Charles University in Prague
Faculty of Mathematics and Physics

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Martin Kupec

Bounds for distance constrained labeling

Department of Applied Mathematics

Supervisor of the master thesis:	Doc. RNDr. Jiří Fiala, Ph.D.
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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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Název práce: Meze pro vzdálenostně podmíněné značkování grafů

Autor: Martin Kupec

Katedra: Katedra Aplikované Matematiky

Vedoucí diplomové práce: Doc. RNDr. Jiří Fiala, Ph.D., Katedra Aplikované Matematiky

Abstrakt: Problém $\lambda - L(p, q)$-značkování je přiřadit vrcholům grafu značky $\{0, \ldots, \lambda\}$ tak, aby sousední vrcholy měly značky od sebe vzdálené alespoň p a vrcholy se společným sousedem značky od sebe vzdáleny alespoň q. Zabýváme se výpočení složitostí tohoto problému a stanovujeme hraniční hodnoty λ, p a q, pro které se tento problém stává NP těžký. Důkaz je veden promocí dvou různých redukcí. Jedna je z NAE-3SATu, druhá z problémů hranového dobarvení předbarveného grafu.

Klíčová slova: značkování, teorie grafů, výpočetní složitost

Title: Bounds for distance constrained labeling

Author: Martin Kupec

Department: Department of Applied Mathematics

Supervisor: Doc. RNDr. Jiří Fiala, Ph.D., Department of Applied Mathematics

Abstract: We study the complexity of the λ–$L(p, q)$-labelling problem for fixed λ, p, and q. The task is to assign vertices of a graph labels from the set $\{0, \ldots, \lambda\}$ such that labels of adjacent vertices differ by at least p while vertices with a common neighbor have different labels. We use two different reductions, one from the NAE-3SAT and the second one from the edge precoloring extension problem.

Keywords: labeling, graph theory, complexity

Contents

1. History

The labelling problem is kind of a distance constrained vertex coloring. But first we look at its ancestor, the channel assignment problem.

The channel assignment problem is to assign a channel (nonnegative integer) to each radio transmitter so that interfering transmitters are assigned channels whose difference is big enough.

Roberts [7] proposed a variant of the channel assignment problem in which "close" transmitters must receive different channels and "very close" transmitters must receive channels that are at least two channels apart. To formulate the problem in graphs, the transmitters are represented by vertices of a graph. Two vertices are "very close" if they are adjacent in the graph and "close" if they are at distance two. This is the definition of $L(2,1)$-labeling. More precise definition will follow later.

This definition provides a natural motivation of this problem.

2. Labeling

Labeling of a graph $G = (V, E)$ is a mapping of vertices to nonnegative integers. It is similar to a proper coloring of a graph, just additional constrains are introduced.

Definition 1 ($L(p, q)$-labelling)**.** An $L(p, q)$-labelling is a mapping l of vertices to non negative integers with the following constrains.
Condition p: vertices connected by an edge are labelled by labels that differ by at least p.

$$\forall u, v \in V, (u, v) \in E : |l(u) - l(v)| \geq p$$

Condition q: vertices with a common neighbour are labelled by labels that differ by at least q.

$$\forall u, v, w \in V, (u, w), (w, v) \in E : |l(u) - l(v)| \geq q$$

We can also define such labelling in a more general way.

Definition 2 (Distance)**.** Distance between vertices u and v in a graph G is the length of the shortest path between these vertices. This distance is denoted by the symbol $dist(u, v)$.

Definition 3 (Span of a labelling)**.** Let $G = (V, E)$ be a graph and $l : V \to N$ be a labelling function. The span of the labelling l is defined as

$$\lambda = \max_{v \in V} l(v) - \min_{v \in V} l(v).$$

It is good to point out that any labelling of the span λ may actually use $\lambda + 1$ labels. It is common, that the labels are chosen as $0, 1, \ldots, \lambda$.

There is also a variant of the labelling problem with more constrains:

Definition 4 ($L(c_1, c_2, c_3, ..., c_k)$-labelling)**.** Let $G = (V, E)$ be a graph and $l : V \to \{0, 1, \ldots\}$ be a function. The function l is an $L(c_1, c_2, c_3, ..., c_k)$-labelling if the following holds:

$$\forall u, v \in V, \forall i \in \{1, ..., k\} : dist(u, v) = i \Rightarrow |l(u) - l(v)| \geq c_i$$

Definition 5 ($\lambda - L(p, q)$-labelling)**.** Let $G = (V, E)$ be a graph. We say, that the graph G can be $\lambda - L(p, q)$-labelled if there exists an $L(p, q)$-labelling with the span λ.

3. Our question

We investigate computational complexity of the existence of an $L(p,q)$-labeling. For this purpose we now define various decision problems. These problems differ by the choice of parameters that are fixed, hence do not belong to the input. When a parameter becomes fixed, the problem may become tractable as we will show in some cases.

Definitions of complexity and tractability can be found for example here [4].

L-labeling
Instance: A graph G and integers p, q and λ.
Question: Can G be properly $L(p,q)$-labelled by the set of labels $\{0, 1, \ldots, \lambda\}$?

As p and q are on the input, you can ask whether a graph G admits an $L(1,0)$-labelling and that is equivalent of asking for a chromatic number of G, which is a well known NP-complete problem.

So we need to ask for a bit different problem. This new formulation of the problem gives us more accurate characterization of the $L(p,q)$-labelling problem.

Generally, fixing some portion of the input as a parameter of the problem can help to get better understanding of the problem. First we try to fix just variables p and q. So the problem is changed to:

$L(p,q)$-labeling
Parameters: Integer p and q.
Instance: A graph G and an integer λ.
Question: Can G be properly $L(p,q)$-labelled by the set of labels $\{0, 1, \ldots, \lambda\}$?

It was already shown before, that there are values of λ for which instance of this problem is NP-hard to decide.

The last variable to fix is the span λ.

$\lambda - L(p,q)$-labeling
Parameters: Integer p, q and λ.
Instance: A graph G.
Question: Can G be properly $L(p,q)$-labelled by the set of labels $\{0, 1, \ldots, \lambda\}$?

We study this variant of the problem. The $3 - L(1,0)$-labelling problem is equivalent to the three coloring problem, hence is NP-complete. On the other hand, the $2 - L(1,0)$-labelling problem is equivalent to the two coloring problem, so this problem is polynomial. We seek for which parameters it belongs to which complexity class.

In [1] Bertossi and Bonuccelli proved the NP-completeness result for $L(0,1)$-labelling.

Also the $L(1,1)$-labelling problem was proved to be NP-complete with a reduction from 3-SAT [6]. This means that any $L(p,p)$-labelling problem is NP-complete as we will show in Lemma 6.

The same question for the $L(2,1)$-labelling is also NP-complete [5] to answer. This result was obtained by a double reduction: from Hamiltonian path to the decision problem asking whether there exist an injection $f : V \to [0, n-1]$ such that $|f(x) - f(y)| \geq 2$ whenever $(x,y) \in G(E)$. This auxiliary problem was then reduced to the decision version of the $L(2,1)$-labelling problem. The problem is NP-complete whenever $\lambda \geq 4$ and it is polynomial for $\lambda \leq 3$ (this case occurs only when G is a disjoint union of paths of length at most 3).

In [3] the authors conjectured that for every $p \geq q \geq 1$, there is a λ (depending on p and q) such that the decision whether G admits an $L(p,q)$-labelling of span λ is NP-complete. In support of their conjecture, the authors proved that there is at least one NP-complete instance, namely that it is NP-complete to decide whether G admits an $L(p,q)$-labelling for the span $p + q\lceil p/q \rceil$ for all fixed $p \geq q \geq 1$.

There are also some results considering special classes of graphs, those can be found in a survey by Calamoneri [2].

Considering all the previous results, there are still several open problems, e.g. the smallest unknown problem is $5-L(3,1)$-labelling.

We focus on the case when $p \geq q \geq 1$ and almost fully resolve the $\lambda - L(p,q)$-labelling problem. The only open cases left are for $\lambda \geq p + 3q$ and $p < 2q$.

4. Properties of an $L(p, q)$-labelling

It is enough to focus on p and q being relative primes. Otherwise the λ, p and q can be divided by a common divisor and this new problem is equivalent to the previous one in the following sense.

Lemma 6 (Dividing lemma). *Let G be a graph. For all p, q, λ, and $c \geq 1$. The graph G admits a $\lambda - L(p, q)$-labelling, if and only if it admits a $c\lambda - L(cp, cq)$-labelling.*

Proof. First let us prove the easy implication from left to right. Assume that there exists a labelling l. Define $l'(v) = cl(v)$. The span of l' is $c\lambda$. If u and v are adjacent, we know that $|l(u) - l(v)| \geq p$ and hence $|l'(u) - l'(v)| = c|l(u) - l(v)| \geq cp$. Analogously, we get that the difference between two vertices with a common neighbour is also c times bigger.

The other implication is similar. It is defined $l'(v) = \lfloor \frac{l(v)}{c} \rfloor$. For each two vertices u and v.

$$|l(u) - l(v)| \geq d \Rightarrow |l'(u) - l'(v)| = \left| \left\lfloor \frac{l(u)}{c} \right\rfloor - \left\lfloor \frac{l(v)}{c} \right\rfloor \right| \geq \left| \left\lfloor \frac{l(u) - l(v)}{c} \right\rfloor \right| \geq \left\lfloor \frac{d}{c} \right\rfloor = d'$$

Depending on the distance between u and v we set $d = cp$ or $d = cq$. In both cases the rounding does not violate the distance constraint. So the d' is exactly p or q as needed. The same can be done for the span, decreasing it from $c\lambda$ to λ. □

By the previous lemma we know, that the problems $\lambda - L(p, q)$-labelling and $c\lambda - L(cp, cq)$-labelling are equivalent.

Lemma 7 (Inverted labelling). *Let $l : V \to \{0, 1, \ldots, \lambda\}$ be a $\lambda - L(p, q)$ labelling of G. Let $\bar{l}$ by defined as $\bar{l}(v) = \lambda - l(v)$. Then $\bar{l}$ is also a valid $\lambda - L(p, q)$ labelling of G.*

Proof. As the labels used by the labeling l are $0, 1, \ldots, \lambda$, it is obvious that the labels used by the labelling $\bar{l}$ are also from the same set. Now consider two vertices u and v. If the difference between two labels in the original labeling l is $|l(u) - l(v)| = d$, then the difference between the same two labels in the inverted labelling is $|\bar{l}(u) - \bar{l}(v)| = |\lambda - l(u) - (\lambda - l(v))| = |l(v) - l(u)| = d$. This means that for each pair of vertices the differences of its labels are the same. □

To make our proofs more readable we group labels as follows.

Definition 8. For given q and λ denote by L_i the set of labels $\{i, i+1, \ldots, i+q-1\}$ and by L the set of all the labels $\{0, 1, \ldots, \lambda\}$.

Finally, we need one more simple but useful lemma.

Lemma 9. *Let G be a graph, that allows a $\lambda - L(p, q)$-labelling for $\lambda < 2p$. Then the graph G is bipartite and the partitions are $A = \{v \in V(G) \,|\, l(v) < p\}$ and $B = \{v \in V(G) \,|\, l(v) \geq p\}$.*

Proof. Let l be the $\lambda - L(p,q)$-labelling.

Consider the set $A = \{v \in V(G) \,|\, l(v) < p\}$ and the set $B = V(G) \setminus A$. Any two vertices $u, v \in A$ satisfy, that $|l(u) - l(v)| < p$ and thus there is no edge between u and v. The same holds for $u, v \in B$, since $p \leq l(u), l(v) < 2p$.

The sets A and B are the classes of the bipartition of the graph G. □

5. Overview of our approach

We first develop bounds for λ, p and q where the $\lambda - L(p,q)$-labeling problem becomes an NP-complete problem. We then employ standard method and develop a polynomial reduction of a known NP-complete problem to the discussed problem of $L(p,q)$-labeling with the span λ.

We could not find a single reduction for all values of λ, p and q, so we distinguish two cases according to values of these parameters. We show two reductions from different known NP-complete problems. The first reduced NP-complete problem is a well known variant of SAT, the NAE 3SAT. The other NP-complete problem is a bit less known problem of edge precoloring extension.

For the SAT reduction, it is needed to construct a graph from a given Boolean formula and use that graph as an input of the $\lambda - L(p,q)$-labelling. The solution of this labelling problem is then transformed to a solution of the SAT problem.

The edge precoloring extension problem has a graph at the input. This graph is transformed to an input graph of the $\lambda - L(p,q)$-labelling problem. As before, the solution is interpreted as a solution of the edge precoloring extension problem.

There are also easy cases. For the $\lambda < p + 2q$ we prove that the $\lambda - L(p,q)$-labelling problem is polynomial. For $\lambda \in \{p + 2q, \dots, p + 3q - 1\}$ we use the SAT reduction as our proof. And at last, the edge precoloring extension problem is used for reduction for remaining values of λ. Unfortunately, there still exists several choices of parameters when the problem is open.

6. Satisfiability reduction

One of the basic NP-complete problems is the Satisfiability problem (SAT). In many cases using the SAT problem in general is inconvenient. There are many variants of the SAT problem which are still NP-complete. One of the commonly known equivalent problem to the SAT problem is the NAE-3SAT problem.

"Not all equal"-3SAT – NAE-3SAT
Instance: Formula φ in conjunctive normal form with 3 variables per clause (3-CNF).
Question: Does there exist an assignment of TRUE and FALSE values to variables, such that each clause in the formula φ contains both TRUE and FALSE literal?

This problem is well known to be NP-complete[8].

In this chapter, we are interested in the following setting:

$$\begin{aligned} q &\geq 1 \\ p + 3q > \lambda &\geq p + 2q \\ \lambda &\geq 2p \end{aligned}$$

Combination of the inequalities yields us more implicit conditions. We also define some variables:

$$\begin{aligned} 3q &> p \\ a &= \lambda - p + 2q \\ d &= p - q \\ d > a &\geq d - q \\ q > a &\geq 0 \end{aligned}$$

Through this chapter the variables p, q and λ satisfy the above conditions.

For the SAT reduction we need to express variables and clauses of the formula provided as an input of the SAT problem. We will construct a graph G_Φ for the formula Φ on the input. A solution of the labelling problem on the graph G_Φ, then corresponds to a solution of the SAT problem for the formula Φ. More precisely, we formulate the result as follows.

Theorem 10. *Let Φ be an instance of NAE-3SAT and G_Φ be the associated graph. Then the graph G_Φ has an $\lambda - L(p,q)$-labelling if and only if Φ is satisfiable.*

For the proof we describe few construction blocks and then compose them together. Before we begin, we need a useful lemma.

Lemma 11. *Any vertex of degree 3 can be $\lambda - L(p,q)$-labelled only by a label from the set L_0 or $L_{\lambda-q+1}$.*

Proof. The vertex is denoted by v.

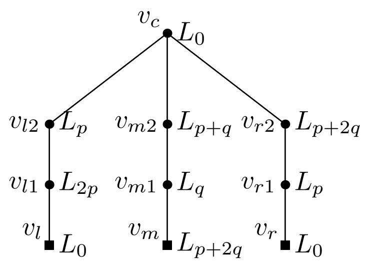

Figure 6.1: Basic block

First, let the vertex v be labelled by the label 0. Then the number of labels allowed to be used on the neighbouring vertices is

$$\lambda + 1 - p = p + 2q + a + 1 - p = 2q + a + 1 \geq 2q + 1.$$

The labels are consecutive and three common neighbours can easily fit into this set.

For the label q, the first label outside the set L_0, the situation is similar. The labels allowed for the neighbours are consecutive, just the number has decreased to

$$\lambda + 1 - p - q = p + 2q + a + 1 - p - q = q + a + 1 \leq 2q.$$

Since this set is consecutive, there can be at most two neighbour of the vertex v labelled by labels from this set.

The situation is different when the vertex v is labelled by at least label p. Then the number of allowed labels is

$$\begin{aligned}\lambda + 1 - 2(p-1) - 1 &= \\ p + 2q + a + 2 - 2p &= \\ 3q + d + a + 2 - 2(q+d) &= \\ q - d + a + 2 &\leq q + 1.\end{aligned}$$

Now, the set of labels is not consecutive. It is divided to the labels smaller than the label of the vertex v and the labels bigger than the label of the vertex v. These two sets are at least $2p$ labels apart. As both these sets are nonempty, there is one label for a neighbour from one of the sets and one from the other set. There can be no more neighbours labelled by a label from any of the sets, since both these sets are too small.

We can do the same from the other side and prove the same for the set $L_{\lambda-q+1}$ by Lemma 7. □

In the following constructions we denote some of the vertices as "endpoints". These vertices are special in two ways. They are used to join building blocks together. In the resulting graph, they will have degree 3, thus they will be allowed to have only labels from the sets L_0 and $L_{\lambda-q+1}$. In the forthcoming lemmas we assume, that these vertices have degree 3 already. This makes the lemmas and proofs much more simple.

The first construction is depicted on Fig. 6.1.

Construction 12 (Basic Block – BB)**.** The basic block is obtained from three paths of length three starting at one common vertex. The common vertex is

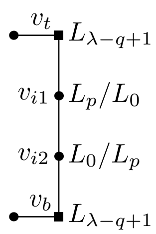

Figure 6.2: Interconnection block

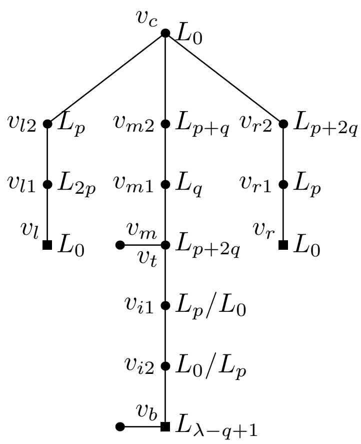

Figure 6.3: Extended block

denoted by v_c. The other ends of these paths are the endpoints for this basic block and denoted by v_l, v_m, and v_r.

We call these paths chains. The path containing the vertex v_l is the left chain, the one containing the vertex v_m is the middle chain, and finally the one containing the vertex v_r is the right chain.

These chain are indistinguishable to each other. The names are used to refer to the labels used on the chains according to the next lemma.

Lemma 13. *Any $\lambda - L(p,q)$-labelling of the graph created by the Construction 12 satisfies that the vertex v_c is labelled by a label from the set L_0 or $L_{\lambda-q+1}$. Moreover, if v_c is labelled by a label from the set L_0, then up to an isomorphism of the building block we may assume without loss of generality that:*

1. *The endpoints v_l and v_r are labelled by a label from the set L_0*
2. *The endpoint v_m is labelled by a label from the set $L_{\lambda-q+1}$*
3. *The vertex v_{m1} is labelled from the set L_q*
4. *The vertex v_{l1} is labelled from the set L_{2p}*
5. *The vertex v_{r1} is labelled from the set L_p*

We point out that the BB construction gives us a graph, where two special vertices have to be labelled by a label from the set L_0 and one has to be labelled by a label from the set $L_{\lambda-q+1}$ (or vice versa).

Definition 14 (Connecting blocks)**.** Let B_1 and B_2 be two blocks and v_1 be a endpoint of B_1, while v_2 belongs to B_2.

Connection of these two blocks by endpoints v_1 and v_2 means that we merge vertices v_1 and v_2 together. We say that we connect also labellings of blocks B_1 and B_2 in the case when v_1 and v_2 have the same label in both labellings.

The following constructions are depicted in Fig. 6.2 and 6.3.

Construction 15 (Interconnecting Block – IB)**.** Let IB be the path of length 5. The second vertex of the path is the endpoint v_t and the neighbour of the last vertex is the endpoint v_b.

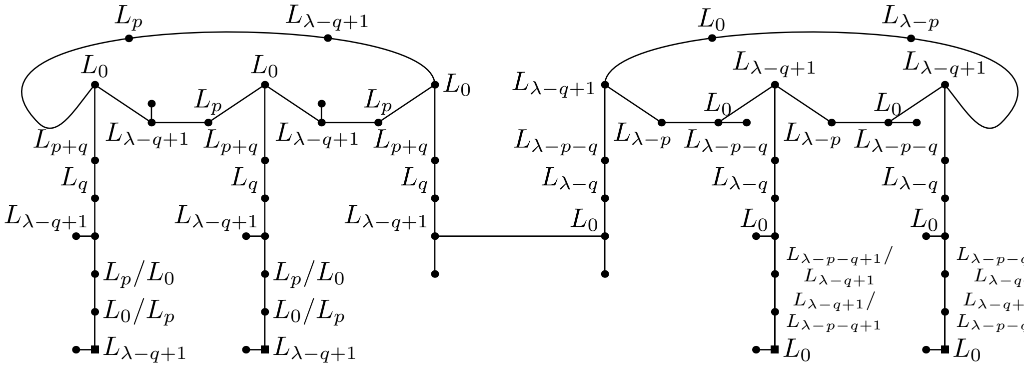

Figure 6.4: Variable circle pair

Construction 16 (Extended block – EB)**.** The EB graph is constructed by connecting the endpoint v_m of a BB and the endpoint v_t of a IB. The endpoints of this block are the endpoints of the BB and IB, except the merged vertices v_m and v_t.

Lemma 17. *Any λ–$L(p, q)$-labelling of the graph created by the Construction 16, such that the labels of the BB chain connected to the IB are the ones of the middle chain and the vertex v_c is labelled by a label from the set L_0. Then the label of the vertex v_t is labelled by a label from the set $L_{\lambda-q+1}$. The label of the vertex v_b is labelled by a label from the set $L_{\lambda-q+1}$. The labels of vertices in between endpoints v_t and v_b can be labelled by labels from the sets L_0 and L_p.*

Lemma 18. *The endpoint v_b of the graph EB can be connected with any endpoint of the graph BB while preserving a $\lambda-L(p, q)$-labelling.*

Construction 19 (Variable circle – VC)**.** Variable circle of size c is a circle in a graph with $3c$ vertices. A path of length 6 is connected to each $3k$-th vertex for $0 \leq k \leq c - 1, k \in \mathbb{N}$. The free end of each of these paths is called an endpoint.

In addition, one extra edge to a new vertex is connected to each $3k + 1$-th vertex and to the third and last vertex on each of the paths.

Construction 20 (Variable circle pair)**.** Let C_1 and C_2 be two variable circles of the same size. The variable circle pair is constructed from these two circles by shortening the path connected to the first vertex on both circles by 3 vertices and connecting this shortened paths.

One of the circles is called positive and the other negative. Endpoints of the positive circle are positive endpoints and endpoints of the negative circle are negative ones.

An example of a variable circle pair is shown on the Fig. 6.4.

Lemma 21. *Any $\lambda - L(p, q)$-labelling of the graph created by Construction 20 satisfies that all positive endpoints are labelled by a label from the set L_0 and all negative endpoints are labelled by a label from the set $L_{\lambda-q+1}$.*

Construction 22. Let Φ be a 3SAT formula. Then a graph G_Φ is constructed as follows.

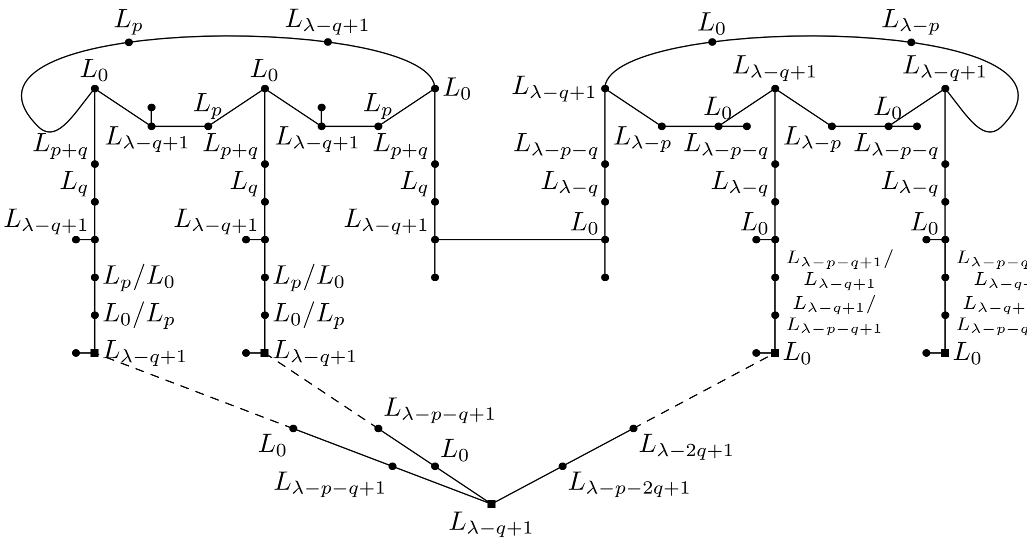

Figure 6.5: Construction example

The formula Φ consists of clauses over variables $x_1, x_2, \dots, x_n$ and clauses $\varphi_1, \varphi_2, \dots, \varphi_m$. In addition we denote by c_i the number of occurrences of a variable x_i.

Each x_i is represented in the graph G_Φ as a VC pair, each circle of size $c_i + 1$ (one pair of paths is used to make the VC pair). These VC pairs are disjoint and disconnected.

Each φ_i is represented in the graph G_Φ as one BB joining three endpoints of VC pairs as follows.

Each φ_i is composed from at most three literals denoted by $x_{\varphi_i(1)}, x_{\varphi_i(2)}, x_{\varphi_i(3)}$.

For each positive literal in φ_i, one of the endpoints of the corresponding BB is connected with a unique positive endpoint of the corresponding variable circle pair.

Similarly, for each negative literal in φ_i, one of the endpoints of the corresponding EB is connected with a unique negative endpoint of the corresponding variable circle pair.

Fig. 6.5 shows how the complete construction looks like.

Now it is time to recall, that through this whole chapter the variables p, q and λ satisfies the inequalities from the beginning of this chapter.

Proof of Lemma 13. The vertex v_c has three neighbours, so it can have only a label from the set L_0 or $L_{\lambda-q+1}$ by Lemma 11. This proves the first condition. From here it is assumed, that the vertex v_c is labelled by a label from the set L_0.

The neighbouring vertices can be labeled only by labels from the sets L_p, L_{p+q} and L_{p+2q}, since $\lambda \in L_{p+2q}$.

Let the vertex v_{l2}, the first vertex in the left chain, be labelled by a label from the set L_p. Then the vertex v_{l1} can be labelled only by a label equal or bigger than $2p$. Meaning that the number of its possible labels is:

$$\begin{aligned} \lambda - 2p &= \\ p + 2q + a - 2p &= \\ 2p - d + q + a - 2p &= \\ q + a - d \end{aligned}$$

That is less than q, thus the label of the vertex v_{l1} has to be from the set L_{2p}. This is the condition 5.

The last vertex, which is the endpoint v_l, cannot be labelled by a label from the set $L_{\lambda-q+1}$, because this would conflict with the label of previous vertex, so it has to be labelled by a label from the set L_0. This proves the v_l part of the condition 2.

Let the vertex v_{m2}, the first vertex in the middle chain, be labelled by a label from the set L_{p+q}. The vertex v_{m1} cannot be labelled by a label equal or bigger than $2p+q$, because there are no such labels as we already know. This vertex also cannot be labelled by a label from the set L_0, as such label is already used on the vertex v_c. The only possible set of labels for that vertex is the set L_q. The last vertex, which is the endpoint v_m, can be labelled just by a label from the set $L_{\lambda-q+1}$. This proves conditions 3 and 4.

At last, let the vertex v_{r2}, the first vertex in the right chain, be labelled by a label from the set L_{p+2q}. The vertex v_{r1} can be labelled by one of the labels from q to at most $\lambda - p$. The last vertex, which is the vertex v_r, cannot be labelled by a label from the set $L_{\lambda-q+1}$, because of the vertex in distance 2. When the previous vertex is labelled by a label p or bigger, then the vertex v_r can be labelled from the set L_0 and that is possible, because $\lambda \geq 2p$. This finished the proof of conditions 2 and 6. □

Proof of Lemma 17. The vertex v_m is identified with the vertex v_t and the vertex v_m cannot have a label from the set L_0 by Lemma 13. Furthermore the vertex v_t now have degree 3, so it can be labelled only by a label from the set $L_{\lambda-q+1}$. The label of the vertex v_{i1} cannot be from the set L_q, as a label from this set is used on the other vertex next to v_t. It also cannot use any label from $\lambda - p$ to λ, because of the label of vertex v_t.

$$\lambda - p = p + 2q + a - p = 2q + a = p + q + a - d$$

So labels from the sets L_0 and L_p are allowed for the vertex v_{i1}.

The vertex v_{l2} can be labelled by the other set than the vertex v_{l1}. The two sets are p labels apart. It would be also possible to label the vertex v_{i2} by a label from the set $L_{\lambda-2q+1}$, but then there would be no possible label for the vertex v_b.

The last vertex, which is the vertex v_b, is an endpoint, meaning that it can be labelled only by a label from the set L_0 or $L_{\lambda-q+1}$. A label from the set L_0 is used on the neighbouring vertex or vertex in distance 2, so that set cannot be used. □

Proof of Lemma 18. First, let the endpoint v_b be connected with the endpoint v_l. Then the vertex next to the endpoint v_l can be labelled by a label from the set

L_{2p} by condition 5 of Lemma 13. In this setting a label from both sets L_0 and L_p can be used on the vertex next to the vertex v_b on the path to the vertex v_t.

In the second case, let v_b be connected with the endpoint v_m. Both label sets L_0 and L_p can be used here. It was already discussed in the proof of Lemma 17.

Finally, in the last case, let v_b be connected with the endpoint v_r. Now the vertex next to the vertex v_b can be labelled by a label from the set L_p, leaving the label set L_0 for the neighbouring vertex. □

Proof of Lemma 21. The VC of size c is actually composed only of EBs, but these EBs overlap at chains.

The vertices on the circle with the long path connected to them are the vertices v_c of a EB. The right chain of each EB overlap with the left chain of the consequent EB. The path connected to each v_c is the middle chain of the BB and the IB block.

By Lemma 13, the endpoints of the chains v_l and v_r are labelled by a label from the set L_0. These endpoints overlap at the vertices v_c of the consequent EB. The vertices next to the endpoints can be labelled by labels from the sets L_p and L_{2p}. Labels of such sets can be used on neighbouring vertices.

The one extra edge added to the vertex next to v_c on the cycle ensures, that chains are as described. The edge raises the degree of the vertex to 3 and neither of the vertices v_{m1} and v_{m2} of the middle chain of a BB cannot by labelled by a label from the set L_0 or $L_{\lambda-q+1}$.

We already know that the vertex v_c can be labelled only by a label from the set L_0 or the set $L_{\lambda-q+1}$. The endpoints v_l and v_r have to be labelled by a label from the same set as the vertex v_c. This concludes, that the vertices v_c of all the EBs have to be labelled by a label from the same set. We also get that all the endpoints of the VC have to be labelled by a label from the other set (L_0 or $L_{\lambda-q+1}$). This is due to Lemma 18.

So far we were dealing with just one VC, but a Variable circle pair is constructed from two circles. The shortening of the path connected to the first vertices of both circles yields an IB stripped from the chain of EB. It can be also viewed as a BB chain with an extra edge connected to the endpoint. Endpoints of both such stripped paths are connected by an edge, raising the degree of both endpoints to 2. The extra edge raise the degree further to 3. Such vertices can be labelled only by a label from the set L_0 or $L_{\lambda-q+1}$. These endpoints are neighbours now, so one has to be labelled by a label from the set L_0 and the other from the set $L_{\lambda-q+1}$. All the endpoints of one VC are labelled by a label from the same set, by Lemma 17 also all the v_t vertices of one VC are labelled by the same set as well.

All together we have that all the endpoints of one VC has to be labelled by a label from the set L_0 and all the endpoints of the other VC by a label from the set $L_{\lambda-q+1}$ or vice versa. Just picking the right VC as positive one and the other as negative one proves the lemma. □

Proof of Theorem 10. Assume that the NAE-3SAT instance is satisfiable and that x'_i is the value of the variable x_i in a satisfying assignment.

For each x'_i being true, the labelling of the corresponding variable circle pair is chosen as depicted on Fig. 6.4. Vertices on the EB parts of VC are depicted with two sets, one of the sets is picked based on the chain connected to this EB. This is described in the proof of Lemma 18. From each set, the first label is used.

For each x_i' being false, the labelling of the corresponding variable circle pair is chosen as inverted labelling to the labelling used in the first case.

Also the labelling of BBs representing clauses is chosen as depicted on Fig. 6.1 and the first label of each set is used. If the clause has two positive and one negative literal the labelling is as described. Otherwise the inverted labelling is used.

On the other hand, assume that a $\lambda{-}L(p, q)$-labelling of the graph G_Φ exists. If a vertex v_c of a variable circle is labelled by a label from the set L_0, then the corresponding variable is assigned true. Otherwise the label is from the set $L_{\lambda-q+1}$ by Lemma 21 and the variable is assigned false.

There are two cases to consider:

In the first case all literals in the clause are positive or all are negative. The BB corresponding to such clause has one endpoint labelled by a label from the set $L_{\lambda-q+1}$ and two by a label from the set L_0 or vice versa. All three variables are connected to positive or all to negative endpoints of variable circles. So one variable circle has to have different labelling than the other two. This means that one has different value than the other two, and this means that this clause is satisfied.

In the second case two literals are positive and one negative or vice versa. Now there are two sub-cases:

The two endpoints labelled with the same label are connected to two positive (or negative) literals. The last endpoint is connected to the negative (or positive) literal. The two variables corresponding to the two positive (or negative) literals thus have the same value and the last variable also has the same value, because it is connected to the endpoint with a label from a different set. So all the variables have the same value, but one literal is negative (or positive). This assignment satisfies the clause.

The second sub-case is when one of the endpoints with different labels is connected to positive literal and the other one to negative literal. These two corresponding variables than have the same value but the literals then have different values so this is satisfying assignment. □

7. Edge precoloring extension reduction

In this chapter, we are interested in $\lambda \geq p+3q$. We have also changed the known NP-complete problem, from well known NAE-3SAT to a bit less known Edge precoloring extension problem on bipartite graphs.

Our situation have changed. The current setting for this chapter is

$$
\begin{aligned}
q &\geq 1 \\
p &\geq 2q \\
\lambda &\geq p+3q
\end{aligned}
$$

and

$$
\begin{aligned}
a &= \lambda - (p+3q) \\
d &= p-q,
\end{aligned}
$$

where

$$
\begin{aligned}
a &\geq 0 \\
d &> a \bmod q.
\end{aligned}
$$

The labelling problem with this setting behaves a bit differently, depending on wherever $\lambda < 2p$ or not. As before, the variables p, q, and λ satisfy the above condition through this whole chapter.

The edge coloring is similar to the vertex coloring. The goal is to assign colors to edges in such a way, that any two edges sharing a vertex have different colors.

Definition 23 (Edge coloring)**.** Let G be a graph, and let S be a set of size χ. A function $c : E(G) \to S$ is a proper edge coloring function if it satisfies the following condition.

$$(\forall e_1, e_2 \in G(E))(|e_1 \cap e_2| \neq \emptyset) \Rightarrow c(e_1) \neq c(e_2)$$

The set S is the set of colors.

In order to properly state the problem we want to use, we need the following definition.

Definition 24 (Regularity of a graph)**.** Graph G is r-regular when all the vertices has degree exactly r.

We also need to properly state our new NP-complete problem. This problem was proved to be NP-complete by Fiala [3].

Edge precoloring extension of 3-regular graph
Instance: A 3-regular bipartite graph G, sets $B, R \subseteq E(G)$, $B \cap R = \emptyset$
Question: Can G be properly edge colored by 3 colors, such that all edges in B have the same color and all edges in R have also the same, but different color?

Theorem 25. *Let G be a connected 3-regular precolored bipartite graph and G' a graph constructed from G by Construction 26. Then the graph G' admits an $\lambda - L(p,q)$-labelling, if and only if the precoloring of the original graph G can be extended to a proper 3-edge-coloring.*

Now it is time to go through the construction of G'. The edge replacement blocks are constructed later, but it is better first to know how they are used.

Construction 26. Consider that we are given a connected 3-regular bipartite graph $G = (V^A \cup V^B, E)$ and disjoint sets of precolored edges $B, R \subseteq E(G)$. The edges of the graph G are possibly marked by b or r according to their incidence to the sets B and R. We construct a new graph G' by replacing each vertex from V^A and its adjacent edges by the result of construction:

1. Construction 28 if $\lambda < 2p$.

2. Construction 29 otherwise.

Observe that this procedure involves all edges of G, as the graph is bipartite. We want both of the edge replacement blocks to behave the same. The next lemma holds for both blocks.

Lemma 27. *Any $\lambda - L(p,q)$-labelling of the block created by Construction 29 or 28 satisfies the following rules:*

1. *The vertex v_m has a label from the set L_0 or $L_{\lambda-q+1}$.*

2. *One vertex $v_i, i \in \{1,2,3\}$ has a label from the set $L_{\lambda-q+1}$*

3. *One vertex $v_i, i \in \{1,2,3\}$ has a label from the set $L_{\lambda-2q+1}$*

4. *One vertex $v_i, i \in \{1,2,3\}$ has a label from the set $L \setminus (L_0 \cup L_q \cup L_{\lambda-q+1} \cup L_{\lambda-2q+1})$*

5. *If there is a vertex v_b, then it has a label from the set $L_{\lambda-q+1}$*

6. *If there is a vertex v_r, then it has a label from the set $L_{\lambda-2q+1}$*

All but the first rule assumes, that the vertex v_m is labelled from the set L_0.

In the following constructions we mean by "add an edge to the vertex v", that we add a new vertex and connect this new vertex and the vertex v by an edge.

We start with the more simple construction for the inequality $\lambda < 2p$.

Construction 28 (Edge replacement block - simple)**.** Given a vertex v_m and its three edges e_1, e_2, e_3, the vertex v_m is kept and each of the edges is replaced by a path of length 2. This paths are denoted by p_1, p_2, and p_3. The middle vertices of the paths are denoted by v_1, v_2, and v_3. In addition, the path representing an edge marked by b (or r) is also denoted by p_b (or p_r) and the corresponding vertex by v_b (or v_r).

There are $\lceil \frac{\lambda+1-p}{q} \rceil - 4$ edges added to each of the vertices v_1, v_2, and v_3.

Another two edges are added to an vertex v_b, if there is such a vertex.

If there is a vertex v_r, then a path of length 2 is added to this vertex and the other end of the path is denoted by v_s. The degree of the vertex v_s is risen up to $\lceil \frac{\lambda+1-p}{q} \rceil$ by adding edges to it.

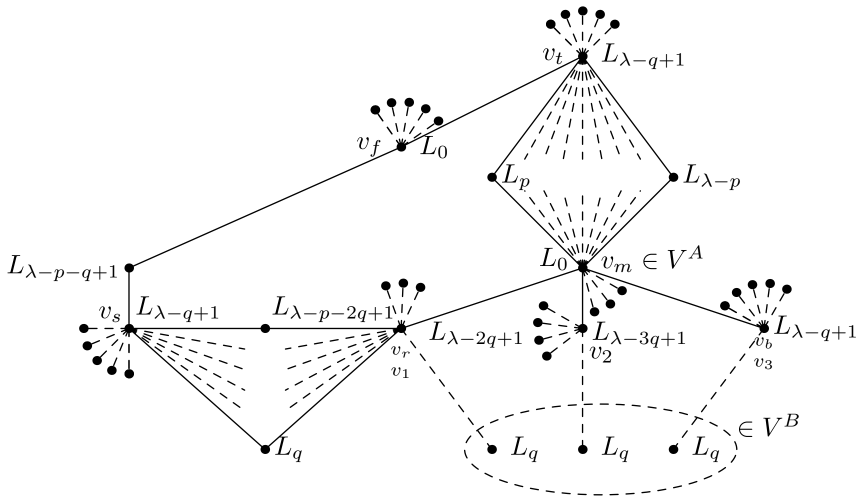

Figure 7.1: Edge replacement block - complex

The edge replacement block for the inequality $\lambda \geq 2p$ is depicted on Fig. 7.1.

Construction 29 (Edge replacement block - complex)**.** Given a vertex v_m and its three edges e_1, e_2, e_3, the vertex v_m is kept and each of the three edges is replaced by a path of length 2. The paths are denoted by p_1, p_2, and p_3. The middle vertices of the paths are denoted by v_1, v_2, and v_3. In addition, the path representing an edge marked by b (or r) is also denoted by p_b (or p_r) and the corresponding vertex by v_b (or v_r).

A new vertex denoted by v_t is added. The vertex v_t is connected by $\lceil\frac{\lambda+1-2p)}{q}\rceil - 2$ paths of length 2 with the vertex v_m. If $p \geq 3p$, then a new similar path is added.

There are $\lceil\frac{\lambda+1-p}{q}\rceil - 2$ edges added to the vertex v_b, if such vertex exists.

There can be also an edge marked by r. In this case, a new vertex v_s is introduced. The vertices v_s and v_t are connected by a path of length 3. The vertex next to v_t on that path is denoted by v_f. There are $\lceil\frac{\lambda+1-p}{q}\rceil - 1$ paths of length 2 added between vertices v_s and v_r.

Finally, the degree of each vertex $v_i, i \in \{1, 2, 3\}$ is risen to $\lceil\frac{\lambda+1-p}{q}\rceil - 2$, and also the degree of vertices v_m, v_t, v_s, and v_f is risen to $\lceil\frac{\lambda+1-p}{q}\rceil$.

Both constructions ensures that each edge of the graph G is represented by a unique path of length 2. Also in both cases the resulting graph G' is still bipartite, but in neither of them it is regular any more.

Before we prove the lemma, we finish the proof of the main theorem of this chapter.

Proof of Theorem 25. The graph G' is constructed from the graph G by Construction 26.

First, assume that an 3-edge-coloring of the graph G is given and that we want to $\lambda - L(p, q)$-label the graph G'. The graph is divided to the Edge replacement blocks. That is one block per vertex in the set V^A. The only vertices belonging to multiple block are the vertices from the set V^B. All this vertices receive label q. Otherwise, each block is labelled independently. The vertices of Edge replacement blocks are labelled as depicted on Fig. 7.1. The first label from each set is used.

A path representing an edge colored by color 1 is labelled by labels of the path containing the vertex v_1. The same holds for colors 2 and 3. The labels don't conflict, since the graph G is properly 3-edge colored.

Now we assume that a $\lambda - L(p,q)$-labelling of the graph G' is given and we want to 3-edge-color the graph G. By the Lemma 27, we know that each vertex representing an edge is labelled by a label from one of three disjoint sets. We color the edges of G according to these sets. Each of the three sets corresponds to one of the three colors. The lemma also ensures that the colors of edges around a vertex from V^A are three different colors.

We now show that edges around vertices from V^B are properly colored. To be able to do that, we have to look back at the labels of vertices representing to those edges. The vertices have labels from the sets $L_{\lambda-q+1}, L_{\lambda-2q+1}, L \setminus (L_0 \cup L_q \cup L_{\lambda-q+1} \cup L_{\lambda-2q+1})$. The difference between the smallest and the biggest labels in the sets $L_{\lambda-q+1}$ and $L_{\lambda-2q+1}$ is strictly smalled than q. So there cannot be two vertices labelled by that sets adjacent to the same vertex. In other words, there are two colors of edges, such that neither of them can be used around one vertex from V^B twice. The last color can be used multiple times around a single vertex, but that is not a problem, since the colors are used evenly on the vertices from V^A. As there is only a single color which is possible to use multiple times, it means that they have to be used evenly also on the vertices from V^B.

At last, we have to assign correct colors to the edges, in order to respect the precoloring. It is not a problem, as the vertices representing edges precolored by one color have a label from a single set among the three possible sets. □

To simplify the next proofs a bit, we define the notion of friends.

Definition 30 (Friend of a label)**.** Friend of a label x is such label y that $|x-y| \geq p$ holds. In other words, x and y can be assigned to neighbouring vertices.

The "number of friends" of a label x is the maximum number of labels, which can be used together (satisfying the q condition) in the neighbourhood of a vertex labelled by label x.

We need a lemma similar to Lemma 11 from the previous chapter.

Lemma 31. *Any vertex of degree $\lceil \frac{\lambda+1-p}{q} \rceil$ can be $\lambda - L(p,q)$-labelled only by a label from the set L_0 or $L_{\lambda-q+1}$.*

Note that according to our choice of p, q and λ we have $\lceil \frac{\lambda+1-p}{q} \rceil \geq 4$.

Proof. First, let the vertex be labelled by the label 0. Then the size of the set of labels allowed for neighbours would be

$$\lambda + 1 - p = p + 3q + a + 1 - p = 3q + a + 1$$

and the set is consecutive. The number of friends possible in such set is

$$\left\lceil \frac{\lambda+1-p}{q} \right\rceil = \left\lceil \frac{3q+a+1}{q} \right\rceil =$$
$$3 + \left\lfloor \frac{a}{q} \right\rfloor + \left\lceil \frac{(a \bmod q)+1}{q} \right\rceil = 4 + \left\lfloor \frac{a}{q} \right\rfloor \geq 4.$$

The first label not in the set L_0, is the label q. When the vertex is labelled by such label, then the set of labels allowed for neighbours is still consecutive. The size of the set is $\lambda + 1 - p - q$, but the number of friends is just $\lceil\frac{\lambda+1-p}{q}\rceil - 1$.

The first label, where the set of labels allowed for neighbours is non consecutive is the label p. When the vertex is labeled by the label p, then the size of the set is

$$\begin{aligned}\lambda + 1 - 2(p-1) - 1 = p + 3q + a + 2 - 2p = \\ 3q + a + 2 - (q+d) = 2q + a + 2 - d.\end{aligned}$$

To count the number of friends, we first assume that the set is consecutive. Then the number of friends would be

$$\left\lceil \frac{2q+a+2-d}{q} \right\rceil = 2 + \left\lfloor \frac{a}{q} \right\rfloor + \left\lceil \frac{(a \bmod q) - d + 2}{q} \right\rceil .$$

The term $(a \bmod q) - d + 2$ is at most 1 by the inequality $d > (a \bmod q) \geq 0$ from the current setting. We know, that the set is split into two parts, one with labels smaller than the label of the vertex and the other with bigger labels than the label. The maximum number of friends is, when one of the sets has just one label (or $kq + 1$, for any suitable k) and the other set the rest of labels. The one label brings in one friend, and the rest $2 + \left\lfloor \frac{a}{q} \right\rfloor$ friends. Having more than one label in the small set is wasting of labels, as the labels are too close to the first one. So the number of friend is at most $\lceil\frac{\lambda+1-p}{q}\rceil - 1$.

As before, by the Lemma 7, the same can be done from the other side and prove the same for the set $L_{\lambda-q+1}$. □

Proof of Lemma 27. We start with Construction 28. Before we prove anything about the construction, we need to prove a simple fact of the setting when this construction is used.

It is used only when $\lambda < 2p$. This provides us one more useful inequality:

$$\begin{aligned} 2p > \lambda &\geq p + 3q \\ 2p &> p + 3q \\ p &> 3q \end{aligned}$$

The vertex v_m is of degree $\lceil\frac{\lambda+1-p}{q}\rceil$, thus it can be labelled only by a label from the set L_0 or $L_{\lambda-q+1}$ by Lemma 31. This is the first rule. From now on, we assume, that the vertex v_m is labelled by a label from the set L_0, since the lemma says so.

If there is a vertex v_b, then it is of the same degree as vertex v_m. So, it also can be labelled only by a label from th set L_0 or $L_{\lambda-q+1}$ by the same lemma. This is the rule 5.

Now we focus on the vertex v_r, if the exist such a vertex. As $\lambda < 2p$ we employ Lemma 9. Let the vertex v_m be in the partition A. Then the vertex v_s is in the partition B. The vertex v_m is labelled by a label from the set L_0, thus the vertex v_s has to be labelled by a label from the set $L_{\lambda-q+1}$ by Lemma 31. The vertex v_r cannot be labelled by a label from the set $L_{\lambda-q+1}$, because of the label of the vertex v_s.

The vertex v_r has degree $\lceil \frac{\lambda+1-p}{q} \rceil - 1$ and $p > 3q$. The second inequality ensures, that $(L_0 \cup L_q \cup L_{2q}) \cap L_p = \emptyset$. Also by the inequality $\lambda < 2p$, the set of labels allowed for neighbours is consecutive. In particular for a vertex labelled by a label at least p, the size of the set would be $\lambda + 1 - 2p \leq 1$.

The number of friends of the vertex labelled by a label from the set L_0 is at most $\lceil \frac{\lambda+1-p}{q} \rceil$, by Lemma 31. The number of friends of the vertex labelled by a label from the set L_q is at most $\lceil \frac{\lambda+1-p}{q} \rceil - 1$, as the set of labels allowed for neighbours is consecutive and smaller by at least q. The same argument still holds for the set L_{2q} and the number of friends is $\lceil \frac{\lambda+1-p}{q} \rceil - 2$. If the vertex would be labelled by a label from any other set, then the size of the set of labels allowed for neighbours would be at most $\lambda + 1 - p - 3q$, thus the number of friends can be at most $\lceil \frac{\lambda+1-p}{q} \rceil - 3$.

All the above also holds for the sets $L_{\lambda-q+1}$, $L_{\lambda-2q+1}$ and $L_{\lambda-3q+1}$, by Lemma 7. So the vertex v_r can be labelled only by a label from the set $L_{\lambda-q+1}$ as its degree is $\lceil \frac{\lambda+1-p}{q} \rceil - 1$.

The last thing to check is, that the vertices $v_i, i \in \{1, 2, 3\}$ have to be labelled by one label from each of the sets $L_{\lambda-q+1}$, $L_{\lambda-2q+1}$, and $L \setminus (L_0 \cup L_q \cup L_{\lambda-q+1} \cup L_{\lambda-2q+1})$. As the vertices are of degree at least $\lceil \frac{\lambda+1-p}{q} \rceil - 2$, we have just proven this.

Now we begin with Construction 29.

The vertices v_m, v_t, v_s, v_f, and v_b are of degree $\lceil \frac{\lambda+1-p}{q} \rceil$, thus each of them can be labelled only by a label from the set L_0 or $L_{\lambda-q+1}$ by Lemma 31. So the first rule is proven. Also now we are assuming, that the vertex v_m is labelled by a label from the set L_0 in the rest of this prove.

If there exists a vertex v_b, it is a neighbour of the vertex v_m, so the only possible set of labels for this vertex is $L_{\lambda-q+1}$. This proves the rule 5.

The vertices v_m and v_t have at least one common neighbour and the sets L_0 and $L_{\lambda-q+1}$ both have range strictly smalled than q, the vertex v_t can be labelled only from the set $L_{\lambda-q+1}$. The common neighbour of vertices v_m and v_t is assured by the inequality $\lambda \geq 2p$.

Now we focus on the vertices on paths between vertices v_m and v_t. There is at most $\lceil \frac{\lambda+1-2p}{q} \rceil$ such vertices. As the vertices v_m and v_t have labels from the sets L_0 and $L_{\lambda-q+1}$, we can consider them labelled by labels 0 and λ, as there is nothing what can restrict the labels from the outer side. Then the number of possible common friends of both vertices is exactly $\lceil \frac{\lambda+1-2p}{q} \rceil$.

The most interesting part, are the vertices v_1, v_2 and v_3. None of the vertices on paths between vertices v_m and v_t can use a label from the set $L_{\lambda-q+1}$, because of the label of the vertex v_t. It is also easy to restrict the labels from the set $L_{\lambda-2q+1}$ from the vertices on that paths, as $p \geq 2q$.

If $p \geq 3q$, then the same holds for the set $L_{\lambda-3q+1}$, otherwise we have erased one of the paths between v_m and v_t, so we have one label, which is allowed to be in the neighbourhood of vertex v_m. This last label is from the set $L \setminus (L_0 \cup L_q \cup L_{\lambda-q+1} \cup L_{\lambda-2q+1})$.

The vertices $v_i, i \in \{1, 2, 3\}$ are of degree $\lceil \frac{\lambda+1-p}{q} \rceil - 2$, this ensures, that if $p \geq 4q$, then only labels from the sets $L_{\lambda-q+1}$, $L_{\lambda-2q+1}$ and $L_{\lambda-3q+1}$ can be used. This proves rules 2, 3, and 4.

The last rule in this case is the rule 6. We know, that vertices v_f and v_s have labels from the sets $L_{\lambda-q+1}$ and L_0. Vertex v_f is a neighbour of the vertex v_t, which is labeled from the set $L_{\lambda-q+1}$, so it has to be labelled from the set L_0. Vertices v_s and v_f have a common neighbour, meaning that the vertex v_s needs to have a label from the set $L_{\lambda-q+1}$.

Vertices v_r and v_s have $\lceil\frac{\lambda+1-p}{q}\rceil - 1$ common neighbours. The vertex v_s restricts the set of labels allowed to be used on the neighbours at least to $\{0, 1, \ldots, \lambda - p\}$. So the vertex v_r need to use at least label $\lambda - p - q + p$, in order to have enough friends. The label is from the set $L_{\lambda-2q+1}$, proving the rule 6 and finishing the proof. □

8. The easy cases

At last, it is time to look at the other cases of p, q and λ, where neither the inequalities from the SAT reduction chapter nor the inequalities from the Edge precoloring extension reduction chapter holds.

First consider the case $\lambda < p$. In this case, if a graph G admits a $\lambda-L(p,q)$-labelling, then it is a set of isolated vertices. This is just by a simple fact that an edge needs two labels at least p apart, and there are none such labels in this narrow span.

A bit more edges are allowed for $\lambda = p$. Now the maximum degree allowed for a vertex is 1. Hence, the graphs admitting a $\lambda-L(p,q)$-labelling are sets of isolated edges and sets of isolated vertices.

In the following cases we are considering only connected graphs. If the graph is disconnected, then each component can be analysed independently.

Setting $p + 2q > \lambda \geq p + q$ allows the graph to have a path of length 3. The path can be labelled by labels p, 0, $p + q$ and q. There cannot be more vertices connected to that path, since there is no more labels to use.

For the setting $p + 3q > \lambda \geq p + 2q$ and $\lambda < 2p$ it is a bit more complicated. The possible values of p are:

$$2p > \lambda > p + 2q$$
$$p > 2q$$

When $\lambda < 2p$ holds, then the graph is bipartite. The inequality $p + 3q > \lambda$ allows the maximal degree to be 3. The inequality $p > 2q$ ensures, that any vertex labelled by a label from the set L_q has degree at most 2. Also any vertex labelled by label $2q$ can be of degree at most 1. As there is no label, which can have a consecutive set of labels allowed for friends, a vertex labelled by any label from between $2q$ and $\lambda - 2q$ can be of degree at most 1.

How does looks like the labels of neighbours of a vertex labelled by a label from the set L_0. There is one vertex labelled by a label from the set L_p, such a vertex can have degree up to 1. Then, there is one vertex labelled by a label from the set L_{p+q}, this vertex can have degree up to 2. Finally the last vertex is labelled by a label from the set L_{p+2q}, which is part of the set $L_{\lambda-q+1}$, so it can have degree up to 3.

This means that if a graph G admits a $\lambda-L(p,q)$-labelling, with this setting, then the graph G is a cycle or a path with an edge connected to some of its vertices.

Observation 32. A simple greedy algorithm starting with any vertex of degree 3 (or any vertex if there is no such vertex) results in a proper $\lambda-L(p,q)$-labelling or answers that the graph does not satisfy the above conditions.

9. Conclusion

We have resolved almost fully the $\lambda - L(p,q)$-labelling problem. The only choice of p, q and λ for further studies is

$$\lambda \geq p + 3q$$
$$p < 2q.$$

We have used two fundamentally different approaches. One using the NAE-3SAT problem, the other using the edge precoloring extension problem. Even using two approaches was not enough, we had to use different constructions to suite different choices of p, q and λ.

At last, here are the constrains for p, q, and λ of the $\lambda - L(p,q)$-labelling problem, where the problem is yet to decide.

The problem is NP-complete when

$$p + 3q > \lambda \geq p + 2q$$
$$\lambda \geq 2p$$

or

$$\lambda \geq p + 3q$$
$$p \geq 2q.$$

It is polynomially solvable when

$$p + 3q > \lambda \geq p + 2q$$
$$\lambda < 2p$$

or

$$\lambda < p + 2q$$
$$p > q \geq 1.$$

The complexity of the $\lambda - L(p,q)$-labelling problem for small values of p and q is illustrated in the following table.

	$\lambda - L(p,q)$-labelling									
	p	q	p	q	p	q	p	q	p	q
	2	1	3	1	3	2	5	2	5	3
SAT	$\lambda = 4$				$7 \leq \lambda \leq 8$		$\lambda = 10$		$11 \leq \lambda \leq 14$	
EPE	$5 \leq \lambda$		$6 \leq \lambda$				$11 \leq \lambda$			
Poly	$\lambda \leq 3$		$\lambda \leq 5$		$\lambda \leq 6$		$\lambda \leq 9$		$\lambda \leq 11$	

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