (N^B \models \gamma)$ and $N^B, e \not\models \varphi$. Because $\|\varphi\|_N[e] \neq 1$, it follows that $\|\neg\varphi\|_N[e] \neq 0$. So let G be an ultrafilter on the complete BA B such that $\|\neg\varphi\|_N[e] \in G$ (such an ultrafilter exists by BPI, Fact 3.20).

Now we consider the two-valued quotient model N/G. By Corollary 3.22 it follows that: $\forall \gamma \in \Gamma (N/G \models \gamma)$ and moreover by Theorem 3.21: $N/G \models \neg\varphi[e']$ which is equivalent to $N/G \not\models \varphi[e']$ and therefore $N/G \not\models \varphi$.

We have show that $\Gamma \not\models \varphi$, which contradicts completeness theorem for standard two-valued predicate logic[14].

$\rightarrow$ We will show that if $\Gamma \models_B \varphi$ then also $\Gamma \models \varphi$. So let M be a two-valued model such that all formulas from Γ are satisfied in M, we show that $M \models \varphi$.

We define a BV-model N^B as follows: M is the universe of N^B, function symbols are interpreted as in M. For predicate symbols:

If $a = b$ in M, then $\|a = b\| = 1$ otherwise $\|a = b\| = 0$.
$R(a_1, ...)$ in M, then $\|R(a_1, ...)\| = 1$ otherwise $\|R(a_1, ...)\| = 0$.

It is easy to verify by use of induction that N^B is a BV-model and that for every evaluation e on M (realize that models M and N^B have the same evaluations) and every φ:

$$M \models \varphi[e] \leftrightarrow N^B, e \models \varphi.$$

This means that N^B satisfies all formulas in Γ and therefore it also satisfies φ, thus we conclude $M \models \varphi$.

□

3.8 Alternative proofs of the two-valued completeness

In this subsection we will introduce an interesting application of ultrafilters on BAs and of quotient models. We will show an alternative proof of the completeness theorem for two-valued first-order predicate calculus. However we do not have enough space to be entirely thorough on this very interesting topic, we only show some general ideas.

The standard way how to prove the completeness theorem for two-valued semantics is to construct the maximally consistent theory for the Henkin extension of a consistent theory T and then to construct a model with universe consisting of closed terms of language $\mathcal{L}$. We show different approach using similar construction as in Theorem 3.21.

We need the notion of Lindenbaum-Tarski algebras[15], here we only recall its universe. So let T be a theory in language $\mathcal{L}$, then

$$\mathrm{B}(T) = \{[\varphi] \mid \varphi \text{ is a formula in } \mathcal{L}\}, \tag{26}$$

[14]For proof see: [3] or [6].
[15]for the definition see for example [5], p. 17.

where $[\varphi]$ is an equivalence class defined by relation $T \vdash \varphi \leftrightarrow \psi$. Moreover realize that φ does not need to be a sentence.

Fact 3.25. Let T be a first-order theory of language $\mathcal{L}$ and let $\varphi(x, x_0, ...)$ be a formula in $\mathcal{L}$. Denote

$$M_\varphi = \{[\varphi(t, x_0, ...)] \,|\, t \text{ is a term in } \mathcal{L}\},$$

where $\varphi(t, x_0, ...)$ denotes the formula created by substitution of t for x (and renaming other variables if necessary). Then

$$\bigvee M_\varphi = [\exists x \varphi(x, x_0, ...)] \text{ and } \bigwedge M_\varphi = [\forall x \varphi(x, x_0, ...)]$$

Proof. See [5] p. 19. □

From Rasiowa-Sikorski theorem[16] and from the previous fact, it follows that if $\mathcal{L}$ is at most countable then any subset F of $\mathrm{B}(T)$ with FIP can be extended to an ultrafilter U such that U preserves quantified formulas, i.e if $[\exists x \varphi] \in U$, then there is a term t such that $[\varphi[^x/_t]] \in U$ (and equivalently for $\forall$).

Now we have everything we need. So let T be a consistent theory in at most countable $\mathcal{L}$. It is obvious that the set F, containing equivalence classes of every formula φ in T, has FIP in $\mathrm{B}(T)$, therefore we can get a Rasiowa-Sikorski ultrafiler U, such that $F \subseteq U$.

Based on this ultrafilter we can define an universe of a model.

$$M = \{[t]_U \,|\, t \text{ is a term in } \mathcal{L}\},$$

where $[t]_U$ is an equivalence class based on relation $[t = s] \in U$, i.e. $[t]_U = \{s \in TERM \,|\, [t = s] \in U\}$. Realize we range over all terms (not only the closed ones), this fact allows us to omit the Henkin extension. The definition of function symbols and predicate symbols is then obvious and it is not very difficult to verify that M is a model of T.

Remark 3.26. The obstacle with the limit for the cardinality of the language $\mathcal{L}$ can be easily overcome. For example by use of ultraproducts[17], compactness[18] can be proven. Because every finite subset of T contains only finite amount of non-logical symbols symbols, by Rasiowa-Sikorski it has a model. The result then follows by compactness.

[16] see [7], p. 35.
[17] See [8], p. 40.
[18] If every finite subset of a theory T has a model, then also T has a model.

Observation 3.27. What we did here can actually be equivalently described within BV-valued theory. The construction is almost identical. For a universe of a model we can take all terms in laguage $\mathcal{L}$ a define BV-model $M^{\mathrm{B}(T)}$ (the definition of functions a predicates is obvious). Because of the Fact 3.25, we do not have a problem with the fact that $\mathrm{B}(T)$ does not necessarily have to be complete. By Rasiowa-Sikorski, (if $\mathcal{L}$ is at most countable) we can easily alter the Theorem 3.21 (the only difference is in (iii)) and thus get the quotient, two-valued model of T.

We can also prove completeness without Rasiowa-Sikorski and moreover for an arbitrary language.

Observation 3.28. Let $\mathcal{L}$ be a language of arbitrary cardinality. We use Theorem 2.36 to construct a BV-model $M^{\mathrm{cm}(\mathrm{B}(T))}$, where $\mathrm{cm}(\mathrm{B}(T))$ is the completion of $\mathrm{B}(T)$. Realize that because $\mathrm{B}(T)$ is a dense subset of $\mathrm{cm}(\mathrm{B}(T))$, we can define the BV-model the same way as in Observation 3.27 (i.e. using only the element of $\mathrm{B}(T)$). Then we can by Theorems 3.15 and 3.21 construct a two-valued model of T.

4 Conclusion

We had two goals. First was to prove the completion theorem for BAs (Theorem 2.36), which we have successfully proven in Section 2. The second goal was to prove generalized completeness theorem for the first-order predicate logic with respect to all complete Boolean algebras (Theorem 3.24), which we have also successfully proven in Section 3.

With the first task, proving of the completion theorem for BAs (Section 2), we proceeded similarly as Balcar and Štěpánek,[1]. Yet there is one important difference between the approach of Balcar and Štěpánek in [1] and the approach used in this thesis. This difference can be found in Theorem 2.21 and Corollary 2.35, which were inspired by Jech, [2]. Moreover we decided to put more emphasis on the proofs than in [1]. Many Lemmas and other statements can not be found in [1] and are products of the author of this thesis.

In the second task, proving of the generalized completeness theorem (Theorem 3.24), we used the definition of Boolean-valued models and the definition of the Boolean-valued semantics from the Handbook of Boolean algebras, vol.3, [4]. In proving of completeness theorem for Boolean algebra $\{0, 1\}$, the more difficult direction is completeness ($\Gamma \models \varphi$, then $\Gamma \vdash \varphi$). However in the proof of the generalized completeness, when we already suppose the completeness for BA $\{0, 1\}$, the more difficult direction was the one usually referred to as correctness ($\Gamma \vdash \varphi$, then $\Gamma \models \varphi$), as one can see in Theorem 3.24. To prove correctness we decided to use the notion of full BV-models, ultrafilters and quotient models (this idea is also inspired by [4]). Nevertheless one can ask if we could use similar approach as we use in proving of correctness for BA $\{0, 1\}$. And the answer is yes, the straightforward way (proof by induction on the length of the proof) is here also possible, however as we said, we decided on different, much more interesting, approach. Most of the Proofs necessary for this theorem can be found in [4], however proofs in [4] are usually dealing only with unary predicate symbols, for the purpose of this work we extended these proofs to all predicate symbols and moreover to all function symbols and constants.

At the end of this thesis have also shown one valuable application of the ultrafilters and of the quotient models, we have described how to use those notions in prooving of the completeness theorem for BA {0,1}.

This work is supposed to help its author to continue in this field of set theory and logic with aim to proceed to forcing.

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