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MASTER THESIS

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Almost optimal trading strategy for multiple risky assets

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Téměř optimální obchodní strategie pro více rizikových aktiv

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Abstrakt: Uvažujeme investora, který má možnost investovat do více rizikových aktiv, nazvaných akcie, a do jednoho nerizikového aktiva, nazvaného opce. Cílem investora je maximalizovat svoje bohatství, a to dlouhodobě. Problematika je zjednodušena zavedením určitých předpokladů, jakož jsou proporcionální transakční náklady, použití vícedimensionálního Brownova pohybu při modelování cen akcií a HARA užitková funkce. Za předpokladu nezávislosti cen akcií jsme schopny najít téměř optimální strategii obchodování.

Klíčová slova: HARA užitková funkce, male transkční náklady, více nezávislých rizikových aktiv

Title: Almost optimal trading strategy for multiple risky assets

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Abstract: Assume that we have an investor who may invests in several risky assets called stocks and in one non-risky asset called bond and that the investor is interested in the expected utility of his/her wealth far in the future. In order to be able to treat the problem, we make several essential simplifying assumptions. First, we assume that the logarithm of the stock market prices is a multidimensional arithmetic Brownian motion, that the investors pays proportional transaction costs and that the utility function is of a HARA type. We are able to propose a strategy that can be called almost optimal in the long run provided that the stock market prices are independent.

Keywords: HARA utility, small transaction costs, multiple and independent risky assets

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Introduction

We consider an agent who does not consume and who invests all his/her wealth in a money market and in several stocks. We assume that he/she pays proportional transaction costs for each trade. The investor's goal is to maximize the long run growth rate of the certainty equivalent of the wealth process

$$\max \lim_{t\to\infty} \tfrac{1}{t} \log \mathrm{CE}_t,$$

provided that the stock market prices follow a multidimensional geometric Brownian motion and provided that he/she faces utility function with hyperbolic absolute risk aversion (HARA) unbounded from below.

We assume that the investor is interested in an almost optimal strategy for small transaction taxes. This leads us to a certain partial differential equation, which can be solved explicitly in the case where the stock prices are independent. Thus, we are able to propose a strategy that should be almost optimal for small values of transaction taxes in case of independent stocks.

In chapter 1 we presented model and described basic properties of the main processes. Chapter 2 briefly discuss existence of the optimal strategy in one-dimensional case. Chapter 3 introduces transactional costs, whose impact on the behaviour of the processes is described in chapters 4 and 5. Finally, chapter 6 is the basis of this paper, since it deals with independent market prices. In this chapter proposed optimal strategy, which is proven to have required features in chapter 7 and not to be worse than any other admissible strategy in chapter 8.

Chapter 1

Model setup

The considered market consists of risky assets (called stocks) and non-risky assets (called bonds). The price of a bond at time $t \geq 0$ is

$$T(t) \triangleq e^{rt},$$

where r is risk-free interest rate. We can restrict this paper to $r = 0$ without any loss of generality of the results, since for $r > 0$ we would base the work on discounted market prices. The stock market prices

$$\mathcal{S}(t) = (\mathcal{S}_1(t), \ldots, \mathcal{S}_n(t))^{\intercal}$$

form an n-dimensional geometric Brownian motion driven by n-dimensional standard Brownian motion $\{W_t, t \geq 0\}$ on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t^W\}_{t\geq 0}, \mathbb{P})$, with

$$\mathrm{d}\mathcal{S}(t) = \mathcal{S}(t) * [\mu \,\mathrm{d}t + \Sigma^{1/2}\,\mathrm{d}W(t)], \quad S(0) = s_0 \in (0, \infty)^n, \tag{1.1}$$

where $x * y = \{x_i y_i\}_{i=1}^n$ if $x, y \in \mathbb{R}^n$ and where $\mu \in \mathbb{R}^n$ and $\Sigma^{1/2} \in \mathbb{R}^{n\times n}$ is a positively definite matrix such that $(\Sigma^{1/2})^2 = \Sigma$. The filtration $\{\mathcal{F}_t^W\}_{t\geq}$ is canonical filtration induced by Brownian motion W augmented by all the $\mathbb{P}$-null sets relative to σ-algebra generated by W.

Let $\mathcal{W}_t$ be the wealth process, i.e., portfolio market price, at time $t \geq 0$ starting from w_0, which is assumed to be positive almost surely for every $t \geq 0$. The changes in the portfolio market price are caused by the changes in stock prices and by payment of transaction costs. An investor benefits from trading with the stocks. The amount of money invested in stocks is equal to $\pi(t)\mathcal{W}_t$, where $\pi(t) = (\pi_1(t), \ldots, \pi_n(t))^{\intercal}$ is ratio process, i.e., $\pi_i(t)$ stands for the proportion of the wealth $\mathcal{W}_t$ held in the risky assets i at time $t \geq 0$.

We measure the utility of the current wealth by HARA (hyperbolic absolute risk aversion) utility function covered by power utility functions and the logarithmic one. Namely, we put

$$\mathcal{U}_0(y) = \log y, \quad \& \quad \mathcal{U}_\gamma(y) = \tfrac{1}{\gamma}\, y^\gamma \quad \text{if} \quad \gamma < 0. \tag{1.2}$$

We restrict ourselves to $\gamma \le 0$. Then the utility function $\mathcal{U}_\gamma$ is unbounded from below on $(0, \infty)$, and this ensures that our investor never risks ruin. Whenever we deal with utility functions, we are interested in so called certainty equivalent

$$\mathrm{CE}_t = \mathcal{U}_\gamma^{-1} E\mathcal{U}_\gamma(\mathcal{W}_t),$$

which is the deterministic value giving the same expected utility as a given random variable $\mathcal{W}_t$. Our assumptions are so special that may expect, for certain type of strategies including the optimal one, that the certainty equivalent grows exponentially at a certain rate and this rate is the value that we would like to maximize. Formally, we can write this effort in the form

$$\max \lim_{t\to\infty} \tfrac{1}{t} \log \mathrm{CE}_t.$$

1.1 Dynamics with no trade

Denote $\mathcal{N}(t) = (\mathcal{N}_1(t), \ldots, \mathcal{N}_n(t))^{\mathsf{T}}$, where $\mathcal{N}_i(t)$ stands for the number of the stocks of the i-th stock in the investor's portfolio. It holds that

$$\pi(t) = \mathcal{N}(t) * \mathcal{S}(t) \cdot \mathcal{W}_t^{-1}. \tag{1.3}$$

Theorem 1.1. *Assume we do not trade. Then $\pi(t)$ and $\mathcal{W}_t$ are Itô process with the following differentials*

$$d\pi(t) = B(\pi(t))\, dt + S(\pi(t)) *\ dW(t), \tag{1.4}$$

$$d \log \mathcal{W}_t = q_0(\pi(t))\, dt + \pi(t)^{\mathsf{T}} \Sigma^{1/2}\, dW(t), \tag{1.5}$$

where

$$\begin{aligned} B(x) &= M(x)(\mu - \Sigma x), \\ S(x) &= M(x)\Sigma^{1/2}, \\ M(x) &= diag(x) - xx^{\mathsf{T}}, \\ q_0(x) &= x^{\mathsf{T}}\mu - \tfrac{1}{2}\, x^{\mathsf{T}}\Sigma x. \end{aligned} \tag{1.6}$$

Proof. Change of the wealth $\mathcal{W}_t$ equals to the change of the stock prices multiplied by number of the stocks in portfolio. Then

$$\begin{aligned} \mathrm{d}\mathcal{W}_t &= \mathcal{N}(t)^{\mathsf{T}}\, \mathrm{d}\mathcal{S}(t) = \mathcal{W}_t \pi(t)^{\mathsf{T}} \mathcal{S}(t)^{-1} * \mathrm{d}\mathcal{S}(t) \\ &= \mathcal{W}_t \pi(t)^{\mathsf{T}} (\mu\, \mathrm{d}t + \Sigma^{1/2}\, \mathrm{d}W(t)). \end{aligned}$$

Using Itô's formula, we obtain

$$\begin{aligned} \mathcal{W}_t \, \mathrm{d}\mathcal{W}_t^{-1} &= -\frac{1}{\mathcal{W}_t} \, \mathrm{d}\mathcal{W}_t + \frac{1}{\mathcal{W}_t^2} \, \mathrm{d}\langle \mathcal{W} \rangle_t \\ &= -\pi(t)^{\intercal} (\mu - \Sigma\pi(t)) \, \mathrm{d}t - \pi(t)^{\intercal} \Sigma^{1/2} \, \mathrm{d}W(t), \end{aligned} \tag{1.7}$$

as well as,

$$\begin{aligned} \mathrm{d} \log \mathcal{W}_t &= \frac{1}{\mathcal{W}_t} \, \mathrm{d}\mathcal{W}_t - \frac{1}{2\mathcal{W}_t^2} \, \mathrm{d}\langle \mathcal{W} \rangle_t \\ &= \left(\pi(t)^{\intercal} \mu - \frac{1}{2} \pi(t)^{\intercal} \Sigma \pi(t) \right) \, \mathrm{d}t + \pi(t)^{\intercal} \Sigma^{1/2} \, \mathrm{d}W(t), \end{aligned} \tag{1.8}$$

which proves (1.5).

Since $\mathcal{N}(t)$ does not change, $\mathrm{d}\mathcal{N}(t) = 0$. From Itô's formula and with $\pi(t) = \mathcal{W}_t^{-1} \mathcal{N}(t) * \mathcal{S}(t)$ we obtain

$$\begin{aligned} \mathrm{d}\pi(t) &= \mathcal{W}_t^{-1} \mathcal{N}(t) * \, \mathrm{d}\mathcal{S}(t) + \mathcal{N}(t) * \mathcal{S}(t) \, \mathrm{d}\mathcal{W}_t^{-1} + \mathcal{N}(t) * \, \mathrm{d}\langle \mathcal{S}, \mathcal{W}^{-1} \rangle_t \\ &= \pi(t) * \mathcal{S}(t)^{-1} * \, \mathrm{d}\mathcal{S}(t) + \pi(t) \mathcal{W}_t \, \mathrm{d}\mathcal{W}_t^{-1} \\ &+ \mathcal{W}_t \pi(t) * \mathcal{S}(t)^{-1} * \, \mathrm{d}\langle \mathcal{S}, \mathcal{W}^{-1} \rangle_t \\ &= \pi(t) * \big(\mu \, \mathrm{d}t + \Sigma^{1/2} \, \mathrm{d}W(t) \big) \\ &- \pi(t)\pi(t)^{\intercal} \left[(\mu - \Sigma\pi(t)) \, \mathrm{d}t + \Sigma^{1/2} \, \mathrm{d}W(t) \right] \\ &- \pi(t)\pi(t)^{\intercal} \Sigma^{1/2} \Sigma^{1/2} \pi(t) \, \mathrm{d}t \\ &= \big(\operatorname{diag} \pi(t) - \pi(t)\pi(t)^{\intercal} \big) \left[(\mu - \Sigma\pi(t)) \, \mathrm{d}t + \Sigma^{1/2} \, \mathrm{d}W(t) \right]. \end{aligned} \tag{1.9}$$

which proves (1.4). □

Chapter 2

One-dimensional case

Let us consider the one-dimensional case, i.e. $n = 1$, and let $\pi(t)$ stand for the proportion of the wealth invested in the stock. Then we are able to find a C^2-function f and $\nu \in \mathbb{R}$ such that

$$\mathrm{e}_\gamma\left(\log(\mathcal{W}_t) - f(\pi(t)) - \nu t\right), \quad \text{where} \quad \mathrm{e}_\gamma(x) \triangleq \mathcal{U}_\gamma(e^x),$$

is an $\mathcal{F}_t$-martingale if we apply a special strategy and it is an $\mathcal{F}_t$-supermartingale in case of any admissible strategy. See [2] for the form of f and the special strategy. Then

$$\nu = \lim_{t\to\infty} \frac{1}{t} \log \widehat{\mathrm{CE}}_t \geq \limsup_{t\to\infty} \frac{1}{t} \log \mathrm{CE}_t, \quad \text{where} \quad \mathrm{CE}_t = \mathcal{U}_\gamma^{-1}\, \mathbb{E}\, \mathcal{U}_\gamma(\mathcal{W}_t)$$

is a certainty equivalent of $\mathcal{W}_t$ and $\widehat{\mathrm{CE}}_t$ is the certainty equivalent corresponding to the special strategy, which can be called also optimal in the above-introduced sense.

In multi-dimensional case we are not able to find an optimal strategy even in the logarithmic case and this is the reason why to restrict ourselves for small transaction costs and to admit some error and to require that the error should be very small if the size of the transaction taxes go to zero.

Chapter 3

Transaction costs

3.1 No transaction costs

Let us assume that $\pi(t)$ is a bounded process. Then $\ln \mathcal{W}_t - \int_0^t q_0(\pi(s))\,\mathrm{d}s$ is a continuous $\mathcal{F}_t$-martingale with Lipschitz quadratic variation. Hence, in order to maximize the expected logarithmic utility of $\mathcal{W}_t$, we should keep the vector of positions $\pi(t)$ on the value $\theta = \arg\max q_0 = \Sigma^{-1}\mu$, which will be called a vector of log-optimal positions. If we are interested in $E\mathcal{U}_\gamma(\mathcal{W}_t)$, we need to express

$$\mathcal{U}_\gamma(\mathcal{W}_t) = \mathcal{E}_t \cdot \mathrm{e}_\gamma(\ln w_0 + \textstyle\int_0^t q_\gamma(\pi(s))\,\mathrm{d}s), \quad \text{where} \quad q_\gamma(x) = x^{\mathsf{T}}\mu - \frac{1-\gamma}{2}\, x^{\mathsf{T}}\Sigma x,$$

with the help of an exponential martingale $\mathcal{E}_t$ with

$$\mathrm{d}\mathcal{E}_t = \gamma\mathcal{E}_t\,\pi(t)\Sigma^{1/2}\,\mathrm{d}W(t), \quad \text{starting from } \mathcal{E}_0 = 1.$$

The second factor is obviously maximal if

$$\pi(t) = \arg\max q_\gamma = \theta/(1-\gamma) \triangleq \Theta$$

holds for every $t \geq 0$. Then it can be easily seen that $E\mathcal{U}_\gamma(\mathcal{W}_t)$ is maximal if $\pi(t)$ is kept on Θ. This value Θ can be called a Merton vector of proportion, since this proportion is also optimal in the classical Merton approach setting in case of no transaction costs.

3.2 Existence of transaction costs

We assume that we pay proportional transaction costs for each transaction. Say that we pay $(1 + \lambda_i^{\uparrow})\mathcal{S}_i(t)$ in order to obtain the i-th stock and that we obtain $(1-\lambda_i^{\downarrow})\mathcal{S}_i(t)$ when we sell it. If we buy the i-th stock, the following values remain the same

$$\mathcal{W}_t\left(1+\lambda_i^{\uparrow}\pi_i(t)\right)=\mathcal{W}_t+\lambda_i^{\uparrow}\mathcal{N}_i(t)\mathcal{S}_i(t)\quad\text{and}\quad\mathcal{W}_t\pi_j(t)=\mathcal{N}_j(t)\mathcal{S}_j(t),\ j\neq i.$$

More obvious is that the values in the second equality do not change, because they describe the investor's wealth invested in the j-th stock, where $j\neq i$. On the other hand, if we buy $\Delta\mathcal{N}_i(t)$ shares of the i-th stock, the value of the transaction costs is $\lambda_i^{\uparrow}\mathcal{S}_i(t)\Delta\mathcal{N}_i(t)$. This is the reason, why also the values in the first equality do not change.

Similarly, if we sell the i-th stock, the following values remain the same:

$$\mathcal{W}_t\left(1-\lambda_i^{\downarrow}\pi_i(t)\right)=\mathcal{W}_t-\lambda_i^{\downarrow}\mathcal{N}_i(t)\mathcal{S}_i(t)\quad\text{and}\quad\mathcal{W}_t\pi_j(t)=\mathcal{N}_j(t)\mathcal{S}_j(t),\ j\neq i.$$

We restrict ourselves to such strategies that enable an investor to withdraw from the market with positive remaining wealth. Assume that the investor with the current wealth $\mathcal{W}_t>0$ and the vector of positions $\pi(t)$ decides to withdraw from the market, i.e., to reach that $\pi(t_+)=0\in\mathbb{R}^n$. Then the following value does not change

$$\mathcal{W}_t\left(1+\sum_{i=1}^{n}\lambda_i\pi_i(t)\right),\tag{3.1}$$

where λ_i is $\lambda_i^{\uparrow}$,$\lambda_i^{\downarrow}$ or 0 if we buy the i-th stock, sell it or do not trade with it, respectively. This means that the remaining wealth is of the form (3.1). We introduce the set of all admissible positions

$$\mathcal{A}=\left\{x\in\mathbb{R}^n:1+\sum_{i=1}^{n}\lambda_i^{\downarrow}x_i^{+}+\sum_{i=1}^{n}\lambda_i^{\uparrow}x_i^{-}>0\right\}.$$

Note that $\mathcal{A}\subseteq\prod_{i=1}^{n}\mathcal{A}_i$, where $\mathcal{A}_i=(-1/\lambda_i^{\uparrow},1/\lambda_i^{\downarrow})$.

Chapter 4

Dynamics with transaction costs

In order to be able to derive the effect of the transactions on the dynamics of the processes $\mathcal{W}_t$ and $\pi(t)$, we introduce differentials $\mathrm{d}_i^{\updownarrow}$ that measure the changes of corresponding processes if we buy ($\uparrow$) or sell ($\downarrow$) the i-th stock. Note that all changes can be viewed as continuous, since the transaction costs are proportional. Further, as the effect of the transactions on $\mathcal{W}_t$ and $\pi_i(t)$ is monotone (and continuous and successive), we get that $\mathrm{d}_i^{\updownarrow} f(X(t)) = f'(X(t))\,\mathrm{d}_i^{\updownarrow} X(t)$ holds if f is a C^1 function, where $X(t)$ stands for $\mathcal{W}_t$ or $\pi_j(t), j \leq n$. As shown in the section (3.2), when trading with the i-th stock, values $\mathcal{W}_t(1 \pm \lambda_i^{\updownarrow}\pi_i(t))$ remain the same, where $1 \pm \lambda_i^{\updownarrow}\pi_i(t)$ are assumed to be positive. Taking the logarithm, we obtain

$$\mathrm{d}_i^{\updownarrow} \log \mathcal{W}_t + \mathrm{d}_i^{\updownarrow} \log \Big(1 \pm \lambda_i^{\updownarrow}\pi_i(t)\Big) = 0,$$

which gives us

$$\begin{aligned}\mathrm{d}_i^{\updownarrow} \log \mathcal{W}_t &= -\,\mathrm{d}_i^{\updownarrow} \log \Big(1 \pm \lambda_i^{\updownarrow}\pi_i(t)\Big) = -\frac{\pm\lambda_i^{\updownarrow}}{1 \pm \lambda_i^{\updownarrow}\pi_i(t)}\,\mathrm{d}_i^{\updownarrow}\pi_i(t) \\ &= -\kappa_i^{\updownarrow}(\pi_i(t))\,\mathrm{d}\pi_i^{\updownarrow}(t),\end{aligned}$$

where

$$\kappa_i^{\updownarrow}(x) \triangleq \frac{\lambda_i^{\updownarrow}}{1 \pm \lambda_i^{\updownarrow}x} \quad \text{and} \quad \mathrm{d}\pi_i^{\updownarrow}(t) \triangleq \pm\,\mathrm{d}_i^{\updownarrow}\pi_i(t) \geq 0. \tag{4.1}$$

The differentials $\mathrm{d}\pi_i^{\updownarrow}(t)$ are differentials of non-decreasing processes $\pi_i^{\updownarrow}(t)$ that are responsible for increasing or decreasing of the i-th position $\pi_i(t)$ thanks to the transactions with the i-th stock, having nothing to do with positive or negative part of $\pi_i(t)$, respectively. Using (1.5), we are now able to describe the full dynamics of the wealth process in the presence of transaction costs in terms of $\mathrm{d}\pi_i^{\updownarrow}(t)$

$$\mathrm{d}\log\mathcal{W}_t = q_0(\pi(t))\,\mathrm{d}t + \pi(t)^{\intercal}\Sigma^{1/2}\,\mathrm{d}W(t) - \sum_{\updownarrow,i=1}^{n}\kappa_i^{\updownarrow}(\pi_i(t))\,\mathrm{d}\pi_i^{\updownarrow}(t), \tag{4.2}$$

where we use $\sum_{\updownarrow} x^{\updownarrow} \triangleq x^{\uparrow} + x^{\downarrow}$.

Similarly, as $\mathcal{W}_t\pi_j(t)$ does not change for $i \neq j$ when trading with the i-th stock, we obtain that

$$\begin{aligned} 0 &= \mathrm{d}_i^{\updownarrow}\log\mathcal{W}_t + \mathrm{d}_i^{\updownarrow}\log\pi_j(t) \\ &= -\kappa_i^{\updownarrow}(\pi_i(t))\,\mathrm{d}\pi_i^{\updownarrow}(t) + \frac{\mathrm{d}_i^{\updownarrow}\pi_j(t)}{\pi_j(t)}, \end{aligned}$$

and therefore

$$\mathrm{d}_i^{\updownarrow}\pi_j(t) = \pi_j(t)\kappa_i^{\updownarrow}(\pi_i(t))\,\mathrm{d}\pi_i^{\updownarrow}(t).$$

This equality describes the effect of transactions with the i-th stock on the j-th position $\pi_j(t)$, where $j \neq i$, caused by the decrease of the wealth process $\mathcal{W}_t$. Then the whole dynamics of $\pi(t)$ is of the form

$$\mathrm{d}\pi(t) = B(\pi(t))\,\mathrm{d}t + S(\pi(t))\,\mathrm{d}W(t) + K^{\uparrow}(\pi(t))\,\mathrm{d}\pi^{\uparrow}(t) - K^{\downarrow}(\pi(t))\,\mathrm{d}\pi^{\downarrow}(t), \tag{4.3}$$

where

$$K^{\updownarrow}(x)_{ji} \triangleq 1_{[i=j]} \pm x_j\kappa_i^{\updownarrow}(x_i)1_{[i\neq j]}. \tag{4.4}$$

When using $\pm$ and $\updownarrow$ in the same formula, $+$ corresponds to $\uparrow$ and $-$ corresponds to $\downarrow$.

4.1 Advanced dynamics

Let $f \in C^1(\mathcal{A})$ have locally absolutely continuous derivatives. If the vector of positions $\pi(t)$ is a continuous process (with values in $\mathcal{A}$), i.e., if $\mathcal{N}(t)$ is continuous (besides having locally finite variations and being $\mathcal{F}_t^W$-adapted), then

$$\mathrm{d}f(\pi(t)) = d_f(\pi(t))\,\mathrm{d}t + \nabla f(\pi(t))^{\intercal}\left[S(\pi(t))\,\mathrm{d}W(t) + \sum_{\updownarrow} \pm K^{\updownarrow}(\pi(t))\,\mathrm{d}\pi^{\updownarrow}(t)\right], \tag{4.5}$$

with

$$d_f(x) \triangleq \nabla f(x)^{\intercal}B(x) + \tfrac{1}{2}\,\mathrm{tr}\{\nabla^2 f(x)S(x)S(x)^{\intercal}\}. \tag{4.6}$$

Let f be as above and $\nu \in \mathbb{R}$, put

$$U_\nu^f(t) \triangleq \log \mathcal{W}_t - f(\pi(t)) - \nu t \quad \text{and} \quad \mathcal{E}_{\gamma,\nu}^f(t) \triangleq \mathrm{e}_\gamma(U_\nu^f(t)).$$

We obtain from (4.2) and (4.6) that $\mathcal{E}_{0,\nu}^f(t) = U_\nu^f(t)$ is an $\mathcal{F}_t$-semimartingale with

$$\begin{aligned} \mathrm{d}\mathcal{E}_{0,\nu}^f(t) &= \mathrm{d}U_\nu^f(t) \\ &= d_{\mathcal{E},f}^{0,\nu}(\pi(t))\,\mathrm{d}t + D_{\mathcal{E},f}(\pi(t))\,\mathrm{d}W(t) - \sum_{\updownarrow} L_f^{\updownarrow}(\pi(t))\,\mathrm{d}\pi^{\updownarrow}(t), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} d_{\mathcal{E},f}^{0,\nu}(x) &\triangleq q_0(x) - d_f(x) - \nu, \\ L_f^{\updownarrow}(x) &\triangleq \Big(\kappa_1^{\updownarrow}(x_1), \cdots, \kappa_n^{\updownarrow}(x_n)\Big) \pm \nabla f(x)^{\intercal} K^{\updownarrow}(x), \\ D_{\mathcal{E},f}(x) &\triangleq x^{\intercal}\Sigma^{1/2} - \nabla f(x)^{\intercal} S(x). \end{aligned} \tag{4.8}$$

Further, we obtain that $\mathcal{E}_{\gamma,\nu}^f(t)$ is an $\mathcal{F}_t$-semimartingale with

$$\begin{aligned} \mathrm{d}\mathcal{E}_{\gamma,\nu}^f(t) &= \exp\left\{\gamma \mathcal{E}_{\gamma,\nu}^f(t)\right\} \cdot \\ &\cdot \left[d_{\mathcal{E},f}^{\gamma,\nu}(\pi(t))\,\mathrm{d}t + D_{\mathcal{E},f}(\pi(t))\,\mathrm{d}W(t) - \sum_{\updownarrow} L_f^{\updownarrow}(\pi(t))\,\mathrm{d}\pi^{\updownarrow}(t) \right], \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} d_{\mathcal{E},f}^{\gamma,\nu}(x) &= q_0(x) - d_f(x) - \nu + \frac{\gamma}{2} D_{\mathcal{E},f}(x) D_{\mathcal{E},f}(x)^{\intercal} = q_\gamma(x) - \nu - d_{f,\gamma}(x), \\ d_{f,\gamma}(x) &= d_f(x) - \frac{\gamma}{2} \nabla f(x)^{\intercal} S(x) S(x)^{\intercal} \nabla f(x)^{\intercal} + \gamma \nabla f(x)^{\intercal} S(x) \Sigma^{1/2} x \\ &= \nabla f(x)^{\intercal} B_\gamma(x) + \frac{1}{2} \operatorname{tr}\left\{ \left[\nabla^2 f(x) - \gamma \nabla f(x) \nabla f(x)^{\intercal}\right] S(x) S(x)^{\intercal} \right\}, \end{aligned} \tag{4.10}$$

with a drift coefficient

$$B_\gamma(x) = B(x) + \gamma S(x)\Sigma^{1/2} x = M(x)\left[\mu - (1-\gamma)\Sigma x\right],$$

of $\pi(t)$ on $[0,T]$ under a new measure Q_T equivalent with P with $\mathrm{d}Q_T = \mathcal{E}_\gamma(T)\,\mathrm{d}P$.

4.2 Asymptotic behaviour of CE

If we find f as above and $\nu \in \mathbb{R}$ and a certain strategy such that $\mathcal{E}_{\gamma,\nu}^f(t)$ is an $\mathcal{F}_t$-martingale, then similarly as in one-dimensional case, we would obtain that

$$\nu = \lim_{t\to\infty} \frac{1}{t} \log \mathcal{U}_\gamma^{-1}\, \mathbb{E}\, \mathcal{U}_\gamma(\mathcal{W}_t) = \lim_{t\to\infty} \frac{1}{t} \log \mathrm{CE}_t$$

holds, which characterizes the asymptotic behaviour of the certainty equivalent. To find such a function $f, \nu \in \mathbb{R}$ and the strategy, it is enough to find f with the smooth properties as written above such that

$$d^{\gamma;\nu}_{\mathcal{E},f}(\pi(t)) = 0, \quad L^{\updownarrow}_f(\pi(t))\,\mathrm{d}\pi^{\updownarrow}(t) = 0, \quad t \geq 0.$$

Obviously, ν is the value that we would like to maximize. We would be happy if we find f with the properties written above, $\hat{\nu} \in \mathbb{R}$ and a special strategy such that $\mathcal{E}^f_{\gamma,\nu}(t)$ is an $\mathcal{F}_t$-martingale when applying the special strategy and such that it is an $\mathcal{F}_t$-supermartingale when applying any other admissible strategy. Then, similarly as in one-dimensional case, we would obtain that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathcal{U}_\gamma^{-1}\,\mathbb{E}\,\mathcal{U}_\gamma(\mathcal{W}_t) \leq \hat{\nu} = \lim_{t\to\infty} \frac{1}{t} \log \mathcal{U}_\gamma^{-1}\,\mathbb{E}\,\mathcal{U}_\gamma(\hat{\mathcal{W}}_t),$$

where $\hat{\mathcal{W}}_t$ stands for the wealth process corresponding to the special strategy here. So we would like to find f smooth as above and $\hat{\nu} \in \mathbb{R}$ such that

$$d^{\gamma;\hat{\nu}}_{\mathcal{E},f}(\hat{\pi}(t)) = 0, \quad L^{\updownarrow}_f(\hat{\pi}(t))\,\mathrm{d}\hat{\pi}^{\updownarrow}(t) = 0, \tag{4.11}$$

$$d^{\gamma;\hat{\nu}}_{\mathcal{E},f}(\pi(t)) \leq 0, \quad L^{\updownarrow}_f(\pi(t))\,\mathrm{d}\pi^{\updownarrow}(t) \geq 0, \tag{4.12}$$

hold for every $t \geq 0$, where $\hat{\pi}(t)$ is a vector of positions corresponding to the special strategy and $\pi(t)$ to another admissible strategy. Such a function $f, \hat{\nu} \in \mathbb{R}$ and a strategy, we are able to find only in one-dimensional case. If $n \geq 2$, we admit some error say $\varepsilon > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathcal{U}_\gamma^{-1}\,\mathbb{E}\,\mathcal{U}_\gamma(\mathcal{W}_t) \leq \hat{\nu} \leq \lim_{t\to\infty} \frac{1}{t} \log \mathcal{U}_\gamma^{-1}\,\mathbb{E}\,\mathcal{U}_\gamma(\hat{\mathcal{W}}_t) + \varepsilon.$$

In order to be able to obtain such inequality, besides certain technical restrictions, it is enough to find a function f smooth as above $\hat{\nu} \in \mathbb{R}$ and an „ε-optimal" strategy such that for every $t \geq 0$ it holds that

$$\varepsilon + d^{\gamma;\hat{\nu}}_{\mathcal{E},f}(\hat{\pi}(t)) \geq 0, \quad L^{\updownarrow}_f(\hat{\pi}(t))\,\mathrm{d}\hat{\pi}^{\updownarrow}(t) = 0 \tag{4.13}$$

$$d^{\gamma;\hat{\nu}}_{\mathcal{E},f}(\pi(t)) \leq 0, \quad L^{\updownarrow}_f(\pi(t))\,\mathrm{d}\pi^{\updownarrow}(t) \geq 0. \tag{4.14}$$

Chapter 5

Small transaction costs

In one-dimensional case the width of the no-trade region is of order $\lambda^{1/3}$ and the risk premium of order $\lambda^{2/3}$ as $\lambda \to 0^+$, where $\lambda = \log \frac{1+\lambda_\uparrow}{1-\lambda_\downarrow}$ describes the real level of the transaction tax. Since the value of the risk premium is connected with the second derivative of f, and we would like to look at the corresponding PDE for very small transaction costs as under microscope, we introduce a new vector variable

$$u \triangleq \frac{(x - \Theta)}{\lambda^{1/3}}$$

and to the function f a new one

$$g(u) \triangleq \lambda^{-4/3} f(\Theta + u\lambda^{1/3}).$$

Since $\nabla f(x) = \lambda \nabla g(u) = \mathcal{O}(\lambda)$ and $\nabla^2 f(x) = \lambda^{2/3} \nabla^2 g(u)$, we get that

$$\begin{aligned} d_{f,\gamma}(x) &= \nabla f(x)^\intercal B_\gamma(x) + \frac{1}{2} \operatorname{tr} \left\{ \left[\nabla^2 f(x) - \gamma \nabla f(x) \nabla f(x)^\intercal \right] S(x) S(x)^\intercal \right\} \\ &= \mathcal{O}(\lambda) + \frac{1}{2} \lambda^{2/3} \operatorname{tr}\{\nabla^2 g(u) S(\Theta + \lambda^{1/3} u) S(\Theta + \lambda^{1/3} u)^\intercal\} \\ &= \mathcal{O}(\lambda) + \frac{1}{2} \lambda^{2/3} \operatorname{tr}\{\nabla^2 g(u)\, \mathbb{S}_\Theta\}, \end{aligned} \tag{5.1}$$

where $\mathbb{S}_\Theta = S(\Theta) S(\Theta)^\intercal$.

We are not able to reach a greater value of ν than $\frac{1-\gamma}{2} \Theta^\intercal \Sigma \Theta$, even in case of zero transaction costs. Therefore, we can restrict ourselves to such $\nu \in \mathbb{R}$, that there exists $\Omega \geq 0$ such that

$$\nu = \frac{1 - \gamma}{2} (\Theta^\intercal \Sigma \Theta - \Omega^2).$$

We introduce $\omega \triangleq \Omega\lambda^{-1/3}$. Since $x^\intercal\mu = x^\intercal\Sigma\Sigma^{-1}\mu = (1-\gamma)x^\intercal\Sigma\Theta$, we have

$$\begin{aligned} q_\gamma(x) - \nu &= x^\intercal\mu - \frac{1-\gamma}{2}x^\intercal\Sigma x - \frac{1-\gamma}{2}(\Theta^\intercal\Sigma\Theta - \lambda^{2/3}\omega^2) \\ &= \frac{1-\gamma}{2}\left(2x^\intercal\Sigma\Theta - x^\intercal\Sigma x - \Theta^\intercal\Sigma\Theta + \lambda^{2/3}\omega^2\right) \\ &= \frac{1-\gamma}{2}\left(\lambda^{2/3}\omega^2 - (x-\Theta)^\intercal\Sigma(x-\Theta)\right) \\ &= \frac{1-\gamma}{2}\lambda^{2/3}\left(\omega^2 - u_x^\intercal\Sigma u_x\right). \end{aligned} \tag{5.2}$$

From (5.1) and (5.2) we obtain

$$\begin{aligned} d^{\gamma,\nu}_{\mathcal{E},f}(x) &= q_\gamma(x) - \nu - d_{f,\gamma}(x) \\ &= \mathcal{O}(\lambda) + \frac{1-\gamma}{2}\lambda^{2/3}\left[\omega^2 - u_x^\intercal\Sigma u_x - \operatorname{tr}\{\nabla^2 g(u)\tilde{\mathbb{S}}_\Theta\}\right], \end{aligned} \tag{5.3}$$

where

$$\tilde{\mathbb{S}}_\Theta = \mathbb{S}_\Theta/(1-\gamma) = S(\Theta)S(\Theta)^\intercal/(1-\gamma).$$

If $\lambda_i^\updownarrow/\lambda = \vartheta_i^\updownarrow \in \mathbb{R}$, then $\kappa_i^\updownarrow(x) = \lambda(\vartheta_i^\updownarrow + \mathcal{O}(\lambda))$. Further, $K^\updownarrow(x) = I + \mathcal{O}(\lambda)$, and therefore

$$L_f^\updownarrow(x) = \lambda\left[(\vartheta_1^\updownarrow, \dots, \vartheta_n^\updownarrow) \pm \nabla g(u)^\intercal\right] + \mathcal{O}(\lambda^2).$$

If we neglect the terms of higher orders, we can redefine requirements (4.11-4.12) for function f, so that

$$\begin{aligned} &\omega^2 - \hat{u}(t)^\intercal\Sigma\hat{u}(t) - \operatorname{tr}\{\nabla^2 g(u(t))\tilde{\mathbb{S}}_\Theta\} = 0, \quad [\vartheta_i^\updownarrow \pm g_i'(u(t))]\,\mathrm{d}\hat{\pi}_i^\updownarrow(t) = 0 \\ &\omega^2 - u(t)^\intercal\Sigma u(t) - \operatorname{tr}\{\nabla^2 g(u(t))\tilde{\mathbb{S}}_\Theta\} \le 0, \quad \vartheta_i^\updownarrow \ge \mp g_i'(u), \end{aligned}$$

with $\hat{u}(t) = \lambda^{-1/3}[\hat{\pi}(t) - \Theta]$ hold for every $i = 1, \dots, n$ and for every $t \ge 0$.

5.1 Change of transaction taxes

Let us assume that the values of the transaction taxes have changed from $\lambda_i^\updownarrow$ to $\tilde{\lambda}_i^\updownarrow, i = 1, \dots, n$, so that there exists $\delta \in (-1, \infty)^n$ such that

$$(1 \pm \tilde{\lambda}_i^\updownarrow)(1 + \delta_i) = 1 \pm \lambda_i^\updownarrow.$$

We are going to show that there is a link between these two cases, so that the solution of the first case gives the solution of the second one.

Let us consider the same model with the changed initial value of the stock prices $\tilde{s}_0 = (\mathbf{1}_n + \delta) * s_0$, where $\mathbf{1}_n = (1, \ldots, 1)^{\intercal} \in \mathbb{R}^n$ and $\delta \in (-1, \infty)^n$. Then the new stock prices are of the form

$$\tilde{\mathcal{S}}(t) = (\mathbf{1}_n + \delta) * \mathcal{S}(t).$$

The number shares in the portfolio does not change, which implies

$$\tilde{\pi}(t)\tilde{\mathcal{W}}_t = \tilde{\mathcal{S}}(t) * \mathcal{N}(t) = (\mathbf{1}_n + \delta) * \mathcal{S}(t) * \mathcal{N}(t) = (\mathbf{1}_n + \delta) * \pi(t)\mathcal{W}_t,$$

where $\tilde{\mathcal{W}}_t$ and $\tilde{\pi}(t)$ stand for the new wealth and new position processes, respectively. The part of the wealth, which is not invested in the stocks does not change, as well. Hence,

$$[1 - \mathbf{1}_n^{\intercal}\pi(t)]\mathcal{W}_t = [1 - \mathbf{1}_n^{\intercal}\tilde{\pi}(t)]\tilde{\mathcal{W}}_t = \tilde{\mathcal{W}}_t - \mathbf{1}_n^{\intercal}(\mathbf{1}_n + \delta) * \pi(t)\mathcal{W}_t,$$

where $*$ is executed always first. Altogether we obtain that

$$\tilde{\mathcal{W}}_t = [1 + \mathbf{1}_n^{\intercal}\delta * \pi(t)]\mathcal{W}_t \quad \text{and} \quad \tilde{\pi}(t) = \xi_\delta(\pi(t)),$$

where $\xi_\delta(x) = \frac{(\mathbf{1}_n+\delta)*x}{1+\mathbf{1}_n^{\intercal}\delta*x}$. Similarly,

$$\mathcal{W}_t = [1 + \mathbf{1}_n^{\intercal}\tilde{\delta} * \tilde{\pi}(t)]\tilde{\mathcal{W}}_t \quad \text{and} \quad \pi(t) = \xi_{\tilde{\delta}}(\tilde{\pi}(t)),$$

where $\tilde{\delta} \in (-1, \infty)^n$ is such that $(\mathbf{1}_n + \delta) * (\mathbf{1}_n + \tilde{\delta}) = \mathbf{1}_n$.

As shown in chapter 4, all the asymptotic analysis of the certainty equivalent is based on the following process

$$U_\nu^f(t) = \log W_t - f(\pi(t)) - \nu t.$$

Put

$$\tilde{U}_\nu^{\tilde{f}}(t) \triangleq \log \tilde{\mathcal{W}}_t - \tilde{f}(\tilde{\pi}(t)) - \nu t.$$

Once f and ν are given, we are searching for $\tilde{f}$ such that $U_\nu^f(t) = \tilde{U}_\nu^{\tilde{f}}(t)$. It follows from the previous relations that it is enough to put

$$\tilde{f}(\tilde{x}) = f(x) + \log(1 + \mathbf{1}_n^{\intercal}\delta * x), \quad \text{where } x = \xi_{\tilde{\delta}}(\tilde{x}).$$

If the solution of the former problem is given by a function $\phi : \mathcal{A} \to \{-1, 0, 1\}^n$ saying that we should buy the i-th stock if $\phi_i(\pi(t)) = 1$ or sell if $\phi_i(\pi(t)) = -1$ or do nothing if $\phi_i(\pi(t)) = 0$, we are able to provide the same strategy in terms of $\tilde{\pi}(t)$ for the changed taxes $\tilde{\lambda}_i^{\updownarrow}$. We just put $\tilde{\phi}(\tilde{x}) = \phi(\xi_{\tilde{\delta}}(\tilde{x}))$.

Chapter 6

Independent stock market prices

We assume that the stock market prices $\mathcal{S}_i = (\mathcal{S}_i(t), t \geq 0), i \leq n$ are independent. It means that $\Sigma = \mathrm{var}(\log \mathcal{S}(1))$ is a diagonal matrix, say $\Sigma = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_n^2)^\intercal$.

In chapter (5) we have introduced function $g(u)$ such that

$$f(x) = \lambda^{4/3} g\left(\frac{x-\Theta}{\lambda^{1/3}}\right), \quad \text{for } x = \Theta + \lambda^{1/3} u.$$

Our aim is to find convenient function g on a compact set, so that we obtain function f with properties described in the section (4.1).

Let $\vartheta_i, \sigma_i, s_i > 0$ hold for $i = 1, \ldots, n$. Let $\omega \geq 0$ and $\bar{u}_i$ for $i = 1, \ldots, n$ be such that

$$\omega^2 \triangleq \sum_{i=1}^{n} \left(\frac{3}{2}\vartheta_i s_i \sigma_i\right)^{2/3}, \quad \bar{u}_i \triangleq \sqrt[3]{\frac{3}{2}\vartheta_i s_i \sigma_i^{-2}}. \tag{6.1}$$

For $u \in \mathrm{R}^n$ we define function g as

$$\begin{aligned} g(u) \triangleq \sum_{i=1}^{n} \Big(& 3^{2/3} 2^{-5/3} (\vartheta_i \sigma_i)^{2/3} s_i^{-1/3} (|u_i| \wedge \bar{u}_i)^2 \\ & - \frac{1}{12} s_i^{-1} \sigma_i^2 (|u_i| \wedge \bar{u}_i)^4 + \vartheta_i (|u_i| - \bar{u}_i)^+ \Big). \end{aligned} \tag{6.2}$$

The following theorem describes main properties of the function g.

Theorem 6.1. *Let g be defind as in (6.2). Then $g \in C^2(\mathbb{R})$ satisfies $|g_i'| \leq \vartheta_i$ on $\mathbb{R}^n$, $g_i'(u) =$ sign$(u_i)\vartheta_i$ if $|u_i| \geq \bar{u}_i$, and*

$$\nabla^2 g(u) = \mathrm{diag}\left\{ s_i^{-1} \left[\left(\frac{3}{2}\vartheta_i s_i \sigma_i\right)^{2/3} - \sigma_i^2 u_i^2 \right]^+ \right\}_{i=1}^{n}. \tag{6.3}$$

In particular, if $\Sigma = \mathrm{diag}\{\sigma_i^2\}_{i=1}^n$ *and* $\mathbb{S} \in \mathbb{R}^{n\times n}$ *is such that* $\mathbb{S}_{ii} = s_i$ *holds for every* $i = 1, \ldots, n$*, then*

$$u^\intercal \Sigma u - \omega^2 + \mathrm{tr}\{\nabla^2 g(u)\mathbb{S}\} = \sum_{i=1}^n \left[\left(\frac{3}{2}\vartheta_i s_i \sigma_i\right)^{2/3} - \sigma_i^2 u_i^2\right]^- \geq 0 \qquad (6.4)$$

holds whenever $u \in \mathbb{R}^n$ *and equality holds if* $u \in U \triangleq \prod_{i=1}^n [-\bar{u}_i, \bar{u}_i]$.

Proof. (i) Let $|u_i| > \bar{u}_i$. Then

$$g_i'(u) = \mathrm{sign}(u_i)\vartheta_i \quad \text{and} \quad g_i''(u) = 0$$

(ii) Let $|u_i| \leq \bar{u}_i$. Then

$$\begin{aligned} g_i'(u) &= \left[\left(\frac{3}{2}\right)^{2/3} (\vartheta_i\sigma_i)^{2/3} s_i^{-1/3}|u_i| - \frac{1}{3}s_i^{-1}\sigma_i^2|u_i|^3\right] \mathrm{sign}(u_i), \\ g_i''(u) &= \left(\frac{3}{2}\right)^{2/3} (\vartheta_i\sigma_i)^{2/3} s_i^{-1/3} - s_i^{-1}\sigma_i^2 u_i^2 \\ &= s_i^{-1}\left[\left(\frac{3}{2}\right)^{2/3} (\vartheta_i\sigma_i s_i)^{2/3} - \sigma_i^2 u_i^2\right]^+ \\ &\geq 0, \end{aligned}$$

where equality holds for $|u_i| = \bar{u}_i$. Therefore $g_i'(u)$ reaches maximum for $u_i = \bar{u}_i$, i.e.,

$$g_i'(u) \leq \left(\frac{3}{2}\right)^{2/3} (\vartheta_i\sigma_i)^{2/3} s_i^{-1/3}\bar{u}_i - \frac{1}{3}s_i^{-1}\sigma_i^2\bar{u}_i^3 = \vartheta_i,$$

Altogether, we proved the first part of the theorem.

Let $u \in \mathbb{R}^n$. Assume that $\Sigma = \mathrm{diag}\{\sigma_i^2\}_{i=1}^n$ and $\mathbb{S} \in \mathbb{R}^{n\times n}$ is such that $\mathbb{S}_{ii} = s_i$ holds for every $i = 1, \ldots, n$. Then

$$\begin{aligned} &u^\intercal \Sigma u - \omega^2 + \mathrm{tr}\{\nabla^2 g(u)\mathbb{S}\} = \\ &= \sum_{i=1}^n \sigma_i^2 u_i^2 - \sum_{i=1}^n \left(\frac{3}{2}\vartheta_i s_i\sigma_i\right)^{2/3} + \sum_{i=1}^n \left[\left(\frac{3}{2}\vartheta_i s_i\sigma_i\right)^{2/3} - \sigma_i^2 u_i^2\right]^+ \\ &= \begin{cases} 0, & \text{when } u_i \in [-\bar{u}_i, \bar{u}_i] \\ \sum_{i=1}^n \left[\left(\frac{3}{2}\vartheta_i s_i\sigma_i\right)^{2/3} - \sigma_i^2 u_i^2\right]^-, & \text{when } u_i \notin [-\bar{u}_i, \bar{u}_i] \end{cases} \\ &\geq 0 \end{aligned}$$

which proves the second part of the theorem. $\square$

Some of the calculations will be initially performed in one-dimensional or two-dimensional space. It would better describe latter definitions and conclusions. As for proves, the multi-dimensional cases are usually carried out analogously to two-dimensional ones.

6.1 f on no-trade region

In previous section we defined function $g(u)$ with some significant properties on the compact set

$$U = \prod_{i=1}^{n} [-\bar{u}_i, \bar{u}_i] = [-\bar{u}, \bar{u}].$$

We are interested in derivations of the function f not only inside a compact set, but on the borders, as well. Therefore, we need to find function $\tilde{f}$, which for small λ has the same properties as function f, but goes beyond borders $\pm\bar{u}$. Put

$$\tilde{f}(x) \triangleq h(\lambda)f(x) = h(\lambda)\lambda^{4/3} g\left(\frac{x_u - \Theta}{\lambda^{1/3}}\right) \tag{6.5}$$

where

$$x_u \triangleq \Theta + \lambda^{1/3} u$$

We are going to demonstrate that with λ small enough we can find function $\tilde{f}$ with required features.

We will first deal with the problem in two-dimensional case and the first coordinate. We denote

$$\tilde{f}(x, y) \triangleq h(\lambda)f(x, y) = h(\lambda)\lambda^{4/3} g\left(\frac{x - \Theta_1}{\lambda^{1/3}}, \frac{y - \Theta_2}{\lambda^{1/3}}\right),$$

where $h(\lambda)$ is defined in the statement of the theorem (6.2). In this case we introduce following designation: Instead of $(\bar{u}_1, \bar{u}_2)$ we will write (u, v). Instead of (x_1, x_2) we will write (x, y). Instead of $x_u = \Theta + \lambda^{1/3} u$ we will write $(x_u, y_v) = (x, y)$.

Theorem 6.2. *There exist constants $\lambda_0 > 0$ and $K > 0$ such that for every $\lambda \in (0, \lambda_0)$ it holds*

$$\kappa_1^{\updownarrow}(x_{\mp\bar{u}}) \pm \nabla\tilde{f}(x_{\mp\bar{u}}, y_v)^T K^{\updownarrow}(x_{\mp\bar{u}}, y_v) \cdot e_1 < 0, \tag{6.6}$$

$$\kappa_2^{\updownarrow}(y_{\mp\bar{v}}) \pm \nabla\tilde{f}(x_u, y_{\mp\bar{v}})^T K^{\updownarrow}(x_u, y_{\mp\bar{v}}) \cdot e_2 < 0, \tag{6.7}$$

where $\tilde{f}(x, y) = h(\lambda)f(x, y)$ with $h(\lambda) = 1 + K\lambda$.

Proof. Since $\kappa_i^{\updownarrow}(x_{\mp\bar{u}}) > 0$, inequality (6.6) is equivalent to

$$\frac{1}{\kappa_1^{\updownarrow}(x_{\mp\bar{u}})}\nabla\tilde{f}(x_{\mp\bar{u}}, y_v)^{\intercal} K^{\updownarrow}(x_{\mp\bar{u}}, y_v)e_1 > 1. \tag{6.8}$$

Let $\lambda > 0$, $K > 0$ and $v \in [-\bar{v}, \bar{v}]$. Then

$$\begin{aligned}
&\frac{1}{\kappa_+^{(1)}(x_{-\bar{u}})}\nabla\tilde{f}(x_{-\bar{u}}, y_v)^{\intercal} K_+(x_{-\bar{u}}, y_v)e_1 = \\
&= \frac{1+\lambda_1 x_{-\bar{u}}}{\lambda_1}(1+K\lambda)\left[f_1'(x_{-\bar{u}}, y_v) + f_2'(x_{-\bar{u}}, y_v)\frac{\lambda_1 y_v}{1+\lambda_1 x_{-\bar{u}}}\right] \\
&= (1+\lambda_1 x_{-\bar{u}})(1+K\lambda)\frac{\lambda}{\lambda_1}\left[g_1'(-\bar{u}, v) + g_2'(-\bar{u}, v)\frac{\lambda_1 y_v}{1+\lambda_1 x_{-\bar{u}}}\right] \\
&= (1+\lambda_1 x_{-\bar{u}})(1+K\lambda)\frac{\lambda\vartheta_{-1}}{\lambda_1}\left[1 + g_2'(-\bar{u}, v)\frac{\lambda_1 y_v}{1+\lambda_1 x_{-\bar{u}}}\vartheta_{-1}^{-1}\right].
\end{aligned}$$

We have that $\lambda\vartheta_1/\lambda_1 = 1$. Following from the properties of the function g, there also exists constant $L > 0$ big enough, such that

$$g_2'(-\bar{u}, v)\frac{\lambda_1 y_v}{1+\lambda_1 x_{-\bar{u}}} = \mathbb{O}(\lambda L),$$

where for function f, g we define

$$f(x) = \mathbb{O}(g(x)) \equiv |f(x)| \leq |g(x)|.$$

Therefore, for $K > 0$ big enough we get

$$\frac{1}{\kappa_+^{(1)}(x_{-\bar{u}})}\nabla\tilde{f}(x_{-\bar{u}}, y_v)^{\intercal} K_+(x_{-\bar{u}}, y_v)e_1 > 1 \quad \text{as } \lambda \to 0^+.$$

The proof of the other part of the inequality (6.6), as well as the proof of the inequality (6.7), is analogous to the proof shown above. □

We will now generalize theorem (6.2). However, we will not prove it, since the proof is similar to one performed for two-dimensional case. We introduce designation $(a, b)_{\langle i\rangle}$: For $a \in \mathbb{R}$, $b = (b_1, \cdots, b_n) \in \mathbb{R}^n$ and $i \in \{1, \cdots, n\}$ we denote

$$(a, b)_{\langle i\rangle} \overset{def}{=} (b_1, \cdots, b_{i-1}, a, b_{i+1}, \cdots, b_n).$$

Theorem 6.3. *There exist constants $\lambda_0 > 0$ (small enough) and $K > 0$ (large enough) such that for every $\lambda \in (0, \lambda_0)$ and for every $i \in \{1, \cdots, n\}$ it holds*

$$\begin{aligned}
&\kappa_i^{\updownarrow}\left(\Theta_i \mp \lambda^{1/3}\bar{u}_i\right) \pm \\
&\pm \nabla\tilde{f}\left(\Theta_i \mp \lambda^{1/3}\bar{u}_i, \Theta + \lambda^{1/3}u\right)_{\langle i\rangle}^{T} K^{\updownarrow}\left(\Theta_i \mp \lambda^{1/3}\bar{u}_i, \Theta + \lambda^{1/3}u\right)_{\langle i\rangle} e_i < 0,
\end{aligned} \tag{6.9}$$

where $\tilde{f}(x) = h(\lambda)f(x)$ with $h(\lambda) = 1 + K\lambda$.

Since we will use the left-hand side of (6.9) in further calculations, we define function

$$\mathcal{L}_i^{\updownarrow}(u) \triangleq \kappa_i^{\updownarrow}\left(\Theta_i + \lambda^{1/3}u_i\right) \pm \nabla\tilde{f}\left(\Theta + \lambda^{1/3}u\right)^{\intercal} K^{\updownarrow}\left(\Theta + \lambda^{1/3}u\right) e_i, \qquad (6.10)$$

for $u \in \mathcal{R}^n$ and $i = 1, \cdots, n$. The (6.9) now becomes

$$\mathcal{L}_i^{\updownarrow}\left(\mp\bar{u}_i, u\right)_{\langle i\rangle} < 0. \qquad (6.11)$$

Let's take a closer look at the function $\tilde{f}$ and its domain, especially at new border points of the domain $x_i^{\updownarrow} = \Theta + \lambda^{1/3}u_i^{\updownarrow}$. For fixed x it should hold that

$$\begin{aligned} 0 &= \kappa_i^{\updownarrow}\left(x_i^{\updownarrow}\right) \pm \nabla\tilde{f}\left(x_i^{\updownarrow}, x\right)_{\langle i\rangle}^{\intercal} K^{\updownarrow}\left(x_i^{\updownarrow}, x\right)_{\langle i\rangle} \cdot e_i \\ &= \kappa_i^{\updownarrow}\left(\Theta_i + \lambda^{1/3}u_i^{\updownarrow}\right) \\ &\pm \nabla\tilde{f}\left(\Theta_i + \lambda^{1/3}u_i^{\updownarrow}, \Theta + \lambda^{1/3}u\right)_{\langle i\rangle}^{\intercal} K^{\updownarrow}\left(\Theta_i + \lambda^{1/3}u_i^{\updownarrow}, \Theta + \lambda^{1/3}u\right)_{\langle i\rangle} \cdot e_i \\ &= \mathcal{L}_i^{\updownarrow}\left(u_i^{\updownarrow}, u\right)_{\langle i\rangle}. \end{aligned}$$

For $u \in \mathcal{R}^n$ we therefore define

$$\begin{aligned} u_i^{\downarrow} &\triangleq \inf\left\{u_i \geq 0 : \mathcal{L}_i^{\downarrow}\left(u_i, u\right)_{\langle i\rangle} \leq 0\right\} \quad \textit{and} \\ u_i^{\uparrow} &\triangleq \sup\left\{u_i \leq 0 : \mathcal{L}_i^{\uparrow}\left(u_i, u\right)_{\langle i\rangle} \leq 0\right\}, \end{aligned}$$

where u_i exceptionally does not depend on i-th component of u. It is obvious that $u_i^{\updownarrow}$ depend on vector u and that $u_i^{\downarrow} \in [0, \bar{u}_i)$ and $u_i^{\uparrow} \in (-\bar{u}_i, 0]$. Using the implicit function theorem (IFT) (see appendix A), we will prove that $u_i^{\updownarrow}$ are locally smooth functions.

We have that $\nabla^2\tilde{f}(x) = \lambda^{2/3}\nabla^2\tilde{g}(u)$, where $\nabla^2\tilde{g}(u)$ is diagonal matrix with positive diagonal components, as shown in the theorem 6.1. It holds that

$$\begin{aligned} \mathcal{L}_i^{\updownarrow}(u) = &\frac{\lambda_i^{\updownarrow}}{1 \pm \lambda_i^{\updownarrow}\left(\Theta_i + \lambda^{1/3}u_i^{\updownarrow}\right)} \pm \\ &\pm \left[\tilde{f}_i'\left(\Theta + \lambda^{1/3}u\right) \pm \sum_{j\neq i} \tilde{f}_j'\left(\Theta + \lambda^{1/3}u\right) \frac{\lambda_i^{\updownarrow}\left(\Theta_j + \lambda^{1/3}u_j\right)}{1 \pm \lambda_i^{\updownarrow}\left(\Theta_i + \lambda^{1/3}u_i\right)}\right], \end{aligned} \qquad (6.12)$$

which equals to the fraction $A(u)/B(u)$, where

$$\begin{aligned} A(u) &= \lambda_i^{\updownarrow} \pm \tilde{f}_i' \left(\Theta + \lambda^{1/3} u\right) \left(1 \pm \lambda_i^{\updownarrow} \left(\Theta_i + \lambda^{1/3} u_i\right)\right) \\ &\quad \pm \lambda_i^{\updownarrow} \sum_{j \neq i} \tilde{f}_j' \left(\Theta + \lambda^{1/3} u\right) \left(\Theta_j + \lambda^{1/3} u_j\right) \\ B(u) &= 1 \pm \lambda_i^{\updownarrow} \left(\Theta_i + \lambda^{1/3} u_i\right). \end{aligned} \tag{6.13}$$

To use the IFT, we need that $\frac{\partial}{\partial u_i} \mathcal{L}_i^{\updownarrow}(u) \neq 0$. The partial derivative of $\mathcal{L}_i^{\updownarrow}$ with respect to u_i equals to

$$\frac{\partial}{\partial u_i} \mathcal{L}_i^{\updownarrow}(u) = \frac{1}{B(u)^2} \left(B(u) \frac{\partial}{\partial u_i} A(u) - A(u) \frac{\partial}{\partial u_i} B(u) \right). \tag{6.14}$$

It would be enough if we calculate partial derivative up to the $\mathcal{O}(\lambda)$. Since $\lambda_i^{\updownarrow} \asymp \lambda$, it holds that

$$B(u) = 1 + \mathcal{O}(\lambda) \quad \text{and} \quad \frac{\partial}{\partial u_i} B(u) \asymp \lambda^{4/3}.$$

Further,

$$\tilde{f}_i' \left(\Theta + \lambda^{1/3} u\right) = \lambda \tilde{g}_i'(u) = \mathcal{O}(\lambda) \text{ and } \tilde{f}_{ij}'' \left(\Theta + \lambda^{1/3} u\right) = \lambda^{2/3} \tilde{g}_{ij}''(u).$$

Thus, $A(u) = \mathcal{O}(\lambda)$ and

$$\begin{aligned} \frac{\partial}{\partial u_i} A(u) &= \pm \lambda^{1/3} \tilde{f}_{ii}'' \left(\Theta + \lambda^{1/3} u\right) + \lambda_i^{\updownarrow} \lambda^{1/3} \tilde{f}_i' \left(\Theta + \lambda^{1/3} u\right) \\ &\quad + \lambda_i^{\updownarrow} \lambda^{1/3} \left(\Theta_i + \lambda^{1/3} u_i\right) \tilde{f}_{ii}'' \left(\Theta + \lambda^{1/3} u\right) \\ &\quad \pm \lambda_i^{\updownarrow} \lambda^{1/3} \sum_{j \neq i} \tilde{f}_{ij}'' \left(\Theta + \lambda^{1/3} u\right) \left(\Theta_j + \lambda^{1/3} u_j\right), \end{aligned}$$

which behaves same as $\lambda \tilde{g}_{ii}''(u) + \mathcal{O}(\lambda) > 0$. We can conclude that

$$\frac{\partial}{\partial u_i} \mathcal{L}_i^{\updownarrow}(u) = \lambda \left(\tilde{g}_{ii}''(u) + \mathcal{O}(\lambda)\right) > 0, \tag{6.15}$$

for every $u \in (0, \bar{u})$, resp. $u \in (-\bar{u}, 0)$, and λ small enough. According to the IFT, $u_i^{\updownarrow}$ are locally smooth functions.

The proof above states that λ is dependent of u, but we need more than this. Let's fix all the components of $u^{\updownarrow}$, but i-th one. We are going to show that

$$u_i^{\updownarrow} \to \mp \bar{u}_i \quad \text{locally uniformly as } \lambda \to 0^+.$$

Calculations are carried out for every $i = 1, \cdots, n$. Function g is separable, i.e., there exist functions $\hat{g}_l$, $l = 1, \cdots, n$ such that $g(u) = \sum_{l=1}^{n} \hat{g}_l(u_l)$, and $\hat{g}_l'$ are continuous increasing with $|\hat{g}_l'| \le \vartheta_l$. Therefore, we will demonstrate one-dimensional case. The multi-dimensional case follows analogously.

Let $K > 0$ and

$$\tilde{g}(u) \triangleq (1 + K\lambda) g(u), \quad \text{for } u \in \mathbb{R}.$$

Let $\tilde{u} \in (0, \bar{u})$ be such that

$$g'(\tilde{u}) = \frac{\vartheta}{1 - \vartheta\lambda(\Theta + \lambda^{1/3}\tilde{u})}, \tag{6.16}$$

which is equivalent to

$$\tilde{f}'(x_{\tilde{u}}) = \kappa^{\downarrow}(x_{\tilde{u}}). \tag{6.17}$$

The first step is to show that $\tilde{u} \to \bar{u}$ as $\lambda \to 0^+$. Since g' is a third grade polynomial increasing function on $(-\bar{u}, \bar{u})$, there exists continuous increasing function $(g')^{-1}$ such that

$$\tilde{u} = (g')^{-1}\left(\frac{\vartheta}{1 - \vartheta\lambda(\Theta + \lambda^{1/3}\tilde{u})} \cdot \frac{1}{1 + K\lambda}\right) \xrightarrow{\lambda \to 0^+} (g')^{-1}(\vartheta) = \bar{u}.$$

The next step is to prove that $\bar{u} - \tilde{u}$ is of order $\lambda^{1/2}$. It follows from the definition of $g(u)$ that there exist constants $B, D > 0$ such that

$$g(u) = \frac{B}{2}u^2 - \frac{D}{24}u^4$$

on the compact set U. Thus

$$\begin{aligned} g''(\bar{u}) &= B - \frac{D}{2}\bar{u}^2, \\ g''(u) &= B(\bar{u}^2 - u^2) \implies g'(u) = B(u\bar{u}^2 - \frac{u^3}{3}) \end{aligned}$$

Let z be $u = \bar{u} - z \to 0$ as $\lambda \to 0^+$. Then

$$\begin{aligned} g''(u) &= Bz(2\bar{u} - z), \\ g'(u) &= B(\bar{u} - z)\left(\bar{u}^2 - \frac{(\bar{u} - z)^2}{3}\right). \end{aligned}$$

It holds that

$$\begin{aligned} g'(\bar{u}) - g'(\tilde{u}) &= \vartheta - g'(\bar{u}) \\ &= [1 - \vartheta\lambda(\Theta + \lambda^{1/3}\tilde{u})]\tilde{g}'(\tilde{u}) - g'(\bar{u}) \\ &= \lambda[-\vartheta(\Theta + \lambda^{1/3}\tilde{u}) + K - \vartheta K\lambda(\Theta + \lambda^{1/3}\tilde{u})]g'(\tilde{u}). \end{aligned} \tag{6.18}$$

On the other hand,

$$g'(\bar{u}) - g'(\tilde{u}) = \int_{\tilde{u}}^{\bar{u}} g''(u)\,\mathrm{d}u = B(\bar{u}-\tilde{u})^2 \frac{2\bar{u}+\tilde{u}}{3}. \tag{6.19}$$

From (6.18) and (6.19) we obtain that $\bar{u}-\tilde{u}$ has the same order as

$$\sqrt{\frac{3}{B}\lambda\left[-\vartheta(\Theta+\lambda^{1/3}\tilde{u}) + K - \vartheta K\lambda(\Theta+\lambda^{1/3}\tilde{u})\right] g'(\tilde{u})},$$

which actually is $\lambda^{1/2}$. Moreover,

$$\tilde{g}''(\tilde{u}) = (1+K\lambda)B(\bar{u}^2-u^2) \asymp \lambda^{1/2}. \tag{6.20}$$

We use similar method to prove that $\tilde{u} \to -\bar{u}$ as $\lambda \to 0^+$ for $\tilde{u} \in (-\bar{u}, 0)$ defined as

$$g'(\tilde{u}) = \frac{-\vartheta}{1-\vartheta\lambda(\Theta+\lambda^{1/3}\tilde{u})}.$$

As mentioned above, these results can be generalized to multi-dimensional space. So far, we obtained that $u_i^\updownarrow \to \mp\bar{u}_i$ as $\lambda \to 0^+$ and $|u_i^\updownarrow - \mp\bar{u}_i|$ is of order $\lambda^{1/2}$, for every $i = 1, \cdots, n$.

Denote

$$u_{\langle i\rangle}^\updownarrow \triangleq \left(u_i^\updownarrow, u\right)_{\langle i\rangle}.$$

Using the definitions (4.1) and (4.4) of $\kappa_i^\updownarrow$ and $K^\updownarrow$, respectively, we obtain

$$\begin{aligned} 0 &= \mathcal{L}_i^\updownarrow u_{\langle i\rangle}^\updownarrow \\ &= \kappa_i^\updownarrow\left(\Theta_i+\lambda^{1/3}u_i^\updownarrow\right) \pm \nabla\tilde{f}\left(\Theta+\lambda^{1/3}u_{\langle i\rangle}^\updownarrow\right)^{\mathsf{T}} K^\updownarrow\left(\Theta+\lambda^{1/3}u_{\langle i\rangle}^\updownarrow\right)e_i \\ &= \frac{\lambda_i^\updownarrow}{1\pm\lambda_i^\updownarrow\left(\Theta_i+\lambda^{1/3}u_i^\updownarrow\right)} \pm \\ &\pm\left[\tilde{f}_i'\left(\Theta+\lambda^{1/3}u_{\langle i\rangle}^\updownarrow\right) \pm \sum_{j\neq i}\tilde{f}_j'\left(\Theta+\lambda^{1/3}u_{\langle i\rangle}^\updownarrow\right)\frac{\lambda_i^\updownarrow\left(\Theta_j+\lambda^{1/3}u_j\right)}{1\pm\lambda_i^\updownarrow\left(\Theta_i+\lambda^{1/3}u_i^\updownarrow\right)}\right]. \end{aligned} \tag{6.21}$$

Since $\nabla\tilde{f}\left(\Theta+\lambda^{1/3}u\right) = \lambda\nabla\tilde{g}(u)$, we can divide both sides with λ to obtain

$$\begin{aligned} 0 = &\frac{\vartheta_i^\updownarrow}{1\pm\lambda_i^\updownarrow\left(\Theta_i+\lambda^{1/3}u_i^\updownarrow\right)} \pm \\ &\pm\left[\tilde{g}_i'\left(u_{\langle i\rangle}^\updownarrow\right) \pm \sum_{j\neq i}\tilde{g}_j'\left(u_{\langle i\rangle}^\updownarrow\right)\frac{\lambda\vartheta_i^\updownarrow\left(\Theta_j+\lambda^{1/3}u_j\right)}{1\pm\lambda\vartheta_i^\updownarrow\left(\Theta_i+\lambda^{1/3}u_i^\updownarrow\right)}\right], \end{aligned}$$

where $\tilde{g}_l' = (1 + K\lambda) g_l'$ for every $l = 1, \cdots, n$. Therefore,

$$\begin{aligned} g_i'\left(u_{\langle i\rangle}^{\updownarrow}\right) = \frac{1}{1+K\lambda}\Bigg[&\mp \frac{\vartheta_i^{\updownarrow}}{1 \pm \lambda_i^{\updownarrow}\left(\Theta_i + \lambda^{1/3} u_i^{\updownarrow}\right)} - \\ &\mp \sum_{j\neq i} \tilde{g}_j'\left(u_{\langle i\rangle}^{\updownarrow}\right) \frac{\lambda \vartheta_i^{\updownarrow}\left(\Theta_j + \lambda^{1/3} u_j\right)}{1 \pm \lambda \vartheta_i^{\updownarrow}\left(\Theta_i + \lambda^{1/3} u_i^{\updownarrow}\right)} \Bigg]. \end{aligned} \tag{6.22}$$

Function g is separable, as well, i.e., there exist functions $\hat{g}_l$, $l = 1, \cdots, n$ such that $g(u) = \sum_{l=1}^n \hat{g}_l(u_l)$, and $\hat{g}_l'$ are continuous increasing with $|\hat{g}_l'| \leq \vartheta_l$. Thus, for every $l = 1, \cdots, n$ there exists limit point $u_{\langle i\rangle}^{\updownarrow\infty}$ such that

$$g_i'\left(u_{\langle i\rangle}^{\updownarrow}\right) \to g_i'\left(u_{\langle i\rangle}^{\updownarrow\infty}\right), \quad \text{as } \lambda \to 0^+.$$

Denote R the right-hand side of (6.22). Since the sum part in R is of the order λ, then for $\lambda \to 0^+$ we obtain

$$g_i'\left(u_{\langle i\rangle}^{\updownarrow\infty}\right) = \mp \vartheta_i^{\updownarrow}.$$

As shown in one-dimensional case, there exist $(\hat{g}_l')^{-1}$ such that

$$u_{\langle i\rangle}^{\updownarrow} = (\hat{g}_l')^{-1}(R).$$

Therefore, for $\lambda \to 0^+$ we obtain

$$u_{\langle i\rangle}^{\updownarrow} \to (\hat{g}_l')^{-1}\left(\mp \vartheta_i^{\updownarrow}\right) = \mp \bar{u}_i,$$

which gives us desired result.

From (6.21) we can acquire even more. Multiplying both sides with $1 \pm \lambda_i^{\updownarrow}\left(\Theta_i + \lambda^{1/3} u_i^{\updownarrow}\right)$ we obtain

$$\begin{aligned} 0 &= \pm\lambda_i^{\updownarrow} + \left[1 \pm \lambda_i^{\updownarrow}\left(\Theta_i + \lambda^{1/3} u_i^{\updownarrow}\right)\right] \tilde{f}_i'\left(\Theta + \lambda^{1/3} u_{\langle i\rangle}^{\updownarrow}\right) \\ &\quad \pm \lambda_i^{\updownarrow} \sum_{j\neq i} \tilde{f}_j'\left(\Theta + \lambda^{1/3} u_{\langle i\rangle}^{\updownarrow}\right)\left(\Theta_j + \lambda^{1/3} u_j\right) \\ &= \tilde{f}_i'\left(\Theta + \lambda^{1/3} u_{\langle i\rangle}^{\updownarrow}\right) \pm \lambda_i^{\updownarrow}\left[1 + \sum_{j=1}^n \tilde{f}_j'\left(\Theta + \lambda^{1/3} u_{\langle i\rangle}^{\updownarrow}\right)\left(\Theta + \lambda^{1/3} u_{\langle i\rangle}\right)_j\right] \\ &= \lambda \tilde{g}_i'\left(\Theta + \lambda^{1/3} u_{\langle i\rangle}^{\updownarrow}\right) \pm \lambda_i^{\updownarrow}\left[1 + \lambda \nabla \tilde{g}\left(\Theta + \lambda^{1/3} u_{\langle i\rangle}^{\updownarrow}\right)^{\mathsf{T}}\left(\Theta + \lambda^{1/3} u_{\langle i\rangle}\right)\right]. \end{aligned}$$

Since $\lambda_i^{\updownarrow} = \vartheta_i^{\updownarrow}\lambda$, we have

$$0 = \tilde{g}_i'\left(u_{\langle i\rangle}^{\updownarrow}\right) \pm \vartheta_i^{\updownarrow}\left[1 + \lambda\,\nabla\tilde{g}\left(u_{\langle i\rangle}^{\updownarrow}\right)^{\intercal}\left(\Theta + \lambda^{1/3}u_{\langle i\rangle}\right)\right]. \tag{6.23}$$

Hence,

$$0 = g_i'\left(u_{\langle i\rangle}^{\updownarrow}\right) \pm \vartheta_i^{\updownarrow}\left[\frac{1}{1+K\lambda} + \lambda\,\nabla g\left(u_{\langle i\rangle}^{\updownarrow}\right)^{\intercal}\left(\Theta + \lambda^{1/3}u_{\langle i\rangle}\right)\right], \tag{6.24}$$

where $g_i'\left(u_{\langle i\rangle}^{\updownarrow}\right) = \hat{g}_i'\left(u_i^{\updownarrow}\right)$. Denote R the right-hand side of the equation (6.24). We derive R with respect to $u_i^{\updownarrow}$ and u_j, respectively. We obtain

$$\begin{aligned}
\frac{\partial R}{\partial u_i^{\updownarrow}} &= \hat{g}_{ii}''\left(u_i^{\updownarrow}\right) \pm \lambda_i^{\updownarrow}\left[\lambda^{1/3}\hat{g}_i'\left(u_i^{\updownarrow}\right) + e_i^{\intercal}\nabla^2 g\left(u_{\langle i\rangle}^{\updownarrow}\right)\left(\Theta + \lambda^{1/3}u_{\langle i\rangle}\right)\right],\\
\frac{\partial R}{\partial u_j} &= \pm\lambda_i^{\updownarrow}\left[\lambda^{1/3}\hat{g}_j'\left(u_j\right) + e_j^{\intercal}\nabla^2 g\left(u_{\langle i\rangle}^{\updownarrow}\right)\left(\Theta + \lambda^{1/3}u_{\langle i\rangle}\right)\right].
\end{aligned}$$

Matrix $\nabla^2 g$ is diagonal, so last two equations reduce to

$$\begin{aligned}
\frac{\partial R}{\partial u_i^{\updownarrow}} &= \hat{g}_{ii}''\left(u_i^{\updownarrow}\right) \pm \lambda_i^{\updownarrow}\left[\lambda^{1/3}\hat{g}_i'\left(u_i^{\updownarrow}\right) + \hat{g}_{ii}''\left(u_i^{\updownarrow}\right)\left(\Theta_i + \lambda^{1/3}u_i^{\updownarrow}\right)\right],\\
\frac{\partial R}{\partial u_j} &= \pm\lambda_i^{\updownarrow}\left[\lambda^{1/3}\hat{g}_j'\left(u_j\right) + \hat{g}_{jj}''\left(u_j^{\updownarrow}\right)\left(\Theta_j + \lambda^{1/3}u_j^{\updownarrow}\right)\right].
\end{aligned}$$

Since $\hat{g}_{ii}''\left(u_i^{\updownarrow}\right) \asymp \lambda^{1/2}$ and $\lambda_i^{\updownarrow} \asymp \lambda$, then $\frac{\partial R}{\partial u_i^{\updownarrow}} \neq 0$. Thus, we can apply the IFT on R and we obtain

$$\begin{aligned}
\frac{\partial u_i^{\updownarrow}}{\partial u_j} &= -\frac{\partial R/\partial u_j}{\partial R/\partial u_i^{\updownarrow}}\\
&= -\frac{\pm\lambda_i^{\updownarrow}\left[\lambda^{1/3}\hat{g}_j'\left(u_j\right) + \hat{g}_{jj}''\left(u_j^{\updownarrow}\right)\left(\Theta_j + \lambda^{1/3}u_j^{\updownarrow}\right)\right]}{\hat{g}_{ii}''\left(u_i^{\updownarrow}\right) \pm \lambda_i^{\updownarrow}\left[\lambda^{1/3}\hat{g}_i'\left(u_i^{\updownarrow}\right) + \hat{g}_{ii}''\left(u_i^{\updownarrow}\right)\left(\Theta_i + \lambda^{1/3}u_i^{\updownarrow}\right)\right]}\\
&\to 0, \quad \text{as } \lambda \to 0^+,
\end{aligned}$$

because numerator is weakly equivalent to λ and denominator is weakly equivalent to $\lambda^{1/2}$.

6.2 f beyond no-trade region

We have defined function

$$\tilde{f}(x) = h(\lambda)f(x) = h(\lambda)\lambda^{4/3}g\left(\frac{x_u - \Theta}{\lambda^{1/3}}\right),$$

for $x \in \mathcal{A}$, where $\mathcal{A}$ is defined in the section (3.2).

$$x \in [-\bar{x}, \bar{x}] = [\Theta - \lambda^{1/3}\bar{u}, \Theta + \lambda^{1/3}\bar{u}].$$

In this section we are going to concentrate on extending $\tilde{f}$ on the space $\mathcal{A}$ with no harm to initial properties of $\tilde{f}$. Denote new extended function $\bar{f}$. We will use "divide and conquer" technique, in sense that we divide space $\mathcal{A}$ on a few areas and define $\bar{f}$ on every of them separately. Denote $A_i^{\updownarrow} \subset \mathcal{A}$ the set of such positions of an investor that trading solely with i-th stock can improve his position and move him to no-trade region.

Our approach will be following: If an investor is in no-trade region, he does not trade. If an investor is in $A_i^{\updownarrow}$ region, then he trades with i-th stock and moves to no-trade region. If an investor is in $\mathcal{A}\backslash(\overline{NT} \cup \bigcup_{i=1}^{n} A_i^{\updownarrow})$ region, then he trades with stocks until he reaches $A_i^{\updownarrow}$ for some $i = 1, \cdots, n$.

As a guide for how to define $\bar{f}$, we will use two-dimensional case. We have that when trading with i-th stock, values $\mathcal{W}_t(1+\lambda_i\pi_i(t))$ and $\mathcal{W}_t\pi_j(t)$ do not change. Denote π and $\bar{\pi}$, respectively $\mathcal{W}$ and $\overline{\mathcal{W}}$, the positions, respectively wealth, of an investor before and after trade with i-th stock. Thus,

$$\frac{1+\lambda_i\pi_i(t)}{\pi_j(t)} = \frac{1+\lambda_i\bar{\pi}_i(t)}{\bar{\pi}_j(t)}.$$

Therefore,

$$\bar{\pi}_j(t) = \pi_j(t)\frac{1+\lambda_i\bar{\pi}_i(t)}{1+\lambda_i\pi_i(t)}. \tag{6.25}$$

We want to find such function f, that $\log(\mathcal{W}_t) - f(\pi(t)) - \nu t$ is a continuous process and

$$\log(\mathcal{W}_t) - f(\pi(t)) = \log(\overline{\mathcal{W}}_t) - f(\bar{\pi}(t)).$$

It follows from $\mathcal{W}_t(1+\lambda_i\pi_i(t)) = \overline{\mathcal{W}}_t(1+\lambda_i\bar{\pi}_i(t))$, that

$$\log\frac{\mathcal{W}_t}{\overline{\mathcal{W}}_t} = \log\frac{1+\lambda_i\bar{\pi}_i(t)}{1+\lambda_i\pi_i(t)}.$$

Therefore, the new function $\bar{f}$ has to satisfy

$$\bar{f}(\pi(t)) = \bar{f}(\bar{\pi}(t)) + \log\frac{1+\lambda_i\bar{\pi}_i(t)}{1+\lambda_i\pi_i(t)}, \tag{6.26}$$

where (6.25) holds. Our aim is to move to no-trade region where $\bar{f} = \tilde{f}$, i.e., we want that $\bar{f}(\bar{\pi}(t)) = \tilde{f}(\bar{\pi}(t))$. Hence, for $x \in \mathcal{A}$ we define $\bar{f}$ as

$$\bar{f}(x) \triangleq \tilde{f}(\bar{x}) + \log\frac{1+\lambda_i\bar{x}}{1+\lambda_i x}. \tag{6.27}$$

Since $\mathcal{L}_i^{\updownarrow}$ describes directional derivative of the function $\tilde{f}$, we require that $\bar{\mathcal{L}}_i^{\updownarrow} = 0$ with respect to corresponding direction, where $\bar{\mathcal{L}}_i^{\updownarrow}$ represents $\mathcal{L}_i^{\updownarrow}$ with respect to the function $\bar{f}$. The first step is to determine the direction along which we have to perform directional derivative of $\bar{f}$. From (6.25) we obtain

$$\bar{\pi}_j(t) - \pi_j(t) = \pi_j(t)\left[\frac{1+\lambda_i\bar{\pi}_i(t)}{1+\lambda_i\pi_i(t)} - 1\right] = \pi_j(t)\frac{\lambda_i(\bar{\pi}_i(t)-\pi_i(t))}{1+\lambda_i\pi_i(t)}.$$

Thus,

$$\mathrm{d}_i^{\updownarrow}\pi_j(t) = \pi_j(t)\frac{\lambda_i}{1+\lambda_i\pi_i(t)}\,\mathrm{d}_i^{\updownarrow}\pi_i(t) = \pi_j(t)\kappa_i^{\updownarrow}\left(\pi_i(t)\right)\,\mathrm{d}_i^{\updownarrow}\pi_i(t).$$

Analogously to the definition (6.10) of $\mathcal{L}_i^{\updownarrow}(u)$, for $x \in \mathbb{R}^n$ we define

$$\mathbb{L}_i^{\updownarrow}(x) \triangleq \kappa_i^{\updownarrow}(x_i) \pm \nabla\tilde{f}(x)^{\intercal} K^{\updownarrow}(x)e_i. \tag{6.28}$$

From the derivation of $\mathcal{L}_i^{\updownarrow}(u)$ with respect to u_i, we have

$$0 = \tilde{f}_i(x) + \lambda_i\left[1 + \nabla\tilde{f}(x)^{\intercal}x\right]. \tag{6.29}$$

Hence, the derivation of $\bar{f}$ depends only on i-th component. Lemma 10.2. in [2] ensures that $\bar{f} \in C^1$. We need

$$\mathrm{d}_i^{\updownarrow}\mathbb{L}_i^{\updownarrow}(\pi(t)) = 0$$

which follows from $\nabla\bar{f}(x)^{\intercal}\cdot\bar{h}_x = \lambda_i$, where $\bar{h}_x = e_i + \lambda_i x$

Put

$$TZ \triangleq \overline{NT} \cup \bigcup_{i=1}^{n} A_i^{\updownarrow}$$

. Denote

$$\alpha_i^{\updownarrow}(x) \triangleq \inf\left\{\alpha \geq 0 :\ \left(x_i \pm \alpha, x\left(1+\alpha\kappa_i^{\updownarrow}(x_i)\right)\right)_{\langle i\rangle} \in TZ\right\}$$

and

$$x^{\updownarrow i} \triangleq \left(x_i \pm \alpha_i^{\updownarrow}(x), x\left(1+\alpha_i^{\updownarrow}(x)\kappa_i^{\updownarrow}(x_i)\right)\right)_{\langle i\rangle} \in TZ.$$

We define the function $\bar{f}$ as

$$\begin{aligned} \bar{f}(x) &\triangleq \tilde{f}(x), \quad \text{for } x \in \overline{NT}, \\ \bar{f}(x) &\triangleq \tilde{f}(\bar{x}) + \log\frac{1+\sum_{i=1}^{n}\lambda_i x_i^{\updownarrow i}}{1+\sum_{i=1}^{n}\lambda_i x_i}, \quad \text{otherwise,} \end{aligned} \tag{6.30}$$

where $\lambda_i \in \{\lambda_i^{\uparrow}, -\lambda_i^{\downarrow}\}$. We assume that $\alpha_i^{\updownarrow}(x)$ is smooth function. Thus, $\bar{f}$ is also smooth. We are going to prove main properties $\bar{f}$ must satisfy. One of them concerns directional derivative of $\bar{f}$ along the direction $\bar{h}_x = e_i + \lambda_i x$. Since we are moving from x to $x^{\updownarrow i}$, the main direction is

$$\begin{aligned}\left(\pm\alpha, \alpha x\kappa_i^{\updownarrow}(x_i)\right)_{\langle i\rangle} \sim \left(1, \pm x\kappa_i^{\updownarrow}(x_i)\right)_{\langle i\rangle} &= \left(1, \frac{\lambda_i x}{1+\lambda_i x_i}\right)_{\langle i\rangle} \\ &\sim (1+\lambda_i x_i, \lambda_i x)_{\langle i\rangle} = e_i + \lambda_i x.\end{aligned} \tag{6.31}$$

Hence, $\overline{x + \delta(e_i + \lambda_i x)} = \bar{x}$. Therefore,

$$\begin{aligned}&\frac{\bar{f}(x+\delta(e_i+\lambda_i x)) - \bar{f}(x)}{\delta} = \\ &= \frac{1}{\delta}\left[\log\frac{1+\lambda_i x_i^{\updownarrow i}}{1+\lambda_i(x_i + \delta(1+\lambda_i x_i))} - \log\frac{1+\lambda_i x_i^{\updownarrow i}}{1+\lambda_i x}\right] \\ &= -\frac{1}{\delta}\log\frac{1+\lambda_i(x_i+\delta(1+\lambda_i x_i))}{1+\lambda_i x_i} \\ &= -\frac{1}{\delta}\log\left(1 + \frac{\delta(1+\lambda_i x_i)}{1+\lambda_i x_i}\lambda_i\right) \\ &\to -\lambda_i, \quad \text{as } \delta \to 0^+.\end{aligned} \tag{6.32}$$

On the other hand

$$\frac{\bar{f}(x+\delta(e_i+\lambda_i x)) - \bar{f}(x)}{\delta} = \nabla\bar{f}(x)^{\intercal}(e_i + \lambda x) = \lambda_i \nabla\bar{f}(x)^{\intercal} x + \bar{f}_i'(x). \tag{6.33}$$

From the definition (6.28) of $\mathbb{L}_i^{\updownarrow}$ with respect to $\bar{f}$ it follows that

$$\bar{\mathbb{L}}_i^{\updownarrow}(x) = \frac{\nabla\bar{f}(x)^{\intercal}(e_i + \lambda_i x_i) + \lambda_i}{1+\lambda_i x_i}. \tag{6.34}$$

Therefore, $\bar{\mathbb{L}}_i^{\updownarrow}(x) = 0$.

We can now generalize these results on multi-dimensional case. The function $\bar{f}$ satisfies

$$\bar{f}(\pi(t)) = \tilde{f}(\bar{\pi}(t)) + \log\frac{1+\sum_{i=1}^n \lambda_i \bar{\pi}_i(t)}{1+\sum_{i=1}^n \lambda_i \pi_i(t)}, \tag{6.35}$$

where

$$\lambda_i = \begin{cases}\lambda_i^{\uparrow}, & \text{if we buy } i\text{-th stock} \\ \lambda_i^{\downarrow}, & \text{if we sell } i\text{-th stock} \\ 0, & \text{if we do not trade with } i\text{-th stock.}\end{cases}$$

If we trade with i-th stock, then

$$
\begin{aligned}
&\frac{\bar{f}(x+\delta(e_i+\lambda_i x))-\bar{f}(x)}{\delta} = \\
&= \frac{1}{\delta}\left[\log\frac{1+\sum_{i=1}^{n}\lambda_i x_i^{\updownarrow i}}{1+\lambda_i(x_i+\delta(1+\lambda_i x_i))+\sum_{i\neq j}\lambda_j(x_j+\delta\lambda_i x_j)}\right. \\
&\left.-\log\frac{1+\sum_{i=1}^{n}\lambda_i x_i^{\updownarrow i}}{1+\sum_{i=1}^{n}\lambda_i x}\right] \\
&= -\frac{1}{\delta}\log\left(1+\delta\lambda_i\right) \\
&\to -\lambda_i, \quad \text{as } \delta\to 0^{+}.
\end{aligned}
\tag{6.36}
$$

It follows that

$$
\bar{\mathbb{L}}_i^{\updownarrow}(x) = \frac{\nabla\bar{f}(x)^{\intercal}(e_i+\lambda_i x_i)+\lambda_i}{1+\lambda_i x_i} = 0. \tag{6.37}
$$

Chapter 7

Gradients of $\bar{f}$

Since we are going to deal with partial derivatives of compound functions, the main calculating tool used in this chapter will be chain rule principle.

Denote $\mathfrak{O} \subset \{1, \cdots, n\}$ the set of the stocks we are trading with at the moment. Let $\lambda \in \mathbb{R}^n$ be such vector, that $\lambda_i \in \{\lambda_i^{\uparrow}, -\lambda_i^{\downarrow}\}$ for $i \in \mathfrak{O}$ and $\lambda_i = 0$ otherwise. We define transformation of the coordinates

$$y \triangleq \frac{x}{1 + \lambda^{\intercal} x}. \tag{7.1}$$

It holds that

$$(1 + \lambda^{\intercal} x)(1 - \lambda^{\intercal} y) = (1 + \lambda^{\intercal} x)\left(1 - \frac{\lambda^{\intercal} x}{1 + \lambda^{\intercal} x}\right) = 1.$$

Therefrom we obtain the inverse transformation

$$x = y(1 + \lambda^{\intercal} x) = \frac{y}{1 - \lambda^{\intercal} y}. \tag{7.2}$$

We then obtain

$$\frac{\partial x_j}{\partial y_i} = \frac{1_{[i=j]}}{1 - \lambda^{\intercal} y} + \frac{\lambda_j y_i}{(1 - \lambda^{\intercal} y)^2}, \tag{7.3}$$

which implies

$$\nabla_y x_i = \frac{e_i}{1 - \lambda^{\intercal} y} + \frac{\lambda y_i}{(1 - \lambda^{\intercal} y)^2}. \tag{7.4}$$

Hence,

$$\begin{aligned} \frac{\partial x}{\partial y}(y) &= \left(\frac{\partial x_j}{\partial y_i}\right)_{i,j}(y) = (\nabla_y x_1(y), \cdots, \nabla_y x_n(y)) \\ &= \frac{(e_1, \cdots, e_n)}{1 - \lambda^{\intercal} y} + \frac{\lambda(y_1, \cdots, y_n)}{(1 - \lambda^{\intercal} y)^2} \\ &= \frac{\mathbb{I}_n}{1 - \lambda^{\intercal} y} + \frac{\lambda y^{\intercal}}{(1 - \lambda^{\intercal} y)^2}. \end{aligned} \tag{7.5}$$

Analogously we obtain

$$\frac{\partial y_j}{\partial x_i} = \frac{1_{[i=j]}}{1+\lambda^\intercal x} + \frac{\lambda_j x_i}{(1+\lambda^\intercal x)^2}, \tag{7.6}$$

as well as

$$\begin{aligned}\frac{\partial y}{\partial x}(x) &= (\nabla_x y_1(x), \cdots, \nabla_x y_n(x)) \\ &= \frac{(e_1, \cdots, e_n)}{1+\lambda^\intercal x} + \frac{\lambda(x_1, \cdots, x_n)}{(1+\lambda^\intercal x)^2} \\ &= \frac{\mathbb{I}_n}{1+\lambda^\intercal x} - \frac{\lambda x^\intercal}{(1+\lambda^\intercal x)^2}.\end{aligned} \tag{7.7}$$

Since we represented x as a function of y, i.e., $x = x(y)$, we define function

$$\begin{aligned}\bar{F}(y) &\triangleq \bar{f}\left(x(y)\right) \\ &= \tilde{f}\left(x(\bar{y})\right) + \log\frac{1+\lambda^\intercal x(\bar{y})}{1+\lambda^\intercal x(y)} \\ &= \tilde{F}(\bar{y}) - \log\frac{1-\lambda^\intercal \bar{y}}{1-\lambda^\intercal y}.\end{aligned} \tag{7.8}$$

It holds that

$$\frac{\partial}{\partial y_i}\bar{F}(y) = \frac{\partial}{\partial y_i}\log(1-\lambda^\intercal y) = -\frac{\lambda_i}{1-\lambda^\intercal y}. \tag{7.9}$$

Thus,

$$-\lambda_i = (1-\lambda^\intercal y)\frac{\partial}{\partial y_i}\bar{F}(y). \tag{7.10}$$

We are now going to calculate partial derivatives of the function $\bar{F}\left(y(x)\right)$ with respect to y_i. We have

$$\begin{aligned}\frac{\partial \bar{F}}{\partial y_i}(y) &= \frac{\partial}{\partial y_i}\bar{f}\left(x(y)\right) = e_i^\intercal \frac{\partial x}{\partial y}\, \nabla_x \bar{f}\left(x(y)\right) \\ &= e_i^\intercal\left[\frac{\mathbb{I}_n}{1-\lambda^\intercal y} + \frac{\lambda y^\intercal}{(1-\lambda^\intercal y)^2}\right]\nabla_x \bar{f}\left(x(y)\right).\end{aligned} \tag{7.11}$$

We then obtain

$$\begin{aligned}\frac{\partial \bar{F}}{\partial y_i}\left(y(x)\right) &= (1+\lambda^\intercal x)e_i^\intercal\left[\mathbb{I}_n + \lambda x^\intercal\right]\nabla_x \bar{f}\left(x(y)\right) \\ &= (1+\lambda^\intercal x)\left[e_i^\intercal + \lambda_i x^\intercal\right]\, \nabla_x \bar{f}\left(x(y)\right).\end{aligned} \tag{7.12}$$

In the points $y = y(x)$ the equation (7.10) becomes

$$\begin{aligned} -\lambda_i &= (1 - \lambda^\intercal y(x)) \frac{\partial}{\partial y_i} \bar{F}\left(y(x)\right) \\ &= \left[e_i^\intercal + \lambda_i x^\intercal\right] \nabla_x \bar{f}\left(x(y)\right). \end{aligned} \tag{7.13}$$

Let's now calculate the second partial derivatives of the function $x(y)$ and $y(x)$. It holds that

$$\frac{\partial^2 x_i}{\partial y_j \partial y_k} = \frac{1_{[i=j]\lambda k}}{(1 - \lambda^\intercal y)^2} + \frac{1_{[i=k]\lambda j}}{(1 - \lambda^\intercal y)^2} + \frac{2\lambda_j \lambda k y_i}{(1 - \lambda^\intercal y)^3} = \mathcal{O}(\lambda) \tag{7.14}$$

and

$$\frac{\partial^2 x_i}{\partial y_j \partial y_k} = -\frac{1_{[i=j]\lambda k}}{(1 + \lambda^\intercal x)^2} - \frac{1_{[i=k]\lambda j}}{(1 + \lambda^\intercal x)^2} + \frac{2\lambda_j \lambda k x_i}{(1 + \lambda^\intercal x)^3} = \mathcal{O}(\lambda). \tag{7.15}$$

We defined the function $\tilde{F}$ as

$$\tilde{F}(y) = \tilde{f}(x(y)). \tag{7.16}$$

We immediately obtain that

$$\nabla_y \tilde{F}(y) = \nabla_y \left[\tilde{f}\left(x(y)\right)\right] = \frac{\partial x}{\partial y}(y)\, \nabla_x \tilde{f}\left(x(y)\right) \tag{7.17}$$

Further,

$$\begin{aligned} \nabla_y^2 \tilde{F}(y) &= \frac{\partial x}{\partial y}(y)\, \nabla_x^2 \tilde{f}\left(x(y)\right)\, \frac{\partial x}{\partial y}(y)^\intercal + \sum_{k=1}^{n} \nabla_y^2 x_k(y)\, \frac{\partial \tilde{f}}{\partial x_k}\left(x(y)\right) \\ &= \left[\left(1 + \lambda^\intercal y\right) \mathbb{I}_n + \lambda y^\intercal\right]\, \nabla_x^2 \tilde{f}\left(x(y)\right)\, \left[\left(1 + \lambda^\intercal y\right) \mathbb{I}_n + \lambda y^\intercal\right] \\ &+ \sum_{k=1}^{n} \left(e_k \lambda^\intercal + \lambda e_k^\intercal\right) \frac{\partial \tilde{f}}{\partial x_k}\left(x(y)\right), \end{aligned} \tag{7.18}$$

where we used results (7.3) and (**??**).

One of the properties of $\tilde{F}$ is that for $i \in \mathfrak{O}$

$$\frac{\partial}{\partial y_i} \tilde{F}(\bar{y}) = -\lambda_i. \tag{7.19}$$

Suppose $i \in \mathfrak{O}$. If we derive (7.19) with respect to y_j, $j = 1, \cdots, n$, we obtain

$$\begin{aligned} 0 &= \frac{\partial^2 \tilde{F}}{\partial y_i^2}(\bar{y})\, \frac{\partial \bar{y}_i}{\partial y_j}(y) + \frac{\partial^2 \tilde{F}}{\partial y_i \partial y_j}(\bar{y})\, \frac{\partial \bar{y}_j}{\partial y_j}(y) \\ &= \frac{\partial^2 \tilde{F}}{\partial y_i^2}(\bar{y})\, \frac{\partial \bar{y}_i}{\partial y_j}(y) + \frac{\partial^2 \tilde{F}}{\partial y_i \partial y_j}(\bar{y}), \end{aligned} \tag{7.20}$$

since from the definition of y_j we have that $\frac{\partial \bar{y}_j}{\partial y_j}(y) = 1$.

We will now concentrate on the computing of the second derivatives of function $\bar{f}$. It is enough to perform calculations up to some error depending solely on λ. Our aim is to show that $\nabla^2 \bar{f}$ is of order lower than λ.

We have that

$$\nabla_x \bar{f}(x) = \nabla_x \left[\bar{F}\left(y(x)\right)\right] = \frac{\partial y}{\partial x}(x)\, \nabla_y \bar{F}\left(y(x)\right) \tag{7.21}$$

and

$$\begin{aligned}\nabla_x^2 \bar{f}(x) &= \nabla_x^2 \left[\bar{F}\left(y(x)\right)\right] \\ &= \frac{\partial y}{\partial x}(x)\, \nabla_y^2 \bar{F}\left(y(x)\right)\, \frac{\partial y}{\partial x}(x)^{\intercal} + \sum_{k=1}^{d} \nabla_x^2 y_k(x)\, \frac{\partial \bar{F}}{\partial y_i}\left(y(x)\right).\end{aligned} \tag{7.22}$$

It holds that

$$\bar{f}(x) = \tilde{f}(\bar{x}) + \log \frac{1 + \lambda^{\intercal} \bar{x}}{1 + \lambda^{\intercal} x}, \tag{7.23}$$

with $\lambda^{\intercal} x = \sum_{i \in \mathfrak{D}} \lambda_i y_i$, i.e., $\lambda_i \in \{\lambda_i^{\uparrow}, -\lambda_i^{\downarrow}\}$ for $i \in \mathfrak{D}$ and $\lambda_i = 0$ for $i \notin \mathfrak{D}$. Therefore,

$$\bar{F}(y) = \tilde{F}(\bar{y}) + \log \frac{1 + \lambda^{\intercal} \bar{y}}{1 + \lambda^{\intercal} y}, \tag{7.24}$$

where $\lambda_i \in \{\lambda_i^{\uparrow}, -\lambda_i^{\downarrow}\}$ for $i \in \mathfrak{D}$ and $\lambda_i = 0$ otherwise. Equation (7.24) implies that

$$\nabla_y \bar{F}(y) = \nabla_y \left[\tilde{F}(\bar{y})\right] - \frac{\lambda}{1 + \lambda^{\intercal} y} = \frac{\partial \bar{y}}{\partial y}(y)\, \nabla_y \tilde{F}(\bar{y}) + \mathcal{O}(\lambda). \tag{7.25}$$

From (7.24) we also obtain

$$\begin{aligned}\nabla_y^2 \bar{F}(y) &= \nabla_y^2 \left[\tilde{F}(\bar{y})\right] + \frac{\lambda \lambda^{\intercal}}{(1 + \lambda^{\intercal} y)^2} \\ &= \frac{\partial \bar{y}}{\partial y}(y)\, \nabla_y^2 \tilde{F}(\bar{y})\, \frac{\partial \bar{y}}{\partial y}(y)^{\intercal} + \sum_{i=1}^{n} \nabla_y^2 \bar{y}_k(y)\, \frac{\partial \tilde{F}}{\partial y_k}(\bar{y}) + \mathcal{O}(\lambda^2)\end{aligned} \tag{7.26}$$

Let's have a look back at the function $\tilde{F}$ and its properties. From (7.17) and (7.18) we get

$$\begin{aligned}\nabla_y \tilde{F}(y) &= \mathcal{O}(\lambda), \\ \nabla_y^2 \tilde{F}(y) &= \mathcal{O}(\lambda^{5/3}) + \lambda^{2/3} \nabla g \left(\frac{x_y - \Theta}{\lambda^{1/3}}\right)\end{aligned} \tag{7.27}$$

since

$$\frac{\partial x}{\partial y}(y) = \mathbb{I}_n + \mathcal{O}(\lambda), \tag{7.28}$$

$$\nabla_y^2 x_k = \mathcal{O}(\lambda), \tag{7.29}$$

$$\nabla_x \tilde{f}\,(x(y)) = \lambda \nabla g\left(\frac{x_y - \Theta}{\lambda^{1/3}}\right) = \mathcal{O}(\lambda), \tag{7.30}$$

$$\nabla_x^2 \tilde{f}\,(x(y)) = \lambda^{2/3} \nabla^2 g\left(\frac{x_y - \Theta}{\lambda^{1/3}}\right). \tag{7.31}$$

Moreover,

$$\frac{\partial^2 \tilde{F}}{\partial y_i \partial y_j}(\bar{y}) = \mathcal{O}(\lambda^{5/3}) \tag{7.32}$$

and

$$\frac{\partial^2 \tilde{F}}{\partial y_i^2}(\bar{y}) = \mathcal{O}(\lambda^{5/3}) + \lambda^{2/3} g_{ii}^{''}\left(\frac{x_{\bar{y}} - \Theta}{\lambda^{1/3}}\right) \asymp \lambda^{7/6}, \tag{7.33}$$

since g is separable function and $g_{ii}^{''}(\frac{x_{\bar{y}}-\Theta}{\lambda^{1/3}}) \asymp \lambda^{1/2}$. Combining these results with (7.20) we can conclude that

$$\left|\frac{\partial \bar{y}_i}{\partial y_j}(y)\right| \asymp \mathcal{O}(\lambda^{1/2}). \tag{7.34}$$

Similarly, we are going to prove that

$$\frac{\partial^2 \bar{y}_i}{\partial y_j^2}(y) = \mathcal{O}(\lambda^{1/6}). \tag{7.35}$$

In this case we need to derive (7.20) with respect to y_j.

We now have all necessary elements for a final calculations. According to (7.25), $\nabla_y \bar{F}(y) = \mathcal{O}(\lambda)$ and therefore

$$\nabla_x \bar{f}(x) = \mathcal{O}(\lambda). \tag{7.36}$$

From () using all previous results we obtain

$$\nabla_x^2 \bar{f}(x) = \mathcal{O}(\lambda^{7/6}) + \lambda^{2/3} \nabla^2 g\left(\frac{\bar{x} - \Theta}{\lambda^{1/3}}\right). \tag{7.37}$$

Chapter 8

Final comparison of the strategies

In this chapter we are going to show that no other strategy is better than the one we have defined in this paper. As described in section 4.2, what we want to prove is

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathrm{CE}_\gamma(t) \le \hat{\nu} \le \lim_{t\to\infty} \frac{1}{t} \log \widehat{\mathrm{CE}}_\gamma(t) + \varepsilon,$$

with some error $\varepsilon > 0$. We defined certainty equivalent as

$$\begin{aligned} \mathrm{CE}_\gamma(t) &= \mathcal{U}_\gamma^{-1}\, \mathbb{E}\, \mathcal{U}_\gamma(\mathcal{W}_t) = \exp\left\{ \mathrm{e}_\gamma^{-1}\, \mathbb{E}\, \mathrm{e}_\gamma(\log \mathcal{W}_t) \right\} \\ &= \exp\left\{ \mathrm{e}_\gamma^{-1}\, \mathbb{E}\, \mathrm{e}_\gamma(U_\nu^f(t) + f(\pi(t)) + \nu t) \right\} \\ &= \exp\left\{ \mathrm{e}_\gamma^{-1}\, \mathbb{E}\, \mathrm{e}_\gamma(U_\nu^f(t) + \nu t + \mathcal{O}(\lambda)) \right\} \end{aligned} \tag{8.1}$$

since $f(\pi(t)) = \mathcal{O}(\lambda)$. From the properties of the function e_γ, we obtain

$$\mathrm{CE}_\gamma(t) = \exp\left\{ \mathcal{O}(\lambda) + \nu t + \mathrm{e}_\gamma^{-1}\, \mathbb{E}\, \mathcal{E}_{\gamma,\nu}^f(t) \right\}. \tag{8.2}$$

where $\mathcal{E}_{\gamma,\nu}^f(t) = \mathrm{e}_\gamma(U_\nu^f(t))$, as defined in chapter 4. Our aim is to estimate $\mathbb{E}\, \mathcal{E}_{\gamma,\nu}^f(t)$.

We have already expressed $\mathrm{d}\mathcal{E}_{\gamma,\nu}^f(t)$ with (4.9), from which we obtain

$$\begin{aligned} \mathcal{E}_{\gamma,\nu}^f(t) = \gamma \mathcal{E}_{\gamma,\nu}^f(0) \exp\left\{ \gamma \int_0^t D_{\mathcal{E},f}(\pi(s))\, \mathrm{d}W(s) - \frac{\gamma^2}{2} \int_0^t D_{\mathcal{E},f}^2(\pi(s))\, \mathrm{d}s \right\} \\ \cdot\, \mathrm{e}_\gamma \left(\int_0^t d_{\mathcal{E},f}^{\gamma,\nu}(\pi(s))\, \mathrm{d}s - \sum_{\updownarrow} \int_0^t L_f^{\updownarrow}(\pi(s))\, \mathrm{d}\pi^{\updownarrow}(s) \right). \end{aligned} \tag{8.3}$$

Since $D_{\mathcal{E},f}$ is a bounded function, then $\int_0^t D_{\mathcal{E},f}(\pi(s))\,\mathrm{d}W(s)$ is martingale. From the theorem 6.3 follows that $\sum_{\updownarrow}\int_0^t L_f^{\updownarrow}(\pi(s))\,\mathrm{d}\pi^{\updownarrow}(s) \leq 0$. Denote $c_0 \triangleq \gamma\mathcal{E}^f_{\gamma,\nu}(0) \in \mathbb{R}^+$. If $d^{\gamma,\nu}_{\mathcal{E},f}(x) \leq \epsilon$, $\epsilon > 0$, then

$$\mathbb{E}\,\mathcal{E}^f_{\gamma,\nu}(t) \leq c_0 e_\gamma\left(\epsilon\int_0^t \mathrm{d}s\right) = c_0 e_\gamma(\epsilon t).$$

In section 5 we showed that

$$d^{\gamma,\nu}_{\mathcal{E},f}(x) = \mathcal{O}(\lambda) + \frac{1-\gamma}{2}\lambda^{2/3}\left[\omega^2 - u_x^{\intercal}\Sigma u_x - \operatorname{tr}\{\nabla^2 g(u)\tilde{\mathbb{S}}_\Theta\}\right]. \tag{8.4}$$

We are going to prove that $d^{\gamma,\nu}_{\mathcal{E},\bar{f}}(x)$ is of order not greater than λ on a small enough neighbourhood of Θ.

Inside the no-trade region we have that

$$\begin{aligned}\nabla_u^2\bar{g}(u) &= \lambda^{-2/3}\nabla_x^2\bar{f}(x_u) = \lambda^{-2/3}\left(\mathcal{O}(\lambda^{7/6}) + \lambda^{2/3}\nabla_u^2 g(u^{\updownarrow})\right)\\ &= \nabla_u^2 g(u^{\updownarrow}) + \mathcal{O}(\lambda^{1/2}).\end{aligned} \tag{8.5}$$

Theorem 6.1 shows that $u^{\intercal}\Sigma u - \omega^2 + \operatorname{tr}\{\nabla^2 g(u)\tilde{\mathbb{S}}_\Theta\} = 0$ holds for

$$u \in U = \prod_{i=1}^n [-\bar{u}_i, \bar{u}_i] = [-\bar{u}, \bar{u}].$$

Using (8.5) we obtain

$$\omega^2 - \operatorname{tr}\{\nabla^2\bar{g}(u)\tilde{\mathbb{S}}_\Theta\} = \omega^2 - \operatorname{tr}\{\nabla^2 g(u^{\updownarrow})\tilde{\mathbb{S}}_\Theta\} + \mathcal{O}(\lambda^{1/2}) = {u^{\updownarrow}}^{\intercal}\Sigma u^{\updownarrow} + \mathcal{O}(\lambda^{1/2}), \tag{8.6}$$

and therefore

$$\begin{aligned}d^{\gamma,\nu}_{\mathcal{E},\bar{f}}(x) &= \mathcal{O}(\lambda) + \frac{1-\gamma}{2}\lambda^{2/3}\left[\omega^2 - u_x^{\intercal}\Sigma u_x - \operatorname{tr}\{\nabla^2 g(u)\tilde{\mathbb{S}}_\Theta\}\right]\\ &= \mathcal{O}(\lambda) + \frac{1-\gamma}{2}\lambda^{2/3}\left[{u_x^{\updownarrow}}^{\intercal}\Sigma u_x^{\updownarrow} - u_x^{\intercal}\Sigma u_x + \mathcal{O}(\lambda^{1/2})\right].\end{aligned} \tag{8.7}$$

Theorem 8.1. *There exists $K > 0$ such that*

$$d^{\gamma,\nu}_{\mathcal{E},\bar{f}}(x) \leq K\lambda.$$

Proof. Let $\|x-\Theta\|_\Sigma \leq K_0\lambda^{1/3}$. Then ${u_x^{\updownarrow}}^{\intercal}\Sigma u_x^{\updownarrow} - u_x^{\intercal}\Sigma u_x = \sum_{i=1}^n \sigma^2(u^{\updownarrow}_{x_i} - u_i)^2 \leq 0$. Hence, from (8.7) we get $d^{\gamma,\nu}_{\mathcal{E},\bar{f}}(x) \leq \mathcal{O}(\lambda)$.

Let $\|x-\Theta\|_\Sigma > K_0\lambda^{1/3}$. let $\mathcal{K} \subset \mathbb{R}^n$. Then for $\lambda > 0$ small enough we have

$$\sup_{x\in\mathcal{K}}\frac{1}{2}\operatorname{tr}[\nabla^2\bar{f}S(x)S(x)^{\intercal}] = \mathcal{O}(\lambda^{2/3})$$

We can also find $K > 0$ big enough, so that

$$q_\gamma(x) - \nu = \frac{1-\gamma}{2}\lambda^{2/3}\left(\omega^2 - u_x^\intercal \Sigma u_x\right) \leq 0.$$

Altogether we obtain $d_{\mathcal{E},\bar{f}}^{\gamma,\nu}(x) \leq \mathcal{O}(\lambda)$. □

We now have enough information to set the upper bound of the certainty equivalent.

Theorem 8.2. *There exist constants $K > 0$ and $C_0 > 0$ such that*

$$\mathrm{CE}_\gamma(t) \leq \exp\left\{\mathcal{O}(\lambda) + \nu t + C_0 + tK\lambda\right\}. \tag{8.8}$$

Proof. The theorem statement directly follows from equality (8.2), the theorem 8.1 and the property of the function e_γ, such that for given $c > 0$ there exists $C > 0$ such that $c\mathrm{e}_\gamma(x) = \mathrm{e}_\gamma(C + x)$ for every $x \in \mathbb{R}^n$. □

The theorem 8.2 holds for every strategy we choose. Now we are going to deal with the special strategy. In that case (8.3) becomes

$$\begin{aligned}\hat{\mathcal{E}}^f_{\gamma,\nu}(t) = \gamma\hat{\mathcal{E}}^f_{\gamma,\nu}(0)\exp\left\{\gamma\int_0^t D_{\mathcal{E},f}(\hat{\pi}(s))\,\mathrm{d}W(s) - \frac{\gamma^2}{2}\int_0^t D^2_{\mathcal{E},f}(\hat{\pi}(s))\,\mathrm{d}s\right\}\\ \cdot\,\mathrm{e}_\gamma\left(\int_0^t d^{\gamma,\nu}_{\mathcal{E},f}(\hat{\pi}(s))\,\mathrm{d}s - \sum_{\updownarrow}\int_0^t L^{\updownarrow}_f(\hat{\pi}(s))\,\mathrm{d}\pi^{\updownarrow}(s)\right).\end{aligned} \tag{8.9}$$

The function $L^{\updownarrow}_f(\pi(t))$ is defined in such way, that when applied to the special strategy, it holds that $\sum_{\updownarrow}\int_0^t L^{\updownarrow}_f(\hat{\pi}(s))\,\mathrm{d}\pi^{\updownarrow}(s) = 0$. For the other parts of the equation (8.9) it can be used same method as for a general case.

Theorem 8.3. *There exist constants $\hat{K} > 0$ and $\hat{C}_0 > 0$ such that*

$$\widehat{\mathrm{CE}}_\gamma(t) = \exp\left\{\mathcal{O}(\lambda) + \nu t + \hat{C}_0 + t\hat{K}\lambda\right\}. \tag{8.10}$$

The proof is analogous to the proof of the theorem (8.2). From the theorems 8.2 and 8.3 it follows that there exist constants $K > 0$ and $\hat{K} > 0$ such that

$$\lim_{t\to\infty}\left|\frac{1}{t}\log\widehat{\mathrm{CE}}_\gamma(t) - \nu\right| = \hat{K}\lambda$$

holds for the special strategy and

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathrm{CE}_\gamma(t) \leq \nu + K\lambda$$

holds for any other admissible strategy. Since $\hat{K}\lambda, K\lambda = \mathcal{O}(\lambda)$ as $\lambda \to \infty$, we altogether obtain

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathrm{CE}_\gamma(t) \le \lim_{t\to\infty} \frac{1}{t} \log \widehat{\mathrm{CE}}_\gamma(t) + \mathcal{O}(\lambda), \tag{8.11}$$

which is exactly what we wanted to prove.

Conclusion

In one-dimensional case we are able to find special strategy, which would maximize long run growth rate of the wealth process.

In multi-dimensional case we are not able to find an optimal strategy, not even in the logarithmic case. But with the specific conditions we have chance to maximize our position on the market and to invest in such a way, it keeps us on the safe ground. We have to restrict ourselves to small transaction costs, to admit some error and to require that the error should be very small if the size of the transaction taxes goes to zero.

We introduced such interval strategy, that trades only to makes an investor's position safe, i.e., to keep it in the previously defined region with certain properties. We even showed that chosen strategy was not worse than any other admissible strategy.

Appendix A

The implicit function theorem

Theorem A.1 (The implicit function theorem). *Let V_1, V_2, W be Banach spaces, $G \subset V_1 \times V_2$ an open set, $(x_0, y_0) \in G$ and $f : G \to W$ be continuously differentiable in G. Assume that*

$$f(x_0, y_0) = 0.$$

Let (the continuous) linear map

$$D_2 f(x_0, y_0) : V_2 \to W$$

be invertible, and let its inverse be likewise continuous. Then there exist open neighborhoods G_1 of x_0 and G_2 of y_0, such that $G_1 \times G_2 \subset G$, and differentiable function $g : G_1 \to G_2$, such that

$$f(x, g(x)) = 0 \tag{A.1}$$

and

$$Dg(x) = -\left(D_2 f\left(x, g(x)\right)\right)^{-1} \circ D_1 f\left(x, g(x)\right)$$

for all $x \in G_1$.

Furthermore, for every $x \in G_1$,$g(x)$ is the only solution of (A.1) in G_2.

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