Univerzita Karlova v Praze

Matematicko-fyzikální fakulta

BAKALÁŘSKÁ PRÁCE

Vojtěch Tůma

Barevnost grafů na plochách

Katedra aplikovanké matematiky

Vedoucí bakalářské práce: Mgr. Dvořák Zdeněk, Ph.D.

Studijní program: Informatika

Studijní obor: obecná informatika

Praha 2011

Poděkování. Nejprve děkuji Zdeňku Dvořákovi za jeho trpělivé a precisní vedení této práce.

Dále děkuji svým rodičům a přátelům za jejich toleranci a trpělivost během psaní této práce.

Prohlašuji, že jsem tuto bakalářskou práci vypracoval(a) samostatně a výhradně s použitím citovaných pramenů, literatury a dalších odborných zdrojů.

Beru na vědomí, že se na moji práci vztahují práva a povinnosti vyplývající ze zákona č. 121/2000 Sb., autorského zákona v platném znění, zejména skutečnost, že Univerzita Karlova v Praze má právo na uzavření licenční smlouvy o užití této práce jako školního díla podle §60 odst. 1 autorského zákona.

V Praze dne .... Vojtěch Tůma

Název práce: Barevnost grafů na plochách

Autor: Vojtěch Tůma

Katedra: Katedra aplikované matematiky

Vedoucí bakalářské práce: Mgr. Dvořák Zdeněk, Ph.D., Katedra aplikované matematiky

Abstrakt: V této práci se zabýváme rozšiřováním 4-obarvení cyklu do zbytku grafu. Ukážeme, že pro rovinný 3-barevný graf takový, že vnější stěna O má velikost 4 nebo 5 a všechny ostatní stěny mají velikost 3, se otázka rozšiřitelnosti předbarvení O redukuje na otázku rozšiřitelnosti předbarvení do 3 malých základních grafů. Dále podobným způsobem klasifikujeme i situaci, kdy je v grafu stěna velikosti 4 různá od O.

Klíčová slova: barevnost, graf, plocha

Title: Coloring graphs on surfaces

Author: Vojtěch Tůma

Department: Department of applied mathematics

Supervisor: Mgr. Dvořák Zdeněk, Ph.D., Department of applied mathematics

Abstract: In this thesis we deal with extending a 4-coloring of a cycle into the rest of the graph. We prove that for plane 3-colorable graphs such that every face except the outer face O is a triangle and O has length 4 or 5, the question of extendability of a precoloring of O reduces to extendability into 3 small basic graphs. We also classify the situation when there is an inner face of size 5 in the graph in similar terms.

Keywords: coloring, graph, surfaces

Contents

1 Introduction **2**
1.1 History and motivation 2
1.2 Preliminaries 3

2 Auxiliary results **4**

3 Main results **7**
3.1 Triangulations 7
3.2 Graphs with inner face of size 5 8

4 Concluding remarks **15**

References **17**

1. Introduction

In this work we are concerned with the coloring extensions in plane graphs. The basic setting is extending a 4-coloring of the vertices of the outer face into the rest of the graph. However, in general this would be a strengthening of the 4-Color Theorem and thus very hard. We shall restrict ourselves to a more specific setting – first of all, all the graphs we deal with will be 3-colorable, as this gives us more freedom in extending of the coloring and also a hint how some parts of the graph can be 3-colored. Similar relaxation was followed for example in [10], which deals with extending of 5-coloring into planar graphs, which are 4-colorable by themselves. Second, we shall restrict the size of the precolored outer face.

Thus the chief object of our study are 3-colorable plane graphs with the outer face O of size 4 or 5. If such a graph G is triangulation except for the outer face, we show that the question of extending a precoloring can be simply deiced – it basically equals the question if G can be contracted to some of the basic graphs, for which extending of a precoloring is straightforward.

If G is not a triangulation, we show that inner faces of size 4 do not allow any extra extendable precolorings in general. However, if the graph also has a face I of size 5, such that I is not separated from O by a cycle of length ≤ 4, the situation is different. Again, we give complete characterization of the extendability of a precoloring – in case the size of O is 4 it can be always extended, in case the size of O is 5 there are two basic graphs B_1, B_2 such that every graph is equivalent to one of these in terms of extendability of a precoloring. Every one of B_1, B_2 allows only one non-extendable precoloring (up to permutation of colors).

The results are derived by describing the properties of a minimal counterexample, whose existence is later shown to be a contradiction by the discharging technique. This approach can be used to classify a broader class of graphs, e.g. with different bounds on the sizes of the faces O and I.

1.1 History and motivation

Coloring extension arguments play an important role in the question of coloring of graphs on surfaces – a notable example of their usage is a theorem of Grötzsch, which states that every triangle-free planar graph can be 3-colored. Both the original proof in [5] and newer ones in [13],[14] extend a precoloring of a cycle. The result was generalized for the torus and the projective plane in [16]. Another well known example in similar spirit is the 5-Choosability of planar graphs, proven in [15]. Extending a coloring of a cycle seems a natural thing to do, as the cycles separating smaller subgraphs of a graph drawn on a surface serve well for induction arguments.

The discharging technique, introduced by Wernicke in [18] and further developed by Heesch, played an important role in proving the 4-Color Theorem as well as in many subsequent problems. For a detailed introduction see [9], for more recent proof of the 4-Color Theorem based on the same ideas but significantly shorter see [12].

Our aim for classifying 3-colorable graphs is not incidental. 3-colorable plane graph correspond to subgraphs of Eulerian triangulations (every vertex has even degree), which are being intensively studied. As for other surfaces, there are Eulerian triangulations of arbitrarily high chromatic number. However, for orientable surfaces it was proven that Eulerian triangulations are 4-colorable provided their representativeness is high enough (see [4] and [8]). On the other hand, for non-orientable surfaces the situation is slightly more complicated, as there are graphs with arbitrarily high face-width which are not 4-colorable – for instance, there are quadrangulations of projective plane, such that for every 4-coloring there is a face such that every vertex gets a different color. Thus, inserting a vertex inside every face and joining it with every vertex of that face yields a 5-chromatic Eulerian triangulation. Still a characterization is available, closely related to quadrangulations – see [1], [7], [11].

1.2 Preliminaries

The reader is refered to [2] or [3] for the basics of graph theory and accompanying notation.

By $\mathcal{G}(a)$ we denote the class of all plane graphs G such that G is 3-colorable, all faces except for the outer face O are triangles, and the length of O is a. By $\mathcal{G}(a,b)$ we denote the class of all plane graphs G such that G is 3-colorable, all faces except for the outer face O and another face I are triangles, the length of O is a, the length of I is b, and I is not separated from O by any cycle of length $b-1$ or less.

By a *precoloring* of O we mean any proper coloring of O using at most 4 colors. A precoloring c is *extendable* if there exist a proper coloring c' of G using at most 4 colors such that $c(v) = c'(v)$ for every $v \in V(O)$.

Let v be a vertex of G of degree 4. A graph H is a *4-contraction of G at v* if there exist non-adjacent neighbors u and w of v and H is obtained from G by contracting the edges uv and vw and suppressing the arising bigon faces (but preserving the non-facial parallel edges, if they are created). Note that if all faces incident with v are triangles and G is 3-colorable, then H is 3-colorable as well. Also such 4-contraction never creates a loop, as the two vertices that were identified have the same color in every 3-coloring of the graph.

A 4-contraction of $G \in \mathcal{G}(a,b)$ is *safe* if the resulting graph H also belongs to $\mathcal{G}(a,b)$, i.e. the 4-contraction did not create a cycle of length at most $b-1$ separating I from O and shortened neither I nor O.

Let v be a vertex of G of degree 2 with neighbors uw, such that v is separated by a 2-cycle uw from the rest of the graph. A graph H is a *2-contraction of G at v*, if H is obtained by replacing vertex v and the edges of the cycle uw by a single edge. Note again that if G is 3-colorable, then H is 3-colorable as well. A 2-contraction is always safe.

We say that G can be *2, 4-contracted to H*, if there exists a sequence of graphs $G = G_0, G_1, \dots, G_k = H$ and vertices $v_i \in V(G_i)$ for $0 \leq i < k$ such that G_{i+1} is a 2-contraction or a 4-contraction of G_i at v_i. Note that for every 4-coloring of a 2, 4-contraction of G there is a 4-coloring of G.

Let b be a 3-coloring of $G \in G(5)$. The restriction of b on $V(O)$ uses all three colors, and one of them appears on exactly one vertex – we call this vertex the *pivot* of b on O.

2. Auxiliary results

Lemma 2.1. *Planar triangulation is 3-colorable iff all its vertices have even degree.*

Proof. See [6] or [17]. □

Lemma 2.1 implies that if a plane graph G, such that all faces of G except for the outer face O are triangles, belongs to $G(|O|)$ then all its vertices of odd degree belong to O. Similarly, if G belongs to $G(a, b)$ then all its vertices of odd degree belong either to O or I.

Observation 2.1. *For a plane graph G with at least 3 vertices we have $|E| \leq 3|V| - 6$, with equality holding iff G is a triangulation.*

Lemma 2.2. *For $G \in \mathcal{G}(4)$, either all of the vertices of O have odd degree, or the vertices of odd and even degree alternate on O. Furthermore, G can be 2,4-contracted to the graph in Figure 2.1 with matching parities of the vertices of O.*

(a) Odd degrees (b) Odd-even

Figure 2.1: Simplifications of $\mathcal{G}(4)$

Proof. First, assume that O is an induced cycle. Then every vertex in $V(O)$ has degree at least 3. We apply the Observation 2.1 to show that there is a vertex $v \in G \setminus V(O)$ of degree 2 or 4. Otherwise we have

$$2|E| = \sum \deg(v) \leq 6 \cdot (|V| - 4) + 3 \cdot 4 = 6|V| - 12,$$

but G is not a triangulation, thus the inequality must be strict. If v has degree 2, we can 2-contract it. Let v have degree 4. If all of its neighbors belong to O, then G is the graph in Figure 2.1(a). Otherwise, there is a neighbor of v not on $V(O)$, and we do the 4-contraction with this vertex and the opposite neighbor. By repeating this procedure, we eventually either 2,4-contract G to the graph in Figure 2.1(a), or create a chord of O.

Suppose now that O is not an induced cycle. Let $O = o_1o_2o_3o_4$ and assume that o_1o_3 is an edge. If $o_1o_2o_3$ and $o_1o_3o_4$ are faces, then G is the graph in Figure 2.1(b). Otherwise, we can again find a vertex of degree 2 or 4 in the interior of one of $o_1o_2o_3$ and $o_1o_3o_4$, since $6 \cdot (|V| - 3) + 3 \cdot 3 > 6|V| - 12$. This vertex can always be 2,4-contracted. □

Lemma 2.3. *For $G \in \mathcal{G}(5)$, exactly two of the vertices of O have odd degree and these vertices are adjacent. If O is an induced cycle, then G can be 2,4-contracted to both graphs in Figure 2.2, otherwise it can be 2,4-contracted to one of them.*

Figure 2.2: Simplifications of $\mathcal{G}(5)$

Proof. Let $O = o_1o_2o_3o_4o_5$ and fix a 3-coloring c of G. Without loss of generality, assume that $c(o_1) = 1, c(o_2) = c(o_4) = 2$, and $c(o_3) = c(o_5) = 3$. Since the faces incident with o_1 distinct from O are all triangles and $c(o_2) \neq c(o_5)$, we conclude that o_1 has even degree. Similarly, o_2 and o_5 have even degree, and o_3 and o_4 have odd degree.

If O is not induced cycle, then we apply Lemma 2.2 to the 4-cycle. Otherwise, let us show that there is a vertex of degree 2 or 4 in $V(G) \setminus V(O)$. We again use the counting argument as in the proof of Lemma 2.2. This time we know that two vertices of O have degree at least 3 and three vertices at least 4:

$$2|E| = \sum \deg(v) \leq 6 \cdot (|V| - 5) + 2 \cdot 3 + 3 \cdot 4 = 6|V| - 12,$$

which is again contradiction as G is not a triangulation.

We shall note that whenever we identify two vertices by a 4-contraction, their color under c has to be the same. Thus, the only vertices of O that could be identified are o_2 with o_4 and o_3 with o_5. Similarly no 4-contraction can create edge o_2o_4 or o_3o_5.

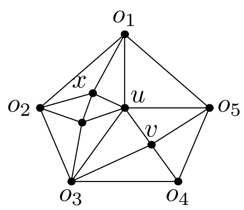


Figure 2.3: Almost 4-contracted graph

Let us first show that we can 2,4-contract G so that either O remains induced or one of the edges o_1o_3, o_1o_4 is created. By repeating such 2,4-contractions, the graph in Figure 2.2(a) is eventually created. We need to exclude the possibility that the 4-contraction identifies o_2 with o_4 or o_3 with o_5, or creates the edge o_2o_5. However, if a 4-contraction at v creates the edge o_2o_5, then the other 4-contraction does not identify any of the vertices of O and thus can be done. If a 4-contraction at v identifies o_2 with o_4, then the other 4-contraction does not identify o_3 with o_5 since this implies would imply that o_2vo_5 is a triangle and o_2o_5 is an edge of G.

Now, we show that we can 2,4-contract G to the graph in Figure 2.2(b). Similarly to the previous case, the only issue is the possible identification of o_2 with o_4 or o_3 with o_5, or the creation of one of the edges o_1o_3, o_1o_4. Thus let v be a vertex of degree 4, such that both 4-contractions at v cause one of the bad situations. It is not possible that both 4-contractions create a chord or that both 4-contractions cause identifications. By symmetry it suffices to consider the situation that v has neighbors o_3, o_4, o_5, u, where u is adjacent to o_1. As the edge o_1o_3 is not present, we can 4-contract G to the graph in Figure 2.3 – apply Lemma 2.2 on the subgraph induced by the interior of the cycle $o_2o_3uo_1$, and let G_i be the graph such that 4-contraction at $x \in G_i$ creates the edge o_1o_3, and y its contracted neighbor distinct from o_1, o_3. Instead of 4-contracting at x, apply Lemma 2.3 to 2,4-contract the graphs induced by the interiors of the cycles o_1xyo_3u and $o_1xyo_3o_2$ to the graph in Figure 2.2(a). This gives the graph in Figure 2.3, which can be 4-contracted at x to get the desired graph.

□

3. Main results

3.1 Triangulations

In this section, we shall derive the basic results concerning the graphs in $\mathcal{G}(4)$ and $\mathcal{G}(5)$, showing that the question of extendability of a precoloring reduces to the question if G can be 2,4-contracted to some specified small graphs.

Theorem 3.1. *If the outer face O of $G \in \mathcal{G}(4)$ has no chord, then a precoloring c of $G \in \mathcal{G}(4)$ can be extended into G if and only if it can be extended into the graph from Figure 2.1(a) or to the graph in Figure 3.1(b), depending on parities of degrees of vertices of O. If the outer face O has chord, then c can be extended into G if and only if it can be extended into the graph from Figure 2.1(b).*

Proof. First, we shall note that if $G \in \mathcal{G}(4)$ has no chord and parities of degree of vertices of O alternate, it can be 2,4-contracted to the graph in Figure 3.1(b) – this observation was proven during the proof of Lemma 2.3. We will refer to the graphs from Figures 2.1(a) and 3.1(b) as *basic graphs*. Let G be in $\mathcal{G}(4)$. Clearly, if G can be 2,4-contracted to H, then every 4-coloring of H extends to a 4-coloring of G. For the other direction we note that for every 4-coloring of G two of the neighbors of v have the same color and thus can be identified while still preserving the 4-coloring. The only issue here is that these two vertices could be on O and the other two could have different colors.

If O has a chord, then no identification of vertices on O can occur during 2,4-contractions, thus G is precisely equal to the graph from Figure 2.1(b). If there is no chord, we try to 2,4-contract the graph. Let $o_1o_2o_3o_4$ denote the vertices of O. If the vertex $v \in G$ with degree 4 has two neighbors o_1 and o_3, and there is a 4-coloring c of G such that $c(o_1) = c(o_3)$, but the colors of the remaining two neighbors m_1 and m_2 of v differ, we shall assume by induction that the subgraphs in the interiors of the 4-cycles $o_1o_2o_3v$ and $o_1o_4o_3v$ equal to one of the basic graphs, or some of o_2v, o_4v are edges – we denote this reduced graph by G'. Now it is straightforward to check that every precoloring that extends to G' extends also to the basic graph with matching parities, since for each of the basic graphs there is only one precoloring (up to a permutation of colors) that does not extend into it, and this precoloring also does not extend into G'. □

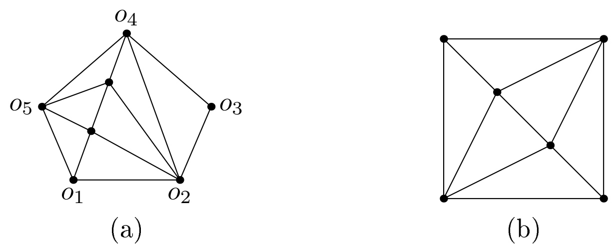


Figure 3.1

Theorem 3.2. *If the outer face O of $G \in \mathcal{G}(5)$ has no chord, then a precoloring c of $G \in \mathcal{G}(5)$ can be extended into G if and only if it can be extended into one of the graphs in Lemma 2.3 with matching parities. If the outer face O has chord, then G is equivalent to exactly one of the graphs in Lemma 2.3 and Figure 3.1(a).*

Proof. We will call the graphs from Lemma 2.3 also basic graphs. We shall again 2,4-contract G. Assume first that O has no chord. If there is a vertex v of degree 4 with two neighbors $o_1, o_3 \in V(O)$ and a 4-coloring of G such that $c(o_1) = c(o_3)$, note that G consists of two cycles $o_1o_2o_3v$ and $o_1o_5o_4o_3v$ which contain some subgraphs inside. By induction, assume that both of these subgraphs correspond to some basic graphs. Note that this implies that O has a chord, so we again use induction to the subgraph bounded by the 4-cycle C with $V(C) \subset V(O)$, which contains this chord. Now either G is one of the graphs from Lemma 2.3, or it is the graph from Figure 3.1(a). But in this case, the only precoloring c which cannot be extended to the graph in Figure 2.2(a) but can be extended to the graph in Figure 3.1(a), i.e. $c(o_1) = a,\ c(o_2) = b,\ c(o_3) = c,\ c(o_4) = a,\ c(o_5) = b$, can be extended to the graph in Figure 2.2(b). If O has two chords, then it is equivalent to 2.2(a). If it has one chord, then we use the previous theorem to the 4-cycle bounded by this chord, which gives the desired result. □

3.2 Graphs with inner face of size 5

Theorem 3.3. *For every $G \in \mathcal{G}(5,5)$, at most one precoloring c of its outer face O is non-extendable. Furthermore, consider a proper 3-coloring b of G. Then either c uses only 3 colors and the pivots of b and c are not adjacent, or c uses all 4 colors and the pivot of b is one of the vertices that have the color that appears twice in c.*

Both outcomes of 3.3 can happen, as demonstrated by graphs B_1 and B_2 in Figure 3.2 (the letters indicate the non-extendable precoloring c, the square vertex denotes the pivot of the 3-coloring b).

Theorem 3.4. *Every precoloring of $G \in \mathcal{G}(4,5)$ is extendable.*

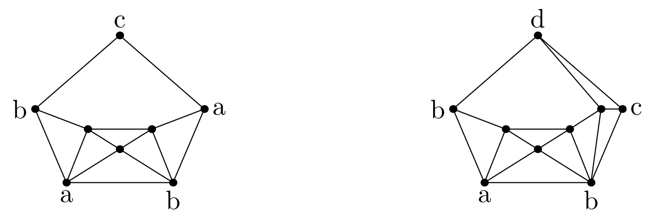
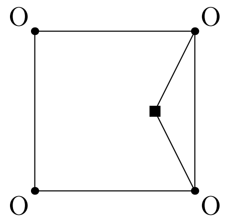


Figure 3.2: Simplifications of $\mathcal{G}(5,5)$

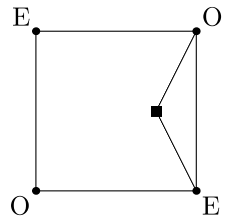
We will prove both Theorem 3.3 and Theorem 3.4 at once. We consider a minimal counterexample to either of the theorems and derive some of its properties. In the second part, we use a discharging argument to derive a contradiction.

Proof. Let $\phi_1(c)$ and $\phi_2(c)$ denote the restrictions on precoloring in the statement of Theorem 3.3, i.e. $\phi_1(c)$ holds if c uses only 3 colors and the pivots of c and the 3-coloring b of G are not adjacent, and $\phi_2(c)$ holds if c uses 4 colors and the pivot of the 3-coloring b of G is one of the vertices that have the color that appears twice in c.

First, we show that every precoloring c such that none of $\phi_1(c), \phi_2(c)$ holds can be extended in G. Let b be a 3-coloring of G. If c uses just 3 colors, and the pivot of c and b is the same vertex, then b is the extension of c. If the pivots



(a) Case 1



(b) Case 2

Figure 3.3: Case when $\deg(v \in C_e) = 2$

are adjacent then by the definition of $G(4,5)$ and $G(5,5)$, O is an induced cycle, hence we can triangulate I and we 2,4-contract the resulting graph to the graph in Figure 2.2(b). Since c extends to a 4-coloring of this graph, it also extends to G. Otherwise, c uses 4-colors, and the pivot of b is colored by a color that appears just once in c. Then we can 2,4-contract G to the graph in Figure 2.2(a), and as c extends to the 4-coloring of this graph, it also extends to G.

Let G be a minimal counterexample – i.e. either $G \in \mathcal{G}(4,5)$ with a nonextendable precoloring or $G \in \mathcal{G}(5,5)$ with two nonextendable precolorings – either both of them satisfy one of ϕ_1, ϕ_2; or one satisfies ϕ_1 and the other ϕ_2.

First, we shall note that all triangles are faces – if a triangle T separates a nonempty subgraph H from the rest of the graph, then every precoloring of $V(T)$ extends to H due to 3-colorability of G. The same holds for a subgraph H separated from O by parallel edges, which could possibly arise in a 4-contraction.

If a cycle of length 5 separates I from O, then by the minimality of G we can assume that its interior is isomorphic to B_1 or B_2. Let C be the outermost such cycle distinct from O, with $C = I$ if no such cycle exists. Let G' be the subgraph of G between O and C. In the rest, we consider the properties of the graph G', and let I' denote the subgraph drawn in the closed disk bounded by C.

Let $O_e = V(O) \setminus V(C)$, and vertices in $C_e = V(C) \setminus V(O)$. No vertex v in O_e or C_e has degree 2, since this would result in a triangle or a 4-cycle separating I from O. The only exception is the case that the resulting 4-cycle is O, i.e. $V(O) \subseteq V(C)$ and $|V(O)| = 4$. In such case, we color the graph directly. Using Lemma 2.2, we distinguish between two cases according to the parities of degree of vertices of O – see Figure 3.3, where the squared vertex stands for the pivot of a 3-coloring of I', and O/E stand for the parities of the vertices of O if I' was triangulated so that we can apply 2.2.

First we deal with the case in Figure 3.3(a). We can assume that all vertices of O have distinct colors, otherwise we can use Lemma 2.2. Thus we have to ways of precoloring C, both of them satisfy ϕ_2. However, for the subgraph separated by C there is at most one nonextendable precoloring. Now we deal with the case in Figure 3.3(b). If the vertices of O with odd degree have different color, we can again triangulate I and 2,4-contract the graph to the graph in Figure 2.1b. Now, if the vertices of O with even degree have the same color, there is just one way to precolor C, and this precoloring satisfies neither ϕ_1 nor ϕ_2. But if the vertices have different color, there are two ways to precolor C, one does not satisfy ϕ_1, the other does not satisfy ϕ_2.

Let $v \in O_e$ have degree 3. If O has length 4, then $G' - v$ belongs to $G(4,5)$, and every precoloring of G' can be extended to a precoloring of outer face of $G' - v$, contradicting the minimality of G'. If O has length 5, let c be a precoloring of G', u_1 and u_2 neighbors of v that are on O and z the third neighbor of v. If the $c(u_1) = c(u_2)$, we can remove v and identify u_1 and u_2, they also have the same color under the 3-coloring of the graph due to odd parity of the degree of v. The resulting graph can be colored with three colors, and if this coloring is not compatible with $c(v)$, we can change the color of z to the fourth color we have not used yet. If $c(u_1) \neq c(u_2)$, then there is only one possible color for z. Since $G' - v$ is in $G(5,5)$, there is only one non-extendable coloring for $G' - v$, thus also at most one non-extendable coloring for G'.

Also, no vertex v in C_e has degree 3, otherwise we can find another 5-cycle C' separating I from O using the neighbor of v not on I. By the maximality of I', we have $C' = O$. However, then a vertex of O_e has degree 3, which we already excluded. Thus we conclude that the vertices in $O_e \cup C_e$ have degree at least 4. Trivially, all vertices in $V(C) \cup V(O)$ have degree at least 2.

Claim 3.1. *For every $v \in V(G') \setminus (V(O) \cup V(C))$ we have $deg(v) \geq 6$, or there is exactly one vertex of degree 4 and G' contains the subgraph depicted in Figure 3.6(b).*

Subproof. Let v be a vertex in $V(G') \setminus (V(O) \cup V(C))$. If $\deg(v) = 2$, then v is separated by parallel edges and can be 2-contracted, contradicting the minimality of G. If $\deg(v) = 4$, and v can be safely 4-contracted with respect to G', then v can be safely 4-contracted with respect to G and we again get a contradiction – each coloring of the contracted graph can be naturally turned into a coloring of $G - v$. Therefore, assume that both 4-contractions at v are unsafe.

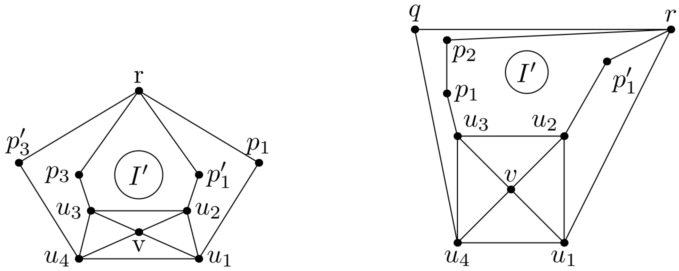
As v has degree 4, there are two possible ways how to 4-contract v – letting $u_1u_2u_3u_4$ denote the neighbors of v in cyclic order, we can either identify u_1 with u_3 or u_2 with u_4. If a 4-contraction at v is not safe, then it creates either a triangle or 4-cycle separating I from O, or shortens C or O. A 2-cycle separating I from O cannot arise, since then there would be a separating 4-cycle in G'.

Suppose that a 4-contraction at v shortens C or O. It follows that say u_1 and u_3 both lie on O or C. The path u_1vu_3 together with a subpath of O or C forms a (≤ 5)-cycle C' separating I from O. As C' is distinct from C and O, this contradicts the fact that C is the outermost 5-cycle separating I from O.

Therefore, we can assume that both 4-contractions at v create a triangle or a 4-cycle separating I from O. Hence, there are two paths P and P' which connect vertices u_1 and u_3, and u_2 and u_4 respectively, such that the number of their edges is equal to the length of the cycle created by the corresponding 4-contraction, i.e. it is either 3 or 4. We will show the existence of a 5-cycle separating I' from O. It suffices to consider the situation that both paths have length 4 – if any of the paths is shorter, the resulting separating cycle has length 4 which is a contradiction.

We denote the vertices of P and P' by p_1, p_2, p_3 and p'_1, p'_2, p'_3 respectively. Note that P and P' share at least one vertex r. By symmetry, we assume that the cycle $C_1 = u_2p'_1 \dots r \dots p_3u_3$ separates I' from O. Therefore, the cycle $C_2 = u_1p_1 \dots r \dots p'_3u_4$ also separates I' from O. By symmetry, assume that C_2 separates C_1 from I'. Note that $|C_1| + |C_2| \leq |P| + |P'| + 2 \leq 10$. Since $|C_1| \geq 5$, we have $|C_2| \leq 5$, and thus $C_2 = O$.

Let us first deal with the case that $|V(O)| = 4$. Let $O = u_1rqu_4$, and by symmetry assume that $r, q \in P$. This implies $p'_2 = r$, $p'_3 = q$ (see Figure 3.4(b)). In this case, we do a 4-contraction at v, identifying u_2 and u_4. This separates I



(a) length$(u_1p_1\dots r) = 2$

(b) $|V(O)| = 4$

Figure 3.4

in a 4-cycle $S = u_2p_1'rq$. Using minimality of G, we know that every precoloring of S can be extended into the separated graph, as long as the 2,4-contraction did create a 4-cycle separating S from I – however, if we choose P_1' such that S is minimal, this case cannot arise. The remaining part of the graph, separated by the cycle $u_1rp_1'u_2$, can be 2,4-contracted to one of the graphs in Figure 2.1. Therefore every proper precoloring of O can be extended to a proper precoloring of S, as S can be created from G just by removing vertices of degree 3.

If $|V(O)| = 5$ then there are up to symmetry only two cases to deal with: either length$(u_1p_1\dots r) = 2$ or length$(u_1p_1\dots r) = 1$. The first case is represented by Figure 3.4(a). We can assume that $ru_1, ru_2, ru_3, ru_4 \notin E(G)$ otherwise I is separated by a 4-cycle. We distinguish between several cases depending on which vertex of O is the pivot of the 3-coloring b of G. By symmetry, there are just three cases:

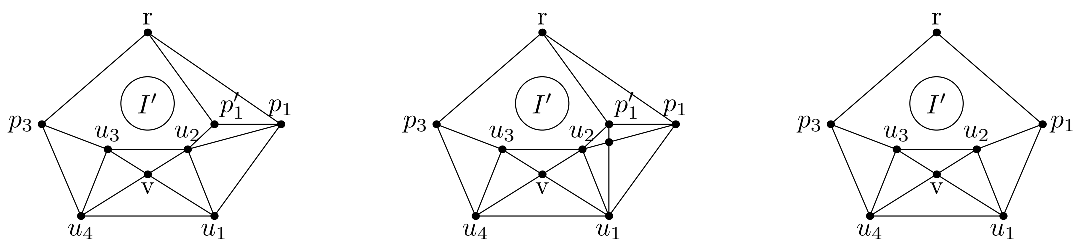
- Suppose that p_1 is the pivot of the 3-coloring b of G. Then $p_3 \neq p_3'$ since u_4 has odd degree. As p_3' has odd degree and $u_3r \notin E(G)$, we can assume that the subgraph drawn in the interior of the cycle $rp_3'u_4u_3p_3$ consists only of edges u_4p_3 and $p_3'p_3$ due to Lemma 2.3. But then p_3' is a vertex of O_e with degree 3 which we have already excluded.

- Suppose that u_1 is the pivot of the 3-coloring b of G. First consider the case that $p_3 \neq p_3'$. As u_4 has even degree and p_3' odd degree, we see that u_3 is the pivot of $u_4u_3p_3rp_3'$. As we assume that G is a minimal counterexample, there is no 2,4-contraction of a vertex in the interior of $u_4u_3p_3rp_3'$. As none of the graphs from Lemma 2.3 breaks the conditions on the graph, i.e. does not separate I from O in a 4-cycle, we can assume that $u_4u_3p_3rp_p'$ was 2,4 contracted to one of them. Either there is the edge u_3p_3' and there is the cycle $u_3p_3'rp_1'u_2$ separating C from O which is a contradiction to the assumptions about C, or p_3' has degree 3 and is in O_e which we have already excluded.

  Hence assume that $p_3 = p_3'$. Now assume $p_1 \neq p_1'$. We distinguish between two more cases, depending on the pivot of $u_1p_1rp_1'u_2$ – it can be either p_1 or u_1, as these two have even degree and are adjacent. If p_1 is the pivot, note that u_1r is not an edge and thus the graph is the Figure 3.5(a) – where C and O are separated by 5-cycle $rp_3u_3u_2p_1$. If u_1 is the pivot, then either there is the edge $p_1'u_1$ and C is separated from O by the cycle $rp_1'u_1u_4p_3$, or there is the edge u_2p_1 and C is separated from O by the cycle $rp_1u_2u_3p_3$.

Therefore, $p_1 = p'_1$, which leads to the case in Figure 3.5(c). In this case, we extend every precoloring of O except at most one in G. Let us fix a 3-coloring b of G. As u_4 and u_1 have even degree, one of them has to be the pivot of O in b, assume it is u_4. Then, u_2 is the pivot of C. Clearly, if we precolor O as in B_1, we get a nonextendable precoloring. The other precoloring c of O satisfying ϕ_1 is such that p_1 is its pivot. But then we can give u_3 the color of p_1 and u_4 the fourth color. Then we have a precoloring of O which satisfies neither ϕ_1 nor ϕ_2.

Now, we have to extend the two precolorings of O satisfying ϕ_2. The first of them is such the precoloring such that u_4 has the color same color as r. Then we give u_2 the color of r and u_3 gets either the color of p_1 or of u_1 – in the first case, this precoloring of O does not satisfy ϕ_2, in the second the precoloring does not satisfy ϕ_1. Thus at least one of them can be extended. Now, let us deal with the precoloring of O such that u_4 gets the same color as p_1. Then we give u_3 the color of u_1, and u_2 gets either the color of r or the color of p_3. Again, at least one of them can be extended.



(a) The case p_1 is the pivot

(b) The case u_1 is the pivot

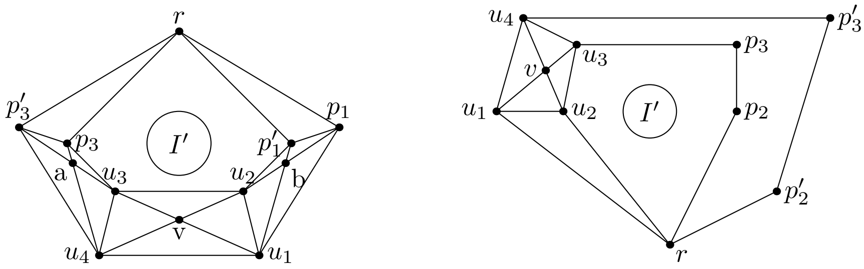
(c) The case $p_1 = p'_1$

Figure 3.5: Unsafe contractions, part 1

- Suppose that r is the pivot of the 3-coloring b of G As there is no edge ru_i for $i = 1 \dots 4$, then by Lemma 2.3 the graph looks as in Figure 3.6(a). Again, we try to extend all precolorings of O. By symmetry, there is just one precoloring c of O satisfying ϕ_1, with say u_4 as a pivot – let $c(u_4) = d,\ c(u_1) = c(r) = a,\ c(p_1) = c(p'_3) = b$. We precolor O as follows: $c(p_3) = d,\ c(u_3) = a, c(u_2) = b, c(p'_1) = d$ or e. Again, at least one of these two can be extended. By symmetry, there is just one precoloring c of O satisfying ϕ_2, with say u_4 having the same color as r. Let $c(p'_3) = a,\ c(u_4) = b,\ c(u_1) = d,\ c(p_1) = e$. We precolor O as follows: $c(u_3) = a,\ c(u_2) = b,\ c(p'_1) = d,\ c(p_3) = e$ or d. Again, at least one of these two can be extended.

The second case is when length$(u_1 p_1 \dots r) = 1$, as seen in Figure 3.6(b). Thus if there is a vertex of degree 4, then it has to be in this configuration. However, to a vertex v in such configuration correspond two vertices in I_e (its neighbors). If there are two vertices of degree 4 in G', they either have disjoint neighbors in I_e and then the graph is the one from 3.6(a) (where a and b are the vertices of degree 4), or they share one – in Figure 3.6(b), it is u_3. But then, u_4 has odd degree but both of its neighbors on O have even degree, a contradiction. Therefore there is only one vertex in this situation.

□



(a) r is a pivot (b) length$(u_1 p_1 \ldots r) = 1$

Figure 3.6: Unsafe contractions, part 2

We have shown that either there is no vertex $v \in G' \setminus V(O) \cup V(C)$ of degree 4 or there is exactly one, and then it is in configuration as in Figure 3.6(b). Now we derive contradiction by a simple discharging argument: assign to every vertex v of G' charge $\phi(v) = \deg(v) - 6$, and to every face F of G' charge $\phi(F) = 2 \cdot |F| - 6$. Now, using Euler formula, we have

$$\sum_v (\deg(v) - 6) + \sum_F (2 \cdot |F| - 6) = -6|V| - 6(\# \text{ of faces of } G') + 2|E| + 4|E| = -12.$$

If we assume that there is no vertex of degree 4 which is not in $V(C) \cup V(O)$, all vertices in $V(G) \setminus V(C) \cup V(O)$ have non-negative charge. Also, all faces except for C and O have charge 0. Thus the charge equals at least

$$\begin{aligned} \sum_v \phi(v) + \sum_F \phi(F) \geq \\ -2 \cdot |O_e| - 2 \cdot |C_e| - 4 \cdot |V(C) \cap V(O)| + \phi(C) + \phi(O) = \\ -2 \cdot |O| - 2 \cdot |C| + \phi(C) + \phi(O) = \\ = -12. \end{aligned} \tag{3.1}$$

If $V(C)$ and $V(O)$ are disjoint, then there is a vertex of odd degree in O_e, thus it has degree at least 5. This increases the left side of Equation (3.1) by 1, and the total charge is strictly greater than -12, a contradiction. If $|V(C) \cap V(O)| > 0$, then there is a vertex u in the intersection with degree greater at least three, as there are vertices $o \in O_e$ and $c \in C_e$ that have common neighbor u, which has another neighbor. Again, left side of Equation (3.1) is increased, a contradiction.

Second case is that $|V(O)| = 5$ – however, the neighbors of the only vertex in O_e are both in $|V(O) \cap C|$ but have degree 3 since they also have neighbor in C_e. Again, left side of $(*)$ is increased.

Now let us assume there is exactly one vertex of degree 4 in $V(G') \setminus (V(O) \cup V(C))$, i.e. the graph is as in Figure 3.6(b). The difference from the previous case is that we have to find at least three units of charge, since v has degree 4. If $|V(C) \cap V(O)| = 3$, then p'_2 and p'_3 are identified with some of the vertices p_2, p_3, u_3. We have that $u_3 \neq p'_2$ and $u_3 \neq p'_3$, as otherwise a (≤ 4)-cycle separates I from O. Thus p_2 equals p'_2 and p_3 equals p'_3, and resulting graph is the one from Figure 3.5(c).

If $|V(C) \cap V(O)| = 1$, then we know that there are at least two vertices of odd degree (one in C_e and one in O_e), and the vertex in $|V(C) \cap V(O)|$ has degree 4, which gives the extra charge.

If $V(C) \cap V(O) = \{r, p_3 = p'_3\}$, then p_3 has odd degree. We can 2,4-contract insides of $rp'_2p'_3p_2$ into either edge $p_2p'_2$ or one universal vertex. In both cases, p'_2 has degree 3 and belongs to O_e, which we have already excluded.

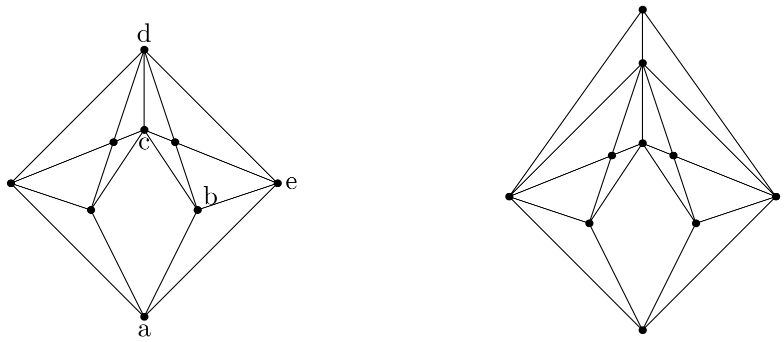
Let $V(C) \cap V(O) = \{r, p_2 = p'_2\}$. We assume that the 5-cycle $u_4u_3p_3p_2p'_3$ contains no vertices of degree 4. Let us further assume that there are no internal vertices at all. Then p'_3 cannot be adjacent to u_3 since this implies p_3 has degree 3 and is in O_e. But p'_3 has degree at least 3, thus is adjacent to p_3. Thus p_3 is adjacent to u_5 and I is separated from O by a 5-cycle $p_3u_4u_1rp_2$, a contradiction. Therefore there is an internal vertex. As the subgraph drawn in the interior of $p_2p'_3u_3u_4p_3$ is in $\mathcal{G}(5)$, we can use the counting argument from Lemma 2.3 to derive a contradiction with nonexistence of vertices of degree 2 or 4.

□

4. Concluding remarks

Although our result is stated for graphs in $\mathcal{G}$, i.e. for 3-colorable triangulations, it shall be noted that it holds for all plane 3-colorable graphs that satisfy the conditions on O and I. Due to Lemma 2.1, a plane graph is 3-colorable iff it is a subgraph of a plane triangulation with all of its vertices having even degree. Thus we shall triangulate all faces except O and I of a graph G, such that I does not get separated from O by a short cycle.

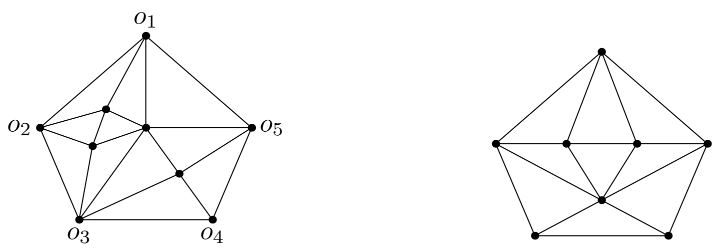
As a further direction, one might ask for similar results for $\mathcal{G}(4,4)$. We cannot hope for a result as nice as Theorem 3.4 for $\mathcal{G}(4,5)$, as there are graphs with non-extendable precoloring – see Figure 4.1.



(a) (b)

Figure 4.1: Some graphs with non-extendable precolorings

However, characterization of graphs that have a non-extendable precoloring seems a rather difficult task. Clearly, the interior of the 4-cycle $bcde$ in Figure 4.1(a) may be replaced by any graph from $\mathcal{G}(4)$ with matching parities. Furthermore, it can be replaced by the graph from Figure 4.1(b), since the graphs from Figure 4.1 are in terms of precoloring equivalent to the graphs from 2.2. Thus we may generate graphs with any number of inner faces of size 4 which are not separated from O by a 3-cycle and have a non-extendable precoloring.



(a) (b)

Figure 4.2: More graphs with non-extendable precolorings

One may be tempted to state similar thing for the interior of the 5-cycle $abcde$, but this is not exactly the case – as replacing it with the graph from Figure 4.2(a), setting vertices $a = o_1,\ b = o_5,\ c = o_4,\ d = o_3,\ e = o_2$ yields a graph such that all precolorings of O are extendable, but setting vertices $a = o_1,\ b = o_2,\ c = o_3,\ d = o_4,\ e = o_5$ yields a graph with a non-extendable precoloring. In the latter case I is separated from O by a 4-cycle, but this is remedied for example by the graph from Figure 4.2(b). It is not even clear if the edge cd must be present in order to get a graph with a non-extendable precoloring.

References

[1] D. Archdeacon, J. Hutchinson, A. Nakamoto, S. Negami, K. Ota: Chromatic numbers of quadrangulations on closed surfaces, Journal of Graph Theory 37 (2001)

[2] B. Bollobas: Modern graph theory, Springer (1998)

[3] R. Diestel: Graph theory, 3rd edition, Springer (2005)

[4] K. L. Collins, J. P. Hutchinson: Four-coloring 6-regular graphs on torus, Graph Colouring and Applications (P. Hansen and O. Marcotte, Eds.), CRM Proceedings and Lecture Notes, Vol. 23 (1999)

[5] H. Grötzsch: Zur Theorie der diskreten Gebilde, VII: Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel, Wiss. Z. Martin-Luther-U., Halle-Wittenberg, Math.-Nat. Reihe 8 (1959)

[6] L. I. Golovina, I. M. Yaglom: Induction in Geometry, D. C. Heath & Co. (1963)

[7] B. Mohar, P. D. seymour: Coloring locally bipartite graphs on surfaces, Proceedings of J. Comb. Theory, Series B, Volume 84 Issue 2 (2002)

[8] J. Hutchinson, B. Richter, P. Seymour: Colouring eulerian triangulations, Journal of Combinatorial Theory Series B, Volume 84 (2002)

[9] P. Kainen, T. Saaty: The Four Color Problem; Assaults and Conquest, McGraw-Hill (1977)

[10] D. Král: Extending partial 5-colorings and 6-colorings in planar graphs, Journal of Combinatorial Theory Series B, Volume 96 Issue 1 (2006)

[11] A. Nakamoto: 5-Chromatic even triangulations on surfaces, Discrete Mathematics 308 (2008)

[12] N. Robertson, D. Sanders, P. Seymour, R. Thomas: The Four-Colour Theorem, Journal of Combinatorial Theory Series B, Volume 70 Issue 1 (1997)

[13] C. Thomassen: 3-list-coloring planar graphs of girth 5, Journal of Combinatorial Theory Series B, Volume 64 Isuue 1 (1995)

[14] C. Thomassen: A short list color proof Grötzsch's theorem, Journal of Combinatorial Theory Series B, Volume 88 Issue 1 (2003)

[15] C. Thomassen: Every planar graph is 5-choosable, Journal of Combinatorial Theory Series B, Volume 62 Issue 1 (1994)

[16] C. Thomassen: Grötzsch's 3-Color Theorem and Its Counterparts for the Torus and the Projective Plane, Journal of Combinatorial Theory Series B, Volume 62 Issue 2 (1994)

[17] M.-T. Tsai, D. B. West: A new proof of 3-colorability of Eulerian triangulations, Ars Mathematicae Contemporanea 4 (2011).

[18] P. Wernicke: Über den kartographischen Vierfarbensatz, Mathematische Annalen 58 (1904).