Univerzita Karlova v Praze
Filozofická fakulta
Katedra logiky

# Diplomová práce

Bc. Petr Švarný

Aplikace temporálních logik ve fyzice
Application of Temporal Logics in Physics

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Chtěl bych poděkovat zejména PhDr. Ondřeji Majerovi, který mi umožnil se tomuto tématu věnovat a pomohl mi v jejím zdárném zakončení. Taktéž Prof. Tomaszi Plackovi, který mi byl zdrojem cenných informací a umožnil mi proniknout do tohoto tématu. Nemohu opomenout ani vedení katedry, které svolilo k práci vedené částečně ze zahraničí a samozřejmě své rodině za pomoc.

Prohlašuji, že jsem diplomovou práci vypracoval samostatně, že jsem řádně citoval všechny použité prameny a literaturu a že práce nebyla využita v rámci jiného vysokoškolského studia či k získání jiného nebo stejného titulu.

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Abstrakt

Práce představuje v krátkosti tři hlavní obory zabývající se zkoumáním času: fyziku, filosofii a logiku. Je podán úvod do obecné teorie relativity, termodynamiky a kvantové fyziky. Jsou vyjmenovány i základní filosofické přístupy k času a jsou probrány ústřední duality této filosofie, jakými jsou např.: eternalismus a presentismus, determinismus a indeterminismus či reálnost a nereálnost času. S ohledem na velkou škálu možných logik, jsou zde popsány různé základní přístupy v nich obsažené, jsou provedeny pro ně typické důkazy či předvedeny jejich zvláštnosti oproti jiným logikám. Zvláště je poté diskutováno užití temporálních logik při formalizaci ve fyzice, však zmíněny jsou i jejich aplikace v jiných oblastech. Následně jsou uvedeny podrobněji systémy zvané Branching space-times (Prostoročasové stromy) a z nich nově odvozené Branching continuations (Stromy pokračování). Tyto logické systémy byly již užitečné v kvantové fyzice. Zde je však vzata základní terminologie spojená s obecnou teorií relativity a také topologie A, P a T. Spolu se zmíněnými logickými systémy jsou užity ke zkoumání možnosti jejich složení.

**Klíčová slova:** čas, obecná teorie relativity, branching space-times, branching continuations, topologie prostoročasu

Abstract

This thesis presents an introduction to the three main fields that study time: physics, philosophy, and logics. A brief introduction to general relativity, thermodynamics and quantum physics is made. Also some of the basic ideas from the philosophy of time are explained and dualities connected to time are described, e.g. eternalism vs. presentism, determinism vs. indeterminism and the reality or unreality of time. As there is a huge number of temporal logics, only the main ideas that differentiate these logics from others are pointed out and some typical proofs are then shown. Special attention is then given to the relation between logics and physics, how the first can be used in the latter. Thereafter, Branching space-times and Branching continuation models are presented, which proved to be useful within quantum physics. Next, some basic terminology connected to general relativity and the A, P and T topologies are introduced . These are used together with the given models to investigate a possible combination.

**Keywords:** time, general theory of relativity, branching space-times, branching continuations, space-time topology

# Contents

# Seznam užitých zkratek

| | |
|---|---|
| BCont | Branching continuations |
| BST | Branching space-times |
| GR | General relativity |
| MBS | Minkowskian branching structures |
| SR | Special relativity |

# Chapter 1

# Introduction

This thesis addresses multiple issues concerning the relation between time and logics. He who is surprised by reading about time and logics being combined will learn that not only this is a vivid field of research but that it represents a very fruitful collaboration of the two subjects. As physics and philosophy are the main endeavours inquiring the nature of this quite odd dimension, they both need to be taken into account when dealing seriously with time. The aim of the thesis is to investigate current relation of temporal logics to physics, present the field of branching structures in a more detailed way and look into a possibility of their connection with general relativity.

Therefore may the reader be kindly asked to decide according to his knowledge of the subject, time available and mood, which part will he read. As the new addition of this thesis is the last part, we can relate to it the preceding ones. The first part speaks generally about the formal representations of time and the notions connected to time. Therefore a reader well educated in physics could learn more about logics and philosophy of time and a reader gifted with knowledge of temporal logic could attain new insights into the physics of time. As temporal logics and philosophy aren't studied at the author's department of logics, this chapter wants to present the context of the work. However, this part plays a secondary role in understanding the final work itself. Nevertheless it tries to be a useful crossroad for anyone interested in the topic, although it maintains a focus on issues relevant to the subject of inquiry.

The second part on the other hand encompasses the introduction to the formal apparatus used in the last part. This chapter is important to read for those who are not familiar with Minkowskian Branching Structures, Branching Continuations or Path topologies in general relativity.

It is only the part called GR branching structures which bears the new addition to the topic of time's formalization in physics. It presents an attempt to combine a system of physics and logics from the point of view of a logician. Two possible ideas are presented in different stages of work. The idea of similarity between Alexandrov topologies and Branching space-time topologies is mentioned. As next, the relation between BCont and $\mathcal{T}$-topology is presented.

# Chapter 2

# Time

As many scientific endeavours also the study of time can trace its origins to the Greek. In Aristotle's *On Interpretation* there is a simple statement that can launch our whole inquiry:

> "Take a sea-battle: it would have neither to happen nor not to happen." [1]

The discussion can now follow in all the paths we are going to peak into. The most common being the question if the statement "There will be a sea-battle tomorrow." is true or false. Thus comes up the question of formalizing temporal statements as seen in [34], for a formal theory permits a more thorough investigation of a given problem. If we already have some formalism, is this clarified statement true or not? Hence we arrived to the philosophy or physics of time. May the reader not be alarmed, this famous example will be discussed later, but for now allow a remark about the organisation of the following text. The topic of time and its less formal introduction is divided into three parts. First, one can learn about the physics of time, then the philosophy behind it is discussed and as last are presented temporal logics. As it is not the aim of this chapter nor of the whole text to give a complete introduction to these topics. The reader is kindly shown relevant literature at appropriate occasions and should regard those for more information about the topics. Herein he finds only a basic and not necessarily complete guide which tries to point out the important buzzwords, relations and problems.

## 2.1 The Arrow of Time

In physics (and natural sciences) time obviously plays a very important role and one could also judge that it is the view on time that differentiates some of the main fields. Although the notion of time used in this work comes from general relativity, it is useful to see how this concept has evolved and on what scientific evidence it is based on.

### 2.1.1 Time in the background

In classical Newtonian physics, time was absolute and worked on the background but presented a foundation for all the apparatus. As the one who dethroned absolute time puts it:

> "...the concept of objective time, without which the formulation of the fundamentals of classical mechanics is impossible..." [15]

Isaac Newton had in the background of his theory absolute space and time. Time flew without being influenced by the happenings in the world and thus two observers should have the same experience concerning time. This view of time is strongly based on our usual experience and seems natural to assume. However, there were some problems connected to this theory. Concerning time, there was the question of the direction of time. As all Newtonian laws seem reversible, how does it happen that there are some processes we do not observe happening in the inverse direction. Puzzling was also the postulate of instant effect of gravity at any distance. Later, when James C. Maxwell's equations for electromagnetism were presented, it was observed that they are not invariant under Galileo's transformation which until then worked in classical physics just fine. This lead to the creation of the Lorentz transformations, preserving invariance also for electromagnetism, and the dethroning of absolute time. However, there was also an opposing view to Newton's physics even at his time. Gottfried Wilhelm von Leibniz denounced not only the concept of absolute space, but also the idea of absolute time. A drawback of his view being that he had no physical theory to build his views on [33].

An interesting addition to Newtonian physics is the discussion of determinism. According to classical mechanics one could, if the state of the system is known, compute step by step how the system looks like in the future (or was in the past). Thus everything is determined and once we get a foothold we cannot be shaken. But some theoretical experiments show that there actually could be indeterminism in the classical theory also and thus would lead to a kind of direction in Newtonian time. The example is the so called Norton's dome [32] and although challenged by others it still represents an interesting view on the topic. As the solution allows for a single past to have multiple futures although in classical mechanics the system's future should be determined by its past.

Classical mechanics however proved to be ineffective when faced with a task seemingly simple, for example the movement and interaction of three bodies (Moon, Sun and Earth as presented in Henri Poincaré's work) and this lead to a variety of new ideas [10]. Among these children of classical mechanics were also thermodynamics and they were the first defendant of a strong arrow of time in physics and thus should be mentioned also.

### 2.1.2 Stirring the thermodynamic pudding

> "When you stir your rice pudding, Septimus, the spoonful of jam spreads itself round making red trails like the picture of a meteor in my astronomical atlas. But if you stir backward, the jam will not come together again. Indeed, the pudding does not notice and continues to turn pink just as before. Do you think this is odd?" [47]

In our daily experience, it is significant that there is an arrow of time and so the world around us evolves in one given direction and we do not observe for example a smashed cup collecting all its pieces and jumping back on the table. One of the first ways how to explain that there is an arrow of time came from thermodynamics. In their second law, all processes are described as irreversible because there always is and will be a loss of energy that turns into heat [10]. Thus any isolated system left unattended inclines to a state of equilibrium but not the other way.

The whole universe is under the rule of entropy, which means everything tends to the thermodynamical equilibrium and thus to higher unorder. As Hermann von Helmholtz observed, this leads to the idea that the evolution of the universe will end in a so called heat death of it. How ever if there is an end of the universe, is there also a beginning? It were the theories of relativity that lead to the view that there is one.

### 2.1.3 Relatively surprising relativity

The simple idea of not having any privileged reference frames not only in mechanics (as the Galileo principle puts it) but in all of physics yields in combination with the assumption of invariant light speed many consequences. This is known as special relativity and with the addition of stronger grativational force it becomes general relativity. The first astronomical proof that something odd is happening with time and observation of events on longer distances was given by Ole Roemer in 1675. He observed Jupiter's moons and suggested that the speed of light is limited and thus creating a delay in our observation of the events happening near Jupiter [10].

However objective time as an important concept in classical Newtonian mechanics still held its position. Einstein claimed [15] that there can be identified two independent statements connected to the introduction of objective time:

1. By using a closed system with a periodical activity (like a clock) we can introduce objective local time.

2. We create time in physics as the enlargement of our local time.

The second statement presents a grave problem in relativistic physics. As the goal of relativity is to eliminate any privileged observers, the original view on time seems to be an illusion.

> “The illusion which prevailed prior to the enunciation of the theory of relativity - that, from the point of view of experience the meaning of simultaneity in relation to happenings distant in space and consequently that the meaning of time in physics is a priori clear-this illusion had its origin in the fact that in our everyday experience, we can neglect the time of propagation of light. We are accustomed on this account to fail to differentiate between “simultaneously seen” and “simultaneously happening”; and, as a result the difference between time and local time fades away.” [15]

However Einstein does remark that it was lucky that these problems were not seen by the early scientists and they could develop their theories without knowing anything about relativity. As we can see in the context of common everyday life relativity is seemingly unimportant[1]. Yet as our goal is to study time itself, Einstein's theory presents a thrilling and new view on time.

It is not space-time that makes relativity what it is, despite Minkowski space-time being very closely tied to special relativity. Also classical mechanics can be represented in a four dimensional graph. The difference would be that in classical mechanics the present could be given as a simultaneity slice without any further discussions. Thus each constant value of time gives us a still image of the universe. The new feature of relativity is that we abandon global time and crucial becomes the notion of light cones and the speed of

[1]Let us remark that this is not true in modern times. For example the GPS system, which is already part of many people's daily activities, has to use calculations based on general relativity.

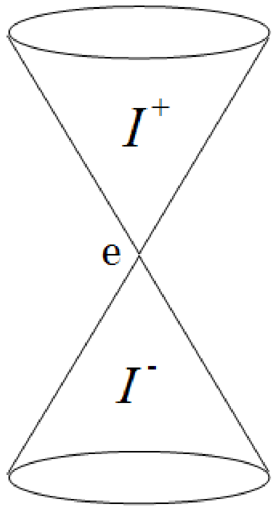


Figure 2.1: Representation of a light cone of the event $e$. $I^+$ marks temporal future and $I^-$ stands for temporal past.

observers. Every spatiotemporal event produces a cone limited by the rays of light. Light-speed is the limit as it represents the highest possible speed of propagation in space-time. The usual representation takes light cones as ordered according to the y axis with the past being downwards and the future being upwards.

The two parts of the light cone represent then either events that have influenced the given event or events that will be influenced by the observed event. Light cones can then form a world line for an object's history or generally represent a course of events, that is a path in the four dimensional space-time. To establish simultaneity in this approach one uses the so called Lorentz transformations which allow us to relate the coordinates if the observers are moving uniformly. These transformations, forming a Lorentz group[2], leave the light cone invaried. Thus we can build on them. Although it might not seem on first glance, special relativity is a much simpler theory than the original Newtonian theory but it still contains a limiting simplification. However, it already presents new results concerning time. This leads us to the abandon of the classical view on time [33].

There are two main results connected with time that come from special relativity - time dilation and relativity of simultaneity. Both concern two

[2]From a mathematical point of view, it is a generalized orthogonal group.

observers with their respective reference frames. First, the time lapse between two events is not invariant with regards to the observers. On the contrary it is dependent on their relative speed. Also they do not profit from absolute simultaneity. Events seemingly simultaneous in one frame, can cease to be in the other [14].

Space-time in special relativity is flat and thus all the light cones are oriented in the same direction. General relativity adds gravity to the universe. This then leads to a space-time that already has a curvature and thus the orientation of light cones can be different. As we saw with special relativity, the question of time is tightly connected with light. As general relativity permits the bending of space-time it thus changes also the way how light signals propagate. They still follow a geodetic but these can now take various shapes and not only straight lines as in the case of special relativity. Also time itself seems slower near massive objects. Thus events closer to a strong gravitational source seem to last longer. This is due to the loss of energy of light in the gravitational field. It even adds new dynamics to the whole system as:

> "Space and time not only affect but also are affected by everything that happens in the universe." [23].

As we mentioned earlier, it is the result of general relativity that the universe must have a beginning [23][3].

It is worth mentioning that general relativity bears also the hopes for a possible machine capable of time travel, as various operations on space-time based on general relativity can lead to many interesting results [22]. As an example can serve Gödel's solution. It was dismissed as not being a solution valid for our universe, but it did show that general relativity allows for time-travels. Another addition to this topic is for example Novikov's self-consistency principle [17]. Contrary to for example Hawking's earlier chronology protection conjecture [23], this principle does not forbid information transfer to the past, it only forbids transfer that could lead to paradoxes by forcing closed timelike curves to obey the usual local laws of physics[4].

Another important result of general relativity is that it adds a new possible arrow of time, called the cosmological arrow of time [23], but which does not have to be in the same direction as the thermodynamic arrow. General relativity lead also to the "Big Bang" singularity theory and depending on the amount of mass present in the universe it leads to two different scenarios

[3] And depending on starting conditions it has also three possible endings. [25]

[4] General relativity itself then brings into play also other consequences (like gravitational waves for example) but here we mention only the main effects on time.

for the distant future of the universe - endless expansion (with Lobachevsky spaces or Euclidean for the limit case) or the "Big Crunch" [36]. However, Hawking argues that here can be used the so called anthropic principle. Assuming that the universe is expanding just so to avoid the "Big Crunch" the universe will expand for a long time and thus after a (long) while:

> "Disorder couldn't increase much because the universe would be in a state of almost complete disorder already. However, a strong thermodynamic arrow is necessary for intelligent life to operate. In order to survive, human beings have to consume food, which is an ordered form of energy, and convert it into heat, which is a disordered form of energy. Thus intelligent life could not exist in the contracting phase of the universe. This is the explanation of why we observe that the thermodynamic and cosmological arrows of time point in the same direction." [23]

At this moment it seems appropriate to mention an idea that gives time in view of global history an interesting twist. The so called Poincaré recurrence theorem states that given a system will, after a sufficiently long time, return to its initial state [10]. Especially in connection with our wish to impose on time a system of logics this presents an interesting view. Much later after Poincaré came with this idea, it received also a quantum mechanical interpretation. So let us then look if quantum mechanics have something to say about time.

### 2.1.4 Quantum and time

The main topic in quantum mechanics with regards to time is the question of causality and determinism. At first glance quantum physics seem to be symmetrical with regards to time, similarly as Newtonian physics are. Erwin Schrödinger's equations seem still to be deterministic, but the collapse of the wave function[5], which they describe, already gives us some interesting results. We won't go into much detail, but there are multiple interpretations of quantum mechanics. One of them suggests that there is the need of an observer who causes the collapse by his observation. A way how to evade the need of a problematic notion of a observer is the solution of Hugh Everett's many worlds interpretation. Then we do not get any collapse but every time a new world is created. Our universe itself is a wave function bearing all the possibilities [44]. This interpretation was then confronted by for example

[5]Strictly speaking a wave function is a probability amplitude, thus a multitude of possible states. Its collapse means that it represents only one state.

Nuel Belnap, who regards our world[6] as sufficient to cover all the possibilities without the need to introduce the concept of many worlds [3]. But generally speaking, how do the two time-relevant problems manifest themselves? Quantum mechanics introduce randomness that strikes down determinism. One still can make reasonable statistical predictions and create interference images for the famous two slit experiment[23]. However, one cannot predict the results of single runs [23]. The second novelty goes against causality. We can formulate for example the Einstein-Podolsky-Rosen paradox. The main idea is the instantaneous influence between two specific particles on in different locations. This violates locality based on special relativity, as information between the particles would be transmitted with a speed superior to the speed of light, and thus it violates also causality[7]. As with relativity theories, our aim was to introduce the main topic relevant to the study of time.

Let us now turn towards the 'observers' and see how philosophy has faced the problem of time.

## 2.2 The Bow of the Philosopher

Time viewed through man's eyes rather than through the lens of science is what this part describes. Still, these are not the eyes of an ordinary man but those of philosophers distributed variously along the time line of human history. This part will, at some moments, slightly venture into the waters of temporal logics. This is only a natural consequence of the clarification effort.[8] Philosophy in connection with time faces two main topics - the ontology of time and the way we relate to it. These raise either questions about determinism, about the nature of time or about the role of free will. As a motivational quote, let us regard:

> "...if it is accepted that every concept, including the concept of time, has to be related to the human mind. Under this perspective it becomes more natural to describe time by means of tenses: past, present and future, than by means of instants (dates, clock-time, etc.). With tenses, we can express that the past is forever lost and the future is not yet here. Without these ideas we cannot hope to grasp the idea of the passing of time." [34]

[6]This world is not understood in the sense of modal logics. It represents a multitude of possibilities and options.

[7]For a deeper insight and study using logics see [5], [39].

[8]If not mentioned otherwise, we reference the history of temporal logics from [34].

It is useful to remember that temporal logics themselves are often a model based on human intuition and thus do not have to reflect in any way reality. While our final hope is to construct a system based on physical models, willing even to sacrifice some amount of our belief what time should look like.

We saw already in the first part that time is hard to pin down and any attempt to define it in its fullness would lead to a failure. However, there have been many tries and approaches which tried to do so. Thus, if we do not regard time simply as being what it is, a sui generis eluding every effort to be bounded in definitions, but try to reason about it, then we might return to the original question:

> Will there be a sea-fight tomorrow or not?

Aristotle, the author of On interpretation where the sea-battle problem was presented, regarded it as an undetermined statement. He judged that we can have necessary truth about the past and even the present but not about the future. It can be questioned whether this meant the introduction of a third truth value or not, but it certainly pointed to the problem of formalizing indeterminism. He also did work that could be related to modern interval logics or work on the different approaches on the description of time [9], later known as the A and B series.

Aristotle points out problems which are being still discussed in the philosophy of time. He criticized his predecessors by saying that time cannot be a certain particular change (clocks or the movement of heavenly bodies). He even claims that it cannot be any change. Time and change can be only correlated notions as we observe them always together. This argumentation ends in claiming the existence of a kind of abstract but change-dependent universal time that does not undergo any changes by itself (as it is for example impossible to state about it that it passes). Although time is not just a form of change, it remains ontologically dependent on change as 'a number of change with respect to the before and after'. Also because time, contrary to change, is not regarded as having a direct connection with substances in the world [9].

Another aspect of Aristotle's approach to, not only, time are puzzles or one could say troubling questions. His first puzzle is about the existence of time itself: future will be and past was, hence there is nothing to exist. One could want to add the 'now' into play but Aristotle dismisses this idea almost right away. The reason is his opinion that 'now' is not a part of time[9]. In

[9] "When divisible things exist," he says, "either one or more of their parts must exist. But no part of time exists. The now is not a part." [9] There it is also argued that these puzzles do not present an illusion of language.

another puzzle, the claim is made that one cannot consistently hold neither that 'now' is always the same nor that it is every time a different one. Ursula Coope [9] argues that a 'now' taken as *"a potential division in all the changes that are going on at it"* can be a solution to the puzzles, even in Aristotle's understanding.

We can see also an interesting discussion about the distinction of things that are in time and things that are not. We won't go into much detail but we just mention that limited existence, like the life of a person, is obviously in time, but the existence of mathematical statements does not depend on time. Connected to this is the view that being in time is interwoven with aging and decay which is caused by time, although time is not the agent producing these changes [9].

A less well documented but as interesting is the addition of Diodorus Cronus from the 3rd century BC. His 'Master Argument' went as follows:

> (D1) Every proposition true about the past is necessary,
> (D2) An impossible proposition cannot follow from (or after) a possible one,
> (D3) There is a proposition which is possible, but which neither is nor will be true.
>
> Assume D1 and D2, then:
> (DPoss) The possible is that which either is or will be true,
> (DNecc) The necessary is that which, being true, will not be false;

According to the Master Argument, the first three statements aren't compatible and Diodorus claims that (D3) is false as (D1) and (D2) are plausible. This could be a case in favour of determinism (as opposed to Aristotle's) and it addresses the relation between modality and time. Yet, sadly, this is all that is known about the argument and so it is up to modern interpretations to judge how the argument can go. There is a great vagueness of the statements, especially from modern temporal logics point of view. The first task of anyone trying to work with the argument is to settle a uniform way how to treat each aspect - what does 'follow mean' or how do we rewrite the premises into a (semi)formal language. But how one does it, is actually up to the given author. The discussion then revolves around the question of truth dependent on the time of utterance or on a metalevel the choice of other premises that support the given reconstruction or generally the interpretation itself. We present here a possible reconstruction, the Mate's one[10]:

[10]For a deeper analysis of the argument, Prior's reconstruction and discussion see [34].

> a) Time is discrete,
>
> b) Diodorean propositions are functions of time. Thus, propositions are functions from instants into truth values and conversely, such functions are propositions. For the function application of a proposition p to an instant t we write T(t,p),
>
> c) The Diodorean implication involved in (D2) can be defined in terms of temporal logic as $(p \Rightarrow q)$ iff $(\forall t)(T(t,p) \rightarrow T(t,q))$.

What actually Diodorus though can hardly be reconstructed but an educated guess can be made about some of his preferences. An interesting point in this reconstruction is that does not involve the problematic modal concepts. And with this interpretation one can show that (D3) contradicts the premises (D1) and (D2).

This presents an opportunity if someone would desire to use the Master Argument against determinism as it is enough to deny (D1) or (D2).

As determinism, another big issue connected with time was presented by Heraclitus and Parmenides. The first took time as perpetual change but the latter regarded it as non existent - there is no time (not even motion). In modern regards, Heraclitean view states that *"the world is made up of 3D objects, which endure and change in time, while retaining their identity from one moment to the next. Parmenideans, on the other hand, believe that the world is a changeless 4D spacetime continuum, containing material objects that are 4D worm-like volumes extended along the time dimension."* [28] These two opinions again echo through the history of temporal logics.

It was mainly in the medieval times when Aristotle's sea-battle got interpreted with the notion of contingent future. A statement about the future judged now can be true or false according to what will actually happen. It is more our limited knowledge, then unsettled future, that does not permit us to know the right truth value[11]. As Peter Abelard in 11th century interprets Aristotle:

> "...while a proposition is necessary when it is true, it is not therefore necessarily true simply and always." [34]

The motivation for this interpretation mainly lays in the Christian faith as it would be unthinkable to have the omniscient God not be able to foretell what is in the future. A solution to this was brought by William of Ockham. As he presented that a statement about the future is true if it will be true. Hence although we have options and possibilities, only one future path amongst them is the 'correct' one.

[11] This marks the birth of the so called 'Ockhamist' view on time that will be discussed also later on and was named so by A. N. Prior.

Henri Bergson was first to clearly formulate the concept of branching time in 1889. Yet, he did not regard it as an appropriate formalisation of time. On the other hand, Charles Sanders Peirce lead the way in reintroducing time into modern logics. In his works, he discussed the need of changeable qualities as time is the universal form of change. As experience played a big role in his philosophy, he mentioned that there is no such a short span of time that would not contain the experience of continuity and that the human mind does perceive the arrow of time in a given direction. But the direction is not arbitrary, as there is a clear distinction between the past made of 'now-unpreventable' facts and the future that is but a subjective human creation (intentions, expectations, etc.).[12]

> "I remember the past, but I have absolutely no slightest approach to such knowledge of the future. On the other hand I have considerable power over the future, but nobody except the Parisian mob imagines that he can change the past by much or by little."

The most interesting addition of Peirce is connected to modality. He already pointed out that time can be viewed as *"a particular variety of objective modality"*. Peirce distinguished three modes of being. From temporal point of view, actuality meant the present and past and then there was possibility and necessity for the future. He did deny truth value to the future, as giving it any value was judged to be meaningless. Only if we admit that laws are real, then the necessary consequents are true as the present on which we base them is. Hence Pearcian approach meant also that one cannot distinguish necessity or non-necessity of future statements.

Peirce supported the view that our will is not free. He came to a similar opinion as Augustin[13] in the question of foreknowledge and free will. He supported that God would need to be beyond time, as if not then there is a contradiction between omniscience and free will. He also discussed the 'doctrine of necessity' in other words determinism. His argumentation went against mechanical determinism as even scientific observations are merely highly probable but not necessary. He assumed there exists probability and even real possibility in the world and so a possibility how to incorporate a limited kind of free will.

> "The freedom lies in the choice which long antecedes the will. There a state of nearly unstable equilibrium is found."

[12]Interesting is that Peirce believed in something as the Poincaré recurrence theorem.

[13]Not very surprising if one takes into account that Peirce was very occupied with Medieval logic.

J. Ellis McTaggart [29] in 1908 judged that time can be regarded in two ways. One either speaks in terms of 'earlier than' and 'later than' or the trinity of 'future','present' and 'past' is used. He called the first B series and the second way of viewing time he called A series. Also an atemporal C series is given. It does not involve any change (as that is the base for time in A and B series) nor does it use any temporal references (e.g. earlier), it only captures non-oriented relations between the events.

These two series can be taken as a basis for the so called eternalist and presentist discussion, although the roots of the two views lie deeper in history[14]. The eternalists claim that every event, past or future, exists in the same way as a present event does. Presentists, on the contrary, say that only the occurring events exist at a given time [45] [38]. These two views can be regarded to some extent as a continuation of the debate between Parmenides and Heraclitus.

If we return to McTaggart's work, with the use of the A and C series, the B series can be defined and so it is argued that the first two are more fundamental. As for the role of the present. Because we search for a time with a unique linear structure, for example the case of multiple presents is dismissed by saying that *"they must be present successively"* and thus they form a linear structure. The argument that A series is more fundamental than the B series is demonstrated also by Novikov. On a closed time curve we cannot distinguish future and past but locally we are still able to tell that an event is earlier or sooner than the other [33].

McTaggart, in the end, argues for the unreality of (objective) time, claiming that its definition, based on the A series, involves a vicious circle. The only consistent and meaningful definition of time would arise from our perception of the present. Which is subjective and also greatly limited by our perception capabilities[15]. The problem is not solved if the present is an interval, and thus rises the question of its duration, nor a single point because both would mean that the objective time is too different from the time as we perceive it to be of any use. Thus according to this argumentation time is unreal and there is only the subjective time with its merits and flaws. Only the C series relations stay unaffected by this result and thus supposedly present a strong enough, but atemporal foundation[16].

[14]For example by Augustine de Hippo in the 11th chapter of his Confessions. He attributes to God the existence outside of time and thus an eternalist view on it.

[15]Here is an interesting link between relativity theory and McTaggart as he describes his idea of 'spacious present' that is the actually perceived environment.

[16]Julian Barbour [2] could be regarded as one of McTaggard's followers. According to his argumentation time does not exist, at least not in the common sense. He denounces time as an illusion created by our minds.

More than relevant to this view is Gödel's model of rotating universe as this, while constructed in accordance with general relativity, does not contain a 'cosmic', i.e. objective and linear, time. Gödel argues that causal time nor cosmic time are guaranteed by general relativity, but they follow from other observed facts (as the earlier mentioned second law of thermodynamics). As these seem more as mere generalizations than laws of nature, the conclusion is that:

> "no ultimate, metaphysical reality can be claimed for time" [8].

An almost opposing opinion, supporting the close collaboration of philosophy and physics, was given by Hans Reichenbach:

> "There is no other way to solve the problem of time than the way through physics. More than any other science, physics has been concerned with the nature of time. If time is objective, the physicist must have discovered that fact, if there is Becoming the physicist must know it; but if time is merely subjective and Being is timeless, the physicist must have been able to ignore time in his construction of reality and describe the world without the help of time." [34]

There are multiple examples contemporary additions to these discussions but let us mention just one. This being from Daniel C. Dennett and Christopher Taylor [11]. They argue against the idea of 'incompatibilism' which states that free will and determinism are incompatible. They employ tools as Quine's possible worlds, Lewis' counterfactuals or Pearl's view on causality.They say that:

> "the truth or falsity of determinism should not affect our belief that certain unrealized events were nevertheless possible, in an important everyday sense of the word." [11]

They give an example of two computer programs playing chess. Their match seems first as a series of reactions and actions but when regarded on the level of the processor, it is a deterministic succession of instructions. The addition of a pseudo-random number generator does not change the situation in the long run. Admitting that this is a simple model, they still take it as a image of how determinism should be treated. In addition to this deterministic model of indeterministic world view, they consider determinism as not a doctrine of necessity. If a condition A leads to some event B, A is only sufficient but not necessary. The event B could be caused by different conditions than A (although probably similar to it).

Summing up the encountered opinions, we can state that there is a multitude or reoccurring problems: determinism vs. indeterminism, actuality vs. eternalism, the existence of objective time vs. the existence of only subjective time. As we already made appeal in some moments to logics, it seems only reasonable to turn ourselves to logics as a tool for analysis. Despite Reichenbachs emphasis on physics, as the tool to investigate time, logics are crucial for the task also. After all, what would be physics without it's formal background?

## 2.3 The Aim of the Logician

Temporal logics take on many forms and reflect either the purpouse for which or views on which they were built. Although they seem less controversial in their basic most known versions [20], some managed to raise a lot of questions. As a significant amount of temporal logics is created based on some assumptions about the structure of time, it is useful to have a general overview of the field. Until now, general ideas about time were discussed, but now we shall regard how exactly logics have struggled with the topic.

In this part some basic temporal logics will be introduced and related to the already mentioned topics. Although temporal logics play a very important role also in computer science or natural language studies, due to the aim of this work, they will be merely mentioned at the end[17]. First the reader will be acquainted with the general history of these logics, followed by a slightly formal presentation of some chosen systems[18].

### 2.3.1 P(temporal logicians)

The modern history of temporal logic starts primarily with Prior[19]. Granting, it was perhaps George Boole in the 19th century who first introduced time into formal logic. In his manuscript Sketch of a Theory and Method of Probabilities Founded upon the Calculus of Logic he came with the idea to interpret symbols as

> "representing the times in which the elementary propositions to which they refer are true." [34]

[17] Let it be mentioned that the herein presented view diverges from the 'main research areas of temporal logics' as in [18].

[18] As good references to the general topic of temporal logics especially the following sources can be recommended: [6], [34]

[19] We follow here to a large extent [34]

Later in his Mathematical Theories of Logic and Probability, he even came close to something as a time interval calculus.

The other 19th century logician who considered time as an option in logics was C. S. Peirce. However, he assumed that logics aren't ready yet for the introduction of time. In his philosophy, we can find already rudiments of temporal logics. Peircean attitude towards logics was that an assertion represents a fact, i.e. a portion of reality. If we work in a changing world this imposes that we need to link our fact with some information about time.

Jan Łukasiewicz also contributed to temporal logics. Starting with Aristotle's sea battle scenario he developed a position already held by the Epicureans. Generally, if we assume for some flow of time the principle of *bivalence*:

$$\forall t, p[(T(t,p) \vee T(t,\neg p)) \wedge \neg(T(t,p) \wedge T(t,\neg p))]]$$

and principle of *logical determinism*:

$$(T(t,p) \wedge (t_1 < t)) \rightarrow T(t_1, T(t,p))$$

and future determinism where $t < t_1$ is the only difference in comparison with logical determinism. If put into words we get for bivalence that either at time t it is true that p or it is true that ¬p but not both, determinism again means that if something is true at time t, it is true at any time before t.

This leads to omnitemporal truth. It can be, by contradiction, shown that future determinism and bivalence imply determinism and thus a proposition true at some t will be true at any t. To solve this Łukasiewicz introduced a third value and so rejected bivalence. The third value then means 'undetermined' and *"is applied to contingent propositions regarding the future"* [34]. Prior later addressed this, being not satisfied that the disjunction of two undetermined propositions is also undetermined and thus does not solve Aristotle's sea battle problem.

In 1957, Time and Modality[20] from Arthur Norman Prior changed the relation of time and logics. He was inspired by the works of his predecessors - Boole and Peirce. Prior analyzed earlier approaches to time and based on them. In collaboration with Charles Hamblin, he developed temporal logic to a modern formal system[21]. Prior's method was to construct different systems and then judge upon the consequences these have, as opposed to the direction other's usually take(i.e. from an ontological idea of time construct a system).

[20]Followed by, the even more significant book Past, Present and Future in 1967.

[21]A contribution worth mentioning stems from Saul Kripke, who in 1958 suggested in a letter to Prior a formal system of branching time [34].

One of the results are two main interpretations of branching time models, which tackle the problem of indeterminism/determinism. First being the so called Ockhamist system which assumes that there is one settled future and thus can define truth of a proposition as:

*'it will be in* n *days that* p*' is true now if and only if* p *will be true in* n *days.*

formally: $t < t_n \rightarrow T(t, T(t_n, p)) \Leftrightarrow T(t_n, p)$

If we use the sea battle scenario: if there will be a battle tomorrow, it would be false to assert that there won't be, despite the fact that we are unable to know today. The other interpretation is called Peircean and was based on Peirce's assumption that the future is not settled yet and thus cannot have any truth value. The result being that it cannot be inferred from the present happening that this was going to happen, but it still holds that it will have happened. A little formally put using the already once applied formalism:

$T(t, p) \not\Rightarrow (\forall t_1 : t_1 < t \rightarrow T(t_1, T(t, p))$

$T(t, p) \Rightarrow (\forall t_1 : t < t_1 \rightarrow T(t_1, T(t, p))$

The only proposition able to hold for the future is a necessary one, thus true for all futures. Prior also introduced other topics still vivid in modern temporal logics, for example he did notice the difference between local and non-local assignments in branching tree structures [6]. As one can assign truth values for variables depending on the branches/histories/worlds or not.

Burgess [8] presents an interesting view on the three valued tense logic as an intersection of the Ockhamist and Peircean view - one supposes that 'it will be that p' as meaningful in the actual course of events, but also one demands that every meaningful statement has a truth-value now that is independent on what future will become actualized. Then **F**p is true when every possible future contains p as true at some point or false when none does. As there are options left, a third value needs to be introduced.

Belnap [3] continued Prior's work by combining the Minkowski space-time and Prior/Thomason branching time into Branching space-time models. These represented a blend of relativity and indeterminism which gave a rigorous theory to work with time. Although inspired by physics, these models were still only formal models with no ties to actual representations of notions found in physics. While BST models look similar to many-worlds models, BST emphasizes the fact that the whole structure represents one universe, which is called our world. As in the original branching structures, the basic entity is a history and its events[22]. Returning to the sea-battle

[22]Out of mere interest, let us remark that Nishimura presented a similar albeit different approach [34].

scenario, for a history $H_1$ and a different history $H_2$, the battle is already settled. Let's say that on the first the battle is going to happen and on the second it is not. They both are constituted from the same events up until this battle and divide at the event before the battle.

The next step was done by Thomas Müller [31], who chose BST models which were isomorphic to Minkowski space-times. These models are called Minkowskian Branching Structures.

The newest addition to this family of branching temporal models is called Branching Continuations. A key feature of this system is the introduction of 'large events' instead of histories. These permit to operate on a more epistemology friendly level as opposed to the infinite histories from earlier models. The last mentioned branching structures have in common that they try to model indeterminism but do not reflect at all the question of eternalism and only a little are they concerned about the objective existence of time. It is more in their collaboration with physics that these features gain on significance.

In semantics of the BCont theory, there is the novelty of adding to the evaluation the distinction of the evaluated point and the point of evaluation. Therefore allowing to judge truth values according to the point of evaluation. As an example from [40] goes: Was Einstein born as a Nobel prize winner? If we judge today, then it is already part of our settled past and thus true. But if we take as the point of evaluation the moment he was born, then the sentence cannot be settled true[23]. This idea was already presented by Reichenbach [8] and was motivated by the will to distinguish the time of utterance and of reference. In this way allowing to analyze differences in present perfect (I have done) and past simple sentences (I did) in English.

Even though it might seem we already mentioned all the ideas on logic and time, there still is at least one option left to discover. We started as usual with the Greek, but logics were present also in other cultures and an interesting contribution to temporal logics comes from ancient China. As Chinese traditional logic works with quite different principles than the one we studied here, we shall only mention the main idea so that we don't get lost in too many details. Jinmei Yuan [52] studied the role of time in logics from the Later Mohist Canon. The novelty[24] with regards to time is that time is subjective and it is observer(human) and context dependent. This observer is also part of the ever changing world. Thus time cannot be pure and without

[23] For a formal presentation, may the reader find the section about BCont.

[24] As far as one can speak of novelty in this case, as the Late Mohist Canon stems approximately from the 5th century B.C.

human experiences and truth must change as the world does. Hand in hand follows that time consists only of a 'now' instant - a particular moment of the event - but this is spread through a multitude of possible worlds (which do not have to correspond fully to our term of possible worlds). These possible worlds serve to show how in the world of constant change nothing remains stable and static, this is also the reason for the different way of reasoning in Chinese traditional logic[25].

At last, may it be said that temporal results were sometimes gained also as a side effect as for example were the prePriorian results of A. G. Walker mentioned in [34].

### 2.3.2 Mainly linear temporal logic

One of the most natural approaches of modeling time is a line, a so called *flow of time*. This is basically a linearly ordered set [49]. This frame of the structure $(T, <)$, where T is a set of time points and $<$ is a strict ordering by the relation of precedence ($p < q$ meaning 'p is earlier than q'), is limited in many aspects. As it pictures the view on time as a simple line it does not allow irreflexivity, wishes for transitivity and does not support parallel time lines. We can regard time as Kripke frames, obviously in this case of very limited choice. The actual linear ordering then might have many possible forms, either being some usual ordered set (like $(\mathbb{N}, <)$) or even a Minkowski spacetime $(\mathbb{R}^4, \lhd)$. The final choice is then dependent on the investigated properties as for example on natural numbers we cannot have a dense ordering.

We list some first order properties that can be imposed on a given flow of time (T,<) (we omit dual formulas):

| | |
|---|---|
| Ending | $\exists p\, \forall q(q < p \vee p = q)$ |
| Right-serial | $\forall p\, \exists q(p < q)$ |
| Denseness | $\forall p, q(p < q \rightarrow \exists r(p < r \wedge r < q))$ |
| Discreteness | $\forall p, q(p < q \rightarrow \exists r(p < r \wedge \neg\exists s(p < s \wedge s < r)))$ |

As we already operate with frames, it is easy to see the connection with classical modal logics. The main difference is, as we saw with Prior, that in temporal logics one *"starts with structures, for which one is trying to find*

[25]One can see in the use similarity to the philosophical use of possible world models. Because as for example the famous argument of Gongsun Long goes - a white horse is not a horse. From a sum over worlds where are white horses I can make an embedding but not an isomorphism into the worlds where I chose all the horses. Thus a white horse is a different thing.

*good modal description languages; whereas in alethic modal logic it has more often been the other way around."* [49]

We add unary temporal operators instead of the classical modal ones of $\Box$ and $\Diamond$. The main difference being that they are not limited in one direction of the accessibility relation. Thus we get the operators **H** and **G** being the future and past equivalent of $\Box$. For the possibility operator the usual notation gives **F** and **P**[26]. There is also an alternative notation based on the modal operators and thus **G** would be noted as [F] and **F** becomes $\langle$F$\rangle$. We can then read them in the following manner:

| | |
|---|---|
| **F**A | *it will sometimes be that* A |
| **P**A | *it was sometimes that* A |
| **G**A | *it always will be that* A |
| **H**A | *it always was that* A |

Similarly as the modal operators, each pair is interdefinable. So it is sufficient to use only one pair to define a tense logic. Let us note that the reading might slightly differ, for example in the necessity operators also the present instant can be added (as **G** being read 'it is and it always will be') [20].

On this basis we can construct either the simplest temporal logic, called $K_t$, or even more complicated logics depending on the choice of axioms. We follow the article of Burgess for the definitions [8].

In *syntax* there aren't any surprising features. We start out with variables p, q, r…... and connectives ¬, →, and operators **G**, **H**. The set of Priorean formulas is the smallest set containing the propositional variables that is closed under formula constructions from the operators and connectives mentioned above. We denote Priorean formulas as $\varphi, \psi, \ldots$. The other connectives are treated as abbreviations (also **F** and **P**). We have in these axiomatic systems three rules of inference: Modus Ponens, Substitution and Temporal Generalization:

| **Modus Ponens** | **Substitution** | **Temporal Generalization** | |
|---|---|---|---|
| $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$ | $\dfrac{\varphi}{\varphi[\psi/q]}$ | $\dfrac{\varphi}{\mathbf{G}\varphi}$ | $\dfrac{\varphi}{\mathbf{H}\varphi}$ |

[26]The memorizing aid for **F**uture and **P**ast is clear. To remember which of the necessity operators addresses which direction, the author uses the word: **H**istory. According to [34] the original idea of Prior could have been : "is always **G**oing to be" and "**H**as always been."

For semantics, we start out on Kripke models $\langle W, R, V \rangle$, well known from modal logic, and add to the valuation the valuation of our new operators. This is done in the following, predictable, way:

$$\begin{array}{lll} \mathcal{M}, p \Vdash \mathbf{G}\varphi & \text{if} & (\forall q)(pRq \rightarrow \mathcal{M}, q \Vdash \varphi) \\ \mathcal{M}, p \Vdash (\mathbf{H}\varphi) & \text{if} & (\forall q)(qRp \rightarrow \mathcal{M}, q \Vdash \varphi) \end{array}$$

There is also another option of introducing a new accessibility relation L. This is then the converse of R such that if pRq then qLp and vice versa [20][27]. Obviously all tautologies from classical propositional logic are true also here. We then have the following axioms to choose from:

$$\begin{array}{llc}
(A1) & a) & \mathbf{G}(p \rightarrow q) \rightarrow (\mathbf{G}p \rightarrow \mathbf{G}q)\ , \\
 & b) & \mathbf{H}(p \rightarrow q) \rightarrow (\mathbf{H}p \rightarrow \mathbf{H}q)\ ; \\
(A2) & a) & p \rightarrow \mathbf{GP}p\ , \\
 & b) & p \rightarrow \mathbf{HF}p\ ; \\
(A3) & & \mathbf{G}p \rightarrow \mathbf{GG}p\ , \\
(A4) & a) & (\mathbf{F}p\ \&\ \mathbf{F}q) \rightarrow [\mathbf{F}(p\ \&\ q) \vee \mathbf{F}(p\ \&\ \mathbf{F}q) \vee \mathbf{F}(\mathbf{F}p\ \&\ q)]\ , \\
 & b) & (\mathbf{P}p\ \&\ \mathbf{P}q) \rightarrow [\mathbf{P}(p\ \&\ q) \vee \mathbf{P}(p\ \&\ \mathbf{P}q) \vee \mathbf{P}(\mathbf{P}p\ \&\ q)]\ ; \\
(A5) & a) & \mathbf{G}p \vee \mathbf{FG}p\ , \\
 & b) & \mathbf{H}p \vee \mathbf{PH}p\ ; \\
(A6) & a) & \mathbf{G}p \rightarrow \mathbf{F}p\ , \\
 & b) & \mathbf{H}p \rightarrow \mathbf{P}p\ \ ; \\
(A7) & & \mathbf{F}p \rightarrow \mathbf{FF}p\ \ , \\
(A8) & a) & (\mathbf{G}p\ \&\ p) \rightarrow \mathbf{PG}p\ , \\
 & b) & (\mathbf{H}p\ \&\ p) \rightarrow \mathbf{FH}p\ ; \\
(A9) & a) & (\mathbf{FG}\neg p\ \&\ \mathbf{F}p) \rightarrow \mathbf{F}(\mathbf{G}\neg p\ \&\ \mathbf{HF}p)\ , \\
 & b) & (\mathbf{PH}\neg p\ \&\ \mathbf{P}p) \rightarrow \mathbf{P}(\mathbf{H}\neg p\ \&\ \mathbf{GP}p)\ ; \\
(A10) & & (A7) \vee [(A8(a))\ \&\ (A8(b))]\ , \\
(A11) & a) & \mathbf{FG}p \rightarrow \mathbf{GF}p\ , \\
 & b) & \mathbf{PH}p \rightarrow \mathbf{HP}p\ ; \\
(A12) & a) & \mathbf{PF}p \rightarrow (\mathbf{P}p \vee p \vee \mathbf{F}p)\ , \\
 & b) & \mathbf{FP}p \rightarrow (\mathbf{F}p \vee p \vee \mathbf{P}p)
\end{array}$$

At first glance it is visible that the pairs arise out of the duality of operators. Now, according to our choice of axioms, we get a different classes of frames given by the theory, similarly as in modal logics. An important role, however, plays the choice of the structure of time.

Let us list according to Burgess [8] some of the properties emerging from the choice of axioms. T denotes the theory with the first four axioms. Actually (A1), (A2), (A3), all the propositional tautologies and the above men-

[27]The axiom (A12) stems from [49].

tioned rules constitute the logic $K_t$[28]. This $K_t$ is sound and complete with respect to the class of all flows of time and it is also decidable. By adding to $K_t$ the axiom non-branching $((A12)(a) \wedge (A12)(b))$, we get an extension for linear flows of time. **Lin**, as the logic is called, is also sound and complete, but now with respect to only the class of linear flows of time. We shall not reproduce any of the proofs as these logics aren't in the main scope of the work, but we mention an important difference when proving the completeness of Lin in comparison with the completeness proof of modal logics. One makes use of canonical models but that is not enough. It is necessary to show that the found **Lin**-consistent set of formulas is also satisfiable in a linear flow of time. This is done by transforming the canonical frame into a strict linear order, but still preserving the truth of formulas. To achieve this, Venema uses a method surnamed bulldozing. The principle is quite simple as the goal is to 'bulldoze' the canonical model (represented by a pseudo line) into the desired strict linear order without losing any information from the original structure. We leave the proof at this point adding only that the points of the original line are first divided into so-called clusters (a cluster being a subset of W), on which already a linear order can be established, and the transformation is actually done with them. For further results and other variations on basic temporal logics we refer to Venema's article [49]. The relation of temporal logics and first order logic (and other here mentioned topics) is also discussed in [6].

For the chosen time structure, we shall get the following properties[29]:

| | | |
|---|---|---|
| **Linear time** | | |
| | Ending | T, (A5)a |
| | Endless | T, (A6)a |
| | Dense | T, (A7) |
| | Discrete | T, (A8) |
| | Dedekind-continuous | T, (A7), (A9) |
| | Homogeneous metrisable | T, (A6), (A10) |
| **Nonlinear time** | | |
| | Arbitrary frames | (A1), (A2) |
| | Transitive | (A1), (A2), (A3) |
| | Forward convergent | (A1), (A2), (A11)a |

One can compare these modal descriptions of some of the properties listed

[28] According to [49], in [8] there is the axiom (A3) omitted when defining $K_t$.

[29] We omit, once again the dual notions. Obviously if we have for example Ending time, then there can be, using the dual axiom, a class of frames having a beginning. Were there is no dual property we simply leave out the distinction.

earlier but in a language for classical propositional logic. There obviously is another possible way how to formalise the wanted properties and it can be seen for example in [49], [20] or [6]. From these one can also gather interesting observations. For example in [20] it is also argued that the question about open future should not be regarded as a condition on the structure of time, but as a interest in the structure of determination. On the other hand in [6], one can witness a hierarchy of linear temporal logics closely related to the structure of the flow of time and thus we can see Rational time based od $(\mathbb{Q}, <)$, Discrete time with $(\mathbb{Z}, <)$ etc. However, it presents more of a different naming convention then anything else.

We can add to this basic language some extensions. One possibility can be the addition of the dyadic operators **S** and **U** meaning 'since' and 'until'.

$$\mathcal{M}, p \Vdash \mathbf{U}\varphi\psi \quad \text{if} \quad (\exists q)(pRq \wedge \mathcal{M}, q \Vdash \varphi) \text{ and } (\forall r)((pRr \wedge rRq) \rightarrow \mathcal{M}, r \Vdash \psi)$$
$$\mathcal{M}, p \Vdash \mathbf{S}\varphi\psi \quad \text{if} \quad (\exists q)(qRp \wedge \mathcal{M}, q \Vdash \varphi) \text{ and } (\forall r)((qRr \wedge rRp) \rightarrow \mathcal{M}, r \Vdash \psi)$$

A worthwhile observation is that the earlier operators can be defined with the new ones if we add to the language a truth constant $\top$. For example $\mathbf{H}\varphi$ is the same as $\mathbf{S}\top\varphi$. It can be even proven that over the class of linear, continuous orderings, every temporal operator can be defined using the language with **S**, **U** and so no other operator adds more expressive power to the language [49]. On the contrary, there is no expressive completeness over the class of all flows of time [6].

Other properties are also worth showing and we follow the lead of [6]. As we have seen earlier, properties from temporal logics can be formulated in first order logic. One can, based on this, create a *translation* of temporal logics into first-order logic[30]. This is closely connected also to the *separation property*. As one would await, separation in temporal logics means that a formula can be formulated with parts using separated future, present or past. A trait of separation is that it depends on the underlying flow of time and that can in some cases lead to surprises. A bit problematic is the *finite model property* a major obstacle being the irreflexivity of flows of time. There are many satisfiable formulas that have only infinite models, as $\mathbf{F}p \wedge \mathbf{G}(p \rightarrow \mathbf{F}p)$.

Also, as we saw only a Hilbert style calculus, we should mention that there are also *Gentzen systems*, *natural deduction*, and even *automata* for temporal logics. As this work is focused not on the formal aspects of temporal logics, it is left to the reader who found interest in such studies to look for the chapter in [6] that gives a overview about the topic.

[30]As we have the standard translation of modal logics.

On our way to nonlinear time models it is important to note that not all properties are definable. Branching frames, for instance, cannot be defined by a Priorean formula. But we also cannot define the class of frames for the flow of time. We do have a formula for transitivity (that is (A3)), but we do not have one for irreflexibility [49].

A different way how to approach linear time is to regard it not as composed of events but to work with *intervals*. Let us sketch some basic ideas about such a system according to [49]. Although one can find also reference to other duration based logics in [34].

First of all one can observe that intervals do not need to be primitive entities. Building on our experience with flows of time $(T, <)$, we can join events according to some criteria. They can be for example taken as closed sets denoted $[s, t] = \{u \in T | s \leq u \leq t\}$. If intervals are taken as primitives, then the question of ordering arises. One of the possible options is to have $s \prec t$ as 'the entire p precedes q' and $p \sqsubset q$ as 'p is a proper part of q'. In this way we can get the structure $\mathcal{P} = (P, \prec, \sqsubset)$. The main difficulty of these logics is, if one tries to completely evade the idea of a point-event. As we are used to them rather than to intervals. For this reason, research in the mutual properties is carried out with much effort. We present now as an example the *Interval Tense Logic* of Halpern and Shoham as in [50].

As for the syntax, to the classical propositional syntax we add the following modal operators:

| | |
|---|---|
| $\langle B \rangle \varphi$ | $\varphi$ holds at a strict beginning subinterval of the current interval |
| $\langle E \rangle \varphi$ | $\varphi$ holds at a strict end subinterval of the current interval |
| $\langle A \rangle \varphi$ | $\varphi$ holds at an interval beginning at the end of the current interval |

And their 'loose' versions denoted $\langle \underline{B} \rangle, \langle \underline{E} \rangle, \langle \underline{A} \rangle$. As en example $\langle \underline{B} \rangle \varphi$ means that $\varphi$ holds at *an* interval which has the current one as a beginning interval. We can as usual define a few useful abbreviations, for us it suffices to notice them : starting points $[[BP]]\varphi$, ending points $[[EP]]\varphi$ and $\Diamond\varphi$ meaning 'somewhere $\varphi$' and $[X]\varphi$ being $\neg \langle X \rangle \neg\varphi$ for whatever operator X.

To define semantics, points are taken as the basic notion. We have a temporal frame $F = (T, <)$, the interval set of F (INT(F)) is the set of all closed intervals $[s, t]$ in T. For a model $(F, V)$ with the valuation $V : L \mapsto 2^{INT(F)}$ we define a truth relation $\models$ inductively on the following page.

F,V $\models$ $p[s,t]$ iff $[s,t] \in V(p)$;
F,V $\models$ $\langle B \rangle\, \varphi[s,t]$ iff there is such u that $s \leq u < t$ and $F, V \models \varphi[s,u]$;
F,V $\models$ $\langle \underline{B} \rangle\, \varphi[s,t]$ iff there is such u that $t < u$ and $F, V \models \varphi[s,u]$;
F,V $\models$ $\langle A \rangle\, \varphi[s,t]$ iff there is such u that $t < u$ and $F, V \models \varphi[t,u]$;
F,V $\models$ $\langle \underline{A} \rangle\, \varphi[s,t]$ iff there is such u that $u < s$ and $F, V \models \varphi[u,s]$;

The connectives and remaining operators are defined as one would expect either according to the usual approach or their definition given earlier. The truth definition implicitly carries with itself relations of intervals based on the given operator $\subset_B, \subset_E, <_A$. As an example let us take the first one:

$$[s,u] \subset_B [s,t] \textit{ iff } s \leq u < t \textit{ iff } [s,u] \text{ is a beginning interval of } [s,t]$$

In succession to this, one can formulate point intervals and thus speak about points from the interval point of view and with this structure we can define $\langle A \rangle\, \varphi = [[EP]]\langle B \rangle \varphi$. The intervals can be presented in the following figure (as in [50]):


We leave interval tense logic and let the reader explore the topic further if he wishes to do so in [50]. We only mention that there is again a translation into first-order predicate logic and many properties of ITL (denseness, use of linear intervals...) can be formulated also by just using first-order language.

### 2.3.3 Other then linear time

If we leave the idea of time as a line, multiple options can arise. The vision of a *circular time* can seem plausible as culturally time often was a cycle[31]. However, unless some deeper analysis is given to the subject, circular time presents a trivial logic with $\mathbf{G}p \leftrightarrow \mathbf{H}p$ and $\mathbf{G}p \rightarrow p$ [8].

We can see a cyclical flow of time $(T, <)$ in [6], where to avoid triviality the $<$ relation can be read as 'a while after, but not too long' reminding us again of Novikov's idea. It satisfies the following axioms:

total order $(\forall x, y)[(x < y) \vee (x = y) \vee (y < x)]$ ;
anti-symmetry $(\forall x, y)\neg[(x < y) \wedge (y < x)]$ ;
future transitivity

$$(\forall x, y, z, u)[(x < y) \wedge (x < z) \wedge (x < u) \wedge (y < z) \wedge (z < u) \rightarrow (y < z)] ;$$

past transitivity

$$(\forall x, y, z, u)[(y < x) \wedge (z < x) \wedge (u < x) \wedge (y < z) \wedge (z < u) \rightarrow (y < z)] ;$$

non-transitivity $(\exists x, y, z)[(x < y) \wedge (y < z) \wedge (z < x)]$

Another option we present are the branching time models. We follow now [8] to demonstrate a formal view on the earlier mentioned Ockhamist and Peircean systems[32]. The basic frame in this case is a tree, i.e. $\mathcal{T} = (T, <)$ such that $<$ is irreflexive, transitive and all the predecessors of any element are linearly ordered. We make use of the following definitions: *x-branch*, $x \in T$, is the maximal linearly ordered subset (chain) in $\{y : x < y\}$. $B \subseteq X$ is a *branch* if it is an x-branch for some x. An x is uniquely determined by its x-branch and in this case is denoted as $x_B$. Let there be B and $y \in B$ then $B_y$ is a *restriction* the y-branch $\{z \in B : y < z\}$. For $y < x$, $B^y$ is an *extension* equal to the y-branch $B \cup x \cup z : y < z \& z < x$. We further denote $\mathcal{B}(\mathcal{X})$ as the set of all branches. Time is then pictured as an upward directed tree, where the past is already settled but in the future we have a diversity of possible outcomes. Here we start to draw the line between Ockhamist and Peircean interpretation and we start out with the first one.

For the Ockhamist syntax, the temporal addition to classical propositional logics is $\mathbf{G}$, $\mathbf{H}$ and $\Box$. As rules we have the ones from linear temporal logic and Gödel's Necessitation Rule saying that from $\Box\alpha$ one can infer $\alpha$.

Concerning semantics, let us remark that valuation is a mapping on the

[31] Also the everyday time can be often understood in this sense as we are going through the weekly cycles of workdays or weekends. Thus it is not only in ancient or agricultural societies where life (to some extent) works in cycles.

[32] Let it be said that sometimes branching time refers to models which are irreflexive and transitive but not necessarily linear [6].

power set of $\mathcal{B}(\mathcal{X})$ for some tree $\mathcal{X}$. It should be kept in mind that a B$\in \mathcal{B}(\mathcal{X})$ determines an x that is thereby chosen as the present and B is the future that will become actual. For the valuation of formulas we present only the temporal part as the rest is as usual:

$$
\begin{aligned}
\mathrm{V}(\mathbf{G}\alpha) &= B \in \mathcal{B}(\mathcal{X}) : \forall y \in X(y \in B \rightarrow B_y \in V(\alpha))\ ;\\
\mathrm{V}(\mathbf{H}\alpha) &= B \in \mathcal{B}(\mathcal{X}) : \forall y \in X(y < x_B \rightarrow B^y \in V(\alpha))\ ;\\
\mathrm{V}(\Box\alpha) &= B \in \mathcal{B}(\mathcal{X}) : \forall B' \in \mathcal{B}(\mathcal{X})(x_{B'} = x_B \rightarrow B' \in V(\alpha))
\end{aligned}
$$

A useful observation is that the pure tense logic is a linear time as we saw it earlier and the pure modal part (omitting **G**, **H**) is the logic S5.

Peircean logic does not have a possible future as future is not settled. Thus Peircean '**F**' actually works as the Ockhamist '$\Box$**F**'[33]. Thus one sees that Ockhamist logic can model Peircean. We note on some of the interesting properties of Ockhamist logic. First, the logic is recursively axiomatizable. Many other properties are not easy to prove or even unknown as finite axiomatization or decidability. Still, there are some partial results. For example the set of valid Peircean formulas is decidable.

Other tree structures can be seen also in [6] as for example bundled tree structures, which allow quantification over branches, or binary trees. As the bundled tree structures are interesting in some aspects we show their differences in comparison with the already presented tree structures.

First, let us define a bundle: a set B of branches of a tree $(T, <)$ is a bundle on the tree $\mathcal{T}$ if: $(\forall x \in T)(\exists b \in B) : x \in b$. $(T, <, B)$ would then be a bundled tree or frame if the B is a bundle. Now what is interesting, these structures correspond to so called Kamp frames, i.e. a triple $(K, <, \equiv)$. The $<$ relates points on the same branch whereas $\equiv$ relates points which represent the same time point paired with different branches and thus allows us to create a tree based on the Kamp frame.

The Kamp frame is defined so that $<$ is a union of linear orders on the points from the set K and $\equiv$ is an equivalence relation for all x,y $\in$ K:
if $x \equiv y$ then we do not have $x < y$ ;
if $x \equiv y$ and $u < x$ then there is $v < y$ such that $u \equiv v$ ;
if $x \equiv y$ and $\forall x < u\ \exists y < v$ with $u \equiv v$ then $x = y$ ;

As the other branching structures ,BST, MBS and BCont, are discussed in great detail later, they are not presented at this moment but the reader should look them up in the next chapter. Thus this concludes our presentation of some formal systems of nonlinear temporal logics.

[33] Burgess [8] compares Peircean approach to the intuitionist's.

### 2.3.4 Physics, philosophy and the logical addition

As we saw now the available tools from logics, we can discuss how and why these tools are useful in physics. As written by Venema [49]:

> "... it is obvious that time plays such a fundamental role in our thinking that there is a clear need for precise reasoning about it".

One of such situations is the relation of temporal logics and special relativity. As shown in [34], even Prior was engaged in a discussion whether there is an objective time given by measures and experiments known from physics or there is a possibility how to regard the problematic *Now* experience of human beings. The conflict could be also described in the McTaggartian terms of a A series vs B series struggle, where A series were held dear by logicians and B series by relativistic physicists. Thus a need to alter either the logics or the physics, eventually the philosophy behind them, arises.

The axioms proposed by Prior for a relativistic space-time were:

$$\mathbf{FG}p \rightarrow \mathbf{GF}p \qquad \mathbf{PH}p \rightarrow \mathbf{HP}p$$

But he noted also that one can form a tense logic out of functors based on the physical ideas, as frames of reference, local proper times and observed events. As we have seen, models developed later by Belnap and further advanced by Müller, Placek, and Wroński allow for a temporal branching time model to be used to formalize special relativity notions and ideas. Even so, Prior's view that this project is a bit strange can still hold.

As mentioned by Burgess [8], the Minkowski frame $\mathbb{M}$ with quadruples and a partial order $<_M$ as:

$$(x_0, y_0, z_0, t_0) <_M (x_1, y_1, z_1, t_1) \leftrightarrow$$
$$(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 < (t_1 - t_0)^2 \& t_0 < t_1$$

and a frame $\mathbb{B}$ consisting of point-events of spacetime and the relation of causal ordering are by the special theory of relativity isomorphic. He (and Prior) argues that the axioms (A1)-(A3), (A11), which we saw earlier, are valid and complete for transitive, convergent frames. However, they do not axiomatize special relativity logic. This is achieved later by the MBS models, who to a large extent present a continuation of Prior's efforts on this subject. Burgess regards also the possibility of general relativistic logic but there it seems even less manageable and even the axiom (A11) does not hold anymore (although the first three still do).

It is necessary to mention also Sharlow's article [46] as he discusses the usefulness of BST models in quantum physics. He points out that BST did already prove being useful with the EPR paradox and it could be a leading tool for the stochastic interpretation of quantum physics or a leading formalism in working with quantum gravity.

Let us fulfill the given promise and mention also temporal logics and their role in Computer science. Often in cooperation with dynamic logics, temporal logics allow to formalize the view on the way how processes function in a given system especially with regards to time sharing [18]. This can be seen for example on the *Temporal Logic of Concurrency* presented in [6], where it is studied how parallel running processes with a shared memory environment interact. The article [35] presents also a good introduction into temporal logics with regards to computer science. The interaction of temporal logics with other logics is presented in [6], e.g. epistemic temporal logic, or also in [51],in this case doxastic logic.

We mentioned that temporal logics are used in natural language analysis too. One of interesting contributions to the topic is for example the article [7]. Therein interval structures are combined with event structures by linking them via a transition preserving function. These so called back-and-forth structures represent the view that temporal constructions are actually describing ways how various information sources are used. Despite its quite general idea, the core of back-and-forth structure analysis uses the English present perfect and so one can wonder if the use of these structures is limited solely to English.

All these logics are fairly similar in their basic ideas as the one's we already saw. In some cases 'physical' temporal logics have something like their 'computer' mirror images. For example the *Computational Tree Logic* which is a branching time type model. Admitting that due to a bit different way of notation and the different motivation, it can take a little while to get accustomed to the computer science versions or vice versa.

This concludes our general chapter and we proceed to the technical part of the thesis.

# Chapter 3

# The Apparatus

In this chapter we introduce both the apparatus from logics and physics which will be used in the next part of the paper. Firstly, we are going to use models called Branching space-time models (BST) and their relation to Minkowskian Branching Structures (MBS) as a parallel of our work. On this we found our effort to relate up-to-date topology models for a general relativistic spacetime with models of Branching Continuations(BCont). The BST and MBS relation serves as a strong source of inspiration and for this it is useful to demonstrate the way how BST gave birth to MBS. In the same time it is unnecessary to venture into too much detail of the procedure, for the reader can see it in [31].

## 3.1 BST and MBS

MBS models were augmented by Placek and Wronski [43]. We follow their article while presenting the basic ideas.

### 3.1.1 BST

For a BST model we need to have a nonempty set $W$ and the partial ordering $\leq$. The set $W$ represents **Our World**[1] with all the possibilities given as point events and $\leq$ is the causal ordering of point events. Where the interpretation is that $e \leq e'$ is read so that $e'$ lies in the possible future of $e$. The crucial term in BST is a history.

**Definition 1**

[1]Let it be again mentioned: this is not one world, i.e. possibility, as known from modal logics.

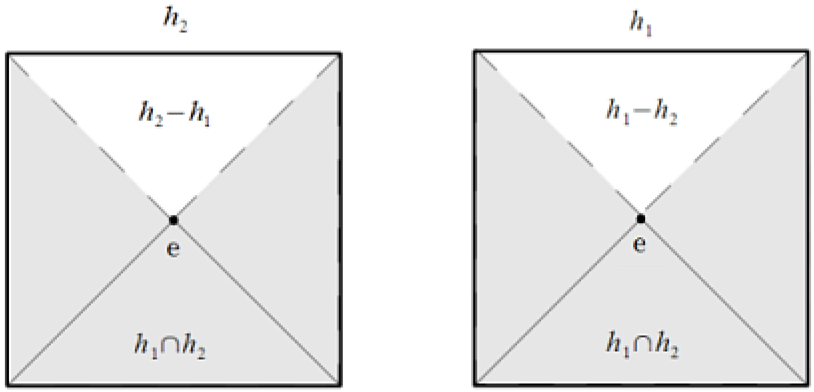


Figure 3.1: Two histories $h_1, h_2$ with a choice point e.

A set $h \subseteq W$ is *upward-directed* iff $\forall e_1, e_2 \in h \exists e \in h$ such that $e_1 \leq e$ and $e_2 \leq e$.

A set $h$ is *maximal with respect to the above property* iff $\forall g \in W$ such that $g \supset h$, $g$ is not upward-directed.

A subset $h$ of $W$ is a *history* iff it is a maximal upward-directed set. $Hist$ is the set of all histories from $W$

For histories $h_1$ and $h_2$, any maximal element in $h_1 \cap h_2$ is called a *choice point* for $h_1$ and $h_2$.

Based on these histories we can prepare the definition for a model of BST.

**Definition 2**

$\langle W, \leq \rangle$ where W is a nonempty set and $\leq$ is a partial ordering on W is a *model of BST* iff it meets the following requirements:

1. The ordering $\leq$ is dense.
2. $\leq$ has no maximal elements.
3. Every lower bounded chain in W has an infimum in W.
4. Every upper bounded chain in W has a supremum in every history that contains it.
5. (Prior choice principle) For any lower bounded chain $O \in h_1 - h_2$ there exists a point $e \in W$ such that e is maximal in $h_1 \cap h_2$ and $\forall e' \in O\ e < e'$.

This concludes our account of BST models, for a more detailed view [4] can be recommended.

### 3.1.2 BST and topology

We follow [42] for this section and list the definition of topology on BST which will be discussed in the last chapter.

**Definition 3** (Diamonds)
Let $\langle W, \leq \rangle$ be a BST model and MC(W) the set of maximal chains in W. For $t \in MC(W)$ let $d_t^{e_1 e_2}$ be *"diamond oriented by t with vertices* $e_1$ *and* $e_2$*"* given as:

$$d_t^{e_1 e_2} := \{y \in W | e_1 < e_2 \wedge \{e_1, e_2\} \subseteq t \wedge e_1 \leq y \leq e_2\}. \tag{3.1}$$

Based on these diamonds we can construct a topology on W, denoted by $\mathfrak{T}(W)$.

**Definition 4** (Diamond topology $\mathfrak{T}$ on W and h)
$Z \in \mathfrak{T}(W)$, iff $Z = W$ or

$$\forall e \in Z \forall t \in MC(W)(e \in t \rightarrow \exists e_1, e_2 \in t(e_1 < e < e_2 \wedge d_t^{e_1 e_2} \subseteq Z))$$

This can be altered to define a topology on a history $h$ as follows: $Z \in \mathfrak{T}(h)$ iff $Z = h$ or

$$\forall e \in Z \forall t \in MC(h)((t \subseteq h \wedge e \in t) \rightarrow \exists e_1, e_2 \in t(e_1 < e < e_2 \wedge d_t^{e_1 e_2} \subseteq Z))$$

We mention also a fact that distinguishes these two topologies and a theorem that shows their relation. We omit here the proof of $\mathfrak{T}(W)$ being a topology on W and direct the reader to [42].

**Fact 5**
If $Z \subseteq h$ for some history $h \subseteq W$ contains a choice point for $h$ and some h', then $Z \notin \mathfrak{T}(W)$. Z may still belong to $\mathfrak{T}(h)$.

**Theorem 6**
$A \in \mathfrak{T}(W)$ iff $(\forall\, h \in Hist)(A \cap h \in \mathfrak{T}(h))$.

Now, to present a base for this topology, some other features need to be defined and also three extra-BST conditions need to be presented. For notation in the following text, may the reader keep in mind that $t^{\geq e_1} := \{x \in t | e_1 \leq x\}$ and similarly $t^{\not\leq e_2} := \{x \in t | x \not\leq e_2\}$. Also we denote the forward light-cone of e in history h as $flc_h(e)$, similarly the backward one.

**Definition 7** (Light-cones)

For history $h \subseteq W$, let $e_1, e_2 \in h$.

$e_2 \in flc_h(e_1)$ iff $(e_1 \leq e_2) \wedge (\exists t \in MC(h))(e_2 \in t \wedge e_2 = inf(t^{\geq e_1}))$

$e_1 \in blc_h(e_2)$ iff $(e_1 \leq e_2) \wedge (\exists t \in MC(h))(e_1 \in t \wedge e_1 = inf(t^{\not\leq e_2}))$

The formula $e \in flc_h(e')$ is read that e lies **on** the forward light cone of e'. Let us note that in general it is not true that $e_2 \in flc_h(e_1)$ iff $e_1 \in blc_h(e_2)$ but we shall impose a condition that partly addresses this issue.

**Condition 8**

**C1** - enough space:
$\forall h \in Hist \forall e_1, e_2 \in h(e_2 \in flc_h(e_1) \rightarrow \exists t \in MC(h)(e_1 \in t \wedge sup_h(t^{\not\leq e_2} = e_1))$, and
$\forall h \in Hist \forall e_1, e_2 \in h(e_1 \in blc_h(e_2) \rightarrow$
$\exists t \in MC(h)(e_2 \in t \wedge \exists e^*(e^* \in t \wedge e_1 \not\leq e^*) \wedge sup_h(t^{\not\geq e_1} = e_2)))$

**C2** - betweenness property
Let $x, y, e \in h, x < y < e : x \in blc_h(e)) \rightarrow y \in blc_h(e)$
and

$e \in flc_h(x)) \rightarrow y \in flc_h(x)$

**C3** - interior of light-cones
Let $h \in Hist$:
$\forall e \in h \exists t \in MC(h)(e \int \wedge t^{<e} \neq \emptyset \wedge t^{<e} \cap blc_h(e) = \emptyset \wedge t^{>e} \cap flc_h(e) = \emptyset)$

**Definition 9** (Borderless diamonds for $h$)

For the set $bd^{e_1 e_2} \subseteq h$ with $h \in Hist$: $bd^{e_1 e_2} \in BD_h$ iff $(\exists d_t^{e_1 e_2} \subseteq h)(bd^{e_1 e_2} = d_t^{e_1 e_2} / (blc_h(e_2) \cup flc_h(e_1)))$

These conditions simplify the matter of topology greatly. For example if C1 holds, then $e_2 \in flc_h(e_1)$ iff $e_1 \in blc_h(e_2)$.

**Theorem 10** (Base for $\mathfrak{T}$(h))

For $h \in Hist$ satisfying C1, C2, and C3:

$$\forall A \in \mathfrak{T}(h) \exists \mathcal{B} \subseteq BD_h \bigcup \mathcal{B} = A \tag{3.2}$$

In [42] it is also proven that this topology has the Hausdorff property and it investigates the location of indeterminism in a given BST model. As it is indeterminism which could lead to the failure of Hausdorff property and thus lead undesired topologies. They formalize also the already mentioned indeterminism without choice.

### 3.1.3 MBS

We proceed now to the correlation of BST models and Minkowskian space-time according to [31]. As a basis we take into account its shape. For the Minkowskian space-time is made out of points from $\mathbb{R}^4$, three coordinates for space and one for time. These, however, do not suffice to represent the point events of BST models. Although we have an ordering of the Minkowski space-time defined for two points of $\mathbb{R}^4$ as:

$$x \leq_M y \text{ iff } D_M^2(x, y) \leq 0 \text{ and } x^0 \leq y^0 \tag{3.3}$$

Here the function $D_M^2 : \mathbb{R}^4 \times \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is the Minkowskian space-time distance - a quite straightforward metric on space-time:

$$D_M^2(x, y) := -(x^0 - y^0)^2 + \sum_{i=1}^{3}(x^i - y^i)^2 \tag{3.4}$$

On these grounds we need to find the counterparts of BST point events and histories. In order to maintain the properties of histories we cannot simply take points from the Minkowski space-time neither is it enough to fill these simple coordinates with some properties[2].

**Definition 11** (Basic MBS definitions)
We define the following:

1. $\Sigma = \{\sigma, \eta, \gamma, \ldots\}$ a set of *labels describing scenarios*;

2. P is a set of *point properties*;

3. S: $\Sigma \times \mathbb{R}^4 \rightarrow P$ is a *state function*;

4. If S is given, then for any $\sigma, \eta \in \Sigma : C_{\sigma\eta} \subset \mathbb{R}^4$ is the set of *splitting points* between $\sigma, \eta$;

5. $R_{\sigma\eta} := \{x \in \mathbb{R}^4 | \neg\exists y \in C_{\sigma\eta}\ y <_M x\}$ is the *region of overlap* between two scenarios;

6. $B := (\Sigma \times \mathbb{R}^4)_{/\equiv_S} = \{[x_\sigma]\ |\sigma \in \Sigma, x \in \mathbb{R}^4\}$;

7. $[x_\sigma] \leq_S [x_\eta]$ iff $x \leq_M y$ and $x_\sigma \equiv_S x_\eta$.

[2]For a more detailed account of the motivation see for example pg.3-4 of [43]

Let us now explain these definitions. The scenarios stand for a simple idea that there are multiple ways how things can happen. How actually things are is then given by $P$ and therefore a state function $S$ describes what happens at a given point of the space-time in the chosen scenario. We could also regard the situation from the point of view of a scenario, when for every label $\sigma$ there is a mapping $S_\sigma : \mathbb{R}^4 \to P$. The concept of a splitting point and overlap between scenarios is quite simple to grasp but as we use them later, we discuss some of their properties.

**Fact 12**

Let $C_{\sigma,\eta} \subset \mathbb{R}^4$ be a set of splitting points in $\mathcal{C}$; $R_{\sigma,\eta}$ region of overlap:

- elements of $C_{\sigma,\eta} \subset \mathbb{R}^4$ are mutually SLR;
- for all $\sigma, \eta$: $C_{\sigma,\eta} \subset \mathbb{R}^4 \neq \emptyset$ iff $\sigma \neq \eta$;
- $(\forall x \in C_{\sigma,\eta})(\exists\, y \in C_{\sigma,\gamma} \cup C_{\gamma,\eta})(y \leq_M x)$
  or $R_{\sigma,\gamma} \cap R_{\gamma,\eta} \subseteq R_{\sigma,\eta}$;
- A given state S is consistent with $\mathcal{C}$ iff
  $(\forall x \in R_{\sigma,\eta})(S_\sigma(x) = S_\eta(x))$ ;
  $\forall x \in C_{\sigma,\eta} \forall y \in \mathbb{R}^4 (x <_M y \to (\exists z \in \mathbb{R}^4)(z \leq_M y \wedge S_\sigma(z) \neq S_\eta(z)))$.

We continue now with defining MBS. First we gain the equivalence relation $\equiv_S$ as[3]:

$$x_\sigma \equiv_S y_\eta \text{ iff } x = y \text{ and } x \in R_{\sigma\eta} \tag{3.5}$$

The defined set $B$ then represents the MBS counterpart of Our World from BST models and the relation $\leq_S$ is a partial ordering on B. What we still lack are histories.

**Theorem 13** (A MBS history)

For any $h$, $h$ is a history in a given MBS iff for any $\sigma \in \Sigma$, the set of equivalence classes $\{[x_\sigma]\,|x \in \mathbb{R}^4\} = h$.

Here we omit the proof itself[4], but let us mention the idea behind it. Proving that a set of the given form is a history needs only to verify that it fulfills upward-directedness and maximality[5]. The opposite direction, in view of our later work, is a bit more interesting. Although the idea is simple,

[3] The fact that it is an equivalence relation is proven in [43]

[4] It can be found on page 5 of [43]

[5] Which, let us recall, is the definition of a history from Definition 1.

the version of the proof from [43] makes use of a topological postulate. A topology based on lower bounded chains from $\mathbb{R}^4$ allows Placek and Wronski to prove the second step of the proof, namely: *"There is a $\sigma \in \Sigma$ such that for every $\eta$, if $[x_\eta] \in h$, then $x_\eta =_S x_\sigma$."* As our work in the last part uses spacetime topological properties, this should not be left unnoticed. By these means we arrive to the point, where $\langle B, \leq_S \rangle$ can be told to be a BST model. A fact, deserving mentioning, is that the splitting points in scenarios generate choice points in given MBS histories but they do not have to represent all the possible choice points in MBS[6].

## 3.2 BCont

The theory of possible Branching Continuations was proposed by Placek in [40] mentioning multiple reasons for its usefulness. The most important for us at the moment said that in general relativistic space-time BST models aren't sufficient to describe the structure of events. The reason being that BST does not allow for two events to belong to one history, if they do not have a common upper bound for the ordering defined by light cones. BCont, as we shall see here, is based on the idea of BST, albeit it differs in an important feature. The goal is to construct local histories and not those chains as seen in the already described approach. For this reason, the crucial term of BCont - l-events - is not to be confused with histories of BST. This, in comparison to the relation between BST and MBS, is not a morphism but a quite new model. Although MBS's satisfy the axioms of BCont. We follow now [40] to define, show and prove the main characteristics of these models.

### 3.2.1 BCont models

In this part we introduce the basic syntactical definitions and notions for BCont. As with BST we start out with $W$ a non-empty set of possible events partially ordered by $\leq$. The basic concept here is a snake-link which represents a generalized way of connecting events in $W$. Snake-links are actually paths as known from graph theory, However, the original terminology from [40] is maintained in this paper.

**Definition 14** (Snake-link)

The properties and basic definitions of *snake-links*:

1. $\langle e_1, e_2, \ldots, e_n, \rangle \subseteq W$ $(1 \leq n)$ is a snake-link iff

$$\forall i : 0 < i < n \rightarrow (e_i \leq e_{i+1} \vee e_{i+1} \leq e_i)$$

[6] As said in [43] this depends on our restrains on splitting points.

2. A snake-link is above (below) $e \in W$ if every element of it is strictly above (below) $e$.

3. Let $W' \subseteq W$ and $x, y \in W'$. $x$ and $y$ are snake-linked in $W'$ iff there is a snake-link $\langle e_1, e_2, \ldots, e_n, \rangle$ such that such that $x = e_1$ and $y = e_n$ and $e_i \in W'$ for every $0 < i \leq n$.

4. For $x, y \in W$, x and y are snake-linked above $e$, $x \approx_e y$, iff there is a snake-link $\langle e_1, e_2, \ldots, e_n, \rangle$ above $e$ such that $x = e_1$ and $y = e_n$.

Obviously the fourth definition is a special case of the third and can be altered for other relations. The relation $\approx_e$ is reflexive, symmetrical and transitive, hence an equivalence relation on the set $W_e = \{e' \in W | e < e'\}$.

**Definition 15** (Set of possible continuations)
*Set of possible continuations* of e, $\Pi_e$, is the partition of $W_e$ induced by the relation $\approx_e$.
$\forall e < x : \Pi_e \langle x \rangle$ is the unique continuation of $e$ to which the given $x$ belongs.

From this we can deduce the following fact and develop some new definitions.

**Fact 16**
$\forall e', e, e_0 \in W : ((e \leq e' \vee e' \leq e) \wedge e_0 < e \wedge e_0 < e' \rightarrow \exists H \in \Pi_{e_0} e, e' \in H)$

**Definition 17** (Set $CE$ of choice events)
For $e \in W$, $e \in CE$ iff $card(\Pi_e) > 1$.

**Definition 18** (Consistency)
For $e, e' \in W$, let there be $W_e := x \in W | \forall c (c \in CE \wedge c < e \rightarrow c < x)$ and a similar for $e'$. Then $e, e'$ are *consistent* iff they are snake-linked within $W_e \cup W_{e'}$. A set $A \subseteq W$ is then consistent if every two elements of $A$ are and it is inconsistent iff it is not consistent.

**Definition 19** (Large events, l-events)
$A \subseteq W$ is an *l-event* iff $A \neq \emptyset$ and A is consistent.

For the definition of a BCont model, the definition of a BST model is used, only altered on places, where the snake-link has its influence. BCont is in many aspects a generalised form of the BST models[7].

[7]For example viz. page 7 in [40].

**Definition 20** (Model of BCont)

$\mathcal{W} = \langle W, \leq \rangle$ is a *model of BCont* if it satisfies:

1. $\mathcal{W}$ is a non-empty partially ordered set;
2. the ordering $\leq$ is dense on $W$;
3. $W$ has no maximal elements;
4. every lower bounded chain $C \subseteq W$ has an infimum;
5. if a chain $C \subseteq W$ is upper bounded and $C \leq b$, then there is a unique minimum in $\{e \in W | C \leq e \wedge e \leq b\}$;
6. for every $x, y, e \in W$, if $e \not< x$ and $e \not< y$, then $x$ and $y$ are snake-linked in the subset $W_{e\not\leq} := \{e' \in W | e \not\leq e'\}$ of $W$;
7. if $x, y \in W$ and $W_{\leq xy} := \{e \in W | e \leq x \wedge e \leq y\} \neq \emptyset$, then $W_{\leq xy}$ has a maximal element;
8. for every $x_1, x_2 \in W$, if $\forall c : c \in CE \rightarrow c \not< x_i$, then $x_1, x_2$ are snake-linked in the subset $W_{\not> CE} := \{e \in W | \forall c \in CEe \not> c\}$ of W.

It should be brought to attention that although some models of BCont do not satisfy the axioms of BST and vice versa, every MBS satisfies the axioms of BCont [40]. The reader should look into the original article for a more detailed discussion of the relation of these models and their relative properties. It suffices for our purpouses to mention that BCont does not permit backward branching and leads to a similar branching structure as BST.

We follow now the paper [40] for some further useful definitions.

The notions of basic transitions, compatibility, space-like related (SLR) are important, especially in connection with physics. For events that are SLR cannot influence each other in a causal manner. The idea comes from physics where space-like separated events are those that are not in each others past nor future, i.e. they cannot be even connected with a light signal because not enough time passes between their occurrences.

**Definition 21** (Basic transitions in BCont)

Let $\langle W, \leq \rangle$ be a model of BCont. A *basic transition* is a pair $\langle e, H \rangle$, where $e \in W$ and $H \in \Pi_e$ is a continuation of $e$.

**Definition 22** (SLR)

$e, e' \in W$ are *SLR* iff they are compatible but incomparable.

**Definition 23** (S-t locations)

We say that a model $\langle W, \leq \rangle$ of BCont has *spatio-temporal locations* iff there is a partition S of W such that

1. For each l-event $A$ and each $s \in S$, the intersection $A \cap s$ contains at most one element;

2. $S$ respects the ordering $\leq$, that is, for all l-events A, B, and all $s_1, s_2 \in S$, if all the intersections $A \cap s_1, A \cap s_2, B \cap s_1$ and $B \cap s_2$ are nonempty, and $A \cap s_1 = A \cap s_2$, then $B \cap s_1 = B \cap s_2$;

3. similarly for the strict ordering;

4. if $e_1 \leq e_2 \leq e_3$, then for every l-event A such that $s(e_1) \cap A \neq \emptyset$ and $s(e_3) \cap A \neq \emptyset$, there is an l-event $A'$ such that $A \subseteq A'$ and $s(e_2) \cap A \neq \emptyset$, where $s(e_i)$ stands for a (unique) $s \in S$ such that $e_i \in s$;

5. if $L$ is a chain of choice events in $\langle W, \leq \rangle$ upper bounded by $e_0$ and such that $\exists s \in S \forall x \in L \exists e \in W : (x < e \wedge s(e) = s)$, then $\exists e*$ $\left(e* \in \bigcap_{x \in L} \Pi_x(e_0) = s\right)$.

$S$ is then called a set of s-t locations for $\langle W, \leq \rangle$.

**Definition 24** (Ordering of s-t locations)

For $s_1, s_2 \in S$, let $s_1 \precsim s_2$ iff $\exists e_1, e_2 (e_1 \in s_1 \wedge e_2 \in s_2 \wedge e_1 \leq e_2)$.

As in the original paper, we also remark the properties of this ordering but we omit the proofs here.

**Fact 25**

If $\langle W, \leq, S \rangle$, a BCont model with a set S of s-t locations, is downward directed, then $\precsim$ is a partial dense ordering on S.

**Fact 26**

Let $\langle W, \leq, S \rangle$ that is downward directed and satisfies the following conditions:

- $\forall e_1, e_2, e_3 \in W (e_1 \leq e_3 \wedge e_2 \leq e_3 \rightarrow e_1 \leq e_2 \vee e_2 \leq e_1)$ surnamed "no backward forks"

- $\forall e, e' \in W$ : if $e, e'$ are incomparable by $\leq$, then there are $H_1, H_2 \in \Pi_m$ such that $H_1 \neq H_2$, $e \in H_1$ and $e' \in H_2$, where $m$ is a maximal element of $W_{\leq ee'} = \{y \leq e \wedge y \leq e'\}$;

Then S is linearly ordered by $\precsim$ and every l-event of $\langle W, \leq, S \rangle$ is a chain.

### 3.2.2 BCont semantics

Semantics of BCont are in [40] inspired by BT semantics. For our purpouse it is sufficient to take into consideration only the basic approach to get an idea of how truth works in BCont models. The crucial idea is taken from Prior/Thomason BT semantics and incorporates the use of event-history pairs to evaluate a formula. However, as there are no histories in BCont, a substitute for them must be found. As this serves in BCont the notion of l-events. Such pair is then written as $e/A$.

**Definition 27** (BT+Instants inspired model)

A model $\langle W, \leq, S\rangle$ is said to be *(BT+Instants)-like* if it satisfies the following conditions:

- downward directedness,
- no backward forks,
- $\forall e, e' \in W$ : if $e, e'$ are incomparable by $\leq$, then there are $H_1, H_2 \in \Pi_m$ such that $H_1 \neq H_2$, $e \in H_1$ and $e' \in H_2$, where $m$ is a maximal element of $W_{\leq ee'} = \{y | y \leq e \wedge y \leq e'\}$;

An important feature of this approach is the possibility to map S on the subset of $\mathbb{R}$, a so called "real coordinalization", denoted X, where $|S| = |\mathbb{R}|$ and thus we can define the following interval relation:

$$\text{int}\left(e_1, e_2, t\right) \text{ iff } X\left(s\left(e_2\right)\right) - X\left(s\left(e_1\right)\right) = t$$

This allows us to state the truth-conditions of metric tenses saying that the two events are $t$ units apart. Sentences will be then judged based on evaluation points, built out of l-events and thus will be event/l-event pairs mentioned already earlier.

**Definition 28** Structure and model

A *structure* for the language $\mathcal{L}$, as defined before, is a pair $\mathfrak{G} = \langle \mathcal{W}, X\rangle$, where $\mathcal{W} = \langle W, \leq, S\rangle$ is a (BT+Instants)-like model of BCont such that $|S| = |\mathcal{R}|$, and $X$ is a real coordinalization of $S$.
A pair $\mathfrak{M} = \langle \mathfrak{G}, \mathcal{I}\rangle$ is a model for language $\mathcal{L}$, where $\mathfrak{G}$ is a structure for $\mathcal{L}$ and $\mathcal{I}$ : *Atoms* $\rightarrow \mathcal{P}(W)$ is an interpretation function and *Atoms* is the set of atomic formulas of $\mathcal{L}$.

**Definition 29** (Evaluation points)

Let $\mathfrak{G} = \langle \mathcal{W}, X\rangle$ be a structure for language $\mathcal{L}$, where $\mathcal{W} = \langle W, \leq, S\rangle$. Then $\langle e, A\rangle$, written as e/A, is an *evaluation point* in $\mathfrak{G}$ for formulas of $\mathcal{L}$ iff $\{e\} \cup A \subseteq W$ and $A \neq \emptyset$.

Noteworthy is the fact that we do not require for a e/A that $e \in A$, also to be mentioned is the fact that Placek [40] suggests a plain ontological reading of the meaning of e/A. Although it is also true that the BCont approach carries with itself less tension between ontology and epistemology as l-events are more accessible than BST histories.

This construction of evaluation points and coordinalization of $X$ allows us to use metric tense operators *F(x)* and *P(x)* with $x \in \mathbb{R}$. For the language $\mathcal{L}$, we assume that its atomic formulas are present-tensed and that it has the two metric tense operators, usual connectives $(\neg, \wedge, \vee, \rightarrow)$ and modal operators *Sett*(as "it is settled"), *Poss*("it is possible") and an operator *Now*.

**Definition 30** (Extensions of an evaluation point)

Let $\mathfrak{G} = \langle \mathcal{W}, X \rangle$ be a structure for language $\mathcal{L}$, $\mathcal{W} = \langle W, \leq, S \rangle$, and e/A be an evaluation point in $\mathfrak{G}$ for $\mathcal{L}$. Then:

- e/A goes *at least x-units-above* e $(0 \leq x)$ iff $\exists e_1 \in W \exists e_2 \in A(e_1 \leq e_2 \wedge \text{int}(e, e_1, x))$;
- e/A' is an *x-units-above-e extension of e/A* $(0 \leq x)$ iff $A \subseteq A' \subseteq W$ and e/A' goes at least x-units-above e.

**Definition 31** (Fan of evaluation points)

Let $\mathfrak{G} = \langle \mathcal{W}, X \rangle$ be a structure for $\mathcal{L}$, $\mathcal{W} = \langle W, \leq, S \rangle$, and e/A be an evaluation point in $\mathfrak{G}$ for $\mathcal{L}$.

Two l-events $A_1$ and $A_2$ of $\mathcal{W}$ are *isomorphic instant-wise* iff
$\forall e_1 \in A_1 \exists e_2 \in A_2 s(e_1) = s(e_2)$ and $\forall e_2 \in A_2 \exists e_1 \in A_1 s(e_1) = s(e_2)$

$e/A' \in \mathcal{F}_{e/A}$, *fan of evaluation points* determined by evaluation point e/A iff e/A' is an evaluation point in $\mathfrak{G}$ and A and A' are isomorphic instant-wise.

In many cases this leads to a single possible A', A itself. An important point is that the evaluation of the formula depends on the moment of use, $e_C$.

**Definition 32** (Point fulfills formula)

For given $e_C, e/A$ and the model $\mathfrak{M} = \langle \mathfrak{G}, \mathcal{I} \rangle$.Then:

1. if $\psi \in$ Atoms:$\mathfrak{M}, e_C, e/A \approx \psi$ iff $e \in \mathcal{I}(\phi)$;
2. if $\psi$ is $\neg\varphi : \mathfrak{M}, e_C, e/A \approx \psi$ iff it is not the case that $\mathfrak{M}, e_C, e/A \approx \varphi$;
3. for $\wedge, \vee, \rightarrow$ also in the usual manner;
4. if $\psi$ is $F_x\varphi$ for $x > 0 : \mathfrak{M}, e_C, e/A \approx \psi$ iff there are $e' \in W$ and $e* \in A$ such that $e' \leq e*$ and int(e', e, x), and $\mathfrak{M}, e_C, e'/A \approx \varphi$;

5. if $\psi$ is $P_x\varphi, x > 0$ : $\mathfrak{M}, e_C, e/A| \approx \psi$ iff there is $e' \in W$ such that $e' \cup A \in$ l-events and int(e',e,x) and $\mathfrak{M}, e_C, e'/A \not\approx \varphi$;

6. if $\psi$ is $Sett : \varphi$ : $\mathfrak{M}, e_C, e/A \not\approx \psi$ iff for every evaluation point $e/A'$ from fan $\mathcal{F}_{e/A}$ and $\mathfrak{M}, e_C, e/A' \not\approx \varphi$;

7. $Poss : \psi := \neg Sett : \neg\psi$;

8. if $\psi$ is Now : $\varphi$ : $\mathfrak{M}, e_C, e/A| \approx \psi$ iff there is $e' \in s(e_C)$ such that $e' \cup A \in$ l-events and $\mathfrak{M}, e_C, e/A' \not\approx \varphi$.

**Definition 33** (Definite truth)

$\mathfrak{M}, e_C, e/A \models \psi$, read as $\psi$ is *definitely true* at $\mathfrak{M}, e_C, e/A$, iff there is an $x \geq 0$ such that for every x-units-above $e$ extension $e/A'$ of $e/A$ : $\mathfrak{M}, e_C, e/A' f' \not\approx \psi$;

$\mathfrak{M}, e_C, e/A \models_{\text{Indef}} \psi$, read as $\psi$ is indefinitely true at $\mathfrak{M}, e_C, e/A$, iff there is no $x \geq 0$ such that for every x-units-above $e$ extension $e/A'$ of $e/A$: $\mathfrak{M}, e_C, e/A'| \approx \psi$ or for every x-units-above-$e$ extension $e/A'$ of $e/A$: $\mathfrak{M}, e_C, e/A' \not\approx \neg\psi$;

**Theorem 34**

For any formula $\psi$ and any evaluation point e/A, exactly one of the following three options must hold: $e/A \models \psi$ or $e/A \models \neg\psi$ or $e/A ?\!\!\models \psi$

We don't go into much more detail but let us list some of the properties from [40].

- if $\psi$ is fulfilled at an evaluation point $e$, it can cease to be fulfilled at an extension of this evaluation point;

- if $\psi$ is definitely true at a evaluation point $e$, then it is definitely true in every extension of $e$;

- if $\psi$ is indefinite at a point, so is its negation;

- if $\psi \wedge \varphi$ is indefinite at a point, $\psi \wedge \varphi$ is either indefinite or definitely false at this point;

- if $\psi \vee \varphi$ is indefinite at a point, $\psi \vee \varphi$ is either definitely true or indefinite at this point;

- if $\psi \rightarrow \varphi$ is indefinite at a point, $\psi \rightarrow \varphi$ is either definitely true or indefinite at this point;

- settled cannot be indefinite: $Sett : \psi$ is definitely true or $\neg Sett : \psi$ is definitely true.

Also in our coordinalization, every sentence becomes definitely true or definitely false at a sufficiently long extension of a initial evaluation point.

The reader might remember the discussion about Peircean and Ockhamist approach to the future. Let us assure the reader that BCont models do not run into troubles with having a Peircean future and thus we can distinguish between what will happen and what will necessarily happen. The BCont semantics allow us to speak about sentences settled in past events but if evaluated in the past, they are not settled. In other words past is not settled if evaluated in the event when it was not settled. Thus it corresponds the way we regard and speak about time.

One formal demonstration was promised in the second chapter - Was Einstein born as a Nobel Prize winner? This example is shown in [40] and allows to see the semantics at work. If we are faced with two possible sentences: $Sett : P(100)F(10)E$ and $P(100)Sett : F(10)E$, where $E$ stands for "Einstein wins the Nobel Prize". We can see that the second does not follow from the first as it is not settled in 1911 that Einstein will get the Nobel prize. There still was the possibility that he would not win. For a clearer idea, the image from [40] was reproduced as figure 3.2.

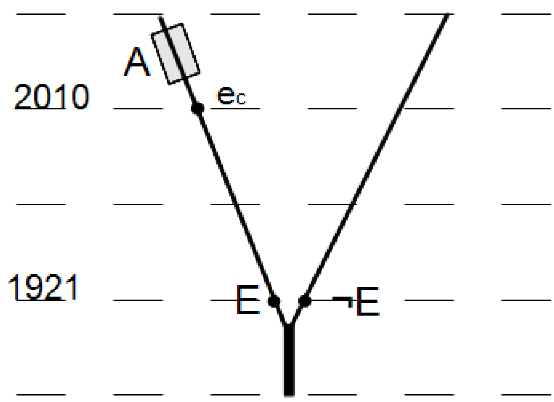


Figure 3.2: Was Einstein born as a Nobel Prize winner?

With this let us end the part about BCont and continue to the second half needed for our final work, the topology of space-time in contemporary physics.

## 3.3 Space-time topology

Obviously the question of space-time topology in general relativity isn't a finished topic. Nevertheless there are some well established and usable results on which our analysis will be based, these were taken primarily from [19] and [27][8]. What now follows is an enumeration of the main ideas which does not aim to cover all the details. Thus proofs are omitted and left for the reader to find in the mentioned literature.

Contemporary ontology of space-time thinks of events as causally connected points. The topology of these points, which is used here, is a development of the path-topology or $\mathcal{P}$-topology that was defined by Hawking, Kind and McCarthy [27]. This topology was based on general relativity but in its construction relies on the original Euclidean topology. The generalized form of $\mathcal{P}$-topology was presented by Kim [27] and is called $\mathcal{T}$-topology. In our final steps both topologies play an important role.

A few ideas should be kept in mind during the whole undertaking. As the reader might have seen in the previous part, the idea of time is bounded with many old and diverse ideas and views, which can make it difficult to grasp the view on time in general relativity. Let us for instance observe that gravity, as a force able to deflect light, determines the causal structure of space-time. For no signal can travel faster than light and thus gravity's impact on it can be felt also in causality [16]. The second important principle is the "democratic" principle of relativity that there are no privileged observers[9]. As a more general argument for our cause serves a quote from Dieks' work:

> "becoming should be conceived as something purely local. Second, I address the question of what becoming consists in. I claim that becoming consists in the coming into being of events, and that this is nothing but the happening of these events at their own spacetime locations." [13]

We shall now introduce some general notions from space-time topology and then the $\mathcal{T}$-topology as seen in [27].

[8] The reader should look into [37] or [16] for a more detailed introduction to this topic.

[9] "For all observers physical processes run the same way, as measured in their own local time (once corrections have been made for the distortions caused by accelerations)" [13].

### 3.3.1 General notions of general relativity space-time topology

To see how $\mathcal{T}$-topology is constructed some basic definitions and properties of space-time are needed. We draw these notions from Penrose [37] with regard to the Kim's article [27].

**Definition 35** (Space-time)
A *space-time* M is to be a real four-dimensional connected $C^{\infty}$ Hausdorff manifold with a globally defined $C^{\infty}$ tensor field g of type (0,2), which is non degenerate and Lorentzian[10].

**Definition 36** (Timelike, spacelike and null)
Let M be a space-time, with $x \in M$. Then any tangent vector X in the tangent space to M at x is said to be: *timelike*, *spacelike*, or *null* according to as g is positive, negative or zero.

**Definition 37** (Time-orientable)
A space-time M is said to be *time-orientable* if it is possible to make a consistent continuous choice all over M, of one component of the set of timelike vectors at each point of M. These are labeled *future-directed* and *past-directed* and the null vectors are termed according to the vectors they limit.

**Definition 38** (Paths, curves, regions)
We use the following definitions:

- a *path* is a continuous map $p : \Sigma \to M$, where $\Sigma$ is a connected subset of $\mathbb{R}$ containing more than one point. This is a *smooth path* if $p$ is smooth with nonvanishing derivative $dp$;
- *(smooth) curve* is an equivalence class of paths equivalent under (smooth) parameter change (i.e., homeomorphisms or diffeomorphisms of the path domains);
- *oriented curve* is if the parameter change is required to be monotonic;
- a smooth path is called *timelike* if its tangent vector is timelike at every point; such a path is future-oriented if its tangent vector is future-directed at every point;

[10]This definition is given here more for reasons of precision and it suffices to remember the topological properties which we shall list later.

- a curve is *causal* or *non-spacelike* if the tangent vector is timelike or null at all points;

- $N \subseteq M$ is a *simple region* iff it is a simply convex subset of M such that $\bar{N}$ is compact and contained in a simply convex open set.

**Proposition 39** (Regions)
From Penrose we note these properties of regions:

- if $N \subset M$ is a simple region then $(\forall\, p, q \in N)(\exists!\ \text{pq} \subset N)$ (pq is a geodesic connecting p,q)[11] ;

- $\partial N$, the boundary of any simple region N, is compact;

- any closed subset of N is compact;

- M can be covered by a locally finite system of simple regions;

- any compact subset of M can be covered by a finite number of simple regions.

**Definition 40** (Ordering and sets I, J)
For $x, y \in M$:

- $x \ll y$ iff there is a future-directed timelike curve from x to y;

- $x \leq y$ iff there is a future-directed causal curve from x to y or $x = y$;

- $I^+(x) = \{y|x \ll y\}$ is called the chronological future of x;

- $J^+(x) = \{y|x \leq y\}$ is called the causal future of x;

- we define respectively chronological or causal past;

- for $S \subseteq M : I^+(S) = \{y|\exists s \in S : s \ll y\}$;

- $I^+(x, N)$ is the set of points that can be reached by a smooth future-directed timelike curve from x in N.

Obvious are the definitions for dual notions and for causal versions of the chronological terms. A useful observation is that $\ll$ allows us to construct an ordering, but $\leq$ does not as it is not transitive [19]. Also the following fact is true:

[11] The geodesic pq is a continuous function of$(p, q) \in NxN$.

**Fact 41**
$\forall x, y \in M, x \neq y$ :

- $x \leq y$ iff $((x \not\ll y) \ \& \ (\forall z)(y \ll z \to x \ll z))$;
- $x \ll y$ iff $((x \not< y) \ \& \ (\exists z)(x < z < y))$[12].

We also list briefly the causality conditions from [16] and [30] for Kim did in his construction use a globally hyperbolic space-time. These conditions However, talk about space-time on a global level. This, so called causal ladder, presents in each step a stronger condition and limitation to causality in order to make the space-time behave more as we know it.

**Definition 42** (Causality conditions)
The causal ladder of space-times and the points x, y from the respective M is given as follows:

1. **non-totally vicious** $\exists x \in M : x \not\ll x$;
2. **chronological** $\forall x \in M : x \not\ll x$;
3. **causal** $\forall x, y \in M : x \leq y \wedge y \leq x \longrightarrow x = y$;
4. **distinguishing** $\forall x, y \in M$
   *past* $I^-(x) = I^-(y) \longrightarrow x = y$;
   *future* $I^+(x) = I^+(y) \longrightarrow x = y$;
5. **strongly causal** $\forall x \in M \ \exists U$ neighbourhood of $x$ such that there exists no timelike curve that passes through U more than once;
6. **stably causal** there exists a global time function ($\exists t$ a scalar field on M with a gradient that is everywhere timelike and future-directed);
7. **causally continuous** $I^{\pm} : M \to \mathcal{P}(M)$ are continuous and one to one;
8. **causally simple** causal and $\forall x \in M : J^+(x), J^-(x)$ are closed;
9. **globally hyperbolic** M is strongly causal and $\forall x, y \in M : J^+(x) \cap J^-(y)$ is compact;

[12] See lemma 1.5 in [19].

Some of the conditions have very strong claims about the global space-like structure, although they are all in accordance with our intuition about the world. As for example the second condition prohibits closed chronological curves. An important note is that the distinguishing condition can be defined using neighbourhoods too, which is more useful for our cause: $\forall x \in M \forall U$ neighbourhood of $x$ : $\exists V \subset U, x \in V$ such that no past(future)-directed causal curve from x intersects V more than once. Also global hyperbolicity has different ways how to be described, one of them being that a globally hyperbolic space-time M has a Cauchy surface [25]. The space-time we are going to refer to is time-oriented and also globally hyperbolic. Although according to Hawking, global hyperbolicity is useful and for example a well-behaved quantum field theory can be formulated on it. It should not be assumed automatically that the space-time must be globally hyperbolic as there might be something that “gravity is trying to tell us” and we would lose by presupposing a given space-time property [25].

### 3.3.2 $\mathcal{A}$ topology

Alexandrov topology is the first formal topology of general relativistic space-time we shall address in this work. It is a basic topology that served as a reference point for the following $\mathcal{P}$ and $\mathcal{T}$ topologies. The main properties we take from [19] and [16].

**Definition 43** (Alexandrov topology)
A topology on the space-time M is a $\mathcal{A}$-*topology* if its basis are sets of the form $(I^+(p) \cap I^-(q))$ for $some p, q \in M$.

An alternative description regards the basis as sets $\{x \in M | p \ll x \,\&\, x \ll q\}$ for $p, q \in M$. An important note is that if the strong causality condition holds, then it is enough to observe causal relationships in order to determine the topological structure of space-time.

### 3.3.3 $\mathcal{P}$ topology

As the $\mathcal{T}$-topology is a generalization of $\mathcal{P}$-topology onto causal boundaries, we look at its basics according to [24][13].

Let us note that for the construction of $\mathcal{P}$-topology it suffices to have a strongly causal space-time. The properties of $\mathcal{P}$-topology as quoted from [24] then are:

[13] This being the original article but [19] and [27] were used as reference and can serve to gain deeper insight, especially the first one.

1. $\mathcal{P}$ is the finest topology on M which induces the Euclidean topology on arbitrary timelike curves;
2. $\mathcal{P}$ incorporates the causal, differential, and smooth conformal structure;
3. the set of $\mathcal{P}$-continuous paths incorporates all timelike paths;
4. $\mathcal{P}$ is Hausdorff, connected and locally connected and every point has a countable neighbourhood basis.[14]

In the manifold $\mathcal{M}$, $I^+(x)$ and $I^-(x)$ are $\mathcal{M}$-open and so $\forall x, y \in M$ : $I^+(x) \cap I^-(y)$ is $\mathcal{M}$-open. A set $E \subset M$ is $\mathcal{P}$-open if and only if for every timelike curve $\gamma$, there is an $O \in \mathcal{M}$ such that $E \cap \gamma = O \cap \gamma$. If $\gamma$ is timelike, it is $\mathcal{P}$-continuous and then it is also $\mathcal{M}$-continuous.

**Definition 44** ($\mathcal{P}$-topology)
$P = \{E \subseteq M | E \text{ is } \mathcal{P}\text{-open}\}$

We omit the proof itself but such a $P$ is a topology on M and finer than $\mathcal{M}$.

The induction of euclidean topology on any timelike curve (which do not have to be smooth [24]) means that each observer views time with the interval topology on the $\mathbb{R}$ line, or simply put: as we are used to it. And she does it for even accelerated observers [19].

As a basis for $\mathcal{P}$ serve the sets of the form $I^+(x, U) \cup I^+(x, U) \cup \{x\}$, where U is a convex normal neighbourhood of x. One of the results being that "for any $\mathcal{P}$-open neighbourhood U of $p \in M$, there is future-directed timelike curve in U with p its future end point" [27] and its dual past version.

This topology was studied by Fullwood [27] showing that we can construct the same using as a basis for $x \ll y \ll z$ the sets $[I^+(x) \cap I^-(y)] \cup [I^+(y) \cap I^-(z)] \cup y$. If we assume that the distinguishing condition holds, this version of the topology, denoted as $\tilde{\mathcal{P}}$, is equal to $\mathcal{P}$.

Another Fullwood's idea is to use the sequence $\{x_i\}$ of points of M to define a path topology $\mathcal{P}'$.

**Definition 45** ($\mathcal{P}'$)
A set $E \subset M$ is $\mathcal{P}'$-closed if every monotonic timelike sequence in E that causally converges has a limit in E.

Where causal convergence of $\{x_i\}$ to $x \in M$ means that either for each subsequence of $\{x_i\}$ $I^-(x) = \bigcup I^-(x_j)$ or for each subsequence of $\{x_i\}$ $I^+(x) = \bigcup I^+(x_j)$. Monotonicity is given as usual, in our case with respect

[14] Other properties are listed for example in [19].

to $\ll$. As Fullwood has proven that $\tilde{\mathcal{P}} = \mathcal{P}'$ we can use this topology also to get insight into the original $\mathcal{P}$-topology.

It is imperative that we cite also some basic definitions and facts as the properties of $\mathcal{P}$-homeomorphisms. Their usefulness will become apparent in the next chapter.

**Fact 46** ($\mathcal{P}$-homeomorphism properties)
Let h be a $\mathcal{P}$-homeomorphism. For a chronological space-time $\mathcal{M}$:

1. for any $x \in \mathcal{M}$ $h(I^-(x) \cup I^+(x)) = (I^-(h(p)) \cup I^+(h(p)))$;
2. let $y \in M$ then $h(I^+(y)) = I^+(h(y))$ or $h(I^+(y)) = I^-(h(y))$;
3. h maps timelike curves on timelike curves.

For a strongly causal space-time $\mathcal{M}$:

1. h is a $\mathcal{M}$-homeomorphism;
2. h maps null geodesic curves on null geodesic curves;

The main drawback of this topology is its foundation on Euclidean topology as seen in [19]. Let us therefore look at its generalization to a causal boundary.

### 3.3.4 $\mathcal{T}$ topology

We first need to introduce the Budic-Sachs causal boundary to which Kim [27] generalizes $\mathcal{P}$-topology. The beginning is the usual terminology connected with boundaries found also in [48].

**Definition 47** (Indecomposable sets)
We define the following notions of *indecomposable sets*:

1. P is a past set iff $(\exists S \subseteq M)(P = I^-(S))$[15];
2. $P \neq \emptyset$ is a *indecomposable past* (IP) set iff for any $Q_1, Q_2$ past sets: $P = Q_1 \cup Q_2 \longrightarrow (P = Q_1 \vee P = Q_2)$;
3. P is a *proper indecomposable past set* (PIP) iff $(\exists x \in M)(P = I^-(x))$;
4. P is a *terminal indecomposable past set* (TIP) iff $(\forall x \in M)(P \neq I^-(x))$;
5. $\hat{M}$ denotes the collection of all IP sets from M.

[15] Past sets are always open as every $I^-(x)$ is open.

**Definition 48** (Equivalence class and boundary)
We present a classical topological boundary definition:

1. $U{\downarrow} = I^-(x|\forall y \in U : x \ll y)$ is the *chronological common past set of* $U$;
2. (U,V) is a hull pair iff $U \in \hat{M}, V \in \check{M} : V = \uparrow U \& U = \downarrow V$
3. $\sim$ is *an equivalence relation on* $\hat{M} \cup \check{M}$ such that $(\forall U, V \in \hat{M} \cup \check{M})(U \sim V$ iff $(U = V \vee ((U, V) \vee (V, U)$ is a hull pair$)))$[16];
4. the *completion* is given as $\bar{M} = \hat{M} \cup \check{M} / \sim$[17];
5. the *boundary* is defined as $\partial M = \bar{M} - I(M)$

These lead in a globally hyperbolic space-time to the observation that for each $x \in M$ there is a corresponding point in $\bar{M}$.

**Definition 49** (Causality on $\bar{M}$)
We define the relations $\ll, \leq$ on $\hat{M} \cup \check{M}$ for any $X, Y \in \hat{M} \cup \check{M}$ and some M',M" such that $(X, Y) \in M' \times M''$ and where the term $(\hat{L}, \check{L})$ stands for some hull pair, according to the following table:

| | $X \leq Y$ | $X \ll Y$ |
|---|---|---|
| $\hat{M} \times \hat{M}$ | $X \subseteq Y$ | $(\uparrow X) \cap Y \neq \emptyset$ |
| $\check{M} \times \check{M}$ | $X \supseteq Y$ | $X \cap (\downarrow Y) \neq \emptyset$ |
| $\hat{M} \times \check{M}$ | $(\check{L} \subseteq X)\ \&\ (\hat{L} \subseteq Y)$ | $X \cap Y \neq \emptyset$ |
| $\check{M} \times \hat{M}$ | $(X \subseteq \hat{L})\ \&\ (Y \subseteq \check{L})$ | $(\uparrow X) \cap (\downarrow Y) \neq \emptyset$ |

This definition of the relations allows us to operate also on $\bar{M}$ as when we have $x, y \in \bar{M}$ and representatives of their equivalence classes which are in relation according to the table (for example $[x] \ll [y]$), then this holds for any of their representatives. And thus permits us to define the earlier defined notions of causal or chronological past and future now with respect to $\bar{M}$. This also leads to the option to define the extended Alexandrov topology $\bar{\mathcal{A}}$ for a causally continuous M. But Kim [27] proves that $\mathcal{T}$-topology is finer then than the extended Alexandrov topology on $\bar{M}$, similarly as the $\mathcal{P}$-topology is finer than $\mathcal{A}$-topology. Let us now proceed with some of the definitions from [27] to reach the topology.

[16]Thus if V=U then either $U \in \hat{M}$ or $U \in \check{M}$.
[17]From this stems the injection $I : M \hookrightarrow U \in \hat{M} \cup \check{M} / \sim$.

**Definition 50** (Sequences)

The basic definitions of sequences for $\mathcal{T}$-topology:

- sequence $x_i \subset \bar{M}$ is *timelike* if $(\forall i \in \mathbb{N})(x_i \ll x_{i+1} \ \vee \ x_{i+1} \ll x_i)$;
- a timelike sequence $x_i$ is *increasing (decreasing)* iff $(\forall i \in \mathbb{N})(x_i \ll x_{i+1})$ (or $x_{i+1} \ll x_i$);
- for any increasing timelike sequence $\{x_i\}$, the corresponding timelike curve has a unique limit x, which is in $\bar{M}$;
- an increasing timelike sequence $x_i$ *converges to* $x$ if $\bigcup I^-(x_i) = I^-(x)$ (similarly for decreasing and $I^+$).

**Proposition 51**

If $x_i$ is an increasing timelike sequence then $\bigcup I^-(x_i)$ is an indecomposable past set.

**Fact 52**

Some useful facts from [27]:

- for any increasing timelike sequence $x_i$: $(\exists\gamma)(\gamma$ is a timelike curve & $\bigcup I^-(x_i) = \bigcup I^-(\gamma)$;
- For a increasing timelike sequence $x_i$: $I^+(x_i) \cap I^-(\gamma) \neq \emptyset \longrightarrow \forall i x_i \ll x$

**Definition 53** ($\mathcal{T}$-topology)

$U \subset \bar{M}$ is *$\mathcal{T}$-closed* if every timelike sequence that converges has a limit in U.

$V \subset \bar{M}$ is *$\mathcal{T}$-open* if its complement is $\mathcal{T}$-closed.

**Proposition 54**

The definition 53 defines a topology on $\bar{M}$.

As the proof is quite simple, only the last part is mentioned here to show how one can work with this topology. Thus we show now, as in [27], that if A, B are closed, then is $A \cup B$ is also closed.

*Proof.* We have an increasing timelike sequence $x_i$ in $A \cup B$ that converges to x. Assume A, B are closed. Then if one of them contains all but finitely many terms of the sequence from $A \cup B$, then it is true. Let us assume that on the contrary they contain infinitely many terms from the sequence. We break $x_i$ into two sequences such that $y_j \in A$ and $z_k \in B$. Thus $\bigcup I^-(x_i) = [\bigcup I^-(y_j)] \cup [\bigcup I^-(z_k)]$. As these unions are all past sets and the one based on our original sequence is an indecomposable past set, then from its definition one of the two constructed unions is equal to $\bigcup I^-(x_i)$. Let it be $\bigcup I^-(y_j)$, then $I^-(x) = \bigcup I^-(x_i) = \bigcup I^-(y_j)$. In other words $y_j$ converges to x. Now as $\forall y_j \in B$ and B is closed, we have $x \in B$ and thus $x \in A \cup B$. $\square$

As already mentioned, this topology is finer than the $\bar{\mathcal{A}}$ and as $\bar{\mathcal{A}}$ is Hausdorff, also $\mathcal{T}$ is Hausdorff. Another fact being that $\mathcal{T}$ extends also one of the Fullwood versions of $\mathcal{P}$, which is proven in the form of a dense imbedding $i : (M, \tilde{\mathcal{P}}) \hookrightarrow (\bar{M}, \mathcal{T})$.

As we can see, we have now two possible topologies to build on, both being on space-time in general relativity. Therefore let us continue to the main part and establish a model of BCont with their use.

# Chapter 4

# GR branching structures

In this part we sketch possible ways how to formalize a topology of general relativistic space-time in branching structures. First we shall address some concerns and questions that might arise in connection with the project, then an approximation attempt made on the basis of the MBS model is presented.

## 4.1 Motivation

Our effort has two starting points. The models of BST and BCont on one hand and general relativity on the other. General relativity, as far as we are concerned here is represented by the two topologies $\mathcal{P}$ and $\mathcal{T}$. It is possible that the reader does not agree on some assumptions that are given here. In that case he should continue reading and see if the results themselves cannot give him something valuable. We present these questions and answers before the formal part so that it is clear with what aim and limits the formal work was done.

Foremost, let us look into the subject of determinism in general relativity. *If the underlying system of physics is deterministic, what are we trying to model?*[1] One possible answer is that we are not postulating any ontological claims. Our sole motivation is to investigate formal connections between the given topologies and branching systems. As the reader may see also in [43]. Thus we are not primarily concerned with modeling some ontology but trying to model indeterminism in the setting of general relativity.

This issue was also addressed recently in [42]. This paper serves as a lighthouse showing that the work presented in this thesis is not in vain and proceeds in good direction. The sections concerning Alexandrov topology

[1]This question was asked multiple times during the presentation of early results on Jagiellonian University in Cracow in April 2011.

were added as a reaction to it. They can serve as an argument for the possibility to build a bridge between general relativity and branching structures.

*Is there then an ontological explanation for our claims?* Yes there could be. Even admitting general relativity, results as Dennett's and Taylor's [11] show that it is reasonable to have an indeterminism modeling system in a deterministic environment (in our case BCont in GR). Simply put, it is our daily experience and the way we perceive the world that call for some notion of indeterminism even in a deterministic system.

*Is there also some formal indeterminism or is it a mere illusion?* The best choice at this point is to recall the earlier mentioned Belnap's "indeterminism without choice" as in [4] or discussed with respect to general relativity in [42].

*Does the herein presented results cover the whole topic of general relativity?* No, they do not. Limit cases like singularities aren't studied in this work. Nonetheless the solutions should be manageable.Let us continue to the formal presentation.

## 4.2 Path branching structures

As seen in [43] or the earlier section on MBS, the idea how to connect the BST model to the physical approach was to take different Minkowski space-times for all the possible scenarios of events. These were then treated as a history in the BST model and hereby connecting indeterminism and special relativity. As noted in the end of [40], this should be done also with general relativity and its manifolds. We present now some observations and attempts to do so.

### 4.2.1 $\mathcal{A}$-branching structure

Our first suggestion takes into account the recent article [42]. The similarity between the basis for BST topology in definition 10 and the $\mathcal{A}$-topology in definition 43 is inviting an attempt of interconnection. We present only a sketch of the approach.

The diamond topology for BST, as mentioned in [42], still permits pathological models and a preliminary investigation is carried out using so called brims. As we are concerned with globally hyperbolic space-time, we could argue for the introduction of some more conditions on the BST topology mimicking partly the causality conditions mentioned in definition 42. Still we point to the fact that the basis for $\mathcal{A}$-topology are sets of $I^+(p) \cap I^-(q)$ and these are borderless diamonds. If one would use the MBS like approach, he could regard a space-time M with given properties assigned to point events as a history in the BST sense. Hence work with $BD_h$ and the topology of $\mathfrak{T}(h)$ on the logics' side and an Alexandrov topology of M on the physics' side.

Still, our aim was to investigate the BCont and topology relations and thus we proceed in that direction. Although one could, especially after [42] hope for a general relativistic solution using only BST.

### 4.2.2 $\mathcal{T}$-branching structure

Let us look at the BCont model constructed on the basis of $\mathcal{T}$-topology. The idea is to work as much as possible on a local level not to rely on any constructs of the BST history type. In other aspects, the procedure follows the MBS creation.

We are presented with a space-time M (or actually with its causal completion $\bar{M}$) and $\mathcal{T}$-topology on M. We also have the two relations - timelike and causal order - on M. What we lack is any sort of choice events, events themselves and also a partial ordering on M. The original relations are not sufficient as $\ll$ is not reflexive and $\leq$ is not transitive.

The space-time we have is just a structure without content. We want to introduce indeterminism as it is usually understood which means that at one space-time location there can be two different events, i.e. different content. Let us have a nonempty $\Sigma = \{\sigma, \eta, \theta, \ldots\}$, as the set of labels for possibilities.[2]

As in MBS, we use a state function $S : \Sigma \times \bar{M} \to \text{Prop}$. We introduce the set of splitting points $C_{\sigma,\eta} \subset \bar{M}$ and the region of overlap $R_{\sigma,\eta}$ as in definition 11 only with respect to $\bar{M}$. One feature to remark is due to the chosen set. As we work with $\bar{M}$, the possibilities tied to a member of this set are connected also to its time-like future or past[3]. The original attempts went in a more local direction, trying to take labels as only local distinctions, as Theseus' ball of thread it would only mark the paths we actually went through. However a local assignment of labels led to too many complications for the construction and thus the MBS method is adopted. Hence, we can only proclaim that scenarios aren't regarded in this model as something ontological. They are a mere tool not to get lost in all the possibilities.

**Definition 55** ($\mathcal{T}$-splitting points and overlaps)
Let $\Sigma = \{\sigma, \eta, \ldots\}$, let $\bar{M}$ from definition 48 and Prop, then we define *the state function* $S : \Sigma \times \bar{M} \to \text{Prop}$.

[2]These labels are called in MBS scenarios, but as a scenario might suggest an arrangement similar to the BST histories, we use simply the term possibilities.

[3]As each member of $\bar{M}$ is actually an equivalence class with two elements, $I^+(x), I^-(x)$.

For $x \in \bar{M}$: x is a *splitting point*, $x \in C_{\sigma,\eta}$ iff
$(\forall y \in \bar{M})((y \ll x \vee x = y) \rightarrow (S_\sigma(y) = S_\eta(y)))$ and
$(\forall y \in \bar{M})(x \ll y \rightarrow (\exists z \in \bar{M})(x \ll z \ll y \wedge S_\sigma(z) \neq S_\eta(z)))$. $\mathcal{C}$ is then the set of all splitting points.

The *region of overlap* is given as $R_{\sigma\eta} = \{x \in M | \neg\exists y \in C_{\sigma,\eta}\, y \ll x\}$.

The properties of these defined terms are the same as in MBS only rephrased with regards to our orderings.

As in MBS we use the equivalence relation to define the set of all events. We take into account that elements from $\Sigma \times \bar{M}$ are suitable as for some locations, those in a region of overlap, we have a redundancy of points. For W, points should be distinguished only if they belong to a different continuation (i.e. are not snake-linked above a given event), which is not fulfilled. Thus the equivalence classes are needed.

**Lemma 56** (Equivalence class)
The relation $\equiv_{\mathcal{T}}$ given as $x_\sigma \equiv_{\mathcal{T}} y_\eta$ iff $x = y \wedge x \in R_{\sigma\eta}$ constitutes an equivalence relation on the set of $\Sigma \times \bar{M}$.

*Proof.* It is enough to prove the properties connected to the state function as the $x = y$ is clear.(1) Reflexivity true out of the definition of overlaps. (2) Symmetry is gained similarly, as $R_{\sigma\eta} = R_{\eta\sigma}$. (3) Transitivity is the result of $R_{\sigma\eta} \cap R_{\eta\theta} \subseteq R_{\sigma\theta}$. □

**Definition 57** ($W_{\mathcal{T}}$)
We introduce $W_{\mathcal{T}} = \{[x_\sigma] | \sigma \in \Sigma \wedge x \in \bar{M}\}$.

Let us note that as in MBS, the scenarios differ if there is a splitting point between them. Splitting points are defined similarly to choice events in BCont. As if we would allow the same properties after a splitting point, then there could be a snake-link. Although the approach is very similar to MBS, we are not attempting to glue together whole space-times and regard them as histories.

**Lemma 58** (Partial ordering on $W_{\mathcal{T}}$)
The relation given as $[x_\sigma] \leq_{\mathcal{T}} [y_\eta]$ iff $((y = x \vee x \ll y) \wedge x_\sigma \equiv_{\mathcal{T}} x_\eta)$ is a partial order on $W_{\mathcal{T}}$.

*Proof.* (1) Reflexivity is proven by definition. (2) Let $[x_\sigma] \leq_{\mathcal{T}} [y_\eta]$ and $[y_\eta] \leq_{\mathcal{T}} [x_\sigma]$ then by the causality condition of chronology of $\bar{M}$ we cannot have $x \ll y \wedge y \ll x$ and thus they must be equal and with the symmetry of $R_{\sigma\eta}$, we have $[x_\sigma] = [y_\eta]$. (3) The proof follows the proof from the original article [31]. Thus let us have $[x_\sigma] \leq_{\mathcal{T}} [y_\eta]$ and $[y_\eta] \leq_{\mathcal{T}} [z_\theta]$. The ordering $x \ll z$ is given by transitivity of $\ll$. As $y_\eta \equiv_{\mathcal{T}} y_\theta$, so $[y_\eta] = [y_\theta]$ we have from the properties of $R_{\sigma\eta}$ that $x_\sigma \equiv_{\mathcal{T}} x_\theta$. □

And thus finally we can show that our system is a BCont model. The proof incorporates also some discussion concerning the given properties and thus we divide it up into lemmas. These are taken from the definition of BCont models in definition 20.

**Lemma 59**
$\langle W_{\mathcal{T}}, \leq_{\mathcal{T}} \rangle$ is a nonempty partially ordered set with the dense ordering on $W_{\mathcal{T}}$.

*Proof.* We make an appeal to the underlying structure. As $\bar{M}$ and $\Sigma$ are taken as nonempty, also here it is so. The order $\leq_{\mathcal{T}}$ is partial ordering according to the lemma 4.2.2. Density is again obtained from the underlying structure. □

**Lemma 60**
W has no maximal elements.

We question the need of no maximal elements. It seemingly does not fulfill any bigger task then to achieve unending future, full with possibilities. Hence it seems more of a preference choice then a necessary logical property of BCont models. And even in physics one can find cases, where maximal elements make sense (e.g. the 'big crunch').

With $\mathcal{T}$, there is another problem. As boundary points are also members of $\bar{M}$, thus they need to be taken into account. However, they can be treated in a specific way. Let us define the state function for boundary points in a slightly different manner, that being that boundary points will receive an infinity of labels, so that there rests no maximum.

*Proof.* If we take into account the changed function S, for $\langle W_{\mathcal{T}}, \leq_{\mathcal{T}} \rangle$ we won't get a maximal element even with $\gamma$, i.e. a boundary point. The relation $\ll$ is dense on $\bar{M}$. □

**Lemma 61**
Every lower bounded chain $C \subseteq W$ has an infimum.

*Proof.* Let us take the sequence $\{lb_i\}$, $lb_i \in W_{\mathcal{T}}$, of lower bounds of C. The underlying coordinates of $lb_i$ are according to our construction points from $\bar{M}$ and form an increasing sequence. From to the definition 50, there is a unique limit to this sequence. As this limit is a IP, it is timelike connected to the underlying sequence of $\{lb_i\}$ and can be labeled from $\Sigma$ and become a member of $W_{\mathcal{T}}$, thus an infimum for C. □

**Lemma 62**
If a chain $C \subseteq W_{\mathcal{T}}$ is upper bounded and $C \leq_{\mathcal{T}} b$, then there is a unique minimum in $\{e \in W_{\mathcal{T}} | C \leq_{\mathcal{T}} e \wedge e \leq_{\mathcal{T}} b\}$.

*Proof.* As we have the chain C, we have also an increasing timelike sequence $\{xi\}$ and thus according to the definition 50, there is a unique limit to it. Obviously, if the chain is in $W_{\mathcal{T}}$ then there is also a chain in $\bar{M}$. The uniqueness of the limit is transmitted to members of $W_{\mathcal{T}}$ simply by the virtue of the fact that the labeling does not change the structure of $\bar{M}$. □

**Lemma 63**

For every $x_\sigma, y_\eta, e \in W_{\mathcal{T}}$, if $e \not< x_\sigma$ and $e \not< y_\eta$, then $x_\sigma, y_\eta$ are snake-linked in the subset $W_{\not\geq e}$.

*Proof.* It is simple to snake-link events in the past of e in the region $W_{\not\geq e}$ as in the worst case one can make a link via e and all the events in e's past need to be in the same regions of overlap as e. Let us assume, wlog, that $x_\sigma$ is in one of the SLR regions of e, i.e. outside of the the region $\{z_\theta \in W_{\mathcal{T}} | z_\theta \leq_{\mathcal{T}} e\}$, and let $y_\eta$ be in the past region of e. Then we either find an intersection between $I^-(x)$ and $I^-(e)$. The case that there would not be any intersection is covered by the last lemma. By the definition of the region of overlap, there is a pair of labels which allow x and e to be snake-linked. □

**Lemma 64**

If $x_\sigma, y_\eta \in W_{\mathcal{T}}$ and $W_{\leq_{\mathcal{T}} x_\sigma, y_\eta} \neq \emptyset$ then $W_{\leq_{\mathcal{T}} x_\sigma, y_\eta}$ has a maximal element.

*Proof.* The given region is in $\bar{M}$ equal to $I^-(x) \cap I^-(y)$. As they are both $\mathcal{T}$-open, their intersection is also and forms for some m $I^-(m)$, i.e. it is a member of $\bar{M}$, who is also maximal as he is an IP. The labels on $m$ must also be in accordance with $x_\sigma, y_\eta$ so that he can be part of their past. Thus we have $ms \in W_{\mathcal{T}}$. □

**Definition 65** (Choice event in $W_{\mathcal{T}}$)

The event $e \in W_{\mathcal{T}}$ is a choice event, CE, if $\exists s \in \mathcal{C}, \exists \sigma \Sigma : e = [s_\sigma]$.

We now formulate also a useful postulate, similar to the mentioned topological postulate of Placek and Wronski. It makes sure that there is some past common for every event from $W_{\mathcal{T}}$.

**Postulate 66**

If the constructed $W_{\mathcal{T}}$ does not have an appropriate region of overlap for all scenarios, then we add to $W_{\mathcal{T}}$ a set of preliminary points, who all have the same properties and thus are part of the same region of overlap and they are also snake-linked.

**Lemma 67**

For every $x_\sigma, y_\eta \in W_{\mathcal{T}}$, if $\neg\exists c : c \in CE \wedge (c < x_\sigma \vee c < y_\eta)$ then $x_\sigma, y_\eta$ are snake-linked in the subset $W_{\not\geq CE}$.

While for many cases it could occur (depending on the original $\bar{M}$) that this property would not be achieved. For this purpouse, was given the postulate 66. Although from a physical standpoint of view, one could assume a 'big bang' singularity that would be present in the beginning and fulfill the lemma's needs.

*Proof.* According to the postulate 66, if there is not enough points at the origin of our universe to snake-link the given points, then we construct enough room for a snake-link to occur. □

At this moment, we can finally prove the theorem.

**Theorem 68** ($\langle W_{\mathcal{T}}, \leq_{\mathcal{T}} \rangle$)
The model $\langle W_{\mathcal{P}}, \leq_{\mathcal{T}} \rangle$ is a model of BCont.

*Proof.* By lemmas 59 – 67, $\langle W_{\mathcal{T}}, \leq_{\mathcal{T}} \rangle$ is a model of BCont. □

As we see, our construction fulfills the properties of a BCont model. At this moment, we could define l-events in $W_{\mathcal{T}}$ and even a BCont model with s-t locations because as it can be easily verified, our method of construction is consistent with the demands on s-l locations. Hence there is a way of combining $\mathcal{T}$-topology with BCont models.

# Chapter 5

# Conclusions

In this paper our goal was to investigate temporal logics, their relation to physics, and look into a specific field of their application from logics' point of view.

We first introduced to the reader who is new to the topic basic findings and notions from physics and philosophy of time. We continued with showing the broadness of temporal logics and their possibilities, presenting some proofs and methods used in the field and multiple temporal logics. We proceeded afterwards to investigate in more detail the systems of branching time - Branching Space-Time, Minkowskian Branching Structures and Branching Continuations. These were presented side by side and their common features were mentioned. Also, we did unveil topologies $\mathcal{A}, \mathcal{P}, \mathcal{T}$ who describg the structure of general relativistic space-time. As last we presented some introductory results combining the $\mathcal{A}$ and $\mathcal{T}$ topologies with branching time structures.

The herein presented construction was a purely logical attempt of combining a given topology and another structure. Thus it still remains an open question is, if, with regards to the physicist's point of view, we can construct physically valid models of GR based on BCont. As the newest articles show, it won't be open for long.

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