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DIPLOMOVÁ PRÁCE

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Nonabsolutely convergent integrals

Katedra matematické analýzy

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Prohlašuji, že jsem tuto diplomovou práci vypracovala samostatně a výhradně s použitím citovaných pramenů, literatury a dalších odborných zdrojů.

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Abstrakt: Cílem práce je zavést pojem neabsolutně konvergentního integrálu na metrickém prostoru s mírou a to tak, aby zahrnoval Lebesgueův integrál. K tomu potřebujeme důkladně popsat vztahy mezi prostory spojitých a lipschitzovských funkcí. Následně vybudujeme tzv. UC-integrál funkce vzhledem k distribuci. Dokážeme, že naše konstrukce má rozumné vlastnosti a vyšetříme vztah k Lebesgueovu integrálu. Dále zavedeme UCN-integrál, který zanedbává množiny Hausdorffovy míry nula. Posléze se v práci věnujeme n-dimenzionálním metrickým currentům. Zavedeme pojem UC-integrálu vzhledem ke currentu a na závěr dokážeme obecnou verzi Gauss-Greenovy věty, jejímž speciálním případem je i Stokesova věta na varietách.

Klíčová slova: Neabsolutní integrály, Vícerozměrná integrace, Gaussova-Greenova věta

Title: Nonabsolutely convergent integrals

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Abstract: Our aim is to introduce an integral on a measure metric space, which will be nonabsolutely convergent but including the Lebesgue integral. We start with spaces of continuous and Lipschitz functions, spaces of Radon measures and their dual and predual spaces. We build up the so-called uniformly controlled integral (UC-integral) of a function with respect to a distribution. Then we investigate the relationship between the UC-integral with respect to a measure and the Lebesgue integral. Then we introduce another kind of integral, called UCN-integral, based on neglecting of small sets with respect to a Hausdorff measure. Hereafter, we focus on the concept of n-dimensional metric currents. We build the UC-integral with respect to a current and then we proceed to a very general version of Gauss-Green Theorem, which includes the Stokes Theorem on manifolds as a special case.

Keywords: Nonabsolutely convergent integrals, Multidimensional integrals, Gauss-Green Theorem

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Introduction

The aim of the theory of nonabsolutely convergent integrals is to build up an integral, which integrates all derivates and includes the Lebesgue integral. In this thesis we study nonabsolutely convergent integration in the framework of measure metric spaces.

The first convincing nonabsolutely convergent integral in $\mathbb{R}$ was introduced by Denjoy [8]. Further, early constructions are due to Lusin [14] and Perron [16]. Lusin's idea is a typical representative of the so-called descriptive approach. He defined the indefinite integral F of a function f on an interval I via derivatives at almost all points of I. The indefinite integral has to obey some condition in spirit of absolute continuity. This is a bit restrictive condition, but it solves the requirement for uniqueness of the integral.

Another approach was chosen by Kurzweil [13] and Henstock [12]. They constructed integral of f on I using Riemann sum of functions. They improved the treatment of partitions of intervals. Their work opened a new era in theory of nonabsolutely convergent integrals, in which also the multidimensional integral was intensively studied.

Let us mention the work of Pfeffer [17], who defines an indefinite integral as a function of BV-set. He applies his integral to establish a very general version of the Gauss-Green Theorem.

In the work [15], Malý defines the so-called UC-integral of a function with respect to a distribution in $\mathbb{R}^n$. The idea is based on the elementary construction given in [5].

In this thesis, we generalise his approach to the setting of metric spaces.

The thesis is divided into four parts. The first part starts with notations and recalls some famous theorems such as Monotone Convergence Theorem or Vitali Theorem. Then we define the ζ-spherical measure $\mathcal{H}^\zeta$, which we will need in the definiton of UCN-integral.

We continue with spaces of continuous functions and by the Riesz Representation Theorem we remind the relationship between Radon measures and functionals on the space $\mathcal{C}_0(X)$.

Afterwards, in the second chapter, we concentrate on spaces of Lipschitz functions and we define the normed linear space $\mathrm{Lip}_b(X)$ of all bounded Lipschitz functions equipped with the norm $\|f\|_{\mathrm{Lip}_b(X)} = \max\{\mathrm{Lip} f,\ \|f\|_{\sup}\}$. This space is fundamental for introducing the UC-integral, since it is space of our "test" functions. Then we move to the dual spaces and define the space of convergent metric distributions $\mathcal{D}'(X)$ as the closure of $\mathcal{C}_0(X)^*$ in the space $\mathrm{Lip}_b(X)^*$. This space is irreplaceable for us, since the UC-integral is defined as an element of $\mathcal{D}'(X)$. Because of the importance of these spaces we would like to obtain an exact description of their properties. Therefore we apply tools of functional analysis and obtain an isometric isomorphism of $\mathrm{Lip}_b(X)$ and $\mathcal{D}'(X)^*$, which allows us to introduce a weak* convergence on the space $\mathrm{Lip}_b(X)$. Equipped with this knowledge, we describe properties of this weak* convergence and we are prepared for introducing the UC-integral.

At the beginning of the third chapter, so-called Key Lemma is situated. It is based on covering of compacts and on the partition of unity and give us technical methods needed in the next section.

Now we are prepared to define the most important notion of this thesis, the UC-integral (for details see Notation 2.11 and Definition 2.6):

Definition. Let (X, ρ) be a locally compact separable metric space equipped with a doubling Radon measure μ and let $f : X \to \mathbb{R}$ be a function. We say that a functional $\mathcal{F} \in \mathcal{D}'_{\mathrm{loc}}(X)$ is an *UC-integral (uniformly controlled integral)* of f with respect to $\mathcal{G} \in \mathcal{D}'_{\mathrm{loc}}(X)$ if there exist $\tau \geq 1$ and a Radon measure Φ such that for all $x \in X$ we have:

$$\|\mathcal{F} - f(x)\mathcal{G}\|^*_{x,r} = o(r\Phi(B(x, \tau r))) \quad \text{as} \quad r \to 0_+,$$

where $\|\cdot\|^*_{x,r}$ denotes the dual norm on the space $\mathrm{Lip}^r_{B(x,r)}(X)^*$.

In the next text we verify the uniqueness and linearity of UC-integral, concentrate on the integration with respect to a measure and then we continue with describing its relationship to the Lebesgue integral.

The last part of this chapter introduces the UCN-integral, which is an analogy of the UC-integral ommiting sets with zero Hausdorff measure, and check its basic properties.

In the last chapter we work with currents, which were studied by Ambrosio and Kirchheim in [2]. We present our definition of currents and introduce basic definitions connected with them. Then we establish the definition of UC-integral with respect to a current and we illustrate this topics on some examples, well known in the theory of curve and surface integrals.

Then we give a very general version of Gauss-Green Theorem. As a bonus of the setting of metric spaces, our generality includes the Stokes Theorem for differential forms on manifolds.

The thesis terminates with some applications of Gauss-Green Theorem, such as the Gauss-Green Theorem for sets with finite perimeter.

Chapter 1

Preliminaries

1.1 Carathéodory construction

In this section we define the ζ-spherical outer measure $\mathcal{H}^\zeta$, which we need for the definition of UCN-integral (see Section 3.3).

Notation 1.1. Let (X, ρ) be a metric space. Then $B(x, r)$ denotes the open ball centered at x with radius r, thus $B(x, r) = \{y \in X : \rho(x, y) < r\}$. Further, $\bar{B}(x, r)$ denotes the closed ball centered at x with radius r, thus $B(x, r) = \{y \in X : \rho(x, y) \leq r\}$.

Definition 1.2. Let μ be an outer measure on a locally compact metric space (X, ρ). We say that μ is *Borel regular*, if every Borel set is μ-measurable and for every set $A \subset X$ there exists a Borel set $B \supset A$ such that $\mu(A) = \mu(B)$.

Definition 1.3. A Borel regular outer measure μ on a metric space (X, ρ) is said to be *doubling*, if $\mu(G) > 0$ for every open set G and if there exists a constant c_D such that for each $B(x, r)$ we have

$$\mu(B(x, 2r)) \leq c_D \mu(B(x, r)).$$

Remark 1.4. We can see that $c_D \geq 1$.

Now we would like to introduce the notion of the ζ-spherical outer measure $\mathcal{H}^\zeta$ according to Carathéodory. For details and proofs see [11, pg.169-173].

Definition 1.5. Let (X, ρ) be a metric space, $\mathfrak{B}$ the family of all closed balls in X and $\zeta : \mathfrak{B} \to [0, \infty]$ be a function. Then for $\delta \in [0, \infty]$ we construct the *size* δ *approximating outer measure* ζ_δ and the ζ*-spherical outer measure* $\mathcal{H}^\zeta$ as follows: for $A \subset X$ we define

$$\zeta_\delta(A) := \inf \left\{ \sum_{j=1}^{\infty} \zeta(B_j) : B_j \in \mathfrak{B}, \bigcup_{j=1}^{\infty} B_j \supset A, \operatorname{diam} B_j \leq \delta \right\}$$

and

$$\mathcal{H}^\zeta(A) := \sup_{\delta > 0} \zeta_\delta(A) = \lim_{\delta \to 0_+} \zeta_\delta(A).$$

Proposition 1.6. (1) $\mathcal{H}^\zeta$ and ζ_δ are outer measures on X.

(2) Let $0 < \delta < \sigma \leq \infty$. Then $\zeta_\delta \geq \zeta_\sigma$, which implies the existence of $\lim_{\delta \to 0_+} \zeta_\delta(A) = \mathcal{H}^\zeta(A)$.

(3) All open subsets of X are $\mathcal{H}^\zeta$-measurable, but not all open sets need to be ζ_δ-measurable.

(4) $\mathcal{H}^\zeta$ is a Borel regular outer measure.

1.2 Spaces of continuous functions

In this section we recall spaces of continuous functions and the well known Riesz Representation Theorem, which will be very useful in the next section. Although some mentioned results work with complex measures, we will concentrate on signed real measures.

Henceforward in this chapter, X will denote a separable and locally compact metric space.

Definition 1.7. Let (X, ρ) be a locally compact separable metric space. Then $\mathcal{C}_b(X)$ denotes the Banach space of all bounded continuous functions on X equipped with the supremum norm

$$\|f\|_{\sup} := \sup_{x \in X} |f(x)|.$$

Further, $\mathcal{C}_c(X)$ denotes the set of all continuous functions on X with compact support and $\mathcal{C}_0(X)$ denotes the closure of $\mathcal{C}_c(X)$ in $\mathcal{C}_b(X)$.

Now let us, just for completeness, recall the Riesz Representation Theorem. For the proof see [18, Theorem 6.19].

Theorem 1.8 (Riesz Representation Theorem)**.** To each bounded linear functional Φ on $\mathcal{C}_0(X)$, where X is a locally compact Hausdorff space, there corresponds a unique complex regular Borel measure μ such that

$$\Phi(f) = \int_X f \, \mathrm{d}\mu, \quad f \in \mathcal{C}_0(X). \tag{1.1}$$

Moreover, if Φ and μ are related as in (1.1), then

$$\|\Phi\| = |\mu|(X).$$

Remark 1.9. The Riesz Representation Theorem allows us to extend the functional Φ on larger classes of functions via integration. We will identify the extended functionals with the original functional Φ.

1.3 Vitali Theorem

We will use the term disjointed for pairwise disjoint.

The proof of the Vitali Theorem mentioned below is an easy consequence of [3, Theorem 2.2.3]. The countability of the subsystem $\mathcal{V}'$ follows from the fact that X is separable.

Theorem 1.10 (Vitali)**.** Let (X, ρ) be a separable metric space and $E \subset X$. Let $\mathcal{V}$ be a system of closed balls in X covering E. Suppose that

$$R := \sup\{r : \bar{B}(x, r) \in \mathcal{V}\} < \infty.$$

Then there exists a countable disjointed subsystem $\mathcal{V}'$ of $\mathcal{V}$ such that

$$E \subset \bigcup_{\bar{B}(x,r) \in \mathcal{V}'} B(x, 5r).$$

Chapter 2

Spaces of Lipschitz functions

In this chapter we start with basic definitions of Lipschitz functions, spaces of Lipschitz functions and their duals and preduals. Then we continue with some technical lemmas leading to the main theorem of this chapter, Theorem 2.9, which provides utilities for introducing the UC-integral. We complete this chapter with the characterisation of weak* convergence in the spaces of Lipschitz functions and we mention also the space of metric distributions.

It is well known that spaces of Lipschitz functions are preduals, see [4], [19]. However, we use a different approach, which we have not found in the literature.

Definition 2.1. Let (X, ρ) be a metric space and $f : X \to \mathbb{R}$. Then

$$\mathrm{Lip}(f) = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{|f(x) - f(y)|}{\rho(x, y)}$$

denotes the *Lipschitz constant of f*. Further $\mathrm{Lip}(X)$ denotes the family of all functions f such that $\mathrm{Lip}(f) < \infty$.

Lemma 2.2. Let (X, ρ) be a metric space and $\omega, \eta : X \to \mathbb{R}$ be bounded Lipschitz functions. Then the product $\omega\eta$ is also a Lipschitz function and $\mathrm{Lip}(\omega\eta) \leq \sup |\omega| \mathrm{Lip}(\eta) + \sup |\eta| \mathrm{Lip}(\omega)$.

Proof. Let us estimate:

$$\begin{aligned} |\omega(x)\eta(x) - \omega(y)\eta(y)| &\leq |\omega(x)\eta(x) - \omega(y)\eta(x)| + |\omega(y)\eta(x) - \omega(y)\eta(y)| \\ &\leq |\eta(x)||\omega(x) - \omega(y)| + |\omega(y)||\eta(x) - \eta(y)| \\ &\leq \sup_{x \in X} |\eta(x)| \mathrm{Lip}(\omega)\rho(x, y) + \sup_{y \in X} |\omega(y)| \mathrm{Lip}(\eta)\rho(x, y). \end{aligned}$$

□

Definition 2.3. Let (X, ρ) be a locally compact separable metric space. Then $\mathrm{Lip}_b(X)$ denotes the normed linear space of bounded Lipschitz functions equipped with the norm $\|f\|_{\mathrm{Lip}_b(X)} = \max\{\mathrm{Lip}(f), \|f\|_{\sup}\}$.

Proposition 2.4. Let (X, ρ) be a locally compact separable metric space. Then $\mathcal{C}_0(X)^* \hookrightarrow \mathrm{Lip}_b(X)^*$ in the sense of Remark 1.9.

Proof. Let $\Phi \in \mathcal{C}_0(X)^*$. Then, by Theorem 1.8, it can be expressed as

$$\Phi(f) = \int_X f \, \mathrm{d}\mu.$$

Since bounded continuous functions are integrable with respect to finite measures, we obtain $\mathcal{C}_0(X)^* \hookrightarrow \mathcal{C}_b(X)^*$ and because Lipschitz functions are also continuous, the proof is done. □

Definition 2.5. The closure of $\mathcal{C}_0(X)^*$ in $\mathrm{Lip}_b(X)^*$ is denoted by $\mathcal{D}'(X)$ and called the space of *convergent metric distributions.*

Definition 2.6. Let (X, ρ) be a metric space and $K \subset X$ be compact. Then $\mathrm{Lip}_K(X)$ denotes the normed linear space of all Lipschitz functions f with $\operatorname{spt} f \subset K$ and is equipped with the norm $\|\cdot\|_{\mathrm{Lip}_b(X)}$.

Further, $\mathrm{Lip}_c(X)$ denotes the topological linear space of all Lipschitz functions on X with compact support endowed with inductive topology generated by the family of functions $\iota_K : \mathrm{Lip}_K(X) \to \mathrm{Lip}(X)$, $\iota_K = \mathrm{Id}$, where K ranges over all compact subsets of X. Analogously, $\mathcal{C}_K(X)$ denotes the space of all $\mathcal{C}_0(X)$ functions with support in K.

Now we can define $\mathcal{D}'_K(X)$ as the closure of $\mathcal{C}_K(X)^*$ in $\mathcal{D}_K(X)^*$. Further, $\mathcal{D}'_{\mathrm{loc}}(X)$ denotes the intersection of $\mathcal{D}'_K(X)$ over all compact $K \subset X$. We will call its elements *(metric) distributions.*

Lemma 2.7. For every $\mu \in \mathcal{C}_0(X)^*$ there exists a sequence (μ_n) such that μ_n are linear combinations of Dirac measures and $\mu_n \to \mu$ in $\mathrm{Lip}_b(X)^*$.

Proof. At first, we will show that the set of Radon measures with compact support is dense in $\mathcal{C}_0(X)^*$ with respect to the norm of $\mathrm{Lip}_b(X)^*$. Let us pick $\mu \in \mathcal{C}_0(X)^*$, we need to find a sequence of measures with compact support (μ_k), $\mu_k \to \mu$ in $\mathrm{Lip}_b(X)^*$.

Since X is separable and locally compact, it can be covered by increasing (countable) sequence of compact sets $(K_k, k \in \mathbb{N})$.

Now, let us define $\mu_k(A) := \mu(A \cap K_k)$ for every A μ-measurable. By the properties of measure, we have $\mu(X) = \mu\left(\bigcup_k K_k\right) = \lim_k \mu(K_k) = \lim_k \mu_k(X)$ and since μ is finite, we have $\lim_k (X \setminus K_k) = 0$.

Then for every $f \in \mathrm{Lip}_b(X)$, $\|f\|_{\mathrm{Lip}_b(X)} \leq 1$ we have

$$\left| \int_X f \, \mathrm{d}\mu - \int_X f \, \mathrm{d}\mu_k \right| \leq \int_{X \setminus K_k} |f| \, \mathrm{d}\mu \leq \mu(X \setminus K_k)$$

and hence $\mu_k \to \mu$ in $\mathrm{Lip}_b(X)^*$.

We have shown that the set of Radon measures with compact support is dense in the set $\mathcal{C}_0(X)^*$. Now, we would like to show that the set of linear combinations of Dirac measures is dense in the set of measures with compact support (both in the $\mathrm{Lip}_b(X)^*$ norm).

Let us choose $\mu \in \mathcal{C}_0(X)^*$ with a compact support K, we need to find a sequence (μ_j), such that $\mu_j \to \mu$ in $\mathrm{Lip}_b(X)^*$.

For every $j \in \mathbb{N}$ we find finite system of pairwise disjoint sets N_i^j such that $K = \bigcup_i N_i^j$ and $\operatorname{diam} N_i^j \leq 2^{-j}$ for all i. Thus, choose $x_i^j \in N_i^j$ for every i and j. We define

$$\mu_j := \sum_i \mu(N_i^j) \delta_{x_i^j}.$$

Now, for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$, such that $2^{-j} < \varepsilon$. Hence, for every $f \in \mathrm{Lip}_b(X)$, $\|f\|_{\mathrm{Lip}_b(X)} \le 1$, and every $x, y \in X$, $\rho(x,y) \le 2^{-j}$, we have $|f(x) - f(y)| \le 2^{-j} < \varepsilon$ and therefore

$$\left|\int_X f \,\mathrm{d}\mu - \int_X f \,\mathrm{d}\mu_j\right| = \left|\int_X f \,\mathrm{d}\mu - \sum_i \mu(N_i^j) f(x_i^j)\right| \le \sum_i \int_{N_i^j} |f - f(x_i^j)| \,\mathrm{d}\mu$$
$$< \varepsilon \sum_i \mu(N_i^j) = \varepsilon \mu(K).$$

Thus, $\mu_j \to \mu$ in $\mathrm{Lip}_b(X)^*$.

We have proved that the set of linear combinations of Dirac measures is dense in the set of measures with compact support. But, since a dense subset of a dense set is also dense in the entire space, we proved that the linear combinations of Dirac measures form a dense subset of $\mathcal{C}_0(X)^*$ (in the $\mathrm{Lip}_b(X)^*$-norm). Hence, for every $\mu \in \mathcal{C}_0(X)^*$ there exists a sequence (μ_n), $\mu_n \to \mu$ (in $\mathrm{Lip}_b(X)^*$-norm). □

Corollary 2.8. The set of linear combinations of Dirac measures is dense in $\mathcal{D}'(X)$. In particular, the space $\mathcal{D}'(X)$ is separable.

Proof. Let us start with the fact that the linear combinations of Dirac measures are dense in $\mathcal{D}'(X)$ (Lemma 2.7). Since the space (X, ρ) is separable, there exists a countable dense subset $Y \subset X$. We can immediatelly see that if $x_n \to x$ in X, then $\delta(x_n) \to \delta(x)$ in $\mathrm{Lip}_b(X)$ and hence that the set of linear combinations of Dirac measures at points of Y is dense in $\mathcal{D}'(X)$. Further, linear combinations of Dirac measures at points of Y with rational coefficients are dense in linear combinations with real coefficients, which proves the separability. □

Theorem 2.9. Let (X, ρ) be a locally compact separable metric space. Then the spaces $\mathrm{Lip}_b(X)$ and $\mathcal{D}'(X)^*$ are isometrically isomorphic.

The proof presented below is based on construction of such isomorphism. In the first step, for fixed $f \in \mathrm{Lip}_b(X)$ we introduce auxiliary mapping $\varepsilon_f : \mathcal{D}'(X) \to \mathbb{R}$. We continue with the definition of canonical embedding $\varepsilon : \mathrm{Lip}_b(X) \to \mathcal{D}'(X)^*$, $\varepsilon(f) = \varepsilon_f$. Then we show that ε is an isometry. Our last task is to prove that ε is onto $\mathcal{D}'(X)^*$. In the third step we for fixed $\mathbb{T} \in \mathcal{D}'(X)^*$ define $f_{\mathbb{T}} : X \to \mathbb{R}$ and then we check that $f_{\mathbb{T}}$ belongs to $\mathrm{Lip}_b(X)$. Finally, we verify the fact that $\varepsilon(f_{\mathbb{T}})(\mu) = \mathbb{T}(\mu)$ for every $\mu \in \mathcal{D}'(X)$ and the proof is done.

Proof. Step 1

Let us start with the mapping $\varepsilon_f : \mathcal{D}'(X) \to \mathbb{R}$ defined for given $f \in \mathrm{Lip}_b(X)$ as

$$\varepsilon_f(\mathscr{T}) := \langle \mathscr{T}, f \rangle, \quad \mathscr{T} \in \mathcal{D}'(X).$$

Now, we can define *canonical embedding* $\varepsilon : \mathrm{Lip}_b(X) \to \mathcal{D}'(X)^*$ as

$$\varepsilon(f) := \varepsilon_f.$$

Obviously, ε is a linear mapping, is well defined and is injective. Now, we would like to show that $\|\varepsilon(f)\|_{\mathcal{D}'(X)^*} = \|f\|_{\mathrm{Lip}_b(X)}$ for every $f \in \mathrm{Lip}_b(X)$. By the definition, we obtain

$$\|\varepsilon(f)\|_{\mathcal{D}'(X)^*} = \|\varepsilon_f\|_{\mathcal{D}'(X)^*} = \sup\{|\varepsilon_f(\mu)|; \mu \in \mathcal{D}'(X), \|\mu\|_{\mathrm{Lip}_b(X)^*} \le 1\}.$$

Since $\mathcal{D}'(X)$ is the closure of $\mathcal{C}_0(X)^*$, we have

$$\begin{aligned}&\sup\{|\varepsilon_f(\mu)|; \mu \in \mathcal{D}'(X), \|\mu\|_{\mathrm{Lip}_b(X)^*} \le 1\}\\ &\quad= \sup\left\{\left|\int_X f \,\mathrm{d}\mu\right|; \mu \in \mathcal{D}'(X), \|\mu\|_{\mathrm{Lip}_b(X)^*} \le 1\right\}\\ &\quad= \sup\left\{|\mu(f)|\,; \mu \in \mathcal{D}'(X), \|\mu\|_{\mathrm{Lip}_b(X)^*} \le 1\right\}.\end{aligned}$$

Hence, we have

$$\|\varepsilon(f)\|_{\mathcal{D}'(X)^*} \le \sup\left\{\|\mu\|_{\mathrm{Lip}_b(X)^*}\|f\|_{\mathrm{Lip}_b(X)}; \mu \in \mathcal{D}'(X), \|\mu\|_{\mathrm{Lip}_b(X)^*} \le 1\right\} \le \|f\|_{\mathrm{Lip}_b(X)}.$$

To prove the second inequality, we need to show that

$$\sup\left\{|\mu(f)|\,; \mu \in \mathcal{D}'(X), \|\mu\|_{\mathrm{Lip}_b(X)^*} \le 1\right\} \ge \|f\|_{\mathrm{Lip}_b(X)} = \max\{\mathrm{Lip}(f), \|f\|_{\sup}\}.$$

Let us denote $s := \sup_{x\in X}|f(x)|$. By properties of supremum, we can find a sequence $(x_n, n \in \mathbb{N})$, such that $|f(x_n)| \to s$. Now, let us consider Dirac measures δ_{x_n}. Obviously, $\delta_x \in \mathcal{C}_0(X)^*$ and $\|\delta_x\|_{\mathrm{Lip}_b(X)^*} \le 1$ for every $x \in X$. Then, we can estimate

$$\sup\left\{|\mu(f)|\,; \mu \in \mathcal{D}'(X), \|\mu\|_{\mathrm{Lip}_b(X)^*} \le 1\right\} \ge \sup_{n\in\mathbb{N}}|\delta_{x_n}(f)| \ge \sup_{n\in\mathbb{N}}|f(x_n)| = s$$

On the other hand, let us consider a functional $\mu_{x,y}(f) := \frac{\delta_x-\delta_y}{\rho(x,y)}$, $x \ne y$. Obviously, $\mu_{x,y} \in \mathcal{D}'(X)$ and $\|\mu_{x,y}\|_{\mathrm{Lip}_b(X)^*} \le 1$. Then, we have

$$\sup_{\mu\in\mathcal{C}_0(X)^*}\left\{|\mu(f)|\,; \mu \in \mathcal{D}'(X), \|\mu\|_{\mathrm{Lip}_b(X)^*} \le 1\right\} \ge \sup_{x,y\in X, x\ne y}|\mu_{x,y}(f)| = \mathrm{Lip}(f)$$

and we have shown that $\|\varepsilon(f)\|_{\mathcal{D}'(X)^*} = \|f\|_{\mathrm{Lip}_b(X)}$ for every $f \in \mathrm{Lip}_b(X)$.

Step 2

Now, we need to show that ε is onto. Let $\mathbb{T} \in \mathcal{D}'(X)^*$. Then let us define $f_{\mathbb{T}}(x) := \mathbb{T}(\delta_x)$ for $x \in X$, where δ_x denotes the Dirac measure.

We can see that $f_{\mathbb{T}}$ is well defined. Further, $f_{\mathbb{T}} \in \mathrm{Lip}_b(X)$, since

$$\begin{aligned}|f_{\mathbb{T}}(x) - f_{\mathbb{T}}(y)| &= |\mathbb{T}(\delta_x) - \mathbb{T}(\delta_y)| = |\mathbb{T}(\delta_x - \delta_y)|\\ &\le \|\mathbb{T}\|_{\mathcal{D}'(X)^*}\|\delta_x - \delta_y\|_{\mathrm{Lip}_b(X)^*} \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*}|x-y|\end{aligned}$$

and

$$|f_{\mathbb{T}}(x)| = |\mathbb{T}(\delta_x)| \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*}\|\delta_x\|_{\mathrm{Lip}_b(X)^*} \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*},$$

hence $\|f_{\mathbb{T}}\|_{\mathrm{Lip}_b(X)} \le \|\mathbb{T}\|_{\mathcal{D}'(X)^*}$.

Step 3

Now, let us choose $\mu \in \mathcal{D}'(X)^*$. Then, by Corollary 2.8, there exists a sequence (μ_k), such that μ_k are linear combinations of Dirac measures and $\mu_k \to \mu$ in $\mathrm{Lip}_b(X)^*$. Since $\mathbb{T} \in \mathcal{D}'(X)^*$ is a continuous functional, we have $\lim_{n\to\infty}\mathbb{T}(\mu_k) = \mathbb{T}(\mu)$. Together, we obtain

$$\varepsilon(f_{\mathbb{T}})(\mu) = \lim_{n\to\infty}\varepsilon(f_{\mathbb{T}})(\mu_k) = \lim_{n\to\infty}\mathbb{T}(\mu_k) = \mathbb{T}(\mu),$$

which concludes the proof.

□

Remark 2.10. Since $\mathrm{Lip}_b(X) \cong \mathcal{D}'(X)^*$, the weak* topology and the weak* convergence on $\mathrm{Lip}_b(X)$ is well defined.

Notation 2.11. Let (X, ρ) be a metric space and M be a bounded nonempty subset of X such that $\operatorname{diam} M \leq 2r$. Then $\mathrm{Lip}^r_M(X)$ denotes the normed linear space of all Lipschitz functions $f : X \to \mathbb{R}$ with $\{f \neq 0\} \subset M$ equipped with the norm

$$\|f\|_{\mathrm{Lip}^r_M(X)} = \max\left\{\mathrm{Lip} f,\ \frac{\|f\|_{\sup}}{2r}\right\}.$$

Especially, let $B(x, r)$ be an open ball in X. Then the space $\mathrm{Lip}^r_{B(x,r)}(X)$ is defined as above. For simplicity, we will denote its norm by $\|\cdot\|_{x,r}$.

Further, the space of all weak*–continuous linear functionals on $\mathrm{Lip}^r_{B(x,r)}(X)$ is denoted by $\mathrm{Lip}^r_{B(x,r)}(X)^*$ and the norm $\|\cdot\|^*_{x,r}$ on this space is defined as

$$\|\mathcal{F}\|^*_{x,r} = \sup\{\langle \mathcal{F}, \eta\rangle; \eta \in \mathrm{Lip}^r_{B(x,r)}(X), \|\eta\|_{x,r} \leq 1\}.$$

Proposition 2.12. Let u, $f_n \in \mathrm{Lip}_b(X)$, $n \in \mathbb{N}$. Then the following assertions are equivalent:

(i) $f_n \to f$ weak* in $\mathrm{Lip}_b(X)$,

(ii) (f_n) is bounded in $\mathrm{Lip}_b(X)$ and $f_n \to f$ pointwise,

(iii) (f_n) is bounded in $\mathrm{Lip}_b(X)$ and $f_n \to f$ locally uniformly.

Proof. (i) $\implies$ (ii). Since (f_n) is a weak* convergent sequence, it is bounded in $\mathrm{Lip}_b(X)$. The pointwise convergence can be easily obtained by applying the weak* convergence on Dirac measures.

(ii) $\implies$ (iii): We need to show that the sequence (f_n) is uniformly convergent on every compact $K \subset X$. Since (f_n) is bounded, there exists M such that for every $n \in \mathbb{N}$ we have $\|f_n\|_{\mathrm{Lip}_b(X)} \leq M$. Let us choose a compact K and $\xi > 0$. Let us construct a covering $\mathcal{V} = \{B(x, \xi), x \in K\}$. Since K is compact, there exists a finite subcovering $\{B(x_i, \xi), i = 1 \dots k\}$, such that

$$\bigcup_{i=1}^{k} B(x_i, \xi) \supset K.$$

Further, since (f_n) is pointwise convergent, for every $i = 1 \dots k$ there exists n_i, such that for every $n \geq n_i$ we have

$$|f_n(x_i) - f(x_i)| < \xi.$$

Let us denote $n_0 := \max\{n_1, \dots n_k\}$. Now let us choose $y \in K$. Since K is covered by $\{B(x_i, \xi), i = 1 \dots k\}$, there exists $i \in (1, \dots, k)$, such that $\rho(y, x_i) < \xi$. Then for every $n > n_0$ we have

$$\begin{aligned}|f_n(y) - f(y)| &\leq |f_n(y) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(y)| \\ &\leq M\rho(x_i, y) + \xi + M\rho(x_i, y) \leq (2M + 1)\xi,\end{aligned}$$

which we needed.

(iii) $\implies$ (ii): This implication is obvious.

(ii) $\implies$ (i): Let us choose $\nu \in \mathcal{D}'(X)$ and $\xi > 0$. Since the set of linear combinations of Dirac measures is dense in $\mathcal{D}'(X)$, there exists $\mu \in \mathcal{C}_0(X)^*$ such that μ is a linear combination of Dirac measures and $\|\mu - \nu\|_{\mathcal{D}'(X)} < \xi$.

We assume that $f_n \to f$ pointwise, which can be expressed as $\varepsilon_{f_n}(\mu) \to \varepsilon_f(\mu)$, where μ is a linear combination of Dirac measures. Hence, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $|\varepsilon_{f_n}(\mu) - \varepsilon_f(\mu)| < \xi$.

Further, let us denote $c := \sup_{n\in\mathbb{N}} \|f_n\|_{\mathrm{Lip}_b(X)}$. Since (f_n) is bounded in $\mathrm{Lip}_b(X)$, we have $c < \infty$.

Now, we obtain

$$\begin{aligned} |\varepsilon_{f_n}(\nu) - \varepsilon_f(\nu)| &\leq |\varepsilon_{f_n}(\nu) - \varepsilon_{f_n}(\mu)| + |\varepsilon_{f_n}(\mu) - \varepsilon_f(\mu)| + |\varepsilon_f(\mu) - \varepsilon_f(\nu)| \\ &\leq \|f_n\|_{\mathrm{Lip}_b(X)}\|\mu - \nu\|_{\mathcal{D}'(X)} + \xi + \|f\|_{\mathrm{Lip}_b(X)}\|\mu - \nu\|_{\mathcal{D}'(X)} \\ &\leq (2c+1)\xi, \end{aligned}$$

which concludes the proof. □

Now let us quote two Theorems related to weak* continuity on Banach spaces. For the proofs see [10, Proposition 3.24] and [10, Corollary 4.46].

Theorem 2.13. Let X be a Banach space. (B_{X^*}, w^*) is metrizable if and only if X is separable.

Theorem 2.14. Let X be a Banach space, and let F be a linear functional on X^*. The following are equivalent.

(1) F is w^*-continuous.

(2) $F \in X$; that is, there is $x \in X$ such that $F(f) = f(x)$ for every $f \in X^*$.

(3) The restriction of F to B_{X^*} is w^*-continuous.

(4) $F^{-1}(0) \cap B_{X^*}$ is w^*-closed.

Proposition 2.15. Let $\mathscr{T} : \mathrm{Lip}_b(X) \to \mathbb{R}$ be a linear functional. Then the following properties are equivalent:

(i) $\mathscr{T} \in \mathcal{D}'(X)$,

(ii) $\mathscr{T}$ is weak* continuous on $\mathrm{Lip}_b(X)$,

(iii) $\mathscr{T}$ is sequentially weak* continuous on $\mathrm{Lip}_b(X)$.

Proof. (i) $\iff$ (ii) follows from Theorem 2.14 applied to the space $\mathcal{D}'(X)$.

(ii) $\implies$ (iii) is obvious.

(iii) $\implies$ (ii) Since $\mathscr{T}$ is sequentially weak* continuous on $\mathrm{Lip}_b(X)$, it is also sequentially weak* continuous on the unit ball in $\mathrm{Lip}_b(X)$. Since the space $\mathcal{D}'(X)$ is separable (see Corollary 2.8), we can apply Theorem 2.13 and we obtain that the unit ball is metrizable. However, the continuity and sequential continuity coincide on metric spaces, hence we obtain, that $\mathscr{T}$ is w^*-continuous on the unit ball in $\mathrm{Lip}_b(X)$. Then the w^*-continuity on the space $\mathrm{Lip}_b(X)$ follows from Theorem 2.14.

□

Definition 2.16. Let (X, ρ) be a metric space. We say that X is *boundedly compact*, if all closed bounded subsets of X are compact.

Proposition 2.17. Let X be a boundedly compact metric space. Then $\mathrm{Lip}_c(X)$ is weak* dense in $\mathrm{Lip}_b(X)$.

Proof. Let us choose $f \in \mathrm{Lip}_b(X)$. We need to find a sequence $(f_n) \subset \mathrm{Lip}_c(X)$ such that $f_n \to f$ in weak* norm. By the Theorem 2.12 it is enough to find a bounded sequence (f_n), such that $f_n \to f$ pointwise.

Let us define functions

$$\kappa_n := \begin{cases} 1, \quad x \in B(x_0, n), \\ 1 - (\rho(x, x_0) - n), \quad x \in B(x_0, n+1) \setminus B(x_0, n), \\ 0, \quad x \notin B(x_0, n+1), \end{cases}$$

where $n \in \mathbb{N}$. Now, let us set $f_n := f\kappa_n$. Since X is boundedly compact, f_n have compact support. Applying Lemma 2.2 we obtain that the functions f_n belong to $\mathrm{Lip}_c(X)$ for every $n \in \mathbb{N}$. Since the sequence (f_n) is obviously bounded in $\mathrm{Lip}_b(X)$ and $f_n \to f$ pointwise, the proof is concluded.

□

Chapter 3

Integral

The aim of this chapter is to present the concepts of UC and UCN-integrals. The chapter starts with some technical utilities and continues with the definition of the UC-integral. We focus on the integral of a function with respect to a metric distribution and study its relation to the Lebesgue integral. The chapter is concluded with the UCN-integral, which neglects sets of Hausdorff measure zero.

3.1 Key lemma

In this section we will prepare some general technical utilities, which are necessary for introduction of the integral in a metric space and which will guarantee such fundamental properties of integral as uniqueness (see Section 3.2). We will start with some basic facts.

Lemma 3.1. Let (X, ρ) be a locally compact metric space equipped with a doubling Radon measure μ and let $\sigma \geq 1$. Then there exists a constant c_T with the following property: for each Radon measure Φ, $x \in X$ and $R > 0$ there exists $0 < r < R$ such that the sum $\Phi + \mu$ satisfies:

$$(\Phi + \mu)(B(x, 10\sigma r)) \leq c_T(\Phi + \mu)(B(x, r)).$$

Proof. By the definition of a doubling measure there exists a constant c_D such that

$$\mu(B(x, 2r)) \leq c_D\mu(B(x, r)), \quad x \in X, \, r > 0.$$

Further let us find $k \in \mathbb{N}$ such that $10\sigma \leq 2^k$. Then for every $x \in X$ and for every $r > 0$ we have

$$\mu(B(x, 10\sigma r)) \leq \mu(B(x, 2^k r)) \leq c_D^k\mu(B(x, r)). \tag{3.1}$$

Now, let us denote $c_T := 2c_D^k$, fix $R > 0$ and assume by contradiction that for each $r \in (0, R]$ we have

$$(\Phi + \mu)(B(x, 10\sigma r)) > 2c_D^k(\Phi + \mu)(B(x, r)).$$

Hence, using (3.1), we obtain

$$\frac{(\Phi + \mu)(B(x, R))}{\mu(B(x, R))} > \frac{2c_D^k(\Phi + \mu)(B(x, R/10\sigma))}{c_D^k\mu(B(x, R/10\sigma))}.$$

Iterating this process and using the fact that $\Phi + \mu \geq \mu$ we get

$$\frac{(\Phi+\mu)(B(x,R))}{\mu(B(x,R))} > \frac{(\Phi+\mu)(B(x,R/(10\sigma)^n))}{\mu(B(x,R/(10\sigma)^n))} 2^n \geq 2^n$$

for each $n \in \mathbb{N}$, which is a contradiction. □

Now we are prepared to formulate and prove key lemma of this section.

Lemma 3.2. Let (X, ρ) be a locally compact metric space equipped with a doubling measure μ. Let $N \subset X$, $\mathcal{F} \in \mathcal{D}'_{\mathrm{loc}}(X)$ and $\tau \geq 1$ be a constant. Let $\mathcal{N} = (B(z_n, d_n), d_n \leq 1)$ be a (finite or infinite) system of balls covering N. Further, let us suppose that $\|\mathcal{F}\|^*_{x,r} = o(r\Phi(B(x, \tau r)))$ as $r \to 0_+$ for each $x \in X \setminus N$. Then there exists a constant $c > 0$ such that

$$\langle \mathcal{F}, \eta \rangle \leq c\|\eta\|_{\mathrm{Lip}_b(X)} \sum_n \frac{\|\mathcal{F}\|^*_{z_n,2d_n}}{d_n}$$

for every $\eta \in \mathrm{Lip}_c(X)$.

Proof. The main idea of the proof is to choose $\eta \in \mathrm{Lip}_c(X)$ and then cover its support K using Vitali Theorem by a system of open balls. Further we construct partition of the unity with respect to this system and afterwards use it in the final estimate.

Step 1

Choose $\eta \in \mathrm{Lip}_c(X)$ and $\varepsilon > 0$. Let us denote $K = \mathrm{spt}(\eta)$.

In this step we construct a covering of the set $K \setminus N$. Choose $x \in K \setminus N$. We have assumed that $\|\mathcal{F}\|^*_{x,r} = o(r\Phi(B(x, \tau r)))$ as $r \to 0_+$. Therefore there exists $0 < r_0(x) < 1$ such that for each $0 < r < r_0(x)$ and for each $\vartheta \in \mathrm{Lip}(X)$, $\|\vartheta\|_{x,r} \leq 1$ we have

$$|\langle \mathcal{F}, \vartheta \rangle| < \varepsilon r \Phi(B(x, \tau r)). \tag{3.2}$$

Furthermore, Lemma 3.1 yields the existence of $r(x) > 0$ such that

$$\Phi(B(x, 10\tau r(x))) \leq c_T \Phi(B(x, r(x))) \tag{3.3}$$

and $10\tau r(x) < r_0(x)$.

Since such $r(x)$ exists for each $x \in K \setminus N$, we obtain a covering

$$\mathcal{C} = \{\bar{B}(x, r(x)), x \in K \setminus N\},$$

where $r(x) < 1$.

Then we apply the Vitali Theorem 1.10 and get a disjointed subsystem $\mathcal{C}'$ of $\mathcal{C}$ such that

$$K \setminus N \subset \bigcup_{\bar{B}(x,r) \in \mathcal{C}'} B(x, 5r).$$

Step 2

Now, we build up

$$\mathcal{V} := \{B(x,r) \in \mathcal{N}\} \cup \{B(x,5r); \bar{B}(x,r) \in \mathcal{C}'\},$$

which covers all points in K by open balls. Further, K is compact, so we can find a finite subcovering: $\{B(x_1, r_1), \dots B(x_k, r_k)\}$ such that $B(x_i, r_i) \in \mathcal{V}$ for

$i = 1, \ldots, k$ and $\bigcup_{i=1}^{k} B(x_i, r_i) \supset K$. Moreover, by a careful choice of $r(x)$ we can guarantee that $\bigcup_{i=1}^{k} B(x_i, r_i) \subset K'$, where K' is a compact set. Without loss of generality we can assume that $r_1 \geq r_2 \geq \ldots \geq r_k$.

Step 3

Now we construct a partition of unity. For $i = 1 \ldots k$ we define

$$\kappa_i := \begin{cases} 1, \quad x \in B(x_i, r_i), \\ 1 - \frac{1}{r_i}(\rho(x, x_i) - r_i), \quad x \in B(x_i, 2r_i) \setminus B(x_i, r_i), \\ 0, \quad x \notin B(x_i, 2r_i). \end{cases}$$

Further, set

$$\omega_1 = \sigma_1 := \kappa_1,$$

$$\sigma_i := \max\{\kappa_1, \ldots \kappa_i\}$$

and

$$\omega_i := \sigma_i - \sigma_{i-1}, \quad i = 2, 3, \ldots \ .$$

Then $\mathrm{Lip}(\omega_i) \leq \frac{2}{r_i}$,

$$\sum_{i=1}^{k} \omega_i(x) = 1, \quad x \in K$$

and η can be written as

$$\eta(x) = \sum_{i=1}^{k} \omega_i(x)\eta(x).$$

Step 4

Now we complete our estimates. Let us recall that $\mathcal{F}$ is additive and let us compute

$$\langle \mathcal{F}, \eta \rangle = \sum_{i=1}^{k} \langle \mathcal{F}, \omega_i \eta \rangle = \sum_{\{i: B(x_i, r_i) \in \mathcal{N}\}} \langle \mathcal{F}, \omega_i \eta \rangle + \sum_{\left\{i: \bar{B}\left(x_i, \frac{1}{5} r_i\right) \in \mathcal{C}'\right\}} \langle \mathcal{F}, \omega_i \eta \rangle.$$

Let us start with balls $B(x_i, r_i) \in \mathcal{N}$. We assumed that η is Lipschitz and since η has a compact support, it is also bounded. Functions ω_i are also Lipschitz and of compact support, which follows from the construction. Using Lemma 2.2, we obtain that $\omega_i \eta$ are also Lipschitz. Furthermore, $\mathrm{spt}(\omega_i \eta) \subset B(x_i, 2r_i)$. Now, since $\mathcal{F}$ is a linear functional we have

$$\sum_{\{i: B(x_i, r_i) \in \mathcal{N}\}} \langle \mathcal{F}, \omega_i \eta \rangle \leq \sum_{\{i: B(x_i, r_i) \in \mathcal{N}\}} \|\mathcal{F}\|^{*}_{x_i, 2r_i} \|\omega_i \eta\|_{x_i, 2r_i}. \tag{3.4}$$

Next, since we have that $r_i \leq 1$, we obtain

$$\begin{aligned} \|\omega_i \eta\|_{x_i, 2r_i} &\leq \max\left\{ \|\eta\|_{\sup} \mathrm{Lip}(\omega_i), \frac{1}{2r_i} \|\eta\|_{\sup} \|\omega_i\|_{\sup} \right\} + \|\omega_i\|_{\sup} \mathrm{Lip}(\eta) \\ &\leq \frac{2}{r_i} \|\eta\|_{\mathrm{Lip}_b(X)} + \|\eta\|_{\mathrm{Lip}_b(X)} \leq \frac{3}{r_i} \|\eta\|_{\mathrm{Lip}_b(X)}. \end{aligned} \tag{3.5}$$

Now, for each $i = 1, \dots, k$ there exists $n \in \mathbb{N}$ (different n for different i) such that $x_i = z_n$ and $r_i = d_n$. Let us write z_n, d_n instead of x_i, r_i and continue

$$\begin{aligned}\sum_{\{i:B(x_i,r_i)\in\mathcal{N}\}} \|\mathcal{F}\|^*_{x_i,2r_i} \|\omega_i\eta\|_{x_i,2r_i} &\le 3\|\eta\|_{\mathrm{Lip}_b(X)} \sum_{\{i:B(x_i,r_i)\in\mathcal{N}\}} \frac{\|\mathcal{F}\|^*_{x_i,2r_i}}{r_i} \\ &\le \sum_n 3\|\eta\|_{\mathrm{Lip}_b(X)} \frac{\|\mathcal{F}\|^*_{z_n,2d_n}}{d_n}.\end{aligned} \tag{3.6}$$

Now, we would like to estimate the second sum of (3.1). Let us recall that $\mathcal{F}$ is linear and that $\|\omega_i\eta\|_{x_i,2r_i} \le \frac{3}{r_i}\|\eta\|_{\mathrm{Lip}_b(X)}$. Let us denote $c := 3\|\eta\|_{\mathrm{Lip}_b(X)}$. Then we have

$$\langle \mathcal{F}, \omega_i\eta\rangle = \frac{c}{r_i}\left\langle \mathcal{F}, \frac{r_i\omega_i\eta}{c}\right\rangle$$

and since $\|\frac{r_i\omega_i\eta}{c}\|_{x_i,2r_i} \le 1$, we can use (3.2) and obtain

$$\sum_{\left\{i:\bar{B}\left(x_i,\frac{1}{5}r_i\right)\in\mathcal{C}'\right\}} \langle \mathcal{F}, \omega_i\eta\rangle \le \sum_{\left\{i:\bar{B}\left(x_i,\frac{1}{5}r_i\right)\in\mathcal{C}'\right\}} c\varepsilon\Phi(B(x_i, 2\tau r_i))$$

and from (3.3)

$$\sum_{\left\{i:\bar{B}\left(x_i,\frac{1}{5}r_i\right)\in\mathcal{C}'\right\}} c\varepsilon\Phi(B(x_i, 2\tau r_i)) \le \sum_{\left\{i:\bar{B}\left(x_i,\frac{1}{5}r_i\right)\in\mathcal{C}'\right\}} c\varepsilon c_T\Phi\left(B\left(x_i, \frac{1}{5}r_i\right)\right).$$

Recall that Φ is Radon measure and $\bar{B}\left(x_i, \frac{1}{5}r_i\right)$ are pairwise disjoint. Hence,

$$\sum_{\left\{i:\bar{B}\left(x_i,\frac{1}{5}r_i\right)\in\mathcal{C}'\right\}} c\varepsilon c_T\Phi\left(B\left(x_i, \frac{1}{5}r_i\right)\right) \le c\varepsilon c_T\Phi(K') = \varepsilon c'. \tag{3.7}$$

Now we can draw the conclusion. The estimates (3.6) and (3.7) give us together

$$\langle \mathcal{F}, \eta\rangle \le \sum_{n=1}^{\infty} 3\|\eta\|_{\mathrm{Lip}_b(X)} \frac{\|\mathcal{F}\|^*_{z_n,2d_n}}{d_n} + \varepsilon c'$$

and if we send $\varepsilon \to 0$ the proof is done. □

3.2 UC-integral

In this section we introduce the UC-integral with respect to a distribution. Then we study the relationship between the UC-integral with respect to a measure and the Lebesgue integral.

Definition 3.3. Let (X, ρ) be a locally compact separable metric space equipped with a doubling Radon measure μ and let $f : X \to \mathbb{R}$ be a function. We say that a functional $\mathcal{F} \in \mathcal{D}'_{\mathrm{loc}}(X)$ is an *UC-integral (uniformly controlled integral)* of f with respect to $\mathcal{G} \in \mathcal{D}'_{\mathrm{loc}}(X)$ if there exist $\tau \ge 1$ and a Radon measure Φ such that for all $x \in X$ we have:

$$\|\mathcal{F} - f(x)\mathcal{G}\|^*_{x,r} = o(r\Phi(B(x, \tau r))) \quad \text{as} \quad r \to 0_+. \tag{3.8}$$

For simplicity, we will use the notation

$$\mathcal{E}_1 \sim \mathcal{E}_2, \quad x \in G$$

if $\mathcal{E}_1(x,\cdot), \mathcal{E}_2(x,\cdot) \in \mathcal{D}'_{\text{loc}}(X)$, $x \in G$, and there exist $\tau \geq 1$ and a Radon measure Φ such that for all $x \in G \subset X$ we have:

$$\|\mathcal{E}_1 - \mathcal{E}_2\|^*_{x,r} = o(r\Phi(B(x,\tau r))) \quad \text{as} \quad r \to 0_+,$$

Especially, the integral above can be written as

$$\mathcal{F} \sim f(x)\mathcal{G}, \quad x \in X.$$

Remark 3.4. The role of the scaling parameter τ is to provide the invariance of the integral with respect to bilipschitz mappings. Especially, in the case $X = \mathbb{R}^n$ we can avoid the dependence on the choice of the norm and the geometry of balls.

Remark 3.5. By Theorem 2.17 we know that $\mathrm{Lip}_c(X)$ is weak* dense in $\mathrm{Lip}_b(X)$. Hence the functional $\mathcal{F}$ is determined by its values on $\mathrm{Lip}_c(X)$.

Lemma 3.6. Let $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathcal{D}'_{\text{loc}}(X)$. Suppose that $\mathcal{F} \sim \mathcal{G}$, $x \in X$ and $\mathcal{G} \sim \mathcal{H}$, $x \in X$. Then $\mathcal{F} \sim \mathcal{H}$, $x \in X$.

Proof. Since $\mathcal{F} \sim \mathcal{G}$, we can find $\tau_{\mathcal{F}} \geq 1$ and a Radon measure $\Phi_{\mathcal{F}}$ such that

$$\lim_{r \to 0+} \frac{\|\mathcal{F} - \mathcal{G}\|^*_{x,r}}{r\Phi_{\mathcal{F}}(B(x,\tau_{\mathcal{F}} r))} = 0.$$

Analogously, since $\mathcal{G} \sim \mathcal{H}$, we can find $\tau_{\mathcal{H}} \geq 1$ and a Radon measure $\Phi_{\mathcal{H}}$ such that

$$\lim_{r \to 0+} \frac{\|\mathcal{G} - \mathcal{H}\|^*_{x,r}}{r\Phi_{\mathcal{H}}(B(x,\tau_{\mathcal{H}} r))} = 0.$$

Let us set $\Phi = \max\{\Phi_{\mathcal{F}}, \Phi_{\mathcal{H}}\}$ and $\tau = \max\{\tau_{\mathcal{F}}, \tau_{\mathcal{H}}\}$. Then we have

$$\begin{aligned}
\lim_{r \to 0+} \frac{\|\mathcal{F} - \mathcal{H}\|^*_{x,r}}{r\Phi(B(x,\tau r))} &\leq \lim_{r \to 0+} \frac{\|\mathcal{F} - \mathcal{G}\|^*_{x,r} + \|\mathcal{G} - \mathcal{H}\|^*_{x,r}}{r\Phi(B(x,\tau r))} \\
&\leq \lim_{r \to 0+} \frac{\|\mathcal{F} - \mathcal{G}\|^*_{x,r}}{r\Phi_{\mathcal{F}}(B(x,\tau_{\mathcal{F}} r))} + \lim_{r \to 0+} \frac{\|\mathcal{G} - \mathcal{H}\|^*_{x,r}}{r\Phi_{\mathcal{H}}(B(x,\tau_{\mathcal{H}} r))} \\
&= 0
\end{aligned}$$

and the proof is done. □

Theorem 3.7 (Uniqueness of UC-integral)**.** Let (X,ρ) and μ be as above. Let $\mathcal{H}_1 \in \mathcal{D}'_{\text{loc}}(X)$ and $\mathcal{H}_2 \in \mathcal{D}'_{\text{loc}}(X)$ be UC-integrals of a function $g : X \to \mathbb{R}$ with respect to $\mathcal{G} \in \mathcal{D}'_{\text{loc}}(X)$. Then $\mathcal{H}_1 = \mathcal{H}_2$.

Proof. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are UC-integrals of g, $\mathcal{H}_1 - \mathcal{H}_2$ is an UC-integral of $h \equiv 0$. So, it is enough to prove that if $\mathcal{F}$ is an UC-integral of $f = 0$, then $\langle \mathcal{F}, \eta \rangle = 0$ for each function $\eta \in \mathrm{Lip}_c(X)$. However, this claim easily follows from Lemma 3.2 by setting $N = \emptyset$. □

Notation 3.8. The (unique) indefinite UC-integral of f with respect to $\mathcal{G}$ will be denoted by

$$\mathcal{G}\lfloor f.$$

Theorem 3.9 (Linearity of UC-integral)**.** Let (X,ρ) and μ be as above and let $\alpha,\beta\in\mathbb{R}$. Let $\mathcal{F}\in\mathcal{D}'_{\mathrm{loc}}(X)$ be the UC-integral of a function $f:X\to\mathbb{R}$ and $\mathcal{G}\in\mathcal{D}'_{\mathrm{loc}}(X)$ be the UC-integral of a function $g:X\to\mathbb{R}$ with respect to $\mathcal{H}\in\mathcal{D}'_{\mathrm{loc}}(X)$. Then $\alpha\mathcal{F}+\beta\mathcal{G}$ is the UC-integral of $\alpha f+\beta g$ with respect to $\mathcal{H}$.

Proof. We need to find a Radon measure Φ and a constant $\tau\geq 1$ such that for every $x\in X$ we have

$$\lim_{r\to 0+}\frac{\|\alpha\mathcal{F}+\beta\mathcal{G}-(\alpha f(x)+\beta g(x))\mathcal{H}\|^*_{x,r}}{r\Phi(B(x,\tau r))}=0.$$

Since $\mathcal{F}$ is an UC-integral of f, there exists $\tau_f\geq 1$ and a Radon measure Φ_f such that for all $x\in X$:

$$\lim_{r\to 0+}\frac{\|\mathcal{F}-f(x)\mathcal{H}\|^*_{x,r}}{r\Phi_f(B(x,\tau_f r))}=0.$$

Analogously, there exists $\tau_g\geq 1$ and a Radon measure Φ_g such that for all $x\in X$ we have

$$\lim_{r\to 0+}\frac{\|\mathcal{G}-g(x)\mathcal{H}\|^*_{x,r}}{r\Phi_g(B(x,\tau_g r))}=0.$$

Let us set $\Phi:=\max\{\Phi_f,\Phi_g\}$ and $\tau:=\max\{\tau_f,\tau_g\}$. Then

$$\begin{aligned}
&\lim_{r\to 0+}\frac{\|\alpha\mathcal{F}+\beta\mathcal{G}-(\alpha f(x)+\beta g(x))\mathcal{H}\|^*_{x,r}}{r\Phi(B(x,\tau r))}\\
&\qquad\leq \lim_{r\to 0+}\frac{\|\alpha\mathcal{F}-\alpha f(x)\mathcal{H}\|^*_{x,r}+\|\beta\mathcal{G}-\beta g(x)\mathcal{H}\|^*_{x,r}}{r\Phi(B(x,\tau r))}\\
&\qquad\leq \lim_{r\to 0+}|\alpha|\frac{\|\mathcal{F}-f(x)\mathcal{H}\|^*_{x,r}}{r\Phi(B(x,\tau_f r))}\\
&\qquad\quad+\lim_{r\to 0+}|\beta|\frac{\|\mathcal{G}-g(x)\mathcal{H}\|^*_{x,r}}{r\Phi(B(x,\tau_g r))}\\
&\qquad\leq \lim_{r\to 0+}|\alpha|\frac{\|\mathcal{F}-f(x)\mathcal{H}\|^*_{x,r}}{r\Phi_f(B(x,\tau_f r))}\\
&\qquad\quad+\lim_{r\to 0+}|\beta|\frac{\|\mathcal{G}-g(x)\mathcal{H}\|^*_{x,r}}{r\Phi_g(B(x,\tau_g r))}\\
&\qquad=0,
\end{aligned}$$

which we needed. ☐

In the sequel, we will estimate the relationship between Lebesgue and UC-integrability with respect to a Radon measure.

Notation 3.10. Let ν be a Radon measure on a locally compact metric space X. Then the functional $\mathcal{G}_\nu\in\mathcal{D}'_{\mathrm{loc}}(X)$ defined as

$$\langle\mathcal{G}_\nu,\eta\rangle=\int_X\eta\,\mathrm{d}\nu,\quad \eta\in\mathrm{Lip}_c(X),$$

is called the metric distribution induced by ν.

Lemma 3.11. Let μ be a Radon measure on X and $N \subset X$ be such that $\mathcal{G}_\mu \sim 0$ for every $x \in N$. Then there exists a Radon measure μ^* which is absolutely continuous with respect to μ and a lower semicontinuous function w on X such that $w \geq 1$, $w = \infty$ on N and $d\mu^* = w\, d\mu$.

Proof. Since N is μ-null, we can for every $n \in \mathbb{N}$ find an open set G_n such that $N \subset G_n$ and $\mu(G_n) < 4^{-n}$.

Then, let us define

$$w := 1 + \sum_n 2^n \chi_{G_n}.$$

Since w is a supremum of lower semicontinuous functions, it is also lower semicontinuous. Obviously, $w = \infty$ on N.

Finally, μ^* defined via $\mathrm{d}\mu^* = w\, \mathrm{d}\mu$ is absolutely continuous with respect to μ and the proof is done. $\square$

Lemma 3.12. Let ν be a Radon measure on a locally compact metric space X. Let $f : X \to \mathbb{R}$ be a function such that $f = 0$ ν-a.e. and let $\mathcal{F} \in \mathcal{D}'_{\mathrm{loc}}(X)$ be an UC-integral with respect to $\mathcal{G}_\nu$. Then $\mathcal{F} = 0$.

Proof. Let us denote $N := \{x \in X, f(x) \neq 0\}$. We need to find a Radon measure Φ and $\tau \geq 1$ such that

$$\lim_{r \to 0+} \frac{\|f(x)\mathcal{G}_\nu\|^*_{x,r}}{r\Phi(B(x, \tau r))} = 0, \quad x \in X.$$

This is obvious for $x \notin N$.

Now, let us choose $x \in N$. Since $\nu(N) = 0$, we can see that $\mathcal{G}_\nu \sim 0$ for every $x \in N$. Then by Lemma 3.11 there exists a Radon measure ν^* and a lower semicontinuous function w on X such that $w \geq 1$, $w = \infty$ on N and $d\nu^* = w\, d\nu$.

Now, let us choose $\varepsilon > 0$. Since w is lower semicontinuous and $w = \infty$ on N, we can find $r > 0$ such that $w(y) > |f(x)|/\varepsilon$ on $B(x, r)$. Now, let us choose $\varphi \in \mathrm{Lip}^r_{B(x,r)}(X)$, $\|\varphi\|_{x,r} \leq 1$. Then we have

$$\begin{aligned}|f(x)\langle \mathcal{G}_\nu, \varphi\rangle| &= \left| f(x) \int_{B(x,r)} \varphi \,\mathrm{d}\nu \right| \leq \varepsilon \left| \|\varphi\|_{\sup} \int_{B(x,r)} w \,\mathrm{d}\nu \right| \leq \varepsilon r \|\varphi\|_{x,r} \nu^*(B(x, r)) \\ &\leq \varepsilon r \nu^*(B(x, r)).\end{aligned}$$

If we set $\Phi = \nu^*$ and $\tau = 1$, the proof is done.

$\square$

Theorem 3.13 (Relation to the Lebesgue integral)**.** Let ν be a Radon measure on a locally compact metric space X and let $f : X \to \mathbb{R}$ be locally ν-integrable. Set

$$\langle \mathcal{F},\ \eta \rangle := \int_X f(y)\eta(y) \,\mathrm{d}\nu(y), \quad \eta \in \mathrm{Lip}_b(X).$$

Then $\mathcal{F}$ is the indefinite UC-integral of f with respect to $\mathcal{G}_\nu$.

Proof. We need to find a Radon measure Φ and a constant $\tau \geq 1$ such that for all $x \in X$ we have

$$\|\mathcal{F} - f(x)\mathcal{G}_\nu\|^*_{x,r} = o(r\Phi(B(x, \tau r))) \quad \text{as} \quad r \to 0_+.$$

In other words, we need to find a Radon measure Φ and a constant $\tau \geq 1$ such that for all $x \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all $r \in (0, \delta)$ and $\eta \in \mathrm{Lip}^r_{B(x,r)}(X)$, $\|\eta\|_{x,r} \leq 1$ we have

$$|\langle \mathcal{F}, \eta \rangle - f(x)\langle \mathcal{G}_\nu, \eta \rangle| < \varepsilon r \Phi(B(x, \tau r)).$$

At first, since continuous functions are dense in $L^1(\nu)$ (see [18, Theorem 3.14]), we can find a sequence (f_n) of continuous functions such that

$$\int_X |f_n - f| \, \mathrm{d}\nu \leq 2^{-n-1}.$$

Then we can define functions

$$g_1 = f_1, \quad g_n = f_n - f_{n-1}$$

and

$$g := \sum_n n|g_n|.$$

Since g can be written as

$$g = \sup_n \sum_{k=1}^n k|g_k|,$$

g is lower semicontinuous and hence Lebesgue integrable with respect to ν. Now, let us consider a measure

$$\Phi(E) = \int_E (1 + g) \, d\nu, \quad E \subset X, E \text{ Borel}.$$

Then, let us define a function

$$\bar{f}(x) = \begin{cases} \lim_{n\to\infty} f_n(x), & \sum_n |g_n(x)| < \infty \\ f(x), & \sum_n |g_n(x)| = \infty. \end{cases}$$

Further, fix $x \in X$ and choose $\varepsilon > 0$. Now, we will consider two cases. At first, let us suppose that $\sum_n |g_n(x)| < \infty$. Then we can find $k \in \mathbb{N}$ such that $\varepsilon > 1/k$ and $|f_k(x) - \bar{f}(x)| < \varepsilon$. Since f_k is continuous, there exists $\delta > 0$ such that $|f_k(y) - f_k(x)| < \varepsilon$ for every $y \in B(x, \delta)$. Fix $0 < r < \delta$ and choose $\eta \in \mathrm{Lip}^r_{B(x,r)}(X)$, $\|\eta\|_{x,r} \leq 1$. Then we have

$$\begin{aligned}
|\langle \mathcal{F}, \eta \rangle - \bar{f}(x)\langle \mathcal{G}_\nu, \eta \rangle| &= \left| \int_X (\bar{f} - \bar{f}(x))\eta \, \mathrm{d}\nu \right| \\
&\leq \int_X |\bar{f} - f_k||\eta| \, \mathrm{d}\nu + \int_X |f_k - f_k(x)||\eta| \, \mathrm{d}\nu + \int_X |f_k(x) - \bar{f}(x)||\eta| \, \mathrm{d}\nu \\
&\leq \int_X \left|\frac{g}{k}\right| |\eta| \, \mathrm{d}\nu + \int_X \varepsilon|\eta| \, \mathrm{d}\nu + \int_X \varepsilon|\eta| \, \mathrm{d}\nu \\
&\leq 3\varepsilon r \|\eta\|_{x,r} \Phi(B(x, r)).
\end{aligned}$$

In the second case, we assume that $\sum_n |g_n| = \infty$. Let us find $k \in \mathbb{N}$ such that $\varepsilon > 1/k$.

Since g is lower semicontinuous we can find $\delta > 0$ such that $|f_k(y) - \bar{f}(x)| < \varepsilon g(y)$ for $y \in B(x, \delta)$. Fix $0 < r < \delta$ and choose $\eta \in \operatorname{Lip}^r_{B(x,r)}(X)$, $\|\eta\|_{x,r} \leq 1$. Then we have

$$\begin{aligned}
|\langle \mathcal{F}, \eta\rangle - \bar{f}(x)\langle \mathcal{G}_\nu, \eta\rangle| &= \left| \int_X (\bar{f} - \bar{f}(x))\eta \,\mathrm{d}\nu \right| \\
&\leq \int_X |\bar{f} - f_k||\eta| \,\mathrm{d}\nu + \int_X |f_k - \bar{f}(x)||\eta| \,\mathrm{d}\nu \\
&\leq \int_X \left|\frac{g}{k}\right| |\eta| \,\mathrm{d}\nu + \int_X \varepsilon g |\eta| \,\mathrm{d}\nu \\
&\leq 2\varepsilon r \|\eta\|_{x,r} \Phi(B(x,r)).
\end{aligned}$$

Thus, for all $x \in X$ we have shown that

$$\|\mathcal{F} - \bar{f}(x)\mathcal{G}_\nu\|^*_{x,r} = o(r\Phi(B(x, \tau r))) \quad \text{as} \quad r \to 0_+$$

and hence $\mathcal{F}$ is the UC-integral of $\bar{f}$. The fact that $\mathcal{F}$ is also the UC-integral of f follows from Lemma 3.12. ☐

Our next aim is to prove that if a function is UC-integrable with respect to $\mathcal{G}_\nu$, then it is measurable with respect to ν. Before introducing this theorem we will concentrate on some technical tools.

Definition 3.14. Let μ be a measure on locally compact separable metric space X. Then the set

$$\operatorname{spt} \mu := \{x \in X; \mu(U) > 0 \text{ for every neighbourhood } U \text{ of } x\},$$

is called the *support of* μ.

Lemma 3.15. Let (X, ρ) be a locally compact metric space and let λ, ν be finite Radon measures on X. Then

$$\limsup_{r \to 0+} \frac{\lambda(B(x,r))}{\nu(B(x,5r))} < \infty$$

holds for ν-a.e. $x \in \operatorname{spt} \nu$.

Proof. Let us set

$$A_n := \left\{ x \in X : \limsup_{r \to 0+} \frac{\lambda(B(x,r))}{\nu(B(x,5r))} > n \right\}, \quad n \in \mathbb{N}$$

and suppose $K \subset A_n$ is compact. Applying the Vitali Theorem on K and using properties of compact set we obtain a pairwise disjoint system of open balls $B(x_i, r_i)$, $i = 1, \ldots, k$ such that $K \subset \bigcup_i B(x_i, 5r_i)$ and

$$n\nu(B(x_i, 5r_i)) \leq \lambda(B(x_i, r_i)).$$

Then we can estimate

$$n\nu(K) \leq n \sum_{i=1}^{k} \nu(B(x_i, 5r_i)) \leq \sum_{i=1}^{k} \lambda(B(x_i, r_i)) \leq \lambda(X).$$

Since the set

$$\left\{x \in \operatorname{spt}\nu;\ \sup_{0<r<r_0} \frac{\lambda(B(x,r))}{\nu(B(x,5r))} > n\right\}$$

is open for each $r_0 > 0$, A_n is a Borel set. Now, because ν is a finite Radon measure, we have

$$n\nu(A_n) = \sup_{K \subset A_n} n\nu(K) \leq \lambda(X),$$

where K denotes a compact set. If $n \to \infty$, then $\nu(A_n) \searrow 0$. Since

$$\left\{x \in \operatorname{spt}\nu; \limsup_{r\to 0+} \frac{\lambda(B(x,r))}{\nu(B(x,5r))} = \infty\right\} = \bigcap_{n\in\mathbb{N}} A_n,$$

we obtain that $\nu(A) = 0$ and the statement is proved.

□

Definition 3.16. Let $\mathcal{F} \in \mathcal{D}'_{\mathrm{loc}}(X)$. We say, that $\mathcal{F}$ is *nonnegative* ($\mathcal{F} \geq 0$) if

$$\langle \mathcal{F}, \eta \rangle \geq 0$$

for every $\eta \geq 0$.

Lemma 3.17. Let (X, ρ) be as above and ν be a Radon measure. Let $f : X \to \mathbb{R}$ be a nonnegative function and let $\mathcal{F}$ be the UC-integral with respect to $\mathcal{G}_\nu$. Then $\mathcal{F} \geq 0$.

Proof. We need to show that for every $\eta \in \operatorname{Lip}_c(X)$, $\eta \geq 0$ we have $\langle \mathcal{F}, \eta \rangle \geq 0$.

Let us choose such η, denote $K := \operatorname{spt}\eta$ and continue analogously to the proof of Lemma 3.2. We obtain a finite system of balls $B(x_i, r_i)$, $i = 1, \dots k$, and functions ω_i, such that $\operatorname{spt}\omega_i \subset B(x_i, 2r_i)$, $\operatorname{Lip}(\omega_i) \leq \frac{2}{r_i}$,

$$\bigcup_i B(x_i, r_i) \subset K',$$

where K' is a suitable compact, and

$$\sum_{i=1}^{k} \omega_i(x) = 1, \quad x \in K.$$

Then we have

$$\langle \mathcal{F}, \eta \rangle = \sum_{i=1}^{k} \langle \mathcal{F}, \omega_i \eta \rangle.$$

Since $\|\omega_i \eta\|_{x_i, 2r_i} \leq \frac{3}{r_i} \|\eta\|_{\operatorname{Lip}_b(X)}$ (see 3.2), we continue analogously to Lemma 3.2 (and use the equivalent of (3.1)). By the definition of UC-integral we can for every $\varepsilon > 0$ find $r_0(i) > 0$, such that for every $r < \min_i\{r_0(i)\}$ we have

$$|\langle \mathcal{F}, \omega_i \eta \rangle - f(x)\langle \mathcal{G}_\nu, \omega_i \eta \rangle| < 3\|\eta\|_{\operatorname{Lip}_b(X)} \varepsilon \Phi(B(x_i, \tau r)).$$

Hence we have

$$\langle \mathcal{F}, \omega_i \eta \rangle > f(x)\langle \mathcal{G}_\nu, \omega_i \eta \rangle - c\varepsilon\Phi(B(x_i, \tau r)),$$

where $c = 3\|\eta\|_{\mathrm{Lip}_b(X)}$, and since $f \geq 0$ and $\mathcal{G}_\nu \geq 0$ we obtain

$$\langle \mathcal{F}, \omega_i \eta \rangle > -c\varepsilon \Phi(B(x_i, \tau r)).$$

Together we have

$$\begin{aligned}\langle \mathcal{F}, \eta \rangle &= \sum_{i=1}^{k} \langle \mathcal{F}, \omega_i \eta \rangle \geq -c\varepsilon \sum_{i=1}^{k} \Phi(B(x_i, \tau r)) \geq -c\varepsilon \sum_{i=1}^{k} c_T \Phi(B(x_i, r)) \\ &\geq -c\varepsilon c_T \Phi(K').\end{aligned}$$

If we send $\varepsilon \to 0$, the proof is done.

□

Corollary 3.18. The UC-integral depends monotonically on the integrand.

Notation 3.19. Let (X, ρ) be a metric space. Then

$$\mathrm{Lip}_c^+(X) = \big\{\varphi \in \mathrm{Lip}_c(X); \varphi(x) \geq 0 \quad \forall x \in X\big\}.$$

Theorem 3.20. Let (X, ρ) be a locally compact separable metric space equipped with a doubling measure μ. Let ν be a Radon measure on X and $\mathcal{G}_\nu$ be as in Remark 3.10. Let $f : X \to \mathbb{R}$ be a function and let $\mathcal{F}$ be its UC-integral with respect to $\mathcal{G}_\nu$. Then

(1) f is ν-measurable,

(2) if $\mathcal{F} = 0$ then $f \equiv 0$ ν-a.e.,

(3) if $f \geq 0$ then for each $\varphi \in \mathrm{Lip}_c^+(X)$ we have

$$0 \leq \int_X f\varphi \,\mathrm{d}\nu < \infty.$$

Proof. Step 1

Since $\mathcal{F}$ is the UC-integral of f with respect to $\mathcal{G}_\nu$, we can find a constant $\tau \geq 1$ and a Radon measure Φ such that for all $x \in X$

$$\lim_{r \to 0+} \frac{\|\mathcal{F} - f(x)\mathcal{G}_\nu\|_{x,r}^*}{r\Phi(B(x, \tau r))} = 0. \tag{3.9}$$

Now, let us fix a $\delta \in \mathbb{R}$ and for $k, l \in \mathbb{N}$ define

$$\begin{aligned}A_{k,l} := \big\{x \in X : \langle \mathcal{F}, \varphi \rangle \geq \delta \langle \mathcal{G}_\nu, \varphi \rangle - 2^{-k} r\Phi(\bar{B}(x, \tau r));& \\ \varphi \in \mathrm{Lip}_{B(x,r)}^r(X), \|\varphi\|_{x,r} \leq 1, 0 < r < 1/l&\big\}.\end{aligned}$$

Further, denote

$$A := \bigcap_{k=1}^{\infty} \bigcup_{l=1}^{\infty} A_{k,l}.$$

Our aim is to show that

$$\{x \in X : f(x) \geq \delta\} = A,$$

which ensures that f is measurable, because $A_{k,l}$ are closed. At first, we will show that

$$\{x \in X : f(x) \geq \delta\} \subset A.$$

Equation (3.9) can be rephrased as follows: For each $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for every $0 < r < 1/l$ and for every $\varphi \in \mathrm{Lip}^r_{B(x,r)}(X)$, $\|\varphi\|_{x,r} \leq 1$ we have

$$-\langle \mathcal{F}, \varphi \rangle + f(x)\langle \mathcal{G}_\nu, \varphi \rangle \leq 2^{-k} r \Phi(B(x, \tau r))$$

and

$$\langle \mathcal{F}, \varphi \rangle - f(x)\langle \mathcal{G}_\nu, \varphi \rangle \leq 2^{-k} r \Phi(B(x, \tau r)).$$

For $f(x) \geq \delta$ we obtain

$$-\langle \mathcal{F}, \varphi \rangle + \delta \langle \mathcal{G}_\nu, \varphi \rangle \leq 2^{-k} r \Phi(B(x, \tau r))$$

and consequently

$$\langle \mathcal{F}, \varphi \rangle \geq \delta \langle \mathcal{G}_\nu, \varphi \rangle - 2^{-k} r \Phi(B(x, \tau r)),$$

which gives

$$\{x \in X : f(x) \geq \delta\} \subset A.$$

In order to get the converse inclusion, let us denote $\lambda = \Phi + \mu$ and suppose that $x \in A$ and

$$\limsup_{r \to 0+} \frac{\lambda(B(x,r))}{\nu(B(x,5r))} < \infty. \tag{3.10}$$

As in the proof of Lemma 3.1, we show that

$$\liminf_{r \to 0} \frac{\lambda(B(x,\tau r))}{\lambda(B(x,r/10))} < \infty.$$

Hence, we can find a sequence $(r_n) \searrow 0$ such that

$$\Phi(B(x, \tau r_n) \leq \lambda(B(x, \tau r_n) \leq C\lambda(B(x, r_n/10)) \leq C\nu(B(x, r_n/2)).$$

Furthermore, we can also find a sequence (φ_n) of nonnegative functions $\varphi_n \in \mathrm{Lip}^{r_n}_{B(x,r_n)}(X)$ such that $\|\varphi_n\|_{x,r_n} \leq 1$ and $\varphi_n \geq 1/2$ on $B(x, r_n/2)$. Hence

$$C\nu(B(x, r_n/2)) \leq C\langle \mathcal{G}_\nu, \varphi_n \rangle.$$

Thus, since $x \in A$, we have

$$\begin{aligned}
f(x) &\geq \limsup_{n \to \infty} \frac{(f(x)\langle \mathcal{G}_\nu, \varphi_n \rangle - \langle \mathcal{F}, \varphi_n \rangle) + (\delta \langle \mathcal{G}_\nu, \varphi_n \rangle - 2^{-m} r_n \Phi(B(x, \tau r_n)))}{\langle \mathcal{G}_\nu, \varphi_n \rangle} \\
&\geq \delta - \frac{1}{C} \limsup_{n \to \infty} \frac{|f(x)\langle \mathcal{G}_\nu, \varphi_n \rangle - \langle \mathcal{F}, \varphi_n \rangle| + 2^{-m} r_n \Phi(B(x, \tau r_n))}{r_n \Phi(B(x, \tau r_n))} \\
&\geq \delta - \frac{2^{-m}}{C}, \qquad m = 1, 2, \ldots.
\end{aligned}$$

Because, using Lemma 3.15, for ν-a.e. $x \in A$ (3.10) is satisfied, we obtain $f \geq \delta$ ν-a.e. on A and thus $A \subset \{x \in X : f(x) \geq \delta\}$. Hence, f is ν-measurable.

Step 2

Now, let us suppose $\mathcal{F} = 0$. If we set $\delta = 0$ we obtain $f \geq 0$ ν-a.e. on X and analogously $f \leq 0$ ν-a.e. on X. Hence, $f = 0$.

Step 3

Suppose $f \geq 0$. Since f is ν-measurable, we can find locally ν-integrable functions $f_n \nearrow f$, $f_n \geq 0$. Let us choose $\varphi \in \mathrm{Lip}_c^+(X)$. By the Monotone Convergence Theorem we have

$$\int_X f\varphi \,\mathrm{d}\nu = \lim_{n\to\infty} \int_X f_n\varphi \,\mathrm{d}\nu$$

and by Theorem 3.13 and Corollary 3.18 we obtain

$$0 \leq \lim_{n\to\infty} \int_X f_n\varphi \,\mathrm{d}\nu = \lim_{n\to\infty} \langle \mathcal{G}_\nu \lfloor f_n, \varphi\rangle \leq \langle \mathcal{F}, \varphi\rangle < \infty,$$

which we needed.

□

Corollary 3.21. Let (X, ρ) be a locally compact separable metric space equipped with a doubling measure μ, let ν be a Radon measure on X and $f : X \to \mathbb{R}$ be a function. Then f is locally ν-integrable if and only if there exist both $\mathcal{G}_\nu \lfloor f$ and $\mathcal{G}_\nu \lfloor |f|$.

3.3 $UCN(\zeta)$-integral

In this section we use the definition of the UC-integral equipped with the idea of omitting a set of measure zero, which allows us to integrate another class of functions. However, this feature is offsetted by a new condition on the integral. Let us start with the definition and then continue with some basic properties.

Definition 3.22. Let (X, ρ) be a locally compact separable metric space equipped with a doubling Radon measure μ and let $f : X \to \mathbb{R}$ be a function. Let $\zeta : \mathfrak{B} \to [0, \infty]$ be a function with following property:

$$\zeta(B(x, 2r)) \leq c_\zeta \zeta(B(x, r)) \tag{3.11}$$

and let $\mathcal{G} \in \mathcal{D}'_{\mathrm{loc}}(X)$ be such that $\|\mathcal{G}\|^*_{x,r} = O(r)\zeta(B(x, r))$, as $r \to 0_+$. We say that $\mathcal{F} \in \mathcal{D}'_{\mathrm{loc}}(X)$ is an *$UCN(\zeta)$-integral (UC-integral with H^ζ-neglection)* of f with respect to $\mathcal{G}$ and ζ, if we have

$$\|\mathcal{F}\|^*_{x,r} = O(r)\zeta(B(x, r)), \quad \text{as} \quad r \to 0_+, \tag{3.12}$$

and if there exists $\tau \geq 1$ and a Radon measure Φ such that for $\mathcal{H}^\zeta$-a.e. $x \in X$ we have:

$$\|\mathcal{F} - f(x)\mathcal{G}\|^*_{x,r} = o(r\Phi(B(x, \tau r))) \quad \text{as} \quad r \to 0_+. \tag{3.13}$$

Theorem 3.23 (Uniqueness of $UCN(\zeta)$-integral)**.** Let (X, ρ), μ, ζ and $\mathcal{G}$ be as above. Let $\mathcal{H}_1 \in \mathcal{D}'_{\mathrm{loc}}(X)$ and $\mathcal{H}_2 \in \mathcal{D}'_{\mathrm{loc}}(X)$ be $UCN(\zeta)$-integrals of a function $g : X \to \mathbb{R}$ with respect to $\mathcal{G} \in \mathcal{D}'_{\mathrm{loc}}(X)$. Then $\mathcal{H}_1 \equiv \mathcal{H}_2$.

Proof. Using the same arguments as in the proof of Theorem 3.7 it is enough to prove that if $\mathcal{F}$ is an indefinite integral of $f \equiv 0$, then $\langle \mathcal{F}, \eta \rangle = 0$ for each test function $\eta \in \mathrm{Lip}_c(X)$.

Let $N \subset X$ denote the set of all $x \in X$ for which (3.13) is not satisfied. At first, we construct a covering of N. Obviously, $\mathcal{H}^\zeta(N) = 0 = \lim_{\delta\to 0+} \zeta_\delta(N)$, which means that for given $\varepsilon > 0$ there exists $1 \geq \delta > 0$ such that $\zeta_\delta(N) < \varepsilon$. Further, ζ_δ is defined as

$$\zeta_\delta(N) := \inf \left\{ \sum_{n=1}^{\infty} \zeta(B_n) : B_n \in \mathfrak{B}, \bigcup_{n=1}^{\infty} B_n \supset N, \operatorname{diam} B_n \leq \delta \right\}.$$

Using properties of infimum and the definition, there exists a countable covering $\mathcal{N} \subset \mathfrak{B}$ of N such that $\operatorname{diam} B \leq \delta$ for $B = B(z_n, d_n) \in \mathcal{N}$ and

$$\sum_{B\in\mathcal{N}} \zeta(B) < \varepsilon. \tag{3.14}$$

Now, we apply Lemma 3.2 and for every $\eta \in \mathrm{Lip}_c(X)$ we obtain

$$\langle \mathcal{F}, \eta \rangle \leq c\|\eta\|_{\mathrm{Lip}_b(X)} \sum_n \frac{\|\mathcal{F}\|^*_{z_n,2d_n}}{d_n}.$$

Since $\|\mathcal{F}\|^*_{x,r} = O(r)\zeta(B(x,r))$, as $r \to 0+$, and $\zeta(2B) \leq c_\zeta \zeta(B)$ (property (3.11)) we have

$$\frac{\|\mathcal{F}\|^*_{z_n,2d_n}}{d_n} \leq c\zeta(B(z_n, 2d_n)) \leq c\zeta(B(z_n, d_n)).$$

Applying (3.14) we obtain

$$\langle \mathcal{F}, \eta \rangle \leq c'\varepsilon,$$

which we needed. □

Theorem 3.24 (Linearity of $UCN(\zeta)$-integral)**.** Let (X, ρ), μ and ζ be as above and let $\alpha, \beta \in \mathbb{R}$. Let $\mathcal{F} \in \mathcal{D}'_{\mathrm{loc}}(X)$ be the $UCN(\zeta)$-integral of a function $f : X \to \mathbb{R}$ and $\mathcal{G} \in \mathcal{D}'_{\mathrm{loc}}(X)$ be the $UCN(\zeta)$-integral of a function $g : X \to \mathbb{R}$ with respect to $\mathcal{H} \in \mathcal{D}'_{\mathrm{loc}}(X)$. Then $\alpha\mathcal{F} + \beta\mathcal{G}$ is the $UCN(\zeta)$-integral of $\alpha f + \beta g$ with respect to $\mathcal{H}$.

Proof. We need to find a Radon measure Φ and a constant $\tau \geq 1$ such that for $\mathcal{H}^\zeta$-a.e. $x \in X$ we have

$$\lim_{r\to 0+} \frac{\|\alpha\mathcal{F} + \beta\mathcal{G} - (\alpha f(x) + \beta g(x))\mathcal{H}\|^*_{x,r}}{r\Phi(B(x, \tau r))} = 0.$$

Since $\mathcal{F}$ is an $UCN(\zeta)$-integral of f, there exists $\tau_f \geq 1$ and a Radon measure Φ_f such that for $\mathcal{H}^\zeta$-a.e. $x \in X$ we have

$$\lim_{r\to 0+} \frac{\|\mathcal{F} - f(x)\mathcal{H}\|^*_{x,r}}{r\Phi_f(B(x, \tau_f r))} = 0. \tag{3.15}$$

Analogously, there exists $\tau_g \geq 1$ and a Radon measure Φ_g such that for $\mathcal{H}^\zeta$-a.e. $x \in X$ we have

$$\lim_{r\to 0+} \frac{\|\mathcal{G} - g(x)\mathcal{H}\|^*_{x,r}}{r\Phi_g(B(x, \tau_g r))} = 0. \tag{3.16}$$

Let us set $\Phi := \max\{\Phi_f, \Phi_g\}$ and $\tau := \max\{\tau_f, \tau_g\}$. Then for $\mathcal{H}^\zeta$-a.e. $x \in X$ (3.15) and (3.16) are satisfied and for such $x \in X$ we have

$$\begin{aligned}
\lim_{r\to 0+} & \frac{\|\alpha\mathcal{F} + \beta\mathcal{G} - (\alpha f(x) + \beta g(x))\mathcal{H}\|^*_{x,r}}{r\Phi(B(x,\tau r))} \\
&\le \lim_{r\to 0+} \frac{\|\alpha\mathcal{F} - \alpha f(x)\mathcal{H}\|^*_{x,r} + \|\beta\mathcal{G} - \beta g(x)\mathcal{H}\|^*_{x,r}}{r\Phi(B(x,\tau r))} \\
&\le \lim_{r\to 0+} \alpha\frac{\|\mathcal{F} - f(x)\mathcal{H}\|^*_{x,r}}{r\Phi(B(x,\tau_f r))} \\
&\quad + \lim_{r\to 0+} \beta\frac{\|\mathcal{G} - g(x)\mathcal{H}\|^*_{x,r}}{r\Phi(B(x,\tau_g r))} \\
&\le \lim_{r\to 0+} \alpha\frac{\|\mathcal{F} - f(x)\mathcal{H}\|^*_{x,r}}{r\Phi_f(B(x,\tau_f r))} \\
&\quad + \lim_{r\to 0+} \beta\frac{\|\mathcal{G} - g(x)\mathcal{H}\|^*_{x,r}}{r\Phi_g(B(x,\tau_g r))} \\
&= 0,
\end{aligned}$$

which we needed. □

Remark 3.25. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, such that its total differential exists for $\mathcal{H}$-a.e. $x \in \mathbb{R}^n$. Then the functional $\mathscr{T}$ defined as $\langle \mathscr{T}, \varphi\rangle := -\int f \frac{\partial\varphi}{\partial x_1}$ is the UCN-integral of $\frac{\partial f}{\partial x_1}$ with respect to the Lebesgue measure. Indeed, we have

$$\begin{aligned}
-\int_{\mathbb{R}^n} f\frac{\partial\varphi}{\partial y_1}\,\mathrm{d}y - \frac{\partial f}{\partial x_1}(x)\,\mathrm{d}y \int_{\mathbb{R}^n}\varphi = \\
-\int_{\mathbb{R}^n} \frac{\partial\varphi}{\partial y_1}\left(f(y) - f(x) - \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x)(y_j - x_j)\,\mathrm{d}y\right).
\end{aligned}$$

Further, let us fix $\varepsilon > 0$ and find $r < \varepsilon$. Then for every $\varphi \in \mathrm{Lip}^r_{B(x,r)}(X)$ we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \frac{\partial\varphi}{\partial y_1}\left(f(y) - f(x) - \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x)(y_j - x_j)\right)\mathrm{d}y\right| &< \varepsilon|y-x| \int_{\mathbb{R}^n}\left|\frac{\partial\varphi}{\partial y_1}\right|\mathrm{d}y \\
&< \varepsilon r \mathrm{Lip}(\varphi)\lambda(B(x,r)) \\
&< \varepsilon r\|\varphi\|_{x,r}\lambda(B(x,r)),
\end{aligned}$$

which we needed.

Now, let us verify the condition (3.12). Let us choose an $r > 0$ and a function $\varphi \in \mathrm{Lip}^r_{B(x,r)}(X)$, $\|\varphi\|_{x,r} \le 1$. Then we have

$$|\langle \mathcal{F}, \varphi\rangle| = \left|\int_{\mathbb{R}^n} f\frac{\partial\varphi}{\partial y_1}\,\mathrm{d}y\right| \le \mathrm{Lip}(\varphi)\left|\int_{B(x,r)} f\,\mathrm{d}y\right| \le \|f\|_{\sup}\lambda(B(x,r)) \le crr^{n-1},$$

which we needed.

This is an example of application of the UCN-integral. We have obtained result on "removable singularities", for which purpose the UC-integral does not seem to work.

Remark 3.26. There is no evident relationship between the UC-integral and UCN-integral. While the condition (3.13) allowed us to integrate functions with removable singularities and generally neglect problem sets of zero measure, we pay for this advantage by the requirement (3.12).

Remark 3.27. For comparison see the integral in [6], where the idea of neglecting sets of σ-finite Hausdorff measure was used . This approach yields some type of Gauss-Green Theorem on sets with finite perimeter. Unfortunately, the UCN-integral brings some complications and does not seem to allow such applications. It is a theme for an additional research to find a kind of integral which would share advantages of both UCN-integral and Pfeffer's integral.

Chapter 4

UC-currents

4.1 UC-currents

The idea of metric currents goes back to De Giorgi [7]. The theory of metric currents with a finite mass was then developed by Ambrosio and Kirchheim [2]. However, for our purposes we need to go further and drop the assumption of finite mass.

In this chapter, k denotes an integer $k \in \mathbb{N}_0$, $X = (X, \rho)$ denotes a locally compact metric space as above, Y denotes a Banach space equipped with a norm $\|\cdot\|_Y$ and $L = L(X, Y)$ denotes the space of all linear functionals $X \to Y$.

Notation 4.1. Let (X, ρ) be a metric space and $k \in \mathbb{N}_0$. Then $\mathcal{D}^k(X)$ denotes the set of all ordered $(k+1)$-tuples $g = \hat{g}\ \mathrm{d}\psi_1 \wedge \cdots \wedge \mathrm{d}\psi_k$, where $\hat{g} \in \mathrm{Lip}_b(X)$ and $\psi_i \in \mathrm{Lip}(X)$ for $i = 1, \ldots, k$.

The support of g is defined as the support of $\hat{g}\psi_1 \cdots \psi_k$ and the space of all functions $g \in \mathcal{D}^k(X)$ with compact support is denoted by $\mathcal{D}^k_c$.

Definition 4.2. Let (X, ρ) be a metric space and $k \in \mathbb{N}_0$. We say that an operator $\mathscr{T} : \mathcal{D}^k(X) \to Y$ is an Y-valued *k-dimensional (metric) current on X* if it satisfies the following conditions:

(C1) $\mathscr{T}$ is multilinear.

(C2) (continuity) Let $(\psi_i^n; n \in \mathbb{N})$, $i = 1, \ldots, k$ be a uniformly bounded sequence of Lipschitz functions such that $\psi_i^n(x) \to \psi_i(x)$ for every $x \in X$. Then

$$\mathscr{T}(\hat{g}\ \mathrm{d}\psi_1^n \wedge \cdots \wedge \mathrm{d}\psi_k^n) \to \mathscr{T}(\hat{g}\ \mathrm{d}\psi_1 \wedge \cdots \wedge \mathrm{d}\psi_k)$$

in Y.

(C3) If a linear combination of ψ_i, $i = 1, \ldots, k$ is constant on a neighbourhood of the set $\{\hat{g} \neq 0\}$, then $\mathscr{T}(\hat{g}\ \mathrm{d}\psi_1 \wedge \cdots \wedge \mathrm{d}\psi_k) = 0$. Especially, $\mathscr{T}$ is alternating in $\psi_1, \ldots, \psi_k$.

(C4) We have

$$\begin{aligned}\langle \mathscr{T}, \hat{g}\ \mathrm{d}\varphi\pi \wedge \mathrm{d}\psi_2 \wedge \cdots \wedge \mathrm{d}\psi_k\rangle &= \langle \mathscr{T}, \hat{g}\varphi\ \mathrm{d}\pi \wedge \mathrm{d}\psi_2 \wedge \cdots \wedge \mathrm{d}\psi_k\rangle \\ &\quad + \langle \mathscr{T}, \hat{g}\pi\ \mathrm{d}\varphi \wedge \mathrm{d}\psi_2 \wedge \cdots \wedge \mathrm{d}\psi_k\rangle.\end{aligned}$$

The vector space of all Y-valued k-dimensional metric currents is denoted by $MC_k(X,Y)$.

We will also use the alternative notation

$$\mathscr{T}(g) = \mathscr{T}(\hat{g}, \psi_1, \dots, \psi_k)$$

for $\langle \mathscr{T}, \hat{g} \, \mathrm{d}\psi_1 \wedge \cdots \wedge \mathrm{d}\psi_k \rangle$.

Definition 4.3. The mapping $\mathrm{d} : \mathcal{D}^k(X) \to \mathcal{D}^{k+1}(X)$ defined as

$$\mathrm{d}(\hat{g} \, \mathrm{d}\psi_1 \wedge \cdots \wedge \mathrm{d}\psi_k) = (1 \, \mathrm{d}\hat{g} \wedge \mathrm{d}\psi_1 \wedge \cdots \wedge \mathrm{d}\psi_k), \quad \hat{g} \, \mathrm{d}\psi_1 \wedge \cdots \wedge \mathrm{d}\psi_k \in \mathcal{D}^k(X),$$

is called the *exterior differential*.

Definition 4.4. Let $\mathscr{T} \in MC_{k+1}(X,Y)$. Then the functional $\partial \mathscr{T} : \mathcal{D}^k(X) \to Y$ defined as

$$\partial \mathscr{T}(g) = \mathscr{T}(\mathrm{d}g), \quad g \in \mathcal{D}^k(X),$$

is called the *boundary of* $\mathscr{T}$.

Remark 4.5. We can ask if $\partial \mathscr{T}$ is also a current. It is easily seen that $\partial \mathscr{T}$ is really a current and $\partial \mathscr{T} \in MC_k(X,Y)$.

Now, let us introduce a definition of finite mass according to [2].

Definition 4.6. Let $\mathscr{T} \in MC_k(X,Y)$ be a k-dimensional current. We say that $\mathscr{T}$ is of *finite mass* if there exists a finite Borel measure ν such that

$$\|\langle \mathscr{T}, g \rangle\|_Y \leq \prod_{i=1}^{k} \mathrm{Lip}(\psi_i) \int_X |\hat{g}| \, \mathrm{d}\nu \tag{4.1}$$

for all $g = \hat{g} \, \mathrm{d}\psi_1 \wedge \cdots \wedge \mathrm{d}\psi_k \in \mathcal{D}^k(X)$.

In the case $k = 0$, we define $\prod_i^k \mathrm{Lip}(\psi_i) = 1$.

The minimal measure satisfying (4.1) is said to be the *mass of* $\mathscr{T}$ and is denoted by $\|\mathscr{T}\|$.

Definition 4.7 (Integral with respect to a current)**.** Let (X, ρ) be a locally compact separable metric space equipped with a doubling Radon measure μ. Let $f : X \to L$ be a function and $\mathcal{F} \in \mathcal{D}'_{k,\mathrm{loc}}(X, Z)$, $\mathcal{G} \in \mathcal{D}'_{k,\mathrm{loc}}(X, Y)$ be currents. We say that $\mathcal{F}$ is an indefinite integral of f with respect to $\mathcal{G}$ if for each k-tuple $(\psi_1, \dots, \psi_k)$ of Lipschitz functions on X we get that $\mathcal{F}(\cdot, \psi_1, \dots, \psi_k)$ is an indefinite integral of f with respect to $\mathcal{G}(\cdot, \psi_1, \dots, \psi_k)$. The indefinite integral is uniquely determined by $\mathcal{G}$ and f, it is denoted by $\mathcal{G} \lfloor f$. The definite integral is defined by

$$\int_{\mathcal{G}} f \, d\psi_1 \wedge \cdots \wedge d\psi_k = \langle \mathcal{G} \lfloor f, 1 \, \mathrm{d}\psi_1 \wedge \cdots \wedge \mathrm{d}\psi_k \rangle,$$

if $\mathcal{G} \lfloor f \in \mathcal{D}'_k(X)$.

Examples 4.8. (1) This example recalls the metric distribution induced by a measure. Let (X, ρ) be a locally compact separable metric space equipped with a Radon measure μ. Then we can define a 0-dimensional current $\mathscr{T}_\mu$ as following:

$$\langle \mathscr{T}_\mu, \hat{g} \rangle = \int_X \hat{g} \, \mathrm{d}\mu.$$

(2) Let M be a 2-dimensional smooth surface. Then we can understand integration over M as 2-dimensional current $\mathscr{T}$:

$$\langle \mathscr{T}, \hat{f}\,\mathrm{d}\psi_1 \wedge \mathrm{d}\psi_2\rangle := \int_M \hat{f}\,\mathrm{d}\psi_1 \wedge \mathrm{d}\psi_2.$$

(3) (Volume integral) Given a measurable set $G \subset \mathbb{R}^n$ we can define an n-dimensional metric current $\mathcal{L}_G$ as follows:

$$\langle \mathcal{L}_G, \hat{\varphi}\,\mathrm{d}\psi_1 \wedge \ldots \wedge \mathrm{d}\psi_n\rangle := \int_G \hat{\varphi}(x)\det(\nabla\psi(x))\,\mathrm{d}x,$$

$\hat{\varphi}\,\mathrm{d}\psi_1 \wedge \ldots \wedge \mathrm{d}\psi_n \in \mathcal{D}^n(\mathbb{R}^n)$ and $\psi = (\psi_1, \ldots, \psi_n)$.

(4) Let $G \subset \mathbb{R}^n$ be an open bounded set with C^1 boundary and let $\mathcal{L}_G$ be as above. Then we can define $(n-1)$-dimensional metric current $\partial\mathcal{L}_G$ as in Definition 4.4.

Especially, set $n = 3$, $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ and $\varphi = \varphi_1\,\mathrm{d}y_2 \wedge \mathrm{d}y_3 + \varphi_2\,\mathrm{d}y_3 \wedge \mathrm{d}y_1 + \varphi_3\,\mathrm{d}y_1 \wedge \mathrm{d}y_2$. Then $\mathrm{d}\varphi = \operatorname{div}\vec{\varphi}\,\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_3$ and using the Gauss–Green theorem we obtain

$$\langle \partial\mathcal{L}_G, \varphi\rangle = \int_G \mathrm{d}\varphi = \int_G \operatorname{div}\vec{\varphi}\,\mathrm{d}y = \int_{\partial G} \vec{\varphi}\cdot\vec{\nu}\,\mathrm{d}S = \int_{\partial G} \varphi.$$

(5) We can define 1-dimensional current

$$\hat{f}\,\mathrm{d}\psi_1 \mapsto \int_{\mathbb{R}^n} \hat{f}(x)\frac{\partial\psi_1(x)}{\partial x_1}\,\mathrm{d}x.$$

Let us notice, that the dimension of the current does not correspond with the dimension of integration.

(6) Let $G \subset \mathbb{R}^3$ be an open set, $f : G \to \mathbb{R}^3$ be a smooth vector field and let $\varphi \in \mathcal{D}_c(\Omega)$. Then

$$\int_G \varphi(x)\operatorname{div} f(x)\,\mathrm{d}x = -\partial(\mathscr{T}\lfloor f)(\varphi),$$

where

$$\mathscr{T}(\varphi,\psi) = \mathbf{e}_1\int_G \varphi\,\mathrm{d}\psi \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathbf{e}_2\int_G \varphi\,\mathrm{d}\psi \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_1 + \mathbf{e}_3\int_G \varphi\,\mathrm{d}\psi \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2.$$

(7) This example deals integrals which appear in the classical Stokes Theorem. We can define $\mathbb{R}^3$ valued 1-current as follows

$$\langle \mathscr{T}, \hat{f}\,\mathrm{d}\psi\rangle := \sum_{i=1}^{3} \mathbf{e}_i \int_M \varphi\,\mathrm{d}\psi \wedge \mathrm{d}x_i.$$

Especially, for $f : M \to \mathbb{R}^3$, $f = (f_1, f_2, f_3)$ a smooth vector field, $g = (g_1, g_2, g_3) = \mathbf{curl}\, f$ and $\varphi \in \mathcal{D}_c(M)$ we obtain

$$\int_M \varphi(g_1\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + g_2\,\mathrm{d}x_3 \wedge \mathrm{d}x_1 + g_3\,\mathrm{d}x_1 \wedge \mathrm{d}x_2) = \sum_{i=1}^{3} \int_M \varphi\,\mathrm{d}f_i \wedge \mathrm{d}x_i$$
$$= -\sum_{i=1}^{3} \int_M f_i\,\mathrm{d}\varphi \wedge \mathrm{d}x_i = -(\mathscr{T}\lfloor f)(1, \varphi) = -\partial(\mathscr{T}\lfloor f)(\varphi).$$

4.2 Gauss-Green Theorem

Definition 4.9. Let $\mathscr{T}$ be a 1-current. We say that a set $N \subset X$ is *$\mathscr{T}$-null* if we have

$$\mathscr{T} \sim 0, \quad x \in N.$$

Definition 4.10. Let (X, ρ) be a metric space and μ a measure on X. Let $G \subset X$ be a μ-measurable set. We say that a point $x \in X$ is a *density point of* G if

$$\limsup_{r \to 0+} \frac{\mu(B(x,r) \cap (X \setminus G))}{\mu(B(x,r))} = 0.$$

Definition 4.11. Let μ be a measure on X and let $G \subset X$ be a μ-measurable set. Then the *μ-density topology boundary* $\partial_\mu G$ is defined by

$$\partial_\mu G := \left\{ x \in G; \limsup_{r \to 0+} \frac{\mu(B(x,r) \cap (X \setminus G))}{\mu(B(x,r))} > 0 \right\}$$
$$\cup \left\{ x \in X \setminus G; \limsup_{r \to 0+} \frac{\mu(B(x,r) \cap G)}{\mu(B(x,r))} > 0 \right\}.$$

Let us mention Vitali-Carathéodory Theorem. For the proof of the following Theorem, see [18, Theorem 2.24].

Theorem 4.12 (Vitali-Carathéodory)**.** Suppose $f \in L^1(\mu)$, f is real valued, and $\varepsilon > 0$. Then there exist functions u and v on X such that $v \le f \le u$, v is upper semicontinuous and bounded above, u is lower semicontinuous and bounded below, and

$$\int_X (u - v)\,\mathrm{d}\mu < \varepsilon.$$

Lemma 4.13. Let $\mathscr{T}$ be a 1-current, $f : X \to \mathbb{R}$ a function and let $\mathscr{T}\lfloor f$ exists. Let $g, \psi \in \mathrm{Lip}_c(X)$. If $f\psi = 0$ on X, then $\langle \mathscr{T}\lfloor f, g\,\mathrm{d}\psi \rangle = 0$.

Proof. Given $\psi \in \mathrm{Lip}_c(X)$, we would like to prove that

$$\mathscr{S}\lfloor f = 0,$$

where $\mathscr{S}$ is the metric distribution $\varphi \mapsto \langle \mathscr{T}, \varphi\,\mathrm{d}\psi \rangle$. Then we obtain that $\langle \mathscr{T}\lfloor f, g\,\mathrm{d}\psi \rangle = 0$ by the uniqueness of the UC-integral.

Since we can write $\psi = \psi^+ - \psi^-$ and $\mathscr{T}\lfloor f$ is multilinear, we can suppose that $\psi \ge 0$.

Now, we consider two possibilities. At first, let $\operatorname{spt} f \cap \operatorname{spt} \psi = \emptyset$. By the definition of UC-integral there exists a Radon measure Φ and a constant $\tau \geq 1$ such that for every $x \in X$ and every $\varepsilon > 0$ there exists r_0 such that for every $r < r_0$ and every $\varphi \in \mathrm{Lip}^r_{B(x,r)}(X)$ we have

$$|\langle \mathscr{T} \lfloor f, \varphi \, \mathrm{d}\psi \rangle| < |f(x) \langle \mathscr{T}, \varphi \, \mathrm{d}\psi \rangle| + \varepsilon r \Phi(B(x, \tau r)).$$

Now we distinguish two cases. At first, let $x \notin \operatorname{spt} f$. Then $f(x) = 0$ and we have

$$|\langle \mathscr{T} \lfloor f, \varphi \, \mathrm{d}\psi \rangle| < \varepsilon r \Phi(B(x, \tau r)).$$

Now, let $x \in \operatorname{spt} f$. Then there exists $r > 0$ such that $\psi = 0$ on a neighbourhood of $\bar{B}(x, r)$. Hence, for $\varphi \in \mathrm{Lip}^r_{B(x,r)}(X)$ the condition (C3) is satisfied and $\langle \mathscr{T}, \varphi \, \mathrm{d}\psi \rangle = 0$. Thus

$$|\langle \mathscr{T} \lfloor f, \varphi \, \mathrm{d}\psi \rangle| < \varepsilon r \Phi(B(x, \tau r)).$$

Now, let us suppose that $\operatorname{spt} f \cap \operatorname{spt} \psi \neq \emptyset$. Then let us define a sequence (ψ_n), $\psi_n := (\psi - 2^{-n})^+$. Now, if $x \notin \operatorname{spt} f$, we continue as above. On the other hand, if $x \in \operatorname{spt} f$, we can find a neighbourhood of x such that $\psi_n = 0$ and hence there exists $r_0 > 0$ such that for all $r < r_0$ and for every $\varphi \in \mathrm{Lip}^r_{B(x,r)}(X)$ the condition (C3) is satisfied (as above) and hence $\langle \mathscr{T}, \varphi \, \mathrm{d}\psi_n \rangle = 0$.

Finally, since $\psi_n(x) \to \psi(x)$ for every $x \in X$, we obtain the statement by (C2).

□

Lemma 4.14. Let $\mathscr{T}$ be a 1-current and let μ be a doubling Radon measure majorizing the mass of $\mathscr{T}$. Let $\varphi, \psi \in \mathrm{Lip}_c(X)$ and let $f \in L^1_{\mathrm{loc}}(X)$. Then

$$|\langle \mathscr{T} \lfloor f, \varphi \, \mathrm{d}\psi \rangle| \leq \|\psi\|_{x,r} \int_W |f \varphi| \, \mathrm{d}\mu,$$

where $W = \{\psi \neq 0\}$.

Proof. Applying Lemma 4.13, we can consider $f = 0$ on $X \setminus W$. Then we will use the same approach as in Lemma 3.2.

Step 1

At first, we would like to construct a covering of $K := \operatorname{spt}(\varphi)$.

Let us pick an $\varepsilon > 0$. Applying Theorem 4.12 on $f\varphi$, we can find a lower semicontinuous function $u : X \to \mathbb{R}$ such that $u \geq |f|$ and

$$\int_X u \varphi \, \mathrm{d}\mu < \int_X |f \varphi| \, \mathrm{d}\mu + \varepsilon. \tag{4.2}$$

Since u is lower semicontinuous, for every $x \in X$ we can find $1 > \delta(x) > 0$ such that

$$|f(x)| < u(y) \tag{4.3}$$

for all $y \in B(x, 10\delta(x))$.

By the Definition of UC-integral, we can for every $x \in X$ find $r_0(x) > 0$, $\delta(x) \geq r_0(x)$ such that for every $0 < r < r_0(x)$ we have

$$|\langle \mathscr{T} \lfloor f, \eta \, \mathrm{d}\psi \rangle - f(x) \langle \mathscr{T}, \eta \, \mathrm{d}\psi \rangle| \leq \varepsilon r \Phi(B(x, \tau r)) \tag{4.4}$$

for every $\eta \in \mathrm{Lip}^{10r}_{B(x,10r)}(X)$, $\|\eta\|_{x,10r} \leq 1$.

Furthermore, Lemma 3.1 yields the existence of a constant c_T and $r(x) > 0$ such that

$$\Phi(B(x, 10\tau r(x))) \leq c_T \Phi(B(x, r(x))) \tag{4.5}$$

and $10\tau r(x) < r_0(x)$.

Thus, we obtain a covering $\mathcal{C} := \{\bar{B}(x, r(x)); x \in K\}$. Moreover, since X is locally compact, a suitable choice of $r(x)$ ensures the existence of a compact K' such that $\bigcup_{\mathcal{C}} \bar{B}(x, r(x)) \subset K'$.

Step 2

Now we apply the Vitali Theorem and get a disjointed subsystem $\mathcal{C}'$ of $\mathcal{C}$ such that

$$K \subset \bigcup_{\bar{B}(x,r) \in \mathcal{C}'} B(x, 5r).$$

Since K is compact, we can find a finite subcovering: $\{B(x_1, r_1), \ldots, B(x_k, r_k)\}$ such that $B(x_i, r_i/5) \in \mathcal{C}'$ for $i = 1, \ldots, k$ and $\bigcup_{i=1}^k B(x_i, r_i) \supset K$. Without loss of generality we can assume that $r_1 \geq r_2 \geq \ldots \geq r_k$.

Now we construct a partition of unity. Define

$$\kappa_i := \begin{cases} 1, & x \in B(x_i, r_i), \\ 1 - \frac{1}{r_i}(\rho(x, x_i) - r_i), & x \in B(x_i, 2r_i) \setminus B(x_i, r_i), \\ 0, & x \notin B(x_i, 2r_i), \end{cases}$$

where $i = 1 \ldots k$.

Further, set

$$\omega_1 = \sigma_1 =: \kappa_1,$$

$$\sigma_i := \max\{\kappa_1, \ldots, \kappa_i\}$$

and

$$\omega_i := \sigma_i - \sigma_{i-1}, \quad i = 2, 3, \ldots \; .$$

Then $\mathrm{Lip}(\omega_i) \leq \frac{2}{r_i}$,

$$\sum_{i=1}^k \omega_i(x) = 1, \quad x \in K$$

and φ can be written as

$$\varphi(x) = \sum_{i=1}^k \omega_i(x)\varphi(x).$$

Step 3

Using the linearity of $\mathscr{T}$, we have

$$|\langle \mathscr{T} \lfloor f, \varphi \, \mathrm{d}\psi \rangle| = \left| \left\langle \mathscr{T} \lfloor f, \sum_{i=1}^k \omega_i \varphi \, \mathrm{d}\psi \right\rangle \right| \leq \sum_{i=1}^k |\langle \mathscr{T} \lfloor f, \omega_i \varphi \, \mathrm{d}\psi \rangle| \, .$$

Analogously to Lemma 3.2 we obtain that $\mathrm{spt}(\omega_i \varphi) \subset B(x_i, 10r(x_i))$ and $\|\omega_i \varphi\|_{x_i, 10r(x_i)} \leq \frac{3}{r(x_i)} \|\varphi\|_{\mathrm{Lip}_b(X)}$. Let us denote $c := 3\|\varphi\|_{\mathrm{Lip}_b(X)}$. Then we use

(4.4) and (4.5) and we obtain

$$\begin{aligned}\sum_{i=1}^{k} |\langle \mathscr{T}\lfloor f, \omega_i\varphi\,\mathrm{d}\psi\rangle| &\le \sum_{i=1}^{k} |f(x_i)\,\langle \mathscr{T}, \omega_i\varphi\,\mathrm{d}\psi\rangle| \\ &\quad + \frac{c}{r(x_i)}\sum_{i=1}^{k} \varepsilon r(x_i)\Phi(B(x_i, 10\tau r(x_i))) \\ &\le \sum_{i=1}^{k} |f(x_i)\,\langle \mathscr{T}, \omega_i\varphi\,\mathrm{d}\psi\rangle| + c\varepsilon \sum_{i=1}^{k} c_T\Phi(B(x_i, r(x_i))) \\ &\le \sum_{i=1}^{k} |f(x_i)\,\langle \mathscr{T}, \omega_i\varphi\,\mathrm{d}\psi\rangle| + c\varepsilon c_T\Phi(K').\end{aligned}$$

Therefore, we can apply (4.3). Hence, since μ majorizes the mass of $\mathscr{T}$, we estimate

$$\begin{aligned}\sum_{i=1}^{k} |f(x_i)|\,|\langle \mathscr{T}, \omega_i\varphi\,\mathrm{d}\psi\rangle| &\le \sum_{i=1}^{k} \|\psi\|_{x,r}\int_X |f(x_i)|\omega_i(y)\varphi(y)\,\mathrm{d}\mu \\ &= \|\psi\|_{x,r}\int_X \sum_{i=1}^{k} |f(x_i)|\omega_i(y)\varphi(y)\,\mathrm{d}\mu \\ &\le \|\psi\|_{x,r}\int_X u(y)\varphi(y)\,\mathrm{d}\mu \le \|\psi\|_{x,r}\int_X |f(y)\varphi(y)|\,\mathrm{d}\mu + \varepsilon.\end{aligned}$$

Together these estimates give us

$$|\langle \mathscr{T}\lfloor f, \varphi\,\mathrm{d}\psi\rangle| \le \|\psi\|_{x,r}\int_X |f(y)\varphi(y)|\,\mathrm{d}\mu + \varepsilon + \varepsilon c c_T\Phi(K'),$$

which we needed.

□

Remark 4.15. Let us fix $\mathscr{T} \in MC_k(X)$, then

$$\langle \mathscr{T}, \varphi\rangle - 1\langle \mathscr{T}, \varphi\rangle \sim 0$$

and hence from the uniqueness of the UC-integral we have

$$\langle \mathscr{T}\lfloor 1, \varphi\rangle = \langle \mathscr{T}, \varphi\rangle.$$

Theorem 4.16 (Gauss–Green–Stokes)**.** Let $\mathscr{T}$ be a boundary-free 1-current, $G \subset X$ be a μ-measurable set and f be a continuous function on X. Let χ be the characteristic function of G. Further, let the following assumptions hold:

(G1) μ majorizes the mass of $\mathscr{T}$,

(G2) there exists an "isoperimetric measure" ν such that

$$\min\{\mu(B(x,r)\cap G), \mu(B(x,r)\setminus G)\} = O(r)\,\nu(B(x,r)), \qquad r \to 0+,\ x \in X, \tag{4.6}$$

(G3) we have

$$0 < \liminf_{r\to 0_+} \frac{\mu(B(x,r)\cap G)}{\mu(B(x,r))} \le \limsup_{r\to 0_+} \frac{\mu(B(x,r)\cap G)}{\mu(B(x,r))} < 1, \tag{4.7}$$

$\partial(\mathscr{T}\lfloor f)$-a.e. on the μ-density boundary of G.

Then the Gauss-Green-Stokes formula

$$\partial(\mathscr{T}\lfloor \chi f) = \partial(\mathscr{T}\lfloor f)\lfloor \chi + \partial(\mathscr{T}\lfloor \chi)\lfloor f \tag{4.8}$$

holds if at least one of the integrals on the right makes sense.

Proof. The proof is divided into several steps. In the first two steps we will show that it does not matter, which integral on the right side of (4.8) makes sense and find out the condition, which we need to prove:

$$\partial(\mathscr{T}\lfloor \chi(f - f(x))) - \partial(\chi(x)\mathscr{T}\lfloor (f - f(x))) \sim 0.$$

In the third step we will express the final condition, which will be proved in the last, fourth, step, in which we will consider various $x \in X$ and, applying assumptions of the Theorem, we finish the proof.

Step 1

Theorem 3.13 shows the existence of $\mathscr{T}\lfloor \chi f$. Hence, the boundary $\partial(\mathscr{T}\lfloor \chi f)$ is well defined according to Remark 4.4.

Now, let us assume that $\partial(\mathscr{T}\lfloor f)\lfloor \chi$ exists. Then we need to show that the functional

$$\partial(\mathscr{T}\lfloor \chi f) - \partial(\mathscr{T}\lfloor f)\lfloor \chi$$

is the UC-integral of f with respect to $\partial(\mathscr{T}\lfloor \chi)$. In other words, that for every $x \in X$ we have

$$\partial(\mathscr{T}\lfloor \chi f) - \partial(\mathscr{T}\lfloor f)\lfloor \chi \sim f(x)\partial(\mathscr{T}\lfloor \chi). \tag{4.9}$$

Our aim is to formulate an equivalent condition, which, however, will be easier to prove. To do this, let us apply some facts. Using the definition of the boundary of current and linearity of $\mathscr{T}$, we have

$$\langle \partial(\mathscr{T}\lfloor \chi f), \varphi\rangle = \langle \mathscr{T}\lfloor \chi(f - f(x)), \mathrm{d}\varphi\rangle + f(x)\langle \mathscr{T}\lfloor \chi, \mathrm{d}\varphi\rangle$$

and

$$\langle \partial(\mathscr{T}\lfloor f)\lfloor \chi, \varphi\rangle = \langle (\mathscr{T}\lfloor f)\lfloor \chi, \mathrm{d}\varphi\rangle.$$

Next, by the Definition of UC-integral for every $x \in X$ we obtain

$$\partial((\mathscr{T}\lfloor f)\lfloor \chi) \sim \chi(x)\partial(\mathscr{T}\lfloor f).$$

Further, since $\partial\mathscr{T} = 0$, we have

$$\begin{aligned}\chi(x)\langle \mathscr{T}\lfloor f, \mathrm{d}\varphi\rangle &= \chi(x)\langle \mathscr{T}\lfloor (f - f(x)), \mathrm{d}\varphi\rangle + f(x)\chi(x)\langle \mathscr{T}\lfloor 1, \mathrm{d}\varphi\rangle \\ &= \chi(x)\langle \mathscr{T}\lfloor (f - f(x)), \mathrm{d}\varphi\rangle + f(x)\chi(x)\langle \partial\mathscr{T}, \varphi\rangle \\ &= \chi(x)\langle \mathscr{T}\lfloor (f - f(x)), \mathrm{d}\varphi\rangle.\end{aligned}$$

Now, applying Lemma 3.6, we can reformulate (4.9) as

$$\partial(\mathscr{T}\lfloor \chi(f - f(x))) + f(x)\partial(\mathscr{T}\lfloor \chi, \mathrm{d}\varphi) - \chi(x)\partial(\mathscr{T}\lfloor (f - f(x))) \sim f(x)\partial(\mathscr{T}\lfloor \chi),$$

hence

$$\langle \mathscr{T}\lfloor\chi(f-f(x)),\,\mathrm{d}\varphi\rangle - \chi(x)\langle \mathscr{T}\lfloor(f-f(x)),\,\mathrm{d}\varphi\rangle \sim 0.$$

Step 2

On the other hand, let us assume the existence of $\partial(\mathscr{T}\lfloor\chi)\lfloor f$. Then we need to prove that

$$\partial(\mathscr{T}\lfloor\chi f) - \partial(\mathscr{T}\lfloor\chi)\lfloor f$$

is the UC-integral of χ with respect to $\partial(\mathscr{T}\lfloor f)$. Hence, we need to show that for every $x \in X$ we have

$$\partial(\mathscr{T}\lfloor\chi f) - \partial(\mathscr{T}\lfloor\chi)\lfloor f \sim \chi(x)\partial(\mathscr{T}\lfloor f).$$

We use the same considerations as above and we obtain

$$\partial(\mathscr{T}\lfloor\chi(f-f(x))) + f(x)\partial(\mathscr{T}\lfloor\chi) - f(x)\partial(\mathscr{T}\lfloor\chi) \sim \chi(x)\partial(\mathscr{T}\lfloor f),$$

which is the same as in (4.9).

Step 3

In the first two steps we have shown, that if there exists either $\partial(\mathscr{T}\lfloor f)\lfloor\chi$ or $\partial(\mathscr{T}\lfloor\chi)\lfloor f$ we need to prove that

$$\partial(\mathscr{T}\lfloor\chi(f-f(x))) + f(x)\partial(\mathscr{T}\lfloor\chi) - f(x)\partial(\mathscr{T}\lfloor\chi) \sim \chi(x)\partial(\mathscr{T}\lfloor f).$$

Now, the following process is the same in both cases. Let us denote by χ^c the characteristic function of G^c. Then, since the UC-integral is linear, we can write

$$\chi(x)\partial(\mathscr{T}\lfloor(f-f(x))) = \chi(x)\partial(\mathscr{T}\lfloor\chi(f-f(x))) + \chi(x)\partial(\mathscr{T}\lfloor\chi^c(f-f(x))).$$

Thus, we need to prove that there exists Φ and τ such that for every $x \in X$ there exists $r_0 > 0$ such that for every $r < r_0$ and $\varphi \in \mathrm{Lip}^r_{B(x,r)}(X)$, $\|\varphi\|_{x,r} \leq 1$ we have

$$\begin{aligned}\|\langle \mathscr{T}\lfloor\chi(f-f(x)),\,\mathrm{d}\varphi\rangle &- \chi(x)\langle \mathscr{T}\lfloor\chi(f-f(x)),\,\mathrm{d}\varphi\rangle \\ &+ \chi(x)\langle \mathscr{T}\lfloor\chi^c(f-f(x)),\,\mathrm{d}\varphi\rangle\|^*_{x,r} < \varepsilon\Phi(B(x,\tau r)).\end{aligned}$$

Step 4

Let us fix $\varepsilon > 0$ and $x \in X$. Now, we will consider several cases depending on x. They give us estimates of Φ and τ, which complete the whole proof.

(1) Let $x \in G$ and let x be a density point of G. Then we need to show that there exists $r_0 > 0$ such that for every $r < r_0$ we have

$$\|\langle \mathscr{T}\lfloor\chi^c(f-f(x)),\,\mathrm{d}\varphi\rangle\|^*_{x,r} < \varepsilon r\Phi(B(x,r)).$$

Since f is continuous, we can find $\delta > 0$ such that for every $y \in B(x,\delta)$ we have $|f(y)-f(x)| < \varepsilon$. Further, since x is a density point of G, we find $r_1 > 0$, $\delta > r_1$ such that for every $r > 0$, $r < r_1$ the estimate $\mu(B(x,r)\cap G^c) \leq \mu(B(x,r)\cap G)$ holds. Then, using Lemma 4.14, for every $\varphi \in \mathrm{Lip}^r_{B(x,r)}(X)$, $\|\varphi\|_{x,r} \leq 1$ we obtain

$$\begin{aligned}|\langle \mathscr{T}\lfloor\chi^c(f-f(x)),\,\mathrm{d}\varphi\rangle| &\leq \|\varphi\|_{x,r}\int_{B(x,r)} |\chi^c(f-f(x))|\,\mathrm{d}\mu \\ &\leq \varepsilon\|\varphi\|_{x,r}\mu(B(x,r)\cap G^c).\end{aligned}$$

Now, by the assumption (4.6), there exist $r_0 > 0$, $r_0 < r_1$ and a constant $C > 0$ such that for every $r > 0$, $r < r_0$ we have

$$\mu(B(x,r) \cap G^c) \leq Cr\nu(B(x,r)).$$

Together

$$|\langle \mathscr{T} \lfloor \chi^c(f - f(x)), \mathrm{d}\varphi \rangle| \leq \varepsilon \|\varphi\|_{x,r} Cr\nu(B(x,r)).$$

(2) Let $x \in G^c$ and let x be a density point of G^c. Then we need to show that there exists $r_0 > 0$ such that for every $r < r_0$ we have

$$\|\langle \mathscr{T} \lfloor \chi(f - f(x)), \mathrm{d}\varphi \rangle\|^*_{x,r} < \varepsilon r \Phi(B(x,r)).$$

Analogously, for suitable $\delta, r_1 > 0$ we have for every $r < r_1$ and $\varphi \in \mathrm{Lip}^r_{B(x,r)}(X)$, $\|\varphi\|_{x,r} \leq 1$

$$\begin{aligned}\langle \mathscr{T} \lfloor \chi(f - f(x)), \mathrm{d}\varphi \rangle &\leq \|\varphi\|_{x,r} \int_{B(x,r)} |\chi(f - f(x))| \,\mathrm{d}\mu \\ &\leq \varepsilon \|\varphi\|_{x,r} \mu(B(x,r) \cap G)\end{aligned}$$

and since x is a density point of G^c, we have for every $r > 0$, where $r < r_0$ for some suitable r_0

$$\mu(B(x,r) \cap G) \leq Cr\nu(B(x,r)).$$

Together we obtain

$$\langle \mathscr{T} \lfloor \chi(f - f(x)), \mathrm{d}\varphi \rangle \leq \varepsilon \|\varphi\|_{x,r} Cr\nu(B(x,r)).$$

(3) Now, consider $x \in \partial_\mu G$, such that (4.7) is satisfied. Let us find $\delta > 0$ such that for every $y \in B(x,\delta)$ we have $|f(y) - f(x)| < \varepsilon$.

If either $x \in G$ and $\mu(B(x,r) \setminus G) = O(r)\nu(B(x,r))$, $r \to 0+$, or $x \in G^c$ and $\mu(B(x,r) \cap G) = O(r)\nu(B(x,r))$, $r \to 0+$, we proceed as above.

In the other cases, we will consider the estimate, which follows from (4.7): there exist $r_0 > 0$ and $M > 0$ such that for every $r > 0$, $r < r_0$ we have

$$\begin{aligned}&\max\{\mu(B(x,r) \cap G), \mu(B(x,r) \setminus G)\} \\ &\leq M \min\{\mu(B(x,r) \cap G), \mu(B(x,r) \setminus G)\}.\end{aligned}$$

Applying this, we obtain

$$\begin{aligned}\langle \mathscr{T} \lfloor \chi(f - f(x)), \mathrm{d}\varphi \rangle &\leq \|\varphi\|_{x,r} \int_{B(x,r)} |\chi(f - f(x))| \,\mathrm{d}\mu \\ &\leq \varepsilon \|\varphi\|_{x,r} \mu(B(x,r) \cap G) \leq M\varepsilon \|\varphi\|_{x,r} \mu(B(x,r) \setminus G) \\ &\leq \varepsilon MCr \|\varphi\|_{x,r} \nu(B(x,r))\end{aligned}$$

or

$$\begin{aligned}\langle \mathscr{T} \lfloor \chi^c(f - f(x)), \mathrm{d}\varphi \rangle &\leq \|\varphi\|_{x,r} \int_{B(x,r)} |\chi^c(f - f(x))| \,\mathrm{d}\mu \\ &\leq \varepsilon \|\varphi\|_{x,r} \mu(B(x,r) \setminus G) \leq M\varepsilon \|\varphi\|_{x,r} \mu(B(x,r) \cap G) \\ &\leq \varepsilon MCr \|\varphi\|_{x,r} \nu(B(x,r)).\end{aligned}$$

(4) Finally, consider $x \in \partial_\mu G$ such that (4.7) is not satisfied. As above, we can find $\delta > 0$ such that for every $y \in B(x,\delta)$ we have $|f(y) - f(x)| < \varepsilon$.

If either $x \in G$ and $\mu(B(x,r) \setminus G) = O(r)\nu(B(x,r))$, $r \to 0+$, or $x \in G^c$ and $\mu(B(x,r) \cap G) = O(r)\nu(B(x,r))$, $r \to 0+$, we proceed as above.

If not, we use the following estimate

$$|\langle \mathscr{T}\lfloor\chi(f - f(x)), \mathrm{d}\varphi\rangle - \langle \mathscr{T}\lfloor\chi^c(f - f(x)), \mathrm{d}\varphi\rangle| \le |\langle \mathscr{T}\lfloor(f - f(x)), \mathrm{d}\varphi\rangle|.$$

Hence

$$|\langle \mathscr{T}\lfloor\chi(f - f(x)), \mathrm{d}\varphi\rangle| \le |\langle \mathscr{T}\lfloor\chi^c(f - f(x)), \mathrm{d}\varphi\rangle| + |\langle \mathscr{T}\lfloor(f - f(x)), \mathrm{d}\varphi\rangle|$$

or

$$|\langle \mathscr{T}\lfloor\chi^c(f - f(x)), \mathrm{d}\varphi\rangle| \le |\langle \mathscr{T}\lfloor\chi(f - f(x)), \mathrm{d}\varphi\rangle| + |\langle \mathscr{T}\lfloor(f - f(x)), \mathrm{d}\varphi\rangle|.$$

Next, by Definition 4.9, there exists a Radon measure Φ_f and a constant τ_f such that there exist $r_0 > 0$, $\delta > r_0$ such that for every $r > 0$, $r < r_0$ and $\varphi \in \mathrm{Lip}^r_{B(x,r)}(X)$, $\|\varphi\|_{x,r} \le 1$ we have

$$|\langle \mathscr{T}\lfloor(f - f(x)), \mathrm{d}\varphi\rangle| = |\langle \mathscr{T}\lfloor f, \mathrm{d}\varphi\rangle| \le \varepsilon r \Phi_f(B(x,\tau_f r)).$$

Now, we can estimate

$$\begin{aligned} |\langle \mathscr{T}\lfloor\chi(f - f(x)), \mathrm{d}\varphi\rangle| &\le |\langle \mathscr{T}\lfloor\chi^c(f - f(x)), \mathrm{d}\varphi\rangle| + |\langle \mathscr{T}\lfloor(f - f(x)), \mathrm{d}\varphi\rangle| \\ &\le \varepsilon\|\varphi\|_{x,r}\mu(B(x,r) \setminus G) + \varepsilon r \Phi_f(B(x,\tau_f r)) \\ &\le \varepsilon C r\|\varphi\|_{x,r}\nu(B(x,r)) + \varepsilon r \Phi_f(B(x,\tau_f r)). \end{aligned}$$

Conclusion: Let us define $\Phi := \nu + \Phi_f$ and $\tau := \tau_f$. Obviously, for such Φ and τ the condition (4.9) is satisfied for all $x \in X$ and hence the proof is done. □

4.3 Applications

In the sequel, we will need some definitions related to the sets of finite perimeter. For details see [1, Definition 3.35], [1, Definition 3.54] and [1, Definition 2.57].

Definition 4.17. Let $E \subset \mathbb{R}^n$ be a $\mathcal{H}^k$-measurable set. We say that E is *countably k-rectifiable* if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$E \subset \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^k).$$

Definition 4.18. Let G be an λ^n-measurable subset of $\mathbb{R}^n$. We say that G is a *set of finite perimeter* if $\vec{\nu} := -D_{\chi_G}$ is a vector Radon measure.

Definition 4.19. Let G be an λ^n-measurable subset of $\mathbb{R}^n$. We call *reduced boundary* $\mathcal{F}G$ the collection of all points $x \in \operatorname{spt}|D_{\chi_G}|$ such that the limit

$$\mathbf{n}_G(x) := \lim_{r\to 0} \frac{\vec{\nu}(B(x,r))}{|\vec{\nu}|(B(x,r))}$$

exists and satisfies $|\mathbf{n}_G(x)| = 1$. The function $\mathbf{n}_G : \mathcal{F}G \to S^{n-1}$, where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$, is called the *generalized exterior normal* to G.

Theorem 4.20. Let $G \subset \mathbb{R}^n$ be a set of finite perimeter. Then there exists a isoperimetric measure ν carried by the reduced boundary.

Proof. Let Γ be the reduced boundary to G.

By [9, 5.6.2] we have that Γ is $(n-1)$-dimensional countably rectifiable and by [1, 3.54] we obtain that $|\nu|(\Omega \setminus \Gamma) = 0$.

Then we use the local isoperimetric inequality [1, pg. 149, (3.37)] and we obtain

$$\begin{aligned}
&\min\{\lambda(B(x,r) \cap G), \lambda(B(x,r) \setminus G)\} \\
&\quad \leq \min\{\lambda(B(x,r) \cap G), \lambda(B(x,r) \setminus G)\}^{1-1/n} \lambda(B(x,r))^{1/n} \\
&\quad = Cr \min\{\lambda(B(x,r) \cap G), \lambda(B(x,r) \setminus G)\}^{1-1/n} \\
&\quad \leq Cr|\nu|(B(x,r))
\end{aligned}$$

for every $x \in \Omega \setminus \Gamma$, which we needed. □

Proposition 4.21. Let $G \subset \mathbb{R}^n$ be a relatively compact set with a finite perimeter. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. Let $\partial(\mathscr{T}\lfloor f)$-a.e. x of the density boundary also belong to reduced boundary. Let $\mathscr{T}$ be a 1-dimensional $\mathbb{R}^n$ current defined as follows

$$\langle \mathscr{T}, \varphi \, \mathrm{d}\psi \rangle = \begin{pmatrix} \int_{\mathbb{R}^n} \varphi \frac{\partial \psi}{\partial x_1} \, \mathrm{d}x \\ \vdots \\ \int_{\mathbb{R}^n} \varphi \frac{\partial \psi}{\partial x_n} \, \mathrm{d}x \end{pmatrix}.$$

Then

$$\int_{\partial(\mathscr{T}\lfloor \chi_G)} f = \int_{\partial(\mathscr{T}\lfloor f)} \chi_G.$$

Proof. Our aim is to apply the Gauss-Green Theorem 4.8 setting $X = \mathbb{R}^n$ and $\mu = \lambda$. Now we check the conditions (G1)-(G3).

At first we show, that $\mathscr{T}$ is majorized by the Lebesgue measure. We start with the fact, that the euclidean and maximum norm are equivalent, hence there exist a constant $c > 0$ such that

$$|\langle \mathscr{T}, \varphi \, \mathrm{d}\psi \rangle| = \begin{vmatrix} \int_{\mathbb{R}^n} \varphi \frac{\partial \psi}{\partial x_1} \, \mathrm{d}x \\ \vdots \\ \int_{\mathbb{R}^n} \varphi \frac{\partial \psi}{\partial x_n} \, \mathrm{d}x \end{vmatrix} \leq c \max_{i=1,\dots,n} \left\{ \left| \int_{\mathbb{R}^n} \varphi \frac{\partial \psi}{\partial x_i} \, \mathrm{d}x \right| \right\}.$$

Now, let us concentrate on the partial derivative. Since $\left| \frac{\partial \psi}{\partial x_i}(x) \right| \leq \mathrm{Lip}(\psi)$, we obtain

$$\left| \int_{\mathbb{R}^n} \varphi \frac{\partial \psi}{\partial x_i} \, \mathrm{d}x \right| \leq \|\psi\|_{\mathrm{Lip}_b(X)} \int_{\mathbb{R}^n} |\varphi| \, \mathrm{d}x,$$

hence the Lebesgue measure majorizes $\mathscr{T}$ and (G1) is satisfied.

Further, by the Theorem 4.20 there exists an "isoperimetric measure" satisfying (4.6) and hence also the condition (G2) is satisfied.

Now, we need to show (G3). By [1, Lemma 3.58] we obtain that (4.7) is satisfied for all x in reduced boundary of G. Since we assumed, that $\partial(\mathscr{T}\lfloor f)$-a.e. x of density boundary are in the reduced boundary, (G3) is satisfied.

We verified all conditions of Theorem 4.8 and we have

$$\partial(\mathscr{T}\lfloor\chi f) = \partial(\mathscr{T}\lfloor f)\lfloor\chi + \partial(\mathscr{T}\lfloor\chi)\lfloor f.$$

In other words,

$$\langle \mathscr{T}\lfloor\chi f,\, \mathrm{d}\varphi\rangle = \langle(\mathscr{T}\lfloor f)\lfloor\chi,\, \mathrm{d}\varphi\rangle + \langle(\mathscr{T}\lfloor\chi)\lfloor f,\, \mathrm{d}\varphi\rangle \tag{4.10}$$

for every $\varphi \in \mathrm{Lip}_b(X)$. Further, we can find a function $\varphi \in \mathrm{Lip}_c(X)$ such that $\varphi = 1$ on a neighbourhood of G. By the definition of integral we have

$$\langle\partial(\mathscr{T}\lfloor\chi f), \varphi\rangle = \langle\mathscr{T}\lfloor\chi f,\, \mathrm{d}\varphi\rangle \sim \chi(x)f(x)\langle\mathscr{T},\, \mathrm{d}\varphi\rangle, \quad x \in X.$$

Since φ has compact support, we obtain

$$\langle\mathscr{T},\, \mathrm{d}\varphi\rangle = \begin{pmatrix} \int_{\mathbb{R}^n} \frac{\partial\varphi}{\partial x_1}\,\mathrm{d}x \\ \vdots \\ \int_{\mathbb{R}^n} \frac{\partial\varphi}{\partial x_n}\,\mathrm{d}x \end{pmatrix} = 0$$

and hence $\langle\partial(\mathscr{T}\lfloor\chi f), \varphi\rangle = 0$. Then, by (4.10) we have

$$\langle(\mathscr{T}\lfloor f)\lfloor\chi,\, \mathrm{d}\varphi\rangle = -\langle(\mathscr{T}\lfloor\chi)\lfloor f,\, \mathrm{d}\varphi\rangle.$$

Further, since $\varphi = 1$ on $\bar{G}$, we have

$$\langle(\mathscr{T}\lfloor f)\lfloor\chi,\, \mathrm{d}\varphi\rangle \mapsto \int_{\partial(\mathscr{T}\lfloor f)} \chi_G$$

and

$$-\langle(\mathscr{T}\lfloor\chi)\lfloor f,\, \mathrm{d}\varphi\rangle \mapsto \int_{\partial(\mathscr{T}\lfloor\chi_G)} f.$$

Thus we obtain

$$\int_{\partial(\mathscr{T}\lfloor f)} \chi_G = \int_{\partial(\mathscr{T}\lfloor\chi_G)} f,$$

which we needed.

□

Remark 4.22. Theorem above, in fact, says that

$$\int_{\partial G} \vec{f}\cdot\mathrm{d}\vec{\nu} = \int_G \operatorname{div}\vec{f}$$

in some generalized sense.

Proposition 4.23. Let $G \subset \mathbb{R}^n$ be an relatively compact open set such that there exists the exterior normal for every x in topological boundary. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Let $\mathscr{T}$ be a 1-dimensional $\mathbb{R}^n$ current defined as in Proposition 4.21. Then

$$\int_{\partial(\mathscr{T}\lfloor\chi_G)} f = \int_{\partial(\mathscr{T}\lfloor f)} \chi_G.$$

Proof. The condition (G1) can be obtained similarly as above in Proposition 4.21.

Further, by Theorem [1, Theorem 2.61] we obtain that G is countably $(n-1)$-rectifiable.

Now, in [15] it is shown that there exists a measure ν on $\mathbb{R}^n$ such that

$$\liminf_{r \to 0+} \frac{\nu(B(x,r))}{r^{n-1}} > 0$$

for every $x \in \partial G$. Since $\lambda(B(x,r)) = cr^n$ and since G is open, the condition (G2) follows.

The condition (G3) follows from the fact that there exists the exterior normal at each $x \in \partial G$. Then we have

$$\liminf_{r \to 0_+} \frac{\mu(B(x,r) \cap G)}{\mu(B(x,r))} = \limsup_{r \to 0_+} \frac{\mu(B(x,r) \cap G)}{\mu(B(x,r))} = \frac{1}{2}$$

for every x in topological boundary.

Next we proceed similarly as in Proposition 4.21.

□

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