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MASTER THESIS

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Instantons and Unitarily Inequivalent Quantum Vacua

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I hereby declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Instantóny a unitárni neekvivalentní kvantová vakua

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Abstrakt: V předložené práci se zabýváme vztahem mezi topologicky odlišnými instantonovými vakui a unitárně neekvivalentními vakui kvantové teorie pole. Zaměřujeme se na kvantově mechanické případy, kde jsou instantony přítomny, ale nevyskytují se zde komplikace spojené s kalibračními poly kvantové teorie pole. Model kvantové disipace a teorie částice unikajíci z metastabilního minima byly porovnány. System dvojité jámy byl vybudován pomocí harmonických oscilátorů a interakčního členu s cílem se přiblížit modelu kvantové disipace, ve kterém se vyskytují neekvivalentní reprezentace. Identifikovali jsme hračkový model, kvantovou částici na kružnici jako vhodny model pro výzkum vztahu mezi unitárně neekvivalentními vakui a topologickými vakui.

Klíčová slova: Instantony, unitárni neekvivalentní reprezentace, vakua, kanonické komutční relace.

Title: Instantons and Unitarily Inequivalent Quantum Vacua

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Abstract: In the presented thesis we investigate the relationship between the topologically distinct instantonic vacua and the unitarily inequivalent vacua of the quantum field theory. We focus on quantum mechanical examples, where instantons appear but the complications due to quantum gauge field theory are absent. A model for quantum dissipation and the theory of one particle escaping from a metastable minimum were compared, what led to some observations. A double well system was build from harmonic oscillators and an interaction term to get closer to the quantum dissipation model, where inequivalent representations are involved. We identified the particularly simple model of a quantum particle constrained on a circle to be the ideal toy model for spotting the relation among unitarily inequivalent vacua and topologically distinct vacua we were seeking for.

Keywords: instantons, unitarily inequivalent representations, vacuum, canonical commutation relations.

Contents

1. Introduction

1.1 Motivation

Unification of electricity and magnetism led to electromagnetism. Unification of electromagnetism and classical mechanics, by extending the validity of Lorentz transformations from the former to the latter theory, gave us the Special Theory of Relativity (STR) which unifies space and time, mass and energy. The equivalence principle stemming from the equivalence of gravitational and inertial mass led to the discovery of General Theory of Relativity, yet another unification. The hunt for unification is behind the birth of Quantum Field Theory (QFT) as an attempt to consistently accommodate STR into Quantum Mechanics (QM). Within that theory we were able to unify the electromagnetic and weak interactions. Needless to mention $SU(2) \times U(1)$ unification which brought us electroweak theory. An example of unification that we are still looking for is quantum gravity.

In this thesis we follow the idea of unification. We investigate the relationship between the topologically distinct instantonic vacua and the unitarily inequivalent vacua of QFT. The latter are a fundamental feature of any QFT, relativistic or not, due to the infinite number of degrees of freedom associated with the description of a field. A typical physical situation where such inequivalent vacua appears is that of spontaneously broken symmetries, [1]. On the other hand, such vacua are infinite in number, hence many of them are inessential (redundant) to the physical description of a finite number of phases of the physical system. The elucidation of the link between those vacua and the instantonic vacua could shed a light on the way to select the sub-sectors of physical vacua using super-selection rules based on topological constraints. Another natural environment to investigate the problem is QFT in curved spacetime, where the Hawking and Unruh phenomena rely on the inequivalent vacua.

In this thesis we focus on examples in QM, where instantons appear but the complications due to quantum gauge field theory are absent. Although in QM (i.e. finite degree of freedom) we do not expect the Unitarily Inequivalent Representations (UIRs) to play any role (Stone-von Neumann theorem (SvNT)). We will focus on two cases where such an instance might indeed be there. Those are quantum dissipative system and a quantum particle constrained on a circle.

1.2 The main topics at a glance

We have chosen two kinds of inequivalence. At first we will comment on the instantonic one. Instantons are localized finite-action solutions of the classical Euclidean field equations. It can be said that they are solitons localized in time. One very well known example of solitons is given in the ϕ^4 model, where such solutions exists and are divided into sectors, characterized by an integer (the winding number). These sectors are topologically distinct in the sense that fields from one sector cannot be distorted continuously into another sector without violating the requirement of finite energy. In particular, since time evolution is

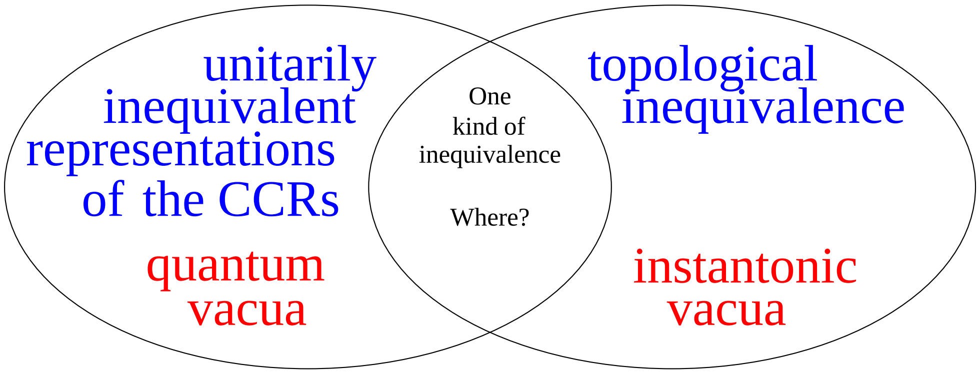

Figure 1.1: The unification we are looking for

an example of continuous distortion, a field configuration from any one sector stays within that sector as time evolves.

Now we comment on the second inequivalence. In QFT, the number of degrees of freedom is infinite and this fact alone ensures that there are representations of the CCRs, which are unitarily non-equivalent. This means that exist quantum vacua, that cannot be related by a unitary transformation. This fact is by now well established and used in many contexts, QFT in curved spacetime as well as applications of QFT methods to condensed matter physics. In what follows, we will be more explicit on the mathematical meaning of such inequivalence.

In this thesis we studied the relation between those two different inequivalences. An issue that, to our knowledge, was never addressed before.

The thesis is organized as follows. In Section 2.1 we present the Stone-von Neumann theorem (SvNT), which establishes the relation between the number of degrees of freedom of a quantum system and the representations of the CCRs. We then introduce the Weyl form of the CCRs, which provides us with a connection to the phase space. In Section 2.2 we summarize the formalism of the Bogoliubov transformations (BTs). In Section 2.3 we discuss a model to treat dissipation in a quantum context based on a paper by Celeghini, Rasetti, Vitiello (CRV) [2]. In there, a system made by one damped harmonic oscillator and one amplified harmonic oscillator is considered and it is derived that the time evolution of physical observables and of the vacuum is generated by BT. In the (necessary) field limit, i.e. for infinite number of degrees of freedom, the vacua parametrized by time are unitary non-equivalent. In this case the time evolution of the vacuum is a "tunneling" through the set of inequivalent quantum vacua (IQV). In Section 3.1 we review ϕ^4 theory to introduce topologically distinct sectors. In Section 3.2 we introduce the Yang-Mills instantons. Then we discuss vacua (pure gauge) and their relation to the Yang-Mills instantons. Also Pontryagin index, which labels the sectors of field configurations. In Section 3.3 we introduce θ vacua in both a QM system and in the Yang-Mills theory. In the QM system a total derivative term in the Lagrangian is related to the different realizations of the UIRs. In Section 4.0 we list a strategy to find a link we are looking for. In Section 4.1 the theory of one particle escaping from a metastable minimum is discussed. In the calculations, the techniques of Wick rotation, stationary phase approximation and gas diluted method are employed. Instantons are present in the description of the system. The system was connected with CRV model through an identification of the decay constants in the systems. In Section 4.2 we start with double well system and move to the system built from two harmonic oscillators, which is closer to the CRV model where UIRs and IQV are present. In Section 4.3 the particle on the circle, its path integral description and geometrical quantization is discussed. The UIRs are present. In section 4.4 a correspondence between particle on circle and a QFT system is discussed. Finally we draw our conclusions.

2. Inequivalent Quantum Vacua

2.1 Stone-von Neumann theorem, Weyl form of the CCRs

The main results of the Stone-von Neumann theorem (SvNT) can be summarized as follows:

- For a quantum system with a finite number of degrees of freedom and trivial topology, all representations of the CCRs are unitarily equivalent.

There are at least two different origins for the violation of the SvNT. The first one is the infinite number of degrees of freedom, see [3], [4]. The second one is the non-trivial topology arising in the quantization of the particle on the circle [5]. In both cases, there exist representations of the CCRs which are unitary nonequivalent.

If we deal only with one particle and trivial topology then there are just two basic operators: the coordinate operator $\hat{q}$ and the momentum operator $\hat{p}$. Those operators satisfy the Heisenberg CCRs

$$[\hat{q}, \hat{p}] = i\hbar\hat{I} \ , \quad [\hat{q}, \hat{I}] = [\hat{p}, \hat{I}] = 0 \tag{2.1}$$

where $\hat{I}$ is the identity operator, $\hbar$ is the Planck constant.

Schroedinger representation of the Heisenberg CCRs is:

$$\begin{aligned} (\hat{q}\psi)(x) &= x\psi(x) \\ (\hat{p}\psi)(x) &= -i\hbar\frac{d\psi}{dx}(x) \end{aligned} \tag{2.2}$$

where $x \in \mathbb{R}$, $\psi \in L^2(\mathbb{R})$. $\hat{q}$ and $\hat{p}$ are not bounded operators. To overcome this problem the following set of operators, which are bounded, is defined:

$$U(Q) = e^{iQ\hat{p}/\hbar} \ , \ \ V(P) = e^{iP\hat{q}/\hbar} \ ; \ \ Q, P \in \mathbb{R} \tag{2.3}$$

In Schroedinger representation:

$$(U(Q)\psi)(x) = \psi(x + Q) \ , \ \ (V(P)\psi)(x) = e^{iPx/\hbar} \tag{2.4}$$

From the Heisenberg CCRs and by employing Baker-Campbell-Hausdorff formula we obtain the Weyl form of the CCRs:

$$U(Q)V(P) = \exp\left(\frac{iPQ}{\hbar}\right)V(P)U(Q) \tag{2.5}$$

We started with $\hat{q}$, $\hat{p}$ and Heisenberg CCRs then U, V were defined and Weyl form of the CCRs was derived. This can be done in the opposite direction therefore we say that there is a one-to-one correspondence between the Heisenberg CCRs on one hand and U, V which satisfy the Weyl form of the Heisenberg CCRs on other hand.

The following lines and the theorem are from [4]. A representation of the Weyl form of the CCRs is said to be irreducible if the only subspaces of the Hilbert space $\mathcal{H}$ of states left invariant by the operators $\{U(a) \mid a \in \mathbb{R}\} \cup \{V(a) \mid a \in \mathbb{R}\}$ are $\{0\}$ and $\mathcal{H}$ itself.

The Stone-von Neumann Uniqueness Theorem:
If $\{\tilde{U}(a) \mid a \in \mathbb{R}\}$ and $\{\tilde{V}(a) \mid a \in \mathbb{R}\}$ are (weakly continuous) families of unitary operators acting irreducibly on a (separable) Hilbert space $\mathcal{H}$ such that

$$\tilde{U}(a)\tilde{U}(b) = \tilde{U}(a+b) \ , \quad \tilde{V}(a)\tilde{V}(b) = \tilde{V}(a+b)$$

$$\tilde{U}(a)\tilde{V}(b) = e^{iab/\hbar}\tilde{V}(b)\tilde{U}(a)$$

then there exists a Hilbert space isomorphism W: $\mathcal{H} \to L^2(\mathbb{R})$ such that

$$W\tilde{U}(a)W^{-1} = U(a) \ , \quad W\tilde{V}(a)W^{-1} = V(a)$$

for all $a \in \mathbb{R}$, where $U(a)$ and $V(a)$ are the Weyl unitaries in the Schroedinger representation defined in Eq. (2.4).

The theorem holds for the finite number of degrees of freedom with trivial topology.

The annihilation operator $\hat{a}$ and creation operator $\hat{a}^\dagger$ are defined and obey following relations:

$$\hat{a} \equiv \frac{\hat{q} + i\hat{p}}{\sqrt{2\hbar}} \ , \quad \hat{a}^\dagger \equiv \frac{\hat{q} - i\hat{p}}{\sqrt{2\hbar}} \tag{2.6}$$

$$[\hat{a}, \hat{a}^\dagger] = \hat{I} \ , \quad [\hat{a}, \hat{I}] = [\hat{a}^\dagger, \hat{I}] = 0 \tag{2.7}$$

This can be rewritten in a more beautiful way by introducing $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$.

$$\hat{e}_1 \equiv i(\hbar)^{-1/2}\hat{p} \ , \quad \hat{e}_2 \equiv i(\hbar)^{-1/2}\hat{q} \ , \quad \hat{e}_3 \equiv i\hat{I} \tag{2.8}$$

$$[\hat{e}_1, \hat{e}_2] = \hat{e}_3 \ , \quad [\hat{e}_1, \hat{e}_3] = [\hat{e}_2, \hat{e}_3] = 0 \tag{2.9}$$

Those operators form the Heisenberg-Weyl algebra $\mathbb{W}$. The elements of $\mathbb{W}$ are written as

$$\begin{aligned} x &\equiv (s; x_1, x_2) = x_1\hat{e}_1 + x_2\hat{e}_2 + s\hat{e}_3 \\ &= \hat{I} + \frac{i}{\hbar}(P\hat{q} - Q\hat{p}) = is\hat{I} + (\alpha\hat{a}^\dagger - \tilde{\alpha}a^\dagger) \end{aligned} \tag{2.10}$$

where

$$x_1 = -(\hbar)^{-1/2}Q \ , \quad x_2 = (\hbar)^{-1/2}P \ , \quad s \in \mathbb{R} \tag{2.11}$$

$$\begin{aligned} \alpha &\equiv (2\hbar)^{-1/2}(Q + iP) = 2^{-1/2}(-x_1 + ix_2) \\ \tilde{\alpha} &\equiv (2\hbar)^{-1/2}(Q - iP) = 2^{-1/2}(-x_1 - ix_2) \end{aligned} \tag{2.12}$$

The elements of the group are:

$$\exp(x) = \exp(is\hat{I})D(\alpha) \ , \quad D(\alpha) = \exp(\alpha a^\dagger - \tilde{\alpha}a) \tag{2.13}$$

By employing Baker-Campbell-Hausdorff formula we obtain the multiplication law for $D(\alpha)$:

$$D(\alpha)D(\beta) = \exp(2iIm\{\alpha\bar{\beta}\})D(\beta)D(\alpha) \tag{2.14}$$

We expressed the Weyl form of the CCRs in terms of $\hat{a}$, $\hat{a}^\dagger$ to see a connection to the phase space. $Im\{\alpha\bar{\beta}\} = 2A(0, \beta, \alpha + \beta)$, where $A(\alpha, \beta, \gamma)$ is the area of the triangle with vertices at the points α, β, γ i.e. $Im\{\alpha\bar{\beta}\}$ is proportional to the area of the triangle on the phase plane [6]. In general Q, P are not any real numbers, but they are related to the phase space.

2.2 The Bogoliubov transformation

When we deal with a scalar field we often use the Fock representation, which is unitarily equivalent to the Schroedinger representation. Let us remind here that a vector in Fock space is:

$$|n_1, n_2, ..\rangle \quad , \quad \sum_i n_i = finite \tag{2.15}$$

where n_i means that the i-th single particle state is occupied by n_i particles. Those vectors form a countable set, basis in Hilbert space. If the n is infinite then the summation over n_i is not restricted to be a finite number and those vectors form a non-countable set. It is often believed that to describe a physical system you need just a countable set of vectors which form the basis of a Hilbert space or in other words physicists use countable basis. But there are infinitely many ways to choose a countable set from the set of the non-countable vectors. To have two different bases means that any basis vector from one space differs from any basis vector in the other space in infinitely many places [7]. Two different bases chosen from this non-countable set are orthogonal to each other. It means that any vector from one space, which is a linear combination of the basis of this space, does not belong to the other space. To be more precise, when we expand such a vector in the basis of the other space we see that all coefficients are zero, see [8]. We can explicitly see this in the next lines.

Let us consider two sets of boson annihilation operators, $\alpha(\mathbf{k})$ and $\beta(\mathbf{k})$. We can build the Fock space, denoted by $\mathscr{H}(\alpha, \beta)$. The vacuum $|0\rangle$ is defined by

$$\alpha(\mathbf{k})|0\rangle = 0 \qquad \beta(\mathbf{k})|0\rangle = 0 \tag{2.16}$$

The CCRs for these operators are

$$[\alpha(\mathbf{k}), \alpha^\dagger(\mathbf{l})] = \delta(\mathbf{k} - \mathbf{l}) \qquad [\beta(\mathbf{k}), \beta^\dagger(\mathbf{l})] = \delta(\mathbf{k} - \mathbf{l}) \tag{2.17}$$

Other commutators are zero. Let us now introduce the operators, $a(\mathbf{k})$ and $b(\mathbf{k})$ through the following Bogoliubov transformations (i.e. the quantum counterpart of the canonical transformations of classical mechanics, see Eq. (2.21)).

$$a(\mathbf{k}) = c_k\alpha(\mathbf{k}) - d_k\beta^\dagger(-\mathbf{k}) \tag{2.18}$$

$$b(\mathbf{k}) = c_k\beta(\mathbf{k}) - d_k\alpha^\dagger(-\mathbf{k}) \tag{2.19}$$

Here the c-number coefficients c_k, d_k are real functions of $\mathbf{k}^2$ and if they satisfy the relation

$$c_k^2 - d_k^2 = 1 \tag{2.20}$$

then $a(\mathbf{k})$ and $b(\mathbf{k})$ satisfy the same CCRs as $\alpha(\mathbf{k})$ and $\beta(\mathbf{k})$:

$$[a(\mathbf{k}), a^\dagger(\mathbf{l})] = \delta(\mathbf{k}-\mathbf{l}) \qquad [b(\mathbf{k}), b^\dagger(\mathbf{l})] = \delta(\mathbf{k}-\mathbf{l}) \tag{2.21}$$

Other commutators vanish. The transformation is canonical. From Eq. (2.20) one sees that a way to represent c_k and d_k is

$$c_k = \cosh(\theta_k) \qquad d_k = \sinh(\theta_k) \tag{2.22}$$

$G(\theta)$ and $A(\theta)$ are defined:

$$G(\theta) = \exp[A(\theta)] \quad , \quad A(\theta) = \int d^3k\theta_k[\alpha(\mathbf{k})\beta(-\mathbf{k}) - \beta^\dagger(-\mathbf{k})\alpha^\dagger(\mathbf{k})] \tag{2.23}$$

The following relations can be derived, see Eq. (A.13)

$$G^{-1}(\theta)\alpha(\mathbf{k})G(\theta) = \alpha(\mathbf{k})\cosh\theta_k - \beta^\dagger(-\mathbf{k})\sinh\theta_k \tag{2.24}$$

$$a(\mathbf{k}) = G^{-1}(\theta)\alpha(\mathbf{k})G(\theta) \quad , \quad b(\mathbf{k}) = G^{-1}(\theta)\beta(\mathbf{k})G(\theta) \tag{2.25}$$

The following important result can be found, see Eq. (A.40)

$$|0\rangle\rangle \equiv G^{-1}(\theta)|0\rangle = f_0(\theta)\exp\Big(\delta^{(3)}(0)\int d^3k\alpha^\dagger(\mathbf{k})\beta^\dagger(-\mathbf{k})\tanh\theta_k\Big)|0\rangle \tag{2.26}$$

$$f_0(\theta) = \exp\Big(-\delta^{(3)}(0)\int d^3k\log\cosh\theta_k\Big) \tag{2.27}$$

where

$$\delta^{(3)}(0) = \lim_{\mathbf{k}\to 0}\delta(\mathbf{k}) \ , \quad \delta(\mathbf{k}) = (2\pi)^{-3}\int d^3xe^{i\mathbf{kx}} \tag{2.28}$$

The $\delta^{(3)}(0)$ can be seen as

$$\delta^{(3)}(0) = (2\pi)^{-3} \times (\textit{volume of the system}) \tag{2.29}$$

Eg. (2.16, 2.25, 2.26) give us that $|0\rangle\rangle$ is the vacuum of $a(\mathbf{k})$ and $b(\mathbf{k})$

$$a(\mathbf{k})|0\rangle\rangle = 0 \quad , \quad b(\mathbf{k})|0\rangle\rangle = 0 \tag{2.30}$$

$\delta^{(3)}(0)$ is infinite, therefore $f_0(\theta) = 0$. We can draw a conclusion that when $|0\rangle\rangle$ is expressed in terms of the basis vectors of $\mathscr{H}[\alpha,\beta]$, every expansion coefficient is zero, $|0\rangle\rangle$ does not belong to $\mathscr{H}[\alpha,\beta]$. We can build the Fock space $\mathscr{H}[a,b]$ from $a^\dagger(\mathbf{k})$, $b^\dagger(\mathbf{k})$, $|0\rangle\rangle$. It can be proven that vectors of $\mathscr{H}[a,b]$ do not belong to $\mathscr{H}[\alpha,\beta]$. We say $\mathscr{H}[\alpha,\beta]$, $\mathscr{H}[a,b]$ are orthogonal to each other. For more details see [8].

2.3 The CRV model for Quantum Dissipation

This section is based on [2]. We are interested in the CRV model for Quantum Dissipation as its description contains IQV and later we compare this system with the path integral description of the particle escaping from metastable minimum because it involves instantons.

In this section the CRV system made by one damped harmonic oscillator and one amplified harmonic oscillator is discussed and an important fact is shown that the time evolution of physical observables and of the vacuum is given by BT. In the (necessary) field limit, i.e. for infinite number of degrees of freedom, the vacua parametrized by time are unitary non-equivalent. In this case, the time evolution of the vacuum can be seen as a "tunneling" through the set of IQV.

The classical equation of motion of the damped harmonic oscillator is

$$m\ddot{x} + \gamma\dot{x} + \kappa x = 0 \tag{2.31}$$

where m is mass, γ is a damping constant, κ is a spring constant.

Let us mention that the canonical quantization of this system is pathological [2] i.e. the CCRs are not preserved by time evolution due to the damping term in the present system.

Consider the following system given by the Lagrangian L

$$L = m\dot{x}\dot{y} + \frac{1}{2}\gamma(x\dot{y} - \dot{x}y) - \kappa xy \tag{2.32}$$

By employing Euler–Lagrange equation of motion we obtain the classical equations of damped and amplified harmonic oscillators

$$m\ddot{x} + \gamma\dot{x} + \kappa x = 0\ , \quad m\ddot{y} - \gamma\dot{y} + \kappa y = 0 \tag{2.33}$$

For completeness we state the canonical momenta p_x and p_y

$$p_x = m\dot{y} - \frac{1}{2}\gamma y\ , \quad p_y = m\dot{x} + \frac{1}{2}\gamma x \tag{2.34}$$

The Hamiltonian of the system is:

$$H = \frac{1}{m}p_x p_y + \frac{1}{2m}\gamma(yp_y - xp_x) + (\kappa - \frac{\gamma^2}{4m})xy \tag{2.35}$$

The system is quantized by a symmetrical procedure, it means that the term $yp_y - xp_x$ in the Hamiltonian is replaced by $1/2(yp_y + p_y y - xp_x - p_x x)$ and the variables are replaced by their associated operators. The quantization is performed as follows:

$$[\hat{x}, \hat{p}_x] = i\hbar = [\hat{y}, \hat{p}_y]\ , \quad [\hat{x}, \hat{y}] = 0 = [\hat{p}_x, \hat{p}_y] \tag{2.36}$$

Here are definitions of the annihilation and creation operators:

$$\hat{a} \equiv \Big(\frac{1}{2\hbar\Omega}\Big)^{1/2}\Big(\frac{\hat{p}_x}{\sqrt{m}} - i\sqrt{m}\Omega\hat{x}\Big)\ , \quad \hat{a}^\dagger \equiv \Big(\frac{1}{2\hbar\Omega}\Big)^{1/2}\Big(\frac{\hat{p}_x}{\sqrt{m}} + i\sqrt{m}\Omega\hat{x}\Big) \tag{2.37}$$

$$\hat{b} \equiv \left(\frac{1}{2\hbar\Omega}\right)^{1/2}\left(\frac{\hat{p}_y}{\sqrt{m}} - i\sqrt{m}\Omega\hat{y}\right) , \quad \hat{b}^\dagger \equiv \left(\frac{1}{2\hbar\Omega}\right)^{1/2}\left(\frac{\hat{p}_y}{\sqrt{m}} + i\sqrt{m}\Omega\hat{y}\right) \tag{2.38}$$

$\hat{a}$, $\hat{a}^\dagger$, $\hat{b}$, $\hat{b}^\dagger$ obey the following CCRs:

$$[\hat{a}, \hat{a}^\dagger] = 1 = [\hat{b}, \hat{b}^\dagger] , \quad [\hat{a}, \hat{b}] = 0 = [\hat{a}, \hat{b}^\dagger] \tag{2.39}$$

Ω, Γ are defined here:

$$\Omega \equiv \left[\frac{1}{m}\left(\kappa - \frac{\gamma^2}{4m}\right)\right]^{1/2} , \quad \Gamma \equiv \gamma/2m \tag{2.40}$$

Γ is the decay constant for the classical variable $x(t)$ we need to introduce operators $\hat{A}$, $\hat{B}$:

$$\hat{A} \equiv (1/\sqrt{2})(\hat{a} + \hat{b}) , \quad \hat{B} \equiv (1/\sqrt{2})(\hat{a} - \hat{b}) \tag{2.41}$$

$\hat{A}$, $\hat{A}^\dagger$, $\hat{B}$, $\hat{B}^\dagger$ obey the following CCRs:

$$[\hat{A}, \hat{A}^\dagger] = 1 = [\hat{B}, \hat{B}^\dagger] , \quad [\hat{A}, \hat{B}] = 0 = [\hat{A}, \hat{B}^\dagger] \tag{2.42}$$

It is also useful to define $\hat{J}_+$, $\hat{J}_-$, $\hat{J}_3$ and the operator $\hat{\mathscr{C}}$:

$$\hat{J}_+ = \hat{A}^\dagger\hat{B}^\dagger , \quad \hat{J}_- = \hat{J}_+^\dagger = \hat{A}\hat{B} , \quad \hat{J}_3 = \frac{1}{2}(\hat{A}^\dagger\hat{A} + \hat{B}^\dagger\hat{B} + 1) \tag{2.43}$$

$$\hat{\mathscr{C}}^2 \equiv \frac{1}{4} + \hat{J}_3^2 - \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) = \frac{1}{4}(\hat{A}^\dagger\hat{A} - \hat{B}^\dagger\hat{B})^2 \tag{2.44}$$

Finally the Hamiltonian can be written as:

$$\hat{H} = \hat{H}_0 + \hat{H}_I \tag{2.45}$$

where

$$\hat{H}_0 = \hbar\Omega(\hat{A}^\dagger\hat{A} - \hat{B}^\dagger\hat{B}) = 2\hbar\Omega\hat{\mathscr{C}} \tag{2.46}$$

$$\hat{H}_I = i\hbar\Gamma(\hat{A}^\dagger\hat{B}^\dagger - \hat{A}\hat{B}) = i\hbar\Gamma(\hat{J}_+ - \hat{J}_-) = -2\hbar\Gamma\hat{J}_2 \tag{2.47}$$

Notice that $\hat{J}_1$ and $\hat{J}_2$ are defined by the following relationship:

$$\hat{J}_\pm \equiv (\hat{J}_1 \pm i\hat{J}_2) \tag{2.48}$$

and it is easy to check that those operators satisfy the $su(1,1)$ algebra:

$$[J_1, J_2] = -iJ_3 \quad , \ [J_3, J_1] = iJ_2 \ , \quad [J_2, J_3] = iJ_1 \tag{2.49}$$

$\hat{\mathscr{C}}$ is the casimir operator, as can be easily checked, hence, in particular:

$$[\hat{H}_0, \hat{H}_I] = 0 \tag{2.50}$$

Simultaneous eigenvectors of $\hat{A}^\dagger\hat{A}$ and $\hat{B}^\dagger\hat{B}$ are denoted by $|n_A, n_B\rangle$ where n_A, n_B are non-negative integers. The eigenvalue of $\hat{H}_0$ is the constant quantity $2\hbar\Omega(n_A - n_B)$. The eigenstates of $\hat{H}_I$ can be written in the standard basis, in terms of the eigenstates of $(\hat{J}_3 - \frac{1}{2})$ in the representation labeled by the value $j \in \mathbb{Z}_{1/2}$ of $\mathscr{C}$, $\{|j,m\rangle; m \geq |j|\}$:

$$\mathscr{C}|j,m\rangle = j|j,m\rangle , \quad j = \frac{1}{2}(n_A - n_B) \tag{2.51}$$

$$(J_3 - \frac{1}{2})|j, m\rangle = m|j, m\rangle \ , \quad m = \frac{1}{2}(n_A + n_B) \tag{2.52}$$

This representation is not a unitary irreducible representation, look at the discussion in [2] (below eq.12).

The vacuum state $|0\rangle$ is defined by the following relationships, it means $|0\rangle = |n_A = 0, n_B = 0\rangle$, i.e. $j = 0$, $m_0 = 0$.

$$\hat{A}|0\rangle = 0 \ \ , \ \hat{B}|0\rangle = 0 \tag{2.53}$$

The evolution operator is given by Eq. (2.23), up to a factor $-i/\hbar$. We just need to take a finite volume of Eq. (2.26-2.29). This is formally achieved by the following replacement $1/(2\pi)^3 \to 1/V$, integration $\to$ summation, $\theta_k \to \Gamma t$ and $\alpha(\mathbf{k})$, $\beta(-\mathbf{k})$ by $\hat{A}_k$, $\hat{B}_k$. In our case, there is no summation over k because we deal only with one pair of creation and annihilation operators. A time evolution of the vacuum can now be easily found

$$|0(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_I t}|0\rangle = \frac{1}{\cosh(\Gamma t)} \exp(\tanh(\Gamma t) J_+)|0\rangle \tag{2.54}$$

$|0(t)\rangle$ has unit norm

$$\langle 0(t)|0(t)\rangle = 1 \tag{2.55}$$

As $t \to \infty$, $|0(t)\rangle$ becomes an asymptotic state which is orthogonal to the $|0\rangle$

$$\lim_{t\to\infty} \langle 0(t)|0\rangle = \lim_{t\to\infty} \exp(-\ln\cosh(\Gamma t)) \to 0 \tag{2.56}$$

It means that time evolution leads out of the original Hilbert space. Despite the fact that the number of degrees of freedom is finite, time evolution of the vacuum give us IQV which breaks the SvNT. This breaking could come from non-trivial topology at finite degree of freedom but we have not investigated this. The approach followed in [2] reopened infinite number of degrees of freedom as the natural setting to treat this problem.

We work with the following QFT system, all is quite similar to the situation above, hence we only very quickly go through the main results. k labels a spatial momentum. We work at finite volume of the system V, and at the end we make the limit $V \to \infty$.

$$\hat{H} = \hat{H}_0 + \hat{H}_I \tag{2.57}$$

$$\hat{H}_0 = \sum_k \hbar\Omega_k(\hat{A}_k^\dagger \hat{A}_k - \hat{B}_k^\dagger \hat{B}_k) \tag{2.58}$$

$$\hat{H}_I = i\sum_k \hbar\Gamma_k(\hat{A}_k^\dagger \hat{B}_k^\dagger - \hat{A}_k \hat{B}_k) \tag{2.59}$$

The CCRs are:

$$[\hat{A}_k, \hat{A}_\lambda^\dagger] = \delta_{k,\lambda} = [\hat{B}_k, \hat{B}_\lambda^\dagger] \ , \quad [\hat{A}_k, \hat{B}_\lambda^\dagger] = 0 = [\hat{A}_k, \hat{B}_\lambda] \tag{2.60}$$

Once again the evolution operator is given by Eq. (2.23), up to a factor $-i/\hbar$. We use Eq. (2.26-2.29), what was proven in Appendix A. Finally we just need to replace θ_k by $\Gamma_k t$ and $\alpha(\mathbf{k})$, $\beta(-\mathbf{k})$ by $\hat{A}_k$, $\hat{B}_k$ to obtain the time evolution of the vacuum.

$$|0(t)\rangle = e^{(-i/\hbar)H_I t} = \prod_k \frac{1}{\cosh(\Gamma_k t)} \exp(\tanh(\Gamma_k t) J_+^{(k)})|0\rangle \tag{2.61}$$

$|0\rangle$ has a unit norm:

$$\langle 0(t)|0(t)\rangle = 1 \quad \forall t \tag{2.62}$$

It is easy to obtain the following relations too, so we can conclude that in the infinite volume limit time t parametrizes IQV, $|0(t)\rangle$.

$$\langle 0(t)|0\rangle = \exp\Big(- \sum_k \ln\cosh(\Gamma_k t)\Big) \tag{2.63}$$

if $\sum_k \Gamma_k > 0$

$$\lim_{t\to\infty} \langle 0(t)|0\rangle \propto \lim_{t\to\infty} \exp\Big(- t \sum_k \Gamma_k \Big) = 0 \tag{2.64}$$

$$V \to \infty : \quad \langle 0(t)|0\rangle \longrightarrow 0 \quad \forall t \tag{2.65}$$

$$V \to \infty : \quad \langle 0(t)|0(t')\rangle \longrightarrow 0 \quad \forall t, t'; \quad t \neq t' \tag{2.66}$$

The time evolution of $\hat{A}_k$ and $\hat{B}_k$ is easily obtained from Eq. (2.24). We only need to replace the integration by a summation, θ_k by $\Gamma_k t$ and $\alpha(\mathbf{k})$, $\beta(-\mathbf{k})$ by $\hat{A}_k$, $\hat{B}_k$.

$$\hat{A}_k(t) = e^{-i(t/\hbar)\hat{H}_I} \hat{A}_k e^{i(t/\hbar)\hat{H}_I} = \hat{A}_k \cosh(\Gamma_k t) - \hat{B}_k^\dagger \sinh(\Gamma_k t) \tag{2.67}$$

$$\hat{B}_k(t) = e^{-i(t/\hbar)\hat{H}_I} \hat{B}_k e^{i(t/\hbar)\hat{H}_I} = -\hat{A}_k^\dagger \sinh(\Gamma_k t) + \hat{B}_k \cosh(\Gamma_k t) \tag{2.68}$$

At every time t the system is build from $\hat{A}_k(t), \hat{A}_k^\dagger(t), \hat{B}_k(t), \hat{B}_k^\dagger(t); |0(t)\rangle$. As time evolves the system is described by UIRs of the CCRs.

3. Instantons

In this chapter we introduce instantons and some of their features which seem to us to be relevant for our investigation. Let us start with a definition:

An instanton is a solution of the Euclidean equations of motion such that the action of that solution is finite.

Why do we deal with Euclidean space despite the fact that our space is Minkowskian? In the path integral formulation of QM it is often very useful to go to Euclidean space by Wick rotation. This effectively means that we replace the real time by an imaginary one. To employ the stationary point approximation to predict the probability to find a particle at x_f that initiated its motion at x_i, we need the classical solution of this motion from x_i to x_f. Such a solution, as well known, is missing for tunneling, i.e. there is no classical counterpart. This problem is solved by Wick rotation. When we go to Euclidean space the potential is effectively reversed with respect to the Minkowski (real) situation and the consequence is that we can find a classical solution that allows us to employ the stationary point approximation.

On the other hand, finite action solution is demanded, mainly because we want to perform the saddle point approximation [9]. If the action of a field configuration were infinite, then the contribution to a probability amplitude would be zero. To satisfy the requirement of a finite action solution we can see that this often brings a non trivial topology into play. A typical example is the ϕ^4 model.

A theta vacuum is build from topological vacua, i.e. quantum vacuum states, which live around classical vacua and are connected by instantons. Theta vacua are objects of QM (QFT) where UIRs are present.

Some basic relationships between Euclidean and Minkowskian coordinate, vectors are in [10]. A vector in Minkowski space is $V_m(x_0, x_1, x_2, x_3)$, the Minkowski metric is $(-1, 1, 1, 1)$, $V_M^2 = -x_0^2 + x_1^2 + x_2^2 + x_3^2$. A vector in the 4-dimensional Euclidean space is $V_e(x_4, x_1, x_2, x_3)$, the 4-dimensional Euclidean metric is $(1, 1, 1, 1)$, $V_e^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$. If we replace real time x_0 by purly imaginary one $-ix_4$ then the Minkowski metric becomes Euclidean and some problems are easier to solve.

3.1 Kink solution in ϕ^4 theory

In this section we review a simple model, where solitons exist. Although solitons are not instantons, they share important features with them. They are solutions of the equation of motion of finite action and they are divided into sectors. These sectors are topologically distinct. We will be more specific now.

The ϕ^4 theory is given by the following Lagrangian density:

$$\mathscr{L}(x,t) = \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\phi')^2 - \frac{1}{4}\left(\phi^2 - 1\right)^2 \tag{3.1}$$

where $\phi(x, t)$ is a single scalar field, a dot (prime) is differentiation with respect to time (space) variable t (x). c, the velocity of light is set to be one. The potential is minimized by field values:

$$\phi = \pm 1 \tag{3.2}$$

we call them classical vacua. The equation of motion is:

$$\ddot{\phi} - \phi'' = \phi - \phi^3 \tag{3.3}$$

We are interested in non-singular, finite-energy, localized energy density, static (not crucial) solutions. In order for the energy of the system to be finite at any given time the solution of equation of motion has to be $\phi = \pm 1$ at spatial infinity ($x = \pm\infty$). The time evolution is the continuous distortion of the field. Therefore the field say at $x = +\infty$ can not jump from value $\phi = +1$ to $\phi = -1$ during the time evolution because at some moment the field at infinity would have to have a different value as one that minimizes the potential and hence the energy of the system would be infinite and the energy conservation law would be violated. Therefore, the values of the fields at infinity are preserved during time evolution. Hence this solutions are divided in to topological sectors, labeled by two indices $(\phi(x = -\infty), \phi(x = +\infty))$ i.e. $(-1, -1)$, $(-1, 1)$, $(1, -1)$, $(1, 1)$. To get a field from one sector to the other, requires a non continuous transformation, a field would have to jump from one sector to another.

There are two kinds of static solutions. They can begin from $\phi = -1$ at $x = -\infty$ and end up with $\phi = +1$ at $x = \infty$, or vice versa. The solution is

$$\phi(x) = \pm \tanh[(1/\sqrt{2})(x - x_0)] \ , \quad x_0 \in \mathbb{R} \tag{3.4}$$

and it is called the 'kink' ('antikink') for the plus (minus) sign. Kink has its values at spatial infinity $(-1, +1)$, therefore we say that Kink connects these vacua.

For more details see [10], [11] and [12]. In [10] you can find review of the more general case (sine-Gordon) and a definition of an analogue of the Pontryagin index. In [11] you can find a physical model of sine-Gordon system, i.e. a string with pegs.

3.2 The Yang-Mills instantons

Here we want to introduce true instantons, the Yang-Mills instantons. We deal with the Euclidean theory. The Lagrangian is:

$$\mathscr{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2} Tr\big(F_{\mu\nu} F^{\mu\nu}\big) \tag{3.5}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{3.6}$$

$$A_\mu \equiv \frac{1}{2}\sigma^a A^a_\mu \ , \quad F_{\mu\nu} \equiv \frac{1}{2}\sigma^a F^a_{\mu\nu} \tag{3.7}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.8}$$

where A_μ is the matrix field, called the non-Abelian gauge field corresponding to the gauge group $SU(2)$. $F_{\mu\nu}$ is the field strength, g denotes a coupling constant, σ^a

are the standard Pauli matrices. A_μ is transformed by the gauge transformation as follows and consequently $F_{\mu\nu}$ is transformed too.

$$A_\mu \to A'_\mu = U A_\mu U^{-1} - \frac{i}{g} U \partial_\mu U^{-1} \ , \quad F_{\mu\nu} \to F'_{\mu\nu} = U F_{\mu\nu} U^{-1} \tag{3.9}$$

The equation of motion is:

$$D_\mu F^{a\mu\nu} = 0 \tag{3.10}$$

where $D_\mu = \partial_\mu - igA_\mu$ is the covariant derivative. This is a non linear equation whose solutions are difficult to find. By employing the Bianchi identity

$$D_\mu F^{*a\mu\nu} = 0 \tag{3.11}$$

where $F^{*a\mu\nu}$ is

$$F^{*a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\kappa} F^a_{\lambda\kappa} \tag{3.12}$$

To find solutions of the equation of motion we just need to find a field configuration which satisfies the following relationship:

$$F^{a\mu\nu} = \pm F^{*a\mu\nu} \tag{3.13}$$

such fields are called self dual (+) or anti-self dual (-).

Let us now focus on the objects we are interested in, i.e. vacua: $A_\mu = 0$ (pure vacuum) and $F_{\mu\nu} = 0$. A gauge transformation gives us

$$A_\mu = -\frac{i}{g} U \partial_\mu U^{-1} \tag{3.14}$$

For such field configuration (pure gauge), again $F_{\mu\nu} = 0$. Every finite-action configuration must be pure gauge at $S_3^{(phy)}$[1] for $r = \infty$ [10]. These field configurations $A_\mu(x)$ are assigned to a group function U, which is defined on $S_3^{(phy)}$. This group function maps to a group space of $SU(2)$, which is $S_3^{(int)}$[2] sphere, so there is a mapping $S_3^{(phy)} \to S_3^{(int)}$. A homotopy theory (see Appendix B) says that we deal with $\pi_3(S_3)$ homotopy group, hence the maps $S_3^{(phy)} \to S_3^{(int)}$ are divided into a discrete infinity of homotopy classes labeled by an integer Q, Pontryagin index. Mappings from one class cannot be continuously deformed into a mapping from another class hence a field $A_\mu(x)$ belonging to a given sector Q cannot be continuously deformed to another sector because the field would have to become not pure gauge at $S_3^{(phy)}$ during such a continuous deformation and consequently the action would be infinite. To get a field from one sector to another one, a discontinuous transformation would be needed. Our configurations at infinity are pure gauge (vacua) and form a set of topologically distinct objects.

[1] $S_3^{(phy)}$ is the physical space, a three dimensional sphere in four dimensional Euclidean space. As we already mentioned we moved from Minkovskian space to Euclidean one by Wick rotation. An analytic equation of the sphere is $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$.

[2] $S_3^{(int)}$ is the $SU(2)$ group space, a three dimensional sphere in four dimensional euclidean space with radius one, because any element of $SU(2)$ can be written as $\sigma.\mathbf{a} + a_4\mathbb{I}$, where a_i and a_4 are any real number such that a formula $\mathbf{a}^2 + a_4^2 = 1$ holds. σ_i are Pauli matrices and $\mathbb{I}$ is the identity matrix 2×2.

The Yang-Mills instanton, i.e. the solution of the equation of motion can be found by solving the (anti) self-dual equation, which can be found in [10], [13] or in the original paper [14]. In [10], [15] you can see a more general N-instanton solution. The Yang-Mills instantons:

$$A_\mu = \frac{x^2}{x^2+\lambda^2}\Big(\frac{-i}{g}\Big)U\partial_\mu U^{-1} \tag{3.15}$$

where

$$U = \frac{x_4 + i\mathbf{x}\cdot\sigma}{\sqrt{x^2}} \quad , \quad x^2 = x_1^2+x_2^2+x_3^2+x_4^2 \tag{3.16}$$

The parameter x_4 is the Euclidean time. When $x_4 \to +\infty$ the Yang-Mills reduces to a pure gauge vacuum, which belongs to some topological sector[3]. When $x_4 \to -\infty$ it reduces to a pure gauge vacuum which belongs to a topological sector, which is assigned an integer which differs by one[4] from the integer assigned to the sector of the former field at $x_4 = +\infty$. The instanton connects pure gauge vacua which are members of different homotopy classes. Let us see how [11]:

$$A_i \to i g_n \partial_i g_n^{-1} \quad , \quad x_4 \to +\infty \quad , \quad i = 1,2,3 \tag{3.17}$$

$$A_i \to i g_{n-1} \partial_i g_{n-1}^{-1} \quad , \quad x_4 \to -\infty \quad , \quad i = 1,2,3 \tag{3.18}$$

where

$$g_1 = \exp\Big[-i\pi\frac{\mathbf{x}\cdot\sigma}{(x^2+\lambda^2)^{1/2}}\Big] \quad , \quad g_n \equiv g_1^n \quad , \quad n\in\mathbb{Z} \tag{3.19}$$

It is the Pontryagin index that assigns a homotopy class to a field configuration. Let us define,

$$\frac{1}{4}Tr\big(F_{\mu\nu}F^{*\mu\nu}\big) = \partial_\mu K^\mu \tag{3.20}$$

where we have used $F_{\mu\nu} = F^*_{\mu\nu}$ and

$$K_\mu \equiv \epsilon_{\mu\nu\lambda\kappa} Tr\Bigg(\frac{1}{2}A_\nu\partial_\lambda A_\kappa - ig\frac{1}{3}A_\nu A_\lambda A_\kappa\Bigg) \tag{3.21}$$

K_0 is called Chern-Simons term. Pontryagin index P can now be defined and through Stokes' theorem expressed as:

$$P \equiv \frac{g^2}{16\pi^2}\int_{\mathbb{R}^4} dx^4 Tr\big(F_{\mu\nu}F^{*\mu\nu}\big) = \frac{g^2}{4\pi}\int_{\partial\mathbb{R}^4 = S^3} d^3\sigma K_\mu n_\mu \tag{3.22}$$

As we deal with pure gauge on the boundary, and we require a finite action, it can be written as

$$\begin{aligned} K_\mu &= \epsilon_{\mu\nu\lambda\kappa}\Big[-\frac{1}{2g^2}Tr\Big(UU^{-1}_{,\nu}(UU^{-1}_{,\kappa})_{,\lambda}\Big) - \frac{1}{3g^2}Tr\Big(UU^{-1}_{,\nu}UU^{-1}_{,\lambda}UU^{-1}_{,\kappa}\Big)\Big] \\ &= \frac{1}{g^2}\epsilon_{\mu\nu\lambda\kappa}\Big(\frac{1}{2}-\frac{1}{3}\Big)Tr\Big(UU^{-1}_{,\nu}UU^{-1}_{,\lambda}UU^{-1}_{\kappa}\Big) \end{aligned} \tag{3.23}$$

[3]The $S_3^{(phy)}$ can be deformed to a such cylinder that at $x_4 = \infty$ we are at the upper disk of the cylinder. For the temporal gauge $A_4 = 0$ at fixed x_4 the 3-dimensional plane can be compactified into sphere S_3 [13]. For $x_4 \to \infty$ A_μ is a pure gauge hence there is $S_3 \to S_3^{(int)}$.

[4]The integer, Pontryagin index, of our Yang-Mills instanton is 1 [10] and in temporal gauge it can be written as difference between two integrals of the field configurations over disks (of the cylinder) at $x_4 \pm \infty$ [11] which is difference between the integers assigned to the field configurations at $\pm\infty$. Hence the integers of those sectors differ by one.

which gives

$$P = \frac{1}{24\pi}\int_{S^3} d^3\sigma\ n_\mu \epsilon_{\mu\nu\lambda\kappa} Tr\Big(UU^{-1}_{,\nu}UU^{-1}_{,\lambda}UU^{-1}_{,\kappa}\Big) \tag{3.24}$$

P is the winding number of the mapping from $S_3^{(phy)}$ into $S_3^{(int)}$ i.e. the number of the times the group space of $SU(2)$ is wrapped around $S_3^{(phy)}$. It can be shown that Pontryagin index is proportional to the integral over the group measure [16].

$$P \propto \int d\mu(U) \tag{3.25}$$

The constant$(-16\pi^2)^{-1}$ in Eq. (3.22) was chosen so that P gives us precisely the winding number. Let us here define: small gauge transformations leave the field configuration in its sector, large gauge transformations do not.

The Euclidean action is

$$S = -\frac{1}{2g^2}\int d^4x Tr[F_{\mu\nu}F^{\mu\nu}] \tag{3.26}$$

Let us consider the identity

$$-\int d^4x Tr[(F_{\mu\nu} \pm F^*_{\mu\nu})^2] \geq 0 \tag{3.27}$$

$$-\int d^4x Tr[F_{\mu\nu}F^{\mu\nu}] \geq \mp \int d^4x Tr[F^*_{\mu\nu}F^{\mu\nu}] \tag{3.28}$$

Therefore for any finite-action configuration

$$S \geq (8\pi^2/g^2)|P| \tag{3.29}$$

Fields which minimize the action in any given homotopy sector are solutions of the Euclidean Yang-Mills equation falling in that sector. We see that the absolute minimum value of $S(S = (8\pi^2/g^2)|P|)$ in a sector P is reached by a (anti) self-dual configuration i.e. (anti) self-dual configuration minimize S and hence solve the Euclidean Yang-Mills equation.

3.3 θ vacua

This section is based mainly on [10] and [11], [9], [17]. Up to now we have mainly discussed vacua of a classical theory with emphasis on instantons. IQV of the CCRs are those of QM (QFT). To investigate the topic it is more than relevant to review theta vacua which often appear when we move from a classical to a quantum theory. They exist due to tunneling phenomena among classical vacua which are connected by instantons.

Let us have a classical system which is given by the Hamiltonian:

$$H = \frac{p^2}{2m} + 1 - \cos q \tag{3.30}$$

The system has degenerate minima (classical vacua) at points $2\pi n$, $n \in \mathbb{Z}$. Around each minima we can employ a harmonic oscillator and build quantum mechanical wave functions, ground states $\psi_0(q)$, which are centered at $2\pi n$. We have a set of groundstates $\psi_0(q-2\pi n)$. We turn on QM, quantum tunneling and we are interested in the true vacuum. To find the true vacuum we compare two amplitudes of probability, that particle will tunnel from $|q_i = 0\rangle$ to $|q_f = 2\pi\rangle$, given by two computation methods.

The first one is the Feynman path integral. By Wick rotation we go to Euclidean theory where classical solution, the instanton, exists. We employ a saddle point approximation and diluted gas method as well. We have to deal with zero mode. It is very similar to that calculation which was carried out in the Section Escape from a metastable minimum with all details for the case of escaping from a metastable minima. Some calculation steps are in [10]. The second method is: we insert identity in terms of eigenvectors, say ϕ_n, of the Hamiltonian between the evolution operator and the initial state. You can find the following relationships in [10]

$$\lim_{\tau\to\infty}\langle 2\pi|e^{-\hat{H}\tau/\hbar}|0\rangle = \lim_{\tau\to\infty}\sum_n \langle 2\pi|\phi_n\rangle\langle\phi_n|0\rangle e^{-E_n\tau/\hbar} \tag{3.31}$$

$$\lim_{\tau\to\infty}\langle 2\pi|e^{-\hat{H}\tau/\hbar}|0\rangle = \int_0^{2\pi}\frac{d\theta}{2\pi}e^{i\theta}\Big(\frac{\omega}{\pi\hbar}\Big)^{\frac{1}{2}}\exp\big(-\frac{1}{2}\omega\tau + 2JK\cos(\theta)e^{-S_0/\hbar}\tau\big) \tag{3.32}$$

where ω is the angular frequency of the harmonic oscillators which were employed. S_0 is the action of the instanton solution which connects these two minima. Definitions of constants J and K are in the Section Escape from a metastable minimum and they can be found in [10], [9], [18]. Terms with the eigenstates which are not the lowest state are suppressed for large τ. By comparing those two amplitudes of probability we find energies of Hamiltonian which are parametrized by parameter θ

$$E(\theta) = \frac{1}{2}\hbar\omega - 2\hbar JK\cos(\theta)e^{-S_0/\hbar} \tag{3.33}$$

Those energies correspond to the eigenvalues of the Hamiltonian, which are denoted by $|\theta\rangle$.

The Hamiltonian is invariant under a transformation $q\to q+2\pi$. Hence it is required that the eigenvectors of the Hamiltonian are invariant under this transformation, up to a phase. It seems to be plausible to expect that those states are a linear combination of the degenerate quantum vacua $\psi_0(q-2\pi n)$, call them topological vacua. Since we deal here with a periodic potential the Bloch's theorem must be satisfied. The following set of vectors parametrized by θ satisfies requirements which we have just mentioned and therefore these vectors are eigenvectors of the operator which replaces q by $q+2\pi$. Since this operator commutes with the Hamiltonian we know that $|\theta(q)\rangle$ are eigenvectors of the Hamiltonian with eigenvalues which we have already found

$$|\theta(q)\rangle = \sum_{n=-\infty}^{\infty} e^{in\theta}|\psi(q-2\pi n)\rangle \tag{3.34}$$

For $\theta = 0$ we have the ground state. And indeed:

$$|\theta(q+2\pi)\rangle = e^{i\theta}|\theta(q)\rangle \tag{3.35}$$

For the case of the particle on the circle with same periodic potential, any wave function has to satisfy $\psi(q+2\pi) = \psi(q)$. From the set of $|\theta(q)\rangle$ only $|\theta = 0(q)\rangle$ satisfies this condition.

Let us consider the particle on the line with the periodic potential again, but a new term $-\theta dq/2\pi dt$ is added to the Lagrangian

$$L = \frac{1}{2}\Big(\frac{dq}{dt}\Big)^2 - (1-\cos q) - \frac{\theta}{2\pi}\frac{dq}{dt} \tag{3.36}$$

The term is a total derivative, hence the equation of motion is not changed. By the procedure which was mentioned above we can find that the ground state has the same energy as the state $|\theta(q)\rangle$ when the new term was not present [10].

For the system we started with, we usually take a Hilbert space which consists of 2π periodic functions, but this is not the only option. We can consider Hilbert space build from quasiperiodic functions [17].

$$\psi(\phi + 2\pi) = e^{2\pi i\theta}\psi(\phi) \tag{3.37}$$

For those functions, eigenvalues of operator $p = -i\partial_\phi$ are $n + \theta$. We can leave a function to be periodic and redefine the momentum operator, in other words we make a following transformation [17]:

$$\begin{aligned} p \to p &= -i\partial_\phi + \theta \\ \phi \to \phi & \end{aligned} \tag{3.38}$$

This transformation is canonical, i.e. the CCRs are preserved. Here we can see different realizations of the CCRs, which are parametrized by θ. This transformation brings total derivative $\theta dq/2\pi dt$ to the Hamiltonian.

Let us now introduce a θ vacuum of the Yang-Mills theory. This theory possesses a set of classical vacua (pure gauge) which are divided into homotopical sectors and are connected by the Yang-Mills instantons. Around each classical vacuum, belonging to a sector N, we can build a perturbative quantum vacuum state (ground state) denoted by $|N\rangle$, called topological vacuum. A Hamiltonian is invariant under gauge transformation. Classical vacua belonging to the N and $N-1$ sector are related by some large gauge transformation. The Corresponding transformation which relates $|N\rangle$ and $|N-1\rangle$ is denoted by T_1, $T_1|N\rangle = |N-1\rangle$. Since T_1 commutes with the Hamiltonian and it is a unitary transformation, eigenstates of the Hamiltonian and the true vacuum are eigenstates of T_1 too, with eigenvalue $e^{i\theta}$. This is satisfied by $|\theta\rangle$, see [10], [11], or the original paper [15]

$$|\theta\rangle = \sum_{N=-\infty}^{\infty} e^{iN\theta}|N\rangle \tag{3.39}$$

Indeed

$$T_1|\theta\rangle = e^{i\theta}|\theta\rangle \tag{3.40}$$

Let us mention that

$$0 = \langle\theta|[B, T_1]|\theta'\rangle = \langle\theta|B|\theta'\rangle(e^{i\theta'} - e^{i\theta}) \tag{3.41}$$

Since any physical operator B commutes with T_1 we see that vectors $|\theta\rangle$ and $|\theta'\rangle$ are not connected by a physical operator, so in this sense they belong to different, unconnected sectors.

$$\langle\theta|B|\theta'\rangle = 0 \qquad for \ \ \theta \neq \theta' \tag{3.42}$$

When the term L_θ,

$$L_\theta = \frac{\theta}{16\pi^2} Tr[F_{\mu\nu}F^{*\mu\nu}] \tag{3.43}$$

L_θ is a total divergence hence equations of motion (instantons) are not effected, is added to the Yang-Mills Lagrangian, then $|\theta\rangle$ is the true vacuum [10]. For each θ we have different quantum theory with the vacuum $|\theta\rangle$.

4. Linking Inequivalent Quantum Vacua and Instantons?

Here we show what has been done in attempt to find a link between topologically distinct vacua and unitarily inequivalent quantum vacua, as follows.

1. The theory of one particle escaping from a metastable minimum is discussed because the path integral description of the system contains desired instantons. We make identification of the decay constants present in this system and in the CRV model for quantum dissipation, which contains IQV, UIRs.

2. We start with a double well system, where instantons connect topological vacua, and we try to move to the system which is build from two harmonic oscilators because the CRV model, where UIRs of the CCRs are present, is build from damped and amplified harmonic oscilators. The idea is nicely captured in the chain of Fig. 4.1, 4.2, 4.3.

3. We have found a QM system, a particle on a circle, which possesses UIRs.

4. We discuss a correspondence between particle on a circle and a QFT system.

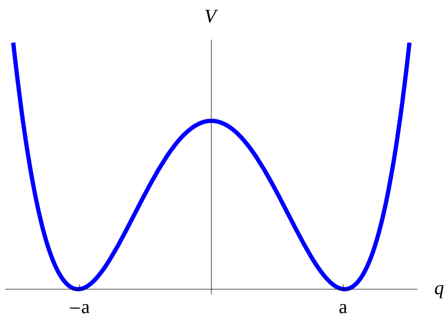

Figure 4.1: Double well system.

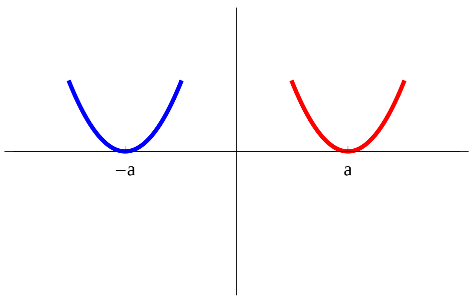

Figure 4.2: Two LHOs.

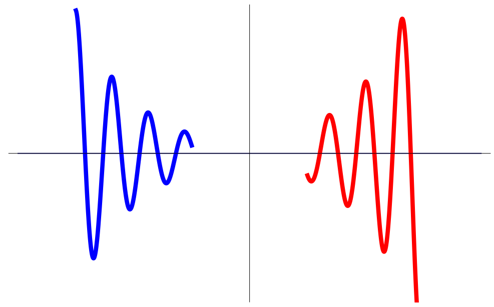

Figure 4.3: The CRV model.

4.1 Escape from a metastable minimum

Here we follow the steps and formulas from [18], [9] where they deal mainly with the double well system, but we handle the decay of false vacuum. The Lagrangian is

$$L = \frac{m}{2}\dot{q}^2 - V(q) \tag{4.1}$$

where m is a mass of the particle, the potential is given by the blue curve in Fig. 4.4. The potential possesses a minimum at q_m. By a Wick rotation $t \rightarrow$

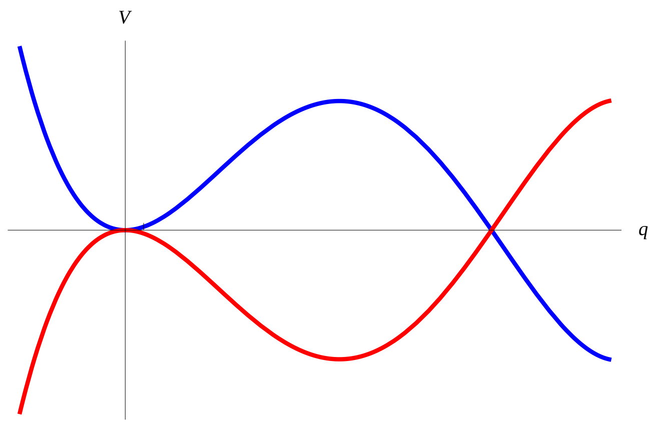

Figure 4.4: Metastable minimum. The potential is given by the blue curve, which is effectively changed to the red one by Wick rotation.

$-i\tau$ we replace real time t by a purely imaginary parameter $-i\tau$. One of the consequences is that the potential is effectively reversed i.e. $V(q) \rightarrow -V(q)$ because $m(dq/dt)^2/2 - V(q) \rightarrow -(m(dq/d\tau)^2/2 + V(q))$. The equation of motion is:

$$-m\ddot{q} + \partial_q V(q) = 0 \tag{4.2}$$

where the dot means a derivation with respect to τ. We are interested in solutions which satisfy $q(\tau_i) = q_m$ and $q(\tau_f) = q_m$. One solution is that the particle stays at q_m at all times. The second solution, let us denote by q_{inst}, is that the particle leaves q_m, accelerates through the "new" minimum, reaches the turning point and bounces back to the q_m at time τ. Let us call this solution the one-instanton solution (bounce, quasi particle), which is very close to q_m except for the moment of the bounce when it changes its position q very quickly. Hence there are approximative solutions, n-instanton solutions built from one-instanton solutions which are centered (the moment of bounce) at $(\tau_1, \tau_2, ..., \tau_n)$.

We want to calculate an amplitude of probability $G(q_m, q_m; \tau)$. We will employ the stationary phase approximation, see Appendix C, despite the fact that the potential is not positively definite and gas diluted method. Later, we will take into account this fact and we will derivate the correct solution. After the calculation, the real time amplitude is obtained by an analytic continuation. $G_n(\tau_1, ..., \tau_n)$ denotes the contribution from this n-instanton solution $(\tau_1, \tau_2, ..., \tau_n)$.

The gas diluted method means that we sum over all instanton sectors (n-instanton solutions) and integrate over all instanton solutions within the sectors

$$G(q_m, q_m; \tau) \simeq \sum_n K^n \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 .... \int_0^{\tau_{n-1}} d\tau_n G_n(\tau_1, ..., \tau_n) \tag{4.3}$$

K is present to compensate dimensions of the time integrations. Due to stationary phase approximation $G_n = G_{n,cl} \times G_{n,qu}$, see Eq. (C.3), $G_{n,cl}$ is related to the action of the n-instanton solution, $G_{n,qu}$ is related to the determinant.

We multiply eq. of motion $-m\ddot{q} + \partial_q V(q) = 0$ by $\dot{q}_{inst}$, integrate over time and use: $\partial_\tau q_{inst} = V = 0$ at $q_{inst} = q_m$, we obtain $\frac{m}{2}\dot{q}_{inst}^2 = V(q_{inst})$. Hence the action of a one-instanton solution is

$$S_{inst} = \int_0^\tau d\tau' \Big(\frac{m}{2} \dot{q}_{inst}^2 + V(q_{inst}) \Big) = \oint_{q_m} dq \Big(2mV(q) \Big)^{1/2} \tag{4.4}$$

S_{inst} is determined by V, i.e. does not depend on the structure of the solution q_{inst}.

For $\tau_i \ll \tau' \ll \tau_{i+1}$ the particle rests at q_m, no action accumulates. An action of one-instanton is S_{inst}. n-instanton is composite from n one-instantons. An action of n-instanton is nS_{inst}, hence

$$G_{n,cl}(\tau_1, ..., \tau_n) = e^{-nS_{inst}/\hbar} \tag{4.5}$$

$G_{n,cl}$ does not depend on τ_i.

$G_{n,qu}$ possesses two kinds of contributions. The first one is when the particle rests on the hill at $\tau_{i+1} - \tau_i$, which is a particle in a well but "Wickly rotated". Hence we replace $t \to -i\tau$ in $\sqrt{1/\sin(\omega t)}$, see equatin Eq. (C.9), which is a contribution for a particle in a well, to obtain the contribution

$$\sqrt{\frac{1}{\sin(-i\omega(\tau_{i+1} - \tau_i))}} \sim e^{-\omega(\tau_{i+1} - \tau_i)/2} \tag{4.6}$$

The second one contributes at the moments of bounces, this time period is short hence it contributes nothing. We set $\tau_0 \equiv 0, \tau_{n+1} \equiv \tau$. $G_{n,qu}$, hence

$$G_{n,qu}(\tau_1, ..., \tau_n) = \prod_{i=0}^{n} e^{-\omega(\tau_{i+1} - \tau_i)/2} = e^{-\omega\tau/2} \tag{4.7}$$

$G_{n,qu}$ does not depend on $\{\tau_i\}$, finally

$$\begin{aligned} G(q_m, q_m; \tau) &\simeq \sum_{n=0}^{\infty} K^n e^{-nS_{inst}/\hbar} e^{-\omega\tau/2} \int_0^{\tau} d\tau_1 \int_0^{\tau_1} d\tau_2 ... \int_0^{\tau_{n-1}} d\tau_n \\ &\simeq \sum_{n=0}^{\infty} K^n e^{-nS_{inst}/\hbar} e^{-\omega\tau/2} \tau^n / n! \\ &\simeq e^{-\omega\tau/2} \exp[\tau K e^{-S_{inst}/\hbar}] \end{aligned} \tag{4.8}$$

The real time amplitude is recovered by the analytic continuation $\tau \to it$.

$$G(q_m, q_m; t) = Ce^{-i\omega t/2} \exp\Big[-\frac{\Gamma}{2}t\Big] \tag{4.9}$$

the decay rate is $\Gamma/2 = |K|e^{-S_{inst}/\hbar}$.

Here we compute K. By differentiating $-m\ddot{q} + V'(q) = 0$ for q_{inst} we find

$$\big(-m\partial_\tau^2 + V''(q_{inst})\big)\partial_\tau q_{inst} = 0 \tag{4.10}$$

The function $\partial_\tau q_{inst}$ is a zero mode of the determinant with respect to the one-instanton sector. We normalize zero mode by normalization factor $S_{inst}^{-1/2}$, hence the zero mode is $q_1 = S_{inst}^{-1/2}\partial_\tau q_{inst}$. A general function obeying the boundary conditions can be written as $q(\tau) = q_{inst}(\tau) + \sum_n c_n q_n(\tau)$, q_n are a complete set of real orthonormal functions vanishing at the boundaries, $\int_{q_i}^{q_f} dt q_n(t) q_m(t) = \delta_{mn}$. We had to integrate over the centers of the instantons in Eq. (4.3). If we make a perturbative change in the τ_1 then the change of $q(\tau)$ is $dq = (dq_{inst}/d\tau)d\tau_1$. The change induced by a small change of c_1 is $dq = q_1 dc_1$. We find

$$dc_1/\sqrt{2\pi\hbar} = d\tau_1\sqrt{S_{inst}/2\pi\hbar} \tag{4.11}$$

Therefore, in evaluating the determinant, we should not include the zero eigenvalue, but we should include into K a factor of $(S_{inst}/2\pi\hbar)^{1/2}$. Hence the one-instanton contribution to G is

$$\langle q_m|e^{-HT}|q_m\rangle_{one-inst.} = NT(S_{inst}/2\pi\hbar)^{1/2}e^{-S_{inst}/\hbar}(\det{}'[-\partial_\tau^2 + V''(\bar{x})])^{-1/2} \tag{4.12}$$

$\det'$ means that zero eigenvalue is excluded. Comparing this to the one-instanton term in Eq. (4.9), up to the prefactor

$$\langle q_m|e^{-HT/\hbar}|q_m\rangle = \Big(\frac{\omega}{\pi\hbar}\Big)^{1/2} e^{-\omega T/2}\sum_n (Ke^{-S_{inst}/\hbar}T)^n/n! \tag{4.13}$$

and by employing Eq. (4.6, C.6-C.9), we get up to the prefactor

$$N[\det(-\partial_\tau^2 + \omega^2)]^{-1/2} = \big(\omega/\pi\hbar\big)^{1/2} e^{-\omega\tau/2} \tag{4.14}$$

from Eq. (4.12-4.13) we find

$$K = (S_{inst}/2\pi\hbar)^{\frac{1}{2}}\left[\frac{\det(-\partial_\tau^2 + \omega^2)}{\det'(-\partial_\tau^2 + V''(\bar{x}))}\right]^{\frac{1}{2}} \tag{4.15}$$

Let us show that K is a pure imaginary number. q_{inst} (a bounce) has a maximum, it looks like the function denoted by $z = 1$ in the Fig. 4.5. Therefore the eigenfunction q_1 has a node, because it is proportional to the time derivative of the bounce. Hence q_1 is not the eigenfunction of the lowest eigenvalue, so there must be a negative eigenvalue. Therefore, we conclude that K is a pure imaginary number.

We have computed $G(q_m, q_m; \tau) \equiv \langle q_m | e^{-H\tau/\hbar} | q_m \rangle$ in Eq. (4.8). Therefore

$$E_0 = \frac{1}{2}\omega\hbar + \hbar K e^{-S_{inst}/\hbar} \tag{4.16}$$

We deal with an unstable state, its energy has an imaginary part. Hence $Im\ E_0 = \Gamma/2 = \hbar|K|e^{-S_{inst}/\hbar}$, Γ is the decay rate. We have ignored that $\hat{A}$ is not positive-definite. Hence a factor of $1/2$ was missing. The correct Γ is

$$\Gamma = \hbar|K|e^{-S_{inste}/\hbar} \tag{4.17}$$

An explanation is in [9]. The energy of an unstable state is defined by an analytic continuation, it is not an eigenvalue of $\hat{H}$. They consider an integral over some path in function space, which is sketched in Fig. 4.5, parametrized by z, $z \in \mathbb{R}$.

$$I = \int dz (2\pi\hbar)^{-\frac{1}{2}} e^{-S(z)/\hbar} \tag{4.18}$$

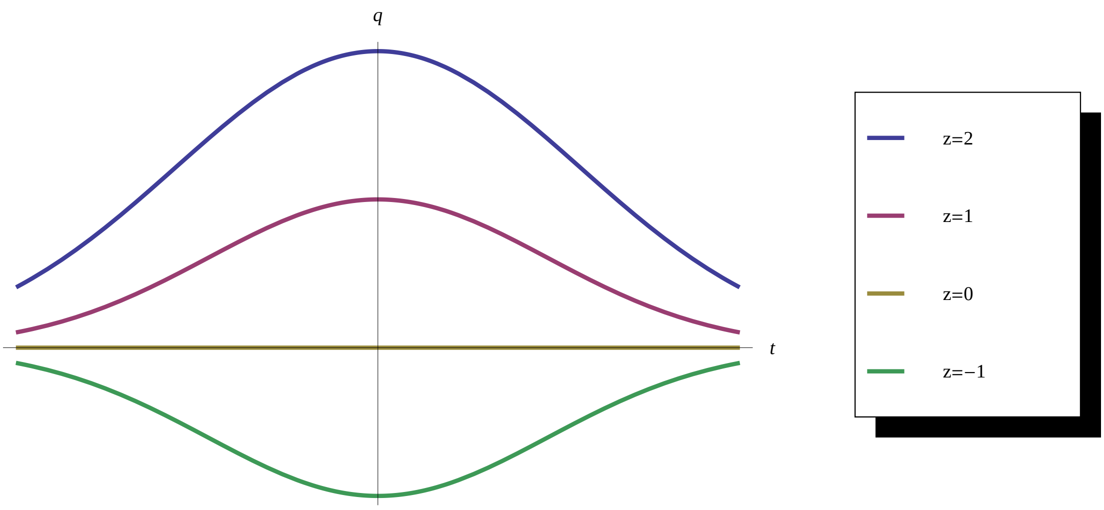

Figure 4.5: The integral over the functions.

This path includes function $q(\tau) = 0$ at $z = 0$ (resting particle) and the bounce (one-instanton) at $z = 1$. $z = 1$ maximizes S. For $z \to \infty$ $S \to -\infty$, because the function spends more and more time in the region where V is negative, hence Eq. (4.18) is divergent. If q_m were the absolute minimum of V then S would never be negative during the integration and the integral would be convergent. We analytically change the potential with absolute minimum into our potential and the steepest descent approximation is employed to keep the integral convergent.

The curve goes along the real axis to $z = 1$ and then out along a line parallel to the imaginary axis. Hence the integral contains an imaginary part

$$Im\ I = Im \int_1^{1+i\infty} dz (2\pi\hbar)^{-\frac{1}{2}} e^{-S(1)/\hbar} e^{-\frac{1}{2}S''(1)(z-1)^2/\hbar} \tag{4.19}$$

$$= \frac{1}{2} e^{-S(1)/\hbar} |S''(1)|^{-\frac{1}{2}} \tag{4.20}$$

We integrated only over half of the Gaussian peak hence 1/2 is present. We have only one negative eigenvalue of the determinant so just one 1/2 was "added". All other one-dimensional integrations give us positive or zero eigenvalues near the stationary point and another 1/2 is not "needed".

For $x \in \mathbb{R}_+$, the terms in the series, $\sum x^n/n!$, increase with n until $n \simeq x$, then they fall fast. Therefore, the important terms in Eq. (4.8) are

$$n \lesssim K\tau e^{-S_{inst}/\hbar} \tag{4.21}$$

The typical summation index of a sum $\sum_n c_n$ of $c_n > 0$ is defined as $\langle n \rangle \equiv \sum_n c_n n / \sum_n c_n$. This gives us a typical number of instantons contributing to $G(q_m, q_m; \tau)$, see Eq (4.8), so we have

$$\langle n \rangle \equiv \frac{\sum_n n X^n/n!}{\sum_n X^n/n!} = X \tag{4.22}$$

where

$$X \equiv \tau K e^{-S_{inst}/\hbar} \tag{4.23}$$

To employ the gas diluted method the instantons must be widely separated, that corresponds with the low density of instantons $\langle n \rangle/\tau$.

Let us remark here that the n and $n+1$-instanton solutions are separated by an energetic barrier. This means they belong to the different sectors.

The description of the CRV model for quantum dissipation includes UIRs of the CCRs and IQV, to be more specific time evolution of the physical observables and of the vacuum is given by BT. Vacua parametrized by time are unitarily non-equivalent in the field limit. The time evolution of the vacuum goes through the set of IQV, see Eq. (2.61, 2.67-2.68). The system contains the decay constant of the classical variable x. The path integral description of the particle escaping from a metastable minimum contains instantons and the decay constant of the escaping particle, see Eq. (4.9). We suggest the identification of the decay constant present in both systems. By this we obtain the equation where on the one side we have the IQV description, on the other side instantonic vacua are explicitly present. So let us denote by Γ_{QD} the decay constant of the classical variable x in the CRV model for quantum dissipation and by Γ_I we denote the decay constant of the escaping particle from metastable minima. By setting up $\Gamma_{QD} \equiv \Gamma_I$ we obtain

$$\Gamma_{QD} = \hbar |K| e^{-S_{inst}/\hbar} \tag{4.24}$$

In Eq. (2.61), we see that inequivalent quantum vacua $|0(t)\rangle$ are not just parametrized by time but rather by $\Gamma_{QD} t$, $|0(\Gamma_{QD} t)\rangle$. Since we have set the equivalence between Γ_{QD} and Γ_I, we can fix time and go through inequivalent quantum vacua by "varying" the potential in the system of a metastable minimum. We can replace $\Gamma_{QD} t$ by $\hbar\langle n \rangle$, see Eq. (4.22, 4.23, 2.61). Inequivalent quantum vacua $|0(\hbar\langle n \rangle)\rangle$ are parametrized by a typical number of instantons contributing to $G(q_m, q_m; \tau)$.

4.2 Double well

Here we review the double well system in the context of the thesis. One particle in the potential, which is given by the blue curve. To see $G(a, \pm a; \tau)$ in the language

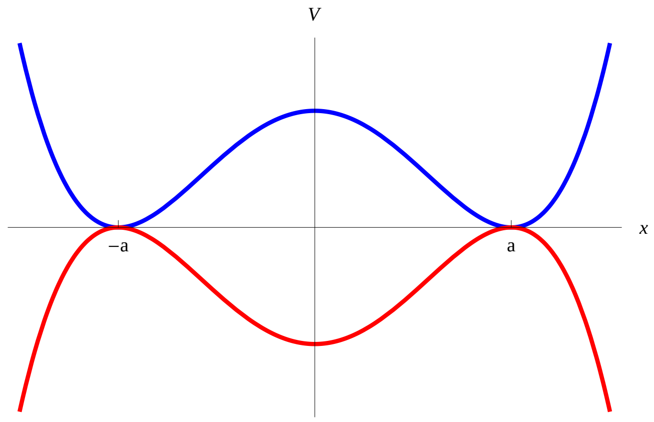

Figure 4.6: Double well.

of path integral we follow procedure in Section 4.1. By Wick rotation the potential is reversed into the red curve hence an instanton which conects the local minima (in Minkowski spacetime) exists. We employ saddle point approximation and gas dilute technique. In Eq. (4.8) we sum over $n = odd$ for $G(a, \pm - a; \tau)$ and $n = even$ for $G(a, \pm a; \tau)$

$$G(a, a; \tau) \simeq Ce^{-\omega\tau/2} \cosh\left(\tau K e^{-S_{inst}/\hbar}\right) \tag{4.25}$$

$$G(a, -a; \tau) \simeq Ce^{-\omega\tau/2} \sinh\left(\tau K e^{-S_{inst}/\hbar}\right) \tag{4.26}$$

Let us deal with two identical but independent linear harmonic oscillators. The eigenvectors and the corresponding eigenvalues of the Hamiltonian are

$$|\psi_0^i\rangle, \ |\psi_1^i\rangle, \ldots \quad \frac{1}{2}\hbar\omega, \ \frac{3}{2}\hbar\omega, \ldots \tag{4.27}$$

with $i = 1, 2$. Let us consider a Hilbertspace made of the two ground states:

$$\mathbb{I} = |\psi_0^1\rangle\langle\psi_0^1| + |\psi_0^2\rangle\langle\psi_0^2| \tag{4.28}$$

The system is described by the Hamiltonian $\hat{H}_0$

$$\hat{H}_0 = E_0^1|\psi_0^1\rangle\langle\psi_0^1| + E_0^2|\psi_0^2\rangle\langle\psi_0^2| \tag{4.29}$$

and $E_0^1 = E_0^2 = \hbar\omega/2$ Let us now turn on an interaction so that the Hamiltonian becomes $\hat{H} = \hat{H}_0 + \hat{H}_I$. In the energy representation we have

$$\hat{H} = \begin{pmatrix} \langle\psi_0^1|\hat{H}|\psi_0^1\rangle & \langle\psi_0^1|\hat{H}|\psi_0^2\rangle \\ \langle\psi_0^2|\hat{H}|\psi_0^1\rangle & \langle\psi_0^2|\hat{H}|\psi_0^2\rangle \end{pmatrix} \equiv \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{2}\hbar\omega + E_D & E_{AD} \\ E_{AD} & \frac{1}{2}\hbar\omega + E_D \end{pmatrix} \tag{4.30}$$

where

$$E_D = \langle\psi_0^1|\hat{H}_I|\psi_0^1\rangle = \langle\psi_0^2|\hat{H}_I|\psi_0^2\rangle \ , \quad E_{AD} = \langle\psi_0^1|\hat{H}_I|\psi_0^2\rangle = \langle\psi_0^2|\hat{H}_I|\psi_0^1\rangle \tag{4.31}$$

If the interaction is turned off, $\hat{H}$ is reduced to

$$\hat{H}_0 = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix} \equiv \begin{pmatrix} E_0^1 & 0 \\ 0 & E_0^2 \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{2}\hbar\omega & 0 \\ 0 & \frac{1}{2}\hbar\omega \end{pmatrix} \tag{4.32}$$

Let us now use a unitary transformation to diagonalize the Hamiltonian and obtain new eigenvectors and new eigenvalues.

$$\hat{H}_{New} = U\hat{H}U^\dagger \ , \quad |\psi^i_{0New}\rangle = U|\psi^i_0\rangle \tag{4.33}$$

where $i = 1, 2$ and

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{4.34}$$

$$\hat{H}_{NEW} \equiv U \begin{pmatrix} \frac{1}{2}\hbar\omega + E_D & E_{AD} \\ E_{AD} & \frac{1}{2}\hbar\omega + E_D \end{pmatrix} U^\dagger = \begin{pmatrix} \frac{1}{2}\hbar\omega + E_D - E_{AD} & 0 \\ 0 & \frac{1}{2}\hbar\omega + E_D + E_{AD} \end{pmatrix} \tag{4.35}$$

$$\begin{pmatrix} |S\rangle \\ |A\rangle \end{pmatrix} \equiv U \begin{pmatrix} |\psi_0^1\rangle \\ |\psi_0^2\rangle \end{pmatrix} \tag{4.36}$$

the eigenvalues and the eigenvectors are

$$E_S = \frac{1}{2}\hbar\omega + E_D - E_A \ , \quad E_A = \frac{1}{2}\hbar\omega + E_D + E_A \tag{4.37}$$

$$|S\rangle = \frac{1}{\sqrt{2}}(|\psi_0^1\rangle + |\psi_0^2\rangle) \ , \quad |A\rangle = \frac{1}{\sqrt{2}}(|\psi_0^1\rangle - |\psi_0^2\rangle) \tag{4.38}$$

Now the identity is

$$\mathbb{I} = |S\rangle\langle S| + |A\rangle\langle A| \tag{4.39}$$

therefore

$$\langle -a|S\rangle\langle S|a\rangle = \langle -a|a\rangle - \langle -a|A\rangle\langle A|a\rangle \tag{4.40}$$

$|\pm a\rangle$ are minima of the double well. We can realize that

$$\langle a|S\rangle = \langle -a|S\rangle \tag{4.41}$$

$$\langle a|A\rangle = \langle -a| - A\rangle = -\langle -a|A\rangle \tag{4.42}$$

we denote

$$\langle a|S\rangle\langle S|a\rangle = \frac{C}{2} \tag{4.43}$$

$\langle -a|a\rangle = \delta(-2a) = 0$ for $a \neq 0$, and Eq. (4.40-4.43) finally give us

$$\langle a|A\rangle\langle A|a\rangle = \langle a|S\rangle\langle S| \pm a\rangle = -\langle a|A\rangle\langle A| - a\rangle = \frac{C}{2} \tag{4.44}$$

now we can calculate

$$\begin{aligned} G(a, \pm a; \tau) &= \langle a|e^{-\frac{1}{\hbar}\hat{H}\tau}| \pm a\rangle \\ &= \langle a|(|S\rangle\langle S| + |A\rangle\langle A|)e^{-\frac{1}{\hbar}\hat{H}\tau}(|S\rangle\langle S| + |A\rangle\langle A|)| \pm a\rangle \\ &= \langle a|(|S\rangle e^{-\frac{1}{\hbar}E_S\tau}\langle S| + |A\rangle e^{-\frac{1}{\hbar}E_A\tau}\langle A|)| \pm a\rangle \\ &= e^{-\frac{1}{\hbar}\tau(\frac{1}{2}\hbar\omega + E_D)}\{\langle a|S\rangle\langle S| \pm a\rangle e^{-\frac{1}{\hbar}E_{AD}\tau} + \langle a|A\rangle\langle A| \pm a\rangle e^{\frac{1}{\hbar}E_{AD}\tau}\} \end{aligned} \tag{4.45}$$

$$G(a,a;\tau) = Ce^{-\frac{1}{\hbar}\tau(\frac{1}{2}\hbar\omega + E_D)} \cosh\left(\frac{E_{AD}\tau}{\hbar}\right) \tag{4.46}$$

$$G(a,-a;\tau) = Ce^{-\frac{1}{\hbar}\tau(\frac{1}{2}\hbar\omega + E_D)} \sinh\left(\frac{E_{AD}\tau}{\hbar}\right) \tag{4.47}$$

On the one hand we have a double well system and on the other we have two harmonic oscillators. In classical theory the double well has two degenerate classical vacua, a topological one in the meaning that if the energy of the particle is small then the particle cannot move from one sector to the other. Here by a sector we mean an area around a classical minima. We started with two isolated linear harmonic oscillators. To manage this we could say that these oscillators are widely separated. We build a system in the way that we take the lowest states from the harmonic oscillators. We would like to build a double well system. We turn on an interaction to effectively gain a double well from two widely separated oscillators. In Euclidean space, reached by Wick rotation, we have instanton which connects those classical vacua. The instanton is the most important path which contributes to the tunneling, which is a QM effect. In our QM system build from two isolated oscilators, a tunneling is present due to the interaction term. Someone could investigate the relation between the interaction term on one hand and instanton on the other. Here the quantum vacuum is a linear combination of the original lowest states of the oscillators and the coefficients do not depend on anything here, hence it does not depend on the interaction term. Only the energy splitting of those two lowest states is proportional to the exponential of an action of the instanton, which connects those classical vacua. Let us find the interaction term which is added to the Hamiltonian of two isolated harmonic oscillators in low energy approximation to get the Hamiltonian which resembles double well system. In the following we drop the diagonal lifting of the energy E_D because it is not essential and we also use the notation

$$|1\rangle \equiv |\psi_0^1\rangle \ , \quad |2\rangle \equiv |\psi_0^2\rangle \tag{4.48}$$

$$\begin{aligned} \hat{H} &= (\frac{1}{2}\hbar\omega - E_{AD})|S\rangle\langle S| + (\frac{1}{2}\hbar\omega + E_{AD})|A\rangle\langle A| \\ &= \frac{1}{2}\Big(\frac{1}{2}\hbar\omega - E_{AD}\Big)(|1\rangle + |2\rangle)(\langle 1| + \langle 2|) + \frac{1}{2}\Big(\frac{1}{2}\hbar\omega + E_{AD}\Big)(|1\rangle - |2\rangle)(\langle 1| - \langle 2|) \\ &= \frac{1}{2}\hbar\omega|1\rangle\langle 1| + \frac{1}{2}\hbar\omega|2\rangle\langle 2| - E_{AD}(|1\rangle\langle 2| + |2\rangle\langle 1|) \end{aligned} \tag{4.49}$$

Let us view the first two terms in the Hamiltonian as two isolated harmonic oscillators, where the first (second) term is related to the Hamiltonian of linear harmonic oscillator with respect to variable x (y)

$$\frac{P_x^2}{2m} + \frac{m}{2}\omega^2 x^2 \tag{4.50}$$

$$\frac{P_y^2}{2m} + \frac{m}{2}\omega^2 y^2 \tag{4.51}$$

and the last term is related to an interaction Hamiltonian $\hat{H}_I$. Let us now show to which expression in terms of x and y $\hat{H}_I$ corresponds. We start with:

$$\mathbb{I} = |S\rangle\langle S| + |A\rangle\langle A| = |1\rangle\langle 1| + |2\rangle\langle 2| \tag{4.52}$$

$$\mathbb{I}\hat{x}\mathbb{I}\hat{y}\mathbb{I} = (|1\rangle\langle 1| + |2\rangle\langle 2|)\hat{x}(|1\rangle\langle 1| + |2\rangle\langle 2|)\hat{y}(|1\rangle\langle 1| + |2\rangle\langle 2|) \tag{4.53}$$

$$\begin{aligned}\mathbb{I}\hat{x}\mathbb{I}\hat{y}\mathbb{I} + \mathbb{I}\hat{y}\mathbb{I}\hat{x}\mathbb{I} &= |1\rangle\langle 1|\Big(\langle 1|\hat{x}|2\rangle\langle 2|\hat{y}|1\rangle + \langle 1|\hat{y}|2\rangle\langle 2|\hat{x}|1\rangle\Big) + \\ &+ |2\rangle\langle 2|\Big(\langle 2|\hat{x}|1\rangle\langle 1|\hat{y}|2\rangle + \langle 2|\hat{y}|1\rangle\langle 1|\hat{x}|2\rangle\Big) + \\ &+ |1\rangle\langle 2|\Big(\langle 1|\hat{y}|1\rangle\langle 1|\hat{x}|2\rangle + \langle 1|\hat{y}|2\rangle\langle 2|\hat{x}|2\rangle\Big) + \\ &+ |2\rangle\langle 1|\Big(\langle 2|\hat{x}|1\rangle\langle 1|\hat{y}|1\rangle + \langle 2|\hat{x}|2\rangle\langle 2|\hat{y}|1\rangle\Big)\end{aligned} \tag{4.54}$$

where we applied well known relations

$$\langle 1|\hat{x}|1\rangle = 0 \ , \quad \langle 2|\hat{y}|2\rangle = 0 \tag{4.55}$$

and we will use

$$\langle x|\psi_0^1\rangle = \psi_0^1(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \tag{4.56}$$

$$\langle y|\psi_0^2\rangle = \psi_0^2(y) = \sqrt[4]{\frac{m\omega}{\pi\hbar}}e^{-\frac{1}{2}\frac{m\omega}{\hbar}y^2} \tag{4.57}$$

We would like to see this oscillators living on the same axis hence both described by a same variable. We now make the link with one variable system, explicitely

$$\hat{x} = (\hat{q} + a) \ , \quad \hat{y} = (\hat{q} - a) \tag{4.58}$$

$$V_x = \frac{1}{2}m\omega^2(q+a)^2 \ , \quad V_y = \frac{1}{2}m\omega^2(q-a)^2 \tag{4.59}$$

$$\begin{aligned}\langle 1|\hat{x}|2\rangle &= \frac{m\omega}{\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{m\omega}{\hbar}(q+a)^2}(q+a)e^{-\frac{1}{2}\frac{m\omega}{\hbar}(q-a)^2}dq \\ &= \frac{m\omega}{\pi\hbar}e^{-\frac{m\omega}{\hbar}a^2}\int_{-\infty}^{\infty}(q+a)e^{-\frac{m\omega}{\hbar}}q^2dq \\ &= a\sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega}{\hbar}a^2}\end{aligned} \tag{4.60}$$

functions are real hence:

$$\langle 1|\hat{x}|2\rangle = \langle 2|\hat{x}|1\rangle \ , \quad \langle 1|\hat{y}|2\rangle = \langle 2|\hat{y}|1\rangle \tag{4.61}$$

we can find that

$$\begin{aligned}\langle 1|\hat{x}|2\rangle &= a\sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega}{\hbar}a^2} \\ \langle 2|\hat{x}|2\rangle &= 2a\sqrt{\frac{m\omega}{\pi\hbar}} \\ \langle 1|\hat{y}|2\rangle &= -\langle 1|\hat{x}|2\rangle \\ \langle 1|\hat{y}|1\rangle &= -\langle 2|\hat{x}|2\rangle\end{aligned} \tag{4.62}$$

$$\hat{x}\hat{y}+\hat{y}\hat{x} = -2(|1\rangle\langle 1|+|2\rangle\langle 2|)(\langle 1|\hat{x}|2\rangle\langle 1|\hat{x}|2\rangle) - - 2(|1\rangle\langle 2|+|2\rangle\langle 1|)(\langle 1|\hat{x}|2\rangle\langle 2|\hat{x}|2\rangle) \quad (4.63)$$

Thus the $-\hat{H}_I/E_{AD}$ can be expressed

$$|1\rangle\langle 2|+|2\rangle\langle 1| = \frac{1}{2}e^{-m\omega a^2/\hbar}(|1\rangle\langle 1|+|2\rangle\langle 2|) - \frac{1}{4a^2}\left(\frac{\pi\hbar}{m\omega}\right)e^{m\omega a^2/\hbar}(\hat{x}\hat{y}+\hat{y}\hat{x}) \quad (4.64)$$

and the Hamiltonian is

$$\hat{H} = \left(\frac{1}{2}\hbar\omega|1\rangle\langle 1| + \frac{1}{2}\hbar\omega|2\rangle\langle 2|\right)\left(1 - \frac{E_{AD}}{\hbar\omega}e^{-m\omega a^2/\hbar}\right) - \frac{E_{AD}}{4a^2}\left(\frac{\pi\hbar}{m\omega}\right)e^{m\omega a^2/\hbar}(\hat{x}\hat{y}+\hat{y}\hat{x}) \quad (4.65)$$

A consequence of adding the $\hat{H}_I$ to the $\hat{H}_0$ is that the $\hat{H}_0$ was effectively multi-

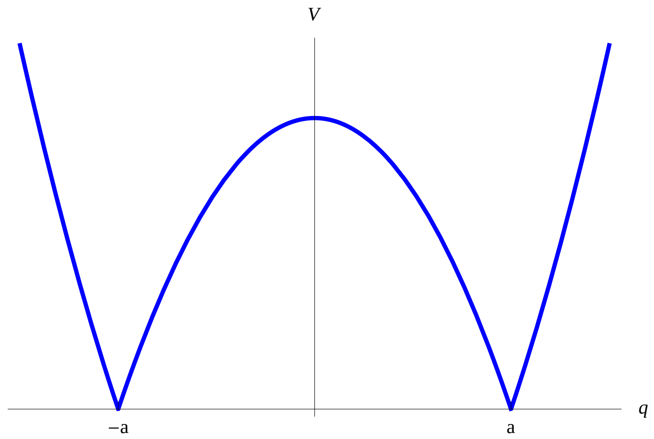

Figure 4.7: Pure interaction term.

plied by a factor, this resembles us a renormalization. It would be interesting to investigate a case where we would start with infinitely many isolated harmonic oscillators, what is QFT, where UIRs are present. In this models we could investigate a relationship between instantons and an interaction term and its connection to the renormalization procedure, which is so often present and needed in QFT. We obtained a term proportional to $\hat{x}\hat{y}+\hat{y}\hat{x}$, let us call it pure interaction. In the Fig. 4.7 we plot this term after the substitution $\hat{x}\to\hat{q}+a$, $\hat{y}\to\hat{q}-a$, the plot is very similar to the double well. This term connects two harmonic oscillators, which we started with.

$$E_{AD} = \langle\psi_0^1|\hat{H}_I|\psi_0^2\rangle \propto \langle\psi_0^1|\hat{x}\hat{y}+\hat{y}\hat{x}|\psi_0^2\rangle \propto \hbar K e^{-S_{Inst}/\hbar} \quad (4.66)$$

Here we can see a relationship between the exponential of the action of the instanton and the pure interaction term surrounded by the ground states of originally isolated harmonic oscillators, which ground states are connected by the instanton.

We could have been even more accurate for example we could start with system built from the states

$$|\psi_0^1\rangle\ ,\ |\psi_1^1\rangle\ ,\ |\psi_0^2\rangle\ ,\ |\psi_1^2\rangle \quad (4.67)$$

And proceed as we did. It is questionable whether it would lead to any interesting observation.

4.3 UIRs and Particle on the circle

We deal with a bead moving without friction on a circular wire with radius r_0. A global topology of a phase space is $S^1 \times \mathbb{R}$. An angle $\varphi \in \mathbb{R}$ mod 2π determines the position of the bead. The angular momentum, $p_\varphi \in \mathbb{R}$, is positive if the bead moves anticlockwise, negative if the bead moves clockwise. The Lagrangian and the Hamiltonian are:

$$L = \frac{1}{2} m r_0^2 \dot{\varphi}^2 \ , \quad H = \frac{p_\varphi^2}{2mr_0^2} \ , \quad p_\varphi \equiv m r_0^2 \dot{\varphi} \tag{4.68}$$

Let us start with path integral quantization of particle on circle, based on [19] and [12]. Paths are divided into the sectors given by the winding number. Any path from any sector cannot be continuously deformed to a path which belong to a different sector. φ_n is a path that belongs to a sector which is labeled by a winding number n. φ_i (φ_f) is an initial (final) position. A probability amplitude is Eq. (C.2)

$$\begin{aligned} \langle \varphi_f | e^{-\frac{i}{\hbar}\hat{H}(t_f - t_i)} | \varphi_i \rangle &= \int_{\substack{\varphi(t_f) = \varphi_f \\ \varphi(t_i) = \varphi_i}} D\varphi \ \exp\left(\frac{i}{\hbar} S[\varphi]\right) \\ &= \sum_{n\in\mathbb{Z}} \int_{\substack{\varphi_n(t_f) = \varphi_f \\ \varphi_n(t_i) = \varphi_i}} D\varphi_n \ \exp\left(\frac{i}{\hbar} S[\varphi_n]\right) \end{aligned} \tag{4.69}$$

It is claimed that Schroedinger equation of motion is satisfied by path integral contribution from any sector, see [19]. Therefore a linear combination of path integral contribution from this sectors should solve the Schroedinger equation too. When we move with φ_f to $\varphi_f + 2\pi$ predictions of a theory should not be effected. The probability amplitude can by multiplied by phase, say $e^{i\tilde{\delta}}$, where $\tilde{\delta} \in [0, 2\pi)$. By moving the final position φ_f by 2π, trajectories connect the initial position with the final position change their winding numbers by one. A path contribution from some sector, say n, becomes a path contribution from sector $n+1$. In order to the consequence of this was that the whole amplitude was just multiplied by phase $e^{i\tilde{\delta}}$ the relationship between coefficients, denote by c_n, of the linear combination of the path integral contributions from each sector must be: $c_{n+1} = e^{i\tilde{\delta}} c_n$. If we set $c_o = 1$ we get $c_n = e^{in\tilde{\delta}}$. The contribution from n-th sector is a propagation of free particle on the line between points φ_i and $\varphi_f + 2\pi n$. Hence:

$$\begin{aligned} \langle \varphi_f | e^{-\frac{i}{\hbar}\hat{H}(t_f - t_i)} | \varphi_i \rangle &= \sum_{n\in\mathbb{Z}} e^{in\delta} \int_{\substack{\varphi_n(t_f) = \varphi_f \\ \varphi_n(t_i) = \varphi_i}} D\varphi_n \ \exp\left(\frac{i}{\hbar} S[\varphi_n]\right) \\ &= \sum_{n\in\mathbb{Z}} \left(\frac{m r_0^2}{2\pi i (t_f - t_i)}\right)^{1/2} \exp\left(in\tilde{\delta} + \frac{i m r_0^2 \big((\varphi_f - \varphi_i) - 2n\pi\big)^2}{2(t_f - t_i)}\right) \end{aligned} \tag{4.70}$$

Schroedinger equation is

$$i\partial_t \psi = -\frac{\hbar}{2mr_0^2} \partial_{\varphi\varphi} \psi \tag{4.71}$$

If we do not require $\psi(0) = \psi(2\pi)$ then the eigenvectors and the eigenvalues of the Hamiltonian are

$$\psi_n = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}(n\varphi + \frac{\tilde{\delta}\varphi}{2\pi})} \tag{4.72}$$

$$E_n = \frac{\hbar}{2mr_0^2}\left(n + \frac{\tilde{\delta}}{2\pi}\right)^2, \ n \in \mathbb{Z}, \ \tilde{\delta} \in [0.2\pi) \tag{4.73}$$

A canonical quantization of the system is based on [5]. The solution of the equation of motion of Eq. (4.68) is

$$\varphi(t) = \omega t + \varphi_0 \tag{4.74}$$

where $\omega = \frac{p_\varphi}{mr_0^2}$ and $\varphi_0 \in [0, 2\pi)$.

As the classical particle moves around the circle, we map this on the real line. The real line is called the universal covering space of the circle. If the bead circles around q times, we say that it runs through a q-fold covering.

Functions $\tilde{h}_1, \tilde{h}_2$ and $\tilde{h}_3$ are defined on the phase space $S_{\varphi,p_\varphi} = \{s = (\varphi, p_\varphi); \varphi \in \mathbb{R} \bmod 2\pi, p_\varphi \in \mathbb{R}\}$. These functions obey the Lie algebra $e(2)$ of the Euclidean group $E(2)$ with respect to the Poisson brackets on S_{φ,p_φ}, hence to quantize the system we replace $\tilde{h}_1, \tilde{h}_2$ and $\tilde{h}_3$ by operators $\hat{X}_1$, $\hat{X}_2$, $\hat{L}$ and the Poisson brackets by commutators, as follows

$$\tilde{h}_1(\varphi, p_\varphi) = \cos\varphi, \quad \tilde{h}_2(\varphi, p_\varphi) = \sin\varphi, \quad \tilde{h}_3(\varphi, p_\varphi) = p_\varphi \tag{4.75}$$

$$\{\tilde{h}_3, \tilde{h}_1\}_{\varphi,p_\varphi} = \tilde{h}_2, \quad \{\tilde{h}_3, \tilde{h}_2\}_{\varphi,p_\varphi} = -\tilde{h}_1, \quad \{\tilde{h}_1, \tilde{h}_2\}_{\varphi,p_\varphi} = 0 \tag{4.76}$$

$$\frac{1}{\hbar}[\hat{L}, \hat{X}_1] = i\hat{X}_2, \quad \frac{1}{\hbar}[\hat{L}, \hat{X}_2] = -i\hat{X}_1, \quad [\hat{X}_1, \hat{X}_2] = 0 \tag{4.77}$$

The relations above are the Lie $e(2)$ algebra. The definitions of $E(2)$ group, the group multiplication law, the action of the group element of $E(2)$ to a point of the phase space are in [5].

If the two-dimensional Euclidean plane is represented by complex numbers z then action of rotation $R(\alpha)$ by angle α, $\alpha \in [0, 2\pi)$ and displacement $T_2(t)$ by t, $t = a + ib$; $a, b \in \mathbb{R}$ is given by

$$\begin{aligned} R(\alpha) &: z \to e^{i\alpha} z \\ T_2(t) &: z \to z + t \end{aligned} \tag{4.78}$$

Our Hilbert space is $L^2(S^1, d\varphi/2\pi)$, scalar product is

$$(\psi_2, \psi_1) = \int_0^{2\pi} \frac{d\varphi}{2\pi} \psi_2^*(\varphi)\psi_1(\varphi) \tag{4.79}$$

The irreducible unitary representations are parametrized by two real numbers (ρ, δ), $\rho > 0$, $\delta \in [0, 1)$. Representations are given by, see [5].

$$[U^{\rho,\delta}(\alpha)\psi](\varphi) = e^{-i\delta\alpha}\psi[(\varphi - \alpha) \bmod 2\pi] \tag{4.80}$$

$$[U^{\rho,\delta}(t = a + ib)\psi](\varphi) = e^{-(i/\hbar)\rho(a\cos\varphi + b\sin\varphi)}\psi(\varphi) \tag{4.81}$$

δ labels the irreducible unitary representations. $\delta = 0$ for the irreducible unitary representations of the group $E(2)$. For the q-fold covering of $E(2)$ $\delta = \frac{p}{q}$, where $p, q \in \mathbb{N}$ and no common divisor. For a representation of the universal covering group $\tilde{E}(2)$ δ is an irrational number.

We define

$$U^{\rho,\delta}(\alpha) = e^{-(i/\hbar)\hat{L}_\delta \alpha}, \quad U^{(\rho,\delta)}(t) = e^{-[i/(\hbar\lambda_0)](\hat{X}_1 a + \hat{X}_2 b)} \tag{4.82}$$

where $\lambda_0 = (\hbar/m\omega)^{1/2}$ from Eq. (4.80, 4.81) we find that

$$\frac{1}{\hbar}L_\delta = \frac{1}{i}\partial_\varphi + \delta, \quad X_1 = r\cos\varphi, \quad X_2 = r\sin\varphi \tag{4.83}$$

where $r = \rho\lambda_0$. $e_n(\varphi) = e^{in\varphi}, n \in \mathbb{Z}$, are eigenfunctions of the orbital angular momentum: $L_\delta e_n = \hbar(n+\delta)e_n, n \in \mathbb{Z}$ and those vectors form the orthonormal basis of our Hilbert space. Now the Hamiltonian and its eigenvalues are

$$H_\delta = \frac{L_\delta^2}{2mr^2} = \frac{1}{2}\epsilon\hbar\omega\Big(\frac{1}{i}\partial_\varphi + \delta\Big)^2 \tag{4.84}$$

$$E_{n,\delta} = \frac{1}{2}\epsilon\hbar\omega(n+\delta)^2, \quad n \in \mathbb{Z} \tag{4.85}$$

The lowest energy depends on whether $\delta \in [0, 1/2)$ or $\delta \in (1/2, 1)$, as follows. If $\delta = 1/2$ the ground state is degenerate

$$E_{n=0,\delta} = \frac{1}{2}\epsilon\hbar\omega\delta^2 \quad or \quad E_{n=-1,\delta} = \frac{1}{2}\epsilon\hbar\omega(1-\delta)^2 \tag{4.86}$$

For different δ the operators L_δ, H_δ have different spectra and observables, such operators are not unitary equivalent.

By the unitary transformations

$$e_n(\varphi) = e^{in\varphi} \to e_{n,\delta}(\varphi) = e^{i\delta\varphi}e_n(\varphi) = e^{i(n+\delta)\varphi} \quad \forall n \in \mathbb{Z} \tag{4.87}$$

we shift dependence of operators on δ to the set of Hilbert spaces labelled by δ. we define a Hilbert space $L^2(S^1, d\varphi/2\pi, \delta)$ for each δ. Now the generators in Eq. (4.83) have the form [5]

$$\frac{1}{\hbar}L_\delta = \frac{1}{i}\partial_\varphi, \quad X_1 = r\cos\varphi, \quad X_2 = r\sin\varphi \tag{4.88}$$

and the operators are independent of δ. The δ-dependence is shifted to the basis $e_{n,\delta}(\varphi)$.

We define

$$U(a_1, a_2) \equiv \exp(a_1\hat{X}_1 + a_2\hat{X}_2)\ , \ V(b) = \exp(b\hat{L}) \tag{4.89}$$

where $\hat{X}_1$, $\hat{X}_2$, $\hat{L}$ are generators of $e(2)$ algebra, $a_1, a_2, b \in \mathbb{R}$. The Weyl form of the CCRs of the $e(2)$ Eq. (4.77) is

$$\begin{aligned} U(a_1,a_2)U(c_1,c_2) &= U(a_1+c_1, a_2+c_2) \\ V(b)V(c) &= V(b+c) \\ V(b)U(a_1,a_2) &= U(a_1 - i\hbar b a_2/2, a_2 + i\hbar b a_1/2)V(b) \end{aligned} \tag{4.90}$$

We will prove only the last one by using Eq. (4.77) and Baker-Campbell-Hausdorff formula

$$\begin{aligned} V(b)U(a) &= e^{[b\hat{L},a_1\hat{X}_1+a_2\hat{X}_2]/2}e^{(a_1\hat{X}_1+a_2\hat{X}_2)}e^{b\hat{L}} \\ &= e^{(i\hbar ba_1\hat{X}_2-i\hbar ba_2\hat{X}_1)/2}e^{(a_1\hat{X}_1+a_2\hat{X}_2)}e^{b\hat{L}} \\ &= U(a_1-i\hbar ba_2/2, a_2+i\hbar ba_1/2)V(b) \end{aligned} \tag{4.91}$$

Now let us quantize our system, i.e. the bead on a circle in a more usual way by canonical quantization, but instead of $[\hat{q},\hat{p}]=i\hbar\hat{I}$ we have

$$[\hat{\varphi},\hat{L}]=i\hbar\hat{I}\ ,\ [\hat{\varphi},\hat{I}]=0\ ,\ [\hat{L},\hat{I}]=0 \tag{4.92}$$

Weyl form of the CCRs is

$$\exp\left[\frac{i}{\hbar}Q\hat{L}\right]\exp\left[\frac{i}{\hbar}P\hat{\varphi}\right]=\exp\left[\frac{i}{\hbar}QP\right]\exp\left[\frac{i}{\hbar}P\hat{\varphi}\right]\exp\left[\frac{i}{\hbar}Q\hat{L}\right] \tag{4.93}$$

we briefly mention the squeezing transformation [20]

$$Q\to\frac{Q}{S},\quad P\to PS,\quad S\in\mathbb{R} \tag{4.94}$$

from Eq. (4.93, 4.94) we get

$$\exp\left[\frac{i}{\hbar}\frac{Q}{S}\hat{L}\right]\exp\left[\frac{i}{\hbar}SP\hat{\varphi}\right]=\exp\left[\frac{i}{\hbar}QP\right]\exp\left[\frac{i}{\hbar}SP\hat{\varphi}\right]\exp\left[\frac{i}{\hbar}\frac{Q}{S}\hat{L}\right] \tag{4.95}$$

The term QP is invariant under the squeezing transformation.

4.4 QM on circle - QFT correspondence

There exists a correspondence between a particle on a vertical circle in a uniform gravitational field and the sine-Gordon field theory. We will start with a Lagrangian which describes the former system

$$L_{QM}=\frac{1}{2}mr^2\dot{\theta}^2-mgr(1-\cos\theta) \tag{4.96}$$

and we will proceed very quickly to the sine-Gordon model by the following procedure as was done in [21]. Particle on a circle is a QM system, it has one degree of freedom. On the other hand we have learned already that for a particle on a circle there are UIRs due to a non trivial topology, where SvNT can not be applied. We can make a correspondence with the sine-Gordon system which is a field theory, hence the degree of freedom is infinity and UIRs are present as well. Radius of the circle is r, mass of the particle is m, g is the gravitational constant. The gravity can be replaced by a magnetic field as was done in [22]. The system is described by the position of the bead on the circle i.e. by the angle θ, $\theta\in[0,2\pi)$. At first we replace θ by q, $q\in(-\infty,+\infty)$

$$L'_{QM}=\frac{1}{2}\dot{q}^2-(1-\cos q) \tag{4.97}$$

This can be done due to a fact that the potential is 2π-periodic. The Lagrangian we obtained is a one-dimensional analog of $1+1$ dimensional field theory, namely sine-Gordon model

$$L_{QFT} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - (1 - \cos\phi) \tag{4.98}$$

where field ϕ, $\phi \in (-\infty, +\infty)$.

A static solution of our field theory is a solitonic solution of our corresponding QM system. We find the static solution of the sine-Gordon model when ϕ is replaced by θ, x by $\omega_0\tau$ we get instanton for our QM system.

$$\theta(\tau) = \pm 4\tan^{-1}[\exp(\omega_0\tau)] \tag{4.99}$$

Due to this correspondence we have access to both QM system with non trivial topology and QFT. Here instantons are seen and UIRs have to be present.

Conclusion

We have introduced the topologically distinct instantonic vacua and the unitarily inequivalent quantum vacua. We investigated the relationship between those two different inequivalences and we would like to unify them under just one kind of inequivalence, possibly at some area. An issue that, to our knowledge, was never addressed before.

By identification of the decay constant present in the CRV model for quantum dissipation, the description of which contains UIRs of the CCRs, and the particle escaping from a metastable minimum, where the path integral description includes instantons, we compared these two systems to see some possible connection we are looking for. We obtained an equation

$$\Gamma_{QD} = \hbar |K| e^{-S_{inst}/\hbar} \tag{4.100}$$

where on the one side we have the inequivalent quantum vacua description, on the other instantonic vacua are explicitly present. Through this "door", the link we are looking for should be closer. In this particular case, a typical number of instantons contributing to a probability amplitude that the particle remains at the metastable minimum, parametrizes IQV in the CRV model.

We considered a double well system, which contains instantons and tried to see this as a system build from two harmonic oscillators and an interaction term because the CRV model is build from damped and amplified harmonic oscillators, where UIRs and IQV are involved. The Hamiltonian which we obtained, see Eq. (4.65, 4.66) contains action of the instanton and term proportional to xy. Someone could somehow try to build the CRV model Eq. (2.35, 2.57-2.59), i.e. to get terms $(yp_y - xp_x)$, $p_x p_y$ involved and still keep the instantons from the double well.

Quantum particle on the circle is a place where UIRs are present too.

Let us make some comment on [21] with respect to our research. In [22] a correspondence between a particle on vertical rotating circle in a uniform gravitational field and double-sine-Gordon field theory is discussed. In the case of a particle on rotating circle due to the fact that if the speed of rotation is bigger than some critical rotation speed than in this system the particle can occupy two degenerate local minima, therefore instantons are present. The situation is very similar to a double well case. But a double well system is a QM system, hence UIRs can not be expected. We have already mentioned that for a particle on a circle there are UIRs due to non trivial topology, where Stone-von Neumann theorem is not applicable. Therefore some relation between instantons and UIRs could exist there.

List of Figures

List of Abbreviations

- STR-Special Theory of Relativity
- QM-Quantum Mechanics
- QFT-Quantum Field Theory
- CCR-Canonical Commutation Relation
- SvNT-Stone-von Neumann Theorem
- UIR-Unitarily Inequivalent Representations
- CRV-Celeghini, Rasetti, Vitiello
- IQV-Inequivalent Quantum Vacua

A. Appendix Calculation of $|0\rangle\rangle$

Here we proove the formula Eq. (A.40). It is based on the guide in [8] (section 2.4.)

We deal with a set of creation and annihilation operators $\alpha(\mathbf{k}), \alpha^\dagger(\mathbf{k}), \beta(\mathbf{k}), \beta^\dagger(\mathbf{k})$, which satisfy the following CCRs. The commutators, that are not explicitly mentioned are equal to zero

$$[\alpha(\mathbf{k}), \alpha^\dagger(\mathbf{l})] = \delta(\mathbf{k}-\mathbf{l}) \ , \quad [\beta(\mathbf{k}), \beta^\dagger(\mathbf{l})] = \delta(\mathbf{k}-\mathbf{l}) \tag{A.1}$$

The Bogoliubov transformation defines operators $a(\mathbf{k}), a^\dagger(\mathbf{k}), b(\mathbf{k}), b^\dagger(\mathbf{k})$

$$a(\mathbf{k}) = \cosh\theta_k \alpha(\mathbf{k}) - \sinh\theta_k \beta^\dagger(-\mathbf{k}) \tag{A.2}$$

$$b(\mathbf{k}) = \cosh\theta_k \beta(\mathbf{k}) - \sinh\theta_k \alpha^\dagger(-\mathbf{k}) \tag{A.3}$$

The transformation is canonical, as we can see from the following relationships.

$$[a(\mathbf{k}), a^\dagger(\mathbf{l})] = \delta(\mathbf{k}-\mathbf{l}) \ , \quad [b(\mathbf{k}), b^\dagger(\mathbf{l})] = \delta(\mathbf{k}-\mathbf{l}) \tag{A.4}$$

It is an easy task to show that the line above holds

$$\begin{aligned}
[a(\mathbf{k}), a^\dagger(\mathbf{l})] &= \cosh\theta_k \cosh\theta_l [\alpha(\mathbf{k}), \alpha^\dagger(\mathbf{l})] + \sinh\theta_k \sinh\theta_l [\beta^\dagger(-\mathbf{k}), \beta(-\mathbf{l})] \\
&= (\cosh\theta_k \cosh\theta_l - \sinh\theta_k \sinh\theta_l)\delta(\mathbf{k}-\mathbf{l}) \\
&= \delta(\mathbf{k}-\mathbf{l})
\end{aligned} \tag{A.5}$$

$A(\theta), G(\theta)$ are defined as follows

$$G(\theta) = \exp\left[A(\theta)\right] \ , \ A(\theta) = \int d^3\mathbf{k}\theta_k \left[\alpha(\mathbf{k})\beta(-\mathbf{k}) - \beta^\dagger(-\mathbf{k})\alpha^\dagger(\mathbf{k})\right] \tag{A.6}$$

We will need to employ the expressions

$$[\alpha(\mathbf{k}), A(\theta)] = -\theta_k \beta^\dagger(-\mathbf{k}) \tag{A.7}$$

$$[\beta^\dagger(-\mathbf{k}), A(\theta)] = -\theta_k \alpha(\mathbf{k}) \tag{A.8}$$

Let us calculate only one of them

$$\begin{aligned}
[\alpha(\mathbf{k}), A(\theta)] &= [\alpha(\mathbf{k}), \int d^3\mathbf{l}\ \theta_l(\alpha(\mathbf{l})\beta(-\mathbf{l}) - \beta^\dagger(-\mathbf{l})\alpha^\dagger(\mathbf{l}))] \\
&= -\int d^3\mathbf{l}\ \theta_l [\alpha(\mathbf{k}), \beta^\dagger(-\mathbf{l})\alpha^\dagger(\mathbf{l})] \\
&= -\theta_k \beta^\dagger(-\mathbf{k})
\end{aligned} \tag{A.9}$$

We want to proove the following statement

$$G^{-1}(\theta)\alpha(\mathbf{k})G(\theta) = \alpha(\mathbf{k})\cosh\theta_k - \beta^\dagger(-\mathbf{k})\sinh\theta_k \tag{A.10}$$

It is an easy task to compute the following terms, $n \in 1, 2, 3, ..$

$$\begin{aligned}
[A(\theta), \alpha(\mathbf{k})] &= \theta_k \beta^\dagger(-\mathbf{k}) \\
\frac{1}{2!}[A(\theta), [A(\theta), \alpha(\mathbf{k})]] &= \frac{1}{2!}[A(\theta), \theta_k \beta^\dagger(-\mathbf{k})] = \frac{1}{2!}\theta_k^2 \alpha(\mathbf{k}) \\
\frac{1}{3!}[A(\theta), [A(\theta), [A(\theta), \alpha(\mathbf{k})]]] &= \frac{1}{3!}\theta_k^2 [A(\theta), \alpha(\mathbf{k})] = \frac{1}{3!}\theta_k^3 \beta^\dagger(-\mathbf{k}) \\
&\vdots \\
\frac{1}{(2n-1)!}[A(\theta), ..[A(\theta), \alpha(\mathbf{k})]] &= \frac{1}{(2n-1)!}\theta_k^{2n-1} \beta^\dagger(-\mathbf{k}) \\
\frac{1}{(2n)!}[A(\theta), ..[A(\theta, \alpha(\mathbf{k}))]] &= \frac{1}{(2n)!}\theta_k^{2n} \alpha(\mathbf{k})
\end{aligned} \tag{A.11}$$

We will need the Hausdorff formula

$$e^{-A} B e^{A} = B - \frac{1}{1!}[A, B] + \frac{1}{2!}[A, [A, B]] - ... + ... \tag{A.12}$$

Now we are ready to calculate Eq. (A.10), where we use Eq. (A.11, A.12).

$$\begin{aligned}
G^{-1}(\theta)\alpha(\mathbf{k})G(\theta) &= e^{-A(\theta)}\alpha(\mathbf{k})e^{A(\theta)} \\
&= \alpha(\mathbf{k}) - \frac{1}{1!}[A(\theta), \alpha(\mathbf{k})] + \frac{1}{2!}[A(\theta), [A(\theta), \alpha(\mathbf{k})]] - .. \\
&= \alpha(\mathbf{k}) - \theta_k \beta^\dagger(-\mathbf{k}) + \frac{1}{2!}\theta_k^2 \alpha(\mathbf{k}) - \frac{1}{3!}\theta_k^3 \beta^\dagger(-\mathbf{k}) + ... \\
&= \alpha(\mathbf{k})(1 + \frac{1}{2!}\theta_k^2 + \frac{1}{4!}\theta_k^4 + ...) - \beta^\dagger(\mathbf{k})(\theta_k + \frac{1}{3!}\theta_k^3 + \frac{1}{5!}\theta_k^5 + ..) \\
&= \alpha(\mathbf{k}) \cosh\theta_k - \beta^\dagger(\mathbf{k}) \sinh\theta_k
\end{aligned} \tag{A.13}$$

Eq. (A.2, A.3, A.10) give us

$$a(\mathbf{k}) = G^{-1}(\theta)\alpha(\mathbf{k})G(\theta) \ , \quad b(\mathbf{k}) = G^{-1}(\theta)\beta(\mathbf{k})G(\theta) \tag{A.14}$$

$f_0(\theta)$ is defined as follows:

$$f_0(\theta) \equiv \langle 0|G^{-1}(\theta)|0\rangle \tag{A.15}$$

$\delta f(\theta, \mathbf{l})$ denotes the change of $f_0(\theta)$ caused by the following substitution

$$\theta(\mathbf{k}) \rightarrow \theta(\mathbf{k}) + \epsilon\delta(\mathbf{k} - \mathbf{l}) \tag{A.16}$$

A definition of a derivative is

$$\frac{\delta}{\delta\theta_l} f_0(\theta) \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon}\delta f(\theta, \mathbf{l}) \tag{A.17}$$

Here we calculate the derivation of $f_0(\theta)$

$$\frac{\delta}{\delta\theta_l} f_0(\theta) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\delta f(\theta,\mathbf{l})$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[\langle 0|G^{-1}(\theta+\epsilon\delta(\mathbf{k}-\mathbf{l}))|0\rangle - \langle 0|G^{-1}(\theta)|0\rangle\Big]$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[\langle 0|\exp\left(-A(\theta+\epsilon\delta(\mathbf{k}-\mathbf{l}))\right)|0\rangle - \langle 0|\exp\left(-A(\theta)\right)|0\rangle\Big]$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[\langle 0|\exp\Big(-\int d^3\mathbf{k}(\theta(\mathbf{k})+\epsilon\delta(\mathbf{k}-\mathbf{l}))(\alpha(\mathbf{k})\beta(-\mathbf{k})-\beta^\dagger(-\mathbf{k})\alpha^\dagger(\mathbf{k}))\Big)|0\rangle -$$
$$-\langle 0|\exp\Big(-\int d^3\mathbf{k}'\theta(\mathbf{k}')(\alpha\beta-\beta^\dagger\alpha^\dagger)\Big)|0\rangle\Big]$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Bigg[\langle 0|\Bigg(\exp\Big(-\int d^3\mathbf{k}\ \epsilon\delta(\mathbf{k}-\mathbf{l})(\alpha\beta-\beta^\dagger\alpha^\dagger)\Big)-1\Bigg)\times$$
$$\exp\Big(-\int d^3\mathbf{k}\ \theta(\mathbf{k})(\alpha\beta-\beta^\dagger\alpha^\dagger)\Big)|0\rangle\Bigg]$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Bigg[\langle 0|\Big(1+\epsilon\Big[\int\Big]+\frac{\epsilon^2}{2!}\Big[\int\int\Big]+\ldots\quad -1\Big)G^{-1}(\theta)|0\rangle\Bigg]$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Bigg[\langle 0|\Bigg(-\epsilon\int d^3\mathbf{k}\delta(\mathbf{k}-\mathbf{l})\Big(\alpha(\mathbf{k})\beta(-\mathbf{k})-\beta^\dagger(-\mathbf{k})\alpha^\dagger(\mathbf{k})\Big)+o(\epsilon^2)\Bigg)G^{-1}(\theta)|0\rangle\Bigg]$$
$$= -\langle 0|\alpha(\mathbf{l})\beta(-\mathbf{l})G^{-1}(\theta)|0\rangle \tag{A.18}$$

So we have

$$\frac{\delta}{\delta\theta_l} f_0(\theta) = -\langle 0|\alpha(\mathbf{l})\beta(-\mathbf{l})G^{-1}(\theta)|0\rangle \tag{A.19}$$
$$\frac{\delta}{\delta\theta_l} f_0(\theta) = \langle 0|G^{-1}(\theta)\beta^\dagger(-\mathbf{l})\alpha^\dagger(\mathbf{l})|0\rangle \tag{A.20}$$

The following expression can be rewritten using the fact that $G^{-1}(\theta) = G(-\theta)$, Eq. (A.10, A.14)

$$\alpha(\mathbf{l})\beta(-\mathbf{l})G^{-1}(\theta) = G^{-1}(\theta)G^{-1}(-\theta)\alpha(\mathbf{l})G(-\theta)G^{-1}(-\theta)\beta(-\mathbf{l})G(-\theta)$$
$$= G^{-1}(\theta)[\alpha(\mathbf{l})\cosh\theta_l+\beta^\dagger(-\mathbf{l})\sinh\theta_l]\times$$
$$[\beta(-\mathbf{l})\cosh\theta_l+\alpha^\dagger(\mathbf{l})\sinh\theta_l] \tag{A.21}$$

From Eq. (A.19, A.20, A.21) we get

$$\frac{\delta}{\delta\theta_l} f_0(\theta) = -\langle 0|\alpha(\mathbf{l})\beta(-\mathbf{l})G^{-1}(\theta)|0\rangle$$
$$= -\langle 0|G^{-1}(\theta)[\alpha(\mathbf{l})\cosh\theta_l+\beta^\dagger(-\mathbf{l})\sinh\theta_l][\beta(-\mathbf{l})\cosh\theta_l+\alpha^\dagger(\mathbf{l})\sinh\theta_l]|0\rangle$$
$$= -\sinh\theta_l\cosh\theta_l\langle 0|G^{-1}(\theta)\alpha(\mathbf{l})\alpha^\dagger(\mathbf{l})|0\rangle - \sinh^2\theta_l\langle 0|G^{-1}(\theta)\beta^\dagger(-\mathbf{l})\alpha^\dagger(\mathbf{l})|0\rangle$$
$$= -\delta^3(0)\sinh\theta_l\cosh\theta_l\langle 0|G^{-1}(\theta)|0\rangle - \sinh^2\theta_l\frac{\delta}{\delta\theta_l}f_0(\theta) \tag{A.22}$$

Eq. (A.22) can be simplified

$$\frac{\delta}{\delta\theta_l} f_0(\theta) = -\delta^{(3)}(0)\tanh(\theta_l)f_0(\theta) \tag{A.23}$$

It is an easy task to find the solution of Eq. (A.23)

$$f_0(\theta) = \exp\Big(- \delta^{(3)}(0) \int d^3\mathbf{k} \log\cosh\theta_k \Big) \tag{A.24}$$

The definition of $f_n(\theta, \mathbf{l})$ will be useful

$$f_n(\theta; \mathbf{l}) \equiv \langle 0|[\alpha(\mathbf{l})\beta(-\mathbf{l})]^n G^{-1}(\theta)|0\rangle \tag{A.25}$$

We calculate the derivative

$$\frac{\delta}{\delta\theta_l} G^{-1}(\theta) = -G^{-1}(\theta)(\alpha(\mathbf{l})\beta(-\mathbf{l}) - \beta^\dagger(-\mathbf{l})\alpha^\dagger(\mathbf{l})) \tag{A.26}$$

We will prove the following statement because it will be used later

$$\begin{aligned}\langle 0|[\alpha(\mathbf{l})\beta(-\mathbf{l})]^n \beta^\dagger(-\mathbf{l})\alpha^\dagger(\mathbf{l}) G^{-1}(\theta)|0\rangle &= n^2[\delta^{(3)}(0)]^2 f_{n-1}(\theta; \mathbf{l}) \\ &= n^2[\delta^{(3)}(0)]^2 \langle 0|[\alpha(\mathbf{l})\beta(-\mathbf{l})]^{n-1} G^{-1}(\theta)|0\rangle\end{aligned} \tag{A.27}$$

At first we show that Eq. (A.27) holds for $n = 2$

$$\begin{aligned}&\langle 0|[\alpha(\mathbf{l})\beta(-\mathbf{l})]^2 \beta^\dagger(-\mathbf{l})\alpha^\dagger(\mathbf{l}) G^{-1}(\theta)|0\rangle = \\ &= \langle 0|\alpha\beta(\alpha^\dagger\alpha + \delta(0))(\beta^\dagger\beta + \delta(0))G^{-1}|0\rangle \\ &= \langle 0|(\alpha^\dagger\alpha + \delta(0))(\beta^\dagger\beta + \delta(0))\alpha\beta G^{-1}|0\rangle + \delta(0)\langle 0|(\alpha^\dagger\alpha + \delta(0))\alpha\beta G^{-1}|0\rangle + \\ &+ \delta(0)\langle 0|(\beta^\dagger\beta + \delta(0))\alpha\beta G^{-1}(\theta)|0\rangle + \delta^2(0)\langle 0|\alpha\beta G^{-1}(\theta)|0\rangle \\ &= 2^2[\delta^{(3)}(0)]^2 f_1(\theta, \mathbf{l})\end{aligned} \tag{A.28}$$

It is an easy task to compute the following three relationships

$$\langle 0|[\alpha\beta]^n \alpha^\dagger\alpha G^{-1}(\theta)|0\rangle = n[\delta^{(3)}(0)]\langle 0|[\alpha\beta]^n G^{-1}(\theta)|0\ \rangle \tag{A.29}$$

$$\langle 0|[\alpha\beta]^n \beta^\dagger\beta G^{-1}(\theta)|0\rangle = n[\delta^{(3)}(0)]\langle 0|[\alpha\beta]^n G^{-1}(\theta)|0\ \rangle \tag{A.30}$$

$$\langle 0|[\alpha\beta]^n \beta^\dagger\beta\alpha^\dagger\alpha G^{-1}(\theta)|0\rangle = n^2[\delta^{(3)}(0)]^2\langle 0|[\alpha\beta]^n G^{-1}(\theta)|0\ \rangle \tag{A.31}$$

Let us assume that Eq. (A.27) holds for some n and we will prove that it holds for $n + 1$. We need to calculate

$$\langle 0|[\alpha(\mathbf{l})\beta(-\mathbf{l})]^{n+1} \beta^\dagger(-\mathbf{l})\alpha^\dagger(\mathbf{l}) G^{-1}(\theta)|0\rangle = (n+1)^2[\delta^{(3)}(0)]^2 \langle 0|[\alpha(\mathbf{l})\beta(-\mathbf{l})]^n G^{-1}(\theta)|0\rangle \tag{A.32}$$

Indeed, here we can see that if Eq. (A.27) holds for some n then it holds for $n+1$ too. To calculate this we employed Eq. (A.29, A.30, A.31)

$$\begin{aligned}&\langle 0|[\alpha\beta]^n \alpha\beta\beta^\dagger\alpha^\dagger G^{-1}(\theta)|0\rangle = \\ &= \langle 0|[\alpha\beta]^n \beta^\dagger\beta\alpha^\dagger\alpha G^{-1}(\theta)|0\rangle + \delta^{(3)}(0)\langle 0|[\alpha\beta]^n \beta^\dagger\beta|0\rangle + \\ &+ \delta^{(3)}(0)\langle 0|[\alpha\beta]^n \alpha^\dagger\alpha G^{-1}|0\rangle + [\delta^{(3)}(0)]^2\langle 0|[\alpha\beta]^n G^{-1}(\theta)|0\rangle \\ &= (n+1)^2[\delta^{(3)}(0)]^2\langle 0|[\alpha\beta]^n G^{-1}(\theta)|0\rangle\end{aligned} \tag{A.33}$$

Eq. (A.27) has been just proven.

We calculate the following relationship because it will be usefull, using Eq. (A.25, A.26, A.27)

$$\begin{aligned}\frac{\delta}{\delta\theta_l}f_n(\theta,\mathbf{l}) &= -\langle 0|[\alpha(\mathbf{l})\beta(-\mathbf{l})]^n(\alpha(\mathbf{l})\beta(-\mathbf{l})-\beta^\dagger(-\mathbf{l})\alpha^\dagger(\mathbf{l}))G^{-1}(\theta)|0\rangle \\ &= -\langle 0|[\alpha(\mathbf{l})\beta(-\mathbf{l})]^{n+1}G^{-1}(\theta)|0\rangle + \langle 0|[\alpha(\mathbf{l})\beta(-\mathbf{l})]^n\beta^\dagger(-\mathbf{l})\alpha^\dagger(\mathbf{l})G^{-1}(\theta)|0\rangle \\ &= -f_{n+1}(\theta;\mathbf{l}) + n^2[\delta^{(3)}(0)]^2 f_{n-1}(\theta;\mathbf{l}) \end{aligned} \tag{A.34}$$

Let us remind that Eq. (A.23, A.25) give us

$$\frac{\delta}{\delta\theta_l}f_0(\theta) = -f_1(\theta;\mathbf{l}) = -\delta^{(3)}(0)\tanh\theta_l f_0(\theta) \tag{A.35}$$

From Eq. (A.34) we have

$$\begin{aligned} f_2(\theta;\mathbf{l}) &= [\delta^{(3)}(0)]^2 f_0(\theta;\mathbf{l}) - \frac{\delta}{\delta\theta_l}f_1(\theta,\mathbf{l}) \\ f_3(\theta;\mathbf{l}) &= 2^2[\delta^{(3)}(0)]^2 f_1(\theta;\mathbf{l}) - \frac{\delta}{\delta\theta_l}f_2(\theta,\mathbf{l}) \\ &\vdots \\ f_n(\theta;\mathbf{l}) &= (n-1)^2[\delta^{(3)}(0)]^2 f_{n-2}(\theta;\mathbf{l}) - \frac{\delta}{\delta\theta_l}f_{n-1}(\theta,\mathbf{l}) \end{aligned} \tag{A.36}$$

By employing Eq. (A.24, A.35, A.36) we find

$$\begin{aligned} f_2(\theta;\mathbf{l}) &= 2[\delta^{(3)}(0)]^2\tanh^2(\theta_l)f_0(\theta;\mathbf{l}) \\ f_3(\theta;\mathbf{l}) &= 6[\delta^{(3)}(0)]^3\tanh^3(\theta_l)f_0(\theta;\mathbf{l}) \\ f_4(\theta;\mathbf{l}) &= 24[\delta^{(3)}(0)]^4\tanh^4(\theta_l)f_0(\theta;\mathbf{l}) \\ &\vdots \end{aligned} \tag{A.37}$$

Someone could guess that $f_n(\theta)$ is

$$\begin{aligned} f_n(\theta;\mathbf{l}) &= n![\delta^{(3)}(0)]^n f_0\tanh^n\theta_l \\ &= n![\delta^{(3)}(0)]^n\exp\Big(-\delta^{(3)}(0)\int d^3\mathbf{k}\log\cosh\theta_k\Big)\tanh^n\theta_l \end{aligned} \tag{A.38}$$

We assume that this holds for some n and we show that the statement is correct for $n+1$ too. To calculate this, Eq. (A.36, A.38) were used

$$\begin{aligned} f_{n+1}(\theta;\mathbf{l}) &= n^2[\delta^{(3)}(0)]^2 f_{n-1}(\theta;\mathbf{l}) - \frac{\delta}{\delta\theta_l}f_n(\theta;\mathbf{l}) \\ &= n^2(n-1)![\delta^{(3)}(0)]^{n+1}f_0(\theta;\mathbf{l})\tanh^{n-1}(\theta_l) - n![\delta^{(3)}(0)]^n\times \\ &\Big(n\tanh^{n-1}\theta_l(1-\tanh^2\theta_l)\delta^{(3)}(0)f_0(\theta;\mathbf{l}) - \delta^{(3)}(0)\tanh^{n+1}(\theta_l)f_0(\theta;\mathbf{l})\Big) \\ &= (n+1)!\tanh^{n+1}(\theta_l)f_0(\theta;\mathbf{l})[\delta^{(3)}(0)]^{n+1} \end{aligned} \tag{A.39}$$

Eq. (A.38) is proven.

Finally from Eq. (A.25, A.38) we see that the following important statemet holds

$$|0\rangle\rangle \equiv G^{-1}(\theta)|0\rangle = f_0(\theta)\exp\Big(\delta^{(3)}(0)\int d^3\mathbf{k}\alpha^\dagger(\mathbf{k})\beta^\dagger(-\mathbf{k})\tanh\theta_k\Big)|0\rangle \tag{A.40}$$

B. Appendix Homotopy

This Appendix is based on [18]. Fields are defined as mappings

$$\begin{aligned} \phi : M &\to T, \\ z &\to \phi(z), \end{aligned} \tag{B.1}$$

where M is a manifold, usually M is $\mathbb{R}^d$ and T is some target space. Often, we require field configurations to have a constant value on the boundary of M so that the action of the system is finite. Hence M can be compactified to a large sphere and we identify $M \simeq S^d$, S^d is the d-dimensional unit sphere. Often it is sufficient to focus on $\phi :\ S^d \to T$.

Two fields ϕ_1 and ϕ_2 are by definition topologically equivalent if they can be continuously deformed into each other i.e. there is a continuous mapping called a homotopy

$$\begin{aligned} \phi : S^d \times [0,1] &\to T \\ (z,t) &\to \phi(z,t) \end{aligned} \tag{B.2}$$

such that $\phi(.,0) = \phi_1$ and $\phi(.,1) = \phi_2$.

The equivalence class of all fields topologically equivalent to a given representative ϕ is denoted by $[\phi]$. The set of all topological equivalence classes $\{[\phi]\}$ of mapping $\phi :\ S^d \to T$ is called the d-th homotopy group, $\pi_d(T)$.

Here we list a few examples of homotopies. Mappings $S^1 \to S^1$ can be classified in terms of winding numbers, W,: $\pi_1(S^1) = \mathbb{Z}$ i.e. the number of times $\phi(z)$ winds around the unit circle as z progresses from 0 to β: $\phi(\beta) - \phi(0) = 2\pi W$. It is not possible to change W by continuous deformation of ϕ.

It is easy to see that any mapping $S^1 \to S^2$ i.e. a closed curve on the 2-sphere can be continuously contracted to a point. Hence $\pi_1(S^2) = 0$. An short list of homotopies:

	π_1	π_2	π_3	π_4	π_5	π_6
S^0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}
S^3	0	0	Z	Z_2	Z_2	Z_{12}
S^4	0	0	0	Z	Z_2	Z_2

To learn more about this topic yhe book [23] is recommended.

C. Appendix The path integral, well

Here we introduce the Feynman path integral and the Stationary phase approximation.

For the Hamiltonian of the form $\hat{H} = \hat{T} + \hat{V}$, i.e. the sum of a kinetic energy $\hat{T} = \hat{p}^2/2m$ and some potential energy operator $\hat{V}$, we can construct the Hamiltonian formulation of the path integral from the "operator QM" (the key is that the eigenstates of the factors $e^{-i\hat{T}\triangle t/\hbar}$, $e^{-i\hat{V}\triangle t/\hbar}$ are known independently)

$$\langle q_f|e^{-i\hat{H}t/\hbar}|q_i\rangle = \int_{\substack{q(t) = q_f \\ q(0) = q_i}} Dx \exp\Big[\frac{i}{\hbar}\int_0^t dt'\,(p\dot{q} - H(p,q))\Big] \tag{C.1}$$

The integration is over all paths through the phase space which connects q_i and q_f.

As $\hat{T}(p) = \hat{p}^2/2m$ then by Gaussian integration we obtain

$$\langle q_f|e^{-i\bar{H}t/\hbar}|q_i\rangle = \int_{\substack{q(t) = q_f \\ q(0) = q_i}} Dq \exp\Big[\frac{i}{\hbar}\int_0^t dt' L(q,\dot{q})\Big] \tag{C.2}$$

where $L(q,\dot{q}) = m\dot{q}^2/2 - V(q)$ and $Dq = \lim_{N\to\infty}(\frac{Nm}{it2\pi\hbar})^{N/2}\prod_{n=1}^{N-1} dq_n$. The integral is over all paths, which begin (end) at the initial (final) point, through coordinate space.

Stationary phase approximation:

$$\int Dx\; e^{-F[x]} \simeq \sum_i e^{-F[\bar{x}_i]} \det\left(\frac{\hat{A}_i}{2\pi}\right)^{-1/2} \tag{C.3}$$

where the "'points" of stationary phase $\bar{x}_i$ are given by the condition of vanishing functional derivative i.e. $\forall t : \frac{\delta F[x]}{\delta x(t)}\Big|_{x=\tilde{x}_i} = 0$ and $\hat{A}_i \equiv A_i(t,t') = \frac{\delta^2 F[x]}{\delta x(t)\delta x(t')}\Big|_{x=\tilde{x}_i}$ denotes the second functional derivative. We can apply the stationary phase approximation if the operator $\hat{A}$ is positive-definite [18].

Let us deal with a quantum particle in a well. The Hamiltonian is

$$\hat{H} = \hat{p}^2/2m + V(\hat{q}) \tag{C.4}$$

The potential is a symmetric well with the minimum at $q = 0$ and $V(0) = 0$. The equation of motion is $m\ddot{q} = -\partial_q V(q)$. We are looking for the solution that satisfies $q(t) = q(0) = 0$. The solution is $q_{cl} = 0$, therefore $S[q_{cl}] = 0$. The Eq. (C.3) gives us

$$G(0,0;t) \simeq J \det\Big(-m(\partial_t^2 + \omega^2)/2\Big)^{-1/2} \tag{C.5}$$

where $\omega^2 = \frac{1}{m}\partial_q^2 V(q)|_{q=0}$, J absorbs constants. The determinant is a product over eigenvalues, so we need to solve $-\frac{m}{2}(\partial_t^2+\omega^2)r_n = \epsilon_n r_n$ where $r_n(t) = r_n(0) = 0$. The solution is $r_n(t') = \sin(n\pi t'/t)$, $n = 1, 2..$, and $\epsilon_n = m[(n\pi/t)^2 - \omega^2]/2$ and

$$\det\Big(-m(\partial_t^2+\omega^2)/2\Big)^{-1/2} = \prod_{n=1}^{\infty}\Big[\frac{m}{2}\Big(\Big(\frac{n\pi}{t}\Big)^2-\omega^2\Big)\Big]^{-1/2} \tag{C.6}$$

what for some t seems to be divergent. If $V \equiv 0$ then our G reduces to G_{free}, i.e. the propagator of a free particle

$$G_{free}(q_i, q_f; t) \equiv \langle q_f|e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t}|q_i\rangle\Theta(t) = \Big(\frac{m}{2\pi i\hbar t}\Big)^{1/2} e^{\frac{i}{\hbar}\frac{m}{2t}(q_f-q_i)^2}\Theta(t) \tag{C.7}$$

$\Theta(t)$ is Heaviside function[18]. Both G and G_{free} have J. We regularize the transition amplitude as [18]

$$G(0,0;t) \equiv \frac{G(0,0;t)}{G_{free}(0,0;t)}G_{free}(0,0;t) = \prod_{n=1}^{\infty}\Big[1-\Big(\frac{\omega t}{n\pi}\Big)^2\Big]^{-1/2}\Big(\frac{m}{2\pi i\hbar t}\Big)^{1/2}\Theta(t) \tag{C.8}$$

by employing the identity $\prod_{n=1}^{\infty}[1-(x/n\pi)^2]^{-1} = x/\sin x$ we find

$$G(0,0;t) \simeq \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega t)}}\Theta(t) \tag{C.9}$$

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